ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1998
Seria: BUDOWNICTWO z. 85 Nr kol. 1404

Jerzy SKRZYPCZYK

# ON EXISTENCE OF SOLUTIONS OF INTERVAL FREDHOLM INTEGRAL EQUATIONS WITH DEGENERATE KERNELS

**Summary.** In the paper basic concepts of the structure of the solutions to the interval Fredholm integral equations are considered, where a free term is taken to be an interval square-integrable function and non-interval kernel is degenerate and square-integrable in [a,b]x[a,b]. At first, the existence of the exact set-valued solution is investigated. In addition the hull of the solution set is obtained. For including a set of solutions of the interval integral equation we apply interval calculus. At the end the theory is illustrated by a simple analytical example.

# O ISTNIENIU ROZWIĄZAŃ LINIOWYCH PRZEDZIAŁOWYCH RÓWNAŃ CAŁKOWYCH FREDHOLMA O JĄDRACH SPECJALNYCH

**Streszczenie.** W pracy badane są istnienie i struktura zbioru rozwiązań przedziałowego równania całkowego Fredholma II rodzaju z niejednorodnością, która jest funkcją przedziałową, całkowalną z kwadratem, natomiast jądro równania całkowego jest jądrem specjalnym, całkowalnym z kwadratem na zbiorze [a,b]x[a,b]. W pierwszej kolejności badane jest zagadnienie dokładnego zbioru rozwiązań, a następnie problem wyznaczenia najmniejszego zbioru przedziałowego, zawierającego dokładny zbiór rozwiązań. W celu wyznaczenia tej aproksymacji zastosowano analizę przedziałową. Teoria zilustrowana jest prostym przykładem analitycznym.

## 1. Introduction

The theory of Fredholm integral equations is very well developed and has a great bibliography [11,12,18,19,25]. Such equations play a great role in investigation of many technical problems in which boundary problems are of the greatest importance, cf. [4]. In recent years the analysis for problems of integral equations with random parameters has also been discussed intensively, cf. [9,22,24]. However, in many particular cases probabilistic analysis is very difficult from mathematical point of view, as well as sometimes the necessary knowledge about probabilistic characteristic of parameters is very poor. On the over hand, especially in engineering sciences, because of manufacturing errors, values of structural parameters of many materials are uncertain and the character of that uncertainty is interval, i.e., unknown and bounded.

If the structural parameters are interval the equations describing the system become of interval character. Following, the corresponding solutions are of set character. Since detailed calculations of the shape of that set are very difficult, approximate methods are needed.

For engineers it is often sufficient to estimate the upper and lower bounds of the solutions of system equations under considerations i.e. to find upper and inner interval approximations of solution sets respectively. We are interested in real solutions only, but the complex case is handled by the method used, since it is more appropriate from mathematical point of view.

Interval integral equations were investigating earlier in some papers [7,8,10,13,21].

Section 2 is devoted to notations and terminology and in Section 3 we discuss the existence of solutions of interval Fredholm integral equations of the second kind with degenerate kernels, as well as the method of investigating exact and approximate solutions. In section 4 we study the simple example which can be solved analytically. Presented methods give new results to study uncertain boundary problems with interval or fuzzy parameters, which are very important in an engineering practice, cf. [5,6,22,23]

## 2. Elementary Concepts and Results

In this report, the following concepts and notations will be used. $R^n$ $\left(R^{n\times m}\right)$ was reserved for the set of n-dimensional vectors (nxm matrices), $R$ the set of reals.

Let the symbols $P(R^n)$, $P(R^{n\times m})$ denote the power sets of $R^n$ and $R^{n\times m}$ respectively. Let $I(R)$ denote the set of all closed bounded intervals $\bar{z}=\left[z^-,z^+\right]$ on the real line R, where $z^-$ and $z^+$ denote the end points of $\bar{z}$. We call further elements of $I(R)$ interval numbers. In the similar way we introduce $I(R^n)$ - the space of interval vectors, and $I\left(R^{n\times m}\right)$ - the space of interval matrices.

The elementary operations on elements from $I(R^n)$ and $I\left(R^{n\times m}\right)$ are described in monographs [1,3, 15-17,20].

The complex plane will be denoted by C, complex vectors by $C^n$ and complex matrices as $C^{n\times m}$. Similarly the symbols $P(C), P(C^n), P(C^{n\times m})$ denote the power sets of C, $C^n$ and $C^{n\times m}$ respectively.

