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A NOTE ON CHARACTERISTIC POLYNOMIAL OF INTERVAL MATRIX

Summary. In the paper, we are concerned with interval - oriented methodology to model uncertainties of eigenvalues of an nxn interval real matrix. We investigate methods of calculation for characteristic polynomials of interval matrices. Presented methodology is probably the simplest way to model and to approximate vibration properties of systems with uncertain parameters.

UWAGI O WIELOMIANACH CHARAKTERYSTYCZNYCH MACIERZY PRZEDZIAŁOWYCH

Streszczenie. W pracy przedstawiono nowe pojęcia i wyniki dotyczące metodologii analizy zagadnień związanych z wartościami własnymi i wyznaczaniem współczynników wielomianu charakterystycznego macierzy rzeczywistych o współczynnikach interwałowych. Przedstawione ujęcie jest prawdopodobnie najprostszym sposobem aproksymacji w modelowaniu drgań i ich własności w systemach o parametrach niepewnych.

1. Introduction

The vibration theory for structures with deterministic parameters is very well developed and has a great bibliography. In recent years the vibration analysis for problems with random parameters has also been discussed intensively. However, in many particular cases probabilistic analysis is very difficult from mathematical point of view, as well as sometimes the necessary knowledge about probabilistic characteristic of parameters is very poor. On the over hand, because of manufacturing errors, values of structural parameters of many materials are uncertain and the character of that uncertainty is interval, i.e., unknown and bounded.

If the structural parameters are interval the system matrices become interval matrices. Following, the corresponding eigenvalues and eigenvectors are of set character. Since detailed calculations of the shape of that set are very difficult, approximate methods are needed [2,4, 10-12,19].

For engineers it is often sufficient to estimate the upper and lower bounds of the frequencies of systems under considerations i.e. to find upper and inner interval approximation of eigenvalue sets and eigenvectors sets respectively cf. [5].

We are interested in the analysis of characteristic polynomial of real interval matrix only, but the complex case can be handled by the method used.

Characteristic polynomials of matrices play an important role in the analysis of vibration techniques, stability and control problems etc., cf. [5,16,19].

2. Elementary Concepts of Interval Calculus

In this report, the following concepts and notations will be used. R^n $(R^{n\times m})$ was reserved for the set of n-dimensional vectors (nxm matrices), R the set of reals.

Let $I(R)$ denote the set of all closed bounded intervals $[z]=[z^-,z^+]$ on the real line R, where z^- and z^+ denote the end points of $[z]$. We call further elements of $I(R)$ interval numbers. In the similar way we introduce $I(R^n)$ - the space of interval vectors, and $I(R^{n\times m})$ - the space of interval matrices.

Further information on interval analysis and the detailed notion the reader can found in papers [1,3,10-12,16], cf. [17,18] in this journal.

3. Eigenvalues and Characteristic Polynomials of the Interval Matrix

In this paper, we are concerned with eigenvalues and eigenvectors of an nxn interval matrix of the form $[\mathbf{A}]=\{\mathbf{A}:\mathbf{A}_c-\Delta\le\mathbf{A}\le\mathbf{A}_c+\Delta\}\in I(R^{n\times n})$.

For an interval matrix in the above form we shall consider the set of complex eigenvalues

$$\Lambda([\mathbf{A}]):=\{\lambda\in C:\mathbf{Ax}-\lambda\mathbf{x}=\mathbf{0},\ \mathbf{A}\in[\mathbf{A}],\ \mathbf{x}\neq\mathbf{0}\} \tag{1}$$

historically first and undoubtedly the most popular of the eigenvalue sets, comp.[5,12,14,19]. In general $\Lambda([A])$ is not an interval complex number.

For any quadratic matrix $\mathbf{A} \in R^{n\times n}$ denote

$$D(\mathbf{A},\lambda) := \det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + p_3\lambda^{n-3} + \dots\dots + p_n \tag{2}$$

which is obviously called the characteristic polynomial of the matrix **A**.

For any interval matrix $[A] \in I(R^{n\times n})$ we introduce the set of characteristic polynomials - which is the range of $D(A,\cdot)$ treated as the function of A:

$$R(D([A],\lambda)) := \{D(A,\lambda): A \in [A]\}, \tag{3}$$

It is naturally isomorphic to the set of polynomial coefficients

$$R(D) = \{(p_1, p_2, \dots, p_n): A \in [A]\}, \tag{4}$$

Naturally R(D) is not an interval. Since $(p_1, p_2, \dots, p_n)$ are algebraic functions of matrix coefficients [9] and $A \in [A]$ varies over bounded interval, it follows that the set (1) is bounded, then $\inf(R(D([A],\lambda)))$ and $\sup(R(D([A],\lambda)))$ exist, and the hull of the characteristic polynomials which we denote as

$$\square R(D([A],\lambda)) := [\inf(R(D([A],\lambda))), \sup(R(D([A],\lambda)))] \tag{5}$$

is defined and is the tightest interval complex vector enclosing $R(D([A],\lambda))$. Notice that $\square R(D([A],\lambda)) \subseteq I(R^{n\times n}) = I(R) \otimes I(R) \otimes \overset{n-times}{\otimes \dots\dots} \otimes I(R)$. The projection of $\square R(D([A],\lambda))$ onto the i-th coordinate we call $\square R_i$.

