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## CONTENTS



## DISCUSSION

Discussion closes March 1, 1946Sept. /4 45 45
Concrete Construction in the National Forests-Clifford $A$. BettsLapped Bar Splices in Concrete Beams-Rolph W. Kluge and Edward C. TumaTests of Prestressed Concrefe Pipes Containing A Steel Cylinder-Culbertson W. RossField Use of Cement Containing Vinsol Resin-Charles E. Wuerpel
Nov. J. 45
Maintenance and Repair of Concrete Bridges on she Oregon Highway Sysfem
-G. S. Paxson
Should Portland Cement Be Dispersed?-T. C. Powers
An Investigation of the Strength of Welded Stirrups in Reinforeed Concrete Beams-Oreste Moretto
Discussion closes April 1, 1946Jon. Jl. '46
Shrinkage Stresses in Concrete-Gerald Pickeff
Floating Block Theory in Structural Analysis-Stanley U. Benscoler
Shrinkage and Plastic Flow of Pre-Stressed Concrefe-Howard R. Staley and DeanPeabody, Jr.
Proposed Minimum Standard Requirements for Precast Conerefe Floor Unils- AClCommittee 711
Proposed Recommended Practice for the Construction of Concrete Farm Silos-AClCommitlee 714

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## AMERICAN CONCRETE INSTITUTE

NEW CENTER BUILDING
DETROIT 2, MICHIGAN


# Shrinkage Stresses in Concrete* 

By GERALD PICKETT $\dagger$<br>Member American Concrele Institule

## SYNOPSIS

Theoretical expressions for deformations of concrete beams and slabs that occur during the course of drying and expressions for distribution of the accompanying shrinkage stresses are derived in Part 1. These expressions are derived on the assumption that the laws governing the development of shrinkage stresses in concrete during drying are analogous to those governing the development of thermal stresses in an ideal body during cooling. Three cases are considered:
(a) slab or beam drying from one face only;
(b) slab or beam drying from two opposite faces; and
(c) prism drying from four faces.

The applicability of the equations to concrete is considered in Part 2 (to appear ACI Journal, February 1946). It is shown that the course of shortening of prisms is in very good agreement with the theorctical equations and that from a test on one prism the shortening versus period of drying of other prisms of the same material differing in size and number of sides exposed to drying can be predicted with fair accuracy if the differences in size are not too great. However, it is shown that the theory must be modified to take into account inelastic deformation and to permit the supposed constants of the material to vary with moisture content and size of specimen if the theory is to be in agreement with all results on all types of specimen of a given concrete.

Various tests are described which, when used in conjunction with the theory, provide a means for studying some of the more fundamental properties of concrete and for predicting the performance of concrete under some conditions in the field.

## INTRODUCTION

Concrete, like many other materials, gains or loses water with changes in ambient conditions. With each change in water content the concrete

[^1]tends to shrink or swell. As a result of these changes in volume, stresses are produced that may affect the performance of the concrete structure concerned.
In the design of concrete structures some consideration is usually given to the possibility of subsequent shrinkage or swelling, but the computations for stresses usually include only the stresses produced by loads. The computations are made by means of formulas taken from the science of mechanics of materials and the computed stresses so obtained are compared with allowable stresses, considering the type of structure and location of the concrete in the structure. The basic assumptions for the formulas are that the material is homogeneous, isotropic, free from self-strain and obeys Hooke's law.

Concrete is not homogeneous; by nature it is heterogeneous, even including the binding medium itself, hardened cement paste. It is not isotropic. Factors such as sedimentation before hardening tend to destroy what isotropy there might have been. It does not obey Hooke's law except perhaps under instantaneous strains. It is not free from selfstrain at any time; the hardened paste may be shrinking while the aggregate may be resisting a change in volume; the regions near the surface may be fairly dry and tending to shrink whereas those farther inward may be much wetter and tending to resist a reduction in volume. In addition, concrete may change with time, becoming stronger and more rigid if conditions are such as to promote additional hydration.

Not only is concrete a much different material from that assumed in the derivations of elementary formulas but the conditions of loading, the tendency toward redundancy (statically indeterminatc), and the structural shapes of concrete members are often such as to make the elementary formulas only rough approximations compared with what the same formulas would be for the usual conditions of loading and structural shapes of steel structures which are usually less redundant.

It is not to be inferred that concrete would necessarily be a better material were all of its properties like those assumed by the design formulas. On the contrary, its ability to relieve stress by creep or plastic flow, for example, partly compensates for its inherently low tensile strength and for uncertainties arising from redundancy. To be remembered also is the fact that, in spite of the deficiencies of design-formulas, concrete structures on the whole perform their intended function. Nevertheless, it should be evident that concrete cannot be used as intelligent'y as it might be and cannot be studied effectively without a better knowledge as to the magnitude and distribution of stresses within it.

The purpose of this paper may be stated as follows:
First, to derive on the basis of simplifying assumptions in regard to the properties of concrete-expressions for: (a) deformations of, and (b)
the distribution of shrinkage stresses in, concrete beams and slabs during the course of drying.
Second, to show by means of data from specimens under controlled conditions the manner and degree to which the equations apply to concrete.

Third, to suggest methods for studying some of the more fundamental properties of drying concrete.

No attempt will be made here to give a complete analysis of stresses in concrete. In particular, the effect of aggregate particles on the stresses within the hardened paste will not be considered.

Before expressions for shrinkage stresses in concrete can be derived, assumptions must be made in regard to the relation between shrinkage and moisture content and the laws controlling the flow of moisture in concrete as well as the relation between stress and strain.

The actual relationships are not as simple as could be desired. If the flow of water were entirely by vapor diffusion, if the vapor pressure of the water in the concrete were proportional to the moisture-content, and if permeability were independent of the moisture-content, then the differential equation for the flow of water would be a partial-differential equation known in physics and mathematics either as the diffusion equation or as the equation of heat conduction. Carlson, ${ }^{1 *}$ in a study of distribution of moisture in concrete, assumed that this equation applies. If the flow of water could be expressed by the diffusion equation and if the shrinkage (or swelling) tendency $\dagger$ of each elemental volume were linearly related to the moisture-content, the unrestrained shrinkage (or swelling) could also be expressed by the diffusion equation. This possibility was also considered by Carlson. ${ }^{1}$ But the flow of water is different from that indicated by the diffusion equation, and the relationship between the change in moisture-content and unrestrained shrinkage is not linear as required by these equations. Moreover, satisfactory expressions for either the flow of water or the moisture-shrinkage relation have not been found.

It is believed that moisture in concrete flows partly as liquid in capillaries, partly as vapor, and partly as adsorbed liquid on the surface of the colloidal products of hydration. While drying progresses, the vapor pressure of the water remaining in the region losing water decreases progressively with the moisture content. This change in vapor pressure with change in moisture content is not linear with respect to moisture

[^2]content. Neither is the rate of flow proportional to the gradient in vapor pressure. The shapes and relative proportions of the spaces occupied by liquid and by vapor change as drying proceeds. This fact, as well as the non-uniformity of the spaces, is believed to be partly responsible for the way in which vapor pressure depends on moisture content and the way in which rate of flow depends on gradient of vapor pressure.

The volume-change-vs.-weight-loss relation is different for different concretes depending on the composition of the concrete and the conditions of curing. For the same concrete it is different during first shrinkage from what it is during the second or subsequent volume changes. If a saturated prism of concrete is allowed to dry, the ratio of change of length to loss of water increases as drying proceeds. At first, comparatively small changes of volume occur per unit loss of weight. The higher the water-cement ratio and the shorter the period of curing the smaller the change during the initial stages of drying. Later the ratio becomes much larger and remains almost constant for some time, after which it may either increase or decrease as the specimen approaches its final weight. It is believed that this ratio, at any stage of drying, depends upon (a) the shape, size, and degree of uniformity of the spaces that hold the water; (b) the shape, size, rigidity, and spacing of the solid particles; and (c) the strength of the bonds between particles.

The relations between stress and strain must be considered in any study of volume changes resulting from moisture changes in concrete because any tendency for a change in volume that progresses from the surface inward always develops stresses. The stresses in turn, through the stressstrain relation, modify the resultant deformations. For low stresses both the elastic and inelastic strain produced by stress are approximately proportional to the stress, permitting Hooke's law to be assumed, but under most conditions of drying the shrinkage stresses, either alone or in combination with stresses from other sources, may be large enough to cause cracks and structural damage within the concrete and for such stresses the proportionality does not hold. Moreover, the apparent plasticity of an element* is greater during the time the element is drying for the first time than at any other time. The relative positions of the colloidal gel particles are no doubt changed by drying and while the changes are taking place small resultant stresses on an element will produce relatively large inelastic deformations.

In spite of the apparent difficulties of obtaining a satisfactory solution to the problem of deformations and stresses in concrete exposed to changes in ambient conditions, a relatively simple procedure has proved to be rather successful. The procedure is to assume as Carlson did that

[^3]the diffusion equation applies to shrinkage even though the simple relations that are implied by that assumption are contrary to fact. It is further assumed that concrete follows Hooke's law. The derivations given in Part 1 are based upon these assumptions.

Since in Part 1 the derivations for deformations and stresses are based on the assumptions that shrinkage follows the diffusion equation and the material follows Hooke's law, the equations are even more applicable to thermal stresses in metals than to shrinkage stresses in concrete. In fact, much of the mathematical work given here was taken from the literature on diffusion of heat and on thermal stresses, as the references will show. However, certain corresponding coefficients in the two problems are of an entirely different order of magnitude. For example, the numerical value of the thermal diffusivity for steel expressed in square inches per second is approximately the same as the numerical value of the shrinkage diffusivity of concrete expressed in square inches per day. Because of the relatively slow diffusion of shrinkage the application of the hypothesis to the shrinkage of concrete necessitates the study of early transient conditions (usually ignored in the treatment of heat).

## PART 1-SHRINKING (OR SWELLING), ITS EFFECT UPON DISPLACEMENTS AND STRESSES IN SLABS AND BEAMS OF HOMOGENEOUS, ISOTROPIC, ELASTIC MATERIAL

## Notation

$S=$ free, unrestrained unit linear shrinkage-strain
$-S=$ free, unrestrained unit linear swelling-strajn
$S_{\infty}=$ final shrinkage-strain under fixed ambient conditions, value of $S$ when $t=\infty$
$S_{a v}=$ average shrinkage over the volume of the specimen, the same as average shortening per unit length if the material follow's Hooke's law
$t$ - time in days
$k=$ diffusivity coefficient of shrinkage in sq. in. per day
$f=$ surface factor, characteristic of the material and the boundary conditions, in in. per day
$a, b, c, d, l=$ distances related to the dimensions of the specimen in inches
$B=f b / k$, a non-dimensional parameter
$T=k t / b^{2}$, a non-dimensional parameter
$B_{c}$ and $T_{c}$, non-dimensional parameters corresponding to $B$ and $T$ and used when a second characteristic dimension of the specimen must be considered
$x, y, z=$ rectangular coordinates
$\beta_{n}=n$th root of $\beta \tan \beta=B$
$\beta_{m}=$ same as $\beta_{n}$ except used in connection with $c$, whereas $\beta_{n}$ is used in connection with $b$
$A_{n}=$ Fourier coefficient
$F_{n}=\frac{2 B}{B^{2}+B+\beta_{n}^{2}}, \bar{F}_{m}=\frac{2 B_{c}}{B_{c}^{2}+B_{c}+\beta_{c}^{2}}$
$H_{n}=\frac{B}{\beta_{n}^{2}} F_{n} \quad H_{m}=\frac{B_{c}}{\beta_{m}^{2}} F_{m}$
$G_{n}=\left(\frac{1}{\cos \beta_{n}}-\frac{B}{2}-1\right) \frac{F_{n}}{\beta_{n}^{2}}$
$\sigma_{x}, \sigma_{y}, \sigma_{z}=$ normal components of stress parallel to $x-, y-$, and $z$-axes-positive if tensile, negative if compressive.
$e_{x}, e_{y}, e_{z}=$ elongations in $x-, y-$, and $z$-directions
$r_{x y}, \tau_{x z}, \tau_{y z}=$ shearing-stress components
$\gamma_{x y}, \gamma_{x z}, \gamma_{y z}=$ shearing-strain components
$E=$ Young's modulus in psi.
$\mu=$ Poisson's ratio
$v=$ deflection in inches, displacement of the elastic line in the $y$-direction
$N=$ the normal to the surface directed outward
$P(x)=\frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-x^{2}} d x$, the probability integral
$\phi(x)=\frac{2}{\sqrt{\pi}} \int^{\infty} e^{-x^{2}} d x=1-P(x)$
$\phi_{b}=1-\sum_{1}^{\infty} e^{-T \beta_{n}^{2}} F_{n} \frac{\cos \beta_{n} \frac{y}{b}}{\cos ^{\rrbracket} \beta_{n}}$
$\phi_{c}=1-\sum_{1}^{\infty} e^{-T_{c} \beta_{m}^{2}} F_{m} \frac{\cos \beta_{m} \frac{z}{c}}{\cos \beta_{m}}$
$H_{s}=1-\sum_{1}^{\infty} e^{-T \beta_{n}^{2}} H_{n}$
$H_{0}=1-\sum_{1}^{\infty} e^{-T_{c} \beta_{n}^{2}} H_{m}$

## Equation for diffusion of unrestrained shrinkage

The diffusion equation is a mathematical statement of the fact that for each infinitesimal volume of a body the excess of the substance in question flowing in over that flowing out per unit of time is equal to the rate of increase of the substance in that volume. When similar assumptions are made in regard to shrinkage, shrinkage thus being treated as if it were a "substance" just as heat is so treated, the result is ${ }^{2}$

$$
\begin{equation*}
k\left[\frac{\partial^{2} S}{\partial x^{2}}+\frac{\partial^{2} S}{\partial y^{2}}+\frac{\partial^{2} S}{\partial z^{2}}\right]=\frac{\partial S}{\partial t} . \tag{1}
\end{equation*}
$$

where $k$ is the diffusivity of shrinkage.
The equation becomes

$$
\begin{equation*}
k \frac{\partial S}{\partial N}=f\left(S_{\infty}-S\right) \tag{2}
\end{equation*}
$$

at exposed boundaries and

$$
\begin{equation*}
\frac{\partial S}{\partial N}=0 \tag{3}
\end{equation*}
$$

at sealed boundaries
where
$N$ is the normal to the surface, directed away from the body
$f$ is the surface factor
$S_{\infty}$ is the value that $S$ will eventually reach under fixed ambient conditions.

Equations 2 and 3 correspond, respectively, to Newton's law of cooling at exposed boundaries and to no flow of heat past perfectly insulated boundaries in the analogous problem of flow of heat.

If the boundaries of the body are not parallel planes, a transformation of Equation 1 from an expression in rectangular coordinates to some other form is usually desirable. For example, if the body is a circular cylinder, Equation 1 is best transformed to

$$
k\left[\frac{\partial^{2} S}{\partial r^{2}}+\frac{1}{r} \frac{\partial S}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} S}{\partial \theta^{2}}+\frac{\partial^{2} S}{\partial z^{2}}\right]=\frac{\partial S}{\partial t}
$$

where $r, \theta$, and $z$ are cylindrical coordinates. Frequently, the condition $\frac{\partial S}{\partial N}=0$ at some boundaries or some other conditions will make $S$ independent of certain coordinates and thereby simplify Equation 1.

Since the form of the solution for $S$ depends upon the form of the differential equation, the form of the solution is dependent upon the boundary condition and the shape of the body under investigation. The initial conditions (values of $S$ at $t=0$ ) and any variation in boundary couditions with time will also affect the form of the solution.

## Assumptions as to elastic properties

After a satisfactory solution for $S$ has been obtained, then displacements and stresses will be found by the application of certain fundamentals of the theory of elasticity. The solutions for stresses are here restricted to homogeneous isotropic solids that follow Hooke's law. Also, as will be brought out below, the effect of Poisson's ratio will be neglected in some cases.

## Effect of shape of body on relative values of principal stresses

The state of stress at any point in a body is defined by the directions and magnitudes of the three principal stresses. The three principal stresses in wide slabs and in narrow beams will be in the directions of length, width, and depth, respectively, if the bodies are under uniform exposure either from one or from two opposite faces and are without external restraint. The principal stress in the direction of depth (normal
to the exposed face) will be zero. The other two principal stresses will be equal in a wide slab (width large compared to depth) if the slab is either not restrained or is restrained equally in the directions of length and width; but in a narrow beam (width small compared to depth) the principal stress in the direction of width will be negligible, and the stress in the direction of length will be ( $1-\mu$ ) times the corresponding stress in a corresponding slab. A corresponding slab differs from the beam just described primarily only in the matter of width. Beams whose widths are not small compared to their depths will have longitudinal stresses intermediate between those of slabs and narrow beams. Corresponding slabs and beams would, of course, have the same longitudinal stresses regardless of widths if Poisson's ratio were zero. Mathematical analyses will be made for the three cases shown in Fig. 1.*

To simplify the mathematical work the effects of Poisson's ratio will be neglected in most of the derivations for Cases I and II. When these effects are neglected, the results will be strictly correct only if Poisson's ratio is zero or if the beam is very narrow.

## CASE I-SLAB OR BEAM DRYING FROM ONE FACE ONLY

## Shrinkage

Solution by Fourier series. The exposed face of the slab or beam will be taken as the plane $y=b$, and the opposite face will be taken as the plane $y=0$ as shown in Fig. 1 for Case I. For this case the diffusion equation reduces to

$$
\begin{equation*}
k \frac{\partial^{2} S}{\partial y^{2}}=\frac{\partial S}{\partial t} \ldots \tag{1a}
\end{equation*}
$$

The equation becomes

$$
\begin{equation*}
\frac{\partial S}{\partial y}=\frac{f}{k}\left(S_{\infty}-S\right) . \tag{2a}
\end{equation*}
$$

at the exposed boundary $y=b$ and

$$
\begin{equation*}
\frac{\partial S}{\partial y}=0 . \tag{3a}
\end{equation*}
$$

at the sealed boundary $y=0$.
A general solution for $S$ satisfying Equations 1a and 3 a is

$$
\begin{equation*}
S=S_{\infty}-\sum^{\infty} A_{n} e^{-\frac{k t}{b^{2}} \beta_{n}^{2}} \cos ^{\beta_{n}} \frac{y}{b} \tag{4}
\end{equation*}
$$

[^4]Equation 2a is also satisfied if $\beta_{n}$ is the $n$th root of

$$
\begin{array}{ll} 
& \beta \tan \beta=\frac{f b}{k} \ldots \\
\text { i.e., } & \beta_{n} \tan \beta_{n}=\frac{f b}{k} .
\end{array}
$$

The above statements may be verified by substituting $S$ from Equation 4 into Equations 1a, 2a and 3a.

For time $t=\infty$, Equation 4 reduces to $S=S_{\infty}$, which is in accord with the definition of $S_{\infty}$. An infinite series of terms such as the trigonometric series in Equation 4 is necessary to give an arbitrary distribution of shrinkage at time $t=0$.

If the initial conditions are such that $S=0$ when $t=0$, then the Fourier cocfficients $A_{n}$ are given by*

$$
A_{n}=\frac{S_{\infty}}{\cos \beta_{n}} \frac{2 \frac{f b}{k}}{\left(\frac{f b}{k}\right)^{2}+\frac{f b}{k}+\beta_{n}^{2}}
$$

It therefore follows that

$$
\begin{equation*}
\frac{S}{S_{-}}=1-\sum_{1}^{\infty} e^{-T \beta_{n}^{2}} F_{n} \frac{\cos \beta_{n} \frac{y}{b}}{\cos \beta_{n}} \tag{5}
\end{equation*}
$$

where

$$
\begin{gathered}
T=\frac{k l}{b^{2}} \\
F_{n}=\frac{2 B}{B^{2}+B+\beta_{n}^{2}} \\
B=\frac{f b}{k}
\end{gathered}
$$

Equation 5 (in slightly different form) and similar equations for other shapes and other conditions, applied to analogous phenomena, may be found in the literature of mathematical physics such as the textbooks of Byerly, Carslaw, and Ingersoll and Zobel. Various tables and diagrams have been prepared from which the numerical relationship of the four non-dimensional quantities $S / S_{\infty}, y / b, B$, and $T$ may be found, such as Fig. 4, page 841 of Perry's Chemical Engineer's Handbook (1934).

[^5]To use more than a few of the terms in Equation 5 for the evaluation of $S / S_{\infty}$ is very laborious because of the difficulty in evaluating $\beta_{n}$ and $F_{n}$. The number of terms required for a given degrec of precision will depend somewhat upon the parameters $B$ and $y / b$ but is chiefly controlled by the parameter $T$. Computations show that very little error is introduced by neglecting all terms in the series except the first if $T$ is more than about 0.2 ; but many terms are needed for the usually desired precision if $T$ is less than 0.02 ,-the smaller the value of $T$ the greater the number of terms needed. Very precise values of $S / S \infty$ for small values of $T$ may be found without the tedious computation indicated in Equation 5 by using another expression which will now be derived.

Solution in terms of the probability integral. As long as the shrinkage at the sealed surface remains negligible, the distribution of shrinkage from the exposed surface inward will be nearly independent of the distance between the two surfaces. Suppose that instead of considering the surface at $y=0$ to be sealed, the body is considered to be extended to infinity in a negative $y$-direction. Then instead of the boundary condition $\frac{\partial S}{\partial y}=0$ at $y$
$=0$, the requirement will be

$$
\begin{equation*}
S=0 \tag{6}
\end{equation*}
$$

at $y=-\infty$.
The solution* satisfying Equations 1a, $2 a$ and 6 and giving $S=0$ when $t=0$ is

$$
\begin{equation*}
\frac{S}{S_{\infty}}=\phi\left[\frac{1-\frac{y}{b}}{2 \sqrt{T}}\right]-\phi\left[\frac{1-\frac{y}{b}}{2 \sqrt{T}}\right] e^{B\left(1-\frac{y}{b}\right)+B^{2} T} \tag{7}
\end{equation*}
$$

where $\phi(x)$ is $\frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-x^{2}} d x$ and $T$ is again used in place of $\frac{k t}{b^{2}}$.
The quantity $1-\phi(x)$, or $\frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-x^{2}} d x$, is known as the proba-
bility integral. Values of $\phi(x)$ may be readily found by using a table of the probability integral.

Numerical calculations show that Equation 7 gives values that differ from those given by Equation 5 by an amount less than the value of $S / S_{\infty}$ at $y=0$; therefore, Equation 7 may be used in place of Equation 5 whenever $T$ is so small that $S / S \infty$ at $y=0$ is less than the permissible error.

[^6]Table 2 and Fig. 7, showing $S / S_{\infty}$ in terms of $y / b, k t / b^{2}$, and $f b / k$, were prepared from Equations 5 and 7.

## Stresses and strains

Continuity, Hooke's law and equilibrium. As stated previously, the solutions for stresses are here restricted to homogeneous, isotropic solids that follow Hooke's law. Equations for the stresses that would be produced in such a body by the shrinkages given by Equations 5 or 7 will now be derived.

The shrinkage $S$ has been defined as the linear unit deformation that would occur if each infinitesimal element werc unrestrained. However, the properties of a continuous solid will not permit an arbitrary distribution of deformations; therefore, unless the distribution of shrinkage given by Equation 5 happens to be compatible with the conditions of continuity, stresses will be produced that will modify the deformations so as to make them compatible. Although in general six partial differential equations are required for a complete mathematical statement of the conditions of compatibility, ${ }^{3}$ these are reduced to

$$
\begin{equation*}
\frac{\partial^{2} e_{x}}{\partial y^{2}}=0 . \tag{8}
\end{equation*}
$$

for either long narrow beams (plane stress) or slabs (plane strain) if the stresses are considered to be independent of the longitudinal coordinate $x$.

The term $e_{x}$ is defined as the resultant unit deformation in the $x$-direction (the direction of length). It is therefore the algebraic sum of shrinkage, $S$, and the strain produced by stresses. $\sigma_{y}$ is obviously zero; and if Poisson's ratio is zero or if the discussion is confined to narrow beams, $\sigma_{z}$ is negligible. Therefore,

$$
\begin{equation*}
e_{x}=\frac{\sigma_{x}}{\widetilde{E}}-\mathrm{S} \tag{9}
\end{equation*}
$$

or, solving for stress,

$$
\begin{equation*}
\sigma_{z}=E\left(e_{x}+S\right) . \tag{10}
\end{equation*}
$$

where $E$ is Young's modulus.
The restriction imposed by Equation 8 requiring that the expression for longitudinal deformation contain no terms in $y$ other than the first power (second derivative equal to zero) is equivalent to the assumption usually made in the elementary theory of beams that "plane cross-sections remain plane." If longitudinal restraint is complete, then $e_{x}$ is zero and it follows from Equation 10 that $\sigma_{x}=E S$. If, however, longitudinal shortening is permitted but complete restraint against bending is provided, then $e_{x}$ is not zero but is still independent of $y$. If the beam has no external restraint, the non-symmetrical distribution of shrinkage causes it to warp, making $e_{x}$ a linear function of $y$. For no external restraint the equations of
equilibrium (summation of forces in the $x$-direction equal to zero and summation of moments about the $z$-axis equal to zero) become

$$
\int_{0}^{b} \sigma_{x} d y=0
$$

and

$$
\begin{equation*}
\int_{0}^{b} \sigma_{x} y d y=0 \tag{12}
\end{equation*}
$$

It may be shown by substituting Equation 10 into both Equations 11 and 12 that if shrinkage $(S)$ is either independent of or a linear function of $y$, an unrestrained beam will be free of stress ( $e_{x}=-S$ and $\sigma_{x}=0$ ). For any other variation of shrinkage a stressed condition must result because the restriction on $e_{x}$ (Equation 8) will not permit it to be equal and opposite to $S$ if shrinkage is a non-linear function of $y$.

The only solution for $e_{x}$ that satisfies Equations 8,10, 11 and 12 is

$$
\begin{equation*}
e_{x}=\left(6 \frac{y}{b}-4\right) \frac{1}{b} \int_{0}^{b} S d y+\left(6-12 \frac{y}{b}\right) \frac{1}{b^{2}} \int_{0}^{b} S y d y \tag{13}
\end{equation*}
$$

When this value of $e_{x}$ is substituted into Equation 10, the result is

$$
\begin{equation*}
\sigma_{x}=E\left[S+\left(6 \frac{y}{b}-4\right) \frac{1}{b} \int_{0}^{b} S d y+\left(6-12 \frac{y}{b}\right) \frac{1}{b^{2}} \int_{0}^{b} S y d y\right] \tag{14}
\end{equation*}
$$

Finally, $S$ from Equation 5 may be substituted into Equation 14, thus giving stress in an unrestrained beam as a function of the parameters $y / b, k t / b^{2}, f b / k, S_{\infty}$ and of Young's modulus. This substitution will not be made until later, because it seems advisable at this time to consider another approach.

Solution by superposition. Although the above derivation is short and is in the simplest form for checking the mathematical correctness of the equation, a derivation in which elementary solutions are superposed is also desirable because it will be easier in general to understand and because the final expressions are in more usable forms. In this second derivation the resultant stress $\sigma_{x}$ is considered as consisting of three parts. The first part is that stress which would be produced by complete restraint against longitudinal deformation; the second part is a uniform stress equal to and opposite in sign to the average of the first part; the third part is a stress
resulting from a simple moment that is equal to and opposite in sign to the moment produced by the sum of the first two parts. That is, the first part alone $\sigma_{x}{ }^{\prime}$ would result from complete restraint, the sum of the first and second parts $\sigma_{x}{ }^{\prime \prime}$ would result from restraint against warping only; the sum of all three parts, i.e., $\sigma_{x}$, would result if no external restraint were applied during shrinkage.

Although in this derivation an expression for $\sigma_{x}$ appears to be the ultimate goal, expressions for $\sigma_{x}{ }^{\prime}$ and for $\sigma_{x}{ }^{\prime \prime}$ are also desirable. The stress $\sigma_{x}^{\prime}$ may be representative of the stress in pavement slabs or building walls that are restrained from shortening and the stress $\sigma_{x}^{\prime \prime}$ is representative of an unrestrained wall drying equally from two opposite sides (Case II discussed later).

Since for complete longitudinal restraint $e_{x}=0$, it follows from Equation 10 that the first part of the stress is

$$
\begin{equation*}
\sigma_{x}^{\prime}=E S \tag{15}
\end{equation*}
$$

Since the average value of $\sigma_{x}{ }^{\prime}$ is $\frac{1}{b} \int_{0}^{b} \sigma_{x}^{\prime} d y$, the second part of
$\sigma_{x}$ is $-\frac{E}{b} \int_{0}^{b} S d y$; therefore, the sum of the first and second parts $\left(\sigma_{x}{ }^{\prime \prime}\right)$ is given by

$$
\begin{equation*}
\sigma_{z}{ }^{\prime \prime}=E\left[S-\frac{1}{b} \int_{0}^{b} S d y\right] \tag{16}
\end{equation*}
$$

The moment produced by $\sigma_{x}{ }^{\prime \prime}$ is the moment necessary to prevent warping. This moment per unit width of beam is found by multiplying Equation 16 by $y d y$ and integrating. This gives

$$
\begin{equation*}
M=\int_{0}^{b} \sigma_{x}{ }^{\prime} y d y=E\left[\int_{0}^{b} S y d y-\frac{b}{2} \int_{0}^{b} S d y\right] \tag{17}
\end{equation*}
$$

For no external restraint this moment must be removed by superposing an equal and opposite moment. The stress resulting from a moment $-M$ is given by the elementary theory of beams as

$$
\frac{M(y-b / 2)}{1 / 12 b^{a}}
$$

This becomes

$$
E\left(6-12 \frac{y}{b}\right)\left[\frac{1}{b^{2}} \int_{0}^{b} S y d y-\frac{1}{2 b} \int_{0}^{b} S d y\right]
$$

when $M$ from Equation 17 is substituted. When this stress, the third part of $\sigma_{x}$ is added to the sum of the first and second parts of Equation 16, the result is Equation 14 previously derived.

Stresses in terms of shortening and warping. The shortening of the beam can be considered as due to the second part of the stress since it is the addition of this part that removes longitudinal restraint and thereby permits the shortening of the beam. From these considerations it follows that

$$
\begin{equation*}
\text { unit shortening }=S_{a v}=\frac{1}{b} \int_{0}^{b} S d y \text {. } \tag{18}
\end{equation*}
$$

where $S_{a v}$ is the average value of $S$.
The bending (warping) of the beam can be considered as due to the third part of the stress since it is the addition of this part that removes the remaining external restraint and thereby permits warping of the beam. The deflection $v$ caused by the moment $-M$ is given by the elementary theory of the bending of beams as $v=\frac{6 M x(l-x)}{E b^{2}}$ where $l$ is the span and $v$ is the deflection of points within the span with respect to either of the end points $x=0$ and $x=l$. The maximum deflection $v_{\max }$ (warping) that occurs at $x=l / 2$ is therefore

$$
\begin{equation*}
v_{\max }=\frac{6 M}{E b^{3}}\left(\frac{l}{2}\right)^{2}=\frac{3 l^{2}}{2 b}\left[\frac{1}{b^{2}} \int_{0}^{b} S y d y-\frac{1}{2 b} \int_{0}^{b} S d y\right]^{*} \tag{19}
\end{equation*}
$$

By writing the second part of $\sigma_{\omega}-\frac{E}{b} \int_{0} S d y$, in terms of the unit shortening $S_{a v}$ it produces, and by writing the third part of $\sigma_{x}^{\prime}$,

$$
E\left(6-12 \frac{y}{b}\right)\left[\frac{1}{b^{2}} \int_{0}^{b} S y d y-\frac{1}{2 b} \int_{0}^{b} S d y\right]
$$

[^7]in terms of the deflection $v_{\max }$ it produces, the expressions for the stresses are put into more usable forms. When this is done, the following equations for stresses are obtained.

For complete longitudinal restraint (first part of $\sigma_{x}$ ),

$$
\begin{equation*}
\sigma_{x}^{\prime}=E S \tag{15}
\end{equation*}
$$

For restraint against warping only (sum of first and second parts of $\sigma_{x}$ ),

$$
\begin{equation*}
\sigma_{x}^{\prime \prime}=E\left(S-S_{a v}\right) \tag{20}
\end{equation*}
$$

For no external restraint (sum of all three parts of $\sigma_{z}$ ),

$$
\begin{equation*}
\sigma_{x}=E\left[S-S_{a v}+\left(6-12 \frac{y}{b}\right) \frac{2 b v_{n a x}}{3 l^{2}}\right] \tag{21}
\end{equation*}
$$

Evaluation of the parameters $\frac{S_{a r}}{S_{\infty}}$ and $\frac{2 b v_{\max }}{3 l^{2} S_{\infty}}$. When Equation 5 for $S / S_{\infty}$ is substituted in Equation 18 for shortening and in Equation 19 for warping and the indicated integrations are performed, the result is

$$
\begin{align*}
& \frac{S_{a v}}{S_{\infty}}=\frac{1}{b} \int_{0}^{b} \frac{S}{S_{\infty}} d y=1-\sum_{1}^{\infty} e^{-T \beta_{n}^{2}} H_{n} \ldots \ldots  \tag{22}\\
& \frac{2 b v_{\max }}{3 l^{2} S_{\infty}}=\frac{1}{b^{2}} \int_{0}^{b} \frac{S}{S_{\infty}} y d y-\frac{S_{a \theta}}{2 S_{\infty}}=\sum_{1}^{\infty} e^{-T \beta_{*}^{2}} G_{n} . \tag{23}
\end{align*}
$$

where

$$
H_{n}=\frac{2 B^{2}}{\beta_{n}^{2}\left(B^{2}+B+\beta_{n}^{2}\right)}
$$

and

$$
G_{n}=\left(\frac{1}{\cos \beta_{n}}-\frac{B}{2}-1\right) \frac{F_{n}}{\beta_{n}^{2}}
$$

If $T$, the non-dimensional time-factor, is small, the series in Equations 22 and 23 converge rather slowly, and in that case it is convenient to use the following equations obtained by substituting Equation 7 into Equations 18 and 19 , respectively. *

$$
\begin{equation*}
\frac{S_{a v}}{S_{c}}=\frac{1}{B}\left[e^{B^{2} T} \phi(B \sqrt{T})-1+\frac{2 B \sqrt{T}}{\sqrt{\pi}}\right] \tag{24}
\end{equation*}
$$

[^8]$\frac{2 b v_{\max }}{3 l^{2} S_{\infty}}=\left\{\left[\frac{1}{2 B}+\frac{1}{B^{2}}\right]\left[e^{B^{2} T} \phi\left(B \sqrt{T)}-1+\frac{2 B \sqrt{T}}{\sqrt{\pi}}\right]-T\right\}\right.$
Furthermore, if the parameter $B \sqrt{T}$ is very small, it is still better to use the following equations obtained by expanding the expressions in the brackets of Equations 24 and 25.
\[

$$
\begin{align*}
\frac{S_{a v}}{S_{\infty}}=B T & {\left[1-\frac{4}{3 \sqrt{\pi}} B \sqrt{T}+\frac{1}{2}(B \sqrt{T})^{2}-\frac{8}{15 \sqrt{\pi}}(B \sqrt{T})^{3}+. .\right] }  \tag{24a}\\
& \frac{2 b v_{\max }}{3 l^{2} S_{\infty}}=\left[\frac{B T}{2}-T\left(1+\frac{B}{2}\right)\left(\frac{4}{3 \sqrt{\pi}} B \sqrt{T}\right.\right. \\
& \left.-\frac{1}{2}(B \sqrt{T})^{2}+\frac{8}{15 \sqrt{\pi}}(B \sqrt{T})^{2}\right) \cdots \cdots \cdots \cdots \cdots \cdots \tag{25a}
\end{align*}
$$
\]

In general the following rules will be found applicable for rapid evaluation of the parameters $\frac{S_{o r}}{S_{\infty}}$ and $\frac{2 b v_{\max }}{3 l^{2} S_{\infty}}$ to a fair degree of accuracy.