The symbols $P_c(R^n)$, $P_c(R^{n\times m})$, $P_c(C), P_c(C^n), P_c(C^{n\times m})$ denote the families of all non-empty compact convex subsets of corresponding spaces.

If $z_1, z_2 \in C$, $Re(z_1) \le Re(z_2)$ and $Im(z_1) \le Im(z_2)$ then we call $[z_1, z_2]$ a rectangular complex interval. The set of rectangular complex intervals, will be denoted by IR(C), similarly IR($C^n$) - rectangular complex vectors, IR($C^{n\times m}$) - rectangular complex matrices. The operations on elements from IR(C), IR($C^n$) and IR($C^{n\times m}$) are described in [1,14].

Rectangular values are called further simply interval complex values and will be denoted as I(C), I($C^n$) and I($C^{n\times m}$) respectively.

Let T be one of the sets: $R^n$, $R^{n\times m}$, $C^n$, $C^{n\times m}$. The operations in the power set $P(T)$ are as usually defined by

$$A*B := \{a*b | a \in A, b \in B\}, \quad A, B \in P(T), \quad * \in \{+,-,\cdot,/\} \tag{1}$$

with well known restrictions for "/". Naturally for matrices the symbols denote corresponding matrix operations [1,3,16,17].

Further informations on interval analysis the reader can be found in papers [1,3, 15-17,20]. Interval numbers are naturally a special kind of fuzzy numbers.

For interval numbers $\bar{a}, \bar{b} \in I(R)$, we obtain from eq. (1):

(A1) $\bar{a} \le \bar{b}$ iff $a^- \le b^-$ and $a^+ \le b^+$;

(A2) $\bar{a} * \bar{b} := \{a*b : a \in \bar{a}, b \in \bar{b}, "*" = "+,-,\cdot,/"\}$;

where (A2) follows from (**) and gives a general method to determine obvious algebraic operations on interval numbers which results in the following formulas:

(B1) $\bar{a}+\bar{b}=\bar{c}$ iff $c^-=a^-+b^-$ and $c^+=a^++b^+$;

(B2) $\bar{a}-\bar{b}=\bar{c}$ iff $c^-=a^-+b^+$ and $c^+=a^+-b^-$;

(B3)

$\bar{a}\cdot\bar{b}=\bar{c}$ iff $c^-=\min(a^-b^-,a^-b^+,a^+b^-,a^+b^+)$ and $c^+=\max(a^-b^-,a^-b^+,a^+b^-,a^+b^+)$;

(B4) $1/\bar{a}=\bar{c}$ iff $c^-=1/a^+$ and $c^+=1/a^-$ whenever $0\notin[a^-,a^+]$;

(B5) $\bar{a}/\bar{b}=\bar{c}$ iff $\bar{c}=\bar{a}\cdot(1/\bar{b})$ whenever $0\notin[b^-,b^+]$.

Recall that the value

(B6) $d(\bar{a},\bar{b}):=|a^--b^-|\vee|a^+-b^+|$ is called the distance between interval numbers $\bar{a}$ and $\bar{b}$. It is easy to see that, if $\bar{a}=a$ and $\bar{b}=b$ are real numbers, then $d(\bar{a},\bar{b})=|a-b|$. For further information see refs. [1,13,14,16,17,20]. It is the Hausdorff metric specified for I(R).

Recall that the Hausdorff metric is defined as

$$H(A,B):=\inf\{\varepsilon: A\subseteq N(B,\varepsilon), B\subseteq N(A,\varepsilon)\}, \tag{2}$$

where

$$A,B\in P_c(T) \text{ and } N(A,\varepsilon)=\{x\in T: \|x-y\|<\varepsilon \text{ for some } y\in A\} \tag{3}$$

We have: $H(A+C,B+C)=H(A,B),\quad \forall\, A,B,C\in P_c(T)$, $\lambda N(A,\varepsilon)=N(\lambda A,|\lambda|\varepsilon)\quad \forall\,\lambda\in C^1$.

