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CONTROLLABILITY OF CONTINUOUS – TIME JUMP LINEAR SYSTEMS

Summary. In this paper we consider a problem of controllability of continous time linear system with randomly jumping parameters which can be described by finite state Markov chain. Different kinds for controllability of such a system are analysed and the sufficient and neccesery conditions for them are established.

STEROWALNOŚĆ LINIOWYCH UKŁADÓW ZE SKOKAMI PARAMETRÓW

Streszczenie. W pracy rozważa się problem sterowalności ciągłych liniowych układów ze skokowo zmieniającymi się parametrami, które mogą być opisane jednorodnym łańcuchem Markowa o skończonej liczbie stanów. Analizowane są różne koncepcje sterowalności takich układów oraz są wyprowadzone dla nich konieczne i wystarczające warunki sterowalności.

1. Introduction

The concept of controllability of dynamical system was introduced to literature by R. E. Kalman in 1960. Since then the problem of controllability has become an object of intensive researches and now there exists huge literature devoted to this problem. For the linear dynamical systems with Markovian jumps in parameter values, which have recently attracted a great deal of interest, the problem of stochastic controllability has been studied in the literature in the following papers: [2], [3], [5], [6]. Generally speaking, the previous results can be classified into two groups depending on the definition of the time in which the system reaches the desired target. This time can be a random variable or a given number. The first type of controllability has been considered in [2] and [5]. In [4] a definition of controllability in given time for general stochastic systems has been proposed and then in [6] this definition has been examined in the context of linear dynamical systems with Markovian jumps in parameter values. Another concept of controllability has been discussed in [3]. This paper deals with conditions of controllability in given time for continuous-time linear system with Markovian jumps in parameter values.

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Consider continuous-time linear system with Markovian jumps, modeled by

$$\dot{x}(t) = A(r(t))\,x(t) + B(r(t))\,u(t), \tag{1}$$

where $t \in [0,\infty)$, $x(t) \in R^n$ is the system state, $u(t) \in R^m$ is the control, $\{r(t), t \geq 0\}$ is a finite state homogeneous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ with values in the set $S = \{1, ..., s\}$ and the infinitesimal generator $Q = (q_{ij})_{i,j\in S}$. Moreover we assume that the states of r form one close class of communicating states. Here $A : S \to R^{n\times n}$, $B : S \to R^{n\times m}$ and we denote $A(i)$ by A_i and $B(i)$ by B_i, for each $i \in S$. In (1) we take the initial state x_0 as a fixed nonrandom constant vector. We consider two classes of admissible control $U_1([0,T])$ and $U_2([0,T])$. The set $U_1([0,T])$ consists of all processes $(u(t))_{t\in[0,T]}$ defined on $(\Omega, \mathcal{F}, P)$ such that $u(t)$ is a random variable measurable with respect to $\sigma-$field generated by $r(s)$, $s \in [0,t)$ for each $t \in [0,T]$, $u(t)$ is such that

$$\int_0^T \|u(s)\|^2\, ds < \infty \text{ a.s.} \tag{2}$$

The set $U_2([0,T])$ is a subset of $U_1([0,T])$ consisting of all $u \in U_2([0,T])$ such that

$$E_i \int_0^T \|u(s)\|^2\, ds < \infty, \tag{3}$$

where E_i is the conditional expectation under condition $r(0) = i$. The solution of (1) with control u, initial condition x_0 and initial distribution $P(r(0) = i_0) = 1$ will by denoted be $x(t, x_0, i_0, u)$.

Through this paper the following concepts of controllability are investigated

Definition 1 *The system (1) is U_1-stochastically controllable (U_2-stochastically controllable) over the time T if for all x_0, $x \in R^n$, $i_0 \in S$ and $\delta \in (0,1)$ there exists a control $u \in U_1([0,T])$ ($u \in U_2([0,T])$) such that*

$$P_{i_0}(x(T, x_0, i_0, u) = x) > \delta\ .$$

Definition 2 *The system (1) is U_1-directly controllable (U_2-directly controllable) over the time T if for all x_0, $x \in R^n$, $i_0 \in S$ and $\delta \in (0,1)$ there exists a control $u \in U_1([0,T])$ ($u \in U_2([0,T])$) such that*

$$P_{i_0}(x(T, x_0, i_0, u) = x) = 1.$$

2. Main results

We start from the following two theorems which show that U_1-direct controllability, U_1-stochastic controllability, U_2-stochastic controllability and deterministic controlability of each pair (A_i, B_i) are equivalent.