From the definition we have for the exact eigenvalue set

$$\begin{aligned}\Lambda([A]) &= \{\lambda \in C: D(A,\lambda) = 0, A \in [A]\} = \\ &= \{\lambda \in C: \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + p_3\lambda^{n-3} + \dots\dots + p_n = 0, (p_1, p_2, \dots, p_n) \in R(D([A],\lambda))\}\end{aligned} \tag{6}$$

and for upper approximation of eigenvalue set

$$\begin{aligned}\Lambda_1([A]) &= \\ &= \{\lambda \in C: \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + p_3\lambda^{n-3} + \dots\dots + p_n = 0, (p_1, p_2, \dots, p_n) \in \square R(D([A],\lambda))\}\end{aligned} \tag{7}$$

From inclusion isotonicity it follows that $\Lambda_1([A])$ is the outer approximation of the exact eigenvalue set $\Lambda([A])$. Naturally $\Lambda_1([A])$ is not an interval complex vector too, but probably it is the best outer approximation within characteristic polynomials with interval coefficients.

Naturally if we take $p_i \in P_i \in I(R)$ and $P_i \supseteq \square R_i$ we obtain

$$\begin{aligned} \Lambda_2([A]) = \{\lambda \in C: \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + p_3\lambda^{n-3} + ...+ p_n = 0, \\ = (p_1, p_2, ..., p_n) \in P_1 \otimes ... \otimes P_n \supseteq \square R(D([A],\lambda))\} \end{aligned} \tag{8}$$

and

$$\Lambda([A]) \subseteq \Lambda_1([A]) \subseteq \Lambda_2([A]) \tag{9}$$

Let $y \in I(R^n)$ be arbitrary non-zero interval vector. Then

$$\begin{aligned} \{(p_1, p_2, ..., p_n): D(A,\lambda) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + p_3\lambda^{n-3} + ...+ p_n, A \in [A]\} = \\ = \{(p_1, p_2, ..., p_n): A^n + p_1 A^{n-1} + p_2 A^{n-2} + p_3 A^{n-3} + ...+ p_n I = 0, A \in [A]\} = \\ = \{(p_1, p_2, ..., p_n): A^n y + p_1 A^{n-1} y + p_2 A^{n-2} y + p_3 A^{n-3} y + ...+ p_n y = 0, A \in [A], y \neq 0\} \end{aligned} \tag{10}$$

Denote $c_k = A^k y, k = 1,2,3,...,n$. Then $R(c_k) = \{c_k : c_k = A^k y, A \in [A]\}$. In the case if $y \in R$, then $R(c_k)$ is the interval, but when $y \in I(R)$, then $R(c_k)$ is only the parallelelepid, cf. [12].

Then eq.(10) takes the form

$$\begin{aligned} \{(p_1, p_2, ..., p_n): D(A,\lambda) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + p_3\lambda^{n-3} + ...+ p_n, A \in [A]\} = \\ = \{(p_1, p_2, ..., p_n): p_1 c_{n-1} + p_2 c_{n-2} + p_3 c_{n-3} + ... p_{n-1} c_1 + p_n y = -c_n, c_k \in R(\{c_k\})\ y \neq 0\} \end{aligned} \tag{11}$$

Let $C := [c_{n-1}, c_{n-2}, c_{n-3}, ..., c_1, y]$, $t := -c_n$. Thus we can determine the solution set of $p = [p_1, p_2, ..., p_n]^T$ from the following system of linear equations with set-valued coefficients

$$Cp = t, \quad c_k \in R(\{c_k\}), \quad t \in R(\{t\}) \tag{12}$$

The system of equations (12) has more complicated structure than well known systems of linear interval equations, since it has set-valued coefficients. Denote the solution set of eq. (12) by $\Sigma(\{C\},\{t\})$. We can show that this solution set $\Sigma(\{C\},\{t\})$ of is not an interval complex function, and need not even be convex. In general, $\Sigma(\{C\},\{t\})$ has a very complicated structure. Introduce $C_1 := \square\{C\}$ and $t_1 := \square\{t\}$. Further we discuss the approximating system of linear interval equations

$$C_1 p_1 = t_1, \quad (c_k)_1 \in \square\{c_k\}, \quad t_1 \in \square\{t\} \tag{13}$$