Denote by C(J;I(T)) the set of all continuous mappings from J to I(T), where J⊆R. We metricize C(J;I(T)) by setting

$$D_1(\bar{x}(t),\bar{y}(t)):=\sup_{t\in J} d(\bar{x}(t),\bar{y}(t)), \tag{4}$$

where $\bar{x}(t),\bar{y}(t)\in C(J;I(T))$ and d is the Hausdorff metric defined in $I(T)$. Since (I(T),d) is a complete metric space, a standard procedure is applied to show that C(J;I(T)) is complete too, cf. [13].

Further we denote by $L^2$(J;I(T)) the set of all measurable interval-valued mappings $\bar{x}(\cdot)$ from J to I(T), where J⊆R and $x^-(.),x^+(.)\in L^2(J;T)$. We metricize $L^2$(J;I(T)) by setting

$$D_2(\bar{x}(t),\bar{y}(t)):=\left(\int_J d^2(\bar{x}(t),\bar{y}(t))dt\right)^{1/2}, \tag{5}$$

where $\bar{x}(t),\bar{y}(t)\in L^2(J;I(T))$. Similar arguments as above apply to show that also $L^2$(J;I(T)) is complete.

An interval-valued real (complex) function is a special closed-valued set valued function $\bar{f}: R \to I(T)$. It is usually written as $\bar{f}(x) = \left[f^-(x), f^+(x)\right]$, where

$$f^-(x) = \inf \bar{f}(x), \quad f^+(x) = \sup \bar{f}(x). \tag{6}$$

Then $\bar{f}$ is measurable iff $f^-$ and $f^+$ are measurable [2,7,13].

### 3. Interval Fredholm Integral Equations with Degenerate Kernels

A linear interval Fredholm integral equation of the second kind with the kernel $K(\cdot,\cdot)$ and the interval-type free term $\bar{g}(\cdot)$ is defined as the family of linear non-homogeneous Fredholm integral equations

$$f(x) = g(x) + \lambda \int_a^b K(x,y)f(y)dy, \quad g(x) \in \bar{g}(x), \lambda \in C^1, \quad a \le x \le b \tag{7}$$

Thus we consider a system of eqs. (7) in which the functions take unknown values ranging in certain intervals. Consider the non-homogeneous interval Fredholm integral equation of the second kind with degenerate non-interval kernel

$$K(x,y) = \sum_{j=1}^{r} u_j(x)\overline{w_j(y)}, \quad a \le x \le b, a \le y \le b \tag{8}$$

We are interested in investigating of the exact solution set of eq. (7), given by

$$\begin{aligned} \Sigma(K,\bar{g})(x) := \Big\{ f(x) : f(x) = g(x) + \lambda \sum_{j=1}^{r} \int_a^b u_j(x)\overline{w_j(y)}f(y)dy, \\ f(\cdot) \in L^2(a,b;I(C)), g(\cdot) \in \bar{g}(\cdot) \Big\}, \quad a \le x \le b \end{aligned} \tag{9}$$

We show that the solution set $\Sigma(K,\bar{g})(x), a \le x \le b$ is not an interval complex function, and need not even be convex; in general, $\Sigma(K,\bar{g})(\cdot)$ has a very complicated structure.

Further a linear interval Fredholm integral equation of the second kind is written in the form

$$\bar{f}(x) = \bar{g}(x) + \lambda \int_a^b K(x,y)\bar{f}(y)dy, \quad a \le x \le b \tag{10}$$

Denote $s_i = \int_a^b f(y)\overline{w_i(y)}dy = \langle f, w_i \rangle$, i=1,2,...,r. Thus

$$\begin{aligned} \Sigma(K,\bar{g})(x) := \Big\{ f(x) : f(x) = g(x) + \lambda \sum_{j=1}^{r} s_j u_j(x), \\ f(\cdot) \in L^2(a,b;I(R)), g(\cdot) \in \bar{g}(\cdot) \Big\}, \quad a \le x \le b \end{aligned} \tag{11}$$

If we multiply both sides of the integral equation $f(x)=g(x)+\lambda\sum_{j=1}^{r}s_j u_j(x)$, by $w_i$ and integrate from a to b, we obtain $s_j$ on the left side. If we define integrals $t_i=\langle g(\cdot),w_i\rangle$ then the solution set (11) becomes

$$\Sigma(K,\bar{g})(x):=\left\{f(x):f(x)=g(x)+\lambda\sum_{j=1}^{r}s_j u_j(x),s_j\in\Sigma(C,\bar{t}),g(\cdot)\in\bar{g}(\cdot)\right\},\quad a\le x\le b \quad (12)$$

where

$$\Sigma(\mathbf{C},\bar{\mathbf{t}}):=\left\{\mathbf{s}:\mathbf{s}=\mathbf{t}+\lambda\mathbf{C}\mathbf{s},\mathbf{t}\in\bar{\mathbf{t}},t_i=\langle g(\cdot),w_i\rangle,c_{ij}=\langle u_j,w_i\rangle\right\} \quad (13)$$

is the solution set of linear interval complex-valued equation with coefficient matrix **C** and interval free term **t**.