Theorem 1 *The following conditions are equivalent*

1. *The system (1) is U_1-directly controllable over the time T.*

2. *The system (1) is U_1-stochastically controllable over the time T.*

3. *The pair (A_i, B_i) is controllable in deterministic sense for each $i \in S$.*

Proof.

The implication $1 \Rightarrow 2$ is obvious. Suppose now that the system (1) is U_1-stochastically controllable over the time T and there exists a state $j \in S$ such that (A_j, B_j) is not controllable in deterministic sense. Denote

$$C = \{\omega \in \Omega : r(t) = j \text{ for all } t \in [0, T]\},$$

then we have

$$P_j(C) := \alpha > 0. \tag{4}$$

By the asumption for all $x_0,\ x \in R^n$ there exists a control $u \in U_1([0,T])$ such that

$$P_j(D) > 1 - \alpha,$$

where

$$D = \{\omega \in \Omega : x(T, x_0, j, u) = x\}. \tag{5}$$

From (4) and (5) it follows that

$$C \cap D \neq \emptyset.$$

Fix an $\omega \in C \cap D$ and consider the deterministic function u_ω. This control has a property that in the deterministic linear system with coefficients equal to A_j and B_j governs the initial condition x_0 to the final value x in time T. This contradicts the assumption that (A_j, B_j) is not controllable in deterministic sense.

To prove the implication $3 \Rightarrow 1$ fix $x_0,\ x \in R^n$ and denote by

$$u(s, j, x_0, x, \cdot) : [s, T) \to R^m$$

any control which governs the initial condition x_0 at time s to the final value x in time $T - s$ for the deterministic system with coefficients equal to A_j and B_j. Now we define the control for (1). Denote by $\tau_1, \tau_2, ..., \tau_l$ the times of jump of the process $r(t)$ on the interval $[0,T]$ (l is an a.s finite random variable), $\tau_0 = 0$, $\tau_{l+1} = T$ and put

$$u(t) = \{u(\tau_k, r(t), x(\tau_k), x, t) \text{ for } t \in [\tau_k, \tau_{k+1}),\ k = 0, ..., l$$

This control satisfies the condition (2), because the process $r(t)$ has only finite number of jumps on the interval $[0,T)$. This control is also such that $u(t)$ is measurable with respect to $\sigma-$field generated by $r(s),\ s \in [0,t)$. Moreover it is clear that for this control we have

$$P_{i_0}(x(T, x_0, i_0, u) = x) = 1,$$

for any $i_0 \in S$.

Theorem 2 *The following conditions are equivalent*

1. *The system (1) is U_2–stochastically controllable over the time T.*
2. *The pair (A_i, B_i) is controllable in deterministic sense for each $i \in S$.*

Proof.

It is clear that U_2–stochastic controllability implies U_1–stochastic controllability and then implications $1 \Rightarrow 2$ follows from Theorem 1. To prove the inverse implication we have to change slightly the definition of the control from the proof of Theorem 1 to ensure that condition (3) holds. Fix x_0, $x \in R^n$ and $\delta \in (0,1)$. For $\Delta > 0$ and $i \in S$ denote by $p(i, \Delta)$ the probability that the process $r(t)$ with $r(0) = i$ has no jumps on the interval $[T - \Delta, T]$. Now for δ let Δ_0 be such that $p(i, \Delta_0) > \delta$ for all $i \in S$ (it is always possible to choose such a Δ_0). Define control u as follows

$$u(t) = \begin{cases} 0 \text{ for } t \in [0, T - \Delta) \\ \underline{u}(t) \text{ for } t \in [T - \Delta, T] \end{cases},$$

where $\underline{u}$ is the control which governs the initial condition $x(T - \Delta)$ to x in time Δ for the deterministic system with coefficients equals to $A_{r(T-\Delta)}$ and $B_{r(T-\Delta)}$. This control is such that $u(t)$ is measurable with respect to σ–field generated by $r(s)$, $s \in [0, t)$ and the condition (3) is satisfied. Moreover it is clear that for this control we have

$$P_{i_0}(x(T, x_0, i_0, u) = x) > \delta,$$

for any $i_0 \in S$.

