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This periodical is published under the sponsorship of Brown University; it prints original papers in applied mathematics which have an intimate connection with application in industry or practical science. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

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FOREWORD

The Quarterly of Applied Mathematics has been founded primarily to meet the needs of certain mathematicians and engineers whose interests extend beyond the accepted boundaries of their respective groups. These mathematicians find their greatest interest in the application of mathematics to physical problems, and these engineers seek solutions of practical problems by advanced mathematical methods. Thus they meet on the common ground of applied mathematics with a stimulating variety of interest.

It is not desirable to attempt too precise a definition of the boundaries of the field to which the Quarterly will be devoted. The mathematical solution of one problem often throws light on another problem in an entirely different field; indeed, the peculiar strength of the mathematical method lies in its power to cut across those lines of demarcation which seem to divide science into separate compartments.

Nevertheless, it is necessary to give an outline of policy for, within fairly wide limits, the pages of the Quarterly should appeal to a common interest. It seems best to start with the common ground of mathematics and engineering as a nucleus, and to build around it a wider circle of interest, embracing mathematical theory related to engineering problems. Thus certain subjects—fluid mechanics, elasticity, plasticity, thermodynamics, and classical mechanics in its engineering applications—are to be regarded as lying within the scope of the Quarterly, and to these must be added electrical engineering, which has been one of the most fruitful fields of mathematical application.

While it is not the purpose of the Quarterly to publish experimental results, we shall welcome mathematical contributions which have an intimate connection with application in industry or practical science. Indeed, the ideal contribution to our pages would be one in which advanced and general mathematical methods lead speedily to results which are in close agreement with experiment, and which are of high importance, either in direct practical application or as an illumination of interesting phenomena hitherto unexplained.

THE EDITORS.

TOOLING UP MATHEMATICS FOR ENGINEERING*

BY

THEODORE von KÁRMÁN

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It has often been said that one of the primary objectives of Mathematics is to furnish tools to physicists and engineers for solution of their problems. It is evident from the history of the mathematical sciences that many fundamental mathematical discoveries have been initiated by the urge for understanding nature's laws and many mathematical methods have been invented by men primarily interested in practical applications. However, every true mathematician will feel that a restriction of mathematical research to problems which have immediate applications would be unfair to the "Queen of Sciences." As a matter of fact, the devoted "minnesingers" of the Queen have often revolted against degradation of their mistress to the position of a "handmaiden" of her more practical minded and temporarily more prosperous sisters.

It is not difficult to understand the reasons for the controversial viewpoints of mathematicians and engineers. They have been pointed out more than once, by representatives of both professions.

The mathematician says to the engineer: I have built a building on a sound foundation: a system of theorems based on well defined postulates. I have delved into the analysis of the process of logical thinking to find out whether or not there are any statements which could be considered true or at least potentially true. I am interested in functional relations between entities which are well defined creations of my own mind and in methods which enable me to explore various aspects of such functional relations. If you find any of the concepts, logical processes or methods which I have developed useful for your daily work, I am certainly glad. All my results are at your disposal, but let me pursue my own objectives in my own way.

Says the engineer: Your great forbears, who were mathematicians long before you, talked a different language. Did not Leonhard Euler distribute his time between discoveries in pure mathematics and in the theory of engineering devices? The fundamentals of the theory of turbines, the theory of buckling of columns, the theory of driving piles into soil were contributions of Euler. The development of mathematical analysis cannot be separated from the development of physics and especially of mechanics. It is doubtful whether a human mind would ever have conceived the idea of differential equations without the urge to find a mathematical tool for the computation

* Received March 15, 1943.

of the path of moving bodies. If one assumes that the motion is determined by certain fundamental mechanical or geometrical relations, which are valid at every instant of the motion, one naturally is led to the idea of the differential equation. Also, the calculus of variations was invented mainly for solution of physical problems; some of which were of teleological, some of practical nature. The eighteenth century and the first decades of the nineteenth were perhaps the period of the most glorious progress in mathematical science; at that time, there was no distinction between pure and applied mathematicians. The abstract minded mathematicians stepped in after the big job was done; they endeavored to fill certain logical gaps, to systematize and codify the abundance of methods and theorems which the giants of the foregoing period created by a combination of logical thinking and creative intuition.

The mathematician: It seems to me that you underestimate the importance of what you call systematization and codification. Don't you think that in order to assure the correct application of calculus and differential equations, there was an absolute necessity to define exactly what we mean by a limiting process; or, was it not absolutely necessary to give a real sense to such terms as infinitely small and infinitely large? You may remember that Galileo—whom you hardly can call an abstract or pure mathematician—pointed out the contradictions which are unavoidable if you try to apply the notions of equality and inequality to infinite quantities. He noticed that you can say either that the number of the integers is larger than the number of the squares, since every square is an integer, but not every integer is a square; or you can say with the same justification that there are as many squares as integers, since every number has a square. The notions of commensurability, denumerability, the logical analysis of the continuum, the theory of sets, and in more recent times, topology, were fundamental steps in the development of the human mind. Many of these developments were conceived independently of any conscious physical applications. But even for the sake of applications, it was necessary to improve the foundations of our own house, that is to improve the logical structure of mathematics. Without exact analysis of the conditions for the convergence of series (the conditions which allow carrying out the processes of differentiation and integration), nobody could feel safe in handling series. It is not correct that the tendency to seek for a solid foundation of the new discoveries began after the men endowed with imagination and intuition did the big job. D'Alembert already demanded that the calculus be founded on the methods of limits. Cauchy, Legendre and Gauss certainly were among the creative mathematical geniuses in your sense; they effectively contributed to the transition from intuition to rigor. In the second half of the nineteenth century this development continued toward the great goal that the mathematicians of that age—perhaps optimistically—considered as perfect logic or absolute rigor. However, in addition to the clarification of the fundamentals, that period also opened new paths for applied mathematics. You mentioned, for example, differential equations. Don't you believe that the theory of functions of complex variables, the classification of differential equations according to their singularities, and the investigation of these singularities, all developed in the period that you call the period of codification, were most important steps in building up the very branch of mathematics from which you engineers derive so much benefit? These theories changed the primitive way of finding solutions of differential equations by trial into a systematic method of mastering the whole field.

The engineer: I agree, especially with what you say about complex variables. Indeed, the conformal transformation is one of the most powerful and most elegant methods for the solution of innumerable physical and engineering problems. I also agree with you on the fundamental importance of the analysis of singularities. In fact, our graphical and numerical methods necessarily fail or become awkward near the irregular points and we have to take recourse to analytical methods. However, you mathematicians unfortunately are somewhat like a physician who is less interested in the laws of normal functioning of the human body than in its diseases, or like the psychologist who instead of investigating the laws of normal mental processes concentrates his attention on the pathological aberrations of the human mind. We have to deal in most cases with "sound functions" and would like to have efficient methods to determine with fair accuracy their behavior in certain definite cases.

Answers the mathematician: Can you not apply the general methods that we developed for the solution of differential and integral equations? If the solutions are given by "sound functions," as you please to call them, I do not see any great difficulty nor do I see what more you expect us to do.

The engineer: Your general theorems deal mostly with the existence of solutions and the convergence of your methods of solution. You may recall the wisecrack of Heaviside: "According to the mathematicians this series is divergent; therefore, we may be able to do something useful with it." You people spend much time and much wit to show the existence of solutions whose existence often is evident to us for obvious physical reasons. You seldom take the pains to find and discuss the actual solutions. If you do so, then you restrict yourself mostly to simple cases, as for example, problems involving bodies of simple geometrical shapes. I refer to the so-called special functions. I concede that a great many such functions were investigated by mathematicians. Their values have been tabulated, their developments in series and their representations by definite integrals have been worked out in great detail. Unfortunately, such functions have only a restricted field of application in engineering. The physicist in his search for fundamental laws may choose specimens of simple geometrical shapes for his experimentation. The engineer has to deal directly with structures of complicated shapes; he cannot give to a structure a simple geometrical form just because the stress distribution in such a structure can be calculated by special functions. Furthermore, most special functions are applicable only to linear problems. In the past, physicists and engineers often linearized their problems for simplicity's sake. Mathematicians liked this simplification because it furnished a beautiful hunting ground for the application of elegant mathematical methods. Unfortunately, as engineering science progressed, the need for more exact information and the necessity to get nearer and nearer to physical reality, forces us to grapple with many nonlinear problems.

The mathematician: Well, many modern mathematicians are extremely interested in non-linear problems. It seems your primary need is the development of appropriate methods of approximation. However, you are not right in your criticism of our proofs of existence. Many proofs of existence in modern mathematics go far beyond the limits of intuition. Then, too, I understand you engineers have good success with various iteration methods. Now, if we want to prove for example the existence of a solution of a boundary value problem, very often we use the iteration method. In other words, we really construct a sequence of approximate solutions exactly as you do. The whole difference is that we prove and you only assume that the process of iteration leads to a unique solution. Also, your so-called "energy method" used for the solution of your problems in elasticity and structures appears to me closely related to the direct methods of the calculus of variations, i.e., to methods which try to construct directly the minimizing function for given boundary values, without referring to the Euler-Lagrange differential equation. It seems to me that after all there are many common elements in pure analysis and applied mathematics.

The engineer: I shall not deny that; as a matter of fact, I have always felt that analysis is the backbone of applied mathematics. However, if you really start to apply analysis to actual cases you will see that there is a long way from the general idea of a method of approximation to a successful application of the same method. There is, for example, the question of available time and manpower. For certain types of work, we have ingenious mechanical or electrical devices such as the differential analyzer or electric computers. However, in most cases we have to do the computation without such help. Then it is not sufficient to know that the process of approximation converges. We have to find out which method requires the least time for a given degree of approximation; we have to have a fair estimate of the improvement of accuracy by successive steps. All such practical questions require difficult mathematical considerations. I think we definitely need mathematicians who help us to refine and, if you wish to say so, criticize and systematize our intuitive methods. In fact, successful applications of mathematics to engineering require the close cooperation of mathematicians and engineers. It is by no means a routine job to recognize the underlying common mathematical relations in apparently very different fields. The mathematician who intends to do applied mathematical research has to have a pretty good sense for the physical processes involved. On the other hand, the engineer has to go into the fundamentals of analysis to a considerable depth in order to use the mathematical tools properly. An arbitrary assembly of machine tools does not constitute an efficient machine shop. We know there are powerful machine tools in your mathematical arsenal. The task before us is to know how to adapt and apply them.

The mathematician: I think you've got something there. To carry your analogy further, in order to get the solution of engineering problems into production, you need some kind of tool designers. These are the real applied mathematicians. Their original backgrounds may differ; they may come from pure mathematics, from physics or from engineering, but their common aim is to "tool up" mathematics for engineering.

A REVIEW OF THE STATISTICAL THEORY OF TURBULENCE*

BY

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1. Introduction. The irregular random motion of small fluid masses to which the name turbulence is given is of such complexity that there can be no hope of a theory which will describe in detail the velocity and pressure fields at every instant. Existing theories may be classified as either empirical or statistical.

In the empirical theories attention is focused only on the distribution of mean speed and mean pressure, and assumptions are made as to the dependence of the shearing stresses required to satisfy the equations of motion of the mean flow. These assumptions involve one or more empirical constants. While the type of assumption adopted is often selected on the basis of some hypothesis as to the character of the fluctuations of speed and pressure, the theory rests on the final assumption rather than on the hypothesis as to the fluctuations. The various "mixing length" theories are of this type.

In the statistical theories consideration is given to the frequency distribution and mean values of the pressure and of the components of the velocity fluctuations, i.e. to the statistical properties of the fluctuations, and to the relation between the mean motion and these statistical properties.

Some attempts have been made to apply the methods of statistical mechanics of discrete particles. In all such attempts it is necessary to select certain discrete elements corresponding to the particles, and to make some assumption as to the probability of occurrence of various values of associated properties or more directly the frequency distribution of the associated properties. Difficulties are encountered at both points. The best known theory of this type is that of Burgers[1] who selected as elements in two-dimensional flow the points in a square network of equally spaced points and as associated property the value of the stream function. This theory has not as yet led to useful results and is not satisfactory to Burgers himself. Other attempts of

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* Received Nov. 19, 1942.

[1] Burgers, J. M., *On the application of statistical mechanics to the theory of turbulent fluid motion*, I to VII, inclusive, Verh. Kon. Akad. v. Wetensch. Amsterdam **32**, 414, 643, 818 (1929); **36**, 276, 390, 487, 620 (1933). Summarized by Trubridge in Reports Phys. Soc. London, 1934, p. 43.

this nature have been made by von Kármán,[2] Noether,[3] Tollmien,[4] Gebelein,[5] Dedebant, Wehrlé and Schereschewsky,[6] and Takahasi.[7]

Many of the statistical theories just mentioned do not require the turbulent fluctuations to satisfy the equations of motion nor do they require the fluid motion to be continuous. A statistical theory of turbulence which is applicable to continuous movements and which satisfies the equations of motion was inaugurated in 1935 by Taylor[8] and further developed by himself and by von Kármán.[9] It is the object of this paper to give a connected account of the present state of this particular statistical theory of turbulence.

2. Turbulent fluctuations and the mean motion. As in other theories of turbulent flow, the flow is regarded as a mean motion with velocity components, U, V, and W, on which are superposed fluctuations of the velocity with components of magnitude u, v, and w at any instant. The mean values of u, v, and w are zero. In most cases U, V, and W are the average values at a fixed point over a definite period of time, although in certain problems it is more convenient to take averages over a selected area or within a selected volume at a given instant. The rules for forming mean values were stated by Reynolds[10] and some further critical discussion by Burgers and others has been recorded in connection with a lecture by Oseen.[11]

When the turbulent motion is produced in a pipe by the action of a constant pressure gradient or near the surface of an object in a wind tunnel in which the fan is operated at a constant speed, there is considerable freedom

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[2] Kármán, Th. von, *Über die Stabilität der Laminarströmung und die Theorie der Turbulenz*, Proc. 1st Inter. Congr. Appl. Mech., Delft, 1924, p. 97.

[3] Noether, F., *Dynamische Gesichtspunkte zu einer statistischen Turbulenztheorie*, Z. angew. Math. u. Mech. 13, 115 (1933).

[4] Tollmien, W., *Der Burgersche Phasenraum und einige Fragen der Turbulenzstatistik*, Z. angew. Math. u. Mech. 13, 331 (1933). Brief abstract of this paper entitled, *On the turbulence statistics in Burgers' phase space*, Physics, 4, 289 (1933).

[5] Gebelein, H., *Turbulenz: Physikalische Statistik und Hydrodynamik*, Julius Springer, Berlin, 1935.

[6] Dedebant, G., Wehrlé, Ph., and Schereschewsky, Ph., *Le maximum de probabilité dans les mouvements permanents. Application à la turbulence*, Comptes Rendus Ac. Sci. Paris 200, 203 (1935). Also Dedebant, G., and Wehrlé, Ph., *Sur les équations aux valeurs probables d'un fluide turbulent*, Comptes Rendus Ac. Sci. Paris 206, 1790 (1938).

[7] Takahasi, K., *On the theory of turbulence*, The Geophysical Magazine 10, 1 (1936).

[8] Taylor, G. I., *Statistical theory of turbulence*, I–V inclusive, Proc. Roy. Soc. London Ser. A, **151**, 421 (1935) and **156**, 307 (1936). Also, *The statistical theory of isotropic turbulence*, Jour. Aeron. Sci., **4**, 311 (1937).

[9] Kármán, Th. von, *On the statistical theory of turbulence*, Proc. Nat. Acad. Sci. **23**, 98 (1937). Also *The fundamentals of the statistical theory of turbulence*, Jour. Aeron. Sci. **4**, 131 (1937). Also with Howarth, L., *On the statistical theory of isotropic turbulence*, Proc. Roy. Soc. London Ser. A, **164**, 192 (1938).

[10] Reynolds, O., *On the dynamical theory of incompressible viscous fluids and the determination of the criterion*, Phil. Trans. Roy. Soc. London **186**, 123 (1895).

[11] Oseen, C. W., *Das Turbulenzproblem*, Proc. 3rd Inter. Congr. Appl. Mech., Stockholm, 1931, vol. **1**, p. 3.

in selecting the time interval for which mean values are taken. So long as the time interval is longer than some fixed value dependent on the scale of the apparatus and the speed, the mean values are independent of the magnitude of the time interval selected and there is a clear separation between the turbulent fluctuations and the mean motion. If the mean motion itself is "slowly" variable, as in the case of the natural wind, difficulty arises; the separation becomes imperfect and arbitrary. The slowly variable mean may be taken over time intervals of five minutes, one day, or ten years according to the object of the study and the magnitude of the turbulent fluctuations varies accordingly. Even in flows under constant pressure gradient, there will usually be some experimental difficulty in maintaining the conditions absolutely constant, and the question will naturally arise as to how the fluctuations arising from this source may be eliminated from the "true" turbulent fluctuations.

3. Vortex trails. For a long time every flow in which "fast" fluctuations of velocity occurred was regarded as a turbulent flow but experimental measurements of fluctuations show several identifiable types. The experimental results suggest the limitation of the term "turbulent fluctuation" to one of these types characterized by the random nature of the fluctuations. This random characteristic is in marked contrast with the regularity and periodicity noted in a second type of fluctuation associated with vortex trails.

It is well known that when a cylinder or other object of blunt cross section is exposed to a fluid stream, a vortex trail appears under certain circumstances, vortices breaking away with a regular periodicity. The speed fluctuations observed in the trail are periodic and in themselves do not produce turbulent mixing. At comparatively short distances the regular pattern transforms into an irregular turbulent motion, but the fluctuations within the trail itself do not have the character of the final turbulent fluctuations.

The fluctuations of turbulence are irregular, without definite periodicity with time. The amplitude distribution corresponds to the Gaussian distribution, i.e. the number of times during a long time interval that a given magnitude of fluctuation is reached varies with the magnitude according to the "error" curve.

If this randomness is regarded as an essential feature of the turbulent fluctuations, turbulence is not equivalent to any regular vortex system however complex. The equivalent vortex picture is a large family of vortex systems, whose statistical properties only, not individual histories, are significant.

4. Space and time averages. The speed fluctuations u, v, and w, though designated the fluctuations at a point, are in reality averages throughout a certain volume and over a certain time as are the speed components in the usual hydrodynamic theory. The volume is small in comparison with the dimensions of interest in the flow but large enough to include many molecules. A cube of size 0.001 mm, containing at atmospheric pressure about 2.7×10^7 molecules, satisfies this condition. The time interval is short in comparison with any time interval of interest in the mean properties of the flow but long in comparison with the time required for a molecule to traverse the mean free path. The number of collisions at atmospheric pressure is of the order of 5×10^9 per second and hence a time interval of 10^{-6} seconds would suffice.

No instruments have yet been constructed to give values averaged over so small a volume or so short a time interval. The best performance obtained to date is that of hot wire anemometers which have been developed to the point where average values over a cylindrical volume perhaps 0.01 mm in diameter and 1 mm long and over a time interval of approximately 0.5×10^{-3} seconds can be obtained. Experimental results show that averages over these space and time intervals are not appreciably different from those for somewhat larger space and time intervals and suggest that averages over smaller intervals would not be appreciably different. The results also suggest that measuring equipment that does not approach these space and time intervals gives results which largely reflect the properties of the measuring instrument rather than the properties of the turbulent fluctuations. In other words the measurement is that of a variable mean velocity over space and time intervals fixed by the characteristics of the instrument, rather than measurements of the turbulent fluctuations. If the frequency spectrum of the turbulent fluctuations is known, the effect of the instrument characteristics can be estimated, as discussed in section 19.

5. Pulsations. Reference has previously been made to the difficulty in certain cases of making a clear separation between the mean motion and the turbulent fluctuations, because of the difficulty of defining a time interval long enough to include many fluctuations but small enough so that the mean varies only slowly. The difficulty is often increased by the presence of a fairly rapid variation of the mean speed over large areas, perhaps the entire cross section of the fluid stream, to which the name pulsation may be given. Such a fluctuation is recognizable by the fact that there is a regularity in the space distribution of the fluctuations such that definite phase relations exist. Pulsations have been observed in laminar flow in boundary layers. An essential characteristic of the turbulent fluctuations is an irregularity and randomness in the space distribution as well as in the time distribution.

It is often possible to eliminate the effect of pulsations on the measurements by a low frequency cut-off in the equipment for measuring u, v, and w. The choice of the cut-off frequency is equivalent to a selection of the time interval over which averages are taken to obtain the mean speed and by this device the pulsations are regarded as variations of the mean speed.

6. Continuity of the turbulent motion. It is well known that the structure of a fluid is in the final analysis discontinuous, the fluid consisting of individual molecules. Nevertheless the usual hydrodynamic theory regards the fluid as a continuum. Such an assumption can be justified when the dimensions of the flow system are very large compared to the mean free path of the molecules. The velocity of the fluid at any point is then defined as the vector average of the velocities of the molecules in a small volume surrounding the point, the value obtained being independent of the magnitude and shape of the volume within certain limits.

Some investigators[12] have concluded that the phenomena of turbulence require the assumption of discontinuity in the instantaneous components. The Taylor-von Kármán statistical theory retains the assumption that the fluctuations are continuous functions of space and time as in Reynolds' theory.

The applicability of this assumption is a matter for experimental determination. If experimentally a volume and time interval can be selected which may be regarded as large in comparison with molecular distances and periods but small as compared to the volumes and time intervals of interest in the turbulent fluctuations, the fluctuations may be safely regarded as continuous. As described in section 4, the experimental data perhaps do not prove but do definitely suggest that such a choice is possible and to that extent the assumption of continuity is experimentally justified.

7. The Reynolds stresses. If in the Navier-Stokes equations of motion the components of the velocity are written as $U+u$, $V+v$, $W+w$, thus regarding the motion as a mean motion U, V, W, with fluctuations u, v, w superposed, and mean values taken in accordance with the rules mentioned in section 2, a new set of equations is obtained which differs from the first only in the presence of additional terms added to the mean values of the stresses due to viscosity. These additional terms are called the Reynolds stresses or eddy stresses. The eddy normal stress components are $-\rho\overline{u^2}$, $-\rho\overline{v^2}$, $-\rho\overline{w^2}$ and the eddy shearing stress components are $-\rho\overline{uv}$, $-\rho\overline{vw}$, $-\rho\overline{uw}$. Each stress component is thus equal to the rate of transfer of momentum across the corresponding surface by the fluctuations.

In the light of kinetic theory the eddy stresses closely parallel in origin the viscous stresses. It has been explained how u, v, and w are themselves the mean speeds of many molecules. The effect of the molecular motions appears in the smoothed equations of the continuum as a stress, the components of which are equal to the rate of transfer of momentum by the molecules across the corresponding surfaces.

8. Correlation. If the fluctuations were perfectly random, the eddy shearing stress components $-\rho\overline{uv}$, $-\rho\overline{vw}$, $-\rho\overline{uw}$ would be zero. The existence of eddy shearing stresses is dependent on the existence of a correlation between the several components of the velocity fluctuation at any given point. The coefficient of correlation between u and v is defined as

[12] Kampé de Fériet, J., *Some recent researches on turbulence*, Proc. Fifth Inter. Congr. Appl. Mech., Cambridge, Mass., 1938, p. 352.

$$R_{uv} = \frac{\overline{uv}}{\sqrt{\overline{u^2}}\sqrt{\overline{v^2}}} \cdot \tag{8.1}$$

The mean values $\sqrt{\overline{u^2}}$, $\sqrt{\overline{v^2}}$, and $\sqrt{\overline{w^2}}$ are often called the components of the intensity of the fluctuations.

The eddy shearing stress may be written in terms of the correlation coefficient as

$$-\rho\overline{uv} = -\rho R_{uv}\sqrt{\overline{u^2}}\sqrt{\overline{v^2}} \tag{8.2}$$

and similarly for the other components.

In addition to the correlation between the components of the velocity fluctuations at a given point, the Taylor-von Kármán theory makes much use of correlations between the components of the velocity fluctuations at neighboring points. Denote the components of the fluctuations at one point by u_1, v_1, w_1, and at another point by u_2, v_2, w_2. The coefficient of correlation between u_1 and v_2 is defined as

$$R_{u_1v_2} = \frac{\overline{u_1v_2}}{\sqrt{\overline{u_1^2}}\sqrt{\overline{v_2^2}}} \tag{8.3}$$

and similarly for any other pair. These correlation coefficients form useful tools to describe the statistical properties of the fluctuations with respect to their spatial distribution and phase relationships.

9. Scale of turbulence. The earliest attempt to describe the spatial characteristics of turbulence was the introduction of the mixing length concept, the mixing length being analogous to the mean free path of the kinetic theory of gases. Logical difficulties arise because there are no discrete fluid particles in the turbulent flow which retain their identity. A method of avoiding these difficulties was suggested by Taylor[13] many years ago. He showed that the diffusion of particles starting from a point depends on the correlation R_t between the velocity of a fluid particle at any instant and that of the same particle after a time interval t. If the functional relationship between R_t and t is of such a character that R_t falls to zero at some interval T and remains so for greater intervals, it is possible to define a length l_1 by the relation:

$$l_1 = \sqrt{\overline{v^2}} \int_0^T R_t dt = \sqrt{\overline{v^2}} \int_0^\infty R_t dt \tag{9.1}$$

in which v is the component of the velocity fluctuations transverse to the mean flow and in the direction in which the diffusion is studied.

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[13] Taylor, G. I., *Diffusion by continuous movements*, Proc. London Math. Soc. Ser. A, 20, 196 (1921).

This method of assigning a scale to turbulence is of value in the study of diffusion as described in section 22. It is based on the Lagrangian manner of describing the flow by following the paths of fluid particles. It is more common to use the Eulerian description by considering the stream lines existing in space at any instant. Taylor later[8] suggested a method of describing the scale in the Eulerian system based on the variation of the correlation coefficient R_y between the values of the component u at two points, separated by the distance y in the direction of the y coordinate, as y is varied. The curve of R_y against y represents the statistical distribution of u along the y axis at any instant. If R_y falls to zero and remains zero, a length L may be defined by the relation:

$$L = \int_0^\infty R_y dy. \tag{9.2}$$

The length L is considered a possible definition of the average size of the eddies present and has been found to be a most useful measure of the scale of the turbulence, especially for the case of isotropic turbulence. Correspondingly, a length L_x may be defined by the relation:

$$L_x = \int_0^\infty R_x dx \tag{9.3}$$

where R_x is the correlation between the values of the component u at two points separated by distance x in the direction of the x coordinate.

10. Isotropic turbulence. The simplest type of turbulence for theoretical or experimental investigation is that in which the intensity components in all directions are equal. More accurately, isotropic turbulence is defined by the condition that the mean value of any function of the velocity components and their derivatives at a given point is independent of rotation and reflection of the axes of reference. Changes in direction and magnitude of the fluctuations at a given point are wholly random and there is no correlation between the components of the fluctuations in different directions. Thus $\overline{u^2} = \overline{v^2} = \overline{w^2}$ and $\overline{uv} = \overline{vw} = \overline{uw} = 0$.

There is a strong tendency toward isotropy in all turbulent motions. The turbulence at the center of a pipe in which the flow is eddying or in the natural wind at a sufficient height above the ground is approximately isotropic. A grid of round wires placed in a uniform fluid stream sets up a more or less regular eddy system of non-isotropic character which very quickly transforms into a field of uniformly distributed isotropic turbulence.

The assumption of isotropy introduces many simplifications in the statistical representation of turbulence. The two quantities, intensity and scale, appear to give a description of the statistical properties of the turbulent field which is sufficient for most purposes. Turbulent fields of this type can readily be produced experimentally and studied. The intensity may be varied from less than 0.1 to about 5.0 percent of the mean speed and the scale independently from a few mm to 25 mm.[14]

11. Decay of isotropic turbulence. The kinetic energy of the turbulent fluctuations per unit volume is equal to $\frac{1}{2}\rho(\overline{u^2}+\overline{v^2}+\overline{w^2})$ which for isotropic turbulence becomes $(3/2)\rho\overline{u^2}$. The rate of decay is therefore $-(3/2)\rho d(\overline{u^2})/dt$. If the isotropic turbulence is superposed on a stream of uniform speed U, we may write $dt=dx/U$ and hence the rate of decay with respect to distance x as $-(3/2)\rho U d(\overline{u^2})/dx$.

In a fully developed turbulent flow the Reynolds stresses are proportional to the squares of the turbulent fluctuations. The work done against these stresses, which in the absence of external forces must come from the kinetic energy of the system, is proportional to $\rho u'^3/L$ where u' is written for $\sqrt{\overline{u^2}}$ and L is a linear dimension defining the scale of the system, which may be taken as the L defined by (9.2). Equating the two expressions for the dissipation and designating the constant of proportionality as $3A$, we find:

$$-(3/2)\rho U d(u'^2)/dx = 3A\rho u'^3/L \qquad (11.1)$$

or

$$L d(U/u')/dx = A. \qquad (11.2)$$

Integrating:

$$U/u' - U/u_0' = A\int_{x_0}^{x} dx/L \qquad (11.3)$$

where U/u_0' is the value of U/u' at $x=x_0$. This equation has been found to give a very good representation of the experimental data. The essential features of the derivation were given by Taylor. To evaluate the integral, L must be known as a function of x. Taylor's first proposal was to assume that L is independent of x and proportional to the mesh M of the grid giving rise to the turbulence. If L is constant,

$$U/u' - U/u_0' = A(x-x_0)/L \qquad (11.4)$$

giving a linear variation of U/u' with x. Assuming $L/M=k$, Taylor found values of A/k for data from various sources varying between 1.03 and 1.32.

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[14] Dryden, H. L., Schubauer, G. B., Mock, W. C., Jr., and Skramstad, H. K., *Measurements of intensity and scale of wind-tunnel turbulence and their relation to the critical Reynolds number of spheres*, Tech. Rept. Nat. Adv. Comm. Aeron. No. 581 (1937).

When measured values of L became available it was found that L increased as x increased, the results being represented empirically within the accuracy of the measurements by the relation $L=L_0+c(x-x_0)$, whence

$$U/u' - U/u_0' = (A/c) \log_e [1 + c(x - x_0)/L_0]. \tag{11.5}$$

Taylor[15] found values of A for data from various sources varying between 0.43 and 0.19.