So the solution to interval Fredholm equation of the second kind (7) with degenerate kernel reduces to solving for $s_i$ from the system of the r linear interval equations (13), since $s_i$ will then be used in the series (12) to obtain $\Sigma(K,\bar{g})(\cdot)$, the solution set of (7).

Denote $\square\Sigma(\mathbf{C},\bar{\mathbf{t}}):=\left[\inf(\Sigma(\mathbf{C},\bar{\mathbf{t}})),\sup(\Sigma(\mathbf{C},\bar{\mathbf{t}}))\right]$ the hull of $\Sigma(\mathbf{C},\bar{\mathbf{t}})$ i.e. the tightest interval set enclosing $\Sigma(\mathbf{C},\bar{\mathbf{t}})$. Then

$$\begin{aligned}\square\Sigma(K,\bar{g})(x)&:=\left\{f(x):f(x)=g(x)+\lambda\sum_{j=1}^{r}s_j u_j(x),s_j\in\bar{s}_j^*,\bar{s}^*=\square\Sigma(\mathbf{C},\bar{\mathbf{t}}),g(\cdot)\in\bar{g}(\cdot)\right\}=\\&=\bar{g}(x)+\lambda\sum_{j=1}^{r}\bar{s}_j^* u_j(x),,\quad a\le x\le b\end{aligned} \quad (14)$$

Naturally $\Sigma(K,\bar{g})(x)\subseteq\square\Sigma(K,\bar{g})(x),\quad\forall\, a\le x\le b$.

## 4. Example

Solve the interval Fredholm integral equation of the second kind

$$\bar{f}(x)=[0,1]x+\lambda\int_0^1\left(xy^2+x^2y\right)\bar{f}(y)dy,\quad 0\le x\le 1 \quad (15)$$

This interval Fredholm integral equation has a degenerate kernel of the form (8),

$$K(x,y)=\sum_{j=1}^{r}u_j(x)\overline{w_j(y)}=xy^2+x^2y,\quad 0\le x,y\le 1, \quad (16)$$

where $u_1(x)=x,u_2(x)=x^2,w_1(y)=y^2,w_2(y)=y$. In order to obtain values of $\Sigma$ and $\square\Sigma$ we must calculate $t_i$, $c_{ij}$ for i,j=1,2 from eqs. (13). We obtain

$$\bar{f}_1(x)=\left[0,\frac{1}{4}\right],\bar{f}_2(x)=\left[0,\frac{1}{3}\right],c_{11}=\frac{\lambda}{4},c_{11}=\frac{\lambda}{4},c_{11}=\frac{\lambda}{4},c_{11}=\frac{\lambda}{4}, \quad (17)$$

Hence the system of interval equations $\bar{s} = \bar{t} + C\bar{s}$ becomes

$$\begin{bmatrix} \bar{s}_1 \\ \bar{s}_2 \end{bmatrix} = \begin{bmatrix} \left[0, \frac{1}{4}\right] \\ \left[0, \frac{1}{3}\right] \end{bmatrix} + \lambda \begin{bmatrix} \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \bar{s}_1 \\ \bar{s}_2 \end{bmatrix} \tag{18}$$

We can transfer this equation to more compact form

$$\begin{bmatrix} 1-\frac{\lambda}{4} & \frac{\lambda}{5} \\ -\frac{\lambda}{3} & 1-\frac{\lambda}{4} \end{bmatrix} \begin{bmatrix} \bar{s}_1 \\ \bar{s}_2 \end{bmatrix} = \begin{bmatrix} \left[0, \frac{1}{4}\right] \\ \left[0, \frac{1}{3}\right] \end{bmatrix} \tag{19}$$