If $T$ is more than about 0.05 , use Equations 22 and 23.
If $T$ is less than about 0.05 and $B$ is more than about 5 , use Equations 24 and 25.

If $T$ is less than about 0.05 and $B$ is less than about 5 , use Equations 24a and 25a.

Forces and moments necessary for complete restraint. The force necessary for longitudinal restraint is $\int \sigma_{x}{ }^{\prime} d A$. Therefore, the average force $b$ per unit area is $\frac{1}{b} \int_{0} \sigma_{x}^{\prime} d y$. From Equations 15, 5, and 22 this becomes

$$
\begin{equation*}
\text { force per unit area }=E S_{\infty}\left[1-\sum_{1}^{\infty} e^{-T \beta_{n}^{2}} H_{n}\right] \tag{26}
\end{equation*}
$$

From Equations 17, 5, and 23, the moment per unit width necessary for restraint against warping is found to be

$$
\begin{equation*}
M=E S_{\infty} b^{2} \sum_{1}^{\infty} e^{-T \beta_{n}^{2}} G_{n} \tag{27}
\end{equation*}
$$

Simplification by taking $B$ as equal to infinity. The principal equations derived above reduce to simpler forms and the computation of numerical values is less tedious if the assumption is made that $B$, i.e., $f b / k$, equals infinity. If $B$ is large, say 100 or more, the error introduced by assuming it to be infinity is negligible. However, if $B$ is less than about 5 , the error introduced by considering it to be infinity may be appreciable as is shown, for example, by Fig. 8, 9, and 14 . Whether justifiable or not, the assumption that $B=\infty$ is frequently made in analogous problems to which the diffusion equation applies. This assumption was made by Terzaghi and Fröhlich ${ }^{4}$ in developing the theory of settlement of foundations due to consolidation of underlying material, by Glover ${ }^{5}$ in a study of distribution of temperature in concrete dams, and by Carlson ${ }^{1}$ in a study of distribution of moisture and shrinkage in concrete. The more important of the above equations for the special case of $B=\infty$ are given below:

Equation 5 becomes

$$
\frac{S}{S_{\infty}}=1-\sum_{1}^{\infty} \frac{4(-1)^{n-1}}{(2 n-1) \pi} e^{-(2 n-1)^{2} \frac{\pi^{2}}{4} T} \cos (2 n-1) \frac{\pi y}{2 b}
$$

Equation 7 becomes

$$
\frac{S}{S_{\infty}}=\phi\left(\frac{1-y / b}{2 \sqrt{T}}\right) .
$$

Equation 22 becomes

$$
\frac{S_{a 0}}{S_{\infty}}=1-\frac{8}{\pi^{2}} \sum \frac{1}{(2 n-1)^{2}} e^{-(2 n-1)^{2}\left(\pi^{2} / 4\right) T}
$$

Equation 24 becomes

$$
\frac{S_{a v}}{S_{\infty}}=\frac{2}{\sqrt{\pi}} \sqrt{T} .
$$

Equation 23 becomes

$$
\frac{2 b v_{\max }}{3 l^{2} S_{\infty}}=\sum_{1}^{\infty} \frac{4}{\pi^{2}(2 n-1)^{2}}\left[\frac{(-1)^{n-1} 4}{\pi(2 n-1)}-1\right] e^{-(2 n-1)^{2} \frac{\pi^{2}}{4} T} .
$$

Equation 25 becomes

$$
\frac{2 b v_{\max }}{3 l^{2} S_{\infty}}=\frac{\sqrt{T}}{\sqrt{\pi}}-T .
$$

Tables, curves, and computations.* Tables and diagrams such as those by Newman ${ }^{6}$ are available from which values of $S / S_{\infty}$ and $S_{a v} / S_{\infty}$ may be determined. However, such published tables are in general not adequate for the present problem. The smallest value of the parameter $T$ used by Newman in his computations was 0.1 , whereas the stresses in concrete may be desired for a much earlier period. The tables given here were prepared for $T$ as low as 0.001 . Moreover, so far as is known, the parameter $\frac{2 b v_{\max }}{3 l^{2} S_{\infty}}$ had not previously been evaluated for this or any analogous problem and its evaluation is necessary for the problem here considered.

A step in the evaluation of Equations 15, 20, and 21 for the theoretical stresses in a beam drying from one side under the different modes of restraint is the evaluation of the three quantities $\frac{S}{S_{\infty}}, \frac{S_{u v}}{S_{\infty}}$ and $\frac{2 b v_{\max }}{3 l^{2} S_{\infty}}$. The first quantity $S / S_{\infty}$ as a function of the three parameters $y / b, B$ and $T$, is given in Table 2 and shown graphically in Fig. 7. The second quantity $S_{a v} / S_{\infty}$ as a function of $B$ and $T$ is given in Table 3 and shown graphically in Fig. 8. $\dagger$ The third quantity $\frac{2 b v_{\max }}{3 l^{2} S_{\infty}}$ as a function of $B$ and $T$ is given in Table 4 and shown graphically in Fig. 9. Tables 5 and 6 giving the stresses $\sigma_{z}^{\prime \prime}$ and $\sigma_{x}$ (Equations 20 and 21) as functions of the three parameters $y / b, B$, and $T$, were readily prepared after the three primary quantities had been evaluated (Tables 2, 3, and 4). Results for $B=5$ are shown graphically in Fig. 10, 11, 12, and 13. Fig. 14 shows maximum values of $\frac{\sigma_{x}^{\prime \prime}}{E S_{\infty}}$ and $\frac{\sigma_{x}}{E S_{\infty}}$ and of $\frac{2 b v_{\max }}{3 l^{2} S_{\infty}}$ versus the parameter $B$.

The computations made for the preparation of the tables and diagrams are explained in part below.

If the parameter $T$ is so small that the equations based upon the assumption that the body extends to infinity may be used instead of the theoretically correct equations, no difficulty is encountered. For example, let $y / b=0.8, B=5$, and $T=0.01$. Equation 7 then becomes

$$
\begin{align*}
\frac{S}{S_{\infty}} & =\phi\left(\frac{0.2}{2 \times 0.1}\right)-\phi\left(\frac{0.2}{2 \times 0.1}+5 \times 0.1\right) e^{5 \times 0.2+25 \times 0.01} \\
& =\phi(1)-\phi(1.5) e^{1.25}
\end{align*}
$$

[^9]From mathematical tables

$$
\phi(1)=0.15730 ; \phi(1.5)=0.03389 ; e^{1.25}=3.4903
$$

Therefore $S / S_{\infty}=0.1573-0.03389 \times 3.4903=0.0390$.
Note that this is the value given in Table 2 for the above values of $B, T$, and $y / b$. Also note that for the same $B$ and $T$ the table gives zero for $y / b=0$, showing that it was permissible to use Equation 7 instead of Equation 5.

When the theoretically correct equations are used, the computations are more involved. For instance, let $T=0.1$ instead of 0.01 in the above example. $T$ will then be so large that $S / S_{\infty}$ will have an appreciable value at $y / b=0$. Therefore, Equation 7 will not be applicable and Equation 5, the exact equation, must be used. A substitution of values for $T$ and $y / b$ into Equation 5 gives

$$
\frac{S}{S_{\infty}}=1-\sum_{1}^{\infty} e^{-0.1 \beta_{n}^{2}} F_{n} \frac{\cos 0.8 \beta_{n}}{\cos \beta_{n}}
$$

The first step in evaluating the above expression is to determine $\beta_{n}$ which Equation 2c shows to be a function of $f b / k$ and $n$, i.e., $B$, and the integer $n$. The determination of $\beta_{n}$ by interpolation is simplified by the introduction of $\alpha_{n}$ where $\alpha_{n}$ depends on $B$ and $n$. The equation for $\beta_{n}$ is then written

$$
\begin{equation*}
\beta_{n}=\left(n-1+\alpha_{n}\right) \pi \tag{28}
\end{equation*}
$$

Curves of $\alpha_{n}$ versus $B$ for the first six values of $n$ and for $n=21$ are shown in Fig. 2. By means of Fig. 2 and Equation 28 any desired $\beta_{n}$ may be found with reasonable accuracy for any value of $B$. The first six values of $\beta_{n}$ for several different values of $B$ are given in Table 1.

After finding $\beta_{n}$ for the given values of $B$ and $n$, the factors $F_{n}$, $\cos \beta_{n}, \cos \left(\beta_{n} \frac{y}{b}\right)$ and $e^{-T \beta_{n}^{2}}$ are determined. $F_{n}$ and $\cos \beta_{n}$ as functions of $B$ are shown in Fig. 3 and 4, respectively, for the first six values of $n$. The functions $\cos \left(\beta, \frac{y}{b}\right)$ and $e^{-T \beta_{n}^{2}}$ are readily obtained from mathematical tables after the products $\beta_{s} \frac{y}{b}$ and $T \beta_{n}^{2}$ have been determined. When the proper values of the four factors listed above are substituted, the above equation for $\frac{S}{S_{\infty}}$ becomes

$$
\begin{gathered}
\frac{S}{S_{\infty}}=1-\frac{0.84147 \times 0.3152 \times 0.4966}{0.2541}-\frac{0.1965 \times 0.2161 \times 0.9963}{0.6277} \\
-\frac{0.00844 \times 0.1286 \times 0.7280}{0.8101}+
\end{gathered}
$$

This reduces to $\frac{S}{S_{\infty}}=1-0.5183-0.0674-0.0009=0.4134$
The values for $S / S_{\infty}$ given in Table 2 were computed by one or the other of the methods illustrated above.

In preparing Table 3 from which Fig. 8 was constructed (Fig. 8 shows shortening as a function of $\sqrt{T}$ for various values of $B$ ), values of $H_{n}$ in Equation 22 were needed. Values of $H_{n}$ as a function of $B$ for the first six values of $n$ are shown in Fig. 5. In like manner, Fig. 6 showing $G_{n}$ as a function of $B$, served in the preparation of Table 4 from which Fig. 9 was constructed. Of course, for small values of T, Fig. 5 and 6 are not necessary since either Equation 24 or Equation $24 a$ is used instead of Equation 22 and either Equation 25 or Equation 25a is used instead of Equation 23, depending on the value of $B$.

Application to beams or slabs of any width-to-depth ratio when Poisson's ratio is not zero. The effect of Poisson's ratio was neglected in the preceding derivations. Its effect stated in general terms in the introductory remarks in regard to Case I will now be analyzed in more detail. If Poisson's ratio is not zero, Equation 9 for $e_{x}$ and Equation 10 for $\sigma_{x}$ will be modified to include the effect of $\sigma_{2}$. That is,

$$
\begin{align*}
& e_{x}=\frac{\sigma_{x}}{E}-\frac{\mu \sigma_{z}}{E}-S \ldots  \tag{9a}\\
& \sigma_{x}-\mu \sigma_{z}=E\left(e_{x}+S\right) \ldots \tag{10a}
\end{align*}
$$

However, if the ratio of width to depth is small, $\sigma_{z}$ will be negligible and Equation 10a reduces to Equation 10. On the other hand, if the ratio of width to depth is very large (a slab), the width being comparable with the length, then $\sigma_{z}$ will be equal to $\sigma_{x}$. If $\sigma_{x}=\sigma_{z}$, then Equation 10a reduces to

$$
\begin{equation*}
\sigma_{x}=\frac{E}{1-\mu} \cdot\left(e_{x}+S\right) \tag{10b}
\end{equation*}
$$

The only difference between Equation 10 b for a wide slab and Equation 10 for a narrow beam is the factor $\mu$, which occurs in Equation 10b but not in Equation 10. Therefore, for stresses in a slab, $E$ in Equations 15,20 , and 21 is replaced by $\frac{E}{1-\mu}$. The stresses in beams whose width-to-depth ratio is intermediate will have stresses intermediate between those of narrow beams and of slabs. Since $E$ does not appear in Equa-
tion 22 for average shrinkage nor in Equation 23 for warping, these quantities are the same for narrow beams and wide slabs.

## CASE II-SLAB OR BEAM DRYING FROM TWO OPPOSITE SURFACES

Equations taken from those derived for Case I. Since the flow of moisture in a slab drying from only one surface is believed to be the same as that in either half of a slab of twice the depth drying from two opposite surfaces, it will be assumed that the theoretical equations derived for shrinkage of a beam or slab drying from only one surface will apply equally well for either half of a beam or slab drying from two opposite surfaces. The plane midway between the drying surfaces will be taken as the plane $y=0$ as shown in Fig. 1 for Case II. Since the two halves of the beam will mutually restrain each other from warping, the equations for stresses, strains, and shortening in each half will be the same as those given previously in Case I for a beam restrained against warping and drying from one surface.

## CASE III-RECTANGULAR PRISM DRYING FROM FOUR FACES

## Shrinkage

The differential equation and boundary conditions. For a prism drying from four faces but not from the ends the diffusion equation reduces to

$$
\begin{equation*}
k\left(\frac{\partial^{2} S}{\partial y^{2}}+\frac{\partial^{2} S}{\partial z^{2}}\right)=\frac{\partial S}{\partial t} \tag{1b}
\end{equation*}
$$

The exposed faces of the prism will be taken as the planes $y= \pm b$, $z= \pm c$, as shown in Fig. 1 for Case III. The boundary conditions then become

$$
\begin{equation*}
\frac{\partial S}{\partial y}= \pm \frac{f}{k}\left(S_{\infty}-S\right) \tag{2a}
\end{equation*}
$$

at the boundaries $y= \pm b$ and

$$
\begin{equation*}
\frac{\partial S}{\partial z}= \pm \frac{f}{k}\left(S_{\infty}-S\right) \tag{3b}
\end{equation*}
$$

at the boundaries $z= \pm c$.
The solution satisfying Equations 1b, 2a, 3b, and giving $S=0$ at $t=0$ and $S=S_{\infty}$ at $t=\infty$ is the following:

$$
\begin{equation*}
\frac{S}{S_{\infty}}=1-\left[\sum_{1}^{\infty} e^{-T \beta_{n}^{2}} F_{\mathrm{a}} \frac{\cos \beta_{n} \frac{y}{b}}{\cos \beta_{n}}\right]\left[\sum_{1}^{\infty} e^{-T_{c} \beta_{m}^{2}} F_{m} \frac{\cos \beta_{m} \frac{z}{c}}{\cos \beta_{m}}\right] \tag{5a}
\end{equation*}
$$

where $\beta_{m}, F_{m}$, and $T_{c}$ correspond to $\beta_{n}, F_{n}$, and $T$, respectively, the difference being that the dimension $b$ has been replaced by $c$.

Shrinkage expressed in terms of the solutions given for a prism drying from one face or two opposite faces. Since the infinite series in the first bracket in Equation 5a is identical with the one given in Equation 5 and the infinite series in the second bracket is like the first except that $y$ is replaced by $z, b$ by $c$, etc., and since Equation 5 applies to either half of a slab exposed on two opposite surfaces (Case II), it follows that the brackets have the following values:

$$
\begin{gather*}
\Sigma e^{-T \beta_{n}^{2}} F_{n} \frac{\cos \beta_{n} \frac{y}{b}}{\cos \beta_{n}}=1-\phi_{b}  \tag{29}\\
\Sigma e^{-T_{c} \beta_{m}^{2}} F_{m} \frac{\cos \beta_{m} \frac{z}{c}}{\cos \beta_{m}}=1-\phi_{c} \tag{30}
\end{gather*}
$$

where $\phi_{b}$ is the value $S / S_{\infty}$ would have if only the surfaces $y= \pm b$ were exposed and $\phi_{c}$ is the value $S / S_{\infty}$ would have if only the surfaces $z= \pm c$ were exposed.

A substitution of Equations 29 and 30 into Equation 5 a gives

$$
\begin{equation*}
\frac{S}{S_{\infty}}=\phi_{b}+\phi_{c}-\phi_{b} \phi_{c} . \tag{5b}
\end{equation*}
$$

Equation 5 b shows that the evaluation of shrinkage for a prism drying from four surfaces becomes a problem of adding the independent effects of drying from surfaces that are perpendicular to each other and then subtracting a term proportional to the product of the separate effects.

For example, consider the shrinkage tendency at the point $y=0.4 b$, $z=0.8 c$ in a prism for which $c=2 b$ (width equal to twice the thickness). Let $f, k$, and $t$ be such that $f b / k=5.0$ and $k t / b^{2}=0.20$; then $f c / k=$ 10.0 and $k t / c^{2}=0.05$. $\phi_{b}$ is found in Table 2 or from Fig. 7 to be 0.2398 . Since Table 2 was prepared for $B$ equal to $0.1,1.0,5.0$, and $\infty$ only, and since $f c / k=10, \phi_{c}$ can be found from Table 2 only by interpolation. However, examination of Table 2 indicates that for $k t / c^{2}$ equal to 0.05 Equation 7 can be used instead of Equation 5 without appreciable error and therefore the equation rather than the table will be used to obtain $\phi_{c}$. From Equation 7

$$
\phi_{c}=\phi\left(\frac{0.2}{2 \sqrt{0.05}}\right)-\phi\left(\frac{0.2}{2 \sqrt{0.05}}+10 \sqrt{0.05}\right) e^{2+5}
$$

From tables giving probability integrals and the exponential function

$$
\phi_{c}=0.5273-0.000156 \times 1097=0.355
$$

Therefore at $y=0.4 b, z=0.8 c$, and $t=0.20 b^{2} / k$,

$$
\frac{S}{S_{\infty}}=0.2398+0.355-0.2398 \times 0.355=0.510
$$

Shortening expressed in terms of the shortening of a prism drying from one face or from two opposite faces. The average shrinkage $S_{a v}$ is given by

$$
\begin{equation*}
S_{a v}=\frac{1}{b c} \int_{0}^{b} \int_{0}^{c} S d y d z \tag{18a}
\end{equation*}
$$

From Equations 5a and 18a

$$
\frac{S_{a}}{S_{\infty}}=1-\sum_{1}^{\infty} e^{-T \beta_{n}^{2}} H_{n} \sum_{1}^{\infty} e^{-T_{c} \beta_{m}^{2}} H_{m} \ldots \ldots \ldots \ldots(22 a)
$$

or

$$
\begin{equation*}
\frac{S_{a}}{S_{\infty}}=H_{b}+H_{c}-H_{b} H_{c}, \tag{22b}
\end{equation*}
$$

where $H_{b}$ is the value $S_{a v} / S_{\infty}$ would have if only the surface $y= \pm b$ were exposed and $H_{c}$ is the value $S_{a v} / S_{\infty}$ would have if only the surfaces $z= \pm c$ were exposed*. Therefore, the average shrinkage, and consequently the shortening, if the body is elastic, of a prism drying from four sides may be found by considering the separate effects of drying from opposite sides in pairs.

For example, consider the shortening of the prism discussed above. $H_{b}$ is found in Table 3 to be 0.3510 and $H_{c}$ is found to be 0.1753 . Therefore, $\frac{S_{a b}}{S_{\infty}}=0.3510+0.1753-0.3510 \times 0.1753=0.4647$.

## Stresses and strains

Nature of the problem and the method to be used to obtain a solution. In Cases I and II previously discussed, where shrinkage was a function of time $t$ and only one space coordinate $y$ and where the problem was further simplified by neglecting the effects of the length and the width of the specimen on the distribution of stresses, a solution for the one stress involved was readily obtained. However, in the problem now under consideration, a prism drying from four surfaces, shrinkage varies with an additional coordinate $z$. As a result stresses vary with this additional coordinate also, and more than one stress will be involved. The problem will be somewhat simplified by neglecting the effect of the length on the distribution of stresses, i.e., the assumption will be made that stresses do not vary along the length. The distribution of stresses given by the solution based on this assumption will deviate a negligible amount from

[^10]the theoretically correct distribution when stresses in the central portion of a long prism are under consideration (principle of Saint Venant ${ }^{8}$ ).
The solution for shrinkage in terms of two space coordinates was almost as simple as when only one coordinate was involved (Equation 5a for Case III compared with Equation 5 for Cases I and II), because tendency to shrink is considered to be a scalar quantity. On the other hand, since stresses are tensor quantities, the solution for stresses usually becomes much more complicated whenever more than one coordinate is involved. In fact, elasticians have obtained exact solutions meeting all boundary conditions for only a relatively few problems in which stresses were functions of at least two coordinates and then only by considering the body to be infinite in the direction of one of these coordinates. The difficulty is that since stresses are tensors, boundary forces are vectors, and in two-dimensional problems two components of force must be satisfied at each boundary. The specified conditions of stress at any two opposite boundaries can be satisfied by superposing particular solutions of the differential equations in accordance with the usual methods of Fourier analysis. But, in general, solutions satisfying rigorously the boundary requirements at two pairs of opposite boundaries simultaneously cannot be found by the usual methods.

A method of solving such problems after the appropriate differential equations have been derived was explained by the author in a recent paper ${ }^{9}$. That method will be used here. It is about the same as that used previously by Taylor ${ }^{10}$ and by Timoshenko ${ }^{11}$ in analogous problems.

Derivation of the differential equations relating stresses to shrinkage. By neglecting the variation of stresses and strains along the length of a body the problem becomes a two-dimensional problem in plane strain. The following equations taken from the theory of elasticity are then applicable ${ }^{12}$.

Equations of Equilibrium:

$$
\begin{aligned}
& \frac{\partial \sigma_{y}}{\partial y}+\frac{\partial \tau_{y z}}{\partial z}=0 \\
& \frac{\partial \sigma_{z}}{\partial z}+\frac{\partial \tau_{y z}}{\partial y}=0
\end{aligned}
$$

Condition of Compatibility:

$$
\frac{\partial^{2} e_{y}}{\partial z^{2}}+\frac{\partial^{2} e_{z}}{\partial y^{2}}=\frac{\partial^{2} \tau_{y z}}{\partial y \partial z}
$$

Hooke's Law Modified to Include Isotropic Shrinkage:

$$
e_{x}=\frac{1}{E}\left[\sigma_{y}-\mu \sigma_{y}-\mu \sigma_{z}\right]-S=-S_{a v} \text { in this problem }
$$

$$
\begin{aligned}
& e_{y}=\frac{1}{E}\left[-\mu \sigma_{x}+\sigma_{y}-\mu \sigma_{z}\right]-S \\
& e_{z}=\frac{1}{E}\left[-\mu \sigma_{x}-\mu \sigma_{y}+\sigma_{z}\right]-S \\
& \gamma_{y z}=\frac{2(1+\mu)}{E} \tau_{y z}
\end{aligned}
$$

The above seven equations giving relations between the eight unknown stresses and strains can be reduced to the following two equations by eliminating the four strains and the shear stress:

$$
\begin{align*}
& \sigma_{z}=\mu\left(\sigma_{y}+\sigma_{z}\right)+E\left(S-S_{a v}\right) .  \tag{31}\\
& \nabla^{2}\left(\sigma_{y}+\sigma_{z}\right)=\frac{E}{1-\mu} \nabla^{2} S \ldots \ldots \tag{32}
\end{align*}
$$

where $\nabla^{2}$ is written for $\frac{\partial^{2}}{\partial y^{2}}+\frac{\partial^{2}}{\partial y^{2}}$.
These two equations together with the two equations of equilibrium and the boundary conditions that

$$
\begin{aligned}
& \sigma_{y} \text { and } \tau_{y z}=0 \text { at } y= \pm b \\
& \sigma_{z} \text { and } \tau_{y z}=0 \text { at } z= \pm c
\end{aligned}
$$

and Equation 5 a for $S$ constitute the mathematical statement of the problem.

Solution for stresses. In general the stress $\sigma_{x}$ will be larger than either $\sigma_{y}$ or $\sigma_{z}$. The stress $\tau_{y z}$ will be relatively small in all cases. If only the value of the theoretical maximum stress is desired, a fairly good approximation can be obtained by the following simplified formula:

$$
\begin{equation*}
\sigma_{x}=E\left(S-S_{a v}\right) \text { approx. } \tag{31a}
\end{equation*}
$$

where $S$ is given by Equation 5 a and $S_{a v}$ is given by Equation 22a. If, however, an accurate theoretical value of all stresses is desired, a complete solution must be obtained and this is given below:

The solution for $\sigma_{y}, \sigma_{z}$, and $\tau_{y z}$ meeting all of the above requirements is as follows:*

$$
\sigma_{v}=A_{\circ} \frac{c^{2}-3 z^{2}}{12 b c}+\frac{b}{c} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{A_{i j}}{i^{2}} \cos \alpha_{i} y \cos \gamma_{j} z
$$

[^11]$+\sum_{j=1}^{\infty} B_{i} \frac{\cos \gamma_{j} z}{\cosh \gamma_{j} b}\left[\gamma_{i} y \sinh \gamma_{j} y-\left(1+\gamma_{j} b \operatorname{coth} \gamma_{i} b\right) \cosh \gamma_{j} y\right]$
$-\sum_{i=1}^{\infty} C_{i} \frac{\cos \alpha_{i} y}{\cosh \alpha_{i} c}\left[\alpha_{i} z \sinh \alpha_{i} z+\left(1-\alpha_{i} c \operatorname{coth} \alpha_{i} c\right) \cosh \alpha_{i} z\right] \ldots$
$\sigma_{z}=A_{0} \frac{b^{2}-3 y^{2}}{12 b c}+\frac{c}{b} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{A_{i j}}{j^{2}} \cos \alpha_{i} y \cos \gamma_{j} z$
$-\sum_{j=1}^{\infty} B_{j} \frac{\cos \gamma_{j}^{2}}{\cosh \gamma_{j} b}\left[\gamma_{j} y \sinh \gamma_{j} y+\left(1-\gamma_{j} b \operatorname{coth} \gamma_{j} b\right) \cosh \gamma_{j} y\right]$
$+\sum_{i=1}^{\infty} C_{i} \frac{\cos \alpha_{i} y}{\cosh \alpha_{i} c}\left[\alpha_{i} z \sinh \alpha_{i} z-\left(1+\alpha_{i} c \operatorname{coth} \alpha_{i} c\right) \cosh \alpha_{i} z\right]$ (34)
$\tau_{y z}=\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{A_{i j}}{i j} \sin \alpha_{i} y \sin \gamma_{j} z$
$+\sum_{j=1}^{\infty} B_{i} \frac{\sin \gamma_{j}{ }^{2}}{\cosh \gamma_{j} b}\left[\gamma_{j} b \operatorname{coth} \gamma_{j} b \sinh \gamma_{j} y-\gamma_{j} y \cosh \gamma_{j} y\right]$
$+\sum_{i=1}^{\infty} C_{i} \frac{\sin \alpha_{i} y}{\cosh \alpha_{i} c}\left[\alpha_{i} c \operatorname{coth} \alpha_{i} c \sinh \alpha_{i} z-\alpha_{i} z \cosh \alpha_{i} z\right]$.
where
$$
\alpha_{i}=\frac{i \pi}{b}, \quad \gamma_{i}=\frac{j \pi}{c}
$$
\[

$$
\begin{equation*}
\left.-\frac{c}{b} \frac{A_{0}(-1)^{2}}{\pi^{2} j^{2}}+\frac{b}{c} \sum_{i=1}^{\infty}(-1)^{i} \frac{A_{i j}}{i^{2}}-\sum_{i=1}^{\infty}(-1)^{i+j} \frac{4 i c}{j \pi j b}\left[1+\left(\frac{i c}{j b}\right)^{2}\right]^{s} C_{s}\right] \tag{36}
\end{equation*}
$$

\]

$$
C_{i}=\frac{-\frac{c}{b} \frac{A_{\circ}(-1)^{i}}{\pi^{2} i^{2}}+\frac{c}{b}{ }_{j=1}^{\infty}(-1)^{i} \frac{A_{i j}}{j^{2}}-\sum_{j=1}^{\infty}(-1)^{i+j} \frac{4}{i \pi} \frac{j b}{i c} \frac{\tanh \frac{j \pi b}{c}}{\left[1+\left(\frac{j b}{i c}\right)^{2}\right]^{2}} B_{i}}{1+\frac{i \pi c}{b}\left(\operatorname{coth} \frac{i \pi c}{b}-\tanh \frac{i \pi c}{b}\right)}
$$

$$
\begin{align*}
& A_{i j}=-\left[\frac{E S_{\infty} 4(-1)^{i+2} i^{2} j^{2}}{(1-\mu) \pi^{2}\left(i^{2} \frac{c}{b}+j^{2} \frac{b}{c}\right)^{2}}\right] \\
& \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\left(\beta_{n}^{2} \frac{c}{b}+\beta_{m}^{2} \frac{b}{c}\right) \epsilon^{-T \beta_{n}^{2} e^{-T_{c} \beta_{m}^{2}} H_{n} H_{m}}}{\left(1-\frac{i^{2} \pi^{2}}{\beta_{m}^{2}}\right)\left(1-\frac{j^{2} \pi^{2}}{\beta_{m}^{2}}\right)} \tag{38}
\end{align*}
$$

$$
\begin{equation*}
A_{o}=-\frac{E S_{\infty}}{1-\mu} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty}\left(\beta_{n}^{2} \frac{c}{b}+\beta_{m}^{2} \frac{b}{c}\right) e^{-T \beta_{n}^{2}} e^{-T_{c} \beta_{m}^{2}} H_{n} H_{m} \ldots \text { (39) } \tag{39}
\end{equation*}
$$

The above equations, together with Equation 31 for $\sigma_{x}$, constitute a complete solution on the basis of the given assumptions.

In general, if both the parameters $T$ and $T_{c}$ are equal to or greater than 0.1 , the series given above converge very rapidly so that only a few terms need be taken for a good approximation. The example given below will demonstrate the use of the above equations.

Example: Stress at the middle of one side of a square prism for which $B$ equals 5 and at a time for which $T$ equals 0.1 . If only one term in each series is used, the following values are obtained: From Equations 38 and 39 and Table 1.
$A_{o}=-\frac{E S_{\infty}}{1-\mu}\left(\beta_{1}^{2}+\beta_{1}^{2}\right) e^{-0.1 \beta_{1}^{2}} e^{-0.1 \beta_{1}^{2}} H_{1}^{2}$
$A_{o}=-\frac{E S_{\infty}}{1-\mu}\left(1.3138^{2}+1.3138^{2}\right) 0.8415^{2} \times 0.9130^{2}$
$A_{\circ}=-2.0377 \frac{E S_{\odot}}{1-\mu}$
$A_{11}=-\frac{E S_{\infty}}{(1-\mu) \pi^{2}} \frac{\left(1.3138^{2}+1.3138^{2}\right) 0.8415^{2} \times 0.9130^{2}}{\left(1-\frac{\pi^{2}}{1.3138^{2}}\right)^{2}}$
$A_{11}=-0.0093 \frac{E S_{\infty}}{1-\mu}$
When these values are substituted into Equation 36, the result is

$$
B_{1}=\frac{\left(-\frac{2.0376}{\pi^{2}}+0.0093\right) \frac{E S_{\infty}}{1-\mu}-\frac{\tanh \pi}{\pi} C_{1}}{1+\pi(\operatorname{coth} \pi-\tanh \pi)}
$$

or since $C_{1}=B_{1}$ (square prism),

$$
C_{1}=B_{1}=-0.1472 \frac{E S_{\infty}}{1-\mu}
$$

When the above values and $z=c, y=0$ are substituted into Equation 33 , the result is

$$
\left.\begin{array}{rl}
\sigma_{\nu} \quad \mid & =\left[\frac{2.0377}{6}+0.0093-0.1472 \times 0.3583+0.1472 \times 0.9773\right] \frac{E S_{\infty}}{1-\mu} \\
z=c \\
y=0
\end{array}\right] \quad \begin{aligned}
& \sigma_{y} \quad=[+0.3396+0.0093-0.0527+0.1439] \frac{E S_{\infty}}{1-\mu}=0.4401 \frac{E S_{\infty}}{1-\mu} \\
& z=c \\
& y=0
\end{aligned}
$$

When the summation of each series is carried to two terms, the result is $0.4214 \frac{E S_{\infty}}{1-\mu}$ and when the summation is carried to four terms in each series the result is $0.4221 \frac{E S_{\infty}}{1-\mu}$ for this stress.

The above shows that the series converge very rapidly for this example.
The stresses $\sigma_{z}$ and $\tau_{y z}$ are obviously zero at the point under consideration.

From Equations 31, 5b, and 22b, Tables 2 and 3, and the above result for $\sigma_{p}$,

$$
\begin{aligned}
& \sigma_{z} \left\lvert\,=\left[\frac{\mu}{1-\mu} 0.4221\right.\right.+(0.0221+0.6913-0.0221 \times 0.6913) \\
& \begin{array}{c}
z=c
\end{array} \\
& y=0 \\
&-(0.2186+0.2186-0.2186 \times 0.2186)] E S_{\infty} \\
& \sigma_{x} \mid=\left[\frac{\mu}{1-\mu} 0.4221+0.3087\right] E S_{\infty} \\
& z=c \\
& y=0
\end{aligned}
$$

For all values of Poisson's ratio $\mu$ less than 0.212 , the above expressions will give larger values for $\sigma_{y}$ than for $\sigma_{x}$. This fact is of interest because,
in the similar problem of a long cylinder drying or cooling from the curved surface the two corresponding stresses are equal regardless of Poisson's ratio ${ }^{13}$.

## OTHER METHODS

A paper of this kind would not be complete without calling attention to the possibilities of using a difference equation instead of Equation 1. The difference equation that would replace Equation 1 would be a statement of the relation between the values of $S$ at the point $(x, y, z)$ at the time $t$ and what the values of $S$ were at this point and at points distant $\pm \Delta x$, $\pm \Delta y$ and $\pm \Delta z$ in the $x$-, $y$-, and $z$-directions, $\Delta t$ units of time previously. In the limit as $\Delta x, \Delta y, \Delta z$ and $\Delta t$ are made smaller, the difference equation becomes Equation 1. Although methods based upon the substitution of a difference equation for the differential equation usually necessitate considerable work to obtain the desired results, they have the advantage of being applicable to any shape of body, any assumption in regard to boundary conditions, or any variation in the coefficient of diffusivity.

Methods of procedure for solving the difference equation, usually for the analogous problem of heat-flow, may be found in the literature*.

