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# SOME PROPERTIES OF ATTAINABLE SETS FOR SYSTEMS WITH DELAYS

**Summary.** This article contains some definitions and basic theorems concerning the relative controllability with constrained controls for linear, stationary, finite dimensional dynamical systems with delays in control. The influence of perturbations of control constraints in stationary systems with delays and with integral control constraints on the form of attainable set and controllability of the system is also studied in the paper.

# PEWNE WŁASNOŚCI ZBIORÓW OSIĄGALNYCH DLA UKŁADÓW Z OPÓŹNIENIAMI

**Streszczenie.** Niniejszy artykuł zawiera definicje oraz podstawowe twierdzenia dotyczące względnej sterowalności przy ograniczeniach na sterowanie linowych, stacjonarnych, skończenie-wymiarowych układów dynamicznych ze stałymi, wielokrotnymi opóźnieniami w sterowaniu. W pracy bada się ponadto wpływ zakłóceń ograniczeń sterowania w układach stacjonarnych z opóźnieniami i z ograniczeniami całkowymi na sterowanie na postać zbioru osiągalnego oraz sterowalność układu.

## 1. Introduction

Controllability of dynamical systems and, as a consequence, a form of attainable set is one of the basic problems in control theory. The type and values of controls are elements which essentialy influence system's controllability and the form of attainable set. Occuring in practice constraints of control may substantially restrict the attainable set. Certain advantages, partially reducing „losses" resulted from constraints of control, may be achieved by introducing the delays in control in dynamical system without delays. Thanks to properly selected delays one can not only „improve" controllability but also it is possible to obtain the controllability of the system which without delays is not controllable at all. [8]

It is also known that in practice frequently occure perturbations of dynamical system's parameters. In some cases, small changes in dynamical system inconsiderably influence system's characteristics like, for example, controllability or the form of attainable set. Then they can be ignored and in mathematical model „the precise" values of parameters can be assumed. Unfortunatelly, it happens also that small changes of parameters cause huge qualitative and numerical changes in the system.

Perturbations of control constraints in stationary systems with delays and with integral constraints of control influence the form of attainable set and controllability of the system is studied in this paper. Moreover, some theorems which constitute the criteria of controllability with constrained controls of stationary dynamical systems with delays in control will be formulated.

We shall consider linear, stationary, finite dimensional dynamical systems with constant, multiple delays in control described by following ordinary differential equation:

$$\dot{x}(t) = A\,x(t) + \sum_{i=0}^{M} B_i\, u(t - h_i), \qquad t \geq 0, \tag{1}$$

where

$x(t) \in \mathbf{R}^n$ - the instantaneous state vector,

$u \in L^2_{loc}([0,\infty), \mathbf{R}^m)$ - the control,

A- $(n \times n)$ -dimensional matrix with elements $a_{kj} \in \mathbf{R}$, $k,j = 1,2,\dots,n$,

$B_i$ $(i = 0,1,2,\dots, M)$ - $(n \times m)$-dimensional matrices with elements $b_{ikj} \in \mathbf{R}$,

$$k = 1,2,\dots,n,\ j = 1,2,\dots,m,$$

$h_i \in \mathbf{R}$, $i = 0,1,2,\dots,M$ - constant delays in control satisfying the following inequalities:

$$0 = h_0 < h_1 < \dots < h_i < \dots < h_{M-1} < h_M.$$

## 2. Definitions

Let $L^2([0, t_1], \mathbf{R}^m)$ denote the set of square integrable functions defined in the time interval $[0, t_1]$ with values in the set $\mathbf{R}^m$. Any control $u \in L^2([0, t_1], \mathbf{R}^m)$ is said to be admissible control for dynamical system (1). For a given initial conditions $z(0)=\{x(0), u_0\} \in \mathbf{R}^n \times$

$L^2([-h_M, 0], \mathbf{R}^m)$, where $u_0 = u(s)$ for $s \in [0-h_M, 0]$ and admissible control $u \in L^2([0, t_1], \mathbf{R}^m)$, for every $t \geq 0$ there exists a unique, absolutely continous solution $x(t, z(0), u)$ of differential equation (1). This solution has the form [5]:

$$x(t, z(0), u) = e^{At} x(0) + \int_0^t e^{t-\tau} \sum_{i=0}^{M} B_i u(\tau-h_i)\, d\tau, \tag{2}$$

The set of initial conditions z(0) is called the initial complete state of the system (1).