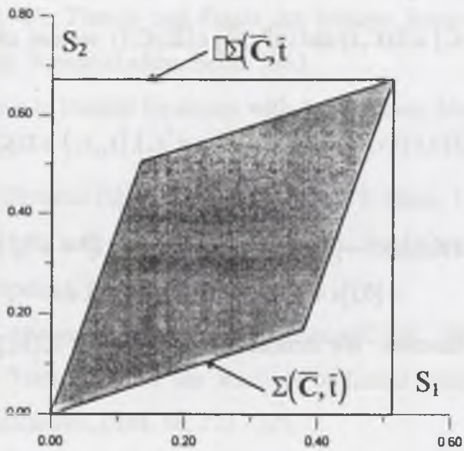


Fig. 1. Exact solution set and its approximation interval hull
Rys. 1. Dokładny zbiór rozwiązań i aproksymujące rozwiązanie przedziałowe

As we have only two equations

$$\begin{aligned} \left(1-\frac{\lambda}{4}\right)\bar{s}_1 - \frac{\lambda}{5}\bar{s}_2 &= \left[0, \frac{1}{4}\right] \\ -\frac{\lambda}{3}\bar{s}_1 + \left(1-\frac{\lambda}{4}\right)\bar{s}_2 &= \left[0, \frac{1}{3}\right] \end{aligned} \tag{20}$$

we can solve them immediately to obtain the exact shape of the solution set $\Sigma(\mathbf{C}, \bar{\mathbf{t}})$, see Fig.1. Since the inverse matrix of the system (13) is easy to calculate we can obtain the hull of solution set in the analytical form. The eigenvalues of the system (13) take values

$$\lambda_1 = -60 - 16\sqrt{15} = -121.9677335, \quad \lambda_2 = -60 + 16\sqrt{15} = 1.967733539 \tag{21}$$

If $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$ then for $\lambda = 4$ we obtain $\Sigma(\mathbf{C}, \bar{\mathbf{t}}) = \square\Sigma(\mathbf{C}, \bar{\mathbf{t}})$

$$\bar{s}_1^H = \left[-\frac{1}{4}, 0\right], \quad \bar{s}_2^H = \left[-\frac{5}{16}, 0\right] \tag{22}$$

and for $\lambda \neq 4$ we have respectively

$$\begin{aligned} \bar{s}_1^H &= \frac{4(60-15\lambda)\left[0, \frac{1}{4}\right] + 48\lambda\left[0, \frac{1}{3}\right]}{240 - 120\lambda - \lambda^2} \\ \bar{s}_2^H &= \frac{80\lambda\left[0, \frac{1}{4}\right] + 60(4-\lambda)\left[0, \frac{1}{3}\right]}{240 - 120\lambda - \lambda^2} \end{aligned} \tag{23}$$

Naturally $\square\Sigma(\mathbf{C}, \bar{\mathbf{t}}) = \left[\bar{s}_1^H, \bar{s}_2^H\right]^T$. The corresponding domains we can compare on Fig. 1.

Once we know $(s_1, s_2) \in \Sigma(\mathbf{C}, \bar{\mathbf{t}})$ and $(\bar{s}_1^H, \bar{s}_2^H) \in \square\Sigma(\mathbf{C}, \bar{\mathbf{t}})$ we can obtain the exact shape of the solution set

$$\Sigma(K, \bar{g})(x) := \left\{f(x) : f(x) = [0,1]x + \lambda\left(xs_1 + x^2 s_2\right), (s_1, s_2) \in \Sigma(\mathbf{C}, \bar{\mathbf{t}})\right\}, \quad 0 \leq x \leq 1 \tag{24}$$

and similarly

$$\begin{aligned} \square\Sigma(K, \bar{g})(x) &= \left\{f(x) : f(x) = [0,1]x + \lambda\left(x\bar{s}_1^H + x^2\bar{s}_2^H\right)\right\} = \\ &= [0,1]x + \lambda\left(x\bar{s}_1^H + x^2\bar{s}_2^H\right), \quad 0 \leq x \leq 1 \end{aligned} \tag{25}$$

which is the interval function. We remember naturally that $\Sigma(K, \bar{g})(x) \subseteq \square\Sigma(K, \bar{g})(x)$ for $\forall\, 0 \leq x \leq 1$.