After a solution for the distribution of shrinkage tendency at a given time has been obtained to the desired accuracy, there still remains the problem of solving for stresses. This may be done by graphical analysis, as illustrated by Buchanan and Schroeder ${ }^{16}$ if the problem is one-dimensional or by more elaborate means if the problem is two- or three-dimensional. $\dagger$

## SUMMARY OF THE THEORETICAL DEVELOPMENT

The theoretical equations for the flow of heat are used for the unrestrained shrinkage of concrete. Unrestrained shrinkage is defined as the unit linear contraction that would occur in an element if it were unrestrained by neighboring elements. This is not the same as unit shortening (commonly called shrinkage) of a so-called unrestrained specimen. Having developed expressions for unrestrained shrinkage for each of three different conditions (Cases I, II and III), equations for shortening, warping, and stresses were derived.

The equations for two conditions-slab or beam drying from one face only, Case I, and slab or beam drying from two opposite faces, Case IIwere found to be very much alike in form. However, the equation for shrinkage stress in the slab or beam drying from one surface contains an additional term, $\left(6-12 \frac{y}{b}\right) \frac{2 E b v_{\text {max }}}{3 l^{2}}$, that is not in the corresponding
equation for a slab or beam drying from two opposite surfaces. Computations show that the equation with the added term gives much less stress for the same size of body and the same period of drying. Compare Tables 6 and 5

Equations are given for all the stresses in a prism drying from all four surfaces, the third condition treated. These equations, though rather complicated in appearance, can be readily evaluated if the desired accuracy is such that only a few terms in each series need be considered. For a rather rough approximation of the stress that is usually the most important in this third condition, the comparatively simple equation $\sigma_{x}=$ $E\left(S-S_{a v}\right)$ is recommended.
Tables and curves are given from which the theoretical shrinkages, stresses, etc., may be obtained, at any point in the specimen after any period of drying, for various values of the physical properties, diffusivity, surface factor, ultimate shrinkage, and dimensions of the specimen.

Examples are given showing how numerical values may be computed from the equations and how the tables and curves may be used.

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Fig. 1-Illustrations of the conditions treated. Shading indicates sealed surfaces. Case I -Beam (or slab) drying from one face only. Case ll-Beam (or slab) drying from two opposite faces. Case III-Prism diying from four faces. The ends of the prism of $x= \pm a$ are sealed.



Fig. 2-Curves for the determination of $\beta_{n}$


Fig. 3-Relationship between $F_{n}$ and B


Fig. 4-Relationship between $\cos \beta_{n}$ and $B$


Fig. 5-Relationship between $H_{n}$ and $B$


Fig. 6-Relationship between $G_{n}$ and B


Fig. 8-Theoretical curves showing the ratio of unit shartening to
final shortening vs. the square root of the time parameter


Fig. 7-Distribution of shrinkage for various values of $B$ and $T$
Fig. 9-Theoretical curves showing warping vs, square rool of the time parameter for a prism drying from one side

Fig. 12-Stress vs. square root of $T$ at various distances from central plane of prisms drying from two opposite sides. $B=5.0$

Fig. 13-Stress vs, square root of $T$ at various values of $y / b$ for
an unrestrained prism drying from only one surface, $B=5.0$


Fig. 14 -Maximum stress (maximum value of stress at exposed surface) and maximum warping vs. the parameter $B$

| TABLE 1- $\beta_{n}$ <br> $\beta_{n} \tan \beta_{n}=B$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $B$ | $\beta_{1}$ | $\beta_{2}$ | $\beta_{3}$ | $\beta_{4}$ | $\beta_{5}$ | $\beta_{6}$ |
| 0.0 | 0 |  | $\pi$ | $2 \pi$ |  | $3 \pi$ |
| 0.1 | 0.311053 | 3.1731 | 6.2991 | 9.4354 | 12.574 | 15.715 |
| 0.5 | 0.653271 | 3.2923 | 6.3615 | 9.4774 | 12.606 | 15.740 |
| 1.0 | 0.860334 | 3.4256 | 6.4372 | 9.5292 | 12.645 | 15.771 |
| 2.0 | 1.076874 | 3.6436 | 6.5783 | 9.6296 | 12.722 | 15.834 |
| 5.0 | 1.31384 | 4.0338 | 6.9097 | 9.8927 | 12.935 | 16.010 |
| 10.0 | 1.42886 | 4.3058 | 7.2281 | 10.200 | 13.213 | 16.260 |
| 100.0 | 1.55525 | 4.6656 | 7.7760 | 10.887 | 13.998 | 17.109 |
| $\infty$ | $\pi / 2$ | $3 \pi / 2$ | $5 \pi / 2$ | $7 \pi / 2$ | $9 \pi / 2$ | $11 \pi / 2$ |
|  |  |  |  |  |  |  |

## TABLE 2-RATIO OF SHRINKAGE (OR SWELLING) TO ULTIMATE SHRINKAGE (OR SWELLING)

| $T$ | $\frac{y}{b}=0$ | $\frac{y}{b}=0.2$ | $\frac{y}{b}=0.4$ | $\frac{y}{b}=0.6$ | $\frac{y}{b}=0.8$ | $\frac{y}{b}=1.0$ | $\frac{y}{b}=0$ | $\frac{y}{b}=0.2$ | $\frac{y}{b}=0.4$ | $\frac{y}{b}=0.6$ | $\frac{y}{b}=0.8$ | $\frac{y}{b}=1.0$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $B=0.1$ |  |  |  |  |  | $B=1.0$ |  |  |  |  |  |
|  | $0$ | $0$ | $0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $0.005$ | $.0000$ | $.0000$ | $.0000$ | . 0000 | . 0002 | . 0080 | 0000 | . 0000 | 0000 | . 0000 | . 0007 | . 0750 |
| 0.010 | . 0000 | . 0000 | . 0000 | . 0000 | . 0009 | 0112 | 0000 | . 0000 | 0000 | . 0002 | . 0093 | . 1035 |
| 0.015 | . 0000 | . 0000 | . 0000 | . 0001 | . 0021 | . 0135 | 0000 | . 0000 | . 0000 | . 0012 | . 0196 | . 1238 |
| 0.020 | . 0000 | . 0000 | . 0000 | . 0003 | . 0033 | . 0158 | . 0000 | . 0000 | 0005 | . 0033 | . 0305 | . 1416 |
| 0.030 | . 0000 | . 0000 | . 0001 | . 0011 | . 0057 | . 0192 | . 0000 | . 0001 | . 0013 | . 0095 | . 0511 | . 1690 |
| 0.040 | . 0000 | . 0000 | . 0003 | . 0020 | . 0080 | . 0222 | . 0001 | . 0003 | . 0033 | . 0182 | . 0700 | . 1910 |
| 0.050 | . 0000 | . 0001 | -0007 | . 0030 | . 0100 | . 0247 | . 00003 | . 0010 | . 0085 | . 0274 | . 0870 | . 2096 |
| 0.075 | . 0002 | . 0005 | . 0020 | . 0060 | . 0146 | . 0302 | . 0021 | . 0050 | . 0178 | . 0516 | . 1236 | . 2471 |
| 0.10 | -0008 | . 0014 | . 0037 | . 0089 | . 0188 | . 0347 | . 0068 | . 0121 | . 0317 | . 0749 | . 1538 | . 2764 |
| 0.15 | . 0029 | . 0040 | . 0077 | . 0146 | . 0260 | . 0423 | . 0244 | . 0333 | . 0623 | . 1168 | . 2025 | . 3218 |
| 0.20 | . 0060 | . 0074 | . 0119 | . 0198 | . 0319 | . 0486 | . 0499 | . 0609 | . 0950 | . 1545 | . 2418 | . 3570 |
| 0.30 | . 0139 | . 0156 | . 0209 | . 0298 | 0.426 | . 0594 | . 1036 | . 1164 | . 1534 | . 2153 | . 3012 | . 4091 |
| 0.40 | . 0228 | . 0246 | . 0301 | . 0393 | .0523 | . 0690 | . 1691 | . 1811 | . 2169 | . 2757 | . 3560 | . 4559 |
| 0.50 | . 0320 | . 0339 | . 0394 | . 0487 | . 0616 | . 0782 | . 2274 | . 2387 | . 2725 | . 3276 | . 4026 | . 4954 |
| 0.75 | . 0550 | . 0569 | . 0623 | . 0714 | . 0841 | . 1004 | . 3575 | . 3670 | . 3953 | . 4413 | . 5038 | . 5810 |
| 1.0 | . 0776 | . 0794 | . 0847 | . 0937 | . 1060 | . 1219 | . 4662 | . 4741 | . 4975 | . 5358 | . 5878 | . 6518 |
| 1.5 | . 1213 | . 1230 | . 1281 | . 1364 | . 1484 | . 1634 | . 6312 | . 6367 | . 6528 | . 6793 | . 7152 | . 7595 |
| 2.0 | . 1627 | . 1643 | . 1691 | . 1772 | . 1885 | . 2028 | . 7453 | . 7490 | . 7603 | . 7785 | . 8033 | . 8339 |
| 3.0 | . 2399 | . 2414 | . 2458 | . 2531 | . 2634 | . 2764 | . 8785 | . 8803 | . 8856 | . 8943 | . 9061 | . 9207 |
| 4.0 | . 3100 | . 3114 | . 31.54 | . 3220 | . 3313 | . 3432 | . 9420 | -9429 | . 94.54 | . 9496 | . 9552 | . 9622 |
| 5.0 | . 3736 | . 3748 | . 3785 | . 3845 | . 3929 | . 4037 | . 9724 | . 9728 | . 9740 | . 9760 | . 9786 | . 9820 |
| 7.5 | . 5082 | . 5092 | . 5120 | . 5167 | . 5234 | . 5318 |  |  |  |  |  |  |
| 10.0 | . 6240 | . 6247 | . 6269 | -6306 | - 6356 | . 6421 |  |  |  |  |  |  |
| 15.0 | . 7620 | 7625 | . 7638 | . 7661 | . 7693 | . 7734 |  |  |  |  |  |  |
| 20.0 | . 8533 | . 8535 | . 8544 | . 8558 | . 8578 | . 8603 |  |  |  |  |  |  |
|  | $B=5.0$ |  |  |  |  |  | $B=\infty$ |  |  |  |  |  |
| 0.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | O | 0 | 000 |
| 0.001 | . 0000 | . 0000 | . 0000 | . 0000 | . 0000 | . 1559 | . 0000 | . 0000 | . 0000 | . 0000 | . 0000 | 1.000 |
| 0.002 | . 0000 | . 0000 | . 0000 | . 0000 | . 0001 | . 2095 | . 0000 | . 0000 | . 0000 | . 0000 | . 0016 | 1.000 |
| 0.005 | . 0000 | . 0000 | . 0000 | . 0000 | . 0072 | . 3019 | 0000 | . 0000 | . 0000 | . 0001 | . 0455 | 1.000 |
| 0.010 | . 0000 | . 0000 | . 0000 | . 0008 | . 0390 | . 3843 | . 0000 | . 0000 | . 0002 | . 0060 | . 1541 | 1.000 |
| 0.015 | . 0000 | . 0000 | . 0000 | . 0048 | . 0770 | . 4378 | . 0000 | . 0000 | .0005 | . 0209 | . 2483 | 1.000 |
| 0.020 | . 0000 | . 0003 | . 0004 | . 0126 | . 1132 | - 4769 | . 0000 | . 0001 | . 0027 | . 0455 | . 3173 | 1.000 |
| 0.030 | . 0001 | . 0010 | . 0043 | . 0353 | . 1762 | . 5332 | .0001 | . 0010 | . 0144 | . 1025 | . 4142 | 1.000 |
| 0.040 | . 0006 | . 0017 | . 0117 | . 0634 | . 2291 | . 5724 | . 0008 | . 0047 | . 0339 | . 1573 | . 4795 | 1.000 |
| 0.050 | . 0020 | . 0042 | . 0225 | . 0918 | . 2723 | . 6026 | . 0031 | . 0115 | . 0578 | . 2059 | . 5271 | 1.000 |
| 0.075 | . 0070 | . 0175 | . 0572 | . 1579 | . 3544 | . 6557 | . 0198 | . 0407 | . 1216 | . 3017 | . 6056 | 1.000 |
| 0.10 | . 0221 | . 0390 | . 0955 | . 2142 | . 4134 | . 6913 | . 0410 | . 0717 | . 1735 | . 3657 | . 6517 | 1.000 |
| 0.15 | . 0725 | . 0962 | . 1706 | . 3025 | . 4947 | . 7379 | . 1358 | . 1726 | . 2839 | . 4687 | . 7160 | 1.000 |
| 0.20 | . 1346 | . 1613 | . 2398 | . 3709 | . 5506 | . 7685 | . 2276 | 2637 | . 3710 | . 5384 | . 7558 | 1.000 |
| 0.30 | . 2637 | . 2882 | . 3607 | . 4771 | . 6304 | . 8106 | . 3933 | . 4227 | . 5085 | . 6425 | . 8119 | 1.000 |
| 0.40 | . 3788 | . 3999 | . 4621 | . 5613 | 6907 | . 8417 | . 5256 | . 5487 | . 6161 | . 7210 | . 8533 | 1.000 |
| 0.50 | . 4769 | . 4948 | . 5474 | . 6311 | . 7401 | . 8670 | . 6294 | . 6475 | . 7001 | . 7821 | . 8855 | 1.000 |
| 0.75 | . 6602 | . 6718 | . 7061 | . 7604 | . 8312 | . 9136 | . 8000 | . 8098 | . 8382 | . 8824 | . 9382 | 1.000 |
| 1.0 | . 7793 | . 7869 | . 8091 | . 8444 | . 8904 | . 9439 | . 8920 | 8972 | . 9126 | . 9365 | .9666 | 1.000 |
| 1.5 | . 9069 | . 9101 | . 9145 | . 9344 | . 9538 | . 9763 | . 9686 | 9700 | . 9746 | . 9815 | -9903 | 1.000 |
| 2.0 | . 9607 | .9621 | . 9660 | . 9723 | . 9805 | . 9900 | . 9908 | 9913 | . 9926 | . 9946 | . 9972 | 1.000 |
| 3.0 | . 9930 | . 9933 | . 9940 | . 99.51 | . 9965 | . 9982 | . 9992 | . 9993 | . 9994 | . 9995 | . 9998 | 1.000 |
| 4.0 | . 9988 | . 9988 | . 9990 | . 9991 | . 9994 | . 9997 |  |  |  |  |  |  |

TABLE 3-UNIT SHORTENING-AVERAGE UNIT SHRINKAGE $\left(\frac{S_{o r}}{S_{-}}\right)$

$$
\frac{S_{w r}}{S_{\infty}}=1-\sum_{1}^{\infty} e^{-T^{T} B^{2} n} H_{n}
$$

| $T$ | $B=0.1$ | $B=0.5$ | $B=1.0$ | $B=2.0$ | $B=5.0$ | $B=10.0$ | $B=\infty$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.001 | 0.0001 | 0.0005 | 0.0010 | 0.0019 | 0.0045 | 0.0080 | 0.0357 |
| 0.002 | 0.0002 | 0.0010 | 0.0019 | 0.0037 | 0.0085 | 0.0147 | 0.0506 |
| 0.003 | 0.0003 | 0.0015 | 0.0029 | 0.0055 | 0.0124 | 0.0206 | 0.0619 |
| 0.005 | 0.0005 | 0.0024 | 0.0047 | 0.0090 | 0.0196 | 0.0323 | 0.0800 |
| 0.010 | 0.0010 | 0.0048 | 0.0093 | 0.0173 | 0.0370 | 0.0556 | 0.1129 |
| 0015 | 0.0015 | 0.0072 | 0.0137 | 0.0252 | 0.0515 | 0.0755 | 0.1383 |
| 0.02 | 0.0020 | 0.0095 | 0.0181 | 0.0317 | 0.0649 | 0.0932 | 0.1596 |
| 0.03 | 0.0030 | 0.0141 | 0.0265 | 0.0473 | 0.0891 | 0.1242 | 0.1954 |
| 0.04 | 0.0039 | 0.0185 | 0.0347 | 0.0611 | 0.1115 | 0.1512 | 0.2257 |
| 0.05 | 0.0049 | 0.0231 | 0.0426 | 0.0742 | 0.1319 | 0.1753 | 0.2523 |
| 0.075 | 0.0074 | 0.0340 | 0.0620 | 0.1050 | 0.1779 | 0.2282 | 0.3090 |
| 0.10 | 0.0098 | 0.0446 | 0.0803 | 0.1336 | 0.2186 | 0.2739 | 0.3506 |
| 0.15 | 0.0146 | 0.0653 | 0.1154 | 0.1860 | 0.2895 | 0.3510 | 0.4370 |
| 0.20 | 0.0193 | 0.0854 | 0.1489 | 0.2327 | 0.3510 | 0.4167 | 0.5041 |
| 0.30 | 0.0288 | 0.1239 | 0.2064 | 0.3190 | 0.4555 | 0.5258 | 0.6133 |
| 0.40 | 0.0382 | 0.1606 | 0.2666 | 0.3939 | 0.5422 | 0.6136 | 0.6979 |
| 0.50 | 0.0474 | 0.1957 | 0.3188 | 0.4604 | 0.6148 | 0.6849 | 0.7641 |
| 0.75 | 0.0702 | 0.2771 | 0.4339 | 0.5962 | 0.7498 | 0.8109 | 0.8726 |
| 1.0 | 0.0924 | 0.3502 | 0.5296 | 0.6979 | 0.8375 | 0.8865 | 0.9313 |
| 1.5 | 0.1353 | 0.4751 | 0.6751 | 0.8306 | 0.9315 | 0.9591 | 0.9800 |
| 2.0 | 0.1761 | 0.5767 | 0.7756 | 0.9052 | 0.9711 | 0.9857 | 0.9942 |
| 3.0 | 0.2521 | 0.7233 | 0.8930 | 0.9625 | 0.9949 | 0.9981 | 0.9995 |
| 4.0 | 0.3211 | 0.8182 | 0.9489 | 0.9907 | 0.9991 | 0.9998 | 1.0000 |
| 5.0 | 0.3837 | 0.8821 | 0.9756 | 0.9971 | 0.9998 | 1.0000 |  |
| 7.5 | 0.5161 | 0.9594 | 0.9962 | 0.9998 | 1.0000 |  |  |
| 10.0 | 0.6301 | 0.9860 | 0.9994 | 1.0000 |  |  |  |
| 15.0 | 0.7658 | 0.9983 | 1.0000 |  |  |  |  |

TABLE 4-WARPING $\left(\frac{2 b v m a r}{33^{2} S=}\right.$ in thousandths $)$

$$
\frac{2 b x_{\text {mex }}}{3 \rho^{2} S_{0}}=\sum_{1}^{\infty} e^{-T \beta^{2} n} G_{n}
$$

| $T$ | $B=0.1$ | $B=0.5$ | $B=1.0$ | $B=2.0$ | $B=5.0$ | $B=10.0$ | $\mathbf{B}=\infty$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.0010 | 0.048 | 0.235 | 0.46 | 0.91 | 2.13 | 3.82 | 16.83 |
| 0.0015 | 0.071 | 0.348 | 0.68 | 1.33 | 3.08 | 5.43 | 19.79 |
| 0.0020 | 0.093 | 0.458 | 0.90 | 1.73 | 3.99 | 6.89 | 23.22 |
| 0.0030 | 0.137 | 0.674 | 1.32 | 2.54 | 5.73 | 9.61 | 27.90 |
| 0.0040 | 0.180 | 0.883 | 1.72 | 3.31 | 7.23 | 12.03 | 31.66 |
| 0.0050 | 0.222 | 1.008 | 2.12 | 4.03 | 8.69 | 14.27 | 34.89 |
| 0.0075 | 0.324 | 1.579 | 3.06 | 5.74 | 13.68 | 19.16 | 41.36 |
| 0.010 | 0.421 | 2.045 | 3.94 | 7.35 | 15.18 | 23.35 | 46.42 |
| 0.015 | 0.606 | 2.920 | 5.59 | 10.25 | 20.45 | 30.38 | 54.12 |
| 0.020 | 0.778 | 3.735 | 7.08 | 12.92 | 24.97 | 35.92 | 59.78 |
| 0.030 | 1.092 | 5.415 | 9.74 | 17.34 | 32.63 | 44.71 | 67.67 |
| 0.040 | 1.375 | 6.518 | 12.05 | 21.07 | 38.32 | 50.93 | 72.80 |
| 0.050 | 1.628 | 7.832 | 14.03 | 24.21 | 42.70 | 55.33 | 76.15 |
| 0.075 | 2. 161 | 10.138 | 17.96 | 30.08 | 49.98 | 62:22 | 79.76 |
| 0.10 | 2.568 | 11.846 | 20.75 | 33.87 | 53.81 | 65.16 | SO. 40 |
| 0.15 | 3.126 | 14.017 | 23.96 | 37.56 | 55.70 | 64.50 | 74.18 |
| 0.20 | 3.457 | 15.128 | 25.29 | 38.32 | 53.89 | 60.56 | 66.85 |
| 0.30 | 3.757 | 15.773 | 25.61 | 36.28 | 47.03 | 50.52 | 52.74 |
| 0.40 | 3.843 | 15.546 | 24.14 | 32.88 | 39.90 | 41.38 | 41.27 |
| 0.50 | 3.851 | 15.043 | 22.59 | 29.43 | 33.64 | 33.77 | 32.25 |
| 0.75 | 3.782 | 13.588 | 18.85 | 22.07 | 21.86 | 20.27 | 17.40 |
| 1.0 | 3.694 | 12.217 | 15.67 | 16.52 | 14.20 | 12.17 | 9.39 |
| 1.5 | 3.519 | 9.869 | 10.82 | 9. 25 | 5.99 | 4.38 | 2.74 |
| 2.0 | 3.353 | 7.959 | 7.47 | 5. 18 | 2.53 | 1.58 | 0.80 |
| 3.0 | 3.044 | 5.203 | 3.57 | 2.05 | 0.45 | 0.20 | 0.07 |
| 4.0 | 2.763 | 3.417 | 1.70 | 0.51 | 0.08 | 0.03 | 0.01 |
| 50 | 2.508 | 2.216 | 0.81 | 0.16 | 0.01 | 0.00 | 0.00 |
| 7.5 | 1.969 | 0.763 | 0.13 | 0.01 | 0.00 |  |  |
| 10.0 | 1.506 | 0.262 | 0.02 | 0.00 |  |  |  |
| 15.0 | 0.953 | 0.031 | 0.00 |  |  |  |  |
| 20.0 | 0.587 | 0.004 |  |  |  |  |  |

TABLE 5-RATIO OF STRESS $\sigma^{\prime \prime} z$ IN AN UNRESTRAINED BEAM DRYING FROM TWO OPPOSITE SIDES (OR IN A BEAM DRYING FROM ONLY ONE SIDE AND RESTRAINED AGAINST WARPING) TO THE ULTIMATE STRESS FOR COMPLETE RESTRAINT ES ${ }_{\infty}$

|  |  |  |  |  | $\frac{\sigma^{\prime \prime} x}{E S_{\infty}}$ | $\frac{S}{S=}$ | $-\frac{S_{a v}}{S_{\infty 0}}$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T$ | $\frac{y}{b}=0$ | $\frac{y}{b}=0.2$ | $\frac{y}{b}=0.4$ | $\frac{y}{b}=0.6$ | $\frac{\boldsymbol{y}}{\boldsymbol{b}}=0.8$ | $\frac{y}{b}=1.0$ | $\frac{y}{b}=0$ | $\frac{y}{b}=0.2$ | $\frac{y}{b}=0.4$ | $\frac{y}{b}=0.6$ | $\frac{y}{b}=0.8$ | $\frac{y}{b}=1.0$ |
|  | $B=0.1$ |  |  |  |  |  | $B=1.0$ |  |  |  |  |  |
| 0.0 | 0 | 0 | 0 |  |  | 0 | 0 | 0 | 0 | 0 |  | 0 |
| 0.005 | -. 00005 | -. 0005 | -.000 | -. 0005 | -. 0003 | . 0075 | -. 0047 | -. 0047 | -. 0047 | -. 0047 | $-.0040$ | $.0703$ |
| 0.010 | -. 0010 | -. 0010 | -. 0010 | -. .0010 | -. 0001 | . 0102 | -. 0093 | -. 00093 | -. 0093 | -. 0091 | . 0000 | . 0942 |
| 0.015 | -. 0015 | -. 0015 | -. 00015 | -.0014 | . 0006 | . 0120 | -. 0137 | -. 0137 | -. 0137 | -.0125 | . 0059 | . 1101 |
| 0.020 | -. 0020 | -. 0020 | -. 0020 | -. 0017 | . 0013 | . 0138 | -. 0181 | -. 0181 | -. 0176 | -. 0148 | . 0124 | 1235 |
| 0.030 | -. 0030 | -. 0030 | -. 0029 | -. 0019 | . 0027 | . 0162 | -. 0265 | -. 0264 | -. 0252 | -. 0170 | .0246 | . 1425 |
| 0.040 | -. 0039 | -. 0039 | -. 00036 | -. 0019 | . 0041 | . 0183 | -. 0346 | -. 0344 | -. 0314 | -. 0165 | . 0353 | 1563 |
| 0.050 | -. 0049 | -. 0048 | -. 0042 | -. 0019 | .0051 | . 0198 | -. 0423 | -. 0416 | -. 0361 | -. 0152 | . 0444 | 1670 |
| 0.075 | -. 0072 | -. 0069 | -. 0054 | -.0014 | . 0072 | . 0228 | -. 0599 | -. 0571 | -. 0442 | -. 0104 | . 0616 | 1851 |
| 0.10 | -. 0090 | -. 0084 | -. 00061 | -.0009 | . 0090 | . 0249 | -. 0735 | -. 0682 | -. 0486 | -. 0054 | . 0735 | 1961 |
| 0.15 | -. 0117 | -. 0106 | -. 0069 | . 0000 | . 0114 | . 0277 | -. 0910 | -. 0821 | -. 0531 | . 0014 | . 0871 | 2064 |
| 0.20 | -. 0133 | -. 01119 | -. 00074 | . 0005 | . 0126 | . 0293 | -. 0999 | -. 0888 | -. 0539 | . 0056 | . 0929 | . 2081 |
| 0.30 | -. 0149 | -. 0132 | -. 0079 | . 0010 | . 0138 | . 0306 | -. 1028 | -. 0900 | -. 0530 | . 0089 | . 0948 | . 2027 |
| 0.40 | -. 0154 | -. 0136 | -.0081 | . 0011 | . 0141 | . 0308 | -. 0975 | -. 0855 | -. 0497 | . 0091 | . 0894 | . 1893 |
| 0.50 | -. 0154 | -. 0135 | -.0080 | . 0013 | . 0142 | . 0311 | -. 0914 | -. 0801 | -. 0463 | . 0088 | . 0838 | . 1766 |
| 0.75 | -. 0152 | -. 0133 | -. 00079 | . 0013 | . 0139 | . 0302 | -. 0764 | -. 0669 | -. 0386 | . 0074 | . 0699 | . 1471 |
| 1.0 | -. 0148 | -. 0130 | -. 00077 | . 0012 | . 0136 | . 0295 | -. 0634 | -. 0555 | -. 0321 | . 0062 | . 0582 | . 1222 |
| 1.5 | -. 0140 | -. 0123 | -. 00072 | . 0011 | . 0131 | . 0281 | -. 0439 | -. 0384 | -. 0223 | . 0042 | . 0401 | . 0844 |
| 2.0 | -. 0134 | -. 0118 | -. 00070 | . 0011 | . 0124 | . 0267 | -. 0303 | -.0266 | -. 0153 | . 0029 | . 0277 | . 0583 |
| 3.0 | -. 0122 | -. 0107 | -. 0063 | . 0010 | . 0113 | . 0243 | -. 0145 | -. 0127 | -. 0074 | . 0013 | . 0131 | . 0277 |
| 4.0 | -. 0111 | -. 0097 | -.0057 | . 0009 | . 0102 | . 0221 | -. 0069 | -. 0060 | -. 0035 | . 0007 | . 0063 | . 0133 |
| 5.0 | -. 0101 | -. 0089 | -. 0052 | . 0008 | . 0092 | . 0200 | -.0032 | -. 0028 | -. 0016 | . 0004 | . 0030 | . 0064 |
| 7.5 | -. 0079 | -. 00069 | -. 0041 | . 0006 | . 0073 | . 0157 |  |  |  |  |  |  |
| 10.0 | -. 0061 | -. 0055 | -. 00033 | . 0005 | . 0055 | . 0120 |  |  |  |  |  |  |
| 15.0 | -. 0038 | -. 0033 | -. 0020 | . 0003 | . 0035 | . 0076 |  |  |  |  |  |  |
| 20.0 | -. 0023 | -. 0021 | -. 0012 | . 0002 | . 0022 | . 0047 |  |  |  |  |  |  |
|  |  |  | $B=$ | 0 |  |  |  |  | $B$ |  |  |  |
| 0.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |  | 0 | O |  |
| 0.001 | -. 0045 | -. 0045 | -. 0045 | -. 0045 | -. 0045 | . 1514 | -. 0357 | -. 0357 | -. 0357 | -. 0357 | -. 0357 | . 9743 |
| 0.002 | -. 0085 | -. 0085 | -. 0085 | -. 0085 | -. 0084 | . 2010 | -. 0506 | -. 0506 | -. 0506 | -. 0506 | -. 0490 | . 9494 |
| 0.005 | -. 0196 | -. 0196 | -. 0196 | -. 0196 | -. 0124 | . 2823 | -. 0800 | -. 0800 | -. 0800 | -. 0799 | -. 0345 | .9200 |
| 0.010 | -. 0370 | -. 0370 | -. 0370 | -. 0362 | . 00220 | . 3473 | -. 1129 | -. 1129 | -. 1127 | -. 1069 | . 0412 | . 88717 |
| 0.015 | -. 0515 | -. 0515 | -.0515 | -. 0467 | . 0255 | . 3863 | -. 1383 | -. 1383 | -. 1378 | -. 1174 | . 1100 | .8617 |
| 0.020 | -. 0649 | -. 0646 | -.0645 | -. 0523 | . 0483 | . 4120 | -. 1596 | -. 1595 | -. 1569 | -. 1141 | . 1577 | . 8404 |
| 0.030 | -. 0890 | -. 0881 | -. 0848 | -. 0538 | . 0871 | . 4441 | -. 1953 | -. 1944 | -. 1810 | -. 0929 | . 2188 | . 8046 |
| 0.040 | -. 1109 | -. 1098 | -. 0998 | -. 0481 | . 1176 | . 4609 | -. 2249 | -. 2210 | -. 1918 | -. 0684 | . 2538 | . 7743 |
| 0.050 | -. 1299 | -. 1277 | -. 1094 | -. 0401 | . 1404 | . 4707 | -. 2492 | -. 2408 | -. 1945 | -. 0464 | . 2748 | . 7477 |
| 0.075 | -. 1709 | -. 1604 | -. 1207 | $-.0200$ | . 1765 | . 4778 | -. 2892 | -. 2683 | -. 1874 | -. 0073 | . 2966 | .6910 |
| 0.10 | -. 1965 | -. 1796 | -. 1230 | -. 0044 | . 1948 | . 4727 | -. 3096 | -. 2789 | -. 1771 | 0151 | . 3011 | . 6494 |
| 0.15 | -. 2169 | -. 1969 | -. 1189 | $-.0130$ | . 2052 | . 4484 | -. 3012 | -. 2644 | -. 1531 | 0317 | . 2790 | . 5630 |
| 0.20 | -. 2164 | -. 1897 | -. 1112 | -. 0199 | . 1996 | .4175 | -. 2765 | -. 2404 | -. 1331 | 0343 | . 2517 | . 4959 |
| 0.30 | -. 1918 | -. 1671 | -. 0948 | . 0216 | . 1749 | . 3551 | -. 2200 | -. 1906 | -. 1048 | 0292 | . 1986 | . 3867 |
| 0.40 | -. 1634 | -. 1422 | -. 0801 | 0191 | . 1485 | . 2995 | -. 1723 | -. 1492 | -. 0818 | 0231 | . 1554 | . 3021 |
| 0.50 | -. 1379 | -. 1193 | -.0674 | 0163 | . 1253 | . 2522 | -. 1347 | -. 1166 | -. 0640 | 0180 | . 1214 | .2359 |
| 0.75 | -. 0896 | -. 0779 | -. 0437 | . 0106 | . 0814 | . 1638 | -. 0727 | -. 0629 | -. 0345 | . 0097 | . 0655 | . 1273 |
| 1.0 | -. 0582 | -. 0506 | -. 0284 | . 0069 | . 0529 | . 1064 | -. 0393 | -. 0341 | -. 0187 | . 0052 | . 0353 | . 0687 |
| 1.5 | -. 0246 | -. 0214 | -. 0170 | . 0029 | . 0223 | . 0448 | -. 0114 | -. 0100 | -. 0054 | 0015 | . 0103 | . 0200 |
| 2.0 | -. 0104 | -.0090 | -.0051 | . 00012 | . 0094 | . 0189 | -. 0034 | -. 0029 | -. 0016 | . 0004 | . 0030 | 0058 |
| 3.0 | -. 0019 | -. 0016 | -. 0009 | 0002 | . 0016 | . 0033 |  |  |  |  |  |  |