Further study suggests another relation for the variation of L with x. A discussion of the general theory will be deferred until section 17 and the question discussed on purely dimensional considerations. If one assumes that du'/dt, the rate of change of intensity, and dL/dt, the rate of change of scale, are determined solely by the values of L and u', i.e. that viscosity and upstream conditions have no influence, it follows from dimensional reasoning that

$$Ld(1/u')/dt = A \quad \text{and} \quad (1/u')dL/dt = B \tag{11.6}$$

or

$$Ld(U/u')/dx = A \quad \text{and} \quad (U/u')dL/dx = B \tag{11.7}$$

where A and B are numerical constants. The first equation of each pair is the same as equation (11.2); the second is a new relation.

Integration of equations (11.6) and (11.7) leads to the relations:

$$\frac{u_0'}{u'} = \left[1 + \frac{(A+B)u_0'(x - x_0)}{L_0 U}\right]^{A/(A+B)} \tag{11.8}$$

and

$$\frac{L}{L_0} = \left[1 + \frac{(A+B)u_0'(x - x_0)}{L_0 U}\right]^{B/(A+B)} \tag{11.9}$$

where u_0' and L_0 are the values at $x=0$.

If it is desired to introduce a reference dimension pertaining to the dimensions of the grid producing the disturbance, this may be done, but according to equations (11.8) and (11.9) any dimension may be used and the decay does not depend on its value. The mesh distance M is often used but certain results reported by von Kármán[16] show that if M/d is not too small, the use

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[15] Taylor, G. I., *Some recent developments in the study of turbulence*, Proc. Fifth Inter. Congr. Appl. Mech., Cambridge, Mass., 294 (1938). See later detailed report of measurements in Hall, A. A., *Measurements of the intensity and scale of turbulence*, Rept. and Memo. No. 1842, Aeronautical Research Committee, Great Britain (1938).

[16] Kármán, Th. von, *Some remarks on the statistical theory of turbulence*, Proc. Fifth Inter. Congr. Appl. Mech., Cambridge, Mass., 1938, p. 347. The grid dimensions are not given in

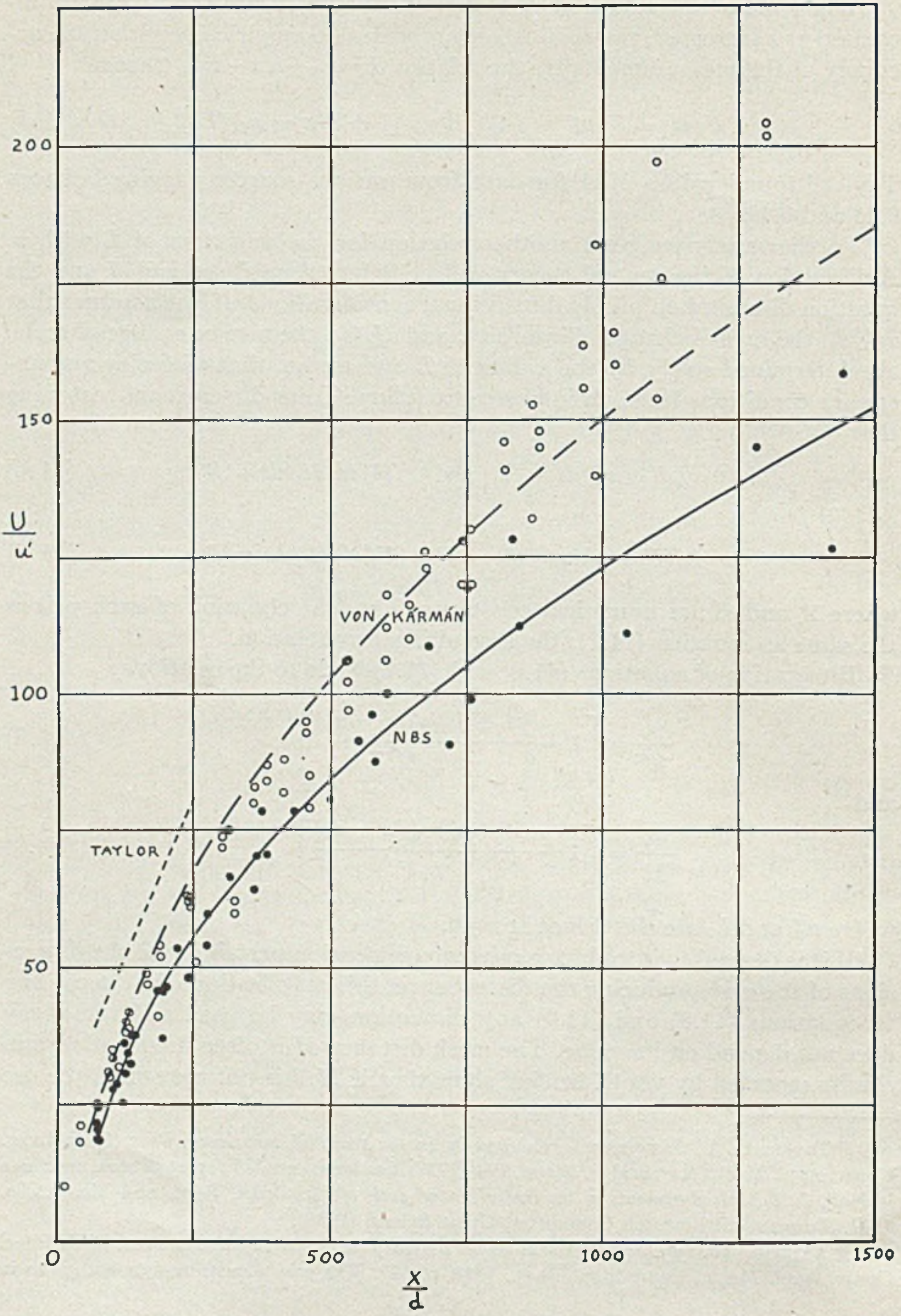

FIG. 1. The turbulent fluctuation u' behind a grid of wires of diameter d as a function of distance x from the grid.

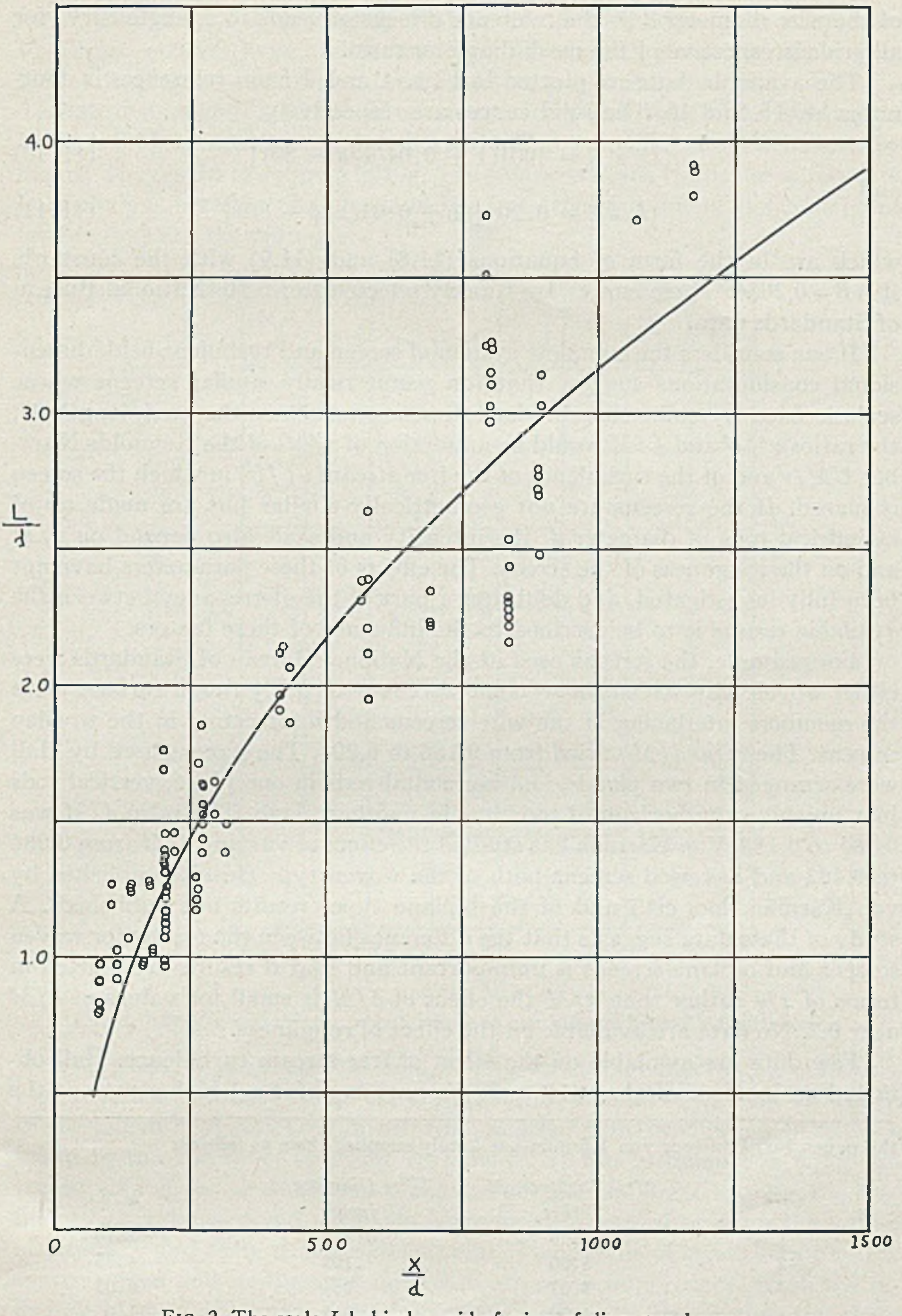

FIG. 2. The scale L behind a grid of wires of diameter d as a function of distance x from the grid.

of the wire diameter d as the reference dimension leads to a single curve for all grids irrespective of the mesh-diameter ratio.

The available data are plotted in Figs. 1 and 2 from references in footnotes 14, 15, and 16. The solid curves are respectively

$$(U/u')^2 = 400[(1 + 0.04(x/d - 80)] \qquad (11.10)$$

and

$$(L/d)^2 = 0.264[(1 + 0.04(x/d - 80)] \qquad (11.11)$$

which are in the form of equations (11.8) and (11.9) with the constants $A = B = 0.2056$. These curves are frankly selected to fit the National Bureau of Standards data.

If one considers the complete system of screen and turbulent field, dimensional considerations suggest that for geometrically similar screens whose scale is fixed by some characteristic dimension, such as the mesh length M, the ratios u'/U and L/M would be a function of x/M, of the Reynolds Number UM/ν and of the turbulence of the free stream u_1'/U, in which the screen is placed. If the screens are not geometrically similar but are made up of cylindrical rods of diameter d, the intensity and scale also depend on d/M and on the roughness of the screen. The effects of these parameters have not been fully investigated, and doubtless a part of the discrepancy between the available results is to be ascribed to the influence of these factors.

For example, the screens used at the National Bureau of Standards were either woven wire screens or wooden screens with fairly rough surfaces with the members interlacing in the wire screens and intersecting in the wooden screens. The ratio d/M varied from 0.186 to 0.201. The screens used by Hall were arranged in two planes, i.e., horizontal rods in one plane, vertical rods just touching the horizontal rods but in another plane. The ratio d/M was 0.184 to 0.188. Von Kármán has studied the effect of varying d/M from 0.086 to 0.462 and has used screens both of the woven type (results published by von Kármán, loc. cit.) and of the biplane type (results not published). A study of these data suggests that the difference between the results for woven screens and biplane screens is unimportant and that if results are plotted in terms of x/d rather than x/M the effect of d/M is small for values of d/M near 0.2. No data are available on the effect of roughness.

Few data are available on the effect of free stream turbulence. Hall obtained an increase of about 10 to 20 percent in u' for a 1-inch screen at the

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the paper, but Professor von Kármán has kindly supplied them as follows:

Grid	*Mesh Distance, M inches*	*Wire Diameter, d inches*	*M/d*
1	4.96	0.230	2.16
2	5.00	.105	4.75
3	5.07	.084	6.03
4	4.99	.043	11.6

same value of x/M by increasing the free stream turbulence from 0.2 percent to 1.3 percent. We have had the opportunity of making some measurements behind the same 1-inch screen used in the measurements described in NACA Technical Report No. 581 in an airstream for which the free stream turbulence is 0.03 percent as compared with 0.85 percent for the older measurements. The results are shown in Fig. 3 as compared with Hall's measurements. It is obvious that the turbulence of the free stream is one of the controlling factors, but not the only one.

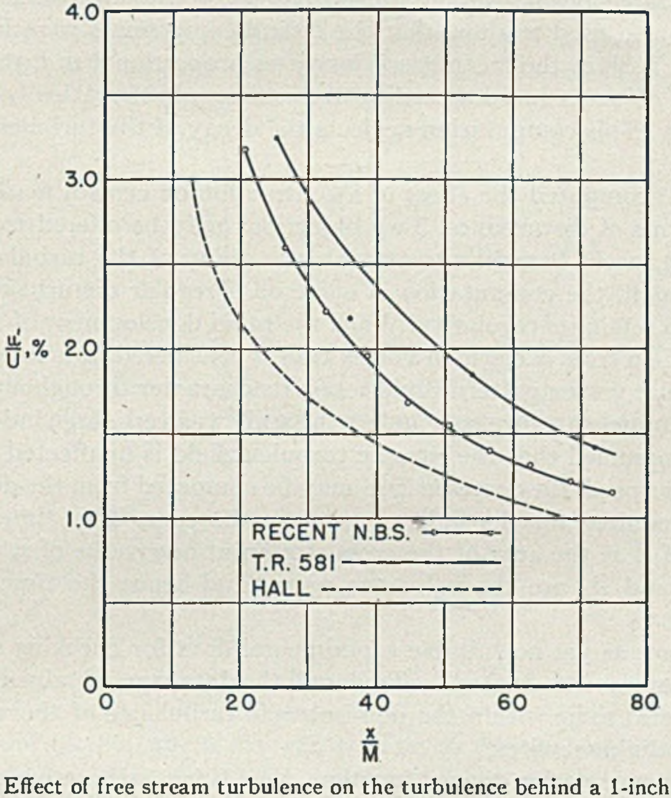

FIG. 3. Effect of free stream turbulence on the turbulence behind a 1-inch screen.

The study of the turbulent field behind screens as affected by numerous parameters is of interest from the standpoint of a study of screens. However, the turbulent field may be regarded from another point of view, i.e. in relation solely to the theory of isotropic turbulence. If the turbulence is truly isotropic, and if its characteristics can be adequately described by the two quantities, intensity and scale, its behavior can depend only on the values of intensity and scale at some given point. The details of construction of the source screen and its distance upstream are of no importance. Even the influence of the turbulence of the free stream should be absorbed in the given values of u' and L at some one point. The decay of isotropic turbulence is considered from this point of view in section 17.

12. Effect of contraction. The behavior of turbulence in a contracting stream is of interest in connection with the flow in the entrance cone of a wind tunnel. Prandtl[17] suggested that the longitudinal components of the fluctuations were reduced in the ratio of 1 to l where l is the ratio of the entrance area to the exit area of the cone. This result was derived on the assumption that the gain in energy is the same for all filaments traversing the cone. The same result was obtained from the Helmholtz vortex theorem, which was also used to show that the lateral components were increased in the ratio $\sqrt{l}$. Since the mean speed increases proportional to l, the values of u'/U and v'/U are reduced according to this theory in the ratios $1/l^2$ and $1/\sqrt{l}$ respectively. This computation neglects the decay of the turbulence because of viscosity.

Taylor[18] computed the effect of a contraction on certain mathematically defined forms of disturbance. Two objections may be offered to this treatment. First, as in Prandtl's treatment, the decay of the turbulence is neglected. Second, the computation is made on a regular disturbance which is assumed to retain its regularity. When the rapid development of an isotropic turbulent field from a Kármán vortex trail is considered, it is hard to believe that a regular vortex pattern could retain its character throughout the length of a wind tunnel entrance cone unless the scale was very large indeed.

If it is assumed that the istropic turbulent field is unaffected by changes in the mean speed, the decrease in u' may be computed from the decay during the time required for the fluid to traverse the cone. This time interval is $\int_{x_0}^{x_1} dx/U$. If A is the area of the cross section at any value of x, $UA = U_0 A_0$ where U_0 and A_0 are the values at $x = x_0$, and hence the time interval is $\int_{x_0}^{x_1} A\, dx/U_0 A_0$.

There are as yet no suitable experimental data for checking any theory. In the measurements quoted by Taylor all the data were obtained sufficiently close to a grid to lie within the non-isotropic turbulence of the vortex trails from the individual wires.

13. The correlation tensor function. Von Kármán[19] introduced the correlation tensor function in the statistical theory of turbulence as a generalization of the particular correlation coefficients discussed by Taylor. The correlation coefficients between any component of the speed fluctuation at a given point and any component of the speed fluctuation at another point

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[17] Prandtl, L., *Herstellung einwandfreier Luftströme (Windkanäle)*, Handbuch der Experimentalphysik, F. A. Barth, Leipzig, 1932, Vol. 4, Part 2, p. 73.

[18] Taylor, G. I., *Turbulence in a contracting stream*, Z. angew. Math. u. Mech. **15**, 91 (1935).

[19] Kármán, Th. von, and Howarth, L., *On the statistical theory of isotropic turbulence*, Proc. Roy. Soc. London, Ser. A, **164**, 192 (1938).

form a tensor. If one point is held fixed and the other varied, the tensor varies as a function of the coordinates of the variable point with respect to the fixed point. We may speak of this function as the correlation tensor function.

In isotropic turbulence the correlation tensor has spherical symmetry and the several components are functions only of the distance r between the two points, and of the time t. Denote by u_1, v_1, w_1 and u_2, v_2, w_2 the components of the velocity fluctuations at the two points P_1 and P_2 having coordinates $(x_1, 0, 0)$ and $(x_2, 0, 0)$ respectively. Suppose that $\overline{u_1^2}$, $\overline{v_1^2}$, $\overline{w_1^2}$, which by isotropy are equal, are independent of position and equal to $\overline{u_2^2}$. Then $\overline{u_2^2}=\overline{v_2^2}=\overline{w_2^2}=\overline{u^2}$.

The correlation coefficients $\overline{v_1v_2}/\overline{u^2}$ and $\overline{w_1w_2}/\overline{u^2}$ will be identical because of isotropy and will be some particular function of the distance r between

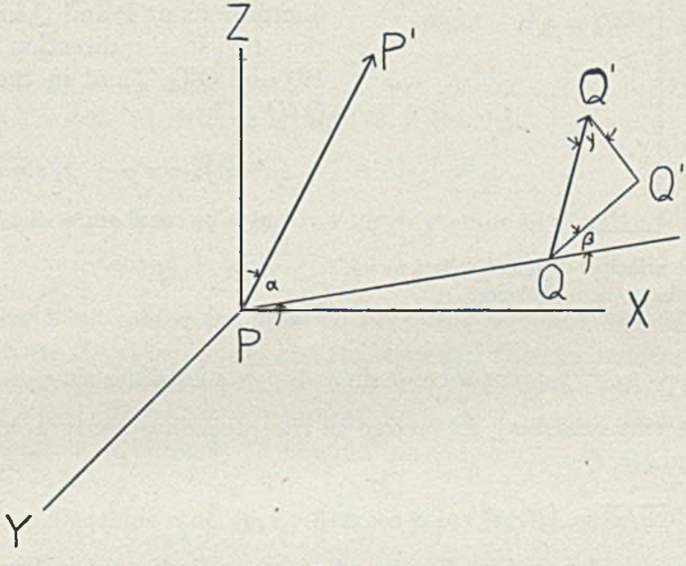

FIG. 4.

P_1 and P_2 and of the time t, say $g(r, t)$. The correlation coefficient $\overline{u_1u_2}/\overline{u^2}$ will also be a function of r and t, say $f(r, t)$. The correlation coefficients $\overline{u_1v_2}/\overline{u^2}$, $\overline{u_1w_2}/\overline{u^2}$, $\overline{v_1u_2}/\overline{u^2}$, $\overline{v_1w_2}/\overline{u^2}$, $\overline{w_1u_2}/\overline{u^2}$ and $\overline{w_1v_2}/\overline{u^2}$ can be shown to be zero. Thus if the Y and Z axes are rotated about the X axis through 180°, the absolute values of all components are unchanged but the signs of the v and w components are reversed. Denoting values referred to the new axes by capital letters, $U_1=u_1$, $U_2=u_2$, $V_1=-v_1$, $V_2=-v_2$, $W_1=w_1$, $W_2=-w_2$, so that, for example, $\overline{U_1V_2}=-\overline{u_1v_2}$. But by isotropy, the value of any function of the components is unchanged by rotation of the axes, and therefore $\overline{U_1V_2}=\overline{u_1v_2}$. To satisfy both relations $\overline{u_1v_2}$ must equal zero. Similarly for the other terms containing u_1 or u_2. By reflection in the XZ plane $\overline{v_1w_2}$ and $\overline{w_1v_2}$ may likewise be shown to be zero.

The correlation coefficient for components of the fluctuations in any arbitrary directions at any two points may be expressed in terms of the functions $f(r, t)$ and $g(r, t)$ and the geometrical parameters. Consider any two points P and Q and components of the fluctuations p in the direction PP′ at P and q in the direction QQ′ at Q. (Fig. 4). Denote by QQ″ the orthogonal projection of QQ′ on the plane PP′Q; by α, β, and γ the angles P′PQ, $\pi -$PQQ″, and QQ′Q″; and by p_1, p_2, p_3 and q_1, q_2, q_3 the components of the fluctuations at P and Q in the direction PQ, in the direction normal to PQ and Q′Q″, and in the direction Q″Q′. Then

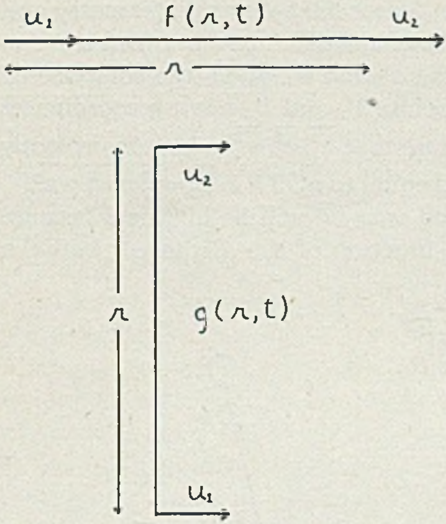

FIG. 5. The principal double correlations in isotropic turbulence.

$$p = p_1 \cos\alpha + p_2 \sin\alpha$$

$$q = q_1 \cos\beta \sin\gamma + q_2 \sin\beta \sin\gamma + q_3 \cos\gamma. \qquad (13.1)$$

Hence

$$\overline{pq} = \overline{p_1q_1} \cos\alpha \cos\beta \sin\gamma + \overline{p_2q_2} \sin\beta \sin\alpha \sin\gamma \qquad (13.2)$$

the other terms vanishing as proved in the preceding paragraph. In terms of $f(r, t)$ and $g(r, t)$

$$\overline{pq}/\overline{u^2} = [f(r, t) \cos\alpha \cos\beta + g(r, t) \sin\alpha \sin\beta] \sin\gamma. \qquad (13.3)$$

The correlations denoted by $f(r, t)$ and $g(r, t)$ are indicated in Fig. 5.

If now any two points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) and speed fluctuations with components u_1, v_1, w_1 and u_2, v_2, w_2 are considered, the nine quantities $\overline{u_1u_2}$, $\overline{u_1v_2}$, $\overline{u_1w_2}$, $\overline{v_1u_1}$, $\overline{v_1v_2}$, $\overline{v_1w_2}$, $\overline{w_1u_1}$, $\overline{w_1v_2}$, and $\overline{w_1w_2}$ are the components of a second rank tensor. Each one may be evaluated by equation (13.2) in terms of $f(r, t)$ and $g(r, t)$ with the result in tensor notation

$$R = \frac{f(r, t) - g(r, t)}{r^2} \mathbf{rr} + g(r, t)I \qquad (13.4)$$

where $\mathbf{r}$ is the vector having components $X = x_2 - x_1$, $Y = y_2 - y_1$, $Z = z_2 - z_1$

r is $|\mathbf{r}|$ and I is the unit tensor $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$.

The velocity fluctuations satisfy the equation of continuity. Hence

$$\frac{\partial u_2}{\partial x_2} + \frac{\partial v_2}{\partial y_2} + \frac{\partial w_2}{\partial z_2} = 0. \tag{13.5}$$

Multiplying by $u_1/\overline{u^2}$ which is independent of x_2, y_2, z_2 and introducing the correlation coefficients $R_{u_1u_2}$, etc. and the components X, Y, Z of $\mathbf{r}$:

$$\frac{\partial R_{u_1u_2}}{\partial X} + \frac{\partial R_{u_1v_2}}{\partial Y} + \frac{\partial R_{u_1w_2}}{\partial Z} = 0. \tag{13.6}$$

From equation (13.4)

$$R_{u_1u_2} = \frac{f-g}{r^2} X^2 + g; \qquad R_{u_1v_2} = \frac{f-g}{r^2} XY; \qquad R_{u_1w_2} = \frac{f-g}{r^2} XZ$$

whence, remembering that $X^2+Y^2+Z^2=r^2$, $\partial r/\partial X = X/r$, $\partial f/\partial X = (\partial f/\partial r)(\partial r/\partial X) = (X/r)(\partial f/\partial r)$, etc., equation (13.6) becomes

$$X[2(f-g) + r(\partial f/\partial r)] = 0. \tag{13.7}$$

The continuity equation must be true for any value of X. Hence

$$2f(r, t) - 2g(r, t) = -r\partial f(r, t)/\partial r. \tag{13.8}$$

The correlation tensor can thus be expressed in terms of a single scalar function, either $f(r, t)$ or $g(r, t)$. The function $g(r, t)$ is the correlation coefficient previously denoted by R_y. The scale $L = \int_0^\infty R_y\,dy = \int_0^\infty g\,dr$. The integral $\int_0^\infty R_x\,dx = \int_0^\infty f\,dr$ is termed the longitudinal scale L_x to distinguish it from the lateral scale L. Obviously from equation (13.8)

$$L - L_x = \tfrac{1}{2}\int_0^\infty r(\partial f/\partial r)dr = \tfrac{1}{2}\int_0^\infty x(\partial R_x/\partial x)dx. \tag{13.9}$$

Since f and g are even functions of r,

$$f = 1 + f_0''r^2/2 + \cdots \tag{13.10}$$

$$g = 1 + g_0''r^2/2 + \cdots . \tag{13.11}$$

From equation (13.8), $2f_0'' = g_0''$, whence for small values of r,

$$\begin{aligned} R &= [1 + (g_0''/2)r^2]I + [(f_0'' - g_0''/2]\mathbf{rr} \\ &= (1 + f_0''r^2)I - (1/2)f_0''\mathbf{rr}. \end{aligned} \tag{13.12}$$

We require later the second derivatives of R at $r=0$, i.e. $X=Y=Z=0$, as follows:

$$\frac{\partial^2 R_{u_1u_2}}{\partial X^2} = \frac{\partial^2 R_{v_1v_2}}{\partial Y^2} = \frac{\partial^2 R_{w_1w_2}}{\partial Z^2} = f_0'' \tag{13.13}$$

$$\frac{\partial^2 R_{u_1u_2}}{\partial Y^2} = \frac{\partial^2 R_{u_1u_2}}{\partial Z^2} \text{ and similar terms obtained by cyclic exchange } = 2f_0'' \tag{13.14}$$

$$\frac{\partial^2 R_{u_1v_2}}{\partial X \partial Y} \text{ and similar terms obtained by cyclic exchange } = -(1/2)f_0''. \tag{13.15}$$

$$\text{All others, e.g. } \frac{\partial^2 R_{u_1v_2}}{\partial X \partial Z} \text{ etc. are zero.} \tag{13.16}$$

Von Kármán points out that the correlation tensor is of the same form as the stress tensor for a continuous medium when there is spherical symmetry. In the analogy $f(r)$ corresponds to the principal radial stress at any point, $g(r)$ to the principal transverse stress, and the several R's to the stress components over planes normal to the coordinate axes. The relation between f and g given by the continuity equation corresponds to the condition for equilibrium of the stresses.

Equation (13.8) has been experimentally checked at the National Physical Laboratory.[15]

14. Correlation between derivatives of the velocity fluctuations. In further developments it will be necessary to know the mean values of the products of the derivatives of the components of the fluctuations at a given point, for example $\overline{(\partial u_1/\partial x_1)(\partial v_1/\partial y_1)}$. These mean values may readily be computed from the correlation tensor. Thus:

$$\frac{\partial(\overline{u_1 v_2})}{\partial x_1} = \overline{u^2}\,\frac{\partial(R_{u_1v_2})}{\partial x_1} = -\,\overline{u^2}\,\frac{\partial(R_{u_1v_2})}{\partial X}\,. \tag{14.1}$$

Since v_2 is not a function of x_1, this may be written

$$\overline{(\partial u_1/\partial x_1)v_2} = -\,\overline{u^2}\partial R_{u_1v_2}/\partial X. \tag{14.2}$$

Differentiating now with respect to y_2

$$\overline{\frac{\partial u_1}{\partial x_1}\frac{\partial v_2}{\partial y_2}} = \frac{\partial}{\partial y_2}\left(\overline{\frac{\partial u_1}{\partial x_1} v_2}\right) = -\,\overline{u^2}\,\frac{\partial}{\partial y_2}\left(\frac{\partial R_{u_1v_2}}{\partial X}\right) = -\,\overline{u^2}\,\frac{\partial^2 R_{u_1v_2}}{\partial X \partial Y}\,. \tag{14.3}$$

Now letting P_1 and P_2 coincide,

$$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} = -\,\overline{u^2}\left(\frac{\partial^2 R_{uv}}{\partial X \partial Y}\right)_{X = Y = 0}. \tag{14.4}$$

The limiting value of the second derivative has previously been computed (equation (13.15)), whence

$$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} = \frac{f_0''}{2}\,\overline{u^2}. \tag{14.5}$$

By similar reasoning it may be shown that

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \overline{\left(\frac{\partial v}{\partial y}\right)^2} = \overline{\left(\frac{\partial w}{\partial z}\right)^2} = -\overline{u^2} f_0'' \tag{14.6}$$

$$\begin{aligned}\overline{\left(\frac{\partial u}{\partial y}\right)^2} = \overline{\left(\frac{\partial u}{\partial z}\right)^2} = \overline{\left(\frac{\partial v}{\partial x}\right)^2} = \overline{\left(\frac{\partial v}{\partial z}\right)^2} = \overline{\left(\frac{\partial w}{\partial x}\right)^2} \\ = \overline{\left(\frac{\partial w}{\partial y}\right)^2} = -2\overline{u^2} f_0\end{aligned} \tag{14.7}$$

and

$$\overline{\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}} = \overline{\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}} = \overline{\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}} = \frac{\overline{u^2}}{2} f_0''. \tag{14.8}$$

The method can be extended to derivatives of higher order.