**Definition 1.** The attainable set $K([0, t_1], z(0))$ for the dynamical system (1) at time $t_1>0$ is given by the following formula:

$$K([0,t_1], z(0))=\{ x(t_1,z(0),u) \in \mathbf{R}^n: x(t_1,z(0),u) = e^{At_1} x(0) + \int_0^{t_1} e^{t_1-\tau} \sum_{i=0}^{M} B_i u(\tau-h_i)\, d\tau, \tag{3}$$

$$u \in L^2([0, t_1], \mathbf{R}^m)\}.$$

With absence of control constraints the attainable set is always a linear subspace of the space $\mathbf{R}^n$.

**Definition 2.** [5] The dynamical system (1) is said to be relatively controllable in $[0, t_1]$ if, for any initial complete state $z(0) \in \mathbf{R}^n \times L^2([-h_M, 0], \mathbf{R}^m)$ and any vector $x_1 \in \mathbf{R}^n$, there exists a control $\tilde{u} \in L^2([0, t_1], \mathbf{R}^m)$ such that the corresponding trajectory $x(t,z(0), \tilde{u})$ of the dynamical system (1) satisfies the following condition:

$$x(t_1, z(0), \tilde{u}) = x_1.$$

The relative controllability can be also defined based on the attainable domain notion.

**Definition 3.** [5] The dynamical system (1) is said to be relatively controllable in $[0, t_1]$, if the equality:

$$K([0,t_1], z(0)) = \mathbf{R}^n$$

is satisfied.

The Definitions 2 and 3 are equivalent.

Next, constraints imposed on controls will be considered. This type of constraints occurs in practical problems connected with optimal control industrial processes very often.

Let $S \subset \mathbf{R}^n$ and $U \subset \mathbf{R}^m$ be arbitrary, nonempty sets. Based on the definition of relative controllability of dynamical systems with delays in control (Definition 2) and the definition of controllability of dynamical systems without delays with constrained controls [5] we can define global controllability of stationary system with delays (1) with constrained controls.

**Definition 4.** Dynamical system (1) is said to be globally relatively U-controllable in $[0, t_1]$ to the set S if it is relatively U-controllable to the set S for every initial complete state $z(0) \in \mathbf{R}^n \times L^2([-h_M, 0], U)$.

In the case when $S = R^n$ we say about global relative U-controllability in time interval $[0, t_1]$. While, when $S=\{0\}$ we say about global relative null U-controllability in $[0, t_1]$.

## 3. Controllability results

We shall formulate basic theorems concerning controllability with constrained controls for stationary dynamical systems with delays in control of the form (1).

**Lemma 1.** Dynamical system (1) without constraints is globally relatively controllable in $[0, t_1]$ if and only if

$$\operatorname{rank}[B_0 \mid B_1 \mid \ldots \mid B_{k-1} \mid AB_0 \mid AB_1 \mid \ldots \mid AB_{k-1} \mid \ldots \mid A^{n-1}B_0 \mid A^{n-1}B_1 \mid \ldots \mid A^{n-1}B_{k-1}] = n,$$

where $k \in \mathbf{N}$ we select so as $0 < k \leq M$ and $t_1 - h_k = 0$.

P r o o f: Lemma 1 follows from Theorems 1.4.1 and 4.6.4 presented in [5]. Based on Theorem 4.6.4 dynamical system (1) without constraints is globally relatively controllable in $[0, t_1]$ if and only if dynamical system without delays and constraints described by following equation:

$$\dot{x}(t) = A\,x(t) + \tilde{B}w(t), \quad t \in [0, t_1]$$

where

$$\tilde{B} = [B_0 \mid B_1 \mid \ldots \mid B_M], \; t \in [0, t_1], \; w \in \mathbf{R}^{(M+1)m}$$

is controllable. Next, from Theorem 1.4.1 follows that the above dynamical system without delays and constraints is controllable if and only if

$$\operatorname{rank}[\tilde{B} \mid A\tilde{B} \mid A^2\tilde{B} \mid \ldots \mid A^n\tilde{B}] = n. \qquad \blacksquare$$