TABLE 6-RATIO OF STRESS $\sigma_{x}$ IN AN UNRESTRAINED BEAM DRYING FROM ONLY ONE SIDE TO ULTIMATE STRESS FOR COMPLETE RESTRAINT ES $\infty_{\infty}$

$$
\frac{\sigma_{z}}{E S_{\infty}}=\frac{S}{S_{\infty}}-\frac{S_{\alpha v}}{S_{\infty}}+\left(6-12 \frac{y}{b}\right) \frac{2 b v_{m a \alpha}}{3 l^{2} S_{\infty}}
$$

| $T$ | $\frac{y}{b}=0$ | $\frac{y}{b}=0.2$ | $\frac{y}{b}=0.4$ | $\frac{y}{b}=0.6$ | $\frac{y}{b}=0.8$ | $\frac{y}{b}=1.0$ | $\frac{y}{b}=0$ | $\frac{y}{b}=0.2$ | $\frac{y}{b}=0.4$ | $\frac{y}{b}=0.6$ | $\frac{y}{b}=0.8$ | $\frac{y}{b}=1.0$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $B=0.1$ |  |  |  |  |  | $B=1.0$ |  |  |  |  |  |
| 0.0 | ${ }_{0}^{0}$ | ${ }_{0}^{0}$ | $\begin{gathered} 0 \\ -0 \cap ก 2 \end{gathered}$ | $\begin{gathered} 0 \\ -.0008 \end{gathered}$ | $\begin{gathered} 0 \\ -.0011 \end{gathered}$ | $\begin{gathered} 0 \\ .0062 \end{gathered}$ | $\begin{gathered} 0 \\ .0080 \end{gathered}$ | $\begin{gathered} 0 \\ .0029 \end{gathered}$ | $\begin{gathered} 0 \\ -.0022 \end{gathered}$ | $\begin{gathered} 0 \\ -.0072 \end{gathered}$ | $\begin{gathered} 0 \\ -.0116 \end{gathered}$ | $\begin{gathered} 0 \\ .0576 \end{gathered}$ |
| 0.005 | . 0008 | . 0003 | $-.0002$ | $-.0008$ | $-.0011$ | .0062 .0078 | . 0080 | . 0029 | $\begin{aligned} & -.0022 \\ & -.0046 \end{aligned}$ | $\begin{aligned} & -.0072 \\ & -.0138 \end{aligned}$ | $\begin{aligned} & -.0116 \\ & -.0142 \end{aligned}$ | $.0576$ |
| 0.010 | . 0015 | . 00005 | -.0005 | -.0015 -.0021 | -. 00016 | . 00078 | . 0198 | . 0044 | -. -0070 | -.0192 | -. 0142 | . 0766 |
| 0.015 | . 0021 | . 00007 | -. 0.0008 | -.0021 -.0026 | -.0016 | . 00091 | . 0243 | . 00074 | -. -0091 | -.0233 | -. 0131 | . 0811 |
| 0.020 0.030 | . 0027 | . 00009 | -.0011 -.0016 | -. | -.0012 | . 0096 | . 0319 | . 0087 | -. 0135 | -.0287 | -.0105 | . 0841 |
| 0.040 | . 0044 | . 0011 | -. 0019 | -. 0036 | -. 0009 | . 0100 | . 0377 | . 0090 | -. 0169 | -.0310 | -. 0081 | . 0840 |
| 0.040 0.050 | . 0048 | . 0011 | -. 0022 | -. 00039 | -. 00008 | . 0100 | . 0419 | . 0089 | -. 0193 | -.0320 | -.0061 | . 0828 |
| 0.075 | . 0058 | . 0009 | -. 0028 | -. 0040 | -. 0006 | . 0098 | . 0479 | . 00077 | -. -.0226 | -. 0.0303 | -.0031 | . 0716 |
| 0.10 | . 0064 | . 0008 | -. 0031 | -. 0040 | -. 0002 | . 00985 | . 0510 | . 00042 | -.0237 | -.0303 | -.0012 | . 0626 |
| 0.15 | . 00071 | . 00007 | -.0031 -.0033 | -.0038 -.0036 | . 00001 | . 00089 | . 0528 | . 00040 | -. 02243 | -.02247 | . 0019 | . 0564 |
| 0.20 0.30 | .0074 .0076 | . 0005 | -.0033 -.0034 | -. 0036 | . 0003 | . 0080 | . 0509 | . 0022 | -. 0223 | -. 0218 | . 0026 | . 0490 |
| 0.40 | . 0077 | . 0002 | -. .0035 | -.0035 | . 0003 | . 0077 | . 0473 | . 0014 | -. 0207 | -. 0199 | . 0025 | . 0445 |
| 0.50 | . 0077 | . 0004 | -. 00034 | -. 0033 | . 0002 | . 0077 | . 0441 | . 00 | -. 0192 | -. 0183 | . 0025 | 0318 |
| 0.75 | . 0075 | . 0003 | -. 0034 | -. 0032 | . 0003 | . 0075 | . 0367 | . 0010 | -.0160 -.0133 | -. 0152 | . 00018 | . 03482 |
| 1.0 | . 00074 | . 0003 | -. 0033 | -. 0032 | . 00003 | .0073 .0070 | . 0306 | . 00009 | -.0133 -.0093 | -.0126 | . 0011 | . 0195 |
| 1.5 | . 0071 | . 00004 | -.0030 -0030 | -.0031 -.0029 | . 00004 | . 00066 | . 0145 | . 0003 | -.0063 | -. 0061 | . 0008 | . 0135 |
| 2.0 3.0 | . 0067 | . 00003 | -.0030 -.0026 | -.0029 -.0027 | . 0003 | . 0060 | . 0069 | . 0002 | -. 00031 | -. .0030 | . 0002 | . 0063 |
| 4.0 | . 0055 | . 0002 | -.0025 | -. 0024 | . 0003 | . 0055 | . 0033 | 0001 | -. 0015 | -. 0013 | . 0002 | . 0031 |
| 5.0 | . 0049 | . 0001 | -. 0022 | -. 00022 | . 0002 | . 0050 | . 0017 | . 0001 | -.0006 | -.0006 | 0001 | . 0015 |
| 7.5 | . 0039 | . 0001 | -. 0017 | -. 0020 | . 0002 | . 0039 |  |  |  |  |  |  |
| 10.0 | . 0030 | . 0001 | -. 0014 | -. 0014 | . 0001 | 0030 |  |  |  |  |  |  |
| 15.0 | . 0019 | . 0001 | -. 0009 | -. 0008 | . 0001 | -0019 |  |  |  |  |  |  |
| 20.0 | . 0012 | . 0000 | -. 0005 | -. 0005 | . 0001 | . 0012 |  |  |  |  |  |  |
|  | $B=5.0$ |  |  |  |  |  | $B=$ |  |  |  |  |  |
| 0.0 0.001 | ${ }_{0}^{0}$ | 0 .0032 |  | $\begin{gathered} 0 \\ -0 n 71 \end{gathered}$ | $\begin{gathered} 0 \\ -.0122 \end{gathered}$ | $\begin{gathered} 0 \\ .1386 \end{gathered}$ | $\begin{gathered} 0 \\ .0653 \end{gathered}$ | $\begin{gathered} 0 \\ .0249 \end{gathered}$ | $\begin{gathered} 0 \\ -.0155 \end{gathered}$ | $\begin{gathered} 0 \\ -.0559 \end{gathered}$ | $\begin{gathered} 0 \\ -0963 \end{gathered}$ | $\begin{gathered} 0 \\ .8733 \end{gathered}$ |
| 0.001 | . 0083 | . 0032 | -. 00019 | -.0071 | -.0122 | . 1386 | . 0653 | . 02439 | -.0155 -.0227 | $\begin{aligned} & -.0559 \\ & -.0785 \end{aligned}$ | .0963 . .1326 | $\begin{array}{r} .8733 \\ .8101 \end{array}$ |
| 0.002 0.005 | .0154 .0325 | .0059 .0117 | -.0037 | -. 0133 | -.0288 | . 1771 | .0887 .1293 | . 0335 | -.0227 -.0381 | -.0785 -.1218 | .1326 . .1601 | . 8101 |
| 0.003 | . 0541 | . 0176 | -. 0188 | -.0544 | -. 0526 | . 2562 | . 1656 | . 0542 | -. 0570 | -. 1626 | -. 1259 | . 6086 |
| 0.015 | . 0712 | . 0221 | -. 0270 | -. 0712 | -. 0481 | . 2636 | . 1864 | . 0565 | . 0729 | -. 1823 | -. 0848 | . 5370 |
| 0.020 | . 0849 | . 0250 | -. 0345 | -. 0822 | -. 0413 | . 2622 | . 1991 | . 0557 | -. 0852 | -. 1858 | -. 0575 | . 4817 |
| 0.030 | 1068 | . 0294 | . .0456 | -. 0930 | -. 0304 | . 2483 | .2109 | . 0493 | -. 0998 | -. 1741 | -. 0249 | . 3984 |
| 0.040 | 1190 | . 0282 | -. 0538 | -. 0941 | -. 0204 | . 2310 | . 2119 | . 0411 | -. 1044 | -. 1558 | -. 0083 | . 3375 |
| 0.050 | . 1263 | . 0260 | -. 0582 | -. 0913 | -. 0133 | . 2145 | . 2080 | . 0335 | . 1031 | -. 1378 | . 0005 | . 2905 |
| 0.075 | . 1290 | . 0195 | -. 0607 | -. 0800 | -. 0034 | . 1779 | . 1896 | . 0190 | -. 0916 | -. 1031 | . 0093 | . 2122 |
| 0.10 | . 1264 | . 0141 | -. 0588 | -. 0690 | . 0011 | . 1498 | . 1728 | . 0105 | -. 0806 | -. 0814 | . 0117 | . 1670 |
| 0.15 | . 1173 | . 0072 | -. 0521 | -. 0538 | . 0047 | . 1142 | . 1439 | . 0027 | . 0641 | -. 0573 | -0119 | . 1179 |
| 0.20 | . 1069 | . 0043 | -. 0465 | -. 0448 | . 0056 | . 0942 | . 1233 | .0001 | -. 0529 | -. 0459 | . 0112 | . 0951 |
| 0.30 | . 0904 | . 0022 | -. 0384 | -. 0348 | . 0056 | . 0729 | . 0962 | -. 0009 | -. 0416 | -. 0340 | . 0089 | . 0705 |
| 0.40 | . 0760 | . 0014 | -. 0322 | -. 0288 | . 0049 | . 0601 | . 0753 | -. 0006 | -.0323 | -. 0264 | . 0068 | . 0545 |
| 0.50 | . 0639 | . 0011 | -. 0270 | -. 0241 | . 0042 | . 0534 | . 0588 | -. 0006 | -. 0253 | -. 0207 | . 0053 | . 0424 |
| 0.75 | . 0416 | . 0008 | -. 0175 | -. 0156 | . 0027 | . 0326 | . 0318 | -. 0002 | -. 0135 | -. 0111 | . 0029 | . 0230 |
| 1.0 | . 0270 | . 00005 | -. 0114 | -. 0101 | . 0018 | . 0212 | . 0170 | -. 00002 | -. 00074 | -. 0061 | . 0015 | $.0124$ |
| 1.5 2.0 | . 0113 | . 0002 | -. 0098 | -. -.0043 | . 00078 | $.0089$ | .0050 .0014 | .0000 .0000 | -.0021 | -.0018 | . 00001 | . 0036 |
| 2.0 3.0 | . 0048 | .0001 .0000 | -.0021 | -. 00018 | . 00003 | $\begin{array}{r} .0037 \\ .0006 \end{array}$ | . 0014 | . 0000 | -. 0006 | -. 0006 | . 0001 | . 0010 |

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# Floating Block Theory in Strucłural Analysis* 

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## S Y NOPSIS


#### Abstract

A process of reaction distribution is developed for the purpose of calculating reactions beneath hinged floating blocks. Application to hinged base slabs is illustrated. The analogous correspondence to the process of moment distribution is explained by using the column analogy.


## INTRODUCTION

Structures such as retaining walls, gravity dams, piers or column footings are designed on the assumption that the base reaction has a linear distribution. The reaction diagram is considered to be of trapezoidal shape (triangular and rectangular diagrams being special cases). Actually a linear base reaction diagram could be valid only if the structure were floating in a very dense liquid. If the foundation material possesses shearing strength and rigidity, which does not break down and disappear with time, the base reaction must be nonlinear. Justification of this statement may be found both in the theory of elasticity and laboratory and field measurements of base pressures.

Another important requirement for the validity of linear reaction diagrams is that the structure must not have a flexible base slab. A flexible base slab will have a non-linear reaction diagram, even when floating in a liquid, if loaded with any distribution of loading other than uniform. Thus it is seen that floating block theory is now in common usage in the analysis of individual, isolated structures. The present paper presents a method of calculating the base reaction distribution for several floating blocks attached by hinges. The computation method may be called "reaction distribution" since it is performed in the same manner as "moment distribution" for the analysis of continuous beams.

## REACTIONS FOR INDIVIDUAL BLOCKS

The reactions for a gravity dam or a cantilever retaining wall, as shown in Fig. 1, are readily calculated by considering a slice of the structure one foot thick and using the flexure formula. Considering the origin of coordinates to be at the center of the base, the toe and heel pressures, $w_{a}$ and $w_{b}$, are given by,

$$
\begin{align*}
& w_{a}=\frac{P}{A}+\frac{M x_{a}}{I} \ldots  \tag{1a}\\
& w_{b}=\frac{P}{A}+\frac{M x_{b}}{I} \cdots \ldots \tag{1b}
\end{align*}
$$

where $P=$ total vertical load on foundation (positive downwards)
$M=$ moment of $P$ about origin
$A=$ area of base of one foot slice (equals length $L$ )
$I=$ moment of inertia of base area (equals $L^{3} / 12$ )
$x_{a}=-L / 2, x_{b}=L / 2$
Fig. 1 -Linear reaction diagrams


The trapezoidal reaction diagrams would be strictly correct only if the structures were floating in a dense liquid. Thus the flexure formula may be considered as a formula for the reaction distribution of a floating block acted upon by forces and moments. The weight of the floating block may be neglected and the flexure formula will then give the reactions caused by the applied loads.

A more general type of floating block is one having a planform which is not rectangular, such as the one shown in Fig. 2. For the purpose of describing the planform of the floating block and the loading distribution, it is convenient to borrow several words from the terminology which is used by the aeronautical engineer in describing the planform and loading distribution of an airplane wing. The width of the planform is called the chord and the length is called the span. The chord c may be variable as shown in Fig. 2. The distribution of loading across
the planform may be described as the chordwise distribution and the distribution along the planform may be described as the spanwise distributon.


Throughout this paper the planform of the floating block is considered to be symmetrical about a center line running in the spanwise direction. When the planform has a variable chord, the area and moment of inertia to be used in the flexure formula for calculating reactions may be compouted from the integral formulas,

$$
\begin{align*}
& A=\int_{0}^{L} c d x \ldots  \tag{2}\\
& I=\int_{0}^{L} c x^{2} d x . \tag{2b}
\end{align*}
$$

where the coordinate $x$ is measured from the centroid of the planform. The chordwise distribution of applied loads and reactions will be considered to be uniform at all sections and such chordwise distribution
diagrams will not be shown. A spanwise distribution diagram and the planform dimensions will serve to define completely the loading magnitude and distribution. The reader who is familiar with the column analogy, as developed by Hardy Cross*, will recognize the similarity in physical concepts between the analogous column and the floating block. Certain fundamental physical properties of an isolated block must be developed for use in the reaction distribution process, just as similar properties of an isolated span of a continuous beam are developed with the aid of the column analogy for use in the moment distribution procedure.

In Fig. 2(a) a floating block is shown with a unit load at the right end. A concentrated load will always be considered as acting on the center line of the planform so that the chordwise distribution of the reaction will always be uniform. The right end reaction ordinate is $K_{b}$ and the left end reaction ordinate is $-R$. The reaction diagram may be regarded as an influence line for reaction at point $b$ at the right end of the block. Consequently the quantities $K_{b}$ and $R$ may be called influence numbers. They may be converted to true deflection influence numbers by dividing by the density of the liquid. However, the actual density of the liquid does not enter into any of the analyses of this paper. In Fig. 2(a) there is also shown a unit load acting at point $a$ at the left end of the block. The left end reaction ordinate is indicated as $K_{a}$ and the right end reaction ordinate is $-R$. The value $R$ arises in both cases in agreement with Maxwell's law of reciprocal deflections.

The quantities $K_{a}$ and $K_{b}$ may be called primary influence numbers and $R$ may be called the secondary influence number. The values of the influence numbers may be computed by using Eqs. (1).

$$
\begin{align*}
K_{a} & =\frac{1}{A}+\frac{x_{a}{ }^{2}}{I} \ldots  \tag{3a}\\
K_{0} & =\frac{1}{A}+\frac{x_{b}{ }^{2}}{I} \cdots  \tag{3b}\\
-R & =\frac{1}{A}+\frac{x_{a} x_{b}}{I} \tag{3c}
\end{align*}
$$

In these equations $I$ is the centroidal moment of inertia and $x_{a}$ and $x_{b}$ are the distances of points $a$ and $b$ from the centroid of the planform. It may prove to be convenient to express the influence numbers in terms of the planform area thus,

$$
\begin{equation*}
K_{a}=\frac{\propto_{a}}{A}, K_{b}=\frac{\propto_{b}}{A}, R=\frac{\beta}{A} . \tag{4}
\end{equation*}
$$

The values of $\propto_{a}, \alpha_{b}$ and $\beta$ can be readily expressed by introducing the radius of gyration $\rho$ of the planform.

[^12]\[

$$
\begin{align*}
& \propto_{a}=1+\left(\frac{x_{a}}{\rho}\right)^{2}  \tag{5a}\\
& \propto_{b}=1+\left(\frac{x_{b}}{\rho}\right)^{2}  \tag{5b}\\
& -\beta=1+\frac{x_{b} x_{b}}{\rho^{2}} \cdots \tag{5c}
\end{align*}
$$
\]

In Fig. 2(b) the same block is shown carrying a continuously distributed load as well as concentrated loads $P_{a}$ and $P_{b}$ at either end. Only the spanwise distribution of the loading is shown since the chordwise distribution is uniform at all points along the span. The reaction is shown as consisting of three parts which may be added, or superposed. The first part is a trapezoidal distribution, having end ordinates $w_{a}^{\prime}$ and $w_{b}^{\prime}$, and is due to the continuously distributed load. The ordinates $w_{a}^{\prime}$ and $w_{b}^{\prime}$ are computed from the flexure formula as given by Eq. (1). If $p$ is the applied load per unit of area, the magnitude of the distributed load and its moment about the center of gravity of the planform, are given by,

$$
\begin{align*}
& P=\int_{0}^{L} p c d x .  \tag{6a}\\
& M=\int_{0}^{L} p c x d x . \tag{6b}
\end{align*}
$$

The second and third parts of the reaction are due to the loads $P_{a}$ and $P_{b}$. They are proportional to the reaction influence diagrams of Fig. $2(a)$. Since the chordwise distribution of the reactions is uniform, only the spanwise distribution is shown. When the three parts of the reaction are added, the resultant reaction has a trapezoidal spanwise distribution with end reaction ordinates given by,

$$
\begin{align*}
& w_{a}=w_{a}^{\prime}+K_{a} P_{a}-R P_{b}  \tag{7a}\\
& w_{b}=w_{b}^{\prime}-R P_{a}+K_{o} P_{b} \tag{7b}
\end{align*}
$$

These equations may be called the reaction-shear equations and may be recognized as having direct analogous correspondence to the slope-deflection equations for a single span of a continuous beam.

## VALIDITY OF THE FLEXURE FORMULA

The flexure formula, as used for the stress analysis of beam-columns, has been given as Eqs. (1) for calculating the reaction distribution. Although the formula is known to be valid for this purpose, a simple proof is of interest. Consider a block with a distributed load $p$ and a linearly distributed reaction $w$, both in lb . per sq. ft . The net load acting on the block is the difference $(p-w)$. The load and reaction together must
satisfy the equation of equilibrium of vertical forces and the equation of equilibrium of moments about any point in the plane of symmetry of the block, say the center of gravity of the planform. These two conditions give the following equations,

$$
\begin{align*}
& \int_{0}^{L}(p-w) c d x=0 \ldots  \tag{8a}\\
& \int_{0}^{L}(p-w) c x d x=0 . \tag{8b}
\end{align*}
$$

These integral equations of equilibrium have direct analogous correspondence to the integral equations of continuity which govern the bending moments in a fixed-ended beam. By separating the integrals into two parts and transposing, Eqs. (8) become,

$$
\begin{align*}
& \int_{0}^{L} w c d x=\int_{0}^{L} p c d x=P \ldots  \tag{9a}\\
& \int_{0}^{L} w c x d x=\int_{0}^{L} p c x d x=M . \tag{9b}
\end{align*}
$$

The formulas of Eqs. (6) have been substituted into Eqs. (9). Since the structural deformation of the floating block is regarded as being negligible, it is known that the reaction $w$ has a linear distribution and may be assumed in the form,

$$
\begin{equation*}
w=a+b x \tag{10}
\end{equation*}
$$

This formula for $w$ may be substituted into Eqs. (9) to obtain two linear algebraic equations which may be solved for $a$ and $b$. The reader should carry out this suggested procedure*. Another approach is to assume values for $a$ and $b$, substitute in Eqs. (9), and thus show the assumption to be correct. This procedure will be illustrated. Assume $w$ to be given by the flexure formula according to Eq. (1).

$$
\begin{equation*}
w=\frac{P}{A}+\left(\frac{M}{I}\right)= \tag{11}
\end{equation*}
$$

This assumption is seen to be a linear function of $x$ in agreement with Eq. (10). Substituting rhis into Eqs. (9) gives,

$$
\begin{align*}
& \int_{0}^{L}\left[\frac{P}{A}+\left(\frac{M}{I}\right) x\right] c d x=P \ldots  \tag{12a}\\
& \int_{0}^{L}\left[\frac{P}{A}+\left(\frac{M}{I}\right) x\right] c x d x=\mathrm{M} \tag{12b}
\end{align*}
$$

[^13]Separating the integrals into two parts and taking the constants outside of the integral signs gives,

$$
\begin{align*}
& \frac{P}{A} \int_{0}^{L} c d x+\frac{M}{I} \int_{0}^{L} c x d x=\mathrm{P} \ldots  \tag{13a}\\
& \frac{P}{A} \int_{0}^{L} c x d x+\frac{M}{I} \int_{0}^{L} c x^{2} d x=\mathrm{M} \tag{13b}
\end{align*}
$$

Since the origin of coordinates is the center of gravity of the planform, one of the integrals in the equations vanishes.

$$
\begin{equation*}
\int_{0}^{L} c x d x=0 \tag{14}
\end{equation*}
$$

Dropping out the terms containing this integral and substituting the formulas of Eqs. (2), the Eqs. (13) reduce to identities. Since the solution is known to be linear, and is also known to be unique, the assumed solution must be the correct solution. A proof, similar to the above, serves to demonstrate the validity of the column analogy in a simple and convenient manner.

## ELEMENTARY EXAMPLES

In Fig. 3 there is shown a number of different loads and reactions acting on a floating block with a rectangular planform. The distributed loadings have a maximum ordinate of $p_{m} \mathrm{lb}$. per sq. ft . The end ordinates of the reaction diagrams may be expressed in terms of $p_{m}$. D. B. Steinman has recently pointed out* a convenient way of remembering the end reaction ordinates for the case of triangular loading. These end ordinates may be computed as the reactions on a simple beam of $\operatorname{span} L$ which is loaded with a concentrated load of magnitude $p_{m}$ acting at the same point as the location of $p_{m}$ on the floating block.

Several examples are shown in Fig. 3 of reactions due to a concentrated load $P$ acting at one end or both ends of the block. In the case of a concentrated load the area $A$ of the planform enters into the reaction calculations. One of the examples shows $P$ at the right end while the left end of the block is attached by a hinge to a rigid wall. This wall may be thought of as the sidewall of a reservoir containing the liquid in which the block is floating. The hinge prevents the block from sinking at the left end and, hence, the reaction ordinate must be zero at that end. For this case the right end reaction ordinate is found from statics to be $\frac{3}{A} P$.

[^14]

The coefficient $3 / A$ may be called a modified influence number. It is three-fourths of the standard influence number for a block with a rectangular planform.

In Fig. 4 two blocks are shown with variable chords. The block of Fig. 4(a) has a planform with a chord which varies linearly. The calculated location of the center of gravity is shown. The block is acted upon by a unit load at the right end and, consequently, the reaction ordinates are influence numbers. The area and section moduli are shown with the calculations for the reactions. The block of Fig. 4(b) has a parabolic variation of the chord. It is symmetrical about the center of the span and, hence, the center of gravity is at mid-span. Due to this symmetry the two section moduli are of equal magnitude but of opposite signs.

## blocks Connected by hinges

The next step in the development of floating block theory is to consider two blocks joined together by a hinge as shown in Fig. 5. Both blocks have the same constant chord but are of unequal spans. The left block has no load but the right block has a load with the spanwise dis-

Fig. 4-Calculation of influence numbers


$$
\begin{gathered}
A=30 \mathrm{ft}^{2} \\
S_{a}=-40 \mathrm{ft}^{3} \\
S_{\mathrm{h}}=+50 \mathrm{ft}^{3} \\
\frac{P}{A}=\frac{1}{30}=.0333 \\
\frac{M}{S_{a}}=\frac{4.44}{-40}=-.1111 \\
\frac{M}{S_{D}}=\frac{4.44}{50}=.0888
\end{gathered}
$$

$$
\begin{gathered}
A=40 \mathrm{ft} .^{2} \\
-S_{a}=S_{b}=67.2 \mathrm{ft}^{3} \\
\frac{P}{A}=\frac{1}{40}=.0250 \\
\frac{M}{S_{a}}=\frac{6}{-67.2}=-.0893 \\
\frac{M}{S_{b}}=\frac{6}{67.2}=.0893
\end{gathered}
$$

tribution as shown. Consider, temporarily, that the pin of the hinge has been removed so that each block is free to move alone into an equilibrium position. The right block will be displaced downward by the load and will develop a trapezoidal reaction as shown. The left end ordinate $w^{\prime}{ }_{a}$ is seen to be two-thirds of $p_{m}$ and the right end ordinate is one-third of $p_{m}$. The values of 10 and 5 lb . per sq. ft. may be called "free end reactions" in analogous correspondence to the "fixed end moments" which enter into a moment distribution analysis.

Consider, now, that a concentrated force $P$ is applied downward on the hinged end of the left block and a force $P$ of equal magnitude is applied upward on the hinged end of the right block. If the magnitude of these artificial forces is gradually increased, the hinge will become realigned at some particular value of $P$ and the pin of the hinge can be replaced. The artificial forces may then be removed and a shear of magnitude $P$ will be transferred through the hinge. In the final equilibrium position of the hinged blocks the left end and right end shears will be,

$$
\begin{align*}
& \text { Left }  \tag{15a}\\
& \text { Block }
\end{align*}\left\{\begin{array}{l}
P_{a}=P_{1}=0 \ldots  \tag{15b}\\
P_{b}=P_{2}=P \ldots
\end{array}\right\} .
$$



Fig 5-Two hinged blocks


The reaction ordinates for the left block in the final position may be indicated as $w_{12}$ and $w_{21}$, while the reaction ordinates for the right block will be $w_{23}$ and $w_{32}$. The use of double subscripts becomes necessary in dealing with more than one block. The influence numbers may also be indicated with double numerical subscripts. The reaction-shear equations for the blocks become (see Eqs. 7),

$$
\begin{align*}
& \text { Left } \quad\left\{\begin{array}{l}
w_{12}=-R_{12} P_{2} \\
w_{21}=K_{21} P_{2}
\end{array}\right.  \tag{16a}\\
& \text { Right }\left\{\begin{array}{l}
w_{23}=w_{23}{ }^{\prime}-K_{23} P_{2}
\end{array}\right.  \tag{16b}\\
& \text { Block }\left\{w_{32}=w_{32}^{\prime}+R_{32} P_{2}\right. \tag{16c}
\end{align*}
$$

The quantities $w_{23}{ }^{\prime}$ and $w_{32}{ }^{\prime}$ are the free end reactions. They are the values of $w_{23}$ and $w_{32}$ when $P_{2}=P_{3}=O$.

In the final position of the blocks, each block will have the same displacement at the hinge. Hence, the reactions $w_{21}$ and $w_{23}$ must be equal. Equating the right sides of Eqs. (16b) and (16c) gives,

$$
\begin{align*}
K_{21} P_{2} & =w_{23}{ }^{\prime}-K_{23} P_{2} .  \tag{17a}\\
P_{2} & =\frac{w_{23}^{\prime}}{K_{21}+K_{23}} . \tag{17b}
\end{align*}
$$

Substituting into Eqs. (16) gives,

$$
\begin{equation*}
w_{12}=-\left(\frac{R_{12}}{K_{21}+K_{23}}\right) w_{23}^{\prime} \tag{18a}
\end{equation*}
$$

$$
\begin{align*}
& w_{21}=\left(\frac{K_{21}}{K_{21}+K_{23}}\right) w_{23}^{\prime} \ldots \ldots  \tag{18b}\\
& w_{23}=w_{23}^{\prime}-\left(\frac{K_{23}}{K_{21}+K_{23}}\right) w_{23}{ }^{\prime} .  \tag{18c}\\
& w_{32}=w_{32}^{\prime}+\left(\frac{R_{32}}{K_{21}+K_{23}}\right) w_{23}^{\prime} \tag{18d}
\end{align*}
$$

The following fractions may be defined as distribution factors,

$$
\begin{equation*}
d_{21}=\frac{K_{21}}{K_{21}+K_{23}} \quad, \quad d_{23}=\frac{K_{23}}{K_{21}+K_{23}} \ldots \tag{19}
\end{equation*}
$$

The first subscript of the distribution factor indicates the joint and the second subscript indicates the block. The distribution factors are proportional to the influence numbers of the blocks adjacent to the hinge. It is only necessary to know the relative values of the influence numbers $K_{21}$ and $K_{23}$ in order to compute $d_{21}$ and $d_{23}$. If both blocks have rectangular planforms with the same chord $c$, the numerical value of $c$ need not be known. If the values of $c$ are unequal, it is only necessary to know their relative values. With equal chords, as in Fig. 5, the distribution factors are inversely proportional to the lengths of the spans.

The following fractions may be defined as carry-over factors,

$$
\begin{equation*}
r_{21}=\frac{R_{21}}{\widetilde{K}_{21}}=\frac{R_{12}}{\widetilde{K}_{21}} \quad, \quad r_{23}=\frac{R_{23}}{\check{K}_{23}}=\frac{R_{32}}{K_{23}} \ldots \tag{20}
\end{equation*}
$$

The carry-over factors are calculated as the ratio of the secondary influence numbers to the primary influence numbers. Thus it is only necessary to know relative values of the influence numbers. In the case of rectangular planforms, all carry-over factors have the value $1 / 2$. The distribution and carry-over factors, as defined, have direct analogous correspondence to the same factors in moment distribution.

Eqs. 18) become,

$$
\begin{aligned}
& w_{21}=d_{21} w_{23}^{\prime} \text {. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (21b) }
\end{aligned}
$$

$$
\begin{align*}
& w_{32}=w_{32}{ }^{\prime}+r_{23} d_{23} w_{23}{ }^{\prime} \tag{21d}
\end{align*}
$$

From these equations it is seen that the complete spanwise distribution of the reaction can be determined from a knowledge of the free end reactions. The chord of the planform does not need to be known. The shear $P_{2}$, which is transmitted through the hinge, is not determined. In order to calculate $P_{2}$, it is first necessary to know the numerical value of the chord. From the calculated reactions the force $P_{2}$ can then be computed from statics.

The reaction distribution calculations are shown in Fig. 5. The distribution factors for the hinged joint are shown in parentheses. The free end reactions are written first at each end of each block. The unbalanced reaction at the center joint is distributed to the blocks in accord with the values of the distribution factors. The changes in reaction thus brought about at the center joint are multiplied by the carry-over factors, with a negative sign prefixed, to give the change in reaction at the outer ends of the blocks. Addition at each joint gives the three reaction ordinates required to determine the reaction distribution. These computations will readily be seen to be in agreement with Eqs. (21). The negative value of reaction at the left end of the left block may be regarded as being physically possible if it is assumed that the block has sufficient dead weight that it does not lift out of the liquid.

An analysis of three hinged blocks is shown in Fig. 6. The blocks are assumed to have rectangular planforms with equal chords and spans. Consequently the distribution factors and carry-over factors are all equal to $1 / 2$. The spanwise loading distributions are shown with numerical values of the maximum ordinates. The free end reactions are also shown.

The numerical computations consist of a series of cycles which must be repeated until the desired accuracy is attained. In the example of Fig. 5 the complete solution was obtained from one cycle of computations. A cycle consists of two steps. In the first step the joints are balanced and in the second step the carry-over is performed in all spans. In the example of Fig. 6, two and one-half cycles have been performed to obtain two significant figure accuracy. As each joint is balanced in the first step of a cycle, a horizontal line is drawn to indicate that the joint has been balanced. At the end of the first cycle the hinged joints are out of balance and hence additional cycles are required. The final reaction diagram is illustrated.

## HINGED BLOCKS WITH NON-RECTANGULAR PLANFORMS

Fig. 7 illustrates a solution for reactions beneath three hinged blocks with non-rectangular planforms. The blocks have a common center line about which they are all symmetrical. The outer blocks have a chord with linear variation while that of the center block has parabolic variation. The first two blocks have a spanwise load distribution as shown in Fig. 7(a). The first block has a parabolic loading while the second block has a triangular loading. The maximum ordinate of the loading diagrams is at mid-span in both cases.

By using calculus, in accord with Eqs. (9), the magnitude and line of action of each block load may be computed to be as shown in Fig. 7 (b).

Fig. 6 (above)-Three hinged blocks
F!g. 7 (right)-Hinged blocks with variable chords

This type of problem could well be used in sophmore calculus courses for engineers to illustrate the application of calculus. It requires the integration of polynomials. The numerical values of the areas and section moduli of these planforms have been given in Fig. 4. From the flexure formula the free end reactions may be computed to be as shown in Fig. 7 (b).

The influence numbers which are needed have been computed in Fig. 4. From these influence numbers the distribution and carry-over factors may be computed by using Eqs. (19) and (20). The reaction distribution is shown in Fig. $\overline{7}$ (c) with a diagram of the final reaction.

## PRACTICAL APPLICATION

An obrious application of reaction distribution is the analysis of a floating bridge which consists of units joined together by hinges. This type of bridge has been used occasionally for both war time and peace time services.

The application which will be illustrated is of more common occurrence. The method of analysis will be applied to concrete structures with hinged base slabs resting on soil. A cross section of a single channel serrage digestion tank is shown in Fig. 8. It is assumed that any reinforcement which crosses the base slab joints is placed at the center of the slab depth and serves only to tie the structure together at these points. Many such structures were built during the war program with no base slab reinforcement at all.

The dimensions of the structure are shown and the magnitudes of the forces have been calculated assuming the concrete to weigh 150 lb per $\mathrm{cu} . \mathrm{ft}$. The reaction is first computed for the dead load weight of the concrete alone. A slice of the structure one foot thick is considered in the analysis so that the floating blocks have rectangular planforms with a unit chord.

The eccentricity of the resultant force on either of the outer blocks may be computed to be 0.722 ft . From the flexure formula the free end reactions may be computed to be as shown in Fig. 8(b). Adrantage may be taken of the symmetry of the structure about its center line. If a reduced influence number for the center block is used in calculating the distribution factors, no carr-over will be required in the center span and the solution will be obtained from one cycle of distribution computations performed in one end span. The standard influence numbers for the spans are inversely proportional to their lengths. The reduced influence number for the center span is one-half of the standard number as illustrated in Fig. 3(f). The distribution calculations are shown in Fig. 7(c) with the final dead load reaction diagram.


Fig. 8-Sewage tank-analysis for dead load


In Fig. 9 the reaction is calculated for the effect of earth loading only. The soil is assumed to weigh 100 lb . per cu. ft. A triangular wedge of soil on the back of each sidewall weighs 0.75 kips and this force has an eccentricity of 2 ft . The horizontal pressure is assumed to have a "rest" value equal to that of an equivalent liquid weighing 50 lb . per $\mathrm{cu} . \mathrm{ft}$. The horizontal thrust in the base slab is assumed to act at mid-depth, giving a moment arm of the horizontal earth pressure of 3.83 ft . The outer floating blocks are acted upon by a thrust and moment from the weight of the soil on the back of the walls and by a pure moment due to the horizontal forces. From the flexure formula the free end reactions due to these loads may be computed to be as shown in Fig. 9(b). The distribution calculations and reaction diagram are shown in Fig. 9(c). The reaction due to earth loads may be added to the dead load reaction to obtain the reaction diagram shown in Fig. $9(\mathrm{~d})$. Internal water loading may be treated in a similar manner. The weight of the water will exert thrusts on each of the blocks and the horizontal water pressure on the walls will exert moments on the outer blocks.

(b)

| -1.79 | +2.89 | $(.2)$ |
| ---: | ---: | ---: |
| +.83 | -1.67 | +.42 |
| -.96 | +.42 | +.42 |


(c)


Fig. 9 Sewage tank analysis for earth loads
(d)

In Fig. 10 is shown a section of a spillway channel which is symmetrical about its center line. The channel floor is constructed of four 25 foot slabs joined by keyed construction joints which act as hinges. The structure rests on soil and does not have reinforcement to carry bending moment across the joints. Concrete and soil weights are assumed as before. The total reaction due to both horizontal and vertical loads is also shown in Fig. 10.