15. Triple correlations. Von Kármán designates the mean values of the product of three components of the velocity fluctuations, two of which are taken at one arbitrary point and the third at a second arbitrary point, as triple correlations. They arise when correlation coefficients are introduced into the equations of motion. He shows that the triple correlations are components of a tensor of third rank designated T which is a function of X, Y, Z and the time. He proves that in isotropic turbulence this tensor can be expressed in terms of three functions $h(r, t)$, $k(r, t)$, and $q(r, t)$ corresponding to the correlations shown in Fig. 6, and that the development of these functions in powers of r begins with the r^3 term. The equation of continuity permits the expression of k and q in terms of h by the relations:

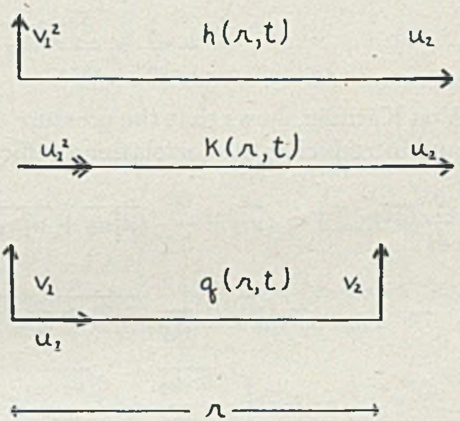

FIG. 6. The principal triple correlations in isotropic turbulence.

$$k = -2h \tag{15.1}$$

$$q = -h - (r/2)(dh/dr). \tag{15.2}$$

Thus the tensor T can be expressed in terms of a single scalar function $h(r, t)$.

16. Propagation of the correlation with time. The fluctuations are assumed to satisfy the equations of motion, namely,

$$\begin{aligned}\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + v_1\frac{\partial u_1}{\partial y_1} + w_1\frac{\partial u_1}{\partial z_1} \\ = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial y_1^2} + \frac{\partial^2 u_1}{\partial z_1^2}\right)\end{aligned} \tag{16.1}$$

and the two equations obtained by cyclic permutation.

Multiplying this equation by u_2, introducing X, Y, and Z, and taking mean values:

$$\overline{u_2 \frac{\partial u_1}{\partial t}} - \frac{\partial(\overline{u_1^2 u_2})}{\partial X} - \frac{\partial(\overline{u_1 v_1 u_2})}{\partial Y} - \frac{\partial(\overline{u_1 w_1 u_2})}{\partial Z}$$

$$= -\frac{1}{\rho}\overline{u_2 \frac{\partial p}{\partial x_1}} + \nu\left(\frac{\partial^2 \overline{u_1 u_2}}{\partial X^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Y^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Z^2}\right). \qquad (16.2)$$

By an analogous procedure, it may be shown that:

$$\overline{u_1 \frac{\partial u_2}{\partial t}} - \frac{\partial \overline{u_2^2 u_1}}{\partial X} - \frac{\partial \overline{u_2 v_2 u_1}}{\partial Y} - \frac{\partial \overline{u_2 w_2 u_1}}{\partial Z}$$

$$= -\frac{1}{\rho}\overline{u_1 \frac{\partial p}{\partial x_2}} + \nu\left(\frac{\partial^2 \overline{u_1 u_2}}{\partial X^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Y^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Z^2}\right). \qquad (16.3)$$

Von Kármán shows that the pressure terms vanish. Adding the two equations and introducing the correlation coefficients, we find

$$\frac{\partial}{\partial t}(\overline{u^2} R_{u_1 u_2}) - (\overline{u^2})^{3/2}\frac{\partial}{\partial X}(\overline{u_1^2 u_2} + \overline{u_2^2 u_1}) - (\overline{u^2})^{3/2}\frac{\partial}{\partial Y}(\overline{u_1 v_1 u_2} + \overline{u_2 v_2 u_1})$$

$$- (\overline{u^2})^{3/2}\frac{\partial}{\partial Z}(\overline{u_1 w_1 u_2} + \overline{u_2 w_2 u_1}) \qquad (16.4)$$

$$= 2\nu\overline{u^2}\left[\frac{\partial^2 \overline{u_1 u_2}}{\partial X^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Y^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Z^2}\right].$$

This equation may be expressed in terms of the functions f, g, k, q and h. Then by using the relations (15.1) and (15.2) between these functions obtained from the equation of continuity, a partial differential equation between f and h is obtained, namely

$$\frac{\partial(f\overline{u^2})}{\partial t} + 2(\overline{u^2})^{3/2}\left(\frac{\partial h}{\partial r} + \frac{4h}{r}\right) = 2\nu\overline{u^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right). \qquad (16.5)$$

This is the equation for the change of the function f with time, but it cannot be solved without some knowledge of the function h.

17. Self-preserving correlation functions. Let us suppose that the functions $f(r, t)$ and $h(r, t)$ preserve the same form as t increases, only the scale varying. Such functions will be termed "self-preserving." If L is some measure of the scale of the correlation curve, f and h will be functions of r/L only, where L is a function of t. The length L may be any measure of the scale such as the radius of curvature of the correlation curve at $r=0$ or any other designated point, the value of r for a given value of the correlation coefficient, or the quantity obtained by integration of the correlation coefficient from $r=0$ to infinity which has previously been termed the scale of the turbulence. Introducing the new variable $\psi=r/L$ and placing $(\overline{u^2})^{1/2}=u'$ in equation (16.5), we obtain

$$\frac{fL}{u'^3}\frac{du'^2}{dt}-\frac{1}{u'}\frac{dL}{dt}\psi\frac{\partial f}{\partial\psi}+2\left(\frac{\partial h}{\partial\psi}+4\frac{h}{\psi}\right)=\frac{2}{N}\left(\frac{\partial^2 f}{\partial\psi^2}+\frac{4}{\psi}\frac{\partial f}{\partial\psi}\right) \qquad (17.1)$$

where N is the Reynolds Number of the turbulence $u'L/\nu$. Since the coefficient of the third term is a numerical constant, the functions f and h will be functions of ψ and t alone only if the coefficients of the other terms are also numerical constants. This requires that

$$\frac{L}{u'^3}\frac{du'^2}{dt}=-L\frac{d(1/u')}{dt}=-A \qquad (17.2)$$

$$\frac{1}{u'}\frac{dL}{dt}=B \qquad (17.3)$$

$$\frac{u'L}{\nu}=N_0 \qquad (17.4)$$

where A, B, and N_0 are independent of u', L, and t. It is readily shown that these relations are consistent only if $A=B$ and that the solutions are

$$\frac{1}{u'^2}-\frac{1}{u_0'^2}=\frac{2A}{N_0\nu}t \qquad (17.5)$$

$$L^2-L_0^2=2AN_0\nu t \qquad (17.6)$$

where u_0' and L_0 are the values for $t=0$ and $u_0'L_0/\nu=N_0$.

These equations are in the form of equations (11.8) and (11.9) with $A=B$ and agree well with the formulation of the experimental data represented by equations (11.10) and (11.11) with $2AN_0\nu/L_0^2=2Au_0^2/N_0\nu=2A\sqrt{u_0^2}/L_0=0.04$, corresponding to $t=0$ at a distance of 80 wire diameters from the grid. The constant A is equal to 0.2056 when L is defined as $\int_0^\infty R_y dy$.

For self-preserving turbulence equation (16.5) becomes

$$-Af-A\psi(\partial f/\partial\psi)+2(\partial h/\partial\psi+4h/\psi)=(2/N_0)\left[(\partial^2 f/\partial\psi^2+(4/\psi)(\partial f/\partial\psi)\right]. \qquad (17.7)$$

This equation determines the shape of the correlation curve. Von Kármán[19] discusses the shape when the function h is neglected. The shape depends on the Reynolds Number N_0 of the turbulence. The shape also depends on the constant A but closer examination shows that A is always associated with L

and is dependent on the method of defining L. If ψ is set equal to rA/L instead of r/L, and the length L/A is used instead of L in the definition of the Reynolds Number of the turbulence, the A disappears from equation (17.7). Whether the values of L defined by $\int g dr$ will yield the same values of A for all shapes of correlation curves described by (17.7) cannot be definitely answered.

Approximate solutions of (17.7) are not easy since it turns out that f varies with N_0 in such a manner that, for small values of ψ at least, the term on the right-hand side is of the order of unity.

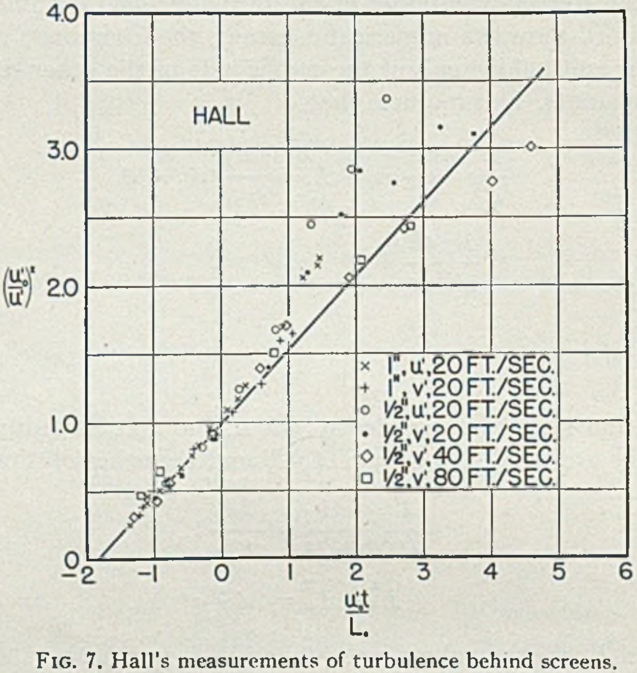

FIG. 7. Hall's measurements of turbulence behind screens.

According to this suggested theory, the shape is self-preserving and the Reynolds Number remains constant during the decay of a given turbulent field. The scale approaches very large values as the intensity approaches very small values. The length λ (which is discussed in section 18) is proportional to L. For different values of the Reynolds Number of the turbulence the constant of proportionality varies inversely as the square root of the Reynolds Number. Likewise the shape of the correlation curve varies with the Reynolds number of the turbulence.

Equation (17.6) shows the same functional relation between the scale and the time as given by Prandtl at the Turbulence Symposium as a result of his analysis of photographs of the decay of isotropic turbulence.

Von Kármán also discusses the case in which the assumption is made that the self-preserving feature applies only to large values of ψ and the Reynolds Number N_0 is sufficiently large that the right-hand term of (17.7) can be neglected. In this case (17.2) and (17.3) are obtained without (17.4) and the solution is identical with that given by (11.8) and (11.9) of section 11. The theoretical equations (17.5) and (17.6) do not involve either U

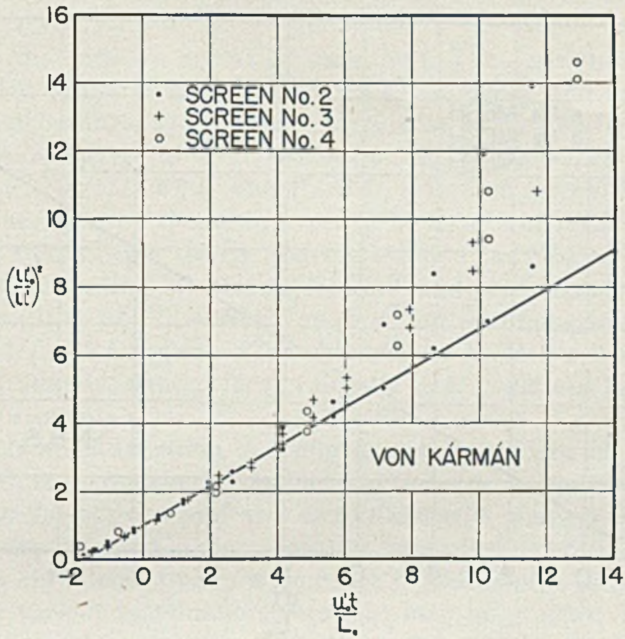

FIG. 8. Von Kármán's measurements of turbulence behind screens.

or M explicitly. However, for comparison with experimental data, they may be written as follows:

$$\left(\frac{u_0'}{u'}\right)^2 = 1 + \frac{2Au_0't}{L_0} = 1 + 2A\,\frac{u_0'}{U}\,\frac{(x - x_0)}{L_0} \tag{17.8}$$

$$\left(\frac{L}{L_0}\right)^2 = 1 + \frac{2Au_0't}{L_0} = 1 + 2A\,\frac{u_0'}{U}\,\frac{(x - x_0)}{L_0}\,. \tag{17.9}$$

Both u_0' and L_0 should be known, but unfortunately L_0 was not measured in all of the experiments.

Figures 7, 8, and 9 show the results of Hall, of von Kármán, and of the author and his associates (designated NBS) plotted in a manner to facilitate comparison with equation (17.8).

The reference position x_0 has been taken as 40 times the mesh length except for von Kármán's results for which x_0 was taken as 212.5 times the rod diameter (equivalent to $x_0/M=40$ for $d/M=0.188$). In the absence of definite information as to L_0, L_0/M was assumed equal to 0.29 except for von Kármán's results for which L_0/d was assumed to be 1.54 (equivalent to $L_0/M=0.29$ for $d/M=0.188$). The value of u_0' was determined by interpolation from the observations of each experimenter near $x/M=40$, giving the following results.

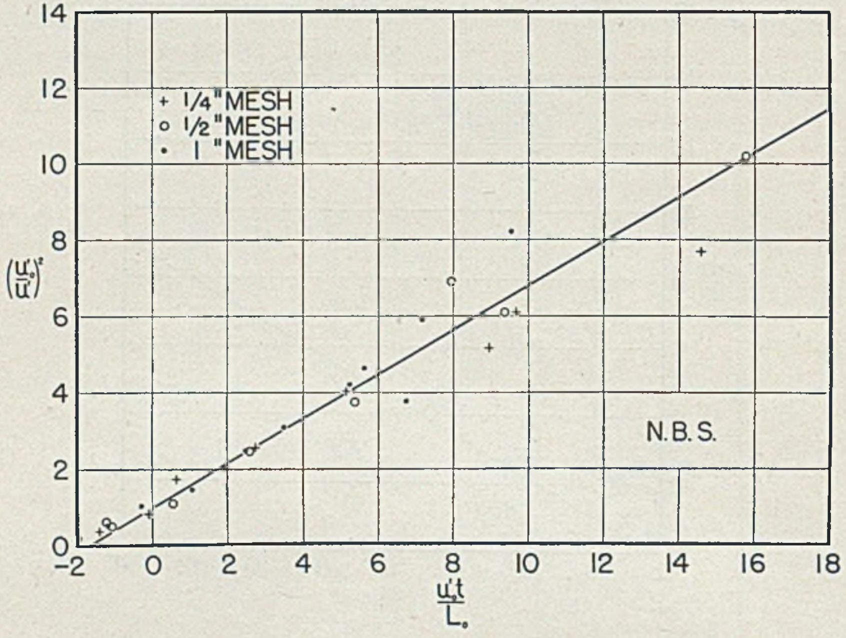

FIG. 9. NBS measurements of turbulence behind screens.

Experimenter	*Mesh Inches*	*Rod Diameter Inches*	*Air Speed ft/sec*	u_0'/U	*Remarks*
Hall	1.0	0.188	20	0.0146	
	0.5	.092	20	.0144	
	0.5	.092	40	.0152	
	0.5	.092	80	.0174	
von Kármán	0.5	0.105	38 and 54	.0201	Screen 2
	0.5	.084	38 and 75	.0201	Screen 3
	0.5	.043	38 and 75	.0299	Screen 4
NBS	0.25	0.050	20–70	0.0250	NACA Tech. Rept. 581
	0.5	.096	20–70	.0221	NACA Tech. Rept. 581
	1.0	.196	20–70	.0224	NACA Tech. Rept. 581
	1.0	0.196	30	0.0188	Recent tests
	1.0	.196	70	.0173	Recent tests

The values of u_0'/U range from 0.0144 to 0.0299; presumably the differences are due mainly to the factors discussed in section 11, although systematic errors may be partly responsible.

In each figure, equation (17.8) with constant A equal to 0.29 is plotted as a straight line. Most of the points would be better fitted by a curve of increasing slope with increasing time. It thus appears that equation (11.9) with $(A+B)/A$ having some value between 1 and 2 fits the experimental data better than (17.8).

However, the data are not at all consistent. The departures are largest for the smaller values of u'/U. In Hall's experiments, the results on the $\frac{1}{2}$-inch screen show little systematic departure at 40 and 80 ft/sec, whereas those on the same screen at 20 ft/sec and on the 1-inch screen begin to rise above the line at $u_0't/L_0=0.5$. Von Kármán's data on screen 2 at 38 ft/sec lie near the line; those on the same screen at 54 ft/sec and on screens 3 and 4 at 38 and 75 ft/sec begin to rise above the line at $u_0't/L_0=4.0$. The older results of the author and his associates, while scattered, agree with the line within 12 percent to $u_0't/L_0=18$; the more recent results begin to rise above the line at $u_0't/L_0=0.5$ and are in fair agreement with Hall's data on a 1-inch screen. Unfortunately, data at large values of x/M could not be obtained in the recent experiments.

Thus, even when attention is confined to the behavior of the isotropic turbulent field, there remain discrepancies in the experimental data such that no definite conclusions can be drawn as to the merits of any theory. Further experiments are required under carefully controlled conditions in an air stream of low turbulence over a wide range of values of x/M and with due regard to the various systematic errors that may be present. These experiments would be of the greatest value if the scale were also measured.

18. The length λ. Relation between λ and L. The general expression for the mean rate of dissipation in the flow of a viscous fluid is:

$$W = \mu\left\{2\overline{\left(\frac{\partial u}{\partial x}\right)^2} + 2\overline{\left(\frac{\partial v}{\partial y}\right)^2} + 2\overline{\left(\frac{\partial w}{\partial z}\right)^2} + \overline{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2} + \overline{\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2} + \overline{\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2}\right\} \tag{18.1}$$

where μ is the viscosity.

For isotropic turbulence this becomes:

$$\frac{W}{\mu} = 6\overline{\left(\frac{\partial u}{\partial x}\right)^2} + 6\overline{\left(\frac{\partial u}{\partial y}\right)^2} + 6\,\overline{\frac{\partial v}{\partial x}\,\frac{\partial u}{\partial y}} \tag{18.2}$$

which, from the relations given in section 14, reduces to:

$$W = -\,7.5\mu\,\overline{u^2}g_0'' = 7.5\mu\,\overline{(\partial u/\partial y)^2}. \tag{18.3}$$

But g_0'' is defined by:

$$g_0'' = -2 \lim_{r \to 0} \left(\frac{1-g}{r^2} \right) \tag{18.4}$$

and has the dimensions of the reciprocal of the square of a length. Let

$$g_0'' = -2/\lambda^2 \tag{18.5}$$

the factor 2 being introduced to conform to Taylor's definition of λ. Then

$$W = 15\mu\overline{u^2}/\lambda^2. \tag{18.6}$$

The length λ may be interpreted in several ways. Equation (18.6) may be considered a definition, λ being regarded roughly as a measure of the diameters of the smallest eddies which are responsible for the dissipation of energy. Or, since $1/\lambda^2 = \lim_{r\to 0} (1-g)/r^2 = \lim_{Y\to 0} (1-R_y)Y^2, \lambda^2$ is a measure of the radius of curvature of the R_y curve at $Y=0$. Or, if a parabola is drawn tangent to the R_y curve at $Y=0$, this parabola cuts the axis at the point $Y=\lambda$.

Since $W = -(3/2)\rho(d\overline{u^2}/dt)$, the decay law may be written:

$$d\overline{u^2}/dt = -10\nu\overline{u^2}/\lambda^2. \tag{18.7}$$

This result can also be derived directly from equation (16.5) as shown by von Kármán.

By comparing this expression for the decay law with that previously given (equation 17.2), namely

$$d\overline{u^2}/dt = -Au'^3/L \tag{18.8}$$

it is seen that

$$Au'/L = 10\nu/\lambda^2 \tag{18.9}$$

or, since $u'L/\nu = N_0$

$$\lambda^2/L^2 = 10/AN_0. \tag{18.10}$$

Introducing the experimental value of A,

$$\lambda/L = 6.97/\sqrt{N_0}. \tag{18.11}$$

A similar relation holds for λ/L_x where L_x is the longitudinal scale. If the Reynolds Number is formed from L_x, the numerical constant is approximately 4.93.

During the decay of self-preserving turbulence N_0 is constant and λ is proportional to L but the constant of proportionality varies inversely as $\sqrt{N_0}$ for turbulent fields of different Reynolds Number.

Although it cannot be expected on physical grounds that these relations hold at very low values of N_0, there is no experimental evidence of any departure from equations (18.7) and (18.8) for values of N_0 as low as 10. There seems to be no difficulty in drawing correlation curves for which λ is greater than L, but no such experimental curves have been measured. However, in an example quoted by Taylor,[15] λ/L is as great as 0.86.

19. The spectrum of turbulence, relation between spectrum and correlation. The description of turbulence in terms of intensity and scale resembles the description of the molecular motion of a gas by temperature and mean free path. A more detailed picture can be obtained by considering the distribution of energy among eddies of different sizes, or more conveniently the distribution of energy with frequency. Just as a beam of white light may be separated into a spectrum by the action of a prism or grating, the electric current produced by a hot wire anemometer subjected to the speed fluctuations may be analyzed by means of electric filters into a spectrum.

The mean value of $\overline{u^2}$ may be regarded as made up of a sum of contributions $\overline{u^2}F(n)dn$, where $F(n)$ is the contribution from frequencies between n and $n+dn$ and $\int_0^\infty F(n)dn=1$. The curve of $F(n)$ plotted against n is the spectrum curve. According to the proof given by Rayleigh and quoted by Taylor[20]

$$F(n) = 2\pi^2 \lim_{T\to\infty} (I_1^2 + I_2^2)/T \tag{19.1}$$

where T is a long time and

$$\begin{aligned} I_1 &= (1/\pi)\int_0^T u \cos 2\pi nt\, dt \\ I_2 &= (1/\pi)\int_0^T u \sin 2\pi nt\, dt. \end{aligned} \tag{19.2}$$

When the fluctuations are superposed on a stream of mean velocity U and are very small in comparison with U, the changes in u at a fixed point may be regarded as due to the passage of a fixed turbulent pattern over the point, i.e., it may be assumed that

$$u = \phi(t) = \phi(x/U) \tag{19.3}$$

where x is measured upstream at time $t=0$ from the fixed point. The correlation R_x between the fluctuations at the times t and $t+x/U$ is defined by

$$R_x = \frac{\overline{\phi(t)\phi(t + x/U)}}{\overline{u^2}}. \tag{19.4}$$

It can be shown[20] that

$$\int_{-\infty}^{\infty} \phi(t)\phi(t + x/U)dt = 2\pi^2 \int_0^\infty (I_1^2 + I_2^2) \cos (2\pi nx/U)dn \tag{19.5}$$

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[20] Taylor, G. I., *The spectrum of turbulence*, Proc. Roy. Soc. London, Ser. A, **164**, 476 (1938).

or, substituting for $I_1^2+I_2^2$ its value in terms of $F(n)$,

$$R_x = \int_0^\infty F(n) \cos (2\pi nx/U)dn \tag{19.6}$$

and

$$F(n) = (4/U) \int_0^\infty R_x \cos (2\pi nx/U)dx \tag{19.7}$$

In other words, the correlation coefficient R_x and $UF(n)/\sqrt{8\pi}$ are Fourier transforms. If either is measured, the other may be computed. R_x is the function denoted by f in section 13. The length λ, which was defined in terms of the function g or R_y, is related to R_x by the equation:

$$1/\lambda^2 = 2 \lim_{x\to 0} (1 - R_x)/x^2. \tag{19.8}$$

When n and x are small, $\cos (2\pi nx/U)$ in (19.6) may be approximated by $1-2\pi^2x^2n^2/U^2$. Hence

$$1/\lambda^2 = (4\pi^2/U^2) \int_0^\infty n^2F(n)dn. \tag{19.9}$$

If the turbulence is self-preserving, the shape of the correlation curve is a function of the Reynolds Number of the turbulence. Hence the spectrum curve is also a function of the Reynolds Number of the turbulence. Introducing the longitudinal scale L_x ($L_x=\int_0^\infty R_x dx$) in equation (19.9),

$$\frac{L_x^2}{\lambda^2} = 4\pi^2 \int_0^\infty \left(\frac{nL_x}{U}\right)^2 \frac{UF(n)}{L_x}\, d\left(\frac{nL_x}{U}\right) \tag{19.10}$$

and in equation (19.7),

$$\frac{UF(n)}{L_x} = 4 \int_0^\infty R_x \cos \frac{2\pi nL_x}{U} \frac{x}{L_x}\, d\left(\frac{x}{L_x}\right) \tag{19.11}$$

both of which are expressed in terms of the non-dimensional variables $UF(x)/L_x$, nL_x/U, x/L_x, λ/L_x, and R_x. The mean speed U enters only in fixing the frequency scale.

Typical spectrum curves determined experimentally[21,22] are shown in Fig. 10. Studies of the relation between the spectrum and the correlation curve have been given by Taylor.[20]

From equation (19.10) it may be inferred that if the curve of $UF(n)/L_x$ vs.

[21] Simmons, L. F. G., and Salter, C., *An experimental determination of the spectrum of turbulence*, Proc. Roy. Soc. London Ser. A, **165**, 73 (1938).

[22] Dryden, H. L., *Turbulence investigations at the National Bureau of Standards*, Proc. Fifth Inter. Congr. Appl. Mech., Cambridge, Mass., 1938, p. 362.

nL_x/U is independent of U, L_x/λ should also be independent of U, which is contrary to the known dependence of L_x/λ on the Reynolds Number of the turbulence. Equation (19.9) shows that the value of λ is determined largely by the values of $F(n)$ at large values of n. The NPL measurements in Fig. 10

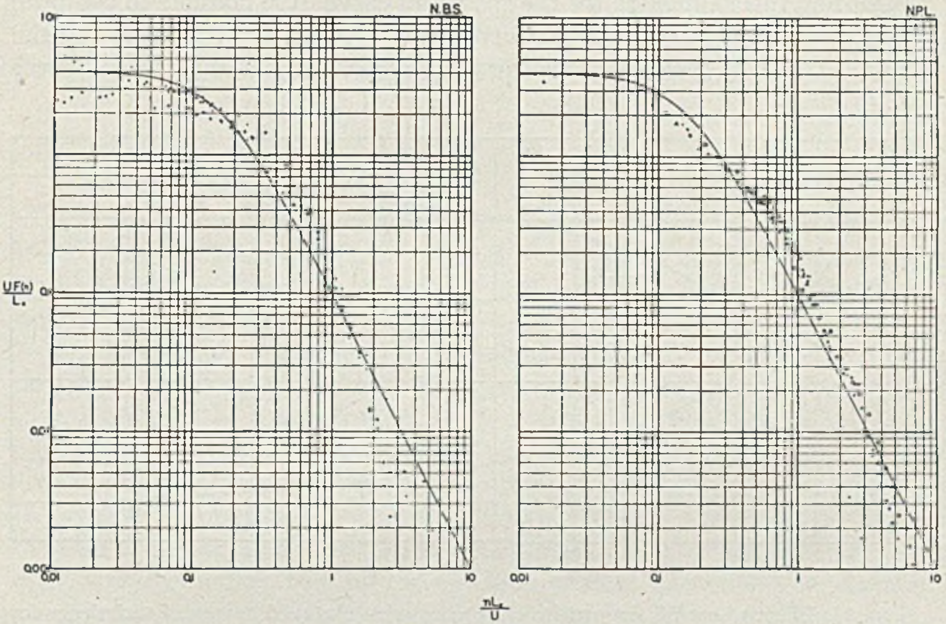

FIG. 10. Comparison of National Bureau of Standards and National Physical Laboratory measurements of the spectrum of turbulence, plotted non-dimensionally.

At left, NBS values 40(·) and 160(+) inches behind 1-inch mesh screen at 40 ft/sec.

At right, NPL values of ($F(n)$ from Table II of reference 21, L_x from reference 20) 82 inches behind 3-inch mesh screen at 15(·), 20(×), 25(+), 30(△), and 35(□) ft/sec.

The reference curve in each case is the curve

$$\frac{UF(n)}{L_x} = \frac{4}{1 + \frac{4\pi^2 n^2 L_x^2}{U^2}}$$

where U is the mean speed, L_x is the integral $\int_0^\infty R_x dx$, R_x is the correlation between the fluctuations at two points separated by the distance x in the direction of flow, n is the frequency, and $F(n)$ is the fraction of the total energy of the turbulence arising from frequencies between n and $n+dn$.

show clearly this dependence of the spectrum curve on U at high frequencies.

When the Reynolds Number of the turbulence is large, λ/L_x becomes small. Experimental measurements show that both R_x and R_y curves approach exponential curves. From integration of equation (13.9) it follows that $2L = L_x$ and equation (19.11) for the corresponding spectrum curve becomes:

$$\frac{UF(n)}{L_x} = \frac{4}{1 + 4\pi^2 n^2 L_x^2/U^2}. \qquad (19.12)$$

This is the reference curve drawn in Fig. 10. As U decreases, λ increases, and the departures at large values of nL_x/U becomes greater. The changes in the total energy of the fluctuations associated with these changes in the spectrum at high frequencies are extremely small.