**Theorem 1.** Dynamical system (1) is globally relatively null U-controllable in $[0, t_1]$ if and only if all the following conditions are satisfied simultaneously:

(1) there exists a $\tilde{w} = [w_0, w_1, \ldots, w_{k-1}]$ such that $\tilde{B}\tilde{w} = 0$, where $\tilde{B} = [B_0 \mid B_1 \mid \ldots \mid B_{k-1}]$, $w_i \in U$, $i = 0, 1, \ldots, k-1$, $k \in \mathbf{N}$ are selected so as $0 < k \leq M$ and $t_1 - h_k = 0$,

(2) the convex hull CH(U) of the set U has a nonempty interior in the space $\mathbf{R}^m$,

(3) $\operatorname{rank}[B_0 \mid B_1 \mid \ldots \mid B_{k-1} \mid AB_0 \mid AB_1 \mid \ldots \mid AB_{k-1} \mid \ldots \mid A^{n-1}B_0 \mid A^{n-1}B_1 \mid \ldots \mid A^{n-1}B_{k-1}] = n$,

(4) there is no real eigenvector $v \in \mathbf{R}^n$ of $A^T$ satisfying $v^T \tilde{B} w \le 0$, i = 0, 1, ... , k-1, for all $w_i \in U$,

(5) no eigenvalue of A has a positive real part.

P r o o f: Theorem 1 follows from Theorem 1.9.1 presented in [5] and from Lemma 1. ■

Let $K([0, t_1], z(0), U)$ denotes the set attainable in time $t_1>0$ for controls with constrained values, i.e. $u(t) \in U$, $t \in [0, t_1]$, given by the following formula:

$$K([0, t_1], z(0), U) = \{x(t_1,z(0),u) \in \mathbf{R}^n: x(t_1,z(0),u) = e^{At_1} x(0) + \int_0^{t_1} e^{t_1-\tau} \sum_{i=0}^{M} B_i u(\tau-h_i)\, d\tau,$$
$$u \in L^2([0, t_1],U)\}.$$

**Corollary 1.** Suppose that the set U is a cone with vertex at zero and nonempty interior in the space $\mathbf{R}^m$. Then the attainable set $K([0, t_1], z(0), U)$ is equal to $\mathbf{R}^n$ if and only if the conditions (3) and (4) of Theorem 1 are satisfied.

P r o o f: It follows directly from Corollary 1.9.4 presented in [5]. ■

**Theorem 2.** Assume that the set U is a cone with vertex at zero and nonempty interior in the space $\mathbf{R}^m$. If dynamical system (1) is globally relatively null U-controllable in $[0,t_1]$ then dynamical system (1) is globally relatively U-controllable in $[0,t_1]$.

P r o o f: Let $x(t_1,z(0),u) = P(z(0)) + Q_{t_1}(u)$, where $u(t) \in U$ and

$$P: \mathbf{R}^n \times L^2([-h_M, 0], \mathbf{R}^m) \rightarrow \mathbf{R}^n$$
$$Q_{t_1}: L^2([0, t_1], U) \rightarrow \mathbf{R}^n$$

are linear and bounded operators. Moreover, the range of the operator $Q_{t_1}$ is equal to the attainable set in the time $t_1$, $K([0, t_1], z(0), U)$, and the range of the operator P is the whole space $\mathbf{R}^n$. From assumption about global relative null U-controllability in $[0, t_1]$ follows that $x(t_1,z(0),u)=0$ for any initial complete state z(0). Hence the range of the operator $Q_{t_1}$ is the whole space $\mathbf{R}^n$. Therefore, for any $\tilde{x}$ $\mathbf{R}^n$ and any initial complete state $\tilde{z}(0)$ there exists an admissible control $\tilde{u}$ such that:

$$\tilde{x} - P(\tilde{z}(0)) = Q_{t_1}(\tilde{u}),$$

what is equivalent to the global relative U-controllability in $[0, t_1]$. ■

**Remark 1.** Since the operator $Q_{t_1}$ is the linear one, from the assumption about global relative null U-controllability follows that $U \subset \mathbf{R}^m$ cannot be bounded.