The total reaction has been determined by calculating the effects of the vertical and horizontal loads separately. The reaction is first calculated in Fig. 11 for the dead weight of the concrete and the weight of the soil fill on the heel of the sidewall. The total load on the first block is 30 kips with an eccentricity of 2.36 ft . The free end reactions are shown in Fig. 11 (a) as given by the flexure formula. The distribution computations and final reactions are shown in Fig. 11 (b) and (c). All influence numbers are inversely proportional to the span lengths and the carryover factors are all one-half. It is only necessary to perform the dis-

Fig. 10 Spillway channel loads and reaction

(a) Free End Reactions

(b) Reaction

Distribution

(c) Reaction
tribution computations on half of the structure since the center joint will always be balanced and will not transmit any shear.

Fig. 12-Analysis for horizontal,loads

(a) Free End Reactions

(b) Reaction Distribution

(c) Reaction

The horizontal earth pressure on the sidewalls is again assumed to have a value equal to that of a liquid weighing 50 lb per cu. ft. Acting over a height of 24 ft . the total force is 14.4 kips per ft . The moment arm of the horizontal force about the center of the floor slab is 10 ft ., giving a moment on the end block of 144 ft . kips. From the flexure formula the free end reactions are $\pm 3.84$ kips per sq. ft. The reaction distribution for the effect of the horizontal earth pressure is shown in Fig. 12. The total reaction diagram due to vertical and horizontal loads is obtained by adding the reaction diagrams of Fig. 11 and 12.

Considering the trapezoid of reaction beneath the sidewall itself, the resultant force represented by the trapezoid may be calculated to be 21.75 kips , or $73 \%$ of the weight of the sidewall and fill on its heel. This means that $27 \%$ of the weight of the sidewall and fill is transmitted through the first hinged joint. The joint key must be designed to transmit this shear. It has been common, but wasteful, practice to neglect this shear force in the past.

## CONTINUOUS BEAM ANALYSIS

In developing the process of reaction distribution a number of instances have been specifically noted in which there is a direct analogous
correspondence with the analysis of continuous beams. In order to analyze a continuous beam it is first necessary to consider each span individually. The primary and secondary stiffness values must be computed. If the method of moment distribution is to be employed, the fixed end moments must be computed. These individual span quantities may readily be computed by using the concepts of the column analogy. For each span there is an analogous column which may also be regarded as a floating block. The chord of the planform of the analogous column, or floating block, is equal to the value of $1 / E I$ for the beam. The spanwise distribution of loading on the blocks corresponds to the simple beam bending moment diagrams for the beam spans. If these blocks are joined by hinges, the calculation of reactions corresponds to the process of moment distribution. The correspondence between hinged block, or analogous column, analysis and continuous beam analysis was noted by Prof. Hardy Cross*. The various elements of correspondence are as follows:
(1) Reaction distribution
(2) Floating Block
(3) Span
(4) Cord c
(5) Spanwise load distribution
(6) Free end reactions
(7) Hinge shear
(8) Influence numbers
(9) Distribution factors
(10) Carry-over factors
(11) Reaction-shear equations
(12) Joint reaction ordinate
(13) Hinge
(14) Free end
(15) Integral equations of equilibrium

|  |  |
| :--- | :--- |
|  | Moment distribution |
|  | Analogous column |
|  | Span |
|  | $1 /$ EI |
|  | Simple beam moment diagram |
|  | Fixed end moments |
|  | Joint rotation |
|  | Stiffness values |
|  | Distribution factors |
|  | Carry-over factors |
|  | Slope-deflection equations |
|  | Joint moment |
|  | Simple support |
|  | Fined end |
|  | Integral equations of continuity |

By substituting the above correspondence the development of reaction distribution as given in this paper becomes a development of moment distribution and the column analogy as applied to beams. The development of hinged block analysis requires only the concept of equilibrium of forces while the development of continuous beam anlysis requires, in addition, the more difficult concept of continuity. Consqeuently it may be possible that the structural engineering student could profitably be introduced to reaction distribution as a prelude to the study of continuous beam analysis.

A few examples of continous beams and the analogous floating blocks will be shown. In Fig. 13 a continuous beam of constant section over three spans is shown. The analogous hinged blocks are also shown with a planform cord of $1 / E I$. The spans and loads are so chosen that the simple beam moment diagrams will be the same as the spanwise distribution diagrams of the example of Fig. 6. Consequently the reaction

[^15]

Fig. 113-Continuous beam and analogous foating_block
distribution calculations of Fig. 6 may be regarded as moment distribution for the beam of Fig. 13.

In Fig. 14(a) is shown a prismatic three span beam with four simple supports. Thus there are four hinges on the three floating blocks as shown in Fig. 14(b). The outer hinges are attached to fixed sidewalls of the channel of liquid. The blocks are loaded with simple beam moment diagrams. The free end reaction for the center block is shown by solid lines and for the end blocks is indicated by dotted lines. All four pins must be removed from the four hinges to allow the blocks to move downwards and develop the free end reactions. It is convenient, before the reaction distribution is performed, to lift the outer ends of the outer blocks until the outer hinges become realigned and insert these two pins so that a shear is transferred into the sidewalls. This reduces the reaction ordinates at these two points to zero. The change from free end value to zero causes a change of half as much as the other end of the block, since the carry-over factor is one-half. The calculations for determining the maximum ordinate of the triangular reaction diagrams of the outer blocks are shown in Fig. 14(b). The computations of Fig. 14(c) may be regarded either as moment distribution for the continuous beam or reaction distribution for the floating blocks. A modified influence number is used for the end spans in computing the distribution factors. This influence number is three-fourths of the standard value as shown in Fig. 3(e). The calculated floating block reaction diagram is also shown in Fig. 14(c). This diagram gives the negative moment diagram to be superimposed on the simple beam moment diagrams to obtain the solution of the continuous beam.

An example of a non-prismatic beam is shown in Fig. 15 with the associated floating blocks. The chord of the planform of each block is proportional to $1 / E I$ of the beam at corresponding points. On the assumption that cross sections of the beam are rectangular, it is only necessary to know the relative depths of the beam at various points. The bending moments can be completely calculated without knowing the actual beam dimensions if the relative values are known.

Fig. 14 -Pismatic beam on simple
supports

(a)


| $\begin{aligned} & 1.92+2.88 \\ & -1.92 \rightarrow \quad .96 \end{aligned}$ | +1.47 | +0.63 | $\begin{aligned} & +1.44 \\ & +.48 \end{aligned}$ | $\begin{aligned} & +0.96 \\ & -0.96 \\ & \hline \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: |
| $0 \quad+3.84$ | $+1.47$ | +0.63 | $+1.92$ | 0 |

(b)

(c)

## SIGN CONVENTION

In order to study the algebra of hinged floating blocks it is desirable to use a different sign convention for the hinge shear than that which was used in wiriting the reaction-shear equations. It is convenient in dealing with hinged blocks to define the end shears, or hinge shears, to be positive when they act as shown in Fig. 16 for joint 3 and block number 34. The reaction-shear equations for block 34 will appear as,

$$
\begin{align*}
& w_{34}=w_{34}^{\prime}-K_{34} P_{3}-R_{34} P_{4} .  \tag{22a}\\
& w_{43}=w_{43}{ }^{\prime}+R_{43} P_{3}+K_{43} P_{4} \tag{22b}
\end{align*}
$$



This sign convention, and form of the equations, is in agreement with a bending moment sign convention for continuous beams if the secondary stiffness $R$ (or influence number) is regarded as a positive constant.

## CONCLUSION

Using the well known formulas for linear base reaction diagrams as a basis, a method of analysis has been developed for hinged base slabs or hinged floating bridges. The numerical distribution process may be called reaction distribution. Its analogous correspondence to moment distribution and the column analogy has been explained. Since the process involves only the simple concept of equilibrium of forces, it may serve usefully as an introduction to a study of continuous beam analysis by moment distribution.

# Shrinkage and Plastic Flow of Pre-Stressed Concrete* 

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## SYNOPSIS

This paper presents the results of shrinkage and plastic flow measurements on pre-stressed and unstressed specimens for the duration of a year. Stored at 70 F and 50 per cent relative humidity the shrinkage of unstressed specimens reached maximum strain values of $8.7 \times 10^{-4} \mathrm{in}$. per in. for the concrete and $6.5 \times 10^{-4}$ for gunite. After the age of 10 days the gunite shrinkage was about 75 per cent of the concrete strains.

The loaded specimens were stressed to approximately 930 psi (low), 1500 psi (intermediate), and 2400 psi (high) for concrete and gunite whose ultimate compressive strengths were 4900 psi and 4500 psi respectively. As shrinkage and plastic flow occurred these stresses decreased until, at the age of 1 year, the stresses in the concrete were 25 to 33 per cent their initial values and the stresses in the gunite were about 43 per cent of the initial. Plastic flow is defined as the difference between the total strain of the loaded specimen and the shrinkage strain of the unstressed specimen during the same time interval. Plastic flow is assumed equal to the stress multiplied by a flow coefficient $c$. At 380 days the flow coefficients varied from $64 \times 10^{-8}$ to $79 \times 10^{-8}$ for the concrete and from $50 \times 10^{-8}$ to $61 \times 10^{-8}$ for the gunite. For the low stress specimens the gunite coefficient is 90 per cent of the concrete; for the other two stresses the gunite coefficient is about 75 per cent that of the concrete.

During the period of plastic flow the shrinkage of the unstressed concrete specimens relative to the total deformation of the loaded specimens was 83 percent for low stress, 64 percent for intermediate, and 50 percent for high stress. The corresponding ratios for the gunite were 65 percent for low stress, 59 percent for intermediate, and 53 percent for the high stressed specimens.

## INTRODUCTION

A large number of tests have been reported on the deformation of concrete under constant loading conditions but very little information is

[^16]available regarding the behavior of prestressed specimens for which the load is not constant but varies with time and conditions of exposure and use. To evaluate the effect of all of the variables that might enter into consideration in the application of pre-stressed concrete would be a tremendous task. However, it is possible to study the behavior of prestressed specimens under conditions that are probably the worst that could be conceived for field usage and thereby obtain information that would be valuable for use in design.

The purpose of the work reported was twofold, (1) the determination of the rate and probable total amount of stress decrease in the concrete and (2) the contribution made by shrinkage to the decrease in prestress. Accordingly measurements of deformation were made on both stressed and unstressed specimens. Specimens of both concrete and gunite were used.

## PROCEDURE IN MAKING TEST SPECIMENS

The gunite was "shot" on the job as a $1: 4$ mixture using standard equipment for this work. The concrete was mixed in the laboratory with the following quantities of materials per cu. yd., 600 lb . cement, 1260 lb . sand, 1920 lb . of $3 / 4-\mathrm{in}$. gravel and 36 gal . of water. The compressive strength at 28 days of 4 in . x 8 in . gunite cylinders was 4900 psi and for 6 in. x 12 in . concrete cylinders was 4500 psi.

Gunite specimens were made by "shooting" the material on to 12 in. planks, 16 ft . long, in a thickness of about 5 in . The test bars were then cut out of this column of material, using cutters devised for the purpose. Brass insert bars had been secured to the plank in proper location and position before shooting started and holes were drilled in proper locations on the other three sides of the bars, before the gunite became completely hardened, for the easy placement of other gage point insert bars after the sections were removed to the laboratory. Twenty four hours after the bars were fabricated, they were placed in damp sand and removed to the laboratory where they were fitted with inserts and gage points for reading of deformations.

The concrete specimens were cast in the laboratory in watertight molds, the gage point inserts being cast into the specimen. These specimens were removed from the molds at 24 hours. Both the gunite and concrete specimens were stored in the damp room at 70 F and $95+$ per cent relative humidity until removal to the constant temperature room. It was thought desirable to have a period of storage in the constant temperature room before and during loading as the specimens would then have a water content at the time of loading more consistent with that of similar material in the field. After removal to the constant tem-

Fig. 1-Method for loading bars


HIGH LOADING
$6-\frac{1}{2}$ rods.
2 plates $6 \times 5 \times 2 \frac{1}{2}$

INTERMEDIATE LOADING
$4-\frac{1}{2}$ rods
2 plates $\quad 6 \times 4 \frac{7}{8} \times 1 \frac{3}{4}$
LOW LOADING

$$
\begin{aligned}
& 4-\frac{3}{8} \text { rods } \\
& 2 \text { plates } 5 \frac{1^{\prime \prime}}{2} \times 4 \frac{1^{-}}{2} \times 1 \frac{1}{4}
\end{aligned}
$$

perature room, maintained at 70 F and 50 per cent relative humidity, the specimens remained there for the balance of the time of reading.

Specimens were loaded to the stresses indicated by means of plates and rods as shown in Fig. 1. A two-day period for application of the load was chosen arbitrarily as being representative of the time for the average prestressing operation in the field, where bars are used. Gage points were provided in the rods used for prestressing, with the same $10-$ inch gage length used for the concrete and gunite bars. At the end of two days the full prestress was restored by a second tightening of the nuts. This corresponds to the final tightening operation in the field. The rods were stressed by tightening the nuts, care being taken not to twist the rod during tightening. Readings were taken with a Whittemore fulcrumplate strain gage, values being estimated to hundred thousandths. Readings were taken on four sides of the bar and on all rods for each specimen at variable times, but sufficiently often to have an ample number of readings to fully identify the curves.

Shrinkage readings were taken on unstressed specimens, of the same size, that were subjected to the same storage conditions as the stressed specimens. As with the stressed specimens, readings were made on four sides of the specimen.

## RESULTS

## Shrinkage of unstressed specimens.

The average shrinkage strains of the unloaded specimens are recorded

## TABLE 1-DATA ON SHRINKAGE SPECIMENS

|  |  |  | SPECIMENS SI2E $4 \times 4 \times 24$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| DAYS | LOC | AVE $\triangle$ L | REVIS | SE SHRIT | NKAGE $D \mathrm{~L}$ |
| AGE |  |  | LOW | INTEFD' | HISH |
| 1 | LAB | 000200 |  | - |  |
| 5 | M R | 0000056 |  | - |  |
| 9 | CTH | 0.000077 | 0.00000 |  |  |
| 12 | CTH | 0000177 | 0.000100 | 000000 |  |
| 16 | CTH | 0000257 | 0000180 | 0000080 | 000000 |
| 19 | CTH | 0.000335 | 0000257 | Q000157 | 0000077 |
| 27 | CTH | 0.000467 | 0000390 | 0.000290 | 0000210 |
| 34 | CTH | 0000536 | 0.000459 | 0.000358 | 0000279 |
| 41 | CTH | 0000537 | 0.0005200 | 0.000419 | 0200340 |
| 48 | Ст | 0.000637 | 0.0005600 | 0000460 | 0000380 |
| 55 | CTH | 0.000664 | 0.000586 .0 | 0000486 | 0.000406 |
| 69 | CTH | 0000709 | 0000632 | 000053 | 0.000452 |
| 97 | СTH | 0.000764 | 0.0006870 | 0020586 | 0000507 |
| 119 | CTH | 0.000785 | 00007080 | 0010607 | 0.000528 |
| 140 | CTH | 0.000801 | 0020723 | 0.000623 | 0000554 |
| 200 | CTH | 0000817 | 0.00074000 | 0000639 | 000c550 |
| 260 | CTH. | 00000845 | 0000768 | 0000667 | 0000588 |
| 400 | Ст | 0000872 | 0000795 | 0000684 | 0000615 |


| GUNITE SPECIMENS |  |  |  | SIZE4×4×24 |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Days | LOC | AVE A: | REVISED | SHRITKA | $\triangle$ GE $\triangle 1$ |
| AGE |  |  | Low | INTER'D | HIGH |
| 5 | LAB. | occoso |  |  |  |
| 6 | M F . | 0,000056 | - |  |  |
| 9 | CTİH | 0.000040 | 000000 | - |  |
| 12 | CTH | 2000129 | a00し089 | 000000 |  |
| 7 | CTH | 0.000200 | 0.000180 | 0000071 | 0.00000 |
| 21 | CTH | 0.00025 | 0.000215 | 0000126 | 0000055 |
| 28 | CTH | 0000326 | 000023 s | 0.000197 | 0000126 |
| 35 | CTH | 0000383 | 0000343 | 0000254 | 0000183 |
| 4.3 | CTH | coon42. | 0.000382 | 0.000291 | 0000222 |
| 50 | CTH | 0000442 | 0.000422 | 0000.333 | 0000262 |
| 64 | CTH | 0000519 | 0000479 | 0000390 | 0000319 |
| 79 | CTH | 000053 | 0000513 | 0000424 | 1000353 |
| 92 | CTH | 0000572 | 0000532 | 0000143 | 0000372 |
| 113 | CTH | 0000582 | 0000542 | 0000453 | 0000382 |
| 140 | CTH | 0000601 | 0.000561 | 0000472 | 0.000401 |
| 200 | CTrl | 0000623 | $0 \cdot 00058$ | 0.000494 | 0.000423 |
| 280 | CTH | 0000638 | 0.000556 | 0000509 | 0.000438 |
| 400 | CTH | 0000654 | 0.000614 | 0000:25] | 0000454 |

Note-Revised shrinkage values are determined from the shrinkage average data using reloading dates of stressed specimens of low-intermediate-high loads to determine zero values.

$$
L O C .=\text { Place of reading } \quad \triangle L \text { is expressed in inches per inch }
$$



Fig. 2-Shrinkage of unstressed specimens
in Table 1 and plotted in Fig. 2. The initial (zero) reading was taken at the age of 1 day for the concrete and at 5 days for the gunite. Both showed an immediate expansion which was rapidly reduced as the specimens dried out in the 50 per cent relative humidity of the storage room. The rate of shrinkage was the same for both types of concrete for 3 or 4 days; thereafter the concrete decreased in length much more rapidly. At the end of 400 days the total shrinkage of the gunite was 75 per cent of that of the concrete.

The shrinkage of the stressed specimens subsequent to loading is assumed to be the same as the unstressed specimens in the same time interval. The bases from which these shrinkages are computed are indicated in Fig. 2.

The maximum shrinkage strains of $8.7 \times 10^{-4}$ for the concrete and $6.5 \times 10^{-4}$ for the gunite appear to be reasonable values for the rather high cement content and the relatively low humidity.

## Deformation of loaded specimens.

The average strains recorded for about 15 specimens of each load are listed in Table 2 and are plotted in Fig. 3 to 5. The concrete strains record shrinkage to the time of loading, the elastic strain due to application of the load and the subsequent deformations in the two-day interval before the reloading. At "reloading" the full prestress is restored by additional tightening of the nuts. The reloading elastic strain is given and the additional strains up to 400 days while the concrete stress decreased. The steel strain readings were commenced when the bars were tightened to give the initial loading. These strains determined the total force in the steel bars and, hence, the stress in the concrete. The modulus of elasticity of the steel was determined by test to be $29.3 \times 10^{6} \mathrm{psi}$ for the $3 / 8$-in. round bars and $28.4 \times 10^{6}$ psi for the $1 / 2$-in. round bars.

These loaded specimens experienced only shrinkage strains until the loads were applied. The concrete shrinkages, both expansion and contraction, closely paralleled those recorded for the unstressed specimens. Upon application of the load, or upon reloading, elastic strains took place immediately. The elastic strains have been used to compute the secant Moduli of Elasticity, which are tabulated below. For comparison the table also includes the tangent Modulus of Elasticity obtained by sonic tests of the specimens immediately prior to critical loading. The tangent moduli are, of course, higher values than the secant moduli.

Subsequent to the reloading, Fig. 3 and 4 show the additional strains as time increased and the concrete stresses decreased. These curves are all similar. If the three concrete curves of Fig. 3 are so placed that their terminals at 400 days are coincident, they are identical between the ages of 70 days and 400 days, though the stresses for the three loadings


Fig. 3-Concrete-strains of loaded specimens


Fig. 4 Gunite--strains of loaded specimens


Fig. 5-Steel strains
are quite different in the time interval. During the same time interval, the shrinkage strains of the unstressed specimens was about 70 per cent as much. When the same procedure is applied to the gunite data, (Fig. 4), the intermediate and high strain curves coincide from 90 to 400 days. The low-stressed curve coincides with the other two only between 280 and 400 days. Between 90 and 280 days the low-stress contractions are about two-thirds as much as the others. The shrinkage of the unstressed specimens was about one-half that of the higher-stressed specimens between 90 and 400 days.

## Deformations subsequent to removal of load.

Table 3 records the average strains of those specimens whose loads were removed at 90 days, or 200 days, or 400 days. These strains are not identical with those listed in Table 2, because Table 2 records the cumulative average of 15 to 9 specimens (including those of Table 3). Using the stress-strain data for the individual specimens, the secant modulus of

## TABLE 2-DATA ON DEFORMATION OF SPECIMENS

| CONCAETE |  | - | LOW STRESS |  |
| :---: | :---: | :---: | :---: | :---: |
| -ME |  | AVERA | GE $\triangle L$ |  |
| READING |  | GONCRETE | STEEL | STRESS |
| TO M A . | 1 | $\bigcirc$ | + 0 | 0 |
| M R YoCTH. | 5 | 10.000046 | + 0 | 0 |
| C.TH. | 7 | 0.000075 | + 0 | 0 |
| CEH.LOAD | 7 | 0.000320 | +0.001123 | 940 |
| C.T. H. | 9 | 0000479 | +0000920 | 770 |
| C.T.H RELC | 9 | 0000514 | +0.001118 | 935 |
| CTH | 18 | 0000784 | +0000850 | 711 |
| C : H . | 34 | 0.001019 | +0000662 | 555 |
| C. T. H. | 76 | -0.0012 67 | +0.000494 | 41.4 |
| C. TH. | 104 | 0.001322 | +0.000455 | 381 |
| CT.H. | 140 | 0.001368 | +0000427 | 358 |
| C.TH. | 200 | 0001409 | $+0000403$ | 337 |
| CTH. | 300 | 0.001444 | +0.000374 | 313 |
| C.T.H. | 400 | 0.001474 | -0.000348 | 291 |
|  |  |  |  |  |


| CONCRETE-INTERMEDIATE STRESS |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| TIME CF READING | DAYS | AVERAGE $\triangle L$ |  |  |
|  |  | CONCRETE | StEEL | STRESS |
| TC N. R | 1 | 0 | + 0 | $\bigcirc$ |
| M R C.I.H. | 5 | 10000064 | + 0 | 0 |
| C.T.H. | 10 | -0.000141 | + 0 | 0 |
| C.T. H LOAD | 10 | 0000624 | +0001090 | 1550 |
| C T. H. | 12 | 0000828 | $+0.000861$ | 1222 |
| C.TH. RELD | 2 | -0.000914 | +0.001090 | 1550 |
| C.T.H. | 18 | -0001171 | +0000870 | 1238 |
| C.T.H. | 31 | -0.001470 | +0000672 | 954 |
| C.T.H. | 66 | -0.001731 | + 0.000462 | 656 |
| C.T.H | 94 | -0001822 | +0.000403 | 572 |
| С.T. H | 122 | -0.001857 | +0.000371 | 526 |
| C.T.H. | 140 | 0001889 | $+0.0003671$ | 509 |
| CT. C . | 200 | 0001920 | +0.0003431 | 484 |
| C T.H. | 280 | -0.001953 | - 0000319 | 454 |
| CTH. | 400 | 0001992 | + 0.000298 | 408 |


| CONCRETE |  | HIGH |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $\begin{array}{r} \text { TIME } \\ \text { OF } \\ \text { REAOING } \end{array}$ | JaYS | AVERAGE $\triangle L$ |  |  |
|  |  | CONCRETE | STEEL | STAESS |
| TO M.R. | 1 | $+0$ | 0 | 0 |
| MA. \&oCT. | 5 | +0.000065 | 0 | 0 |
| CTH | 14 | -0000233. | + 0 | 0 |
| CTTHICAD. | 14 | -0001040 | + 0001120 | 2390 |
| C.T.H. | 6 | -0.001323 | +0000851 | 1811 |
| C.T. C RELD. | 16 | -0.001451 | $+0.001120$ | 2389 |
| C.T.H. | 27 | -0001905 | + 0000768 | 1955 |
| C.T H | 34 | -0,002046. | $+0.000684$ | 1459 |
| C.T. H | 62 | -0.002302 | +0000523 | 1113 |
| C.T. H | 83 | -0002380 | $+0.000470$ | 1004 |
| C.T. H | 111 | -0.002433 | $+0.00045$ | 926 |
| C.T. | 140 | -0.002470 | $+000045^{1}$ | 865 |
| C.T.H. | 200 | -0.002504 | $+0000377$ | 803 |
| C.TH. | 280 | -0.002533 | $+0000352$ | 750 |
| С.T.H. | 400 | -0002570 | +00003 9 | 680 |


| GUNITE |  |  | LOW STRESS |  |
| :---: | :---: | :---: | :---: | :---: |
| $\begin{gathered} \text { TIME } \\ \text { OF } \\ \text { READING } \end{gathered}$ | DAYS | AVERAGE $\triangle L$ |  |  |
|  |  | CONCRETE | STEEL | StRESS |
| TO M.R. | 5 | 0 | 0 | 0 |
| MR-\%oC.T.H. | 6 | 10.000062 | 0 | 0 |
| C.T.H | 7 | -0000096 | 0 | 0 |
|  | 7 | -0000280 | +0.001003 | 790 |
| CTH | 9 | -0000444 | +000085 | 651 |
| CTH RELD | 9 | -0000525 | +0001162 | 916 |
| CT.H | 18 | -0.000815 | -0,000913 | 720 |
| C.T.H. | 35 | -0001002 | +0000760 | 598 |
| C.T.H. | 64 | -0001157 | +0.000646 | 510 |
| CTT.M | 77 | -0.001190 | +0000623 | 491 |
| CTH. | 105 | -0001229 | +0,000594 | 469 |
| С.T.H. | 133 | -0001256 | +0000578 | 455 |
| C.T.H. | 200 | 0.001279 | +0000554 | 445 |
| C.TH | 280 | -0001343 | -0.000529 | 418 |
| С.т.H. | 400 | -0001335 | +0.000511 | 404 |


| GUNITE-INTERMEDIATE S |  |  |  | StRESS |
| :---: | :---: | :---: | :---: | :---: |
| TIME |  | AVERAG | E $\triangle L$ |  |
| READING |  | CONCRETE | STEEL | STRESS |
| TO MA | 5 | 0 | 0 | 0 |
| MR to C.TH | 6 | +0000079 | 0 | 0 |
| C.T.H | 10 | -0,000082 | 0 | 0 |
| CTH LOAO | 10 | -0000641 | +0.00112 | 1483 |
| C.TH | 12 | -0000858 | +0000s01 | 1191 |
| C.T.H.RELD | 12 | -0000969 | +0001171 | 1550 |
| C.T.H. | 18 | -0001198 | +0000992 | 1310 |
| C.T. | 31 | -0001385 | +0000626 | 1092 |
| CT.H. | 63 | -0001618 | +0,000662 | 875 |
| C.T.H. | 83 | -0001672 | +0000620 | 820 |
| C.T.H. | 111 | -0001706 | +0,000591 | 781 |
| CTH. | 140 | -0001740 | +0000572 | 756 |
| CT.H. | 200 | -0001786 | +0.000538 | 711 |
| C.T.H. | 280 | -0001827 | +0000505 | 667. |
| C.T.H. | 400 | -0001859 | +0.000490 | 648 |


| GUNITE |  | HIGM ST |  | RESS |
| :---: | :---: | :---: | :---: | :---: |
| TIME |  | AVERAG | E $\triangle$ L |  |
| READING |  | CONCRETE | StEEL | STRESS |
| TO MR | 5 | 0 | 0 | 0 |
| MR ${ }_{\text {ToCT. }}$ C. | 6 | +0.000091 | 0 | 0 |
| СТ. H . | 15 | -0000195 | 0 | 0 |
| CTH LOAD | 15 | -0000986 | \$0001173 | 2425 |
| C.T.H. | 17 | -00011915 | +0.000956 | 981 |
| CT.H.RELD | 17 | -0.001282 | +0.001158 | 2390 |
| C.T.H. | 37 | -0.001698 | +0.000814 | 1686 |
| CT.H | 58 | -0001851 | +0000695 | 1442 |
| C.T.H. | 79 | -0001937 | +0.000642 | 1330 |
| CT. | 93 | -0001974 | +0000616 | 1279 |
| C.T.H. | 128 | -0002022 | +0.00058c | 1203 |
| C.T. H. | 156 | -0002045 | +00C0568 | 1177 |
| CT.H. | 200 | -0002075 | +0.000547 | 1134. |
| C.T.H | 280 | -0002115 | +0000516 | 1072 |
| C.T. H | 400 | -0002143 | +0000497 | 1030 |

Nore-Reld $=$ Reloaded; $\Delta L=$ Deformation of specimen expressed in inches per inch; Days=Age of specimen; Stress $=$ Pounds per square inch. in concrete; M.R. $=$ Moisture room; C.T.H. $=$ Constant tem-persture-humidity room.

## TABLE 3-DEFORMATION CURVE DATA

Note: These figures represent the average of three specimens for their respective unloading date. Each $\Delta \mathbf{L}$, therefore, is calculated from its respective three specimens.


| TIME OF EAOING | CONCRETE |  |  | SFECIMENS |  |  | GUNITE |  | SPECIMENS |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | IGH STA | AESS | 2390 PSI. |  |  |  | W STRESS |  | 935 |  | PS I |
|  | 90 OAY |  | 200 DAY |  | 400 DAY |  | 90 DaY |  | 200 CAY |  | 400 DAY |  |
|  | ors | $\triangle$ L | Drs | $\triangle 1$ | Ors | $\Delta L$ | ors | $\Delta L$ | DVS | $\Delta L$ | DYs | $\Delta L$ |
| $\begin{aligned} & \text { END OF } \\ & \text { CURING } \\ & \text { FERIOD } \end{aligned}$ | 5 | $0000089$ | 5 | 0000052 | 5 | $00{ }^{+}$ | $G$ | $0000065$ | 6 | $0000056$ | 6 | $0.000053$ |
| SPECIMENS LOADED | 14 | $0001103$ | 14 | 0000975 | 4 | 0001111 | 7 | 0000259 | 7 | 0000279 | 7 | 0000289 |
| SPECIMENS heloadid | 16 | 0001558 | 16 | 0001292 | 16 | 0001550 | 9 | 0000507 | 9 | 0000518 | 9 | 0000527 |
| BCFOAE <br> UNL OADING | 90 | 0002547 | 200 | 0002302 | 4000 | 0002719 | 105 | 0001188 | 200 | $1000-264$ | 400 | 0001358 |
| $\begin{aligned} & \text { ELASIIC } \\ & \text { RECOYERY } \end{aligned}$ | 90 | 0002302 | 200 | 0002069 | 400 | 0002528 | 90 | 0001055 | 200 | 0001140 | 4 CO | 0001234 |
| $\begin{aligned} & \text { ELASTIC } \\ & \triangle F T E R \\ & \text { EFFECT } \end{aligned}$ | 04 | 0002252 | 214 | $0002030$ | 414 | $0002499$ | 104 | $0.001023$ | 214 | $0001107$ | 414 | $0.0 \overline{1171}$ |


$D Y S=$ Age of specimen in days; $\Delta L=$ deformation of specimen in inches per inch.

SECANT MODULUS OF ELASTICITY

|  |  | State of Stress |  |  |
| :---: | :---: | :---: | :---: | :---: |
| CONCRETE | Initial Load | Low | Intermediate | High |
|  | Stress change, psi | $\begin{gathered} 940 \\ 372 \times 10^{6} \end{gathered}$ | $1550$ | $\begin{aligned} & 2390 \\ & 296 \times 10^{5} \end{aligned}$ |
|  | E, psi Reload | $3.73 \times 10^{6}$ | $3.21 \times 10^{6}$ | $2.96 \times 10^{5}$ |
|  | Stress change psi | 165 | 325 | 579 |
|  | E, psi | $4.72 \times 10^{6}$ | $3.81 \times 10^{6}$ | $4.52 \times 10^{6}$ |
| GUNITE | increase, per cent | 27 |  |  |
|  | Stress change, psi | 790* | 1483 | 2425 |
|  | E, psi | $4.29 \times 10^{6}$ | $2.65 \times 10^{6}$ | $3.07 \times 10^{6}$ |
|  | Reload |  |  |  |
|  | Stress change, psi | $3.27 \times 10^{6}$ | $3.23 \times 10^{6}$ | $4.51 \times 10^{6}$ |
|  | increase, per cent | -24* | 22 | 47 |

TANGENT MODULUS OF ELASTICITY

CONCRETE GUNITE
$4.88 \times 10^{6}$
$5.75 \times 10^{6}$
$5.95 \times 10^{6}$
$5.74 \times 10^{6}$
$4.78 \times 10^{6}$
$6.16 \times 10^{6}$
*Initial load is 126 psi less than reload stress.