Adopting this expression for the spectrum curve, it is possible to compute the effect of varying the cut-off frequency of the measuring equipment on the measured value of the energy of the fluctuations. If the equipment passes

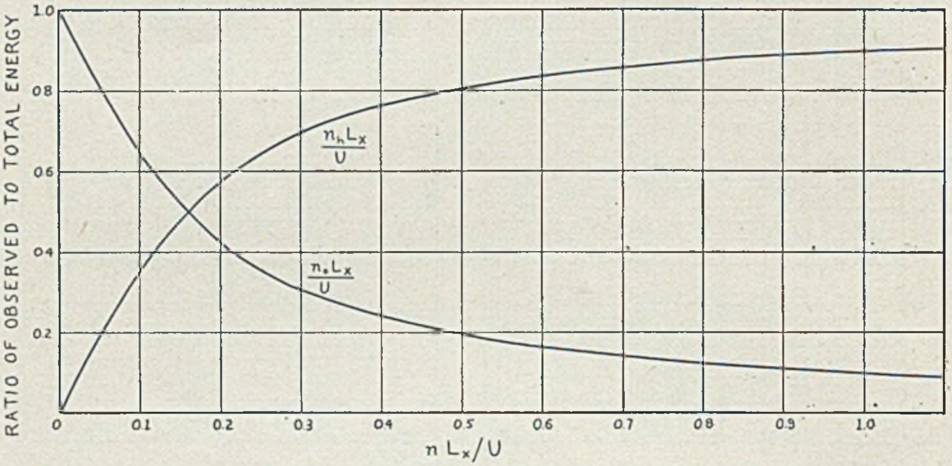

FIG. 11. Effect of cut-off frequencies of apparatus on observed energy of turbulence for spectrum given by reference curve of Fig. 9.

n_0 is the lower cut-off frequency, n_h the upper cut-off frequency, L_x the longitudinal scale, U the mean speed.

high frequencies but cuts off sharply at a lower frequency n_0, the measured total energy is

$$\tfrac{1}{2}\rho\overline{u^2}\int_{n_0L_x/U}^{\infty}\frac{4(L_x/U)dn}{1+4\pi^2n^2L_x^2/U^2}=\left(1-\frac{4}{2\pi}\tan^{-1}\frac{2\pi n_0L_x}{U}\right)\tfrac{1}{2}\rho\overline{u^2}\,. \qquad (19.13)$$

The ratio of the observed to the actual total energy is shown in Fig. 11 for various values of n_0L_x/U.

Similarly, if the equipment passes low frequencies but cuts off sharply at a higher frequency n_h, the measured total energy is $(4/2\pi)\tan^{-1} 2\pi n_hL_x/U(\tfrac{1}{2}\rho\overline{u^2})$. The ratio of the observed to the actual total energy for high frequency cut-off is also shown in Fig. 10.

The fact that the correlation and spectrum curves are of the exponential type has been interpreted[22] as meaning that turbulence is a generalized chance phenomenon, as nearly chance as a continuous curve can be and retain its continuity.

20. Fluctuating pressure gradients. In theories of the effect of turbulence on transition in boundary layers, it is desired to know the value of the root-mean-square pressure gradients, i.e., $\overline{(\partial p/\partial x)^2}$, $\overline{(\partial p/\partial y)^2}$, and $\overline{(\partial p/\partial z)^2}$. Taylor has shown[8] that

$$\sqrt{\overline{(\partial p/\partial x)^2}} = 2\sqrt{2}\,\rho\overline{u^2}/\lambda. \tag{20.1}$$

Combining this with the relation (18.11), i.e., $\lambda/L = 6.97/\sqrt{(\overline{u^2})^{1/2}L/\nu}$

$$\sqrt{\overline{\left(\frac{\partial p}{\partial x}\right)^2}} = \frac{2\sqrt{2}\,\rho\overline{u^2}}{6.97L}\sqrt{\frac{(\overline{u^2})^{1/2}L}{\nu}}\,. \tag{20.2}$$

The quantities $\sqrt{\overline{u^2}}$ and L occur in this expression in the combination $[(\sqrt{\overline{u^2}})/L^{1/5}]^{5/2}$. The ratio $(\sqrt{\overline{u^2}}/U)(D/L)^{1/5}$, where U is the mean speed and D the reference dimension of a body under study is known as the Taylor turbulence parameter.

21. The diffusive character of turbulence. An early experimental distinction between turbulent and non-turbulent flow was based on the observation that a filament of dye introduced into a turbulent fluid stream is rapidly diffused over the entire cross section of the stream whereas in a non-turbulent flow the filament retains its identity although it may show some waviness. It has been pointed out in section 7 that the effect of the turbulent fluctuations on the mean motion is the introduction of eddy stresses associated with the transfer of momentum by the diffusion of fluid particles. Von Kármán[23] has given a useful account of the mechanism of the diffusion of discrete particles and its effect in producing a shearing stress. A theory of diffusion by continuous movements has been developed by Taylor.[13] The process of diffusion has been found helpful in the experimental study of the statistical properties of turbulence.

22. Diffusion by continuous movements. Consider in a uniform isotropic turbulent field the displacement X and velocity u parallel to the arbitrarily selected x axis. The intensity $\sqrt{\overline{u^2}}$ is constant, the field being assumed uniform. Let u_t and $u_{t'}$ be the values of u at times t and t' respectively. Consider the definite integral $\int_0^t u_t u_{t'} dt'$. Introducing the correlation coefficient $R_{t,t'}$ between u_t and $u_{t'}$, remembering that $\overline{u^2}$ is constant,

$$\int_0^t \overline{u_t u_{t'}}\,dt' = \overline{u^2}\int_0^t R_{t,t'}\,dt'. \tag{22.1}$$

Let $t'-t=T$ and place $R_{t,t'}=R_T$. Since R_T is an even function of T, (22.1) may be written:

$$\int_0^t \overline{u_t u_{t'}}\,dt' = \overline{u^2}\int_0^t R_T\,dT. \tag{22.2}$$

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[23] Kármán, Th. von, *Turbulence*, Jour. Roy. Aeron. Soc. 41, 1109 (1937).

But

$$\int_0^t \overline{u_t u_{t'}} dt' = \overline{u_t \int_0^t u_{t'} dt'} = \overline{u_t X} = \overline{uX}. \tag{22.3}$$

Hence

$$\overline{u^2} \int_0^t R_T dT = \overline{uX} = (1/2) d\overline{X^2}/dt. \tag{22.4}$$

When the time t is so small that R_T approximates unity, equation (22.4) becomes:

$$(1/2) d\overline{X^2}/dt = \overline{u^2} t$$

or

$$\sqrt{\overline{X^2}} = \sqrt{\overline{u^2}}\, t. \tag{22.5}$$

If R_T is equal to zero for all times greater than some time T_0

$$\overline{uX} = \overline{u^2} \int_0^{T_0} R_T dT = \text{constant}. \tag{22.6}$$

Define a length l_1 by the relation:

$$l_1 = \sqrt{\overline{u^2}} \int_0^{T_0} R_T dT \tag{22.7}$$

whence

$$l_1 \sqrt{\overline{u^2}} = \overline{uX} = (1/2) d\overline{X^2}/dt \tag{22.8}$$

and

$$\overline{X^2} = 2 l_1 \sqrt{\overline{u^2}}\, t. \tag{22.9}$$

If $R_T = e^{-T/T_0}$, $l_1 = \sqrt{\overline{u^2}}\, T_0$ and the solution of (22.4) yields:

$$\overline{X^2} = 2\overline{u^2} T_0 [t - T_0(1 - e^{-T/T_0})]. \tag{22.10}$$

Equation (22.10) reduces to (22.5) when t is small compared to T_0 and to (22.9) when t is large compared to T_0.

The diffusion in a uniform field is accordingly completely determined by the correlation function R_T.

23. Diffusion in isotropic turbulence. The foregoing theory is directly applicable to diffusion in a uniform isotropic field. However no general statement can be made as to the relation between the length l_1 and the scale L defined in terms of the correlation coefficient R_y in section 9. For the turbulence behind a grid or honeycomb, Taylor found from an analysis of the available experimental results that L was approximately twice l_1.

The essential features of diffusion in isotropic turbulence expressed in equations (22.5) and (22.9) may be summarized as follows:

1. For time intervals which are small in comparison with the ratio of l_1 to $\sqrt{\overline{u^2}}$, the diffusing quantity spreads at a uniform rate proportional to the intensity $\sqrt{\overline{u^2}}$, and the rate is not dependent on the length l_1.

2. For time intervals which are large in comparison with the ratio of l_1 to $\sqrt{\overline{u^2}}$, the diffusing quantity N spreads in accordance with the usual diffusion equation

$$\frac{\partial N}{\partial t} + U\frac{\partial N}{\partial x} + V\frac{\partial N}{\partial y} + W\frac{\partial N}{\partial z} = \frac{\partial}{\partial x}\left(D\frac{\partial N}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial N}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial N}{\partial z}\right)$$

with a coefficient of diffusion D equal to $l_1\sqrt{\overline{u^2}}$, where l_1 is a length defined by $\int_0^\infty R_T dT$.

3. For intermediate time intervals, the diffusion is dependent on the function R_T which represents the correlation between the speed of a particle at any instant and the speed of the same particle after a time interval T.

Consider the diffusion of heat from a hot wire placed in a uniform field of isotropic turbulence in a fluid stream of mean speed U. Observations of the lateral spread of the thermal wake at a distant x downstream may be used to compute the root-mean-square lateral displacement $\sqrt{\overline{Y^2}}$ of the heated particles during a time interval $t=x/U$.

It is convenient to characterize the spread by the angle subtended at the source by the two positions where the temperature rise is half that at the center of the wake. There is a lateral spread of heat produced by the ordinary molecular conduction corresponding to an angle α_0 in degrees of $190.8\sqrt{k/\rho c Ux}$ where k, ρ, and c are thermal conductivity, density, and specific heat (at constant pressure) of the fluid. It may be shown that the total subtended angle α is related to the angle α_t produced by turbulent diffusion and α_0, as follows:

$$\alpha^2 = \alpha_t^2 + \alpha_0^2. \tag{23.1}$$

The temperature distribution in the wake follows an "error" curve as does the amplitude of the turbulent velocity fluctuations, so that the lateral displacement Y also has the same Gaussian frequency distribution. The value of the lateral spread at which the ordinate is half the maximum is $2.354\sqrt{\overline{Y^2}}$ for this distribution. Hence, expressing α_t in degrees,

$$\alpha_t = 134.7\sqrt{\overline{Y^2}}/x \tag{23.2}$$

whence from (22.5) for small values of x,

$$\alpha_t = 134.7\sqrt{\overline{v^2}}/U \tag{23.3}$$

where v is written in place of u in (22.5) since the diffusion in the v direction is being studied.

Thus an experiment on thermal diffusion provides a method of measuring $\sqrt{\overline{v^2}}$. The method was used by Schubauer[21] who showed that α_t was independent of speed over the range 10 to 50 ft/sec and also independent of x over the range 1/2 to 6 inches.

From measurements at large values of x, it is theoretically possible to compute the correlation curve, R_T, vs. T. In any actual experiment, however, the intensity of the turbulence will decrease with x to an extent that must be considered. As discussed by Taylor,[8] R_T may then be considered a function of $\eta = \int_0^T \sqrt{\overline{v^2}} dT = \int_0^x (\sqrt{\overline{v^2}}/U) dx$. The equation analogous to (22.4) for v and Y becomes:

$$(1/2)(U/\sqrt{\overline{v^2}})(d\overline{Y^2}/dx) = \int_0^{\eta_x} R_\eta d\eta. \tag{23.4}$$

The correlation is given by the expression

$$R_\eta = \frac{d}{d\eta}\left(\frac{U}{2\sqrt{\overline{v^2}}} \frac{d\overline{Y^2}}{dx}\right) \tag{23.5}$$

and thus involves a double differentiation of experimental curves, a process which is usually not very accurate.

24. Statistical theory of non-isotropic turbulence. In non-isotropic turbulence the description of the state of the turbulence becomes much more complex. The eddy shearing stresses do not vanish and the eddy normal stresses are not necessarily equal. Six quantities instead of one are required to specify the intensity. Similarly the correlation tensor cannot be expressed in terms of a single scalar function. In general six scalar functions are required. No theoretical investigation using these twelve functions has yet been carried out.

The exploration of this field is still in its earliest stages. Von Kármán[9,16] has given some discussion of energy transport and dissipation and vorticity transport, neglecting the triple correlations, and he has also presented a more detailed discussion of two-dimensional flow with constant shearing stress (Couette's problem). The advance of the theory is definitely handicapped by the absence of reliable experimental data on the twelve functions required to describe the state of turbulence.

25. Diffusion in non-isotropic turbulence. The only theoretical approach at present available for estimating the diffusion in non-isotropic turbulence is to consider the process as approximately equivalent to diffusion in isotropic turbulence of intensity equal to $\sqrt{\overline{v^2}}$ and scale l', where v is the component in the direction in which the diffusion is studied and l' is the length defined by an equation analogous to (22.7), namely,

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[21] Schubauer, G. B., *A turbulence indicator utilizing the diffusion of heat*, Tech. Rept. Nat. Adv. Comm. Aeron., No. 524 (1935).

$$l' = \sqrt{\overline{v^2}} \int_0^\infty R_T dt. \tag{25.1}$$

In most experiments the length l' is not measured. Prandtl defined a mixing length l in terms of the shearing stress τ by the relation:

$$\tau = \rho l^2 \left| \frac{dU}{dy} \right| \frac{dU}{dy} \cdot \tag{25.2}$$

This relation may be interpreted as an equation governing the diffusion of momentum with a coefficient of diffusion equal to $l^2|dU/dy|$. Prandtl in fact assumed $\sqrt{\overline{v^2}}$ proportional to $l|dU/dy|$ and incorporated the factors of proportionality in the length l. It is obvious that

$$l^2 | dU/dy | = l' \sqrt{\overline{v^2}}. \tag{25.3}$$

The length l can be obtained experimentally if the distributions of velocity and shearing stress are known, and, if $\sqrt{\overline{v^2}}$ is also measured, l' may be computed. Sherwood and Woertz[25] have made an experimental study of these relationships.

Taylor[26] pointed out that fluctuating pressure gradients influence the transfer of momentum and suggested that the vorticity be taken as the property undergoing diffusion. The result was the well known vortex transport theory.

Both theories imply diffusion for a time interval long compared to $l'\sqrt{\overline{v^2}}$. When diffusion is studied near the source, experiment shows[27] a behaviour like that discussed in section 23. The spread is nearly linear with x, although unsymmetical in this case. It is probable that the unsymmetrical character cannot be explained on the basis of a single scalar diffusion coefficient.

26. Correlation in turbulent flow through a pipe. Taylor[28] has shown that the correlation between the component of velocity at a fixed point and that at a variable point in the same cross section must be negative for some positions of the variable point, if the applied pressure difference between the ends of the pipe is constant and the fluid may be considered incompressible. Suppose the mean velocity is U and the correlation R has been measured between

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[25] Sherwood, T. K., and Woertz, B. B., *Mass transfer between phases, role of eddy diffusion*, Ind. Eng. Chem. 31, 1034 (1939).

[26] Taylor, G. I., *Transport of vorticity and heat through fluids in turbulent motion*, Proc. Roy. Soc. London Ser. A, 135, 685 (1932).

[27] Skramstad, H. K., and Schubauer, G. B., *The application of thermal diffusion to the study of turbulent air flow*, Phys. Rev., 53, 927 (1938). Abstract only. Full paper not published. A few additional details are given in Dryden, Hugh L., *Turbulence and diffusion*, Ind. Eng. Chem. 31, 416 (1939).

[28] Taylor, G. I., *Correlation measurements in a turbulent flow through a pipe*, Proc. Roy. Soc. London Ser. A, 157, 537 (1936).

the component u_1 of the fluctuations at a fixed point P and u_2 at a variable point Q in the same cross section. Since the mean flow is constant,

$$\int (U + u_2)dydz = \int Udydz = \text{constant} \qquad (26.1)$$

where the integration is taken over the cross section. At any instant,

$$\int u_2 dydz = 0. \qquad (26.2)$$

Multiplying by u_1, which is constant for this integration and may be placed under the integral sign,

$$\int u_1 u_2 dydz = 0. \qquad (26.3)$$

Since (26.3) is true for any instant, it is true for the integral over a time interval T. Hence

$$(1/T)\int_0^T \left[\int u_1 u_2 dydz\right] dt = 0. \qquad (26.4)$$

Changing the order of the integration and remembering that $(1/T)\int_0^T u_1 u_2 dt = \overline{u_1 u_2}$

$$\int \overline{u_1 u_2} dydz = 0. \qquad (26.5)$$

Introducing the correlation R,

$$\int R u_1' u_2' \, dydz = 0. \qquad (26.6)$$

But u_1' is constant with respect to the integration and accordingly

$$\int R u_2' \, dydz = 0. \qquad (26.7)$$

Since u_2' is positive, R must be negative for some positions of Q.

For a circular pipe (26.7) becomes:

$$\int_0^a u_r' Rrdr = 0 \qquad (26.8)$$

where u_r' is the value of $\sqrt{\overline{u^2}}$ at radius r and a is the radius of the pipe.

This relation was experimentally verified in experiments made by Simmons with the fixed point at the center of the pipe.

ON THE MOTION OF A PENDULUM IN A TURBULENT FLUID*

BY

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1. Introduction. In a recent paper, Schumann[1] has investigated the motion of a damped pendulum in a turbulent fluid by considering the effect of the fluid as a continuous fluctuating force. He first considers a damped pendulum which is bombarded by pellets of equal mass mh at equal intervals h of the time, and then treats the case of continuous fluctuations by a limiting process. For this latter case, which "must be regarded as being of far more practical importance," Schumann obtains the very interesting result:[2]

$$r(\xi)=\frac{1}{2\int_0^\infty R(x)e^{-\lambda x}\sin(\beta x+\gamma)dx}\left[\int_0^\infty\{R(x+\xi)+R(x-\xi)\}e^{-\lambda x}\sin(\beta x+\gamma)dx\right.$$
$$\left.+\int_0^\xi\{R(\xi-x)-R(x-\xi)\}e^{-\lambda x}\sin(\beta x+\gamma)dx\right]. \qquad (1.1)$$

The notation is as follows:

$r(\xi)$ = correlation function of the *displacements* of the pendulum at two instants separated by a time interval ξ;

$R(\xi)$ = limiting correlation function of the *velocities* of the impinging pellets;

$\lambda=l+m/M$, M being the mass of the pendulum and l its damping factor;

$\beta^2=\alpha^2+l^2-\lambda^2$, $2\pi/\alpha$ being the (damped) period of the pendulum; $\sin\gamma=\beta/\beta_1$;

$\beta_1^2=\beta^2+\lambda^2=\alpha^2+l^2$.

The analysis used by Schumann is very elegant, but somewhat lengthy. In this article, we shall study the problem from another point of view, and give an alternative derivation of (1.1). This derivation, though unable to cover the case of discontinuous impacts, seems to show the nature of that relation much more clearly.

2. Damped pendulum under the action of a fluctuating force. We shall now investigate the correlation of *displacement* of a damped pendulum in relation to that of the exciting *force*. The notation used in this section should first be regarded as having different (though analogous) interpretations from

* Received Dec. 4, 1942.

[1] Schumann, T. E. W., Phil. Mag. (7), **33**, 138–150 (1942).

[2] Loc. cit., eq. (57), p. 146.

those used in §1. The identification of the two systems of notation will be made in §3.

Consider the equation of motion of the pendulum

$$\frac{d^2y}{dt^2} + 2\lambda \frac{dy}{dt} + (\beta^2 + \lambda^2)y = P, \tag{2.1}$$

where y is the displacement, λ is the damping factor and $2\pi/\beta$ is the (damped) period of the pendulum, P is the exciting force per unit mass of the pendulum, and t is the time. If the force P is quasi-periodic, and is given by the real part of

$$P = \sum A_n e^{i\omega_n t}, \tag{2.2}$$

where ω_n and A_n are real and complex constants respectively, the steady-state displacement of the frequency $\omega_n/2\pi$ is given by the real part of

$$y_n = a_n e^{i\omega_n t}, \qquad a_n = \frac{A_n}{(\beta^2 + \lambda^2 - \omega_n^2) + 2i\lambda\omega_n}. \tag{2.3}$$

Thus, we have

$$|a_n|^2 = \frac{|A_n|^2}{(\beta^2 + \lambda^2 - \omega_n^2)^2 + 4\lambda^2\omega_n^2}. \tag{2.4}$$

This is the relation between the spectrum of the displacement and that of the force in the case of discrete spectra. It is not difficult to generalize this result to the case of continuous spectra by the considerations of generalized harmonic analysis.[3] We have then

$$f(\omega) = A \frac{F(\omega)}{(\beta^2 + \lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2}, \tag{2.5}$$

where $f(\omega)$ and $F(\omega)$ are the spectra of the displacement and the force respectively,

$$\int_0^\infty f(\omega)d\omega = 1, \qquad \int_0^\infty F(\omega)d\omega = 1, \tag{2.6}$$

and A is a constant of normalization,

$$\frac{1}{A} = \int_0^\infty \frac{F(\omega)d\omega}{(\beta^2 + \lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2}. \tag{2.7}$$

These are the well-known relations in the phenomena of resonance.

The correlations $r(\xi)$ and $R(\xi)$ of the *displacement* and the *force* respectively stand in Fourier transform relations to the spectra[4] (apart from conventional numerical factors):

[3] Wiener, N., "The Fourier Integral" (Cambridge, 1933), p. 150.

[4] Wiener, N., loc. cit., eq. (21.21), p. 161, and discussions on p. 163. To be exact, we should follow Wiener in calling $f(\omega)$ and $F(\omega)$ the spectral densities.

$$f(\omega) = \frac{2}{\pi}\int_0^\infty r(x)\cos\omega x\,dx, \qquad r(\xi) = \int_0^\infty f(\omega)\cos\omega\xi\,d\omega; \tag{2.8}$$

$$F(\omega) = \frac{2}{\pi}\int_0^\infty R(x)\cos\omega x\,dx, \qquad R(\xi) = \int_0^\infty F(\omega)\cos\omega\xi\,d\omega. \tag{2.9}$$

From (2.5), (2.8), and (2.9), we have at once

$$r(\xi) = \frac{2A}{\pi}\int_0^\infty \frac{\cos\omega\xi d\omega}{(\beta^2+\lambda^2-\omega^2)^2+4\lambda^2\omega^2}\int_0^\infty R(x)\cos\omega x\,dx. \tag{2.10}$$

It is not difficult to justify a change of the order of integration, since the correlation functions are expected to go to zero at infinity sufficiently rapidly. The above relation then becomes

$$r(\xi) = \frac{2A}{\pi}\int_0^\infty R(x)dx\int_0^\infty \frac{\cos\omega\xi\cos\omega x}{(\beta^2+\lambda^2-\omega^2)^2+4\lambda^2\omega^2}d\omega. \tag{2.11}$$

Since we have

$$\cos\omega\xi\cos\omega x = \tfrac{1}{2}\{\cos\omega(\xi+x)+\cos\omega(\xi-x)\}, \tag{2.12}$$

we can evaluate the integral with respect to ξ in (2.11), if we know

$$I(t) = \int_0^\infty \frac{\cos\omega t}{(\beta^2+\lambda^2-\omega^2)^2+4\lambda^2\omega^2}d\omega. \tag{2.13}$$

This integral is relatively easy to evaluate. We write

$$I(t) = \frac{1}{2}\int_{-\infty}^\infty \frac{e^{i\omega t}}{(\beta^2+\lambda^2-\omega^2)^2+4\lambda^2\omega^2}d\omega, \tag{2.14}$$

and consider the corresponding contour integral in the complex ω-plane, the contour being the usual one composed of the real axis and a semi-circle at infinity. The circle is taken in the upper half-plane if $t>0$, and in the lower half-plane if $t<0$. Since (2.13) shows that $I(t)$ is an even function, we shall carry out the calculations for $t>0$ alone.

There is no difficulty in showing that the integral over the semi-circle goes to zero. For, when the imaginary part of ωt is positive, $|e^{i\omega t}|$ is bounded, and $|(\beta^2+\lambda^2-\omega^2)^2+4\lambda^2\omega^2| = O(|\omega|^4)$ for large values of $|\omega|$. The evaluation of (2.14) then reduces to the calculation of the residues of the integrand at the two simple poles $\pm\beta+i\lambda$ $(\lambda>0)$ inside the contour. The result can be easily verified to be

$$I(t) = \frac{\pi e^{-\lambda t}}{4\beta\lambda} \sin(\beta t + \gamma), \tag{2.15}$$

where

$$\sin\gamma = \frac{\beta}{(\beta^2 + \lambda^2)^{1/2}}. \tag{2.16}$$

With the help of (2.12), (2.13), and (2.15), the equation (2.11) becomes

$$r(\xi) = \frac{A}{4\beta\lambda}\left[\int_0^\infty \left\{R(x+\xi) + R(x-\xi)\right\} e^{-\lambda x} \sin(\beta x + \gamma)dx \right.$$
$$\left. + \int_0^\xi \left\{R(\xi - x) - R(x-\xi)\right\} e^{-\lambda x}\sin(\beta x + \gamma)dx\right]. \tag{2.17}$$

This is Schumann's relation (1.1), if the constants can be identified. There is no difficulty with the normalization coefficient. We have

$$\frac{4\beta\lambda}{A} = 2\int_0^\infty R(x)e^{-\lambda x}\sin(\beta x + \gamma)dx$$
$$= 4\beta\lambda \int_0^\infty \frac{F(\omega)d\omega}{(\beta^2 + \lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2}, \tag{2.18}$$

on putting $\xi = 0$ in (2.17) and recalling (2.7). The second relation is a by-product of our investigation. The limiting case $\lambda \to 0$ reduces to the well-known relation (2.9).

Referring to (2.9), we see that $R(\xi)$ is an even function of ξ, so that the second integral in (2.17) may be dropped.

From the derivation, we see that (2.17) is nothing but the Fourier transform of the well-known resonance relation (2.5).

3. Identification of the results. We shall now identify Schumann's result with ours by showing that in the limiting case, his pendulum has an effective damping factor λ instead of l, and that his correlation of *velocity* of the pellets becomes the correlation of *force*. The equations of motion as given by Schumann are[5]

$$u_r = \frac{M - mh}{M + mh} u_r' + \frac{2mh}{M + mh} v_r, \tag{3.1}$$

at the rth impact at the instant $t = rh$, and

$$\frac{d^2y}{dt^2} + 2l\frac{dy}{dt} + (\alpha^2 + l^2)y = 0, \tag{3.2}$$

[5] Loc. cit., eqs. (1), (2). It seems that there are some misprints in the original paper.

between successive impacts. In (3.1), u_r' and u_r denote respectively the velocities of the mass M just before and just after the rth impact, and v_r is the velocity of the rth pellet.

In the limiting case, the discrete impacts become a continuous force given by

$$P_1 = \lim_{h \to 0} \frac{u_r - u_r'}{h} M = -2mu + 2mv, \tag{3.3}$$

where u and v are the limiting values for u_r and v_r at the instant $t=rh$. Thus u is evidently the velocity of the pendulum dy/dt, and v is the velocity of the infinitesimal impinging pellet in the interval $(t, t+dt)$. The equation of motion (3.2) then becomes

$$\frac{d^2y}{dt^2} + 2l\frac{dy}{dt} + (\alpha^2 + l^2)y = \frac{1}{M}\left(-2m\frac{dy}{dt} + 2mv\right). \tag{3.4}$$

We see that this limiting case carries an inherent damping factor m/M in addition to the damping factor l. The term $2mv/M$ is evidently the exciting force per unit mass, since m is the impinging mass per second, the factor 2 corresponding to the fact that when the impinging mass is infinitesimal compared with the colliding mass, the former rebounds with the colliding speed,—a fact often used in the kinetic theory of gases.

The identification of Schumann's result with that given in §2 is therefore complete.

4. Discussion. In view of the above derivation, we must be a little careful in applying Schumann's relation (1.1) to the study of the motion of a pendulum in a turbulent fluid. The correlation function R obtained (by suitable processes) according to that relation from the correlation function r of the displacements is that for the hydrodynamic *force* (the part corresponding to an extra damping being removed). In a turbulent fluid, the connection between the *velocity* fluctuations of the fluid *before* the introduction of a pendulum and the fluctuating *force* acting upon the pendulum *after* its introduction is not at once evident. A preliminary careful investigation seems to be necessary before the method suggested by Schumann could be used with advantage.

It is also clear that the spectrum of velocity fluctuations of the pendulum will be proportional to $\omega^2 f(\omega)$. It is then clear from (2.8) that the correlation function of the velocity fluctuations will be proportional to $-r''(\xi)$.

There is another point which should be mentioned. The spectrum and correlation function discussed above refer to those observed at a fixed *spatial* point, if we assume the pendulum to be never far from its position of equilibrium. It is not difficult to observe this spectrum with a hot-wire anemometer, as has been done by several observers.[6] The Fourier transform of this spectrum will then give the correlation of the fluctuations at successive instants at the same point in *space.* It is not clear, however, how (as Schumann suggested) the pendulum can be used to observe the correlation function defined with respect to the same *material* point,—the quantity introduced by Taylor[7] in the Lagrangian description of turbulence for the study of the phenomenon of turbulent diffusion.[8]

In conclusion, the author wishes to express his sincere thanks to Professor Theodore von Kármán for suggesting the problem to him and for his invaluable suggestions.

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[6] For example, L. F. G. Simmons and C. Salter, Proc. Roy. Soc. Ser. A, **165**, 73–89 (1938); H. L. Dryden, Proceedings of the Fifth International Congress of Applied Mechanics (Cambridge, Mass., 1938), pp. 362–367; H. Motzfeld, Zeits. f. angew. Math. u. Mech. **18**, 362–365 (1938).

[7] Taylor, G. I., Proc. Lond. Math. Soc., (2), **20**, 196–212, (1921). Clearer statements regarding this point are made in his paper of 1935, Proc. Roy. Soc., A, **151**, 421–478 (1935).