## 4. Examples

In a further passage we assume that admissible controls $u \in L^2([0, t_1], R^m)$ are constrained by the integral condition of the form:

$$\int_0^{t_1} S(\tau)u(\tau)d\tau \in Y, \tag{4}$$

where $Y \in R^n$ is a given closed set and $S(t)$ is a given, $n\times m$-dimensional matrix with elements $s_{kj} \in L^2_{loc}([0,\infty), R^m)$.

The above integral constraint leads to modification of the attainable set notion. Then the attainable set $K_Y([0,t_1], z(0))$ for the initial complete state $z(0) \in R^n \times L^2([-h_M, 0], R^m)$ of the dynamical system (1) with integral constraint (4) is the set of all ends of trajectories $x(t,z(0),u)$ such that the admissible control u satisfies the condition (4), i.e.

$$K_Y([0,t_1], z(0)) = \{ x(t_1,z(0),u) \in R^n : x(t_1,z(0),u) = e^{At_1} x(0) + \int_0^{t_1} e^{t_1-\tau} \sum_{i=0}^{M} B_i u(\tau-h_i)\, d\tau, \tag{5}$$

$$u \in L^2([0, t_1], R^m),\ \int_0^{t_1} S(\tau)u(\tau)d\tau \in Y \}.$$

The main purpose of this article is to show, based on selected examples of dynamical systems with delays in control of the form (1), the basic properties of attainable sets with constraint (4).

Assuming weaker integral condition, i.e. replacing the set Y by its „ε-neighbourhood" $Y^{(\varepsilon)} \subset R^n$, $\varepsilon > 0$ we get a new condition qualifying admissible controls $u \in L^2([0, t_1], R^m)$ of the following form:

$$\int_0^{t_1} S(\tau)u(\tau)d\tau \in Y^{(\varepsilon)},\ \varepsilon > 0. \tag{6}$$

The condition (6) is interpreted as the perturbation of control constraint (4). Based on condition (6) we get a new attainable set for the initial complete state z(0), denoted as $K_Y^{(\varepsilon)}([0,t_1], z(0))$. If this set will be, for a small ε, $\varepsilon > 0$, „near" to attainable set corresponding with condition (4), then we can say that the control perturbation does not influence the attainable set and, as a consequence, the controllability ( see Definition 3 ), and omit this perturbation. However, such desired from practical point of view property appears comparatively seldom.

Examples mentioned below show how essentially the solutions and consequently attainable sets of dynamical system (1) can change under perturbation control constraint of the form (4).

**Example 1.** Let us consider the dynamical system described by following scalar differential equation:

$$\dot{x}(t) = x(t) + u(t) - u(t-\tfrac{1}{2}) \tag{7}$$

defined in the fixed time interval [0, 1]. In this case n = 1 and m = 1. We assume that admissible controls u are nonnegative and satisfying the integral condition given by inequality:

$$\int_0^1 tu(t)dt \le 0, \tag{8}$$

and so $S(t) = t$ for $t \in [0, 1]$ and $Y = \{ y \in \mathbf{R}: y \le 0 \}$.

The unique admissible control satisfying the condition (8) is $u \equiv 0$. Let take $\varepsilon > 0$ and let consider a new integral condition qualifying selection of admissible control u in the following form

$$\int_0^1 tu(t)dt \le \varepsilon. \tag{9}$$

We treat the condition (9) as the perturbation of constraint (8).