## SECANT MODULUS OF ELASTICITY

| State of Stress | Load Procedure | CONCRETE |  |  | GUNITE |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $\begin{aligned} & \text { Age } \\ & \text { days } \end{aligned}$ | Stress Change psi | $\begin{gathered} \mathrm{E} \\ \mathrm{Nsi} \\ \mathrm{No.} \times 10^{6} \end{gathered}$ | $\begin{aligned} & \text { Age } \\ & \text { days } \end{aligned}$ | Stress Change psi | $\begin{gathered} \mathbf{E} \\ \mathrm{psi} \\ \text { No. } \times 10^{6} \end{gathered}$ |
| Low | Reload Removed | $\begin{array}{r} 9 \\ 104 \end{array}$ | $\begin{aligned} & 172 \\ & 336 \end{aligned}$ | $\begin{aligned} & 4.20 \\ & 3.68 \end{aligned}$ | $\begin{array}{r} 9 \\ 105 \end{array}$ | $\begin{aligned} & 250 \\ & 133 \end{aligned}$ | $\begin{aligned} & 3.39 \\ & 3.58 \end{aligned}$ |
|  | Reload <br> Removed | $20{ }^{9}$ | 163 386 | 2.50 4.96 | 9 200 | 240 412 | $\begin{aligned} & 3.47 \\ & 3.32 \end{aligned}$ |
|  | Reload <br> Removed | 9 400 | $\begin{aligned} & 176 \\ & 276 \end{aligned}$ | $\begin{aligned} & 4.09 \\ & 3.33 \end{aligned}$ | 9 400 | $\begin{aligned} & 253 \\ & 381 \end{aligned}$ | 3.44 3.07 |
| Intermediate | Reload Removed | $\begin{aligned} & 12 \\ & 94 \end{aligned}$ | $\begin{aligned} & 311 \\ & 563 \end{aligned}$ | $\begin{aligned} & 3.45 \\ & 3.27 \end{aligned}$ | $\begin{array}{r} 12 \\ 111 \end{array}$ | $\begin{aligned} & 389 \\ & 734 \end{aligned}$ | $\begin{aligned} & 2.47 \\ & 2.89 \end{aligned}$ |
|  | Reload <br> Removed | $\begin{array}{r} 12 \\ 200 \end{array}$ | $\begin{aligned} & 342 \\ & 603 \end{aligned}$ | $\begin{aligned} & 4.37 \\ & 3.91 \end{aligned}$ | $\begin{array}{r} 12 \\ 200 \end{array}$ | $\begin{aligned} & 295 \\ & 731 \end{aligned}$ | $\begin{aligned} & 2.94 \\ & 3.28 \end{aligned}$ |
|  | Reload <br> Removed | $\begin{array}{r} 12 \\ 400 \end{array}$ | $\begin{aligned} & 321 \\ & 408 \end{aligned}$ | $\begin{aligned} & 4.19 \\ & 3.85 \end{aligned}$ | $\begin{array}{r} 12 \\ 400 \end{array}$ | $\begin{aligned} & 318 \\ & 673 \end{aligned}$ | $\begin{aligned} & 3.25 \\ & 3.12 \end{aligned}$ |
| High | Reload <br> Removed | $\begin{array}{r} 16 \\ 111 \end{array}$ | $\begin{aligned} & 284 \\ & 911 \end{aligned}$ | $\begin{aligned} & 3.83 \\ & 3.72 \end{aligned}$ | 17 100 | $\begin{array}{r} 270 \\ 1105 \end{array}$ | $\begin{aligned} & 5.92 \\ & 3.00 \end{aligned}$ |
|  | Reload <br> Removed | $\begin{array}{r} 16 \\ 200 \end{array}$ | $\begin{aligned} & 450 \\ & 820 \end{aligned}$ | $\begin{aligned} & 4.22 \\ & 3.52 \end{aligned}$ | $\begin{array}{r} 17 \\ 200 \end{array}$ | $\begin{array}{r} 387 \\ 1002 \end{array}$ | $\begin{aligned} & 5.69 \\ & 2.87 \end{aligned}$ |
|  | Reload <br> Removed | $\begin{array}{r} 16 \\ 400 \end{array}$ | $\begin{aligned} & 601 \\ & 675 \end{aligned}$ | $\begin{aligned} & 3.62 \\ & 3.54 \end{aligned}$ | $\begin{array}{r} 17 \\ 400 \end{array}$ | $\begin{array}{r} 423 \\ 1042 \end{array}$ | $\begin{array}{r} 3.79 \\ 2.99 \end{array}$ |

elasticity upon removal of the load is shown opposite; and, for the comparison, the earlier secant modulus of elasticity of the same three specimens when reloaded.

## DISCUSSION OF RESULTS

## Correlation of concrete and steel strains.

A test of the consistency of the data is a comparison of the change of length of the concrete $\Delta l_{s}$ with that of the steel bars $\Delta l_{c}$. The two should be equal. Since the steel strains and modulus of elasticity are known, the steel stress and the total force in the bars can be ascertained, and also the concrete stresses. However, at the ends of the bars, the force is reduced as the threads at the bar ends mesh with the threads of the nuts. It is not known just how far each nut was tightened, so the total elongation $\Delta l_{s}$ of the bars cannot be accurately computed.

The concrete strains were measured in the middle of the 24 in . length. It is probable that the strains at the ends were different due to the contact with the bearing plate, which inhibits free lateral deformation. Therefore, the true change in length $\Delta l_{c}$ of the concrete or gunite specimen cannot be computed. During the loading and reloading of the specimens the nuts on the bars were tightened and the length of bar between nuts changed, including the consolidation of nuts on the bearing plates and the bearing plate on the concrete specimen. So the check of the accuracy of the readings is applied only to those readings after reloading. Adopting the arbitrary assumption that the measured steel strains times a length equal to the out-to-out distance between bearing plates plus one inch will give the steel deformation $\Delta l_{s}$, and that the concrete strain times 22.15 in . gives the concrete deformation $\Delta l_{c}$, a comparison of the two deformations subsequent to reloading at the ages of 100,200 and 400 days is tabulated on the next page.

It would seem that the data are consistent, if one accepts some such uniform assumption of arbitrary lengths.

## SHRINKAGE

The plot of shrinkage strains of non-stressed specimens in Fig. 2 has the usual shape of such data. The following shrinkage strains (next page) have been abstracted for comparison of the concrete and gunite shrinkages.

It is apparent that the gunite has much less shrinkage after the first 10 days of drying out.

## PLASTIC FLOW

## Reduction of prestress

The prestressing procedure adopted in these tests consisted of an initial prestress and a restoration of full prestress at 2 days thereafter. After

CHANGE IN LENGTH OF CONCRETE VERSUS STEEL BARS

| State of Stress |  | 100 days |  | 200 days |  | 400 days |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Low |  | $\Delta l_{\text {c }}$ | $\Delta l_{\text {s }}$ | $\Delta l_{c}$ | $\Delta l_{s}$ | $\Delta l_{c}$ | $\Delta l s$ |
| $l_{\text {c }}=22.15 \mathrm{in}$. | Concrete | 0.01790 | 0.01822 | 0.01982 | 0.01965 | 0.02125 | 0.02116 |
| $l_{s}=27.5 \mathrm{in}$. | Gunite | 0.01560 | 0.01561 | 0.1670 | 0.01644 | 0.01795 | 0.01788 |
| Intermediate |  |  |  |  |  |  |  |
| $l_{c}=22.15 \mathrm{in}$. | Concrete | 0.02012 | 0.01960 | 0.02250 | 0.02136 | 0.02385 | 0.02285 |
| $l_{s}=28.5 \mathrm{in}$. | Gunite | 0.01634 | 0.01653 | 0.01810 | 0.01805 | 0.01972 | 0.01940 |
| High |  |  |  |  |  |  |  |
| $l_{c}=22.15$ in. | Concrete | 0.02176 | 0.02060 | 0.02335 | 0.02232 | 0.02476 | 0.02305 |
| $l_{s}=30 \mathrm{in}$. | Gunite | 0.01534 | 0.01623 | 0.01758 | 0.01830 | 0.01908 | 0.01983 |

SHRINKAGE STRAINS

|  | Age (days) <br> After Zero <br> Reading | Strain $=\frac{\text { in. }}{\text { in. }}=$ No. $\times 10^{-4}$ |  | Gunite Strain (as percent of Concrete Strain) |
| :---: | :---: | :---: | :---: | :---: |
|  |  | Concrete | Gunite |  |
| Expansion |  | $+0.56$ | +0.56 | 100 |
| Contraction | 10 | $-1.35$ | -1.68 | 124 |
|  | 20 | -3.72 | $-2.91$ | 78 |
|  | 50 | $-6.47$ | $-4.86$ | 75 |
|  | 100 | -8.13 | $-5.82$ | 72 |
|  | 200 | -8.20 | -6.25 | 76 |
|  | 400 | -8.72 | -6.54 | 75 |

reloading the adjustment of the nuts was not changed and the force in the bars and in the concrete decreased as shrinkage and plastic flow deformations occurred. This data will differ from many other tests performed with a constant force on the concrete. A salient fact obtained from these data is the marked decrease in the concrete and steel stresses as time elapsed. It indicates, as do other sources, that much of the effect of prestressing to such stresses as 30,000 to 35,000 psi is lost in a comparatively short time. The following tabulation shows the decrease in steel stresses during the time interval investigated.

The concrete and gunite stresses vary with the steel stress. While the final gunite stresses are less than 50 per cent of their prestress, they are

STEEL STRESS (psi)

|  | Concrete <br> State of Stress |  |  | Gunite <br> State of Stress |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Low | Inter- <br> mediate | High | Low | Inter- <br> mediate | High |
| at reload | 32,000 | 31,000 | 31,800 | 34,000 | 33,300 | 32,850 |
| 380 days later | 10,460 | 8,260 | 9,020 | $\mathbf{1 4 , 9 8 0}$ | 13,920 | $\mathbf{1 4 , 0 5 0}$ |
| per cent of reload | 32.7 | 25.6 | 28.3 | $\mathbf{4 4 . 0}$ | 41.8 | 42.7 |

markedly higher than the corresponding concrete stresses. The latter are only $1 / 4$ to $1 / 3$ their initial values.

## PLASTIC FLOW COEFFICIENT

The additional strains in concrete or gunite that occur subsequent to reloading can be read from Fig. 3 or 4. At $t$ days after reloading this strain amounts to $e_{t}$. The additional shrinkage strain $e_{s}$ of the nonstressed specimens can also be read at $t$ days after reloading from Figure 2. The term plastic flow, as used in this paper, will be defined as the difference $e_{t}-e_{s}=e_{f}$. This strain $e_{j}$ includes flow strains due to the load and whatever difference there may be between shrinkage of non-stressed and stressed specimens. Within ordinary working stresses the plastic strain $e_{f}$ is usually assumed to vary directly with the stress $f_{c}$, so that $e_{f}=\mathrm{f}_{c} c$, where $c$ is a flow coefficient that is a function of the time $t$. At some time $\ell$ the stress in the concrete is $f_{c}$; a day or so later at the age $(t+\Delta t)$ the stress in the concrete has decreased an amount $\Delta f_{c}$. This decrease in stress corresponds to a tensile stress $\Delta f_{c}$ added to the compression stress $f_{c}$ to produce a resultant stress $\left(f_{c}-\Delta f_{c}\right)$. The immediate elastic effect of the change $\Delta f_{c}$ is an increase of length whose strain is $\frac{\Delta f_{c}}{E_{c}}$, where $E_{c}$ is the modulus of elasticity of the stressed specimen at the age $t$. In addition, during the time $\Delta t$ there is plastic flow amounting to a strain $f_{c}(\Delta c)$, where $\Delta c$ is the increase of the flow coefficient $c$ in the time $\Delta t$. This is a compressive strain. The data indicate that the resultant of elastic and plastic changes in time $\Delta t$ is an increase of the compressive strain $\Delta e_{j}$. Therefore:

$$
\begin{gathered}
\Delta e_{s}=f_{c}(\Delta c)-\frac{\Delta f_{c}}{\mathrm{E}_{\mathrm{c}}} \\
\text { or } \Delta_{s}=\frac{\Delta e_{s}+\frac{\Delta f_{c}}{\mathrm{E}_{c}}}{f_{c}}
\end{gathered}
$$

By use of the test data, $f_{c}, \Delta f_{c}$, and $\Delta e_{f}$ can be computed. The actual value of the modulus of elasticity $E_{c}$ of the stressed specimens at any age $t$ is not known. The discussion on page 238 gives a few values of secant moduli for specimens unloaded at approximately 100,200 and 400 days. The data indicate that $E_{c}$ may be greater or less than the value at the time of reloading, usually less. The following tabulation of the flow coefficient $c$ has been made assuming an average value of $E_{c}=4.35 \times 10^{6}$ psi for concrete specimens and $E_{c}=3.67 \times 10^{6}$ psi for the gunite. These values are the average $E_{c}$ at the time of reloading. The value of the flow coefficient $c$ was obtained by summing up values of $\Delta c$ for a time interval of 5 days after reloading followed by 10 day intervals for the entire range of the tests.

|  | Concrete <br> Age (days) <br> after <br> reloading |  |  | State of Stress |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Low | Intermediate | High | Gunite |  |  |
| 50 | 33.4 | 49.1 | 47.7 | 35.9 | 38.2 | 31.8 |
| 100 | 45.2 | 63.9 | 55.0 | 46.0 | 44.6 | 39.7 |
| 200 | 57.9 | 71.8 | 62.9 | 50.0 | 53.8 | 45.0 |
| 380 | 64.1 | 79.3 | 67.3 | 57.7 | 60.8 | 50.3 |

If $E_{c}$ were accurately determined at each age and the method of analysis were correct, the flow coefficient should be independent of the state of stress and all values of $c$ at the age $t$ should be the same. The results given above indicate that this is roughly true, since for both concrete and gunite the low and high stress ralues check fairly well. For both materials the intermediate values are greater than either low or high stressed specimens. The tabulation emphasizes again the fact that the plastic flow is less for the gunite than for the concrete, as is also true for the shrinkage strains of the unstressed specimens.

## SUMMARY

The purpose of prestressing of concrete is to produce initial compressive stresses in the concrete that may balance tensile stresses caused by application of other loads. The tests described in this report apply to concrete members with a uniform prestress, such as pipes, tanks and columns. These tests indicate that moderate prestresses of 30,000 to 35,000 psi do not fulfill the object of prestressing since shrinkage and flow will reduce this prestress to less than one-half its original value for gunite and to $1 / 3$ or $1 / 4$ the original value for concrete. That there is a decrease
has already been understood, but the magnitude of these reductions has not been generally known. The solution has been to employ high strength steel wires prestressed to 100,000 or $150,000 \mathrm{psi}$. A reduction of 20,000 to 30,000 psi due to shrinkage and flow still leaves a satisfactory prestress.

Unstressed specimens stored at 50 percent relative humidity had shrinkage strains at the end of 1 year of $8.7 \times 10^{-4}$ for concrete and $6.5 \times 10^{-4}$ for the gunite, the gunite shrinkage being about 75 percent of the concrete. After the second application of the full prestress to the loaded specimen (reload) the subsequent strains of the loaded specimens were greater than the corresponding shrinkage of the unloaded specimens. The difference, called plastic flow, is roughly proportional to the stress applied. Subsequent to reloading the shrinkage strains were a considerable part of the total strains of the loaded specimens. At $\mathbf{1}$ year for the different stress conditions they varied from 83 percent for low stress, 64 percent for intermediate, to 50 percent for the high stress for the concrete specimens. For the gunite the ratio of shrinkage to total deformation was 76 percent at low stress, 59 percent for intermediate, and 53 percent for high stress.

## ACKNOWLEDGMENT

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# Proposed <br> Minimum Standard Requirements for Precast Concrete Floor Units* 

REPORTED BY ACI COMMITTEE 711

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## GENERAL

## 10. Scope or limits

(a) These minimum standard requirements for precast concrete floor units, are to be used as a supplement to the ACI "Building Regulations for Reinforced Concrete" (ACI 318-41) $\dagger$.
(b) With respect to design for strength, i.e., for bending moment, bond and shear stresses, all the types referred to in 10 (d) are to be designed in accord with standard reinforced concrete theory and in accord with (ACI 318-41), except that with respect to cover, there is in some cases departure therefrom justified by the greater refinement in the finished product, both as to dimensions and to quality, when made by factory methods and with factory control. See Section 11.
(c) Section $103(a)$, ACI 318-41 $\ddagger$, recognizes and makes provision for special systems of reinforced concrete. With reference to precast floors its provisions may be invoked where necessary by the manufacturer, architect, engineer, building inspector, builder or owner, whereever this report is silent.

[^17]

Fig. 1

Fig. 2 - Note: The mortise shall be not less than $1 / 2$ in.

(d) Two distinct types are now manufactured:

1. I-beam type, with either cast in place or precast slab. (Fig. 1)

See section 50 for definitions. When the slab is cast in place as shown in Fig. 2 the result is a Tee beam and may be computed as such.
2. Hollow core type (Fig. 5)

Others are of channel shaped cross section often used as roof slabs, and the orthodox rectangular beam. Standard reinforced concrete theory is applicable to all.

## 11. Concrete protection for reinforcement

(a) Precast floor and roof units made of high quality factory controlled concrete may, when used in locations protected from the weather or moisture and with minimum fire hazards, be approved with $5 / 8$-in. concrete cover for the reinforcing provided, however, that the concrete cover in all cases shall be at least equal to the diameter of round bars and one and one-half times the side dimension of square bars, and provided that to insure exact final location to the steel, positive and rigid devices for that purpose are employed in the manufacturing process. When the precast-members are exposed to weather, moisture or fire hazard the protective cover shall be increased to conform with section 507, ACI 318-41. $\dagger$

[^18]
## MATERIALS

## 20. Cement

(a) High-early strength concrete as produced with Type III portland cement or with Type I portland cement and accelerated curing is recommended. Portland cement shall conform to the "Standard Specifications for Portland Cement" (A.S.T.M. Serial Designation C150-42) and shall be Type I or Type III.

## 21. Aggregate

(a) Concrete aggregates shall conform to the "Standard Specifications for Concrete Aggregates" (ASTM Serial Designation: C33-44), provided, however, that aggregates which have been shown by test or actual service to produce concrete of the required strength, durability, water-tightness, fire-resistance, and wearing qualities may be used under Section 302(a) Method 2, ACI 318-41* where authorized by the Commissioner of Buildings.
(b) The maximum size of the aggregate for precast joist shall not be larger than one-third of the narrowest dimension between sides of the forms of the member in which the unit is cast nor larger than threefourths of the minimum clear spacing between reinforcing bars and sides of the forms except that where the concrete is placed by means of high frequency vibration the maximum size of the aggregate shall not be larger than one-half the narrowest dimension between sides of the forms.
(c) Aggregate for floor slabs shall conform to Section 21 (a) and in addition the combined aggregate shall be so graded from fine to coarse that not less than one-half nor more than two-thirds by weight of the total, based on dry materials, is retained on the No. 4 standard sieve, except that these proportions do not necessarily apply to light weight aggregates. The maximum size shall not exceed one-third the thickness of the slab.

## 22. Steel

(a) In the unprestressed types, the steel in the joist or floor units shall be intermediate grades Billet-Steel Concrete Reinforcement Bars (ASTM Serial Designation. A15-39) or Rail Steel Concrete Reinforcement Bars (ASTM Scrial Designation: A16-39) or cold drawn steel wire for concrete reinforcement (ASTM Serial Designation: A82-34).
(b) Prestressed steel may be used in any of the types mentioned in section ( $10(\mathrm{~d})$. When used, computations for stresses, moments and allowable loads shall be in accord with the theory outlined in "Prestressed Concrete, Design Principles and Reinforcing Units", by Herman Schorer $\dagger$ or in a more condensed article by the same author in Reinforced Concrete No. 6 (Portland Cement Assn.).

## 23. Strength of concrete

(a) Concrete for floor units made of sand and gravel, crushed stone, slag or other heavy aggregate and of a span of 12 ft . or more shall have a compressive strength of not less than 3750 psi at 28 days when tested in accordance with the applicable current standards of the A.S.T.M.
(b) For roof slabs or for floor units made of light weight aggregate lower compressive strengths may be permitted where the unit stresses used in design for strength and bond will satisfy the requirements of paragraph 24 (a) of these standards.

## 24. Unir stresses in concrete and reinforcement

(a) The allowable design stresses in the concrete shall conform to the requirements set forth in Section 305 (a) and Table 305(a), ACI 318-41. $\dagger$
(b) The allowable stresses in the steel shall conform to the requirements set forth in Section 306(a) and 306(b) ACI 318-41†.

## MANUFACTURE

## 30. Workmanship

(a) The finished product shall be free of honeycomb or rock pockets. The mix, the gradation of the aggregate and the workability shall be such as to insure complete filling of the form and continuous intimate bond between the concrete and all steel. To assist in attaining the latter, vibration is recommended, but any method which will meet the stated requirements and with the strength as required in unit beams or slabs or in the finished floor, is acceptable.
(b) Handling and conveying before curing shall be reduced to a minimum. Machinery for this purpose should be so designed that the unit will not be subject to bending or shock which will produce incipient cracks, broken edges or corners.
31. Curing
(a) The minimum amount of curing of precast units shall consist in keeping the concrete moist for at least 7 days, if made of normal portland cement and for at least 3 days if made of high early strength cement. For each decrement of 5 degrees below 70 F in the average curing temperature these curing periods shall be increased by four days for units made of normal portland cement and by two days for units made of high early strength cement. See Table 1. The average curing temperature in no case shall be less than 50 F .
(b) Curing by high pressure steam, steam vapor, or other accepted processes may be employed to accelerate the hardening of the concrete and to reduce the time of curing provided, however, the compressive

TABLE $1(\sec 31 a)$ - MINIMUM CURING TIME IN DAYS IN MOIST ATMOSPHERE

| Cement | Temperature in Degrees $F$ |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  | 70 | 65 | 60 | 55 | 50 |
| Normal Portland. ....... | 7 | 11 | 15 | 19 | 23 |
| High Early Strength...... | 3 | 5 | 7 | 9 | 11 |

strength of the concrete is at least equal to that obtained with the curing specified in Section 31a and that the 28-day strength meets the requirements of Sec. 23.

## 32. Identification and marking

(a) All joist, beams, girders and other floor units shall show some mark plainly indicating the top of the unit and the size of the bending moment reinforcement. This mark or symbol shall indicate the length, size and type of reinforcing and carrying capacity of the unit, and shall be shown on the placing plans.

## 33. Transportation

(a) After curing, units shall be so stored, stacked, loaded and transported, unloaded and placed, that no transverse or longitudinal cracks will develop. Adequate instructions should be given to handlers and insofar as possible, only experienced men should be put in charge of this phase of the work.
(b) To insure the eventual placement of the units in the structure without cracks, the handling, whether manually or in slings or cradles, shall be done in such a manner that bending about either the vertical or horizontal axis of the cross-section will be reduced to a minimum.

## TESTS

## 40. Beams and floors

(a) Where the individual unit is tested as a simple beam, it shall sustain without complete failure*, a uniformly distributed load of at least 2.25 times the design live load based on allowable stresses in bending moment and shear as given in Section and Table $305(a)$ and Section 306(a), ACI 318-41 $\dagger$
(b) When field tests are made they shall be made as required by and shall meet the requirements of Section 202, ACI 318-41 $\dagger$ ) making use of notation used in Section 200, ACI 318-41. $\dagger$

[^19]
## I-BEAM TYPE JOISTS

## 50. Definitions

(a) Floors made of precast joists mortised or embedded into a monolithic floor placed or poured on the job, producing a T-beam are called Precast joist cast-in-place concrete slab floors, and the resisting moment may be computed as if the joist and slab form a T-beam. See Section 54.
(b) Floors made of precast joists over which precast slabs are laid and bonded to produce T-beam action are called precast joist and slab concrete floors.
(c) Floors made of precast joists over which precast slabs are laid for flooring and not bonded to the joist to produce T-beam action are called independent precast joist and slab concrete floors.

## 51. Sections

(a) The most commonly made joists of I-beam section are as shown in Fig. 1. Other sizes and shapes meeting the regulations of ACI 318-41 as to resistance to bending moment, shear, deflection and bearing, may be used.
(b) Since the shear and bending moment resistances are based on nominal dimensions as well as on area of the steel and allowable working stresses in concrete and steel, the following tolerances shall not be exceeded: plus or minus $1 / 8$-in. as to width and height, and plus or minus $1 / 2$ in. as to length.

## 52. Floor slab thickness

(a) The recommended minimum thickness of cast-in-place reinforced concrete floor slabs with joist heads embedded not less than $1 / 2 \mathrm{in}$. and with joist spacing less than 30 in., is 2 in . For joist spacing of 30 to 36 in . the minimum thickness of slab concrete floors should be $21 / 2 \mathrm{in}$. Greater thickness may be required where unusual loads or spans are encountered. The required thickness of slabs spanning more than 36 in . shall be determined by accepted design methods.
(b) The recommended minimum thickness of precast slab to be used with precast joists is 2 in. with joist spacing up to 30 in . and $21 / 2$ in. with joist spacing from 30 in . to 36 in . In the case of slabs of ribbedor channel-section, the thickness requirement applies to the portion thereof containing the tensile reinforcement.

## 53. Extra or concentrated loads

(a) Where the floor supports partition walls parallel to the joist or where loads heavier than the uniform load for which the floor is designed are known to be expected, joists may be placed side by side, with flanges touching, but under such conditions the joists and cast-inplace floor slabs are not to be considered as a T-beam unless the shear reinforcing loops in the joists extend into the slab.


Fig. 3-Defails of joist hangers

Fig. 4-Tension bar hanger inserted in joist as cast

(b) Where multiple joists are used, their strength shall be at least that of a single beam, multiplied by the number used.

## 54. Design

(a) Floors laid as defined in $50(a)$ and $50(b)$ of these standards may be designed as T-beams, where joists are supported at sufficient intervals to take out the sag while the cast-in-place or precast slab is being laid, the support being left in place until the concrete has hardened. Under this condition the dead load is considered to be the weight of the floor per joist plus the weight of the joist.*
(b) The resistance to longitudinal shear between floor and joist where the joists are embedded $1 / 2 \mathrm{in}$. may be taken as equal to the allowable shear stress for bcams with no web reinforcement, but with special anchorage of longitudinal steel. Table $305(a)$, ACI $318-41 \dagger$.
(c) Where ends of joists cannot be rested on walls, as at stair wells, etc., metal joist hangers made as in Fig. 3 may be used to provide end support, or a preformed tension bar hanger may be inserted in the joist at the time of casting (Fig. 4).

[^20]
## 55. Holes in web

(a) Because they reduce the shearing resistance, holes in the web shall be reduced to a minimum. Where found necessary they should be located as near the center of the beam as possible or at location of minimum shear. They shall be cast when the beam is made or drilled on the job (not punched) and be not more than 2 in . in diameter. No holes should be made by any mechanic on a job except after approval by and under the supervision of the architect or engineer.
56. Installation and construction details
(a) On every job there will be a need for a joist setting plan; prepared by an architect or an engineer and approved by the manufacturer. Only in this way will the owner be assured of unquestionable results. Working stresses based on maximum strength of the materials used, shall be as provided in Section $305(a)$ and Table $305(a)$, ACI $318-41^{*}$, and shall be given the architect or engineer by the manufacturer. Shears and bending moments will be properly taken into account by the architect or engineer and accepted by the manufacturer. $\dagger$
(b) There is a need for standardization of many installation and construction details. This does not mean that innovations should be prohibited or frowned upon, but rather that an acceptable practice in handling and setting, leveling, shoring of joists, placing forms for floors, reinforcement for floors, conduits, bulkheads, stairwells, partition bearing joist, etc., should be approved by the architect, engineer and manufacturer and the building contractor informed thereof by properly drawn plans and through the supervision of the architect or engineer.

## HOLLOW CORE TYPE JOISTS

## 60. Definitions

(a) Floors made of precast concrete units in which some portion of the cross section between top, bottom, and sides is left out at the time of casting for purposes of reducing dead load and quantity of material used in their manufacture are called Hollow-Core-Precast Floors.

## 61. Sections

(a) Sections of hollow units as shown in Fig. 5 are acceptable.
(b) Any other sections which will provide proper protection for the steel reinforcement, and strength sufficient for handling and for carrying the loads for which they are designed and which meet all other requirements of this report will be acceptable, provided the computations for carrying capacity have been verified by a recognized testing laboratory.

[^21]Fig. 5

62. Design
(a) Resisting moments and shear resistance of hollow core units shall be computed by the standard formulas and methods. Allowable unit stresses in concrete shall conform to the requirements of Section 305(a) and Table 305(a) ACI 318-41.*
(b) The allowable stresses in the steel shall conform to the requirements set forth in Section 306 (a) and 306(b) ACI 318-41.*
63. Special conditions-openings known in advance of construction
(a) At stairways or other openings when no wall or girder bearing is available for one end of a floor unit, the long dimension of the opening shall be parallel to the length of the unit. Specially designed reinforced headers or curbs at the short side of such openings shall transfer their dead and live load to the longitudinal units on the side of the opening by devices or means satisfactory to the architect or engineer. Units adjacent to the side of such openings shall be designed to carry the reaction of the headers in addition to their own specified dead and live load.
(b) The requirements of unusual conditions, such as those outlined in $63(a)$ or others, often may be met by making use of special units,

[^22]some of which are wider than the standard, some deeper and some more heavily reinforced.
(c) Holes in the bottom or ceiling sides of units, for conduit or for hangers should be located below the hollow portion of the unit and should be cast at the time of manufacture or drilled under the supervision of the architect or engineer.
(d) Channeling in the top or floor side, except over the support is not permissible and channels placed over the supports shall not reduce the shear resistance below that allowable in this standard.
(e) Cutting of reinforcement for installation of pipes or conduit is permitted only upon approval of the architect or engineer and only after satisfying requirements specified in Section 64(a) (b) and (c).

## 64. Cutting of holes and channels

(a) No openings or channels not provided for in the structural design shall be made on the job without the specific approval of the engineer and in accord with his written, detailed instructions covering such work.
(b) In some cases the section of and reinforcement in the adjacent beam are such that when the span is taken into consideration the resistance to bending moment and shear is greater than that required by the live and dead loads called for by the building code or specifications. In such a case holes may be cut and curbed providing it is done in a manner to insure that the stresses on the transversely cut units will be transferred through the curb or longitudinal key to the adjoining units. In general, such cutting should be located near the quarter point of the span.
(c) Where holes are cut, the load normally carried by the cut units, and by the cutting transferred laterally to adjacent units may be considered to be uniformly distributed laterally for three units one foot wide on either side. With such assumption the computed stresses in the concrete may not esceed $0.45 \mathrm{f}^{\prime}$ nor in any case 1500 psi compression; $0.02 f^{\prime}{ }_{c}$ (for reinforcement without special anchorage) with a maximum allowable unit stress of 75 psi or $0.03 \mathrm{f}^{\prime}{ }_{c}$ (for reinforcement with special anchorage) with a maximum allowable unit stress of 112.5 psi in shear for units without stirrups; or 20,000 psi tension in the reinforcing steel.
(d) Holes in the bottom or ceiling side for conduit or for hangers should be below the hollow portion of the unit; should not be less than $13 / 4$ in. from the longitudinal reinforcement, and may be either cast at the time of manufacture or drilled under the supervision of the architect or engineer. Hangers may be placed in the joints between the units before they are grouted.
(e) There shall be no channeling in the top or floor side except over the support. Channels cut over the supports are permissable only when they do not reduce the shear resistance below that allowable in this standard, and must be approved by the architect or engineer.
(f) Cutting of reinforcement for installation of pipes or conduit is permissible only with approval of the architect or engineer and only after satisfying requirements specified in Section 64 (a) (b) and (c).

## 65. Installation and construction details

(a) An erection or unit setting plan shall be prepared by the architect or engineer and approved by the manufacturer for each job. To provide properly for shear and bending moment stresses, the plan shall indicate all openings, stairways, etc., together with the location of points, if any, where loads are in excess of the general floor loading.

## APPENDIX <br> (Excerpts from Slandard Building Regulations for Reinforeed Concrete-ACI 318.41)

## 103. Special systems of reinforced concrete

(a) The sponsors of any system of reinforced concrete which bas been in successfut use, or the adequacy of which has been shown by test, and the design of which is either in conflict with, or not covered by these regulations shall have the right to present the data on which their design is based to a "Board of Examiners for Special Construction" appointed by the Commissioner of Buildings. This Board shall be composed of competent engineers, architects and builders, and shall have the authority to investigate the data so submitted and to formulate rules governing the design and construction of such systems. These rules when approved by the Commissioner of Buildings shall be of the same force and effect as the provisions of this code.

## 200. Notation

$D=$ Deflection of a floor member under load test.
$L=$ Span of member under load test.
$t=$ The total thickness or depth of a member under load test.

## 202. Load Tests

(a) When a load test is required, the member or portion of the structure under consideration shall be subject to a superimposed load equal to one and one-half times the live load plus one-half of the dead load. This load shall be left in position for a period of twenty-four hours before removal. If, during the test, or upon removal of the load, the member or portion of the structure shows evident failure, such changes or modifications as are necessary to make the structure adequate for the rated capacity shall be made; or, where lawful, a lower rating shall be established. The structure shall be considered to have passed the test if the maximum deflection at the end of the twentyfour hour period does not exceed the value of $D$ as given in the following:

$$
\begin{equation*}
D=\frac{.001 L^{2}}{12 t} \tag{1}
\end{equation*}
$$

all terms expressed in the same units.
If the deflection exceeds the value of $D$ as given in formula (1), the construction shall be considered to have passed the test if within twenty-four hours after the removal of the load the member or portion of the structure shows a recovery of at least seventyfive percent of the observed deflection.

## 302. Defermination of strength-quality of materials

(a) The determination of the proportions of cement, aggregate and water to attain the required strengths shall be made by one of the following methods:

Method 1-Conerete made from average materials:
When no preliminary tests of the materials to be used are made, the water-content per sack of cement shall not exceed the values in Table $302(a)$. Method 2 shall be employed when artificial aggregates or admixtures are used.

## TABLE 302(a)-ASSUMED STRENGTH OF CONCRETE MIXTURES

Water-Content U. S. Gallons

Per 94-lb. Sack of Cement $|$| Assumed Compressive Strength |
| :---: |
| at 28 Days-psi |

## 305. Allowable unit stresses in concrete

(a) The unit stresses in pounds per square inch on concrete to be used in the design shall not exceed the values of Table $305(a)$ where $f^{\prime} c$ equals the minimum specified ultimate compressive strength at 28 days, or at the earlier age at which the concrete may be expected to receive its full load.

## 306. Allowable unit stresses in reinforcement

Unless otherwise provided in these Regulations, steel for concrete reinforcement shall not be stressed in excess of the following limits:
(a) Tension
( $f_{s}=$ Tensile unit stress in longitudinal reinforcement)
and ( $f_{v}=$ Tensile unit stress in web reinforcement)
20,000 psi for Rail-Steel Concrete Reinforcement Bars, Billet-Steel Conerete
Reinforcement Bars (of intermediate and hard grades), Axle-Steel Concrete
Reinforcement Bars (of intermediate and hard grades), and Cold-Drawn Steel
Wire for Concrete Reinforcement.
18,000 psi for Billet-Steel Concrete Reinforcement Bars (of structural grade), and Axle-Steel Concrete Reinforcement Bars (of structural grade).

## 507. Concrete protection for reinforcement

(a) The reinforcement of footings and other principal structural members in which the concrete is deposited against the ground shall have not less than three inches of concrete between it and the ground contact surface. If concrete surfaces after removal of the forms are to be exposed to the weather or be in contact with the ground, the reinforcement shall be protected with not less than two inches of concrete for bars more than $5 / 8$ inch in diameter and one and one-half inches for bars $5 / 8$ inch or less in diameter.

## TABLE 305(a)-ALLOWABLE UNIT STRESSES IN CONCRETE



[^23](b) The concrete protective covering for reinforcement at surfaces not exposed directly to the ground or weather shall be not less than three-fourths inch for slabs and walls; and not less than one and one-half inches for beams, girders and columns. In concrete joist floors in which the clear distance between joists is not more than thirty inches, the protection of metal reinforcement shall be at least three-fourths inch.
(c) If the code of which these regulations form a part specifies, as fire-protective covering of the reinforcement, thicknesses of concrete greater than those given in this section, then such greater thicknesses shall be used.
(d) Concrete protection for reinforcement shall in all cases be at least equal to the diameter of round bars, and one and one-half times the side dimension of square bars.
(e) Exposed reinforcement bars intended for bonding with future extensions shall be protected from corrosion by concrete or other adequate covering.

# AMERICAN CO (copyr ghted) 

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# Proposed Recommended Practice for the Construction of Concrete Farm Silos* 

## Reported by ACI Committee 714

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## AS HERE REVISED REPORT IS PRESENTED FOR ADOPTION AS A STANDARD

This constitutes formal notice to the membership of the Institute that the report of the Committee 714, William W. Gurney, Chairman, published p. 189 of the ACI Journal January 1944 has been revised by the originating committee in the light of discussion p. 204-1 of the November 1944 Journal Supplement.

