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QUARTERLY OF APPLIED MATHEMATICS

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THE ANTENNA PROBLEM*

BY

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1. Introduction. The recent expansion of radio towards ultra short waves has aroused a new interest in theoretical problems of electro-magnetism and especially in the problem of antenna oscillations and radiation properties. The type of approximate discussions used by radio engineers for the case of long wave lengths is of little practical value for ultra short waves, where a more rigorous theory is needed, because the diameter of the antenna wire can no longer be considered as very small when compared to the wave length.

Some older calculations on rather thick antennas have already been found very useful. M. Abraham's[1] discussion of the vibrations of very long ellipsoids has often been referred to. A complete discussion of proper vibrations of ellipsoids of revolution may be found in M. Brillouin's book *Propagation de l'électricité* (Hermann, Paris, 1904, pp. 314–395) with numerical tables for all eccentricities, from the sphere to rather thin ellipsoids. More recently, L. Page and N. I. Adams, and subsequently R. M. Ryder[2] have discussed the free and forced oscillations of all types of prolate ellipsoids of revolution; while Barrow,[3] Schelkunoff,[4] and others have treated the problem of the biconical antenna and its free or forced oscillations. Mie and Debye[5] had formerly discussed the free vibrations of the sphere. In most of these papers, the theory was based on a computation of the whole field distribution around the antenna with the proper boundary conditions on the surface of the antenna. For a perfect metal, for instance, the electric field must be orthogonal to the metal surface.

The aim of the present paper is to emphasize the practical importance of another method based on the use of retarded potentials. The principle of the procedure was indicated a long time ago,[6] and the method was recently applied by Hallen and

* Received May 3, 1943. Part of a research sponsored by the Federal Tel. and Radio Laboratories, New York.

[1] M. Abraham, Ann. d. Physik, **66**, 435 (1898); Math. Ann. **52**, 81 (1899).

[2] L. Page and N. I. Adams, Phys. Rev. **53**, 819 (1938); R. M. Ryder, Appl. Phys. **13**, 327 (1942).

[3] W. L. Barrow, L. J. Chu, J. J. Jansen, Proc. I.R.E., **27**, 769 (1939).

[4] S. A. Schelkunoff, Trans. A.I.E.E., **57**, 744 (1938); Proc. I.R.E., **29**, 493 (1941).

[5] G. Mie, Ann. d. Phys., **25**, 377 (1908); P. Debye, Ann. d. Physik, **30**, 59 (1909).

[6] H. C. Pocklington, Proc. Cambridge Phil. Soc., **9**, 324 (1897); Lord Rayleigh, Proc. Roy. Soc., Ser. A, **87**, 193 (1912); C. W. Oseen, Ark. f. Mat. Astr. Fysik, **9**, No. 12 (1913); L. Brillouin, Radio-électricité, **3**, 147 (1922).

Ronold King[7] to the actual computation of antennas. The finite conductivity of a real metal can be taken into account, but there are still a few basic questions to be discussed, and these will appear more clearly in the problem of a perfect metal with infinite conductivity.

The principle of the method is the following: let us first assume a very thin wire and call s a distance measured along the wire. The problem is to find the current distribution, $I(s, t)$, along the antenna wire. To such a current, I, there corresponds a charge density, $\sigma(s, t)$, by the condition of conservation of electricity

$$-\frac{\partial \sigma}{\partial t} = \frac{\partial I}{\partial s}, \tag{1}$$

or, if we assume the following time dependence $I(s, t) = I(s)e^{i\omega t}$,

$$\sigma(s, t) = \frac{i}{\omega} \frac{\partial I}{\partial s} e^{i\omega t}. \tag{2}$$

Here, real ω means sustained oscillations; while proper oscillations of the antenna array will yield complex proper values ω, the imaginary part corresponding to radiative damping.

An arbitrary current distribution (1), creates an electromagnetic field in the whole space which satisfies Maxwell's equations. This field can be readily computed by the *method of retarded potentials*. In particular, the field on the surface of the metal wire can be obtained in this way; and one may then write the necessary boundary condition, that this electric field shall be orthogonal to the surface. This yields an integro-differential equation which is perfectly rigorous and whose solution is the actual current distribution required.