[8] A discussion of the three types of correlations that may be defined in the study of isotropic turbulence has been made by G. Dedebant and P. Wehrle, Comptes Rendus **208**, 625–628 (1939).

ON PLANE RIGID FRAMES LOADED PERPENDICULARLY TO THEIR PLANE*

BY

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1. Introduction. For purposes of stress analysis, the engineer prefers to consider his structures as consisting of plane systems, each of which is subject to forces acting in its plane. A typical example is furnished by the conventional analysis of a parallel chord bridge span in which the side trusses take the vertical loads and the top and bottom trusses the transverse loads due to wind, etc. In civil engineering this resolution of space systems into plane components is possible in most cases, and only very rarely is a structure considered as a unit in space. Accordingly, the methods of dealing with space structures have not been developed nearly as much as those used in the analysis of plane structures. Of course, the general principles of structural theory, for instance the principle of virtual work or Castigliano's principle, apply to space structures as well as to plane structures but, as is known from

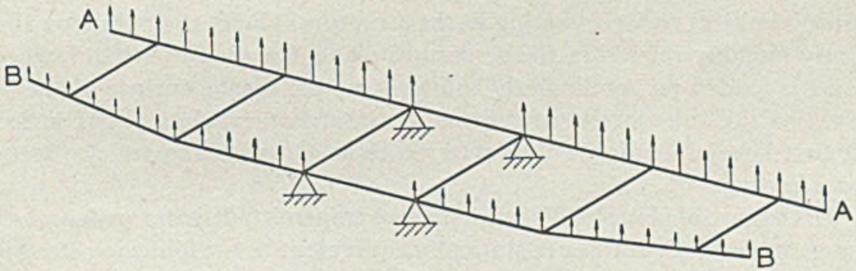


FIG. 1a. Two-sparred wing.

the case of plane structures, these principles frequently do not offer the most convenient approach to the solution of a particular problem. As regards special methods, which have been developed so abundantly in the case of plane structures, little work has as yet been done in the field of space structures. Most of this work is concerned with pin-jointed frameworks. The iteration procedure of R. V. Southwell's relaxation method can be applied to space structures as well as to plane structures,[1] but efficient direct methods for the stress analysis of rigid frames in space are entirely lacking. The present paper, intended as a contribution towards the development of such methods, deals with the particular case of plane rigid frames carrying loads which act per-

* Received Dec. 3, 1942.

[1] See R. V. Southwell: *Relaxation methods in engineering science*, Chap. IV, Oxford 1940.

pendicularly to the plane of the frame. An example of this type of structure is the monoplane wing of Fig. 1a, where the spars A—A and B—B are connected by several main ribs which are fastened rigidly to the spars. Another example is the foot ring of an observatory cupola shown in Fig. 1b.

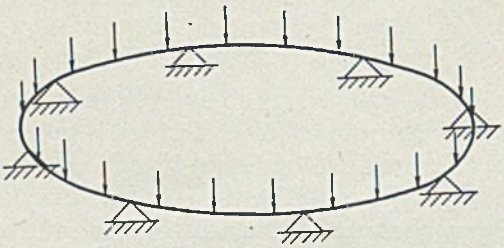


Fig. 1b. Cupola foot ring.

The method proposed in this paper makes extensive use of a dual analogy between plane structures loaded in their plane and plane structures loaded perpendicularly to their plane. In the case of a single straight beam this analogy forms the basis of the method of conjugate beams[2] which can be considered as a particular case of the present method.

2. Definitions, notations and sign conventions. This paper is concerned with rigid frames consisting of straight or curved members whose axes lie in the same plane. This plane is called the *structural plane.* We will consider only frames with members such that every cross section has a principal axis of inertia at its centroid lying in the structural plane. Accordingly, when the frame is subject to forces acting in the structural plane, the points on the axis of any member remain in the structural plane. On the other hand, when the frame is loaded perpendicularly to its plane, the displacements of the points on the axis of any member are normal to the structural plane. For conciseness, the first type of loading will be referred to as *plane loading* and the second as *space loading.*

In the case of plane loading the stresses transmitted across any cross section of a member of the frame are statically equivalent to the following *stress resultants*: 1) an axial force which, for the sake of brevity, will be called the *pull* although it may produce either compression or tension; 2) a transverse force, called the *shear*, which acts in the structural plane normal to the axis of the member under consideration; 3) a couple, called the *bending moment*, which also acts in the structural plane. In the case of space loading the stress resultants are: 1) a twisting couple, called *torque*, acting in the plane of the cross section; 2) a bending couple, called *bending moment*, whose plane is perpendicular to the structural plane as well as to the cross-sectional plane; 3) a transverse force, called *shear*, whose line of action is normal to the structural plane.

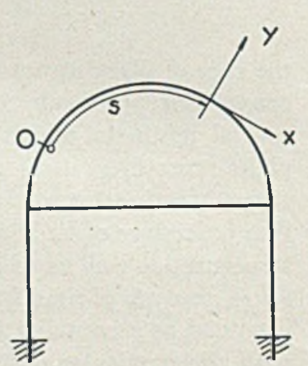


Fig. 2.

[2] H. M. Westergaard, *Deflection of beams by the conjugate beam method*, Journal of the Western Society of Engineers, 26, 369 (1921).

In order to arrive at notations applicable to both types of loading we choose on the axis of each member of the frame an origin O and denote by s the arc length of the axis measured from this point. A cross section B is then specified by giving the corresponding value of s (Fig. 2*). In most cases it will be convenient to choose the origin O at one end of the member, in which case s will have positive values only. In order to establish appropriate sign conventions for the loads, displacements, stress resultants and distortions at the cross section B, we introduce a rectangular right hand triad with origin at B, the x-axis being tangent to the axis of the member at B in the direction of increasing s, and the y-axis lying in the structural plane (Fig. 2).

The loads which the structure carries at B may be forces or couples, either concentrated or distributed, or both. The components of the concentrated force at B, the distributed force at B, the concentrated couple at B and the distributed couple at B we denote by F_x, F_y, F_z, f_x, f_y, f_z, C_x, C_y, C_z, c_x, c_y, c_z respectively, relative to the rectangular triad at B. For example, C_x is a concentrated twisting couple, and is positive if its sense is the same as that of the 90° rotation necessary to move the y-axis into coincidence with the z-axis. Not all the load components thus defined have practical importance; however, the analogies which we intend to establish appear more clearly when the most general case is considered.

The force system transmitted across the cross section at B is equivalent to a force at the centroid plus a couple. These will be referred to as the stress resultants, and we shall denote their components by R_x, R_y, R_z, M_x, M_y, M_z respectively, relative to the rectangular triad at B. R_x is the pull, R_y and R_z the shears parallel and perpendicular to the structural plane, M_x the torque and M_y, M_z the bending moments.

The stress resultants and the loads are connected by the *equations of equilibrium*. If no concentrated forces are applied at the cross section B, the equations of equilibrium for a straight structural member are:

$$\begin{aligned} R_x' + f_x &= 0, & M_x' + c_x &= 0, \\ R_y' + f_y &= 0, & M_y' + c_y - R_z &= 0, \\ R_z' + f_z &= 0, & M_z' + c_z + R_y &= 0, \end{aligned} \tag{1'}$$

where the dashes denote differentiation with respect to the arc length s. If concentrated loads F_x, F_y, F_z and C_x, C_y, C_z are applied at B, we have

$$\begin{aligned} R_x(s+\epsilon) - R_x(s-\epsilon) + F_x &= 0, & M_x(s+\epsilon) - M_x(s-\epsilon) + C_x &= 0, \\ R_y(s+\epsilon) - R_y(s-\epsilon) + F_y &= 0, & M_y(s+\epsilon) - M_y(s-\epsilon) + C_y &= 0, \\ R_z(s+\epsilon) - R_z(s-\epsilon) + F_z &= 0, & M_z(s+\epsilon) - M_z(s-\epsilon) + C_z &= 0, \end{aligned} \tag{1''}$$

where ϵ denotes an arbitrarily small length.

---

* In Fig. 2 the origin of the system x, y should be marked B.

The *displacement* of the cross section B is specified by the components u_x, u_y, u_z of the translation of the centroid and the components θ_x, θ_y, θ_z of the rotation of the cross section.

Finally, the six *distortion components*, g_x, g_y, g_z and h_x, h_y, h_z, of a straight structural member are defined as follows:

$$\begin{aligned} g_x &= u_x', & h_x &= \theta_x', \\ g_y &= u_y' - \theta_z, & h_y &= \theta_y', \\ g_z &= u_z' + \theta_y, & h_z &= \theta_z', \end{aligned} \tag{2'}$$

where the dashes again denote differentiation with respect to the arc length s. g_x will be called the *stretch*, g_y and g_z the *slips*, h_x the *twist* and h_y, h_z the *bends*. Two structural members may be connected by a link which permits some relative displacement of the end sections of the two members, for instance by a hinge permitting a free bend. Such relative displacements can be handled as *concentrated distortions*:

$$\begin{aligned} G_x &= u_x(s+\epsilon) - u_x(s-\epsilon), & H_x &= \theta_x(s+\epsilon) - \theta_x(s-\epsilon), \\ G_y &= u_y(s+\epsilon) - u_y(s-\epsilon), & H_y &= \theta_y(s+\epsilon) - \theta_y(s-\epsilon), \\ G_z &= u_z(s+\epsilon) - u_z(s-\epsilon), & H_z &= \theta_z(s+\epsilon) - \theta_z(s-\epsilon), \end{aligned} \tag{2''}$$

where ϵ is again an arbitrarily small length.

For elastic structural members the stress resultants can be represented as the products of the corresponding distortions and *stiffness factors*:

$$\begin{aligned} R_x &= \alpha_x g_x, & R_y &= \alpha_y g_y, & R_z &= \alpha_z g_z, \\ M_x &= \beta_x h_x, & M_y &= \beta_y h_y, & M_z &= \beta_z h_z, \end{aligned} \tag{3}$$

where $\alpha_x = EA$, $\alpha_y = GA/k_y$, $\alpha_z = GA/k_z$, β_x is the torsional rigidity of the member, $\beta_y = EI_z$, $\beta_z = EI_y$, E being Young's modulus, A the area of the cross section, G the modulus of rigidity, k_y and k_z constants depending on the shape of the cross section, I_y and I_z the moments of inertia of the cross section with respect to the axes of y and z.

3. Analogy between statics of plane loaded and kinematics of space loaded frames. For a plane loaded frame the loads F_z, f_z, C_x, c_x, C_y, c_y and the stress resultants R_z, M_x, M_y vanish. Similarly, for a space loaded frame the displacements u_x, u_y, θ_z and the distortions G_x, g_x, G_y, g_y, H_z, h_z are zero. The remaining equations (1) for the members of the plane loaded frame then are seen to correspond exactly to the remaining equations (2) for the members of the space loaded frame according to the Table I.

TABLE I

Statics of Plane Loaded Frame		Kinematics of Space Loaded Frame	
Loads	F_x, F_y, C_z	Distortions	$-H_x$, $-H_y$, $-G_z$
	f_x, f_y, c_z		$-h_x$, $-h_y$, $-g_z$
Stress resultants	R_x, R_y, M_z	Displacements	θ_x, θ_y, u_z

At a point B where two straight members are rigidly fastened to one another under the angle ϕ (Fig. 3), we have the following relations between the stress resultants of the plane loaded frame at the two sides of B:

$$\overline{R}_x = R_x \cos\phi + R_y \sin\phi, \qquad \overline{R}_y = -R_x \sin\phi + R_y \cos\phi, \qquad \overline{M}_z = M_z.$$

These relations correspond exactly to the following relations for the displacement components of the space loaded frame on the two sides of B:

$$\bar{\theta}_x = \bar{\theta}_x \cos\phi + \theta_y \sin\phi, \qquad \bar{\theta}_y = -\theta_x \sin\phi + \theta_y \cos\phi, \qquad \bar{u}_z = u_z. \tag{5}$$

A curved member may be considered as the limiting case of a polygonal arrangement of straight members. Accordingly, the correspondence between the stress resultants of the plane loaded frame and the displacements of the space loaded frame remains valid in the case of frames containing curved members. Of course, equations (1) and (2) in their present forms do not apply to frames with curved members. Equations applicable to such frames will be developed in a later paper.

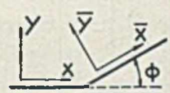


FIG. 3.

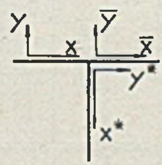


FIG. 4.

If more than two members join at the same point, the analogy breaks down. This can be seen from the example in Fig. 4, where we have for the stress resultants of the plane loaded frame

$$\overline{R}_x = R_x - R_y^*, \qquad \overline{R}_y = R_y + R_x^*, \qquad \overline{M}_z = M_z - M_z^*,$$

and for the displacements of the space loaded from

$$\bar{\theta}_x = \theta_x = \theta_y^*, \qquad \bar{\theta}_y = \theta_y = -\theta_x^*, \qquad \bar{u}_z = u_z = u_z^*.$$

The analogy therefore applies directly only to frames which consist of a simple chain of members without branch points. In spite of this fact the analogy is very useful even in the case of more complex frames since these may be considered as consisting of simple frames to which the analogy can be applied.

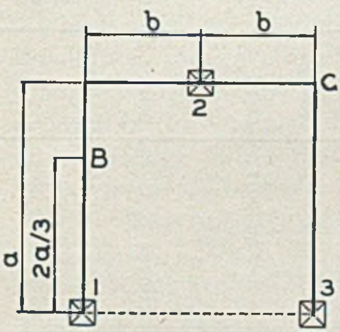
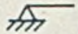
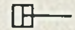
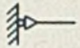
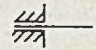
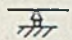
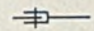
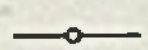
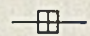


FIG. 5.

In order that the analogy indicated above hold everywhere in the structures, it is necessary that there can be a certain correspondence between the various supports and links in the frame with plane loading and the frame with space loading. For example, the analogy is maintained if to a pin support in the frame with plane loading there corresponds a simple support in the plane with space loading. A consideration of certain types of supports and links leads to the results presented in Table II. This table is by no means complete.

TABLE II

	Statics of Plane Loaded Frame					Kinematics of Space Loaded Frame				
	Description	R_x	R_y	M_z	Symbol	Description	θ_x	θ_y	u_z	Symbol
End Supports	Pin support	—	—	0		Simple support	—	—	0	
	Roller support (axial motion)	0	—	0		Pin support	0	—	0	
	Roller support (transverse motion)	—	0	0		Journal bearing	—	0	0	
	Free end	0	0	0		Clamped end	0	0	0	
	Clamped end	—	—	—		Free end	—	—	—	
Intermediate Supports and Links	Pin support	discont.	discont.	cont.		Spherical hinge	discont.	discont.	cont.	
	Roller support	cont.	discont.	cont.		Hinge	cont.	discont.	cont.	
	Hinge	cont.	cont.	0		Simple support	cont.	cont.	0	
	Extensible hinge	0	cont.	0		Pin support	0	cont.	0	

We shall now use this analogy to determine the influence lines for the stress resultants of a statically determinate space loaded frame. We consider the rectangular frame with three pin supports 1, 2, 3 shown in Fig. 5. According to a well known principle of the theory of structures, the ordinates of the influence line for the bending moment produced at the section B by transverse loads F_z are precisely the displacements u_z produced by a unit bend, $H_y=1$, at B. According to the analogy explained above these displacements can be obtained as the bending moments, M_z, produced in the corresponding plane loaded frame by the load, $F_y=-1$, at B. This latter frame is called the conjugate frame and is the three hinged portal shown in Fig. 6a. The bending moments due to the unit load are easily computed and are shown in Fig. 6b. The diagram showing the bending moments of the conjugate frame is at the same time the influence line for the bending moment produced at the section B of the space loaded frame by transverse loads F_z.

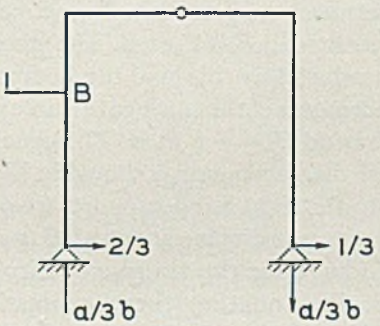


FIG. 6a.

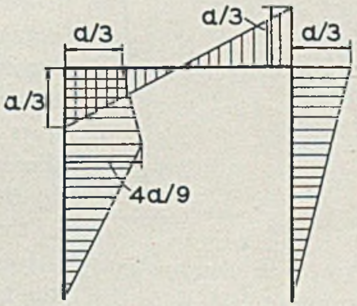


FIG. 6b.

Similarly, the influence line for the torque M_x produced at the section B of the space loaded frame by transverse loads F_z can be obtained as the bending moment diagram of the conjugate frame due to the load $F_x=-1$ at B. Finally, the influence line for the shear R_z produced at the section B of the space loaded frame by transverse loads F_z is found as the bending moment diagram of the conjugate frame due to the couple $C_z=-1$ at B.

4. Analogy between kinematics of plane loaded and statics of space loaded frames. For a plane loaded frame the displacements u_z, θ_x, θ_y vanish and the distortions G_z, g_z, H_x, h_x, H_y, h_y are zero. For a space loaded frame the loads F_x, f_x, F_y, f_y, C_z, c_z and the stress resultants R_x, R_y, M_z vanish. The remaining equations (2) for the members of the plane loaded frame then are found to correspond exactly to the remaining equations (1) for the members of the space loaded frame according to Table III.

The relations between the displacements of the plane loaded frame at both sides of the angle joint of Fig. 3 are easily seen to correspond to the relations between the stress resultants of the space loaded frame. Furthermore, the correspondence between the various types of supports and links shown in Table II is valid also in the present case where we are concerned with the kinematics of the plane loaded frame and the statics of the space loaded frame.

TABLE III

Kinematics of Plane Loaded Frame		Statics of Space Loaded Frame	
Displacements	$u_x, \quad u_y, \quad \theta_z$	Stress resultants	M_x, M_y, R_z
Distortions	$\begin{cases} -G_x, & -G_y, & -H_z \\ -g_x, & -g_y, & -h_z \end{cases}$	Loads	$\begin{cases} C_x, & C_y, & F_z \\ c_x, & c_y, & f_z \end{cases}$

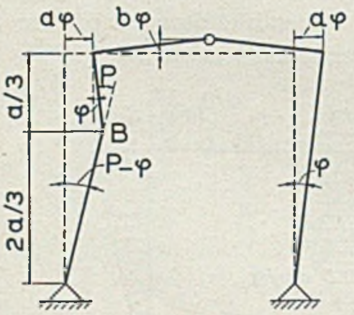


FIG. 7.

The analogy established in this section can be used to find the stress resultants of a statically determinate space loaded frame by determining the displacements of the conjugate plane loaded frame. Let us consider for instance the frame in Fig. 5 carrying a transverse load $F_z = P$ at B. The stress resultants produced by this load can be found as the displacements of the conjugate frame produced by the bend $H_z = -P$ at B. The general trend of these displacements is shown in Fig. 7. Relative to the right hand support, we obtain for the transverse displacement at B: $u_y = 4a\varphi/3$. Relative to the left hand support, we have for the transverse displacement at the same point: $u_y = 2a(P - \varphi)/3$. By equating these expressions and solving for φ, we find that $\varphi = P/3$. The displacements u_x, u_y, θ_z of the conjugate frame are thus known. They correspond to the stress resultants M_x, M_y, R_z of the space loaded frame, the distribution of which is shown in Figs. 8 a–c.

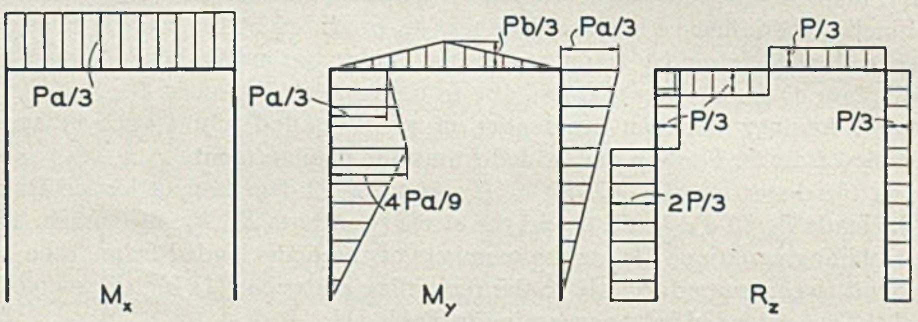


FIG. 8a–c.

5. Elastic deformations of space loaded frames. Indeterminate space loaded frames. When the stress resultants of a space loaded frame have been determined by the method given in the preceding section, the analogy of §3 can be used in order to find the elastic deformations of the space loaded frame. For example, let us consider again the frame in Fig. 5, carrying a transverse load $F_z = P$ at B. The stress resultants have been determined in the preceding section and are shown in Fig. 8. Equations (3) furnish the distortions produced by these stress resultants: $h_x = M_x/\beta_x$, $h_y = M_y/\beta_y$, $g_z = R_z/\alpha_z$. According to the analogy of §3, the displacements θ_x, θ_y, u_z corresponding to these distortions can be found as the stress resultants R_x, R_y, M_z of the conjugate frame carrying the loads $f_x = -h_x$, $f_y = -h_y$, $c_z = -g_z$. In most cases the influences of the shear R_z on the deformations can be neglected, and consequently in the conjugate frame the distributed couples c_z need not be considered. The loads on the conjugate frame then consist of the axial loads shown in Fig. 9a and the transverse loads shown in Fig. 9b, it having been assumed that β_x and β_y have the same values for all members. The reactions can be computed easily and are indicated in Figs. 9a and 9b. If now, for instance, we wish to determine the deflection u_z of the right hand corner C of the space loaded frame, we have only to compute the bending moment M_z at the corresponding corner of the conjugate frame. We find that

$$u_z = -\frac{Pa^3}{162\beta_y}[23 - 18n^3 + 54\gamma n], \tag{6}$$

where $n = b/a$ and $\gamma = \beta_y/\beta_x$.

This method of computing elastic deformations enables us to carry out the stress analysis of indeterminate space loaded frames. Let us suppose, for instance, that the frame in Fig. 5 is given a further simple support at B and carries a transverse load $F_z = Q$ at C. By establishing the condition for vanish-

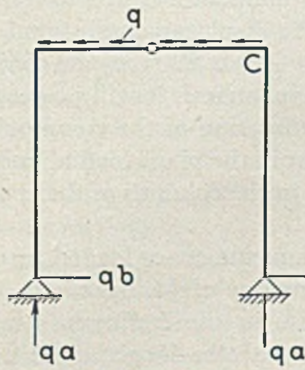


FIG. 9a. $q = Pa/3\beta_x$.

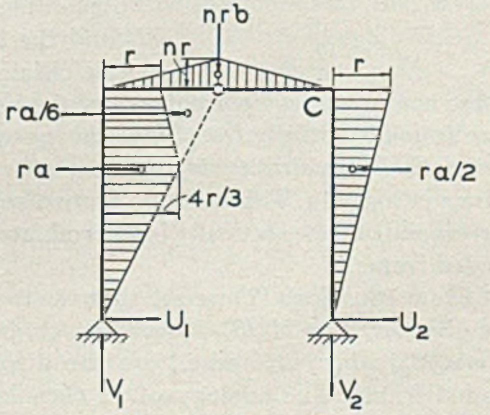


FIG. 9b. $n = b/a$, $r = Pa/3\beta_y$,

$$U_1 = \frac{ra}{54}(40 + 18n^3), \qquad U_2 = \frac{ra}{54}(32 - 18n^3).$$

$$V_1 = \frac{ra}{54n}(23 + 27n^3) \qquad V_2 = \frac{ra}{54n}(23 - 27n^3).$$

ing deflection at B and applying Maxwell's law of reciprocal deflections in the usual way, we find the reaction at B in the form $R=-Qu_C/u_B$, where u_B and u_C denote the deflections which a unit transverse load $F_z=1$ at B produces at the points B and C respectively; u_C can be obtained from (6) by setting $P=1$; u_B can be computed as the bending moment M_z at the point B of the conjugate frame, loaded according to Figs. 9a and 9b. With $P=1$ we find that

$$u_B = \frac{a^3}{243\beta_y}[32 + 18n^3 + 54\gamma n]$$

and therefore

$$R = \frac{3(23 - 18n^3 + 54\gamma n)}{2(32 + 18n^3 + 54\gamma n)}.$$

6. **The inverse column analogy.** The following method of determining the stress resultants of indeterminate space loaded frames is patterned after the column analogy method of H. Cross.[3] In order to avoid lengthy computations which might obscure the essential feature of this new method, we shall consider the simple problem of the frame in Fig. 10 carrying a transverse load $F_z=P$ at B.

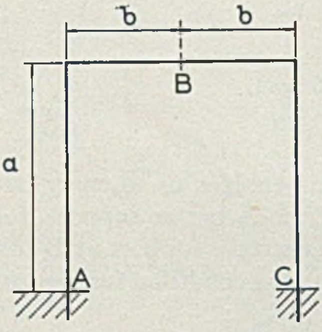


Fig. 10.

We suppose again that the stiffness factors β_x and β_y have the same values for all members. The frame then is symmetrical and is loaded symmetrically.

According to the analogy of §4, the torque M_x and the bending moment M_y of the frame can be obtained as the displacements u_x and u_y of the conjugate frame produced by the bend $H_z=-P$ at B. Now the conjugate frame is entirely free. Since the system is symmetrical, it will possess a *kinematically indeterminate* displacement v in the direction of the columns as shown in Fig. 11a. This indeterminate displacement in the plane loaded frame corresponds to the statically indeterminate torque in the columns of the space loaded frame.

From equations (2) we see that the distortions of the space loaded frame are given by $h_x=M_x/\beta_x$, $h_y=M_y/\beta_y$, or, by the analogy of §4, by $h_x=u_x/\beta_x$, $h_y=u_y/\beta_y$, where u_x and u_y are the displacements in the conjugate plane loaded frame. The analogy of §3 then indicates that the longitudinal and transverse loads on the conjugate frame are given by $f_x=-u_x/\beta_x$, $f_y=-u_y/\beta_y$.

[3] H. Cross, *The column analogy*, Univ. Illinois Engineering Experiment Station, Bull. No. 215, 1930.

The horizontal load is in equilibrium because of symmetry. The vertical load is shown in Fig. 11b, and is in equilibrium if

$$v = \frac{Pb}{4} \frac{n}{n + \gamma},$$

where $n = b/a$, $\gamma = \beta_y/\beta_x$. Thus the torque M_x in the left hand column is given by $M_x = u_x = v$. Also, the bending moment M_y at B is given by

$$M_y = -(v - Pb/2) = \frac{Pb}{4} \frac{2\gamma + n}{\gamma + n}.$$

The procedure outlined above is equivalent to the following procedure. We suppose that the conjugate frame is embedded in an elastic jelly which

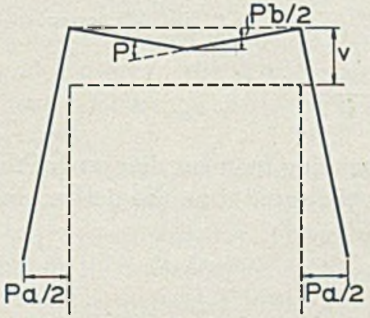


FIG. 11a.

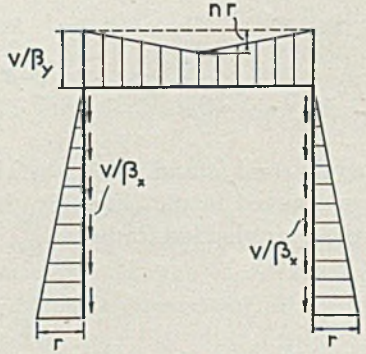


FIG. 11b. $n = b/a$, $r = Pa/2\beta_y$.

offers resistance to the displacement of the frame in such a way that an element ds with displacements u_x and u_y will meet with a resistance consisting of longitudinal and transverse forces of magnitudes $u_x ds/\beta_x$ and $u_y dy/\beta_y$, respectively. We then determine the displacements of this elastically supported frame, and the longitudinal displacement u_x then gives the torque M_x in the space loaded frame, while the transverse displacement u_y gives the bending moment M_y. The above method of procedure is clearly seen to be the counterpart of the column analogy of H. Cross; it will be called the *inverse column analogy*.

The inverse column analogy furnishes a simple method of determining the influence lines of an indeterminate space loaded frame. For example, let us consider the influence line for the bending moment M_y which transverse loads F_z produce at the section B of the symmetric frame in Fig. 10. Following the line of approach of §3, we consider the conjugate frame loaded by a transverse force $F_y = -1$ at B and supported elastically in accordance with the inverse column analogy. Since this frame is symmetrical and loaded symmetrically, the displacements are of the type indicated in Fig. 12a. The forces which the elastic support exerts on the frame are shown in Fig. 12b. From the condition of equilibrium for the vertical forces, we find that

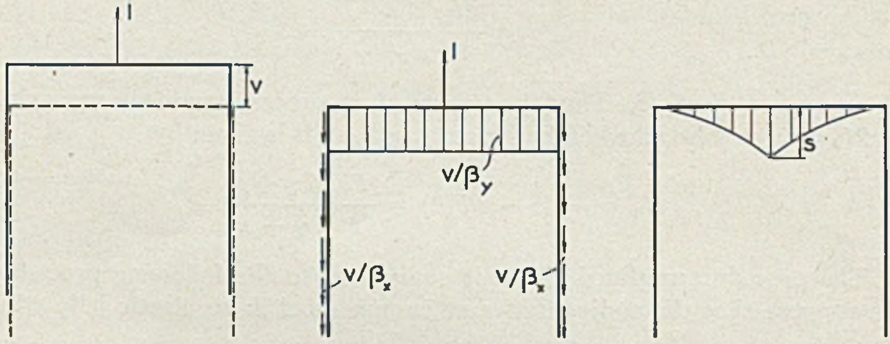


FIG. 12a. FIG. 12b. FIG. 12c. $s = vab/\beta_x + vb^2/\beta_y$.

$$v = \frac{\beta_y}{2b}\,\frac{n}{n+\gamma},$$

where again $n = b/a$ and $\gamma = \beta_y/\beta_x$. The bending moment diagram of this elastically supported frame (Fig. 12c) is at the same time the desired influence line of the space loaded frame.