We will prove that the set of admissible control's values is now the set $[0, \infty)$. [2]

To this end let us fix a number $a \ge 0$ and let us define for $\varepsilon > 0$ a number $b_a^{(\varepsilon)}$ by the following formula:

$$b_a^{(\varepsilon)} = \inf(\{ \tfrac{\varepsilon}{a+1}, 1\}) \in (0, 1].$$

Let the admissible control $u_a^{(\varepsilon)}$ satisfies the conditions:

(i) $\forall_{t \in [0, b_a^{(\varepsilon)})} \; u_a^{(\varepsilon)}(t) = \frac{a}{b_a^{(\varepsilon)}}$

(ii) $\forall_{t \in [b_a^{(\varepsilon)}, 1]} \; u_a^{(\varepsilon)}(t) = 0.$

Then

$$\int_0^1 u_a^{(\varepsilon)}(t)dt = \int_0^{b_a^{(\varepsilon)}} u_a^{(\varepsilon)}(t)dt + \int_{b_a^{(\varepsilon)}}^1 u_a^{(\varepsilon)}(t)dt = \frac{a}{b_a^{(\varepsilon)}}[t]_0^{b_a^{(\varepsilon)}} = a.$$

Moreover

$$\int_0^1 tu_a^{(\varepsilon)}(t)dt = \int_0^{b_a^{(\varepsilon)}} tu_a^{(\varepsilon)}(t)dt \le b_a^{(\varepsilon)} \int_0^{b_a^{(\varepsilon)}} u_a^{(\varepsilon)}(t)dt = b_a^{(\varepsilon)} \int_0^1 tu_a^{(\varepsilon)}(t)dt = ab_a^{(\varepsilon)} \le a\frac{\varepsilon}{a+1} \le \varepsilon.$$

Since the number a we have selected freely from interval $[0, \infty)$ we get:

$$\int_0^1 u(t)dt \in [0,\infty),$$

so the set of admissible control's values is also $[0, \infty)$.

Let us analyse what effect have above facts on controllability of the system (7). With given constraint (8) the system (7) is not controllable ($K_Y([0,t_1], z(0))=\varnothing$), because, how it has been noticed, the unique admissible control satisfying the condition (8) is $u \equiv 0$. While, with appearance of control constraint specified by condition (9) the system (7) is already relatively controllable.

Theorem 1 gives the criterion of controllability for stationary systems of the form (1) with constrained controls $u(t)\in U$. In particular, Corollary 1 gives us the controllability criterion for system (1) with positive controls. In Example 1 we got positive controls, i.e. $u(t)\in[0, \infty)$. Furthermore, there are satisfied conditions (3) and (4) of Theorem 1 for system (7), and so based on Corollary 1 this system is, with constrained controls, globally relatively null U-controllable in $[0,t_1]$. Finally, on the basis of Theorem 2 we conclude that the dynamical system (7) with constraint on control of the form (9) is globally relatively U-controllable in $[0,t_1]$, so $K_Y^{(\varepsilon)}([0,t_1], z(0))=\mathbf{R}^n$.

**Example 2.** Let us consider another constraint on control for the system (7).[2] Assume $s_1 : [0, 1] \to \mathbf{R}$ and $s_2 : [0, 1] \to \mathbf{R}$. Suppose that

$$\forall_{t\in[0,\frac{1}{2})} \left(s_1(t) = \tfrac{3}{2} - t, s_2(t) = -1\right)$$

and

$$\forall_{t\in[\frac{1}{2},1]} \left(s_1(t) = -1, s_2(t) = t + \tfrac{1}{2}\right).$$

Consider the following condition qualifying an admissible control selection

$$\int_0^1 s_1(t)u(t)dt \le 0\ ,\quad \int_0^1 s_2(t)u(t)dt \le 0. \tag{10}$$

We shall prove that $u \equiv 0$ is the unique function satisfying the above system of integral inequalities. Suppose, on contrary to our claim, that $u(t) \neq 0$ for some $t \in [0, 1]$ and the control u satisfies condition (10). If

$$\int_0^{\frac{1}{2}} u(t)dt < \int_{\frac{1}{2}}^{1} u(t)dt,$$

then

$$-\int_0^{\frac{1}{2}} s_2(t)u(t)dt = \int_0^{\frac{1}{2}} u(t)dt < \int_{\frac{1}{2}}^{1} s_2(t)u(t)dt,$$

and the second inequality in (10) is not satisfied. Analogously, if

$$\int_{\frac{1}{2}}^{1} u(t)dt < \int_0^{\frac{1}{2}} u(t)dt$$

then the first inequality in system (10) is not fulfilled. Finally, let

$$\int_0^{\frac{1}{2}} u(t)dt = \int_{\frac{1}{2}}^{1} u(t)dt.$$

We have now

$$-\int_{\frac{1}{2}}^{1} s_1(t)u(t)dt = \int_{\frac{1}{2}}^{1} u(t)dt < \int_0^{\frac{1}{2}} u(t)dt + \int_0^{\frac{1}{2}} (s_1(t)-1)u(t)dt = \int_0^{\frac{1}{2}} s_1(t)u(t)dt,$$

what again contradicts the assumption (10).