Using retarded potentials, one is certain to obtain, at a large distance, a field distribution corresponding to a wave spreading out of the antenna. It should be emphasized, however, that the same method can *not always be used* for the computation of oscillations inside a *closed tank resonator*, where the oscillations are of the type of standing waves and have no outside radiation (advanced potentials may sometimes be needed too).

The proper values of this integral equation give the proper frequencies (including damping) of the antenna. The same method can be used to study forced vibrations, if one assumes an outer electric field acting on the antenna (receiving antenna) or a certain electromotive force inserted in the circuit (transmitting antenna). In the second case, one must take into account, for the computation of the retarded potentials, the field radiated from a dipole representing the electromotive force.

Let us discuss the free vibrations of an antenna. The field at a point P is given by the well known formulae:

$$\begin{aligned} h_x &= -\frac{\partial V}{\partial x} - \frac{\partial F_x}{\partial t}, \cdots, \cdots; \quad V = \frac{1}{\epsilon_0}\int \frac{\sigma^* ds}{r}; \\ \mu_0 H_x &= \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \cdots, \cdots; \quad \vec{F} = \mu_0 \int \frac{\vec{i}^* ds}{r}; \end{aligned} \tag{3}$$

[7] E. Hallen, Uppsala Univ. Arsskrift 1930, No. 1; Nova Acta, Uppsala, Ser. 4, 11, No. 4 (1938); L. V. King, Trans. Roy. Soc. London, 236, 381 (1937); Ronold King and F. G. Blake, Proc. I.R.E., 30, 335 (1942).

$\vec{h}$, electric field; $\vec{H}$, magnetic field; V, scalar potential; $\vec{F}$, vector potential; r distance from the element ds on the circuit to the point P where the field is observed; σ^*, $\vec{i}^*$ charge and the current at the time $t-r/c$; ϵ_0, μ_0 dielectric constant and permeability in vacuum, in non-rationalized units (rationalized units introduce a $1/4\pi$ factor in the formulae for both potentials). Let us assume an antenna consisting of a straight wire along the z axis, extending from $z=0$ to $z=l$. We need the z component, h_z, of the electric field along the wire and must write that this longitudinal component vanishes:

$$h_z = -\frac{\partial V}{\partial z} - \frac{\partial F_z}{\partial t}, \qquad i^* = I(z')e^{i(\omega t - kr)}, \qquad k = \frac{\omega}{c} = \omega\sqrt{\epsilon_0\mu_0}\,,$$
$$V = \frac{ie^{i\omega t}}{\epsilon_0\omega}\int_0^l \frac{\partial I(z')}{\partial z'}\frac{e^{-ikr}}{r}dz', \qquad F_z = \mu_0 e^{i\omega t}\int_0^l I(z')\frac{e^{-ikr}}{r}dz'. \tag{4}$$

The field at point z is the result of integration over all the points, z', of the antenna. Finally, we obtain the condition

$$h_z = \frac{i}{\epsilon_0\omega}\int_0^l \left[-\frac{\partial I(z')}{\partial z'}\frac{\partial G(r)}{\partial z} - k^2 I(z')G(r)\right]dz' = 0,$$
$$\text{putting} \qquad G(r) = \frac{e^{-ikr}}{r}, \qquad r = |z - z'|. \tag{5}$$

This is our fundamental integro-differential equation for the straight antenna.

One difficulty appears immediately: G is infinite for $r=0$, $z=z'$. This means that one must take into account the radius of the wire; but when this radius, a, is explicitly introduced in the calculation, there is an additional condition to be written for both ends of the wire. Here most authors do not attempt to write rigorous conditions; they are satisfied with approximations corresponding to the problem of very thin wires. They neglect a/l but keep terms in Ω^{-1}, Ω^{-2}, $\cdots$ where

$$\Omega = 2\log\frac{l}{a}\cdot \tag{6}$$

Such a procedure is suggested by the similar approximations used by M. Abraham in his discussion of ellipsoids. It should work correctly when $\Omega>14$, which means $l/a>1000$, but could certainly not be relied upon for thicker wires.