ON THE VIBRATIONS OF A CLAMPED PLATE UNDER TENSION*

BY

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The object of the present paper is the computation of the fundamental frequency of a vibrating clamped square plate under uniform tension. It will be seen that the method used here reduces our problem for a plate of any shape to the membrane problem for the corresponding domain. For this reason similar numerical results could be obtained for a number of other shapes. A similar question has been discussed for a circular plate by W. G. Bickley[1] in connection with the problem of reception of acoustic signals in a condenser microphone. The circular plate is an elementary problem from the theoretical viewpoint. However, the actual calculations involving Bessel's functions are rather heavy. Bickley was able to give the frequencies only for a small range of the tension.

The frequencies of a square plate cannot be obtained explicitly in terms of elementary functions. However, the Rayleigh-Ritz method yields an upper bound for these quantities. The result cannot be considered as satisfactory since this method does not give us an estimation of the error. Fortunately, an increasing sequence of lower bounds can be obtained for all frequencies by the application of a variational method already introduced by one of the authors in several vibration and buckling problems. Combining these lower bounds with the upper bounds obtained by Rayleigh-Ritz, we obtain a narrow interval in which our frequencies are located.

Moreover, it is obvious that for questions like that of microphone reception, the lower bounds are the more important data.

The theory of the new variational method has been developed in several papers.[2] The modifications in the present case are slight. For this reason we will omit all theoretical details. The reader can easily reconstruct the proofs of the rules which we are following here.

Let S be the domain of a plate of arbitrary shape, and let C denote its boundary. In the numerical applications we shall assume that S is the square $-\pi/2 \leqq x,\ y \leqq \pi/2$.

We denote by:

$2h$, the thickness of the plate

T, the tension

* Received Dec. 11, 1942.

[1] W. G. Bickley, Phil. Mag. (7), 15, 776–797 (1933).

[2] A. Weinstein, Mémorial des Sciences Mathématiques, No. 88, 1937; A. Weinstein, Portugaliae Mathematica 2, 36 (1941).

E, Young's modulus
σ, Poisson's ratio.
ρ, density of the material
$D=2Eh^3/3(1-\sigma^2)$, the flexural rigidity
ω, the eigenfrequency (number of pulsations in 2π seconds)
w, the transversal displacement.

We put $\tau=T/D$. Our problem admits an infinite sequence of eigenfrequencies $\omega=\omega_1, \omega_2, \cdots$, in place of which we shall use the eigenvalues $\lambda=\lambda_1, \lambda_2, \cdots$, where

$$\lambda = \frac{2h\rho\omega^2}{D}\cdot$$

The displacements corresponding to these eigenvalues will be denoted by $w=w_1, w_2, \cdots$. These transverse displacements w satisfy in S the differential equation:

$$\Delta\Delta w - \tau\Delta w - \lambda w = 0 \tag{1}$$

with the boundary conditions

$$w = 0 \tag{2}$$

$$dw/dn = 0 \tag{3}$$

on C.

The equation (1) may be written as follows:

$$(\Delta + \alpha)(\Delta - \beta)w = 0, \qquad (\alpha > 0, \beta > 0) \tag{4}$$

with

$$\beta - \alpha = \tau, \qquad \alpha\beta = \lambda. \tag{5}$$

or

$$\alpha = -\frac{\tau}{2} + \sqrt{\frac{\tau^2}{4} + \lambda}, \qquad \beta = \frac{\tau}{2} + \sqrt{\frac{\tau^2}{4} + \lambda}. \tag{6}$$

We see that we have the *identity*:

$$w = u + \bar{u} \quad \text{in} \quad S + C, \tag{7}$$

where u and $\bar{u}$ are solutions of

$$\Delta u + \alpha u = 0, \tag{8}$$

$$\Delta\bar{u} - \beta\bar{u} = 0. \tag{9}$$

We have therefore also the following *identity*:

$$\Delta w = \Delta(u + \bar{u}) = \beta\bar{u} - \alpha u \quad \text{in} \quad S + C. \tag{10}$$

The identities (7) and (10) will be useful in the following.

It is well known that the eigenvalues of our plate can be defined by minima problems, the same as could be used in the Rayleigh-Ritz method. For instance, the fundamental eigenvalue $\lambda=\lambda_1$, in which we are interested in this paper, is given by the variational problem:

$$U(w) \equiv \iint_S (\Delta w)^2 dxdy + \tau \iint_S \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dxdy$$
$$= \min = \lambda_1 \tag{11}$$

with the condition

$$H(w) \equiv \iint_S w^2 dxdy = 1 \tag{12}$$

and with the boundary conditions (2) and (3).

Let us note that U is not the potential energy of the plate. Nevertheless our variational problem gives us the correct differential equation and boundary conditions. This variational problem will be denoted by P. The higher eigenvalues $\lambda_2, \lambda_3, \cdots$ can also be defined by similar variational problems. However, we shall not use them in this paper.

The Euler equation of P is the equation (1). This equation together with the boundary conditions (2) and (3) defines a *differential eigenvalue problem* $\overline{P}$ which admits the solutions $w_1, w_2, \cdots$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots$.

In order to obtain an increasing sequence of lower bounds for λ_1 we begin by cancelling in the variational problem P the boundary condition $dw/dn=0$. In this way we obtain a new variational problem P_0:

$$U(w) = \min = \lambda_1^{(0)}; \qquad H(w) = 1 \tag{13}$$

with the boundary condition $w=0$.

The conditions in P_0 being less restrictive than in P, we have $\lambda_1^{(0)} \leqq \lambda_1$. The Euler equation in P_0 is the same as in P, namely the equation (1). However, the boundary conditions for this equation are

$$w = 0 \text{ and } \Delta w = 0 \text{ on } C, \tag{14}$$

the last condition being a so-called *natural boundary condition*, i.e., a condition which is automatically satisfied by the minimizing function in P_0. The corresponding differential eigenvalue problem $\overline{P}_0$ is given by (1) and (14). $\overline{P}_0$ admits a sequence of eigenvalues $\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}, \cdots$ the smallest of which is identical with the minimum $\lambda_1^{(0)}$ in P_0. In the case of a square plate, $\overline{P}_0$ is identical with the problem of the vibrations of a supported plate under tension. The problem $\overline{P}_0$ can be solved, for a domain of any shape, in terms of the membrane problem for the same domain, a fact which has been implicitly used in the elementary theory of a square supported plate.

In order to show this we use the identities (7) and (10) and we obtain from (14) at once

$$u = 0, \ \bar{u} = 0 \text{ on } C.$$

In view of (8) and (9), it follows immediately that $\bar{u}\equiv 0$ in $S+C$ and that (1) and (14) are satisfied by $w=u$, where u is an eigenfunction of the membrane problem $\Delta u+\alpha u=0$ in S, $u=0$ on C. From the eigenvalues α of this problem we can compute the eigenvalues $\lambda^{(0)}$ in $\overline{P}_0$ by using the equations (5).

In order to obtain an increasing sequence of lower bounds for the eigenvalue λ_1 in P we link P_0 with P by a chain of intermediate variational problems P_1, P_2, $\cdots$, the solutions of which can be expressed in terms of the solutions of $\overline{P}_0$. In this way our problem for the clamped plate can be reduced to the solution of $\overline{P}_0$ which is, as we have seen, equivalent to the problem of a vibrating membrane.

In order to show how this can be done, let

$$p_1(s),\ p_2(s), \cdots,\ p_{m-1}(s),\ p_m(s), \cdots \tag{15}$$

be an arbitrarily given sequence of functions defined on the boundary C of the plate, s being the arc, and $p_1(s)$ being positive. The problem P_m ($m=1, 2, \cdots$) is then defined as follows:

Problem P_m: Find the minimum $\lambda_1^{(m)}$ of $U(w)$ with the condition $H(w)=1$ and with the boundary conditions

$$w = 0 \text{ on } C, \tag{16}$$

$$\int_C p_k \frac{dw}{dn}\, ds = 0, \qquad k = 1, 2, \cdots, m. \tag{17}$$

The conditions in P_m are more restrictive than those in P_{m-1} but they are less restrictive than the boundary conditions in P.

We have therefore $\lambda_1^{(0)} \leqq \lambda_1^{(1)} \leqq \lambda_1^{(2)} \leqq \cdots \leqq \lambda_1$. The minimizing function w in P satisfies the same Euler equation (1) as in P_0 (or in P), but the boundary conditions are given now by the equations (16), (17) and by

$$\Delta w = a_1 p_1 + \cdots + a_m p_m \text{ on } C, \tag{18}$$

the last condition being again a natural boundary condition. The constant coefficients $a_1, \cdots, a_m$ are unknowns. In order to solve P_m we have to determine the lowest eigenvalue of the differential problem $\overline{P}_m$ defined by (1), (16), (17) and (18). We use again the identities (7) and (10) in a way described in our previous papers. In order to avoid repetitions which would considerably increase the length of this paper, we will only formulate the rules for the computation. It can be shown that $\lambda_1^{(m)}$ can be computed by the following procedure. Denote by u_i and $\bar{u}_i$ ($i=1, \cdots, m$) the solutions of the equations

$$\Delta u_i + \alpha u_i = 0 \tag{19}$$

$$\Delta \bar{u}_i - \beta \bar{u}_i = 0 \tag{20}$$

with the boundary conditions

$$u_i = - p_i(s) \tag{21}$$

$$\bar{u}_i = p_i(s) \tag{22}$$

where α and β are considered as parameters. These equations can be solved in terms of the solutions of $\overline{P}_0$. Put $w_i = u_i + \bar{u}_i$ and compute the quantities

$$\alpha_{ij}(\lambda) = \int_C p_i \frac{dw_j}{dn} ds; \qquad i, j = 1, \cdots, m, \tag{23}$$

where the parameter λ is defined in terms of α and β by (5) and (6). Then $\lambda_1^{(m)}$ is the smallest root of the determinant equation

$$\|\alpha_{ij}(\lambda)\| = 0; \qquad i, j = 1, \cdots, m \tag{24}$$

provided that the smallest root is smaller than the *second eigenvalue* $\lambda_2^{(0)}$ of the differential eigenvalue problem $\overline{P}_0$, defined by the equations (1) and (14).

The calculation of (23) can be further simplified by introducing a sequence of *harmonic functions*

$$p_1(x, y), p_2(x, y), \cdots, p_{m-1}(x, y), p_m(x, y), \cdots \tag{25}$$

whose boundary values are given as in (15). Then by Green's theorem, (23) can be written as follows

$$\alpha_{ij}(\lambda) = \iint_S p_i(x, y)(\beta \bar{u}_j - \alpha u_j) dx dy. \tag{26}$$

Calculation for a square plate $(-\pi/2 \leqq x, y \leqq \pi/2)$: In this case, we take

$$p_i(x, y) = \frac{\cosh (\beta_{2i-1}\pi/2) \cosh (\alpha_{2i-1}\pi/2)}{\cosh (2i - 1)\pi/2} \cdot \{\cosh (2i - 1)x \cos (2i - 1)y + \cosh (2i - 1)y \cos (2i - 1)x\} \tag{27}$$

where

$$\alpha_{2i-1} = \sqrt{(2i - 1)^2 - \alpha}, \qquad \beta_{2i-1} = \sqrt{(2i - 1)^2 + \beta}. \tag{28}$$

On the boundary, we have

$$\begin{cases} p_i(\pm \pi/2, y) = \cos (2i+1)y \cosh \beta_{2i-1}\pi/2 \cosh \alpha_{2i-1}\pi/2 \\ p_i(x, \pm \pi/2) = \cos (2i - 1)x \cosh \beta_{2i-1}\pi/2 \cosh \alpha_{2i-1}\pi/2. \end{cases} \tag{29}$$

Then the solutions of the problems (19), (21) and (20), (22) are

$$\begin{cases} u_i = -\cosh \beta_{2i-1}\pi/2\,[\cos (2i-1)x \cosh \alpha_{2i-1}y + \cos (2i-1)y \cosh \alpha_{2i-1}x] \\ \bar{u}_i = \quad \cosh \alpha_{2i-1}\pi/2\,[\cos (2i-1)x \cosh \beta_{2i-1}y + \cos (2i-1)y \cosh \beta_{2i-1}x]. \end{cases} \tag{30}$$

Substituting $p_i(x, y)$ from (27), u_i and $\bar{u}_i$ from (30) into (26), we obtain after a little calculation,

$$\alpha_{ij} = 4 \cosh \alpha_{2i-1}\pi/2 \cosh \alpha_{2j-1}\pi/2 \cosh \beta_{2i-1}\pi/2 \cosh \beta_{2j-1}\pi/2 \;\; (A_{ij} + B_{ij}) \tag{31}$$

where

$$A_{ij} = A_{ji} = \frac{2(2j-1)(2i-1)(-1)^{i+j}}{[(2i-1)^2 + (2j-1)^2]}\left[\frac{\beta}{\beta^2_{2j-1} + (2i-1)^2} + \frac{\alpha}{\alpha^2_{2j-1} + (2i-1)^2}\right] \tag{32}$$

$$\begin{cases} B_{ii} = \frac{1}{2}[\beta_{2i-1}\pi \tanh \beta_{2i-1}\pi/2 - \alpha_{2i-1}\pi \tanh \alpha_{2i-1}\pi - 1/2] \\ B_{ij} = 0 \text{ for } i \neq j. \end{cases} \tag{33}$$

It should be noted that the roots of $\|\alpha_{ij}(\lambda)\|$ are equal to the roots of $\|\bar{\alpha}_{ij}(\lambda)\|$, where

$$\bar{\alpha}_{ij} = A_{ij} + B_{ij}. \tag{34}$$

The results of our numerical computations are given in Table I below. The first and second columns give $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ for the supported plate. The next three columns give the smallest root of the determinantal equation (24)

Table I

Supported Plate			Clamped Plate			
τ	1st eigen-value $\lambda_1^{(0)}$	2nd eigen-value $\lambda_2^{(0)}$	$\lambda_1^{(1)}$	$\lambda_1^{(2)}$	$\lambda_1^{(3)}$	Rayleigh-Ritz method
5	14	50	24.982	25.222	25.236	25.509
10	24	75	36.639	36.845	36.862	37.443
15	34	100	48.084	48.253	48.284	49.261
20	44	125	59.289	59.452	59.491	61.008
30	64	175	81.651	81.760	81.809	84.372
50	104	275	125.43	125.56	125.59	130.85
100	204	525	225.56	225.63	225.65	246.58
200	404	1025	443.15	443.24	443.25	477.58

for $m=1, 2, 3$. Since these roots are smaller than the corresponding second eigenvalues $\lambda_2^{(0)}$, they are, according to the general theory, identical with the eigenvalues $\lambda_1^{(1)}$, $\lambda_1^{(2)}$, $\lambda_1^{(3)}$ and give therefore an increasing sequence of lower bounds for the fundamental eigenvalue λ_1 in P. The corresponding upper bounds, obtained by the Rayleigh-Ritz method are tabulated in the last column. They have been obtained from the variational problem P by putting

$$w = A \cos^2 x \cos^2 y + B \cos^3 x \cos^3 y.$$

A comparison with $\lambda_1^{(3)}$ shows us that the error in the values of λ_1 is, for small tensions, less than 1.2 per cent and, for great tensions, less than 7 per cent. The fact that $\lambda_1^{(3)}$ hardly exceeds $\lambda_1^{(2)}$ makes it probable that the lower bounds are much closer to the true value of λ_1 than the upper bound given by the Rayleigh-Ritz method.

In figure 1 are plotted curves of fundamental eigenvalues of clamped circular plate and square plate against the tension τ. The curve I is the values of $\lambda_1^{(3)}$ for the clamped square plate. The curves II and III are respectively the fundamental eigenvalues for a circular plate of equal area and equal circumference as the given square plate. Both of the latter curves are calculated from the Bickley result.

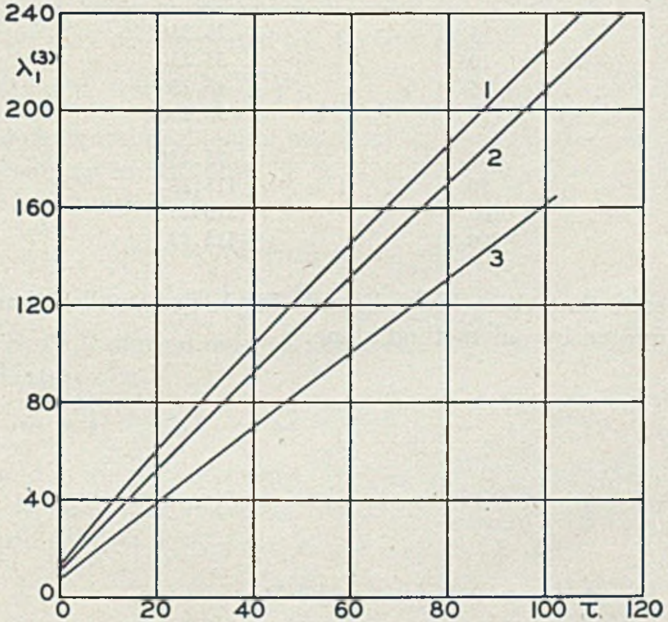

FIG. 1. Curve 1: Clamped Square Plate ($\pi/2 \geqq x, y \geqq -\pi/2$)
Curve 2: Clamped Circular Plate ($r=\sqrt{\pi}$)
Curve 3: Clamped Circular Plate ($r=2$)

Remark. Using our lower bound $\lambda_1^{(3)}$ for a *single value* τ_0 of τ we can easily compute lower bounds for λ_1 for every value of τ. This result can be obtained by combining our method with an idea of R. V. Southwell.[3] In fact, the lowest eigenvalue $\lambda_1 = \lambda_1(\tau)$ is given by the minimum of $U(w)/H(w)$ under the con-

[3] H. Lamb and R. V. Southwell, Proc. Royal Soc. Ser. A, 99, 272 (1921).

ditions (2) and (3). Denoting in (11) the first and second integrals by $J(w)$ and $D(w)$ respectively we have

$$U/H = J/H + \tau D/H = J/H + \tau_0 D/H + (\tau - \tau_0)D/H. \tag{35}$$

Since $D(w)/H(w)$ is obviously greater than 2 (i.e., greater than the lowest eigenvalue of the vibrating membrane) we have for all values $\tau > \tau_0$

$$\lambda_1(\tau) > \lambda_1(\tau_0) + 2(\tau - \tau_0) > \lambda_1^{(3)}(\tau_0) + 2(\tau - \tau_0).$$

Putting $\tau_0 = 5$ we give in Table II the values of $\lambda_1^{(3)}(5) + 2(\tau - 5)$.

TABLE II

τ	$\lambda_1^3(5)+2(\tau-5)$
5	25.236
10	35.236
15	45.236
20	55.236
30	75.236
50	115.23
100	215.23
200	415.23

It will be seen that these lower bounds for λ_1 are smaller than the lower bounds computed by our method alone.

A DIRECT IMAGE ERROR THEORY*

M. HERZBERGER

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1. In a previous paper[1] the author proposed a direct approach to the problem of geometrical optics. In this paper we shall give a new image error theory, to the fifth order, which seems to be more adapted to the practical problems than former theories. We are given a rotationally symmetric system. Let us choose two Cartesian systems, one in object space and the other in image space, such that the x, x' and y, y' axes have the same directions and the z, z' axes coincide with the optical axis of the system.

A ray is given in object (image) space by the coordinates x, y, (x', y') of its intersection point with the plane $z=0$, ($z'=0$). The optical direction cosines (the direction cosines multiplied by the refractive indices n and n', respectively) may be designated by the Greek letters

$$\xi, \eta, \zeta = \sqrt{n^2 - (\xi^2 + \eta^2)}; \qquad \xi', \eta', \zeta' = \sqrt{n'^2 - (\xi'^2 + \eta'^2)}.$$

The fundamental problem of practical optics is to find x', y', ξ', η', when x, y, ξ, η are given. Because of the rotational symmetry, four functions, A, B, C, D, exist such that,

$$\begin{aligned} x' &= Ax + B\xi, \qquad & \xi' &= Cx + D\xi, \\ y' &= Ay + B\eta, \qquad & \eta' &= Cy + D\eta. \end{aligned} \tag{1}$$

where A, B, C, D depend only on the three symmetric functions u_1, u_2, u_3 of our coordinates:

$$u_1 = \tfrac{1}{2}(x^2 + y^2), \qquad u_2 = x\xi + y\eta, \qquad u_3 = \tfrac{1}{2}(\xi^2 + \eta^2). \tag{2}$$

We found in the previous paper[1] that, according to the laws of geometrical optics, A, B, C, D cannot be arbitrary functions, but must fulfill one finite and three differential equations, viz.,

$$AD - BC = 1$$

and

$$\begin{aligned} A\left(\frac{\partial C}{\partial u_2} - \frac{\partial D}{\partial u_1}\right) - C\left(\frac{\partial A}{\partial u_2} - \frac{\partial B}{\partial u_1}\right) + 2u_1\left(\frac{\partial A}{\partial u_1}\frac{\partial C}{\partial u_2} - \frac{\partial A}{\partial u_2}\frac{\partial C}{\partial u_1}\right) \\ + u_2\left(\frac{\partial A}{\partial u_1}\frac{\partial D}{\partial u_2} - \frac{\partial A}{\partial u_2}\frac{\partial D}{\partial u_1} + \frac{\partial B}{\partial u_1}\frac{\partial C}{\partial u_2} - \frac{\partial B}{\partial u_2}\frac{\partial C}{\partial u_1}\right) \\ + 2u_3\left(\frac{\partial B}{\partial u_1}\frac{\partial D}{\partial u_2} - \frac{\partial B}{\partial u_2}\frac{\partial D}{\partial u_1}\right) = 0, \end{aligned}$$

* Received Nov. 26, 1943.

[1] M. Herzberger, *Direct methods in geometrical optics*, Trans. Amer. Math. Soc., **53**, 218 (1943).

$$
\begin{aligned}
A\left(\frac{\partial C}{\partial u_3}-\frac{\partial D}{\partial u_2}\right)&+B\left(\frac{\partial C}{\partial u_2}-\frac{\partial D}{\partial u_1}\right)-C\left(\frac{\partial A}{\partial u_3}-\frac{\partial B}{\partial u_2}\right)\\
&-D\left(\frac{\partial A}{\partial u_2}-\frac{\partial B}{\partial u_1}\right)+2u_1\left(\frac{\partial A}{\partial u_1}\frac{\partial C}{\partial u_3}-\frac{\partial A}{\partial u_3}\frac{\partial C}{\partial u_1}\right)\\
&+u_2\left(\frac{\partial A}{\partial u_1}\frac{\partial D}{\partial u_3}-\frac{\partial A}{\partial u_3}\frac{\partial D}{\partial u_1}+\frac{\partial B}{\partial u_1}\frac{\partial C}{\partial u_3}-\frac{\partial B}{\partial u_3}\frac{\partial C}{\partial u_1}\right)\\
&+2u_3\left(\frac{\partial B}{\partial u_1}\frac{\partial D}{\partial u_3}-\frac{\partial B}{\partial u_3}\frac{\partial D}{\partial u_1}\right)=0,\\
B\left(\frac{\partial C}{\partial u_3}-\frac{\partial D}{\partial u_2}\right)&-D\left(\frac{\partial A}{\partial u_3}-\frac{\partial B}{\partial u_2}\right)+2u_1\left(\frac{\partial A}{\partial u_2}\frac{\partial C}{\partial u_3}-\frac{\partial A}{\partial u_3}\frac{\partial C}{\partial u_2}\right)\\
&+u_2\left(\frac{\partial A}{\partial u_2}\frac{\partial D}{\partial u_3}-\frac{\partial A}{\partial u_3}\frac{\partial D}{\partial u_2}+\frac{\partial B}{\partial u_2}\frac{\partial C}{\partial u_3}-\frac{\partial B}{\partial u_3}\frac{\partial C}{\partial u_2}\right)=0.^*
\end{aligned}
\tag{A}
$$

It is the purpose of this paper to develop from formulae (A) the theory of image errors. Developing A, B, C, D into a series with respect to u_1, u_2, u_3, we can write

$$
\begin{aligned}
A = A_0 + A_1u_1 + A_2u_2 + A_3u_3&\\
+ \tfrac{1}{2}(A_{11}u_1^2 + 2A_{12}u_1u_2 + A_{22}u_2^2 + \cdots + A_{33}u_3^2),&
\end{aligned}
\tag{3}
$$

and for B, C, D, correspondingly. Inserting (3) in (A) and comparing coefficients leads to the first-order equation:

$$A_0D_0 - B_0C_0 = 1; \tag{4}$$

the third-order equations

$$
\begin{aligned}
A_0D_1 + A_1D_0 &= B_0C_1 + B_1C_0,\\
A_0D_2 + A_2D_0 &= B_0C_2 + B_2C_0,\\
A_0D_3 + A_3D_0 &= B_0C_3 + B_3C_0;
\end{aligned}
\tag{5a}
$$

and

$$
\begin{aligned}
A_0(C_2 - D_1) - C_0(A_2 - B_1) &= 0,\\
A_0(C_3 - D_2) + B_0(C_2 - D_1) - C_0(A_3 - B_2) - D_0(A_2 - B_1) &= 0,\\
B_0(C_3 - D_2) - D_0(A_3 - B_2) &= 0;
\end{aligned}
\tag{5b}
$$

and finally the fifth-order equations:

---

* If we differentiate the finite equation above with respect to u_1, u_2, u_3 and subtract from each of the equations (7) of the former paper,[1] equation (A) above results.

$$
\begin{aligned}
A_0D_{11} + D_0A_{11} - B_0C_{11} - C_0B_{11} &= 2(B_1C_1 - A_1D_1),\\
A_0D_{12} + D_0A_{12} - B_0C_{12} - C_0B_{12} &= B_1C_2 + B_2C_1 - A_1D_2 - A_2D_1,\\
A_0D_{13} + D_0A_{13} - B_0C_{13} - C_0B_{13} &= B_1C_3 + B_3C_1 - A_1D_3 - A_3D_1,\\
A_0D_{22} + D_0A_{22} - B_0C_{22} - C_0B_{22} &= 2(B_2C_2 - A_2D_2),\\
A_0D_{23} + D_0A_{23} - B_0C_{23} - C_0B_{23} &= B_2C_3 + B_3C_2 - A_2D_3 - A_3D_2,\\
A_0D_{33} + D_0A_{33} - B_0C_{33} - C_0B_{33} &= 2(B_3C_3 - A_3D_3),
\end{aligned} \tag{6a}
$$

and

$$
\begin{aligned}
&A_0(C_{21}-D_{11})-C_0(A_{21}-B_{11})=C_1(A_2-B_1)-A_1(C_2-D_1)-2(A_1C_2-A_2C_1),\\
&A_0(C_{22}-D_{12})-C_0(A_{22}-B_{12})=C_2(A_2-B_1)-A_2(C_2-D_1)-(A_1D_2-A_2D_1)\\
&\qquad\qquad -(B_1C_2-B_2C_1),\\
&A_0(C_{23}-D_{13})-C_0(A_{23}-B_{13})=C_3(A_2-B_1)-A_3(C_2-D_1)-2(B_1D_2-B_2D_1),\\
&A_0(C_{31}-D_{21})+B_0(C_{21}-D_{11})-C_0(A_{31}-B_{21})-D_0(A_{21}-B_{11})\\
&\quad =-B_1(C_2-D_1)-A_1(C_3-D_2)+D_1(A_2-B_1)+C_1(A_3-B_2)-2(A_1C_3-A_3C_1),\\
&A_0(C_{32}-D_{22})+B_0(C_{22}-D_{12})-C_0(A_{32}-B_{22})-D_0(A_{22}-B_{12})\\
&\quad =-B_2(C_2-D_1)-A_2(C_3-D_2)+D_2(A_2-B_1)+C_2(A_3-B_2)\\
&\quad -(A_1D_3-A_3D_1)-(B_1C_3-B_3C_1),\\
&A_0(C_{33}-D_{23})+B_0(C_{23}-D_{13})-C_0(A_{33}-B_{23})-D_0(A_{23}-B_{13})\\
&\quad =-B_3(C_2-D_1)-A_3(C_3-D_2)+D_3(A_2-B_1)+C_3(A_3-B_2)-2(B_1D_3-B_3D_1),\\
&B_0(C_{31}-D_{21})-D_0(A_{31}-B_{21})=D_1(A_3-B_2)-B_1(C_3-D_2)-2(A_2C_3-A_3C_2),\\
&B_0(C_{32}-D_{22})-D_0(A_{32}-B_{22})=D_2(A_3-B_2)-B_2(C_3-D_2)\\
&\qquad\qquad -(A_2D_3-A_3D_2)-(B_2C_3-B_3C_2),\\
&B_0(C_{33}-D_{23})-D_0(A_{33}-B_{23})=D_3(A_3-B_2)-B_3(C_3-D_2)-2(B_2D_3-B_3D_2).
\end{aligned} \tag{6b}
$$

Moreover, the (6+9) equations (6) are not independent; they are connected by the identity

$$
\begin{aligned}
&[A_0(C_{32}-D_{22})+B_0(C_{22}-D_{12})-C_0(A_{32}-B_{22})-D_0(A_{22}-B_{12})]\\
&\quad +[A_0D_{22}+D_0A_{22}-B_0C_{22}-C_0B_{22}]-[A_0D_{13}+D_0A_{13}-B_0C_{13}-C_0B_{13}]\\
&\quad -[B_0(C_{31}-D_{21})-D_0(A_{31}-B_{21})]-[A_0(C_{23}-D_{13})-C_0(A_{23}-B_{13})]\equiv 0.
\end{aligned} \tag{7}
$$

2. Gaussian optics. Let us consider first the rays in the neighborhood of the axis. Let us assume that u_1, u_2, u_3 are so small that we can assume functions A, B, C, D to be equal to their constant members:

$$
\begin{aligned}
x' &= A_0x + B_0\xi, \qquad & \xi' &= C_0x + D_0\xi,\\
y' &= A_0y + B_0\eta, \qquad & \eta' &= C_0y + D_0\eta;
\end{aligned} \tag{8}
$$

where $A_0D_0-B_0C_0=1$.