## REFERENCES


1. Alefeld G. & Herzberger, J.: Introduction to Interval Computations, Academic Press, New York, 1983.
2. Aumann R.J.: Integrals of Set-Valued Functions, J.Math.Anal.Appl., 1965, 12, 1-12.
3. Bauch H. & Jahn K.U. & Oelschlagel D. & Süsse H. & Wiebigke V.: Interval-mathematik, BSG B.G. Teubner Verlagsgeselschaft, Berlin, 1987.
4. Burczyński T.: Boundary Element Method, WN-T, Warszawa, 1995. (in polish)
5. Burczyński T. & Skrzypczyk J.: The Fuzzy Boundary Element Method: A New Methodology, Sci. Fasc. of Silesian Tech. Univ., ser. Civil Eng., 1996, 83, Gliwice, 25-42

6. Burczyński T. & Skrzypczyk J.: Fuzzy Aspects of the Boundary Element Method, Engineering Analysis with Boundary Elements, 1997, 19, 209-216

7. Caprani O., Madsen K., Rall L. B.: Integration of Interval Functions, SIAM J. Math. Anal.. 1981, 12, 321 - 341

8. Caprani O., Madsen K., Stauning O.: Enclosing Solutions of Integral Equations, Extended Abstracts, APIC'95, El Paso, A Supplement to the Reliable Computing

9. Corduneanu C.: Integral Equations and Stability of Feedback Systems, Academic Press, New york and London, 1973

10. Dobner H. J.: Einschließungsmethoden für Fredholmsche Integralgleichungen, ZAMM. 1990, 70, 592 - 594

11. Fenyö S., Stolle H. W.: Theorie und Praxis der linearen Integralgleichungen 2, VEB Deutscher Verlag der Wissenschaften, Berlin 1983

12. Jerri A. J.: Introduction to Integral Equations with Applications, Marcel Dekker, Inc., New York and Basel 1985

13. Kaleva O.: Fuzzy Differential Equations, Fuzzy sets and Systems, 1987, 24, 301 - 317.

14. Mayer G.: Algebraische und metrische Strukturen in der Intervalrechnung und einige Anwendungen, Computing, 1970, 5, 144 - 162

15. Moore R.E.: Interval Analysis, Prentice Hall, Englewood Cliffs, 1966.

16. Neumaier A.: New Techniques for the Analysis of Linear Interval Equations, Linear Algebra and Its Applications, 1984, 58, 273 - 325.

17. Neumaier A.: Interval Methods for Systems of Equations, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne, Sydney, 1990.

18. Piskorek A.: Integral Equations, WN-T, Warszawa, 1997. (in polish)

19. Pogorzelski W.: Integral Equations and Their Applications, PWN, Warszawa, 1970.

20. Rohn J.: Systems of Linear Interval Equations, Linear Algebra and Its Applications, 1989, 126, 39 - 78

21. Seikkala S.: On the Fuzzy Initial Value Problem, Fuzzy sets and Systems, 1987, 24, 319 - 330

22. Skrzypczyk J.: Dynamics of Complex Mechanical Systems with Random Parameters, Zesz. Nauk. Pol. Śl. Seria Mechanika 120, Gliwice 1994 (in polish)

23. Skrzypczyk J., Burczynski T.: Fuzzy Boundary Element Method in the Analysis of Uncertain Systems, J. of Theoretical and Appl. Mechanics, 1998, 36, 493-512

24. Tsokos c. P., Padgett W., J.: Random Integral Equations with Applications to Stochastic Systems, Springer-Verlag, Berlin Heidelberg new York 1971

25. Widom H.: Lectures on Integral Equations, Van Nostrand - Reinhold Company, New York, Toronto, London, Melbourne 1969

Recenzent: Prof.dr hab.inż. Tadeusz Burczyński

**Abstract**

In the paper basic concepts of the structure of the solutions to the interval Fredholm integral equations

$$f(x) = g(x) + \lambda \int_a^b K(x,y)f(y)dy, \quad g(x) \in \bar{g}(x), \lambda \in C^1, \quad a \leq x \leq b$$

are considered, where g(.) is taken as an interval square-integrable function and non-interval kernel k(.,.) is degenerate and square-integrable in [a,b]x[a,b]. At first, the existence of the exact set-valued solution is investigated. In addition the hull of the solution set is obtained. For including a set of solutions of the interval integral equation we apply interval calculus. At the end the theory is illustrated by a simple analytical example.