Assume now perturbations of control constraints (10) of the following form

$$\int_0^1 s_1(t)u(t)dt \le \varepsilon, \quad \int_0^1 s_2(t)u(t)dt \le \varepsilon, \quad \varepsilon > 0. \tag{11}$$

We shall prove that the set of values of admissible controls increases from the single-element set $\{0\}$ to the ray $[0, \infty)$.

Let us fix an $\alpha \in [0, \infty)$. Define a number $\beta_\alpha^{(\varepsilon)}$ by the formula:

$$\beta_\alpha^{(\varepsilon)} = \inf(\{\tfrac{\varepsilon}{\alpha+1}, \tfrac{1}{2}\}) \in (0, \tfrac{1}{2}]$$

and assume an admissible control $u_\alpha^{(\varepsilon)}$ satisfying the following conditions:

(i) $\forall_{t\in[0,\frac{1}{2}-\beta_\alpha^{(\varepsilon)})} \quad u_\alpha^{(\varepsilon)}(t) = 0$

(ii) $\forall_{t\in[\frac{1}{2}-\beta_\alpha^{(\varepsilon)},\frac{1}{2}+\beta_\alpha^{(\varepsilon)})} \quad u_\alpha^{(\varepsilon)}(t) = \frac{\alpha}{2\beta_\alpha^{(\varepsilon)}}$

(iii) $\forall_{t\in[\frac{1}{2}+\beta_\alpha^{(\varepsilon)},1]} \quad u_\alpha^{(\varepsilon)}(t) = 0.$

Then $\int_0^1 u_\alpha^{(\varepsilon)}(t)dt = \alpha$. Furthermore

$$\int_0^1 s_1(t)u_\alpha^{(\varepsilon)}(t)dt = \int_{\frac{1}{2}-\beta_\alpha^{(\varepsilon)}}^{\frac{1}{2}}(\tfrac{1}{2}-t)u_\alpha^{(\varepsilon)}(t)dt + \int_{\frac{1}{2}-\beta_\alpha^{(\varepsilon)}}^{\frac{1}{2}} u_\alpha^{(\varepsilon)}(t)dt - \int_{\frac{1}{2}}^{\frac{1}{2}+\beta_\alpha^{(\varepsilon)}} u_\alpha^{(\varepsilon)}(t)dt = \int_{\frac{1}{2}-\beta_\alpha^{(\varepsilon)}}^{\frac{1}{2}}(\tfrac{1}{2}-t)u_\alpha^{(\varepsilon)}(t)dt \le$$

$$\le \beta_\alpha^{(\varepsilon)} \int_{\frac{1}{2}-\beta_\alpha^{(\varepsilon)}}^{\frac{1}{2}} u_\alpha^{(\varepsilon)}(t)dt = \beta_\alpha^{(\varepsilon)}\frac{\alpha}{2} \le \frac{\varepsilon}{\alpha+1}\frac{\alpha}{2} \le \frac{\varepsilon}{2} < \varepsilon$$

and

$$\int_0^1 s_2(t)u_\alpha^{(\varepsilon)}(t)dt = \int_{\frac{1}{2}-\beta_\alpha^{(\varepsilon)}}^{\frac{1}{2}}(-u_\alpha^{(\varepsilon)}(t))dt + \int_{\frac{1}{2}}^{\frac{1}{2}+\beta_\alpha^{(\varepsilon)}} u_\alpha^{(\varepsilon)}(t)dt + \int_{\frac{1}{2}}^{\frac{1}{2}+\beta_\alpha^{(\varepsilon)}}(t-\tfrac{1}{2})u_\alpha^{(\varepsilon)}(t)dt = \int_{\frac{1}{2}}^{\frac{1}{2}+\beta_\alpha^{(\varepsilon)}}(t-\tfrac{1}{2})u_\alpha^{(\varepsilon)}(t)dt \le$$