The evaluation of these equations and the investigation of the geometrical meaning of the coefficients form the content of Gaussian optics.

We shall not go into great detail here, but refer the reader to the discussion in the Journal of the Optical Society of America.[2]

Equations (8) can be inverted, and we obtain then

$$\begin{aligned} x &= D_0x' - B_0\xi', & \xi &= -C_0x' + A_0\xi', \\ y &= D_0y' - B_0\eta', & \eta &= -C_0y' + A_0\eta'. \end{aligned} \tag{9}$$

Let us investigate what happens if one of the coefficients vanishes.

$A_0=0$ means that for $\xi=\eta=0$, $x'=y'=0$, which means that the bundle of rays parallel to the axis converges to the image origin. We say that the image origin is at the focal point of the system.

$B_0=0$ means that for $x=y=0$, $x'=y'=0$. The rays through the object origin meet at the image origin. We say then that object and image origin are optically *conjugate*.

$C_0=0$ means that $\xi=\eta=0$ implies $\xi'=\eta'=0$, or, a bundle of rays entering the system parallel to the axis emerges parallel to the axis. The system is a *telescopic* system.

$D_0=0$ means that $x=y=0$ implies $\xi'=\eta'=0$. The rays through the object origin emerge parallel to the axis. The object origin is the object-side (front) *focal point*.

3. Image error functions. Let us for finite rays* project image point and direction back into the object space, according to Gaussian optics. That means we form equations (9) for our finite rays. The ensuing expressions may be called the equivalent object coordinates $\bar{x}$, $\bar{y}$, $\bar{\xi}$, $\bar{\eta}$. We have from (9) and (1)

$$\begin{aligned} \bar{x} &= D_0x' - B_0\xi' = (D_0A - B_0C)x + (D_0B - B_0D)\xi = ax + b\xi, \\ \bar{\xi} &= -C_0x' + A_0\xi' = (-C_0A + A_0C)x + (-C_0B + A_0D)\xi = cx + d\xi; \end{aligned} \tag{10}$$

and analogously,

$$\begin{aligned} \bar{y} &= ay + b\eta, \\ \bar{\eta} &= cy + d\eta. \end{aligned}$$

a, b, c, d are with A, B, C, D functions of u_1, u_2, u_3, and we have

$$\begin{aligned} a_0 &= d_0 = 1, \\ b_0 &= c_0 = 0, \end{aligned} \tag{11}$$

the values a_0, b_0, c_0, d_0, being the limits of a, b, c, d for $u_i=0$. If Gaussian optics were correct, we would have equation (11) for all values of u_i, that is for finite aperture and finite field. The deviation from its constant term as a

[2] M. Herzberger, *On the fundamental optical invariant, the optical tetrality principle, and on the new development of Gaussian optics based on this law*, J. Opt. Soc. Amer. 25, 295–304 (1935).

* The expression finite is used in distinction from paraxial rays, rays near the axis.

function of aperture and field is therefore a measure of the image errors. We call a, b, c, d the error functions.

In the nomenclature of matrix algebra we can express these equations as follows:

Let

$$\begin{pmatrix} x' \\ \xi' \end{pmatrix} = M\begin{pmatrix} x \\ \xi \end{pmatrix}, \qquad \begin{pmatrix} x_0' \\ \xi_0' \end{pmatrix} = M_0\begin{pmatrix} x \\ \xi \end{pmatrix},$$
$$\begin{pmatrix} y' \\ \eta' \end{pmatrix} = M\begin{pmatrix} y \\ \eta \end{pmatrix}, \qquad \begin{pmatrix} y_0' \\ \eta_0' \end{pmatrix} = M_0\begin{pmatrix} y \\ \eta \end{pmatrix}. \tag{12}$$

x_0', y_0', ξ_0', η_0' would be the coordinates of the image ray if Gaussian optics were valid. Equations (12) combine equations (1) and (8), M being the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and M_0 being the matrix $\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$.

We have then

$$\begin{pmatrix} x \\ \tilde{\xi} \end{pmatrix} = M_0^{-1}M\begin{pmatrix} x \\ \xi \end{pmatrix} = m\begin{pmatrix} x \\ \xi \end{pmatrix},$$

where

$$m = M_0^{-1}M, \qquad M = M_0m. \tag{13}$$

From (13) it is obvious that the determinant of m is equal to unity. Therefore,

$$ad - bc = 1. \tag{14}$$

The reader can verify for himself that a, b, c, d fulfill equations (A) and therefore equations (5) and (6), which simplify considerably, owing to the fact that (11) is fulfilled.

Equations (13) can be written explicitly

$$\begin{aligned} a &= D_0A - B_0C, & A &= A_0a + B_0c, \\ b &= D_0B - B_0D, & B &= A_0b + B_0d, \\ c &= -C_0A + A_0C, & C &= C_0a + D_0c, \\ d &= -C_0B + A_0D, & D &= C_0b + D_0d. \end{aligned} \tag{15}$$

Differentiation and substitution in (A) prove our statement.

4. Third-order theory. The third-order image errors are usually called Seidel errors.

From our point of view, we obtain the image errors by inserting (11) into (5). Abbreviating $(\partial a/\partial u_k)_{u_i=0}$ by a_k

$$\begin{aligned} a_1 + d_1 &= 0 & c_2 &= d_1 \\ a_2 + d_2 &= 0 & c_3 &= b_1 \\ a_3 + d_3 &= 0 & a_3 &= b_2. \end{aligned} \tag{16}$$

Equations (16) lead to the conclusion that only six of these twelve coefficients are independent. Equations (16) are identically fulfilled by selecting six parameters k_{ik} with permutable indices such that

$$\begin{array}{llll} a_1 = k_{21} & b_1 = k_{31} & c_1 = -k_{11} & d_1 = -k_{21} \\ a_2 = k_{22} & b_2 = k_{32} & c_2 = -k_{12} & d_2 = -k_{22} \\ a_3 = k_{23} & b_3 = k_{33} & c_3 = -k_{13} & d_3 = -k_{23}. \end{array} \tag{17}$$

Geometrical investigation (which we omit) would show that (if object and image planes are optically conjugated), k_{33} may be interpreted as the coefficient of spherical aberration for the object origin; k_{23} as the coma coefficient; k_{22} and k_{13} as coefficients of the field errors; and k_{12} as the coefficient of the distortion for an object at the origin and an infinite entrance pupil.

On the other hand, k_{11} may be considered as the coefficient of spherical aberration for an infinite object; k_{12} as coma coefficient; k_{13} and k_{22} as field errors; and k_{23} as the coefficient of distortion for an infinite object and the entrance pupil at the object origin.

The connections between these errors are well-known laws of the Seidel theory.

The method developed here differs from the usual methods in that, first, we do not assume the coordinate origins to be in conjugated planes, and, second, we do not restrict ourselves to the consideration of the deviation of the object point, but investigate at the same time the deviation of the direction of the ray. Equations (10) give, within the limits of our Seidel region, the following equations:

$$\begin{aligned} \tilde{x} - x &= (k_{21}u_1 + k_{22}u_2 + k_{23}u_3)x + (k_{31}u_1 + k_{32}u_2 + k_{33}u_3)\xi, \\ \xi - \tilde{\xi} &= (k_{11}u_1 + k_{12}u_2 + k_{13}u_3)x + (k_{21}u_1 + k_{22}u_2 + k_{23}u_3)\xi; \end{aligned} \tag{18}$$

and y and η analogously.

We recommend a detailed study of these equations and their derivatives with respect to x and ξ for meridian rays ($y=\eta=0$), especially in the case where our origins are not conjugated.

5. Fifth-order aberrations. For the fifth-order aberrations we find from (6a) and (6b) the following fourteen independent equations between the twenty-four coefficients $a_{\iota\kappa}$, etc.

Making use of equations (11) and (17) we find that

$$\begin{array}{ll} a_{11}+d_{11}=2(k_{12}^2-k_{11}k_{13}), & a_{22}+d_{22}=2(k_{22}^2-k_{12}k_{23}), \\ a_{12}+d_{12}=2k_{12}k_{22}-k_{12}k_{13}-k_{11}k_{23}, & a_{23}+d_{23}=2k_{22}k_{23}-k_{13}k_{23}-k_{12}k_{33}, \\ a_{13}+d_{13}=2k_{12}k_{23}-k_{13}^2-k_{11}k_{33}, & a_{33}+d_{33}=2(k_{23}^2-k_{13}k_{33}), \\ c_{21}-d_{11}=2k_{12}^2+k_{11}k_{13}-3k_{11}k_{22}, & b_{12}-a_{13}=k_{13}^2+k_{13}k_{22}-2k_{12}k_{23}, \\ c_{22}-d_{12}=2k_{12}k_{13}-k_{12}k_{22}-k_{11}k_{23}, & b_{22}-a_{23}=2k_{13}k_{23}-k_{12}k_{33}-k_{22}k_{23}, \\ c_{23}-d_{13}=k_{13}^2+k_{13}k_{22}-2k_{12}k_{23}, & b_{23}-a_{33}=2k_{23}^2+k_{13}k_{33}-3k_{22}k_{33}, \\ c_{31}+b_{11}=3(k_{12}k_{13}-k_{11}k_{23}), & c_{33}+b_{13}=3(k_{13}k_{23}-k_{12}k_{33}). \end{array} \tag{19}$$

Here again, we can express the twenty-four quantities in terms of the third-order coefficients and nine parameters $k_{\iota\kappa\lambda}$.

6. The single sphere and the plane. In our previous paper we were able to calculate the functions A, B, C, D for the case of a plane and a sphere.

In the case of a plane, we put object and image origins at the point where the axis intersects the plane and found that

$$x' = x, \qquad y' = y,$$
$$\xi' = \xi, \qquad \eta' = \eta; \tag{20}$$

or $A=D=1$, $B=C=0$. In this case we have $a=d=1$, $b=c=0$, and all the image errors vanish.

In the case of a spherical surface, we put the object and image origins at the center, and found that

$$x' = Ax, \qquad y' = Ay,$$
$$\xi' = Cx + D\xi, \qquad \eta' = Cy + D\eta; \tag{21}$$

where

$$C=\frac{1}{r}\left\{\sqrt{n'^2-\frac{2n^2u_1-u_2^2}{r^2}}-\sqrt{n^2-\frac{2n^2u_1-u_2^2}{r^2}}\right\},$$
$$n^2D=\frac{2n^2u_1-u_2^2}{r^2}+\sqrt{\left(n'^2-\frac{2n^2u_1-u}{r^2}\right)\left(n^2-\frac{2n^2u_1-u}{r^2}\right)}-Cu_2, \tag{22}$$
$$A=\frac{1}{D}\cdot$$

If we develop A, B, C as functions of u_1, u_2, u_3, we obtain the Seidel and fifth-order coefficients. Taking care of (5) and observing that

$$A_0=\frac{n}{n'}, \qquad B_0=0,$$
$$C_0=\frac{n'-n}{r}, \qquad D_0=\frac{n'}{n}, \tag{23}$$

we finally find the image-error coefficients:

$$a_1=\frac{n^2}{r^2}\left(\frac{1}{n'}-\frac{1}{n}\right)^2, \qquad a_2=\frac{1}{r}\left(\frac{1}{n'}-\frac{1}{n}\right), \qquad a_3=0,$$
$$b_1=0, \qquad b_2=0, \qquad b_3=0,$$
$$c_1=-\frac{n^4}{r^3}\left(\frac{1}{n'^3}-\frac{4}{n'^2n}-\frac{4}{n'n^2}+\frac{1}{n^3}\right), \qquad c_2=-\frac{n^2}{r^2}\left(\frac{1}{n'}-\frac{1}{n}\right)^2, \qquad c_3=0, \tag{24a}$$
$$d_1=-\frac{n^2}{r^2}\left(\frac{1}{n'}-\frac{1}{n}\right)^2, \qquad d_2=-\frac{1}{r}\left(\frac{1}{n'}-\frac{1}{n}\right), \qquad d_3=0;$$

and the fifth-order coefficients:

$$a_{11}=\frac{n^4}{r^4}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\left(\frac{3}{n'^2}-\frac{2}{n'n}+\frac{3}{n^2}\right), \qquad a_{13}=0,$$

$$a_{12}=\frac{n^2}{r^3}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{2}{n'^2}-\frac{3}{n'n}+\frac{2}{n^2}\right), \qquad a_{23}=0,$$

$$a_{22}=\frac{1}{r^2}\left(\frac{1}{n'}-\frac{1}{n}\right)^2, \qquad a_{33}=0,$$

$$b_{11}=b_{12}=b_{22}=b_{13}=b_{23}=b_{33}=0,$$

$$c_{11}=-\frac{3n^6}{r^5}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{1}{n'^4}-\frac{3}{n'^3n}+\frac{3}{n'^2n^2}-\frac{3}{n'n^3}+\frac{1}{n^4}\right), \qquad c_{13}=0,$$

$$c_{12}=-\frac{n^4}{r^4}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\left(\frac{2}{n'^2}-\frac{3}{n'n}+\frac{2}{n^2}\right), \qquad c_{23}=0, \tag{24b}$$

$$c_{22}=-\frac{n^2}{r^3}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{1}{n'^2}-\frac{1}{n'n}+\frac{1}{n^2}\right), \qquad c_{33}=0,$$

$$d_{11}=-\frac{n^4}{r^4}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^2, \qquad d_{13}=0,$$

$$d_{12}=-\frac{n^2}{r^3}\left(\frac{1}{n'}-\frac{1}{n}\right)\frac{1}{n'n}, \qquad d_{23}=0,$$

$$d_{22}=\frac{1}{r^2}\left(\frac{1}{n'}-\frac{1}{n}\right)^2, \qquad d_{33}=0,$$

equations which fulfill all the conditions of equations (6).

The nonvanishing seventh-order coefficients for one surface would be:

$$a_{111}=\frac{3n^6}{r^6}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\left(\frac{5}{n'^4}-\frac{6}{n'^3n}+\frac{10}{n'^2n^2}-\frac{6}{n'n^3}+\frac{5}{n^4}\right),$$

$$a_{112}=\frac{n^4}{r^5}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{8}{n'^4}-\frac{19}{n'^3n}+\frac{25}{n'^2n^2}-\frac{19}{n'n^3}+\frac{8}{n^4}\right),$$

$$a_{122}=\frac{n^2}{r^4}\left(\frac{1}{n'}-\frac{1}{n}\right)^2\left(\frac{3}{n'^2}-\frac{2}{n'n}+\frac{3}{n^2}\right),$$

$$a_{222}=-\frac{3}{r^2}\left(\frac{1}{n'}-\frac{1}{n}\right)\frac{1}{n'n},$$

$$c_{111}=-\frac{3n^8}{r^7}\left(\frac{1}{n'}-\frac{1}{n}\right)\left(\frac{5}{n'^6}-\frac{17}{n'^5n}+\frac{26}{n'^4n^2}-\frac{33}{n'^3n^3}+\frac{26}{n'^2n^4}-\frac{17}{n'n^5}+\frac{5}{n^6}\right),$$

$$c_{112} = -\frac{n^6}{r^6}\left(\frac{1}{n'} - \frac{1}{n}\right)^2\left(\frac{8}{n'^4} - \frac{19}{n'^3 n} + \frac{25}{n'^2 n^2} - \frac{19}{n' n^3} + \frac{8}{n^4}\right),$$

$$c_{122} = -\frac{n^4}{r^5}\left(\frac{1}{n'} - \frac{1}{n}\right)\left(\frac{3}{n'^4} - \frac{7}{n'^3 n} + \frac{11}{n'^2 n^2} - \frac{7}{n' n^3} + \frac{3}{n^4}\right), \tag{25}$$

$$c_{222} = -\frac{3n^2}{r^4}\left(\frac{1}{n'} - \frac{1}{n}\right)^2 \frac{1}{n' n},$$

$$d_{111} = -\frac{3n^6}{r^6}\left(\frac{1}{n'} - \frac{1}{n}\right)^2\left(\frac{1}{n'^4} + \frac{2}{n'^3 n} + \frac{2}{n'^2 n^2} + \frac{2}{n' n^3} + \frac{1}{n^4}\right),$$

$$d_{112} = -\frac{n^4}{r^5}\left(\frac{1}{n'} - \frac{1}{n}\right)\left(\frac{1}{n'^2} + \frac{1}{n' n} + \frac{1}{n^2}\right)\frac{1}{n' n},$$

$$d_{122} = \frac{n^2}{r^4}\left(\frac{1}{n'} - \frac{1}{n}\right)^2\left(\frac{1}{n'^2} + \frac{2}{n' n} + \frac{1}{n^2}\right),$$

$$d_{222} = \frac{3}{r^3}\left(\frac{1}{n'} - \frac{1}{n}\right)\frac{1}{n' n}.$$

THE IMPEDANCE OF A TRANSVERSE WIRE IN A RECTANGULAR WAVE GUIDE*

BY

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The purpose of this paper is to derive approximate formulae for the impedance of a transverse wire carrying uniform current (Fig. 1).

The total impedance Z to the current through the wire may be defined as

$$Z = \frac{V}{I}, \tag{1}$$

where V is the applied voltage and I is the electric current in the wire. The total electromotive force V is the sum[1] of the internal electromotive force V_i and the external electromotive force V_e

$$V = V_i + V_e. \tag{2}$$

Correspondingly, we have an internal impedance Z_i of the wire and the external impedance Z_e. By (1) these two impedances are in series with each other

$$Z = Z_i + Z_e. \tag{3}$$

If the guide is infinitely long on both sides of the wire, the external impedance (above the absolute cut-off frequency) is complex

$$Z_e = R_e + iX_e. \tag{4}$$

The resistance term represents radiation of energy into the guide. If the frequency is within the principal frequency range and if K is the characteristic impedance of the guide to the dominant wave, as seen from the wire,[2] then evidently

$$R_e = \tfrac{1}{2}K. \tag{5}$$

We shall now calculate the impedance of the wire on the assumption that its radius is small. The current in the wire will generate transverse electric waves in which the field is independent of the y-coordinate. The general form of the field (for $z > 0$) is

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* Received February 8, 1943.

[1] For an explanation, see S. A. Schelkunoff, *Electromagnetic Waves*, D. Van Nostrand Company, Inc., 1943.

[2] And not from a plane current sheet generating a pure dominant wave.

$$
\begin{aligned}
H_x(x, z) &= \sum_{l=1}^{\infty} H_l \sin \frac{l\pi x}{a} e^{-\Gamma_l z}, \\
E_y(x, z) &= -\sum_{l=1}^{\infty} K_l H_l \sin \frac{l\pi x}{a} e^{-\Gamma_l z},
\end{aligned}
\tag{6}
$$

where Γ_l and K_l are respectively the propagation constant and the specific impedance of a typical $TE_{l,0}$-wave

$$
\Gamma_l = \sqrt{\frac{l^2\pi^2}{a^2} - \frac{4\pi^2}{\lambda^2}}, \qquad K_l = \frac{i\omega\mu}{\Gamma_l}. \tag{7}
$$

The propagation constant of the dominant $TE_{1,0}$-wave is

$$
\Gamma_1 = \sqrt{\frac{\pi^2}{a^2} - \frac{4\pi^2}{\lambda^2}} = \frac{2\pi i}{\lambda}\sqrt{1 - \frac{\lambda^2}{4a^2}}. \tag{8}
$$

The dominant wavelength range extends from $\lambda_1 = 2a$ to $\lambda_2 = a$, λ_2 being the cut-off wavelength of $TE_{2,0}$-wave. If $a < \lambda < 2a$, all the propagation constants of secondary waves are real

$$
\Gamma_l = \frac{l\pi}{a}\sqrt{1 - \frac{4a^2}{l^2\lambda^2}}, \qquad l > 1. \tag{9}
$$

For the specific impedances we obtain

$$
\begin{aligned}
K_1 &= \eta\left(1 - \frac{\lambda^2}{4a^2}\right)^{1/2}, \qquad \eta = \sqrt{\frac{\mu}{\epsilon}}, \\
K_l &= \frac{i\omega\mu a}{l\pi}\left(1 - \frac{4a^2}{l^2\lambda^2}\right)^{-1/2} = i\eta\frac{2a}{l\lambda}\left(1 - \frac{4a^2}{l^2\lambda^2}\right)^{-1/2}.
\end{aligned}
\tag{10}
$$

The external electromotive force V_e necessary to support current I through a thin wire of radius r is

$$
V_e = -bE_y(d, r), \tag{11}
$$

where d is the distance shown in Fig. 1. This equation presupposes that the

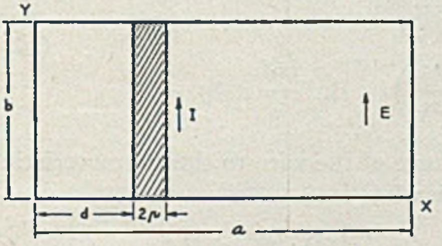

FIG. 1

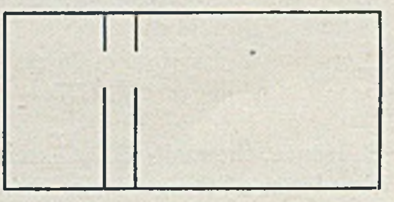

FIG. 2

current is distributed uniformly around the wire. As the radius of the wire increases, the current distribution gradually begins to depart from uniformity. From (1), (6), and (11), we have

$$Z_e = \frac{V_e}{I} = \frac{b}{I} \sum_{l=1}^{\infty} K_l H_l \sin \frac{l\pi d}{a} e^{-\Gamma_l r}. \tag{12}$$

The next step is to calculate the coefficients H_l. We shall assume that the wire is so thin that the field outside the wire could be regarded as nearly equal to that of an infinitely thin electric current filament along the axis of the wire. For an infinitely thin filament, we have

$$H_l = \lim \frac{2}{a} \int_d^{d+2r} \frac{I}{4r} \sin \frac{l\pi x}{a} dx \text{ as } r \to 0. \tag{13}$$

Integrating and passing to the limit, we obtain

$$H_l = \frac{I}{a} \sin \frac{l\pi d}{a} \cdot \tag{14}$$

Substituting (14) in (12), we have

$$Z_e = \frac{b}{a} \sum_{l=1}^{\infty} K_l \sin^2 \frac{l\pi d}{a} e^{-\Gamma_l r}; \tag{15}$$

and, therefore,

$$\begin{aligned} R_e &\doteq \frac{b}{a} K_1 \sin^2 \frac{\pi d}{a}, \\ K = 2R_e &= \frac{2b}{a} K_1 \sin^2 \frac{\pi d}{a}, \\ iX_e &= \frac{b}{a} \sum_{l=2}^{\infty} K_l \sin^2 \frac{l\pi d}{a} e^{-\Gamma_l r}. \end{aligned} \tag{16}$$

Substituting from (10), we obtain

$$\begin{aligned} K &= \eta \frac{2b}{a} \sin^2 \frac{\pi d}{a} \left(1 - \frac{\lambda^2}{4a^2}\right)^{-1/2} \\ X_e &= \eta \frac{2b}{\lambda} \sum_{l=2}^{\infty} \frac{1}{l} \left(1 - \frac{4a^2}{l^2\lambda^2}\right)^{-1/2} \sin^2 \frac{l\pi d}{a} e^{-\Gamma_l r}. \end{aligned} \tag{17}$$

Hence, the ratio of the external reactance of the wire to the characteristic impedance of the guide (as seen from the wire) is

$$\frac{X_e}{K} = \frac{a}{\lambda} \sqrt{1 - \frac{\lambda^2}{4a^2}} \csc^2 \frac{\pi d}{a} \sum_{l=2}^{\infty} \frac{1}{l} \left(1 - \frac{4a^2}{l^2\lambda^2}\right)^{-1/2} \sin^2 \frac{l\pi d}{a} e^{-\Gamma_l r}. \tag{18}$$

It is evident that the total inductance of the wire is a series combination of inductances associated with the individual secondary TE waves, generated by the current in the wire.

For the internal impedance of the wire, we have[3]

$$Z_i = \eta_i \frac{bI_0(\sigma_i r)}{2\pi r I_1(\sigma_i r)},$$

$$\sigma_i = \sqrt{i\omega\mu_i(g_i + i\omega\epsilon_i)}, \qquad \eta_i = \sqrt{\frac{i\omega\mu_i}{g_i + i\omega\epsilon_i}}. \tag{19}$$

This is a general expression applicable to dielectric wires as well as to metal wires. In the case of metal wires, we let $\epsilon_i = 0$. Usually, the radii of metal wires will be sufficiently large to make the modified Bessel functions in (19) nearly equal so that approximately

$$Z_i = \frac{b\eta_i}{2\pi r} = \frac{b}{2\pi r}\sqrt{\frac{i\omega\mu_i}{g_i}} = \frac{b}{2\pi r}\sqrt{\frac{\pi f\mu_i}{g_i}}\,(1 + i). \tag{20}$$

If r/a is small, the series (18) converges slowly. The difficulty may be obviated with the aid of the following device. Let

$$u = \sum_l u_l \tag{21}$$

be a slowly converging series; let

$$v = \sum_l v_l \tag{22}$$

be a series of terms approximating (21) in such a way that the approximation becomes increasingly better as l increases; then

$$u = \sum_l v_l + \sum_l (u_l - v_l) \tag{23}$$

so that (21) can be regarded as the sum of two series, of which the second is more rapidly convergent than the original series. If now the sum of the first series in (23) happens to be known, we have succeeded in replacing the original slowly convergent by a more rapidly convergent series.

We shall apply this device to (18). First we rewrite (18) in the following form

$$\frac{X_e}{K} = \frac{1}{4}\sqrt{\frac{4a^2}{\lambda^2} - 1}\,\csc^2\frac{\pi d}{a}\sum_{l=2}^{\infty}\frac{1 - \cos\dfrac{2l\pi d}{a}}{l\sqrt{1 - \dfrac{4a^2}{l^2\lambda^2}}}\,e^{-\Gamma_l r}; \tag{24}$$

[3] See the book mentioned in footnote 1.

then we note that as l tends to infinity, Γ_l tends to $l\pi/a$ and $\sqrt{1-4a^2/l^2\lambda^2}$ tends to unity. Hence, a typical term of the v-series will be $(1/l)[1-\cos(2l\pi d/a)]e^{-l\pi r/a}$, and (24) may be expressed as

$$\frac{X_e}{K}=\frac{1}{4}\sqrt{\frac{4a^2}{\lambda^2}-1}\,\csc^2\frac{\pi d}{a}\left[\sum_{l=2}^{\infty}\frac{1-\cos\dfrac{2l\pi d}{a}}{l}e^{-l\pi r/a}+T\right],$$

$$T=\sum_{l=2}^{\infty}\frac{1-\cos\dfrac{2l\pi d}{a}}{l}\left(\frac{e^{-\Gamma_l r}}{\sqrt{1-\dfrac{4a^2}{l^2\lambda^2}}}-e^{-l\pi r/a}\right). \tag{25}$$

It is known that

$$\sum_{l=1}^{\infty}\frac{1}{l}e^{-lp}\cos lq=-\frac{1}{2}\log\left(1-2e^{-p}\cos q+e^{-2p}\right); \tag{26}$$

therefore,

$$\frac{X_e}{K}=\frac{1}{4}\sqrt{\frac{4a^2}{\lambda^2}-1}\,\csc^2\frac{\pi d}{a}\left[\frac{1}{2}\log\left(1-2e^{-\pi r/a}\cos\frac{2\pi d}{a}+e^{-2\pi r/a}\right)\right.$$
$$\left.-\log\left(1-e^{-\pi r/a}\right)-e^{-\pi r/a}\left(1-\cos\frac{2\pi d}{a}\right)+T\right]. \tag{27}$$

This can be transformed into

$$\frac{X_e}{K}=\frac{1}{4}\sqrt{\frac{4a^2}{\lambda^2}-1}\,\csc^2\frac{\pi d}{a}\left[\frac{1}{2}\log\left(\cosh\frac{\pi r}{a}-\cos\frac{2\pi d}{a}\right)\right.$$
$$\left.-\frac{1}{2}\log 2-\log\sinh\frac{\pi r}{2a}-e^{-\pi r/a}\left(1-\cos\frac{2\pi d}{a}\right)+T\right]. \tag{28}$$

An entirely different expression for Z_e can be obtained by the image method. Assuming again that r is sufficiently small and that the wire is not too close to the boundaries of the guide and that consequently there is no "proximity effect," we can immediately obtain

$$Z_e=\frac{1}{4}\eta\beta b\left[H_0^2(\beta r)+2\sum_{n=1}^{\infty}H_0^2(2n\beta a)-\sum_{n=0}^{\infty}H_0^2(2n\beta a+2\beta d)\right.$$
$$\left.-\sum_{n=0}^{\infty}H_0^2(2n\beta a+2\beta a-2\beta d)\right]. \tag{29}$$

This is a slowly converging series and is useless for direct numerical computations; on the other hand, it may be useful for other purposes. Thus the difference between the external impedances of two wires of different radii is obtained in the following simple form

$$
\begin{aligned}
Z_e(r_2) - Z_e(r_1) &= \tfrac{1}{4}\eta\beta b\left[\overset{2}{H}_0(\beta r_2) - \overset{2}{H}_0(\beta r_1)\right] \\
&= \tfrac{1}{4}\eta\beta b\left[J_0(\beta r_2) - J_0(\beta r_1)\right] + \tfrac{1}{4}\eta\beta b i\left[N_0(\beta r_1) - N_0(\beta r_2)\right].
\end{aligned} \tag{30}
$$

The first term represents the effect[4] of the radius of the wire on the impedance of the guide as seen by the wire. The second term represents the difference between the external reactances of two wires

$$X_e(r_2) - X_e(r_1) = \tfrac{1}{4}\eta\beta b\left[N_0(\beta r_1) - N_0(\beta r_2)\right]. \tag{31}$$

This equation can be used for numerical calculations in conjunction with (28). The slowly converging part of (29) is the mutual impedance between the wire and the wave guide.