Committee 714 now asks for the adoption of the report as a Standard of the Institute subject to the following revisions:

Under "2. General Design Recommendations," sub-section B1). "Footings" is revised to read as follows:
B. Footings 1) Depth below ground line: The distance from the ground line to the base of the footing should be at least 24 in. and not less than 36 in. in localities in which the ground freezes to a depth of two feet or more.

Page 192, of the report as originally published, the title of Table 3 is revised to read "Table 3-Hoop spacing for concrete stave silos designed for silage with a moisture content not exceeding 75 percent," (the italicized words constituting an addition to the previous title).

[^24]In section "3. Materials," sub-section "E. Hoops," p. 193, is revised to read as follows:
3. Hoop-Spacing: The maximum hoop-spacing for either grass or corn silage should be as given in Table 3. This table does not apply for silage with a moisture content above 75 percent, as closer spacing will be required for wetter silage.

# Current Reviews of Significant Contributions in Foreign and Domestic Publications, prepared by the Institute's Reviewers 

Calculation of raw mixes<br>Artiro Vera, Jr., Rock Products, V. 48, No. 8, p. 109, Aug. 1945.

Reviewed by Roy N. Young
The use of determinants facilitates design of 3-component mixes for specific clinker compositions in the manufacture of portland cement. A practical example of the method is given.

Slurry sampling methods<br>Mark Lintz, Rock Products, V. 48, No. 8, pp. 107-108, Aug. 1945.<br>Reviewed by Roy N. Young

A system for splitting and handling test samples from an automatic sampler is described, involving sufficient agitation to insure uniformity. The use of grab samples from storage tanks gives unreliable results and may cause cyclic errors.

## Natural cement blend improves concrete

D. E. Loveweld, Engineating News Record, V. I35, No. 4 (July 26, 1945) pp. 86-88

Reviewed by S. J. Chamberlin
Improved properties are claimed for concrete made by substituting one bag of airentraining natural cement for a bag of normal cement in a cubic yard mixture. The mixture is adjusted by decreasing the percentage of sand and the water-content, resulting in comparable strengths and workability. The air-content of the concrete can be accurately controlled by changing the amount of natural cement used.

## Large triaxial testing machine built by Bureau of Reclamation

## R. F. Blanks, and Douglas McHenry, Engineeting News Record, V. 135, No. 6 (Aug. 9, 1945) pp. 113-115

Cylindrical specimens, 6 by 12 in ., may be tested both axially and laterally under independent pressures as high as $125,000 \mathrm{psi}$. The walls of the chamber are $15 \frac{1}{2}$ in. thick, made of three concentric shrink-fit cylinders of alloy steel with the inner wall prestressed. Specimens are inserted or removed by unscrewing the smaller of two concentric threaded plugs. The ram for axial loading may be removed by unserewing the plug at the bottom. The machine will be used for determining the shearing strength of concrete and rock, and the behavior of these materials under loadings simulating those obtained in large gravity-type dams.

## Form linings for concrete surfaces

Harrison V. Pitiman, Engineering News Record, V. 135, No. 18 (Nov. 1, 1945), pp. 92-96

Reviewed by S. J. Chamberlins
The lining used at Norfolk Dam was about 0.08 in. thick and consisted of a thin fabric facing adhesively bonded to an absorptive cardboard back. Total cost of the lining was about 12 c. per sq. ft. Backing the absorptive material with a non-absorptive lining, $1 / 4$-in. hard-surfaced manufactured board or plywood, produced better surfaces at a savings of $31 / 2$ c. per sq. ft. over use directly on wood sheathing, which would have required frequent reconditioning. During cold weather, when the application of water for curing was discontinued, the lining was left in place for 21 days as a curing and protective measure. The lining was effective in minimizing surface pits and voids; eliminating sand streaking; and in producing an even-textured, dense appearing surface.

## Paving for the biggest planes in sight

Engineering News Record, V. 135, No. 2 (July 12, 1945), pp. 96-98 Reviewed by S. J. Chamekritw
Designed for a wheel load of $175,000 \mathrm{lb}$., the slabs are 9,12 and 18 in . thick, and are supported on a well compacted $24-\mathrm{in}$. fill. The thickest slab is required on portions of the hangar floor and aprons where there is excessive vibration incidental to warming up the planes. The $20 \times 20-\mathrm{ft}$. panels of the thick concrete contain only temperature steel and $2-\mathrm{in}$. dowels on $12-\mathrm{in}$, centers. Dowels were held in place by split headers and by welded pipe removable jigs. Bayonet-type vibrators were used ahead of the finishing machine. The surfaces were floated, screeded, floated again, belt finished and then broomed. The 12 and 18 -in. slabs were cured by ponding to help dissipate the heat of setting.

## Portland cement dispersion by adsorption of calcium lignosulfonate

Fred M. Enssherger and Weslet G. Frasce (Ohio State Cniversity), Industrial and Engineering Chemistry. F. 37, p. 598, June, 1945 Authons' Smopsis

The results of turbidity tests on portland cements suspended in an aqueous medium, with and without addition of calcium lignosulfonate, as obtained with the Wagner turbidimeter, show that the additional agent disperses the suspended cement particles. In the tests reported, the degree of increase in turbidity for two types of portland cement in an aqueous medium is shown to be a function of the amount of adsorbed dispersing agent. It may be concluded that the mechanism of dispersion involves the positive adsorption of lignosulfonate anions by the cement particles. The determination of an adsorption isotherm for calcium lignosulfonate and portland cement suspended in water is described; data are given for a typical cement.

## New method of revelting old man river

Lt. Col. A B. Prckpmt. Enginearing News Record, V. 135, No. 12 (Sept. 20, 1945), pp. 124-128

The reinforced concrete mats, 24 ft . wide, 60 ft . long, and $11 / 8 \mathrm{in}$. thick are flexible enough to permit their being rolled onto a large timber core at the casting plant. They are then carried by barge to the site, rerolled on a steel drum and unrolled down the river bank under water to final position. Light steel angles were tacked across a tempered wallboard platform to serve as chairs for the wire mesh reinforcement and to serve as spacers for removable metal grid forms. The slotted metal grids, equal in depth to the thickness of the mat, formed $V$-shaped cuts and served as guides for the screed. After initial set, the grids were lifted and moved ahead to the next platform.

## Prefabricated reinforced concrete structures

Hemunne Stoess, Beton u. Stahlbetonbau, V. 43, No. 5/6. page 25, Mer. 15, 1944

This article describes several systems of precast concrete-unit constructions developed in Germany for barrack or industrial housing purposes. The complete framework as well as the footings, floors, and roof slabs are precast elements. For the filler walls either light weight concrete slabs or burned clay units are used. Details of all structural members as well as methods of jointing are described. The quantities of materials required are stated to be 1.86 pounds of steel and 149 pounds of cement per square yard of floor area, or approximately 3.9 pounds of steel and 37 pounds of cement per cubic yard of covered space.

## Temperature effects on mortars

L. R. Forbrice, Rock Products, V. 48, No. 9, pp. 106-7, Aug. $1945 . \quad$ Reviewen by Roy N. Young

Masonry cement mortar cubes were stored at 70 F ., 28 F . or -10 F . during the first 3 days, thereafter in air or water at 70 F . until tested in compression. Strengths of specimens cured in water at 70 F . were 1850,2400 and 2800 psi at 3,7 and 28 days. Freezing reduced strength, the maximum reductions at 28 days being 15 per cent for water curing, 53 per cent for air curing. The greatest strength reductions resulted from -10 F. exposure when specimens were fresh. Storage at 70 F. for 6 or 24 hours before freezing lessened the effect of freezing on strength. For cold weather work, mortar having high water retention and relatively rapid hardening is recommended. If freezing occurs, an adequate supply of curing water after thawing is necessary.

## Stresses in beams near supports

Hermann Bay, Beton u. Stahlbetonbau, V. 43, No. $7 / 8$ page 40.
(April 1944)
Reviewed by Arthua V . Theyr
In this discussion two methods of attacking the involved problem of stresses in a beam near supports are presented in some detail. The first of these methods is an analytical study based on the solution of the problem of plane stress distribution in a wedge as derived by Mitchell in England (London Math. Proc. V. 32, p 35, 1900). The second method of solution follows by means of an application of difference cquations. No special claims are advanced by the author regarding cither method other than that of giving the reader a general indication of the nature of the stress distribution.

The stress distribution as derived by the author is shown by means of graphs for horizontal as well as vertical sections taken at intervals across a short section of beam near a support. The stress trajectories are then plotted and the resulting diagram is compared with similar ones derived by means of photo-elastic studies.

## Multiple grouting of a dam foundation

1. F Harza, Engineeting News Record, V. 135, No. 10 (Sept. 6, 1945). pp 103-105

## Reviewed by S. J. Chamberlin

The first grouting in the basaltic lava foundation of the Rincon del Bonete Dam in Uruguay was carried to about 13 ft . below the concrete cut-off. After 8 or 10 hours the hole was re-drilled to former depth and size and the grouting repeated until the limiting pressure was again reached. The cycle of re-drilling and re-grouting was repeated until negligible additional grout could be injected. The hole was then drilled 13 ft . deeper each time into the rock and the process repeated to a depth of 80 ft . with a rubber "packer" in place to prevent return of the grout above. One group of holes studied were only approximately one-half grouted the first time. The author believes that by several repetitions the various communicating cavities might be filled up in
the order of their resistance of their paths of communication with the grout hole. The author also raises the question of whether the slow rate of grouting due to the equipment and procedure employed accounts for the large re-grouting ratios.

## Effect of heat on portland cements containing Vinsol resin

Technical News Bulletin, National Bureau of Standards,
No. 340, Aug. 1945
Higeway Regearch Abstracts
The use of air-entraining cements is rapidly increasing, principally becanse of their superior resistance to freezing and thawing. One agent commonly used to produce air entrainment is Vinsol resin. Some mills have experienced considerable loss of resin while grinding the flake or powdered material with clinker. This has occurred most frequently when high temperatures (around 300 deg. F.) existed. Leonard Bean and Albert Litvin, of the Bureau, have conducted a study of the effect of heat on Vinsol resin and on its air-entraining properties. Vinsol resin was found to carbonize on heating at 300 F . Heating cement containing the resin for 48 hours at that temperature volatilized as much as two-thirds of the resin and reduced the air entrained by mortars made from the cement. Also, the methoxyl content of Vinsol resin was found to be reduced by heating at 300 F . Heating takes place in the finish grinding of commercial cements, and temperatures of over 250 F . may be found after many days of storage. Methods of determining Vinsol resin are discussed.

## Structural behavior of Fontana Dam revealed by dual-purpose instruments

W. R. Waugh, Engineering News Record, V. 135, No. 20 (Nov. 15, 1945), pp. 94-98

Reviewed by S. J. Chamberlin
A large number of special measuring instruments were located to determine structural behavior as well as data required for construction and data for study of long time behavior. Electrical instruments, including Carlson strain meters and Carlson stress meters, have been embedded in three typical blocks of the dam to determine the stresses and strains near the base of the maximum section and to determine boundary conditions at the upstream and downstream faces. Joint meters were installed across longitudinal and transverse joints to determine changes at contraction joints, to observe movements of adjacent blocks during grouting and to check subsequent changes. Deflection measurements with long plumb lines have proven valuable in indicating the condition of the top grout sections in the longitudinal joints. During construction the embedded electrical instruments which measure temperature were used as the base control for the cooling operation. Measurements of foundation uplift pressures are being compared to assumed pressures as a basis for determining whether additional foundation drains or curtain grouting will be required.

## Wintertime concreting at the Soo

Engineering News Record, V. 135, No. 4 (July 26, 1945), pp. 80-85
Reviewed by S. J. Chamberlin
The wall site was completely enclosed by insulating board, supported on 900 lin . ft. of self-supporting, steel tubular scaffolding about 40 ft . high, to permit concreting operations at temperatures of to -24 F . and snowfall aggregating some 90 in . The timber roof, supported by the scaffolding and the undisturbed bank back of the wall, contained removable panels for installation and removal of the panel forms. Hatches with removable covers were installed to receive the metal hoppers for placing the concrete. Force-blower steam heaters heated the enclosure to a temperature of 50 F . at the bottom and 70 F . under the roof. Steam, water and air lines extended the full length in a sawdust filled timber box conduit. Transit mix concrete made with air-entraining cement was delivered at a minimum temperature of 60 deg. Dumped into buckets, the concrete
was lifted by crane to the hoppers which terminated in elephant trunks for placing. The concrete was cast in alternate monoliths against panel forms lined with an absorptive material consisting of a porous paperboard faced with a ply of light-weight cotton sheeting.

## Special steel forms speed concreting at Chicago's water filtration plant <br> Engineering News Record, V, 134, No. 26 (June 28, 1945), pp. 76-79 <br> Reviewed by S. J. Chamberlin

Special steel forms were used to build the reinforced concrete washwater troughs that had to be level and to within $\frac{1}{32} \mathrm{in}$. of the required elevation. Each trough had a length 25 ft .9 in . and an inside opening 26 by 17 in . with 3 in . side walls. The forms consisted basically of four side plates for forming the walls and a bottom plate bolted to the sides. The side plate assembly carried a heavy steel walkway on each side and had adjustable ties across the top. Forms were supported on timber frames through screw jacks. Reinforcement was welded into cages approximately equal in length to that of the troughs. Concrete was chuted into catch-boxes from a traveling platform that operated on rollers along the top of the filters and raked into the forms. Concrete first was placed to a 2 -in. depth over the bottom and up about three-fourths of the wall height. Excess material that flowed out onto the bottom slab was removed after 30 to 45 min . and the remainder troweled to correct shape and thickness. Additional concrete brought the side walls up to correct elevation. Finished troughs were water-cured and rubbed, if necessary with carborundum stone.

## Construction practices in South America

Arthur J. Boase, Engincering News Record, V. 135, No. 10 (Sept. 6, 1945),
pp. 96-102
Reviewed by S. J. Chamberlin
The author believes that neither the climate nor methods of construction can account for the higher working stresses and the use of more slender members in South American practice. There is some excellent construction and control in individual cases. The concrete is not as strong, strengths for a given water-cement ratio being about 60 per cent of those attained in this country. Control tests of aggregate are seldom made. In general, small, plain bars of structural grade are used. All fabrication is done on the job, mostly by hand labor. Fabricated accessories, such as bar supports, are not commonly used. Form lumber is rougher and produces more and larger fins. Concrete mixtures are generally arbitrary and aggregate is usually measured by volume in wood boxes. Concrete is sometimes distributed by old-fashioned, long, high chutes. Excellent, scientifically controlled concrete was being produced by the modern plant at Volta Redonda, Brazil, for the construction of a steel mill. An interesting feature at the same site was a traveling form-support of reinforced concrete in the shape of an inverted truss.

## Scientific sampling for accuracy

Hubert Woods, Rock Products, V. 48, No. 8, pp. 102, 103, 110, Aug. 1945 Reviewed by Roy N. Young
Sampling of crushed material for process control is frequently conducted in a manner which leads to sampling errors many times greater than the analytical error. The minimum amount of material required for a properly representative sample depends on the maximum particle size of the material and the degree of precision required in the test results. Experimental determination of the minimum quantity is possible, and is well worth while where a material stream is to be sampled periodically. The probable total error is determined by analyses of at least 10 samples cut from a large sample by quartering and riffling, the test sample size being small enough to yield results with
a probable error considerably greater than the desired precision. The probable analytical error is determined by making at least 10 analyses of a single sample. The probable sampling error is computed from the analytical and total error data. The effect of maximum particle size on required sample quantity is determined by repeating the process on portions of the same material crushed to a smaller maximum particle size, a total set consisting of at least 3 samples of different maximum sizes. Formulas are given for computing necessary sample quantities from the probable error data.

## Aggregate production at Fontana Dam

Jack B. McKamey, Engineeting News Record, V. 135, No. 8 (Aug. 23, 1945),

pp. 88-94<br>Reviewed by S. J. Cbamberlin

More than six million tons of quartzite was quarried from a restricted area between two heavy slate beds at a site about a mile downstream from the dam. A natural fault divided the quarry so that it was possible to alternate operations at the two ends. As quarrying progressed, a second face was developed 250 ft . above the floor to eliminate hazards and deep drilling. The second face reached a maximum height of 150 ft . Loading proceeded from the two levels. Large boulders were broken by placing a charge of dynamite on top and covering the charge with plastic clay. Rock was loaded by electric shovels into 10 cu. yd. trucks which delivered to $42-\mathrm{in}$. gyratory crushers. The crushed rock was carried by a 42 -in. conveyor belt to primary storage across the river. Coarse aggregate was prepared in two screening and crushing plants which could be operated independently or concurrently. Fine aggregate was produced in one operation from fine stone in rod mills fed from a surge bin. The wet material flowed by gravity to a hydro-separator and rake classifiers. Finished storage of the four sizes of aggregate and sand was in a natural ravine. Steam coils were installed at the sand and fine rock openings of the reclaiming tunnel to prevent freezing and crusting. The material was conveyed back across the river to the concrete plant by belt.

## Vacuum processing of Shasta Dam spillway

C. S. Rippon, Engineering News Record, V. 134, No. 24 (June 14, 1945)
pp. 93-96
Reviewed by S. J. Chamberlin
The regular form panels for a 5-foot lift were divided into five tiers of five 10 -ft. chambers each by plywood strips, sealed to the form with asphaltic compound. The air space was formed by a layer of heavy base wire covered with a layer of fly screen and then a layer of unbleached muslin. A $2-\mathrm{in}$. pipe extended from the centrally located, reciprocating type compressor and connected by rubber hose to receiving tanks on the back of the forms. A section of hose connected the tanks to a pipe manifold which led off with smaller hose to the vacuum chambers. As soon as enough concrete was compacted against the form to cover one chamber the vacuum was applied. After a short time the concrete near the face was revibrated with the suction maintained. A vacuum of 15 to 20 in . of mercury could be maintained with as many as ten of the chambers operating at one time. An average of 0.58 lb . of water was extracted from each square foot of surface processed at a cost of 20 cents. The indicated compressive strength of the surface was 2.6 times greater, as determined by the "ball impact" test and comparative resistance to sand blast, than that of concrete placed against a wooden form. The vacuum process was also used on the apron floor with movable mats applied to the surface after it had been shaped to line and grade.

## Deck bridge built of precast T-beam units

Engineering News Record, V. 135, No. 16 (Oct. 18, 1945), pp. 96-99 Reviewed by S. J. Ceamberlin
Part of the $26-\mathrm{ft}$. clear roadway bridge consists of thirty-three $40-\mathrm{ft}$. trestle spans supported by 5 -pile concrete bents. The $71 / 4-\mathrm{in}$. thick slab is carried on five lines of
beams 33 in. deep. Six sections are cast side by side in a row perpendicular to the shoreline in plywood forms which are supported on heavy falsework timbers resting on concrete sills. The forms are stripped rather than lifting the slabs out. Five pairs of bolts are embedded in each end of the slab to permit the section to be raised by a gantry crane for loading on barges. Steel bearing plates are laid flat on the bottom near each end of the beam forms to match $3-\mathrm{in}$. deep recesses in the caps of the pile bents. The plates contain holes for extending short $1 / 4-\mathrm{in}$. bolts downward through the forms for the addition of other bearing plates after the slab has been raised. Larger openings in the plates, and cast the depth of the slab, are provided for $3 / 4-\mathrm{in}$. dowels to be placed from above after the slab has been transferred and aligned over the supporting bents. The recesses in the bent caps are filled with mortar and the sections lowered into position by pumping water into the scows. The load is supported on jacks until the mortar has hardened. The $21-\mathrm{in}$. square reinforced piles were cast on a $4-\mathrm{in}$. reinforced concrete slab. Alternate piles were cast first, wooden forms being used for this work only. For the remainder of the piles no forms were used, alternate piles being removed and those left in place used as forms. Caps for the pile bents were cast in place on the driven piles.

## Pressure grouting

Wilhelm Detig, Beton u. Siahlbefonbau, V. 43, No. $7 / 8$, p. 37. (Apr. 1944)

Reviewed by Arthur V. Theur
The author describes a series of laboratory tests to determine the efficiency of pressure grouting methods. The apparatus used consisted of two stecl I-beams 27 ft . long, each with one half of one of its flanges removed so that the webs could be brought together. By means of different thicknesses of spacer plates which could be inserted along the edges between the two beams and a series of closely drilled bolt holes, a closed channel of varying thicknesses could be obtained. Near the end of one of the beams which were laid with webs horizontal, a threaded hole was provided for attaching the grout line. At several locations along the length of the channels, means for measuring pressures were provided.

The experiments were made for three different channel thicknesses, 0 or with the beams bolted together directly, 0.013 , and 0.078 inches. Pressures ranged from 280 to 880 psi . For all pressures greater than 590 psi , a peculiar and decided spreading or bulging of the beams occurred.

Using a neat cement made up with 40 per cent by weight of water, complete filling of the 0.078 inch channel was obtained along the full length of 27 feet under a pressure of 590 psi . For a complete filling of the 0.013 inch channel using this same mix, a pressure of 880 psi was required. With a mix consisting of 80 per cent by weight of water, and with the beams bolted directly together, filling of the channel to a distance of 4.1 feet from point of entry was obtained with a pressure of 500 psi . The addition of 1 per cent of plastiment to the several mixes used, increased the distance to which complete filling of the channel could be obtained in every case.
Additional data relating to the adhesiveness of the hardened grout to the beam surfaces and its flexural tensile strength are included.

## Two-way prestressed concrete water storage tank

I Roland Cabr, Engineering News-Record, V. 135, No. 14 (Oct. 4, 1945),
pp. 108-113
Reviewed by S. J. Ceamberlin
The $4,750,000-\mathrm{gal}$. tank has an average inside diameter of 156 ft ., an average water depth of 33 ft .3 in . when full and a thin-shell concrete dome rising 24 ft .9 in . above the walls. The inner wall, 8 -in. thick at the top and sloping on the inside to a $21-\mathrm{in}$. thick-
ness at the bottom, was cast first and the main horizontal bands placed around it. The outer shell was built to a minimum thickness of 4 in . at the top and stepped out at the pilasters to a total thickness of $93 / 4 \mathrm{in}$. Both the main wall and the outer shell were placed in 26 vertical sections with movable forms. The vertical prestressed rods of $\frac{13}{64}$-in. diam. were placed 4 in . inside of the main hoops at 12 -in. centers and 1 in . outside of the hoops at $24-\mathrm{in}$. centers. The vertical rods were greased, spirally wrapped with building paper and hung from outriggers spiked to the top of the timber forms. The nut and $21 / 2$-in. washer at the top of each rod were countersunk below the top of the wall in a recess formed by attaching a waxed paper cup filled with plaster. After the concrete had been cast and cured the plaster was removed and the inner rods were stressed to $32,000 \mathrm{psi}$. ( 15000 psi in the outer row) by tightening down the nuts to a predetermined amount as measured by an ordinary micrometer depth gage. Theoretically the vertical prestress in the concrete varied from 120 psi on the thinnest section of the wall to 45 psi at the base. Each of the 81 horizontal hoops was made of 13 rods joined together with turnbuckles. Rods were not upset at their ends but had cold rolled threads to develop their tensile strength. The hoops were stressed to $32,000 \mathrm{psi}$, as determined by a 4 -ft, job-made extensometer, by adjusting the turnbuckles. At the end of a 20-day period in which some of the stress had been lost the hoops were retightened to the design stress. Hoops were adjusted in banks of eight, the rods being prevented from twisting by welding short vertical pieces of $1 / 2$-in. reinforcing rods to them. Some $80,000 \mathrm{~b}$. ft. of timber was required for the inside dome form. A beveled hinged joint was formed between the parapet and dome to permit rotation.

## Complete buildings constructed with precast reinforced concrete elements

B. Loesser. Beton u. Stahlbetonbau, V. 43, No. 9/10, 11/12, and 13/14

May, June and July, 1944
Reviewed by Arthur V. Theur
In this series of three articles Professor Loesser describes recent German activity in the field of prefabricated reinforced concrete-unit building construction. While recognizing the considerable number of buildings of this type constructed prior to 1940, little he states, was accomplished in the way of standardization of the units, their dimensions, shapes and detailing. It is in this respect that uniqueness is claimed for the several structures under discussion. For much of the pioneering work in standardization credit is assigned to the I. G. Farbenindustrie whose mill-type of building is given a first rating.

After a discussion of the advantages and disadvantages of the precast unit as against the monolithic structure, the author presents a detailed analysis of the design of footings for the precast job. For the type of footing adopted the column is inserted into a hole provided for the purposc. This hole is slightly larger than the column cross-section and the column is supported on chairs inserted between the footing and column base. After the column has been brought up to proper elevation it is wedged into alignment and the open space around and underneath the column is filled with grout.

Eight different cross-sectional shapes of beams were standardized and nine different types of roof construction developed. The type of roof construction used depends largely on the length and number of spans to be covered, and includes all of those types, - simple beam, tied arch, truss, saw tooth, and cantilevered beam with monitor,commonly found in conventional construction. The author touches briefly the subject of permissible stresses. He also describes the details of various interlocking joints secured by means of dowels and recesses, and devotes considerable space to the subject of erection equipment.

In the third article of the series the author includes a large number of photographs illustrating a number of structures in different stages of erection.

## Brazilian concrete building design compared with United States practice <br> Arthur J Boase, Engineeting News Record, V. 134, No. 26 (June 28, 1945), po. 8 ) <br> pp. $80-88$ Reviewed by S. J. Chamberlin

The author makes a quantitative comparative study of the Brazilian code and the ACI Building regulations by redesigning a 16 -story apartment building recently erected in Rio de Janeiro. As built, the $3.15-\mathrm{in}$. slab is reinforced with $\frac{3}{18}-\mathrm{in}$. plain rounds, interior beams over the partitions are 3.15 or 3.93 in . wide and 22.8 in.. deep, and front and rear girders are 8.2 in . wide. Redesigned. the slab would be 4 in . with $3 / 8-\mathrm{in}$. rounds, the interior beams would be 6 by 24 in . and the spandrel beams would be 8 by 32 in. in front and rear walls and 8 by 24 in . in side walls. Brazilian column design is illustrated in two design charts for tied columns with 2500 -psi concrete. Data are incorporated from the ACI code for comparison. Because of the lower strength cement used, the Brazilian code does not recognize concrete strengths greater than $2,670 \mathrm{psi}$ for building construction. Material costs are higher and labor costs are lower than in the United States. In the relative cost studies the Brazilian costs were computed at 35 c . per cu. ft. for concrete, 15 c . per sq. it. for forms and $11 / 2 \mathrm{c}$. per lb. for reinforcement.

The article summarizes the principal points of difference in the two codes:
"American practice, using intermediate grade steel, would require 26 per cent more reinforcing steel in the floor system than does the Brazilian design, where structural grade was used. Under these conditions the American design uses slightly less reinforcing steel per cubic yard of concrete than does the Brazilian.
"Using the same strength of concrete, the American design requires 32 per cent more concrete in the floor system than does the Brazilian design.
"For axial load on columns the Brazilian code allows 85 per cent higher stress on concrete and 66 per cent higher stress on steel.
"The American tied column has about twice as much concrete, and one and one-half limes as much steel as does the Brazilian tied column. A comparison when bending in the columns is considered as still more in favor of the Brazilian type.
"Size of columns will change less frequently under the Brazilian code than it will under American practice, thus permitting large savings in formwork.
"American columns are approximately 60 per cent more costly than are the Brazilian columns for $2,500-\mathrm{lb}$. concrete. American tied columns of $2,500-\mathrm{lb}$. concrete have approximately 80 per cent more volume than have the Brazilian columns."

## Rigid type pavement and joint spacing

H. F. Clemmer (in reply to the discussion by Fred Grumra, Highway Research Abstracts, Oct. 1944

It is satisfying to know of the general agreement of the California Division of Highways with the data presented. It is of interest to report that data obtained during the year since this report was presented and the performance of the pavements referred to are in further agreement with the conclusions of the paper.

Mr. Grumm states their experiences has not indicated the need for cutting dummy groove joints to one-half the depth of the concrete slab as reported by this Department. It is stated in the paper that "It is desirable to divide the concrete, only enough to cause cracking, so as to obtain maximum shear value from the irregular edges and to provide maximum area in contact should the pavement be subjected to a bigh compressive stress". Further discussion explained that this Department had found it necessary to cut the groove one-half the depth of the slab to be sure of forming the planes of weakness. Where the design depends upon the close spacing of joints it is most improtant
to be sure of producing these planes of weakness and it was desired to caution against assuming the planes of weakness were properly produced. It is preferable to sacrifice a small degree of shear value rather than to fail to produce the plane of weakness. A careful check should be made of the completed pavement slab to determine if separation has occurred at the designated locations.

Longitudinal planes of weakness are produced with less difficulty because warping in this direction is not restrained so much as it is transversely; however failure to cut a deep enough groove to create a center longitudinal joint in one State project, constructed during this emergency, resulted in the cracking of the slab some distance from the groove and at a point directly under traffic loads.

Mr. Grumm suggested that load transfer installations may have had an effect in prohibiting the formation of the planes of weakness as reported in this paper. The particular study reported upon as to the depth of the groove required to create planes of weakness, was of plain concrete, that is without reinforcing or the load transfer devices. Undoubtedly, there are many conditions affecting requirements for producing dummy groove joints - such as richness of mix, temperature at time of placing concrete, rate of setting of the cement, curing conditions, subgrade friction etc.

This Department is not quite in agreement with Mr. Grumm's statement that because of the apparent success of the construction of pavement slabs without mesh or bar reinforcement or the use of load transfer units that the use of these materials has no particular value. As discussed in the paper it is believed that reinforcement is of particular value in distributing shrinkage stresses, developed during hydration of the cement, which tends towards the formation of incipient cracks that may later develop into structural failures. Concrete pavements constructed during the last ten years in the District of Columbia show practically no structural failures, that is no cracks or corner breaks, and it is believed the use of mesh reinforcement and load transfer units has been an important factor in this result.

The use of reinforcement and load transfer units is evidently considered excellent insurance by many engineers as indicated by the number of state highway departments that have adopted this design.

With this great improvement in general design, engineers responsible for the planning and construction of the enormous postwar highway program, should be far more concerned than before as to whether or not a change in design of pavement will increase the length of service of the pavement. It is believed that too often design features are based on satisfactory service for five or ten years rather than on careful study of the ultimate results that may be obtained. It is believed that the added cost of including reinforcement and the use of load transfer units in the construction of pavements is splendid insurance and their use will prove to be a factor in extending the service of a concrete pavement slab to the maximum.

# ACI NEWS LETTER 

## 42 ${ }^{\text {nd }}$ ANNUAL ACI CONVENTION Buffalo-Feb. 18-21, 1946

The program for the Institute's Forty-second Annual Convention to be held Hotel Statler, Buffalo, N. Y., February 19, 20 and 21, has been developed to the point of announcing with a good deal of confidence a promising array of papers on many timely and important subjects. Not all of this material can at this writing be definitely announced; in some instances papers now in the making, are still to be selected. The Publications Committee under the Chairmanship of Robert F. Blanks, with the responsibilities of the convention program, has worked under unusual difficulties-difficulties with the elements of time and indecision. Much work has been done in a short period; the program has not completely "jelled". What is reasonably assured, however, will in itself make a good convention - what is just in the offing, and not at the stage of announcement, is due to strengthen the program by selection from a number of outstanding items.

Scheduled a year ago for Hotel New Yorker, New York City, based on the possibility of a full-fledged convention by the removal of war-time control of meetings, arrangements were confirmed by the Board of Direction in September last. Late in November, a meeting was called in New York City of a few nembers of the Institute in that territory who might be of assistance in connection with the convention, with every expectation of an outstanding meeting in that city. Almost immediately after these arrangements, the problems of transportation and of hotel accommodations raised serious questions, and after careful consideration of the whole situation, the Executive Committee of the Board made a change. We were graciously released from our arrangements at Hotel New Yorker and welcomed to Hotel Statler in Buffalo, with strong assurances of the ability of the Statler to take care of all ACI comers.

With the ballots, mailed to all ACI members, Dec. 19 last, went a corrected announcement with reference to convention plans. It now appears that the transportation situation reached a high point of disorder and uncertainty at the holiday season. As to reservations of accommodations at the Hotel Statler, see p. 9. Perhaps it is not necessary to emphasize the importance of making reservation of transportation accommodations at the earliest possible date under existing regulations.

As these News Letter pages (the last to go to the printer for the January Journal) are "made up," the situation is as follows:

## Board and Committee Meetings

Meetings of the Advisory Committee and of the Board of Direction are scheduled for Monday February 18.

Committee 318, Standard Building Code, A. J. Boase, Chairman, Roy R. Zipprodt, Secretary, is scheduled to meet for a preliminary session, $9: 30 \mathrm{a} . \mathrm{m}$. Tuesday February 19, with the expectation of continuing its labors the afternoon of the 21st following the adjournment of the convention and possibly through Friday, the 22nd.

Committee 711, Precast Floor Systems for Houses, F. N. Menefee, Chairman, is scheduled to meet the morning of February 19 at 10:30 for possible last minute action in reference to the presentation of the Committee's report (published in the Proceedings pages of this Journal) for adoption as an Institute Standard.

John R. Nichols, chairman Sub-committee 5, of Committee 318, has called a meeting of that sub-committee at a time and place to be announced on the convention bulletin board.

Further announcements will be made of technical committee meetings at convention time as their chairmen decide and notify us.

## CONVENTION SESSIONS

General convention registration begins at 9:30 a.m., February 19.
The first general session of the convention will open at 2 o'clock, Tuesday, February 19, and will cover a variety of subject matter. (See note of that variety, p. 4).

## Air Entrainment in Concrete

Tuesday evening, February 19, will be devoted to a consideration of the development of rapidly increasing experience with air entrainment in concrete. Frank H. Jackson will lead that session.

The success of the 1944 convention session on air entraining concrete has prompted the schedule of another session on this most inter-
esting and timely subject. The same general procedure will be followed, that is, there will be one major paper covering certain broad aspects of the problem, followed by a series of short 10 -minute discussions on particular phases. These will be presented by a selected group of participants, all of whom are familiar with the latest developments in the particular fields which they will cover. Subjects include the use of air-entraining concrete in ready-mix concrete operations, in the manufacture of concrete products, in highway and airport runway construction, etc. The comparative merits of various methods of securing air entrainment-by the use of airentraining cements, air-entraining agents added at the mixer, and air-entraining natural cements blended with portland cement will be discussed, as well as various methods of field control, including recently developed procedures for determining air content of concrete by direct volumetric measurement. Opportunity for general discussion will be provided insofar as possible.

## Research

At 9:30 a.m. Wednesday, February 20, the session will be under the auspices of ACI Committee 115, Research, Morton O. Withey, Chairman, S. J. Chamberlin, Secretary. Every year Professor Chamberlin queries the membership of the large Research Committee and other sources of information. From the reports made some research projects are listed for outline presentation. The annual sessions of this committee have increased in interest and attendance throughout the years.