In our further considerations we will use the following concept of stabilizability.

Definition 3 *[3]We say the system (1) is stochastically stabilizable if, for all $x_0 \in R^n$ and $i_0 \in S$, there exists a linear feedback*

$$u(t) = -L(r(t))\, x(t),$$

such that

$$E_{i_0} \int_0^\infty \|x(t, x_0, i_0, u)\|^2 \, dt < \infty.$$

The next result can be easy deduced from Theorem 5 in [3].

Theorem 3 *If for all $x_0 \in R^n$ and $i_0 \in S$ there exists a control $u \in U_2([0, \infty))$ (not necessary in the feedback form) such that*

$$E_{i_0}\left[\int_0^\infty \left(\|x(t, x_0, i_0, u)\|^2 + \|u(t)\|^2\right) dt\right] < \infty, \tag{6}$$

then the system is stochastically stabilizable.

An immediate consequence of U_2–direct controllability and Theorem 3 is the following theorem.

Theorem 4 *If there exists $T > 0$ such the system (1) is U_2–directly controllable over time T then it is stochastically stabilizable.*

Proof.

Take $x = 0$ in the definition of U_2-direct controllability and let

$$\underline{u} \in U_2([0,T])$$

be such that

$$P_{i_0}(x(T, x_0, i_0, \underline{u}) = 0) = 1 ,$$

for each $i_0 \in S$. Condition (6) is now satisfied with

$$u(t) = \begin{cases} \underline{u}(t) & \text{for } t \in [0,T] \\ 0 & \text{for } t \in [T,\infty) \end{cases} ,$$

and consequently system (1) is stochastically stabilizable by Theorem 3.

From Theorem 1 and Theorem 2 we see that U_1- direct controllability, U_1- stochastic controllability , U_2- stochastic controllability and deterministic controlability of each pair (A_i, B_i) are equivalent. The next example shows that U_1- direct controllability does not imply U_2- direct controllability and consequently deterministic controllability of each pair (A_i, B_i), $i \in S$ is not sufficient for U_2- direct controllability.

Example 1 *[3] Consider system (1) with $S = \{1,2\}$,*

$$Q = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \; A_1 = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.5 & 0 \\ 1 & 1.5 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Note that each pair (A_i, B_i), $i \in \{1,2\}$ is controllable and then by Theorem 1 this system is U_1-directly controllable over T for each $T > 0$. From the other hand in [3] it has been shown that this system is not stochastically stabilizable and therefore by Theorem 3 is not U_2-directly controllable.

3. Conclusion

In this paper a concept of controllability on fixed time interval for continuous time linear system with jumping parameters is investigated. Two concepts of controllability are proposed: controllability with probability one and controllability with any positive probability. The sufficient and necessary conditions for each type of controllability are presented. Moreover the efficient algorithms to find the control laws which ensure the realization of the given control goal are given. The proposed definitions of controllability reduce in the deterministic case to the usual concepts of controllability. It is also discussed when such definitions of controllability imply stochastic stabilizability.

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Omówienie

W pracy rozważa się problem sterowalności ciągłych liniowych układów ze skokowo zmieniającymi się parametrami, które mogą być opisane jednorodnym łańcuchem Markowa o skończonej liczbie stanów. Analizowane są różne koncepcje sterowalności takich układów. Pokazano, że U_1 - dokładana sterowalność w czasie T, U_1 - stochastyczna sterowalność w czasie T i U_2 - stochastyczna sterowalność w czasie T są równoważne i warunkiem koniecznym i wystarczającym dla każdej z nich jest sterowalność każdej z par (A_i, B_i) dla $i \in S$. Zaprezentowano również przykład pokazujący, że U_2 - dokładna sterowalność w czasie T jest wymaganiem istotnie silniejszym od sterowalności każdej z par (A_i, B_i) dla $i \in S$. Otwartym problemem jest znalezienie warunków koniecznych i wystarczających dla U_2 - dokładnej sterowalności w czasie T.