An expression for the mutual impedance between two parallel wires in the wave guide can also be obtained by the image method. Thus we have $(d_2 > d_1)$

$$
\begin{aligned}
Z_{12} = \frac{1}{4}\eta\beta b\Bigg[&\overset{2}{H}_0(\beta d_2 - \beta d_1) + \sum_{n=1}^{\infty}\overset{2}{H}_0(2n\beta a + \beta d_2 - \beta d_1) \\
&+ \sum_{n=1}^{\infty}\overset{2}{H}_0(2n\beta a + \beta d_1 - \beta d_2) - \sum_{n=0}^{\infty}\overset{2}{H}_0(2n\beta a + \beta d_1 + \beta d_2) \\
&- \sum_{n=0}^{\infty}\overset{2}{H}_0(2n\beta a + 2\beta a - \beta d_1 - \beta d_2)\Bigg].
\end{aligned} \tag{32}
$$

Next we shall deal briefly with the external impedance of a "split" wire (Fig. 2). Let the electromotive intensity at the surface of the wire and the current in the wire be

$$
\begin{aligned}
E_y &= -\sum_{m=0}^{\infty} E_m \cos\frac{m\pi y}{b}, \\
I &= \sum_{m=0}^{\infty} I_m \cos\frac{m\pi y}{b}
\end{aligned} \tag{33}
$$

The complex flow of power is

$$\Psi^* = \frac{1}{2}\overline{Y}_e V_e V_e^* = \frac{1}{2} b\left[E_0^* I_0 + \frac{1}{2}\sum_{m=1}^{\infty} E_m^* I_m\right], \tag{34}$$

where $\overline{Y}_e$ is the external admittance of the split wire and V_e the voltage across this admittance. Since

$$V_e = bE_0, \tag{35}$$

[4] Which was entirely ignored in the derivation of (28).

we obtain from (34)

$$\overline{Y}_e = \frac{I_0}{V_e} + \frac{1}{2}\sum_{m=1}^{\infty} Y_m \frac{E_m E_m^*}{E_0 E_0^*} \tag{36}$$

where

$$Y_m = \frac{I_m}{bE_m}. \tag{37}$$

The first term in (36) is the external admittance corresponding to a uniform current filament (Fig. 1) and, hence, is equal to the reciprocal of either (15) or (29). The second term is the capacitative admittance (assuming that $\lambda > 2b$) between the external surfaces of the two portions of the transverse wire. The impedance diagram is shown in Fig. 3 where the parallel lines

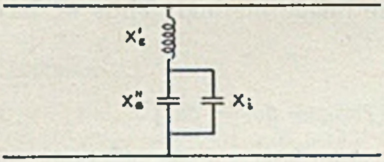

FIG. 3

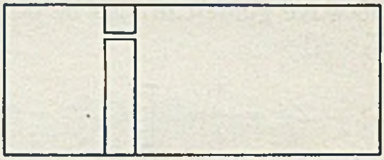

FIG. 4

represent the wave guide, the inductive reactance X_e' is the reactive part of (29), the capacitative reactance X_e'' is the reciprocal of the second term in (36) and X_i is the reactance looking inward from the surface of the gap in the wire. In the case illustrated by Fig. 4 the internal reactance is approximately

$$X_i = \frac{s}{i\omega\epsilon\pi r^2}, \tag{38}$$

where r is the radius of the wire and s is the length of the gap.

The internal reactance of two hollow cylinders as well as the quantity X_e'' will be discussed in a separate paper. Here we shall merely derive general formulae and show that roughly X_e'' is equal to the external capacitative reactance between two sections of a split transverse wire placed across infinite parallel planes.

Each component $I_m \cos(m\pi y/b)$ of the total current I in (33) originates a radial wave of the following type

$$\begin{aligned} H_\phi(\rho) &= \frac{I_m K_1(\gamma_m \rho)}{2\pi r K_1(\gamma_m r)} \cos\frac{m\pi y}{b} \\ E_y(\rho) &= -\frac{\eta\gamma_m I_m K_0(\gamma_m \rho)}{2i\beta r K_1(\gamma_m r)} \cos\frac{m\pi y}{b} \end{aligned} \tag{39}$$

where ρ is the distance from the axis of the current and γ_m is the radial propagation constant of the mth cylindrical wave

$$\gamma_m = \sqrt{\frac{m^2\pi^2}{b^2} - \frac{4\pi^2}{\lambda^2}} = \frac{m\pi}{b}\sqrt{1 - \frac{4b^2}{m^2\lambda^2}}. \tag{40}$$

If $\lambda > 2b$, all the radial propagation constants of order m higher than zero are real. This explains why even the nearest image will have but little effect on the admittance Y_m except when the wire is quite close to the walls of the wave guide, or when λ is nearly equal to $2b$. Even when the wire is close to the walls of the guide only the nearest image will have an appreciable effect on Y_m unless λ is nearly equal to $2b$.

The complete expression for the impedance $Z_m = 1/Y_m$ is

$$Z_m = \frac{\eta\gamma_m b}{2\pi i\beta r K_1(\gamma_m r)}\left[K_0(\gamma_m r) + 2\sum_{n=1}^{\infty} K_0(2n\gamma_m a) - \sum_{n=0}^{\infty} K_0(2n\gamma_m a + 2\gamma_m d) - \sum_{n=0}^{\infty} K_0(2n\gamma_m a + 2\gamma_m a + 2\gamma_m d)\right]. \tag{41}$$

Equation (29) corresponding to the principal cylindrical wave ($m=0$), is of course, a special case of (41). The propagation constant Γ_0 of the principal wave, however, is pure imaginary and, hence, distant images have a pronounced effect on Z_0.

NOTES

ON A. C. AITKEN'S METHOD OF INTERPOLATION*

By WILLY FELLER (*Brown University*)

1. A. C. Aitken[1] has recently devised a method of practical interpolation which is particularly well adapted for computing machines; neither differences nor tables of interpolation coefficients are used, and the necessary operations are most easily performed on modern computing machines. Moreover, the degree of the interpolating polynomial decreases by two at each stage, which minimizes the amount of necessary work. Recent experience has again confirmed that the method is extremely convenient and timesaving. It would nevertheless seem that the method is not sufficiently known, and we propose therefore to give a brief outline. Our proofs seem simpler than the two proofs given by Aitken,[1] or that given by Lidstone.[3] At the same time we shall be led to a procedure which works for an odd number of data as well as for an even number (originally the method appeared to work for an even number only and special computational devices were used to reduce an odd number of data). For most practical arrangements of computations we have to refer to Aitken[1,2] and Lidstone.[3]

2. **Linear cross-means.** The full power of the method appears only with the use of quadratic cross-means, but these are in turn based on linear cross-means. Moreover, with completely unsymmetrical data only linear cross-means can be used.

Let it be required to compute the value $f(\xi)$ of a polynomial of nth degree, $f(x)$, given $f_k = f(x_k)$ for $k = 0, \cdots, n$. We note that

$$f^{(1)}(x) = \begin{vmatrix} f_0 & x_0 - \xi \\ f(x) & x - \xi \end{vmatrix} \div (x - x_0) \tag{1}$$

is a polynomial of degree $n-1$, and that $f^{(1)}(\xi) = f(\xi)$. Hence we are required to find $f^{(1)}(\xi)$ knowing

$$f_k^{(1)} = \begin{vmatrix} f_0 & x_0 - \xi \\ f_k & x_k - \xi \end{vmatrix} \div (x_k - x_0) \tag{2}$$

for $k = 1, \cdots, n$. Thus the problem has been reduced from n to $n-1$. In like manner the problem is further reduced to $n-2$ and so on.

All the computer has to do is to write in a column the "parts" $x_k - \xi$, and

* Received Dec. 18, 1942.

[1] A. C. Aitken: *On interpolation by iteration of proportional parts, without the use of differences.* Proc. Edinburgh Math. Soc. Ser. 2, 3, 56–76 (1932).

[2] A. C. Aitken: *Studies in practical mathematics III: The application of quadratic extrapolation to the evaluation of derivatives, and to inverse interpolation* Proc Roy Soc. Edinburgh, Vol. 58, pp. 161–175, 1938.

[3] G. J. Lidstone: *Notes on interpolation.* J. Inst. Actuar., Vol. 68, pp. 267–296, 1938.

in an adjacent column the corresponding values f_k. Using (2), new columns are successively added to the right, the number of rows decreasing by one each time. The "parts" remain obviously the same throughout the computation. The determinant in (2) is easily computed, and the result appears in the main dials ready for division without clearing. Moreover, on most machines, the divisor $x_k-x_0=(x_k-\xi)-(x_0-\xi)$ will automatically appear on the secondary dials (provided the main keyboard has been used for the factors f_k). Actually in most cases the divisor x_k-x_0 will be a small integer. It should also be noted that as the computation proceeds the entries will tend to agree in an ever increasing number of their more important digits. These, of course, will not be copied down; this reduction of digits of f_k makes it in turn possible to drop some last digits of the "parts."

3. Quadratic cross-means. For these it is necessary that the given data be placed symmetrically with respect to some point $x=a$. Denote, then, two symmetrically placed points by x_k and x_{-k} $(k=1, \cdots, m;\ x_k-a=a-x_{-k})$. The point $x_0=a$ is included among the data only if $n=2m$. Consider

$$\phi(x) = \begin{vmatrix} f(2a-x) & 2a-x-\xi \\ f(x) & x-\xi \end{vmatrix} \div 2(x-a) \tag{3}$$

and

$$\psi(x) = \begin{vmatrix} -f(2a-x) & 2a-x-\xi \\ f(x) & x-\xi \end{vmatrix} \div 2(\xi-a). \tag{4}$$

Obviously $\phi(x)$ and $\psi(x)$ are even functions of $x-a$, and hence polynomials in $t=(x-a)^2$. Moreover, $\phi(\xi)=\psi(\xi)=f(\xi)$.

(a) If $n=2m-1$, the problem is reduced to finding the value for $t=(\xi-a)^2$ of the polynomial of $(m-1)$th degree $\phi(a+\sqrt{t})$ given its values

$$\phi_k = \begin{vmatrix} f_{-k} & x_{-k}-\xi \\ f_k & x_k-\xi \end{vmatrix} \div (x_k-x_{-k}) \tag{5}$$

for $t=(x_k-a)^2$, $k=1, \cdots, m$. Thus a simple application of (5) will reduce the number of data from $2m$ to m. From here we proceed as before using linear cross-means. It should be noticed that the "parts" now to be used are $(x_k-a)^2-(\xi-a)^2=-(x_k-\xi)(x_{-k}-\xi)$, that is to say the product of the parts already used for (5). This invariant property dispenses of the necessity to label the panels. In most practical cases the new "parts" will differ only by integers or multiples of 1/2.

(b) If $n=2m$, we compute the values

$$\psi_k = \begin{vmatrix} -f_{-k} & x_{-k}-\xi \\ f_k & x_k-\xi \end{vmatrix} \div 2(\xi-a) \tag{6}$$

for $t=(x_k-a)^2$, $k=0, \cdots, m$, and proceed as before. Since here the denominator is the same for all k the division may be deferred to the final result. This is a slight simplication.

A NEW DERIVATION OF MUNK'S FORMULAE*

By W. C. RANDELS[1] (*University of Oklahoma*)

Recently M. A. Biot[2] has applied the method of the acceleration potential to some problems of two-dimensional airfoil theory. In this paper this method will be used in order to obtain a short proof of Munk's formulae[3] for the lift and moment of a thin airfoil.

As usual in the theory of thin wing sections we replace the airfoil by its mean camber line supposed to deviate but little from the chord. Studying the plane irrotational flow of an incompressible fluid around this indefinitely thin airfoil, we take its chord as the x-axis of a system of rectangular coordinates x, y, ascribing to the leading and trailing edge the abscissae -1 and $+1$ respectively. Denoting by V the velocity at infinity and by α the angle of attack, supposed to be small, we write the x- and y-components of the velocity vector $\vec{w}$ as $V+u$ and $\alpha V+v$ respectively, where u, v and αV will be small as compared with V. We denote the pressure by p and the density by ρ. Then, by Bernoulli's equation

$$\frac{\rho}{2} w^2 + p = p_0 = \text{const.}$$

Neglecting quantities of the second order, we have

$$\rho(V^2 + 2uV) = -2(p - p_0). \tag{1}$$

The quantity $\Phi = -1/\rho(p-p_0)$ is called the acceleration potential, since the acceleration equals grad Φ.

It is known that $V+u-i(\alpha V+v)$ is an analytic function of $z=x+iy$. Since V is a constant $(\frac{1}{2}V^2+uV)-i(\alpha V^2+vV)$ is also an analytic function of z. The functions $\Phi=(\frac{1}{2}V^2+uV)$, $\Psi=-(\alpha V^2+vV)$ are thus seen to be conjugate harmonic functions.

Let the mean camber line of the airfoil be given by the equation $y=c(x)$, $(-1\leqq x\leqq 1;\ c(1)=c(-1)=0)$. The condition that this be part of a stream line furnishes the condition

$$\frac{\alpha V + v}{V + u} = c'(x) \text{ along } y = c(x), \qquad (-1 \leqq x \leqq 1).$$

---

* Received Oct. 28, 1942.

[1] This note has been prepared at the suggestion of Professor W. Prager while the author was a fellow under the Program of Advanced Instruction and Research in Mechanics at Brown University. The author is indebted to Dr. L. Bers for valuable advice.

[2] M. A. Biot, *Some simplified methods in airfoil theory*, Journal of the Aeronautical Sciences 9, 185–190, (1942).

[3] M. M. Munk, *General theory of wing sections*, N.A.C.A. Techn. Rep. No. 142 (1922).

Neglecting quantities which are small of a higher order than the first, we obtain

$$\Psi = -V^2c'(x) \text{ along } y = c(x), \qquad (-1 \leqq x \leqq 1). \tag{2}$$

Since v vanishes at infinity we have

$$\Psi(\infty) = -\alpha V^2.$$

As the mean camber line deviates but little from the segment $-1 \leqq x \leqq 1$ of the x-axis, we will not commit an appreciable error by fulfilling the condition (2) along this segment rather than along the mean camber line. We set

$$\Psi = -\alpha V^2 + \Psi_1 + \Psi_2$$

where

$$\Psi_1 = \alpha V^2 \text{ and } \Psi_2 = -V^2c'(x) \text{ along } -1 \leqq x \leqq 1, \qquad y = 0$$

and

$$\Psi_1(\infty) = \Psi_2(\infty) = 0.$$

Ψ_1 and the conjugate harmonic function Φ_1 have been determined by Biot. From Φ_1 the lift distribution due to the angle of attack can be obtained. In the following we shall set $\alpha = 0$ and thus obtain the lift distribution due to the curvature of the mean camber line. Within the framework of our linear theory these two influences are additive.

In order to solve the boundary value problem for Ψ_2 we map the exterior of the segment of the real axis between $z = -1$ and $z = +1$ onto the exterior of the unit circle in the ζ plane by the conformal transformation

$$z = 1/2(\zeta + 1/\zeta).$$

The line segment $(-1, 1)$ then is transformed into the circumference of the unit circle and we have $x = \cos\theta$ (Fig. 1). Since a conformal transformation takes a harmonic function into a harmonic function, our problem becomes that of finding a harmonic function having the values $-V^2c'(\cos\theta)$ on the unit circle. If we assume Ψ to be regular on the boundary, the solution is given by the Poisson integral but the resulting function will not vanish at infinity unless

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} c'(\cos\theta)d\theta = 0.$$

Therefore, to satisfy the condition $\Psi(\infty) = 0$ we introduce a singularity corresponding to a source-sink doublet at the leading edge.[4] With the notations of Fig. 1, we obtain

[4] It is natural to assume a singularity at the leading edge since our assumption about u and v being small does not hold here.

$$\Psi(r,\theta) = -2a_0V^2\frac{\cos\theta_1}{r_1} - \frac{1}{2\pi}V_2\int_0^{2\pi}[c'(\cos\tau) - a_0]\frac{r^2-1}{r^2+1-2r\cos(\tau-\theta)}d\tau.$$

This function clearly vanishes at infinity and will satisfy the other boundary condition because

$$\frac{\cos\theta_1}{r_1} = \frac{1}{2} \text{ on the unit circle.}$$

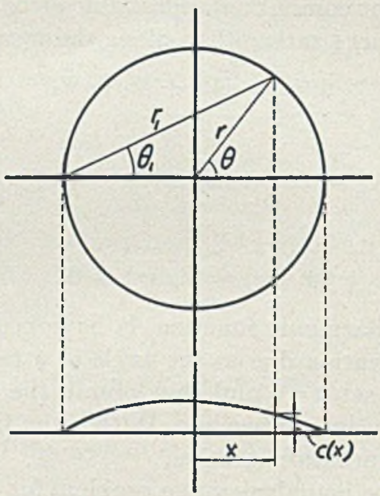

FIG. 1.

The Poisson integral used above is only legitimate if

$$\int_0^{2\pi}|c'(\cos\tau)|\,d\tau = 2\int_{-1}^{1}\frac{|c'(x)|}{[1-x^2]^{1/2}}dx < \infty.$$

This implies a condition on the rapidity with which $c(x)$ tends to zero as x tends to ± 1.

The values of the conjugate function $\Phi(r,\theta)$ on the boundary of the unit circle are given by the formula[5]

$$\Phi(1,\theta) = -2a_0V^2\frac{\sin\theta_1}{r_1} + \frac{1}{2\pi}\int_0^{*2\pi}c'(\cos\tau)\cot\frac{\tau-\theta}{2}d\tau,$$

where $\int^*$ denotes the Cauchy principal value. The total lift L will be given by

[5] J. D. Tamarkin, "Theory of Fourier series," Brown University, 1933, p. 110.

$$L = \rho \int_0^{2\pi} \Phi(1, \theta) \sin \theta d\theta$$

$$= -2a_0 \rho V^2 \int_0^{2\pi} \frac{\sin \theta_1}{r_1} \sin \theta d\theta$$

$$+ \frac{V^2}{2\pi} \int_0^{2\pi} \sin \theta d0 \int_0^{*2\pi} c'(\cos \tau) \cot \frac{\tau - \theta}{2} d\tau.$$

It is easy to calculate:

$$\int_0^{2\pi} \frac{\sin \theta_1}{r_1} \sin \theta d\theta = \pi.$$

The second integral is evaluated by making a formal interchange of the order of integration. This interchange can be easily justified. We then have:

$$\frac{1}{2\pi} \int_0^{2\pi} \sin \theta d\theta \int_0^{*2\pi} c'(\cos \tau) \cot \frac{\tau - \theta}{2} d\tau$$

$$= \int_0^{2\pi} c'(\cos \tau) d\tau \frac{1}{2\pi} \int_0^{*2\pi} \sin \theta \cot \frac{\tau - \theta}{2} d\theta$$

and since the function conjugate to $\sin \theta$ is $-\cos \theta$

$$\frac{1}{2\pi} \int_0^{*2\pi} \sin \theta \cot \frac{(\tau - \theta)}{2} d\theta = -\cos \tau.$$

Using this together with the definition of a_0 we obtain the lift

$$L = -\rho V^2 \left[\int_0^{2\pi} c'(\cos \tau) d\tau + \int_0^{2\pi} c'(\cos \tau) \cos \tau d\tau \right]$$

$$= -2\rho V^2 \int_{-1}^{1} c'(x) \frac{1 + x}{[1 - x^2]^{1/2}} dx$$

$$= -2\rho V^2 \left\{ c(x) \frac{1 + x}{[1 - x^2]^{1/2}} \Big|_{-1}^{1} - \int_{-1}^{1} c(x) \frac{1 + x}{[1 - x^2]^{3/2}} dx \right.$$

$$= 2\rho V^2 \int_{-1}^{1} c(x) \frac{dx}{(1 - x)[1 - x^2]^{1/2}}.$$

We have assumed that $c(x)$ is such that $\lim_{x \to 1} (c(x)(1 - x)^{-1/2} = 0$. The underlined expression is Munk's formula for the total lift, due to the curvature of the wing.

A similar procedure furnishes the moment M of the lift. It is given by

$$M = \rho \int_0^{2\pi} \Phi(1, \theta) \cos\theta \sin\theta d\theta$$

$$= -2\rho V^2 a_0 \int_0^{2\pi} \frac{\sin\theta_1}{r_1} \cos\theta \sin\theta d\theta$$

$$+ \frac{\rho V^2}{2\pi} \int_0^{2\pi} \cos\theta \sin\theta d\theta \int_0^{*\,2\pi} c'(\cos\tau) \cot\frac{\tau - \theta}{2} d\tau.$$

Then

$$\int_0^{2\pi} \frac{\sin\theta_1}{r_1} \cos\theta \sin\theta d\theta = -\frac{\pi}{2},$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \cos\theta \sin\theta d\theta \int_0^{*\,2\pi} c'(\cos\tau) \cot\frac{\tau - \theta}{2} d\tau$$

$$= -\frac{1}{2} \int_0^{2\pi} c'(\cos\tau) \cos 2\tau d\tau$$

so that

$$M = -\rho V^2 \left[-\frac{1}{2} \int_0^{2\pi} c'(\cos\tau) d\tau + \frac{1}{2} \int_0^{2\pi} c'(\cos\tau) \cos 2\tau d\tau \right.$$

$$= -2\rho V^2 \int_{-1}^{1} c'(x) \frac{x^2 - 1}{[1 - x^2]^{1/2}} dx$$

$$= 2\rho V^2 \int_{-1}^{1} c(x) \frac{x dx}{[1 - x^2]^{1/2}}$$

which is Munk's formula for the moment, due to the curvature of the wing.[6]

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[6] After the manuscript of this paper had been completed (August 1942), H. J. Stewart has published an analysis proceeding along similar lines: *A simplified two-dimensional theory of thin airfoils*, Journal of the Aeronautical Sciences 9, 452–456 (Oct. 1942).

CONCERNING THE ACCELERATION POTENTIAL*

By LIPMAN BERS† (*Brown University*)

The following lines aim at indicating the possibility of a more rigorous approach to Prandtl's method of the acceleration potential for two dimensional flow.[1]

We consider a steady incompressible potential flow past an airfoil of infinite span. We assume the profile, P, to be given by

$$z = x + iy = Z(s), \qquad 0 \leqq s \leqq s_T, \tag{P}$$

where the sense of increase of the arc length s corresponds to the counterclockwise direction and the sharp trailing edge, T, is given by $T = Z(0) = Z(s_T)$. The position of the stagnation point, S, near the nose of the airfoil shall be given by $S = Z(s_S)$. We also set

$$-\frac{dZ}{ds} = e^{i\beta(s)},$$

$\beta(s)$ being a continuous function of s and such that on the upper bank of the wing near T, $-\pi/2 \leqq \beta \leqq \pi/2$.

We denote by u and v the velocity components in the x and y directions respectively and assume that

$$u = U > 0, \qquad v = 0, \text{ at infinity.}$$

Then $u - iv$ is an analytic function of $z = x + iy$ and so is

$$\Phi + i\Psi = \log (u - iv).$$

At S the function $\Phi + i\Psi$ possesses a singularity. (There also is a singularity at T, unless the angle there is 0.) $\Phi + i\Psi$ may be determined as a solution of the following boundary value problem:

A. *To determine a one-valued analytic function* $\Phi + i\Psi$ *defined on the region exterior to* P *and satisfying on* P *the boundary condition*

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* Received Jan. 22, 1943.

† This note has been written at the suggestion of Professor W. Prager. The author is indebted to Professor K. O. Friedrichs for criticism.

[1] Cf. L. Prandtl, *Beitrag zur Theorie der tragenden Fläche,* Zeitschrift f. ang. Math. u. Mech. 16, 360–361 (1936); *Über neuere Arbeiten zur Theorie der tragenden Fläche,* Proc. Intern. Congress for Appl. Mech., Cambridge, U.S.A., (1938), pp. 478–482, (See also: N.A.C.A. Technical Memorandum No. 962); M. A. Biot, *Some Simplified Methods in Airfoil Theory,* Journ. of the Aeronaut. Sci., 9, 185–190 (1942); W. C. Randels, *A New Derivation of Munk's Formulae,* this Quarterly 1, 88 (1943).

$$\Psi = \begin{cases} -\beta(s) & \text{for} \quad 0 \leqq s \leqq s_S \\ \pi - \beta(s) & \text{for} \quad s_S \leqq s \leqq s_T \end{cases} \tag{1}$$

as well as the condition

$$\Phi = \log U, \qquad \Psi = 0, \quad \text{at} \quad z = \infty. \tag{2}$$

Equation (1) expresses that P is a streamline of the flow and takes care of the Kutta-Joukowsky condition (no flow around T). The *unknown* position of S is uniquely determined by (2). For instance, let $z=f(\zeta)$ map $|\zeta|>1$ conformally into the exterior of P, taking $\zeta=\infty$ into $z=\infty$ and $\zeta=1$ into $z=T$. Set $f(e^{i\theta})=Z[\sigma(\theta)]$, $S=f(e^{i\tau})$. Then the condition (2) may be written in the form

$$\int_0^{2\pi} \Psi\{Z[\sigma(\theta)]\}\,d\theta = 0.$$

In view of (1) we obtain

$$\tau = 2\pi - \frac{1}{\pi}\int_0^{2} \beta[\sigma(\theta)]\,d\theta. \tag{3}$$

From Φ we can calculate the pressure p. In fact, we have by Bernoulli's equation

$$p + \tfrac{1}{2}\rho(u^2 + v^2) = p_\infty + \tfrac{1}{2}\rho U^2 = p_0, \tag{4}$$

where ρ is the density, p_∞ the pressure at infinity and p_0 the stagnation pressure. Since $\Phi = \log|u - iv|$,

$$p = p_0 - \frac{\rho}{2} e^{2\Phi}. \tag{5}$$

If the wing is infinitely thin, say given by

$$x = x, \qquad y = Y(x), \qquad -1 \leqq x \leqq 1, \tag{P_1}$$

the boundary value problem A takes the form:

B. *To determine a one-valued analytic function* $\Phi + i\Psi$ *defined on the region exterior to* P_1 *and satisfying on* P_1 *the boundary condition*

$$\Psi = \begin{cases} -\text{arc tan } Y'(x) \text{ on the upper bank of } P_1, \\ -\text{arc tan } Y'(x) - \pi \text{ on the lower bank of } P_1, \; -1 \leqq x \leqq x_S \\ -\text{arc tan } Y'(x) \text{ on the lower bank of } P_1, \; x_S \leqq x \leqq 1 \end{cases}$$

($x_S + iY(x_S)$ being the stagnation point) *as well as the condition* (2).

If the wing is very slightly curved and very slightly inclined, the above rigorous but inconvenient treatment can be simplified as follows. The distance between the leading edge $L=-1+iY(-1)$ and the stagnation point S is small of second order as compared with the angle of attack. In fact, the general character of the flow around P_1 will be similar to that of a flow around a straight line, say P_2,

$$y = -x \tan \alpha, \qquad -\cos \alpha \leqq x \leqq \cos \alpha. \tag{P_2}$$

By

$$z = \zeta + \frac{1}{4} e^{-2i\alpha} \frac{1}{\zeta} \tag{6}$$

the exterior of P_2 is mapped into $|\zeta| > \frac{1}{2}$ and T is taken into $\frac{1}{2}e^{-i\alpha}$. S is taken into $-\frac{1}{2}e^{-i\alpha}$ (this follows for instance from (3)). Therefore

$$S = -\tfrac{1}{2}(e^{i\alpha} + e^{-3i\alpha})$$

and, since in this case $L=-e^{-i\alpha}$, we have for small values of α

$$|L - S| \sim 2\alpha^2.$$

Now, $\Phi+i\Psi$ possesses singularities at L and at S. For small angles of attack we may assume that we will make a very slight error if we replace these two singularities by a single singularity situated at L. In order to determine the character of this singularity, we again consider the flow around P_2. The complex potential, say for $U=1$, is given by

$$w = \zeta + \frac{1}{4\zeta} + (i \sin \alpha) \log \zeta \tag{7}$$

so that, by (6) and (7),

$$u - iv = \frac{dw}{dz} = \frac{dw}{d\zeta}\bigg/\frac{dz}{d\zeta} = \frac{\zeta + \frac{1}{2}e^{i\alpha}}{\zeta + \frac{1}{2}e^{-i\alpha}}$$

and

$$\Phi + i\Psi = \log (\zeta + \tfrac{1}{2}e^{i\alpha}) - \log (\zeta + \tfrac{1}{2}e^{-i\alpha}).$$

This is (in the ζ-plane) the complex potential of a source at $-\frac{1}{2}e^{-i\alpha}$ and a sink at $-\frac{1}{2}e^{i\alpha}$. For small values of α we may approximate this source-sink system by a doublet with a vertical axis.

Thus we replace problem B by

C. *To determine a one-valued analytic function $\Phi+i\Psi$ defined on the region exterior to P_1, satisfying on P_1 the boundary condition*

$$\Psi = -\arctan Y'(x) \tag{8}$$

as well as the condition (2) *and possessing at L a singularity which assumes the* *form of a potential of a doublet with a vertical axis when the exterior of* P_1 *is mapped conformally into that of a circle (by a transformation* $f(z)$ *with* $f'(\infty) > 0$).

The actual solution of this problem is still difficult. Therefore we make use of the fact that P_1 is closely approximated by the slit

$$y = 0, \qquad -1 \leqq x \leqq 1, \tag{P_3}$$

and replace the domain of definition of $\Phi + i\Psi$ by the exterior of P_3. Then we obtain the following boundary value problem:

D. *To determine a one-valued analytic function* $\Phi + i\Psi$ *defined on the region exterior to* P_3, *satisfying on* P_3 *condition* (8) *as well as condition* (2) *and possessing at* -1 *a singularity which, in the* ζ*-plane determined by*

$$z = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right),$$

assumes the form of a potential of a doublet with a vertical axis.

This problem can be easily solved. The presence of the singularity enables us to satisfy both conditions, (2) and (8).

It remains to show that the method described above is identical with the method of the acceleration potential, the latter usually being presented as based upon the assumption

$$\Delta p = 0.$$

By virtue of our hypotheses $p - p_\infty$ will be very small as compared to $p_\infty - p_0$ (except at the neighborhood of the leading edge), so that disregarding terms of higher than first order in $(p - p_\infty)/(p_\infty - p_0)$ we have

$$\log (p_0 - p) = \log (p_0 - p_\infty) + \frac{p_\infty - p}{p_0 - p_\infty}$$

and, by (4) and (5),

$$p = -\rho U^2 \Phi + \text{const.}$$

On the basis of the above considerations an estimation of the error (due to replacing the actual problem B first by C and then D) seems to be both desirable and possible.

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