## Repair and Maintenance of Concrete

Wednesday afternoon, February 20, there will be a session under the guidance of Roderick B. Young, past president of the Institute, Chairman ACI Committee Department 800, Repair and Maintenance, as the basis for committee studies in Department 800. Mr. Young, who will lead off with a contribution of his own, has been successful in developing a series of papers on repair and maintenance of concrete in a variety of work. One will discuss "Hydraulic Structure Maintenance in Using Pneumatically Placed Mortar"; another, "Methods of Repair of Two 175-ft. Concrete Chimneys with Minimum Shutdown"; others will consider "Maintenance and Repair of Concrete Pavements"; "Maintenance and Repair of Concrete Bridges on the Oregon Highway System" by G. S. Paxson. (See Nov. 1945 Jourxal.) Still another paper, contrasting workmanship on two hydraulic structures, will report the repair procedure on one
of them; and there is one in the series which describes some of the methods used in repair, restoration and stabilization of structures in need of repair, or strengthening to meet increased use and loads.

## Election Returns-Presentation of Honor Awards-R/C Design

At 8 o'clock Wednesday evening, February 20, a general session will open with a report on the annual election followed by the address of President Parsons, and presentation of awards, and will continue with the development of interesting themes in reinforced concrete design theory and practice under the leadership of A. J. Boase.

A paper by Douglas McHenry will describe a design method for determining internal stresses in concrete structures by means of a lattice analogy which is solved by successive approximations involving only simple arithmetic. The theory of the method was published in a paper by McHenry in the Journal of the Institution of Civil Engineers (London). The present paper will be concerned chiefly with examples of the application of the method.

## Construction Practice-Emphasis on War

The final convention session, Thursday morning, will take up examples of construction practice in the field of concrete with some emphasis on war experience, under the leadership of Lewis H . Tuthill.

## A Variety of Subject Matter Not Yet Placed on Schedule

"Membrane Waterproofing" will be the subject of a paper to come from Bureau of Reclamation studies.

John R. Nichols will discuss the matter of "radiant heat" in its relation to reinforced concrete floors, walls, etc.

New design ideas for precast, reinforced concrete members "welded" with pneumatically placed mortar will be presented and there will be a report of their war-time application to a ship.

Also: A report on atomic bomb effects on concrete in Japan; use of coral aggregates at Pacific bases, and other important items.

Committee 711, F. N. Menefee, Chairman, will present its report on "Proposed Minimum Standard Requirements for Precast Concrete Floor Units," on a motion for adoption as an ACI standard.

Committee 714, William W. Gurney, Chairman, will present "Proposed Recommended Practice for the Construction of Concrete Farm Silos" for adoption as an ACI Standard.

## WHO'S WHO in this JOURNAL

## Gerald K. Pickett

(see p. 165) a comparatively new ACI Member, is nevertheless not new to these pages. Since 1940 he has been identified with the basic research division of the Portland Cement Association Research Laboratory and only recently (September, 1945) returned to the faculty of Kansas State College, Manhattan, Kan., where he is Professor of Applied Mechanics.

## Stanley U. Benscoter

structural engineer with the Navy Department's Bureau of Aeronautics, Washington, though not yet an ACI Member, is not unknown to ACI readers. To the February 1943 Journal he contributed "A Semi-Circular Arched Conduit with Uniform Symmetrical Loading." (See also News Letter p. 6 of that issue). His present contribution, "Floating Block Theory in Structural Analysis" appears p. 205, this Journal.

## Howard R. Staley and Dean Peabody, Jr.

contribute "Shrinkage and Plastic Flow of Prestressed Concrete" to this issue ( $p$. 229).

Howard R. Staley, ACI Member since 1937, is Assistant Professor of Building Construction in the Department of Building Engineering and Construction at the M.I.T. He was born in Iowa nad received the degree of S.B. in 1935 and S.M. in 1937 from M.I.T. He is a member of Tau Beta Pi, Chi Epsilon and Sigma Xi. He has been a member of the instructing staff of M.I.T. since 1937. For the twelve years previous to 1932 be was a member of the building firm of B. S. Staley and Sons, Centerville, Iowa. He has presented papers to the Designers Section of the Boston Society of Civil Engineers and to the American Society for Testing Materials, of both which organizations he is a mem-
ber. He is a member also of A.S.C.E., S.P.E.E., A.A.A.S.

Dean Peabody, Jr., ACI Member since 1914, and member of Committee 318, Standard Building Code, is Associate Professor of Structural Design in the Department of Building Engineering and Construction at the Massachusetts Institute of Technology. He was born in Massachusetts and received his S. B. degree from M.I.T. in 1910 and the S.M. degree in 1938. He is a member of Chi Epsilon and Sigma Xi. He has been a member of the instructing staff of M.I.T. since 1910, including one year as exchange instructor at the Worcester Polytechnic Institute. During the summers of 1911 to 1916 he worked in the field or the design offices of the Aberthaw Construction Co.; New England Concrete Construction Co.; Lockwood, Greene and Co.; and Stone and Webster. He is author of "Reinforced Concrete Structures" and has presented several papers to the Designer's Section of the Boston Society of Civil Engineers of which he is a member. He is also a member of A.S.C.E.; A.S.T.M.; A.A.A.S.

For further convention sub-
ject matter and a convention
time-table see February Jour-
nal - or, about February 1
ask the ACl secretary for an
advance program.

## New Members

The Board of Direction approved 47 applications for Membership ( 42 Individual, 2 Corporation, 1 Junior, 2 Student) received in October and November as follows:

Abel, G. C., 23 View Road, London N. 6, Fingland
Allen, Herbert C., 303 Geneva St., Highland Park 3, Mich.
Anjel, Anton A., Box 112, Cocoli, Canal Zone
Booth, John J., 117 Dilworth Bldgs., Cor. Queen \& Customs Sts., Auckland, New Zealand
Brooks, Samuel H., 326 Center St., East Pittsburgh, Pa.
Cafaro, Martin, 309 East 28th St., New York 16, N. Y.
Cameron, L. S., 3428 Ettie St., Oakland 8, Calif.
Campbell, W. E., 319 S. 12th St., New Castle, Ind.
Cardenas, Victor Morales y de, Compostela 158, Havana, Cuba
Chang, Wan-chiu, Engineering Dept,, Central China, Railways Administration North Station, Shanghai, China
Chen, Manning, c/o Mr. R. A. Kane, Room 1010 U. P. Headquarters Bldg., 1416 Dodge St., Omaha 2, Nebr.
Crowe, George F., 2801 Colonial Ave., Norfolk 8, Va.
Daymude, Charles A., Dept. of Bldgs. \& Safety Engineering, 555 Clinton St., Detroit 26, Mich.
Dayton, Cedric L., 16 Hollywood Ave., Tuckahoe 7, N. Y.
Ernst, E. S., Dewey Portland Cement Co., Box 749, Davenport, Iowa
Flores, Rodrigo, Ramon Nieto 920 of 306, Santiago, Chile
Fox, Carl, 887 W. Division St., Decatur, Ill.

Garing, Athol C., 606 Aspen, Coulee Dam, Wash.

Gundmundsson, Jon Sig., P. O. Box 396, Reykjavk, Iceland

Haggerty, Francis V. J., c/o Warner Co., 200 S. Market St., Wilmington, Del.
Harrison, Marc M., 45 Popham Road, Scarsdale, N. Y.
Holbrook, Albert E., 129 Ivanhoe St., S. W., Washington 20, D. C.

Huerta, Ing. Manuel Castro, Monte Alban No. 39, Col. Narvarte, Mexico
Jeffers, Paul E., 810 West Fifth St., Los Angeles 13, Calif.
Knaust, Adam, Marlboro, N. Y.
Kosseim, Emile, 2F Rue Said, Heliopolis, Egypt
Laboratory of Strength of Materials, Beaucheff 850, Santiago, Chile ( Sr . Edmundo Thomas)

Ledon, Rolando Castaneda, Edificio America, Jovellar y N, Dpto 210, Havana, Cuba
Leduc, Joseph R., Ecole Polytechnique, 1430 St. Denis St., Montreal 18, Que., Canada
Lewis, Elbert F., Box 573, Greensboro, N. C.

Lundteigen, A. Jr., Ash Grove Lime \& Portland Cement Co., Louisville, Nebr.
McKean, J. L., 17 Kelvinside Terrace, Glasgow N. W., Scotland
Mercer Steel Company Inc., 838 N. W. 13th Ave., Portland 9, Ore.
Miles, M. A., c/o Ayres, Lewis, Norris \& May, 506 Wolverine Bldg., Ann Arbor, Mich.

Oliva, Francisco Santos, Tepeji Ňo. 85, Mexico D. F., Mexico
Paredes-Manrique, Luis, Cia de Petroleo Shell Bogota, Colombia, S. A.
Porter, George W., 6729 Sycamore, Ave Seattle 7, Wash.
Rathod, M. P. B. E. (Civil), Jiva Devsi Bldg. No. 2 Room No. 19, Ranade Road, Dadar (B. B.) Bombay 14, India
Sara, Allan M., 29 Church St., Parramatta, N. S. W., Australia

Spencer, Ernest L., Northeastern University, 360 Huntington Ave., Boston, Mass.
Staples, Leroy A., 423-15th St., Alexandria, La.
Suarez, Angel Cano, F y 2 La Sierra, Havana, Cuba
Tajirian, Faraj, M. I. T. Dormitories, Box 60, Cambridge 39, Mass.
Tobalina, Bernardo Vazquez y, Luz Caballero Oeste 464 altos, J. del Monte, Havana, Cuba
Tolley, H. F., U. S. Bureau of Reclamation, Klamath Falls, Ore.
Valdez, Alberto Andrue, 10 de Octubre 1369, J. del Monte, Havana, Cuba
Winner, Walter I., 303 Ferry, Coulee Dam, Wash.

## Honor Roll

February 1 to December 31, 1945
Rene Pulido y Morales, in Havana, Cuba heads the list with 24 new Members proposed since Feb. 1. The current Honor Roll closes with applications received by January 31, 1946.

## Rene Pulido y Morales

Roy Zipprodt
H. F. Gonnerman

Harry B. Dickens.
A. Amirikian
J. A. Croft

Ernst Gruenwald
Dean Peabody, Jr.
J. H. Spilkin

Charles S. Whitney
Francis MacLeay.
D. E. Parsons . .................... . . $21 / 2$

Charles E. Wuerpel $21 / 2$
C. Blaschitz
H. W. Cormack ................... 2
J. W. Kelly .2

Calvin C. Oleson
O. G. Patch . . . . . . . . . . . . . . . . . . 2
C. H. Scholer

2
J. M. Wells

2
J. C. Witt
Ben E. Nutter. ..... $11 / 2$
O. A. Aisher ..... 1
J. B. Alexander . ..... 1
Michel Bakhoun ..... 1
H. C. Bruce ..... 1
H. Victor Carman. ..... 1
W. Fisher Cassie ..... 1
A. R. Collins . .....  1
R. F. Dierking ..... 1
H. F. Faulkner. ..... 1
P. J. Freeman. ..... 1
B. F. Friberg ..... 1
J. K. Gannett. ..... 1
Arturo Gantes ..... 1
Stanley S. Haendel ..... 1
W. S. Hanna ..... 1
G. H. Hodgson ..... 1
F. B. Hornibrook ..... 1
V. P. Jensen ..... 1
L. I. Johnstone . ..... 1
William G. McFarland ..... 1
Denis Matthews. ..... 1
Charles E. Morgan ..... 1
H. W. Mundt ..... 1
Y. G. Patel ..... 1
J. R. Pattilo. ..... 1
A. F. Penny . .....  1
Kenneth Powers. ..... 1
Guy Richards. ..... 1
A. T. Rogers .....  1
Simeon Ross. ..... 1
John A. Ruhling. ..... 1
Herman Schorer. ..... 1
Byram Steel ..... 1
G. W. Stokes ..... 1
H. D. Sullivan ..... 1
Wm. Summers Jr. ..... 1
M. A. Timlin ..... 1
J. W. Tinkler ..... 1
Lewis H. Tuthill ..... 1
Maxwell Upson ..... 1
Stanton Walker ..... 1
The following credits are, in each in-stance, " $50-50$ " with another Member.

| Birger Arneberg | P. M. Ferguson |
| :--- | :--- |
| E. E. Bauer | Alerander Foster |
| E. W. Bauman | G. L. Freeman |
| P. G. Bowie | Grayson Gill |
| C. H. Chubb | E. A. Gramstorft |
| J. H. Chubb | N. M. Hadley |
| Arthur P. Clark | W. C. Hanna |
| R. R. Coghlan | Carl W. Hunt |
| R. B. Crepps | A. Dovali Jaime |
| R.A. Crysler | W.R. Johnson |
| Harreer E. Davis | Paul A. Jonea |
| J. L. Drueke | H.J. McGillivray |

R. E. McLaughlin Adolph Meyer A. F. Moore
O. F. Moore E. Nennigar M. D. Olver C. E. O'Rourke Lucien Perrault Milos Polivka Jerome Raphael F. E. Richart Chas. S. Rippon D. O. Robinson Kanwar Sain J. L. Savage

Oskar Schreier
A. L. Strong
A. C. Trice
K. Tsutsumi
J. H. de W. Waller
S. J. Warberg
A. R. Waters

David Watstein
H. J. Whitten George Winter Harry C. Witter S. H. Woodard
K. B. Woods
L. Zeevaert

## John J. Earley

John Joseph Earley, past president (1938) American Concrete Institute, Henry C. Turner Gold Medalist (1934) for "outstanding achievement in developing concrete as an architectural medium" and one of the Institute's 15 members elected to Honorary membership, died November 25, 1945 at his home 2701 Connecticut Ave. N. W., Washington, D. C., after a long illness.

Mr. Earley was an architectural sculptor fifth in a line of such artists, himself a native of New York. He was an apprentice in his father's studio, specializing chiefly in ecclesiastical sculpture, at the age of 17. His death occurred at the age of 64 .

Entering the contracting business, specializing largely in plaster and stucco, 1906, his work led him to a study of the possibilities of architectural concrete of the exposed aggregate type, and from that, strongly influenced by the lavish color of Byzantine architecture, he made an exhaustive study of the possibilities of what he called plastic mosaic, which at far less cost, would achieve some of the effects of the early mosaics with their employment of high talents of art and workmanship. One of the early records of this development is to be found in Washington's Church of the Sacred Heart, one of a score of United States churches to which he gave his decorative talent-the Baha'i Temple, Evanston, Ill., with its white quartz concrete tracery, perhaps his greatest work.
To the Earley Studio, Rosslyn, Va., he brought as materials for his artist "pal-


John J. Earley
ette", crushed aggregates selected for durability and color-marble, glass, ceramics, arranged by the Munsell system of color notation. He used to say: "Imagine the day before there was any system of recording musical notes, and you have the present condition so far as general practice goes, in the notation of color".

He gained fame in converting the Lorado Taft "Fountain of Time," Chicago, into relatively durable form, when all the usual media of the sculptor had to be passed up because of the enormous cost involved for the symbolic group more than 100 ft . long. Mr. Earley's paper "Building the Fountain of Time," ACI Proceedings Vol. 19, won the Wason medal for "the most meritorious paper" of the year.

A sketchy, outline record of his contributions to the technology of concrete and its application to architectural works, will be found in naming some of Mr. Earley's papers published by the Institute:
"Some Problems in Devising A New Finish for Concrete," V. 14; "New

Developments for Surface Treated Conerete and Stucco" (as co-author with J. C. Pearson) V. 16. Then came "Building the Fountain of Time", with its description of the making and use of a 3000 -piece mold for the placing of concrete using crushed Potomac River gravel as the predominating aggregate; two papers, each under the title "Architectural Concrete" V. 20 and 22: "Time as a Factor in Making Portland Cement Stucco", V. 23; "The Project of Ornamenting the Baha'i Temple Dome" V. 29; "Architectural Concrete of the Exposed Aggregate Type", V. 30; "Architectural Concrete Makes Prefabricated Houses Possible" V. 31; "Mosaic Ceilings, U. S. Department of Justice Building" V. 31; "On the Work of the Committee on Architectural Concrete of the Exposed Aggregate Type and the Thomas Alva Edison Memorial Tower," V. 34; "The Characteristics of Concrete for Architectural Use," V. 35.

These contributions of Mr. Earley to the work of the Institute, and to the field of concrete, were unique. Unique though his contribution was, it brought sharply into focus the necessity for concrete of very bigh quality for all exposed ornament. He had haunted the Bureau of Standards, and other sources of data, to find the path that would lead him to mixes best suited to his special purposes. At the Bureau he found a typical plotting of a large number of tests of quality concrete, with the characteristic curve, following close to the medial line among the dots of the individual test specimens. In these he was not interested, but he found one dot barely inside the top margin of the diagram showing a concrete specimen of unusually high quality, and he could not be satisfied until he had found the record of what that concrete was-what made it so good, neglected as it was in plotting the curve.
Always dominated by the artistic approach and with a keen sense of drama, he could bring himself and endless patience to the study of the engineering approach to the entegrity of a material. With that he was also a salesman of a super type. Be-
cause he maintained that writing was slow and difficult for him his papers are therefore interesting as specimens of English composition. He insisted that to set down a principle is sufficient, and only rarely did he record the details of means to apply the principle to his particular work. As a young man he was no mean athlete--he played handball with the best, was Washington's champion fencer (crossed points with the late President Theodore Roosevelt), and was a strong swimmer.

He was a member of the Board of Direction as Director Fourth District, 1924 and 1925 and again 1932 and 1934; he was elected Vice-President 1926 and resigned in shying from election to the Presidency. Again he was Vice President, 1936 and 1937, and was elected President in 1938; served as past president member of the Board from 1939 to 1943.

## Convention Hotel Reservations

Following is an announcement of rates for room accommodations at Hotel Statler, Buffalo, effective for those attending the convention of the American Concrete Institute February 18 to 21. Make your reservations at once. It is well to ask for a confirmation; it is important to identify yourself with the American Concrete Institute. Address: Mr. Theodore Krueger, Manager, Hotel Statler, Buffalo 5, New York.

Cnless requested otherwise, the hotel will hold your reservation until 9 p.m. of the day of your arrival. Give date arriving, and hour.

| Room and Bath for One Per Day | $\begin{aligned} & 3.30 \\ & 3.85 \\ & 4.40 \end{aligned}$ | $\begin{aligned} & 4.95 \\ & 5.50 \end{aligned}$ | $\begin{aligned} & 6.05 \\ & 6.60 \end{aligned}$ |
| :---: | :---: | :---: | :---: |
| Double-Bed Room with Bath <br> For Two- Per Day | $\begin{aligned} & 5.50 \\ & 6.05 \\ & 6.60 \end{aligned}$ | $\begin{aligned} & 7.15 \\ & 7.70 \end{aligned}$ | $\begin{aligned} & 8.25 \\ & 8.80 \end{aligned}$ |
| Twin-Bed Room with Bath <br> For Two-Per Day | $\begin{aligned} & 6.60 \\ & 7.70 \\ & 8.25 \end{aligned}$ | $\begin{aligned} & 8.80 \\ & 9.35 \end{aligned}$ | $\begin{array}{r} 9.90 \\ 11.00 \end{array}$ |
| Three Persons <br> is One Room | 7.70 | 8.80 | 9.90 |
| SEITTE-Living Room, <br> Bed Room and Bath <br> For One-Per Day <br> For Two-Double Bed <br> -Twin Beds |  | 16.00 | $\begin{aligned} & 12.50 \\ & 14.70 \\ & 19.00 \end{aligned}$ |

[^25]
## Sources of Equipment, Materials, and Services

(A reference list of advertisers who participated in the Fourth Annual Technical Progress Issue of the ACI JOURNALthe pages indicated will be found in the February 1945 issue and (when it is completed) in $\mathrm{V} .41, \mathrm{ACl}$ Proceedings. Watch for the 5th Annual Technical Progress Section in the February 1946 JOURNAL.
Concrete Products Plant Equipment ..... page
Besser Manufacturing Co., 800 45th St., Alpena, Mich ..... 428

- Concrete producis plant equipment, production
Stearns Manufacturing Co., Inc., Adrian, Mich. ..... 387
- Vibration and tamp type block machines, mixers and skip loaders
Construction Equipment
Baily Vibrator Co., 1526 Wood St., Philadelphia 2, Pa. ..... 407
- Concrete vibrators
Blaw-Knox Division of Blaw-Knox Co., Farmers Bank Bldg., Pittsburgh, Pa. ..... 394-5
-Truck mixer loading and bulk cement plants, road building equipment, buckets,batching plants, steel forms
Butler Bin Co., Waukesha, Wis. ..... 421
- Central mix
Chain Belt Co. of Milwaukee, Milwaukee, Wis. ..... 418-9
- Mixers, pavers, pumps
Electric Tamper \& Equipment Co., Ludington, Mich. ..... 410-11
- Concrete vibrators
Flexible Road Joint Machine Co., Warren, Ohio ..... 389
-Pavement joint and joint installers
Fuller Co., Catasauqua, Pa. ..... 377- Unloading and conveying pulverized materials
Helizel Steel Form \& Iron Co., Warren, Ohio ..... 378-9
-Pavement expansion joint beams
Jaeger Machine Co., The, Columbus, Ohio. ..... 392-3
- Concrete paving equipment
C. S. Johnson Co., The, Champaign, III. ..... 429
- Mixing and balching planis, buckels, elevators
Kelley Electric Machine Co., 287 Hinman Ave., Buffalo 17, N. Y. ..... 422-3
-Floor finishing equipment
Koehring Co., Milwaukee, Wis. ..... 380
-Tilting and non-lilting construction mixers
Mall Tool Co., 7703 So. Chicago Ave., Chicago 19, III. ..... 427
- Concrele vibrators
Master Vibrator Co., Dayton 1, Ohio ..... 430-1 - Concrete vibrators
Ransome Machinery Co., Dunellen, N. J. ..... 433
-Mixers-paving, truck
Viber Co., 726 So. Flower St., Burbank, Calif. ..... 381
- Concrete vibrators
Contractors, Engineers and Special Services
Kalman Floor Co., Inc., 110 E. 42nd St., New York 17, N. Y. ..... 390-1-Floor finishing methods
Prepakt Concrete Co., The, and Inirusion-Prepakt, Inc., Union Commerce Bldg, Cleveland 14, Ohio ..... 397-400
- Pressure filled concrete
Raymond Concrete Pile Co., 140 Cedar St., New York 6, N. Y. ..... 396
- Pile foundations
Roberts and Schaefer Co., 307 No. Michigan Ave,, Chicago 1, IIl. ..... 426
-Thin shell concrete roofs
Scientific Concrete Service Corp., McLachlen Bldg., Washington, D. C. ..... 416
- Mix controls and records
Vacuum Concrete, Inc., 4210 Sansom St., Philadelphia 4, Pa. ..... 409
-Suction control of water in the concrete
Materials and Accessories
Calcium Chloride Assn., The, 4145 Penobscot Bldg., Detroit 26, Mich. ..... 420
-Calcium chloride
Concrete Masonry Products Co., 140 W. 65th St., Chicago, III. ..... 385
-Non-shrink metallic aggregate
Dewey and Almy Chemical Co., Cambridge, Mass. ..... 412-5
-Air-entraining and plasticising agents
Horn Co., A. C., Long Isiand City 1, N. Y. ..... 417
-Waterproofing
Hunt Process Co., 7012 Stanford Ave., Los Angeles 1, Calif. ..... 425
-Curing compound
Inland Steel Co., The, 38 So. Dearborn St., Chicago 3, III. ..... 384
-Reinforcing bars
Lone Star Cement Corp., 342 Madison Ave., N. Y. ..... 382-3
- Portland cements
Master Builders Co., The, Cleveland, Ohio, Toronto, Ont. ..... 402-5- Cement dispersing and air-entraining agents
Rail Steel Bar Association ..... 386
-Reinforcement bars
Richmond Screw Anchor Co., Inc., 816 Liberty Ave,, Brooklyn 8, N. Y. ..... 408
--Form tying devices
Sika Chemical Corp., 37 Gregory Ave., Passaic, N. J. ..... 434-5- Waterproofings, plasticizer, and densifier
United States Rubber Co., Rockefeller Center, New York 20, N. Y. ..... 401-Form lining
Testing Equipment
American Machine \& Metals, Inc., East Moline, III. . ..... 388
-Riehle hydraulic testing machines


# ACl publications in large current demand 

## ACI Standards-1945

148 pages, $6 \times 9$ reprinting ACl current standards: Building Regulations for Reinforced Concrete (ACI 318-41), three recommended practices: Use of Metal Supports for Reintorcement (ACI-319-42); Measuring, Mixing and Placing Concrete (ACl 614-42), Design of Concrele Mixes ( ACl 613-44); and iwo specifications: Concrete Pavements and Bases (ACl 617-44) and Cast Stone ( $\mathrm{ACl} 704-44$ )-all between two covers, $\$ 1.50$ per copy-lo ACl Members, $\$ 1.000$.

## Air Entrainment in Concrete (1944)

92 pages of reports of laboratory data and field experience including a 31-page paper by H. F. Gonnerman, "Tests of Concretes Containing Air-entraining Portland Cements or Airentraining Materials Added to Batch at Mixer," and 61 pages of the contributions of 15 participants in a 1944 ACl Convention Symposium, "Concretes Containing Arr-entraining Agents," reprinted (in special covers) from the ACI JOURNAL for June, 1944. \$1.25 per copy; 75 cents to Members.

## ACI Manual of Concrete Inspection (July 1941)

This 140-page book (pocket size) is the work of ACI Committee 611, Inspection of Concrete. It sets up what good practice requires of concrete inspectors and a background of information on the "why" of such good practice. Price \$1.00-10 ACl members 75 cents.

## "The Joint Committee Report" (June 1940)

The Report of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete submifting "Recommended Practice and Standard Specifications for Concrete and Reinforced Concrete," represents the ten-year work of the third Joint Committee, consisting of affiliated committees of the American Concrete Institute, American Institute of Architects, American Railway Engineering Association, American Sociely of Civil Engineers, American Sociery for Testing Materials, Portland Cement Association. Published June 15, 1940, 140 pages. Price $\$ 1.50$-to ACl members $\$ 100$.

## Reinforced Concrete Design Handbook (Dec. 1939)

This report of ACl Committee 317 is in increasing demand. From the Committee's Foreword: "One of the important objectives of the committee has been to prepare tables covering as large a range of unit stresses as may be met in general practice. A second and equally important aim has been to reduce the design of members under combined bending and axial load to the same simple form as is used in the solution of common lexural problems."-132 pages, price $\$ 2.00-\$ 1.00$ to ACl members.

## Concrete Primer (Feb. 1928)

Prepared for ACl by F. R. McMillan, it had five separate printings by the Institute alone (totalling nearly 70,000 copies). By special arrangement it has been translated and published abroad in many different languages. It is still going strong. In the foreword the author said "This primer is an attempt to develop in simple terms the principles governing concrele mixtures and to show how a knowledge of these principles and of the properties of cement can be annlied 10 the production of permanent structures in concrete." 46 pages, 25 cents (cheaper in auantity)

For further information about ACI Membership and Publications (including
presenting Synopsis of recent ACl papers and reports) address: pamphlets
AMERICAN CONCRETE INSTITUTE 742 New Centel Building Detroit 2, Michigan

## THE AMERICAN CONCRETE INSTITUTE

is a non-profit, non-partisan organization of engineers, scientists, builders, manulacturers and representatives of industries associated in their technical interest with the field of concrete. The Institute is dedicated to the public service. Its primary objective is to assist its members and the engineering profession generally, by gathering and disseminating information about the properties and applications of concrete and reinforced concrete and their constituent materials.

For nearly four decades that primary objective has been achieved by the combined membership effort. Individually and through committees, and with the cooperation of many public and private agencies, members have correlated the results of research, from both field and laboratory, and of practices in design, construction and manufacture.

The work of the Institute has become available to the engineering profession in annual volumes of ACl Proceedings since 1905. Beginning 1929 she Proceedings have first appeared periodically in the Journal of the American Concrete Institute and in many separate publications. (Pamphlets presenting brief synopses of Journal papers and reports of recent years, most of them available at nominal prices in separate prints, are available for the asking.)

## ACI Construction-Practice Award

A year ago the American Concrete Institute announced the inauguration of the ACl Construction-Practice Award, to be given for a paper of outstanding merit on concrete construction practice. This award was established to honor the construction man - the man whose resourcefulness comes in between the paper conception and the solid fact of a completed structure.
The token of the award is to be a suitable Certificate of Award accompanied by $\$ 300$ (maturity value) of United States War Bonds Series E . The object is the enrichment of the literature of concrete construction practice. We await word from the Awards Committee on the first year of this award. The second year is under way.

$$
* * *
$$

Five cash awards for contributions to the Job Problems and Practice pages September 1945 to June 1946 are open to all comers.
For further particulars address Secretary American Concrete Institute, New Center Building, Detroit 2, Mich.


[^0]:    dues. (A special dues rale of $\$ 3.00$ per year applies for "a student in residence al a recognized lechnical or engineering school" and includes Journal subscription.
    Bound volumes 1 to 40 of PROCEEDINGS OF THE AMERICAN CONCRETE INSTITUTE (1905 to 1944) are for sale as far as available, at prices 10 be had on inquiry of the Secretary-Treasurer. Special prices apply for members ordering bound volumes in addition to the monithy Journal.

    Publication addiess: 7400 Secand Boulevard, Detroit 2 , Michigan. Copyright, 1946, American Concrete Institute, Prinled in U. S. A. Entered at the Post Office at Detroit, Michipan, as mail of the second class under provisions of the Acl of March 3, 1879.

[^1]:    *Received by the Institute, April 30, 1945.
    $\dagger$ Professor of Applied Mechanics, Kansas State College, Manhattan, Kans, ; formerly Portland Cement Association Research Laboratory, Chicago.

[^2]:    *See references at end of text of Part 1.
    $\dagger$ By shrinkage (or swelling) tendency is meant the unit linear deformation due to any cause other than stress that would occur in an infinitesimal element if the element were unrestrained by neighboring elements. It is not to be confused with the average unit deformation, commonly called shrinkage, of a so-called unrestrained specimen, nor with the resultant linear unit deformation which for the x-direction will be designated $e_{x}$. Hereinafter, the linear unit shrinkage tendency will be referred to either as unreatrained shrinkage, for clarity, or merely as shrinkage, for brevity.

[^3]:    *The reference is to an element of hydrated paste of just sufficient size to be a representative sample of
    the paite.

[^4]:    *All figurea and tables pertaining to Part I will be found on pages 196 to 204.

[^5]:    *The general procedure of obtaining Fourier coefficients to satisfy initial conditions somewhat analogous to the present problem is given in Articles 66 to 68 of Byerly's Fourier Series and Spherical Harmonics Boston: Ginn \& Co., 1893).

[^6]:    *This solution is very similar to that given for an analogous problem by Carslaw in Article 25 of The Conduction of Heat, (Macmillan \& Co., Titd., 2d ed., 1921).

[^7]:    *The part in the brackets is numerically equal to one-sixth of the unit ctrain $\epsilon$ that would be produced in either the top or the bottom of the beam by a moment just sufficient to straighten it. That is,

    $$
    \frac{1}{6} \&=\frac{2 b b_{\max }}{3 l^{2}}
    $$

[^8]:    *The lower integration limit for each integral is decreased from 0 to - $\infty$.

[^9]:    *The values of $\beta_{n}$ given in Table l differ slightly from those given by Newman in some instances. The values in Table 1 are believed to be accurate to the number of places given. The tables other than Table 1 have not been checked by duplicate computations. But, except for the last digit, which may be inaccurate by a point or two, these tables are believed to be reasonably accurate.

    IIn Fig. 8 and 9, where shortening and warping were the dependent variables, the square root of $T$ for the abscissas was found to be better than $T$, because a considerable part of such graphs were approximately straight lines. For this reason the square root of $T$ rather than $T$ was used for the abscissss in the construction of other diagrams.

[^10]:    *If a prism were drying from all six surfaces, the corresponding equation would be $S_{a v} / S_{\infty}=H_{a}+H_{b}$ $+H_{e}-H_{a} H_{b}-H_{b} H_{e}-H_{e} H_{a}+H_{c} H_{b} H_{c}$. Another way of expressing these relations is given by Glover (Ref. 7).

[^11]:    *Equations 33, 34, and 35 for stresses and Equations 36 and 37 for the coefficients $B$; and $C_{2}$ given here are almost the same as Equations 1, 2, and 3 for stresses and Equations 6 and 7 for the coefficients $B_{n}$ and $A_{n}$ respectively given in Ref. 9. The $A_{i j}$-series and the $A_{u}$-terms enter into the equations given here in place of the terms in the atress $S$ given there; otherwise, except for slight differences in notation, the corresponding equations are identical and the equations given here may be derived by the procedures given there.

    Equation 38 for $A_{1}$, and Equation 39 for $A_{0}$ may be verified by substituting Equations 33,34 , and 5 a into Equation 32 and then proving the equality of the two sides of the resulting equation within the domain under consideration by the usual Fourier analysis.

[^12]:    *'"The Column Analogy," by Hardy Cross, Bull. No. 215, Eng. Expt. Sta., 1932, Univ. of Ill.

[^13]:    *The procedure is illustrated for the more general case of unsymmetrical bending in "The Column Analogy,' p. 10.

[^14]:    *"Graphic Methods for Engineers," by D. B. Steinman, Eng. News-Rec. V. 132, No. 18, May 4, 1944, pp. 659-661.

[^15]:    "The Column Analogy," p. 55.

[^16]:    *Received by the Institute Oct. 4, 1945.

    * Received Professor of Building Construction and Associate Professor of Structural Design, Massachusetts Institute of Technology.

[^17]:    *The Committee's report as published ACI Jotrnal. February 1944, Proceedings V. 40, p. 305, is now nublished with revisions as released by the ACI Standards Committee for presentation to the 1946 Convention on a motion for adoption as an ACI Standard.

    + Reference is hereafter referred to as ACI 318-41.
    $\ddagger$ See appendix bereto.

[^18]:    +See appendix hereto.

[^19]:    *Failure in Section 40 is defined as, any behavior of the beam under load which indicates that the yield point of the steel has been exceeded, or that cracking of the concrete is such that it would not be permitted in a structure in regular service.
    +See appendix hereto.

[^20]:    *For further details as to design for Type 1 joists, the Portland Cement Association pahmphlet "How to Design and Build Precast Joist Concrete Floors" is at present the most complete treatise.
    +See appendir hereto.

[^21]:    *See appendix hereto.
    †The manufacturer's supervision over the placing of the reinforcement and over the factory treatment of the joist from planning the mixture to delivery on the job, will be stricter, where he is provided with data on the maximum shear and bending moment stresses as computed by the architect or engineer.

[^22]:    *See appendix hereto.

[^23]:    *See Section 807. **See Section 905 (a) and 808(a).
    the allowable bearing stress on an area greater than one-third but less than the full area shall be interpolated between the values given.
    $\ddagger$ Where special anchorage is provided (see Section $903(\alpha)$ ), one and one-half times these values in bond may be used in bearos, slabs and one-way footinge, but in no case to exceed 200 psi for plain bars and 250 psi for deformed bars. The values given for two-way footings include an allowance for special anchorage.

[^24]:    *The report (published ACI Journal. January, 1944 ; Proceedings V. 40, p. 189) as here revised and with some contemplated revisions of an editorial nature, has been released by the ACL Standards Committee for presentation to the Institute's 1946 Convention on a motion for adoption as an ACI Standard.

[^25]:    If a room at the rate requested is unavailable, one at the nearest available rate will be reserved.