$$\le \beta_\alpha^{(\varepsilon)} \int_{\frac{1}{2}}^{\frac{1}{2}+\beta_\alpha^{(\varepsilon)}} u_\alpha^{(\varepsilon)}(t)dt = \beta_\alpha^{(\varepsilon)}\frac{\alpha}{2} \le \frac{\varepsilon}{2} < \varepsilon,$$

which proves the correctness of control $u_\alpha^{(\varepsilon)}$ selection.

Finally, with perturbations of integral constraints (4) we get positive admissible controls and analogous controllability of dynamical system (7) as in Example 1.

Examples 1 and 2 show that even small perturbations of control constraints may lead to essential changes in the system. The system which, with given constraints, is not controllable at all becomes controllable after the modification of control constraints. Attainable set of the system, from the empty set increases to whole ray $[0, \infty)$.

**Example 3.** We consider stationary system without delays described by following differential equation:

$$\dot{x}(t) = x(t) + u(t) \tag{12}$$

in fixed time interval $[0, t_1]$. We assume that $U = [0, \infty)$, i.e. we have only positive controls. Then, by Corollary 1.9.5 presented in [5] dynamical system (12) is no null U-controllable.

Now, introduce in the above system the delay in control $h=\frac{1}{2}$ with negative coefficient $B_1 = -1$. We get the system described by following differential equation:

$$\dot{x}(t) = x(t) + u(t) - u(t-\tfrac{1}{2}). \tag{13}$$

Based on Corollary 1, in fixed time interval $[0, t_1]$ dynamical system (13) is relatively null U-controllable.

Example 3 shows that thanks to introduction of properly selected delays in control in stationary system without delays, which is no U-controllable in fixed time interval, it is easy to get its relative U-controllability in this interval. This result is the efect of an extending of the control space dimension of the system with delays in control in comparison with corresponding system without delays.

## 5. Final remarks

Occuring in practice perturbations of control constraints generate a question about their influence on the other parameters of given dynamical system, especially on a form of the attainable set and controllability. In some cases, perturbations cause so little changes in the system that one can ignore them. However, in Examples 1 i 2 it has been shown that there are cases, when this influence is huge. Even small deviations of constraints may cause radical, qualitative and numerical changes in the system. Therefore, perturbations cannot be entirely omitted. To the problem of controllability with perturbations of constraints one can approach dually. On the one hand, one can search such additional properties of system's parametrs under which occure only small changes of attainable set, and so of controllability. In this case, the influence of perturbations can be omitted. On the other hand, one should modify the definition of the attainable set for unperturbated model in such a way that perturbations can be taken into account.

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**Streszczenie**

W artykule rozpatrujemy liniowe, stacjonarne, skończenie wymiarowe układy dynamiczne z opóźnieniami w sterowaniu opisane różniczkowym równaniem stanu o postaci (1). Podajemy definicje różnych rodzajów względnej sterowalności przy ograniczeniach na sterowanie układu dynamicznego (1). Formułujemy twierdzenia stanowiące kryteria badania

względnej sterowalności przy ograniczeniach na sterowanie układów z opóźnieniami. W przedstawionych przykładach opisujemy pewne własności zbiorów osiągalnych układów dynamicznych z opóźnieniami w sterowaniu (1) przy ograniczeniu (4). Pokazujemy, że zakłócenia ograniczeń sterowania mogą w radykalny sposób zmienić postać zbioru osiągalnego oraz sterowalność układu (przykład 1 i 2). Zauważamy również, że pewne korzyści mogą wyniknąć dzięki wprowadzeniu opóźnień w sterowaniu w układzie stacjonarnym z ograniczonymi sterowaniami bez opóźnień. W przykładzie 3 pokazujemy, że dzięki właściwie dobranym opóźnieniom jesteśmy w stanie uzyskać względną sterowalność przy dodatnich sterowaniach układu, który bez opóźnień nie był w ogóle sterowalny.