Notice, that from the inclusion isotonicity property it follows, that

$$\{p\} = \Sigma(\{C\},\{t\}) \subseteq \{p_1\} = \Sigma([C_1],[t_1]) \subseteq \square\Sigma([C_1],[t_1]) = [p_1] \tag{14}$$

and

$$\square\Sigma(\{\mathbf{C}\},\{\mathbf{t}\}) \subseteq \square\Sigma([\mathbf{C}_1],[\mathbf{t}_1]) \tag{15}$$

We can obtain this approximation following simple property

$$\{\mathbf{c}_k : \mathbf{c}_k = \mathbf{A}^k\mathbf{y}, \mathbf{A} \in [\mathbf{A}]\} \subseteq \square[\mathbf{c}_k] = [\mathbf{A}^k][\mathbf{y}], \tag{16}$$

Naturally $\{\mathbf{p}\} \subseteq \{\mathbf{p}_1\} \subseteq [\mathbf{p}_1]$.

For more information about polynomials with interval coefficients the reader can compare refs. [6-8,13,15].

4. Example

Consider the simple 2x2 interval real matrix of the form $[\mathbf{A}] = \begin{bmatrix} [2,4] & [-2,-1] \\ [2,5] & [4,5] \end{bmatrix}$. It is easy to calculate that if $\mathbf{y} = [1,0]$, then

$$[\mathbf{A}][\mathbf{y}] = [[2,4],\ \ [2,5]],\ \ [\mathbf{A}]^2[\mathbf{y}] = [[-6,14],\ \ [12,45]] \tag{17}$$

Thus we obtain the approximating linear interval system of equations

$$\begin{aligned} [2,4]\mathrm{p}_1 + [1,1]\mathrm{p}_2 &= [-14,6], \\ [2,5]\mathrm{p}_1 + [0,0]\mathrm{p}_2 &= [-45,-12] \end{aligned} \tag{18}$$

The solution set of the interval linear equation (18) takes the very regular form presented in fig. (1).

As we noticed earlier the exact shape of the solution set $\Sigma(\{\mathbf{C}\},\{\mathbf{t}\})$ is not an interval, it takes the form of parallelelepid. Compare that set with approximated one obtained by interval analysis $\square\Sigma([\mathbf{C}_1],[\mathbf{t}_1])$.

5. Conclusions

This paper deals with the eigenvalue problem of general real interval matrices. Some introductory definitions are given and some mathematical results are presented. A simple numerical example is given to illustrate complexity of presented problems.

Since only particular mathematical problems are solved, further work is needed to obtain more detailed technical results and to use the presented methods to calculate sets of particular eigenvalues.

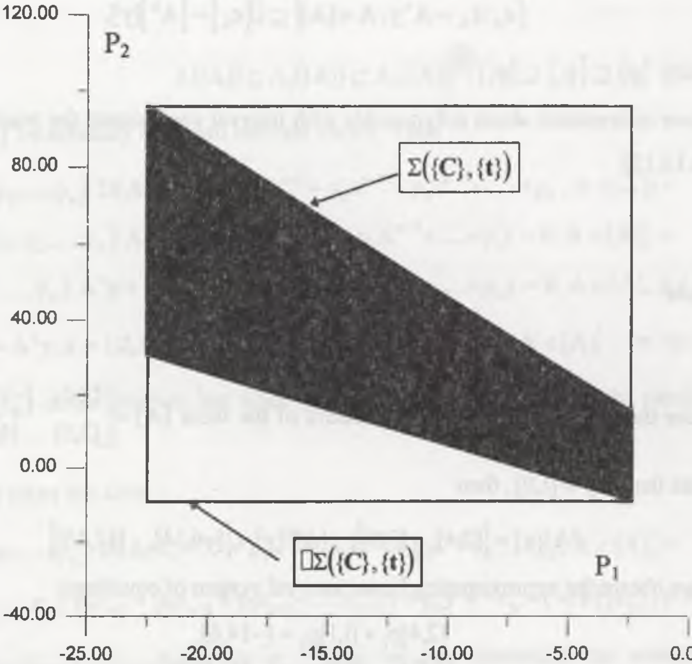

Fig.1. The exact(shaded) and approximated(contour) solutions set of the interval equation
Rys.1. Dokładne(cieniowane) i przybliżone (liniowe) rozwiązania równania przedziałowego

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Abstract

In the paper, we are concerned with interval - oriented methodology to model uncertainties of eigenvalues of an nxn interval real matrix $[\mathbf{A}] = \{\mathbf{A}; \mathbf{A}_c - \Delta \leq \mathbf{A} \leq \mathbf{A}_c + \Delta\}$ where Δ is a measure of uncertainty. We investigate methods of calculation for characteristic polynomials of interval matrices. Presented methodology is probably the simplest way to model and to approximate vibration properties of systems with uncertain parameters. We are interested in the analysis of characteristic polynomial of real interval matrix only, but the complex case can be handled by the method used.

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