Furthermore, Oseen and Hallen both use the following assumptions:

$$\text{A)} \quad I(0) = 0,\ I(l) = 0, \text{ current zero at both ends;}$$
$$\text{B)} \quad r = [(z-z')^2 + a^2]^{1/2}. \tag{7}$$

The first condition, A, is not quite correct, since there must be a small current at both ends in order to charge and discharge the terminal capacities. It is only for the case of a hollow cylinder that the current would be exactly zero at both ends; and this hollow pipe is a very special case, as shall be seen later.

The second assumption, B, is explained differently by both writers. Oseen assumes a *current flowing along the axis* of the cylindrical wire and computes the field h_z, Eq. (5), on the surface. Hence his boundary condition (5) is right, but the assumption about the axial current is certainly wrong. Indeed, owing to the skin effect, the actual electric current, in a perfect conductor, flows along the surface. Oseen assumes that the field created by this actual superficial current could be obtained by a fictitious axial current. This may be right for very thin wires, but the assumption is obviously wrong for thick wires or for cylinders of large radius. Moreover, it cannot be proved that the fictitious axial current satisfies the first assumption A. So Oseen hardly justifies the use of both assumptions A and B.

Hallen takes a different point of view. He starts from the well-known property that the current flows along the surface; but instead of computing the field on the surface of the same cylinder, he takes the *h_z field along the axis.* This field must certainly vanish; and from this fact, Eqs. (5) and (7 B) follow. This necessary condition, however, is not sufficient. One may very well have no longitudinal field along the axis and still find a longitudinal field on the surface of the cylinder. These approximations would probably be all right for very thin wires; but they can certainly not be used for thick wires, where B is wrong and A must be replaced by a more elaborate condition, in order to take account of the electric currents and charges on the flat ends of the cylinder.

2. Complete statement for a cylindrical wire of finite radius. The antenna is a solid cylinder of radius a and height l. The oscillations studied are those of cylindrical symmetry where the current is equally distributed around the cylinder and flows along the surface. $I(z', t)$ is the total current at z', and $(1/2\pi)\ I(z', t)d\varphi$ is the current through a small sector $d\varphi$ (Fig. 1); hence, $\sigma(z', t)dz'$, Eqs. (1), (2), is the charge per length dz', all around the cylinder, and $(1/2\pi)\sigma dz' d\varphi$ the charge for a small angle $d\varphi$. For the flat top of the cylinder $(z=l)$, we call $I_l(\rho)$ the total radial current crossing a circle of radius ρ; while $\sigma_l(\rho)d\rho$ represents the electric charge between ρ and $\rho+d\rho$:

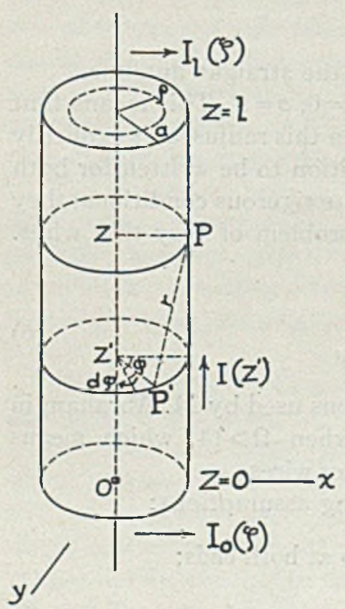

Fig. 1.

$$-\frac{\partial \sigma_l}{\partial t} = \frac{\partial I_l}{\partial \rho}, \qquad \sigma_l = \frac{i}{\omega}\frac{\partial I_l}{\partial \rho}. \tag{8}$$

Similar definitions apply for the bottom of the cylinder $(z=0)$ with a current $I_0(\rho)$ and charge $\sigma_0(\rho)d\rho$. The positive signs correspond to the directions indicated by arrows in Fig. 1. The conditions for continuity of the current around the corners, at $z=0$ and $z=l$ read

$$I_l(a) = -I(l), \qquad I_0(a) = I(0). \tag{9}$$

Let us first study the fields and potentials *at a point $P(z)$ located on the cylindrical surface.* The potentials due to currents and charges along the cylinder are the following ($e^{i\omega t}$ factors have been dropped):

$$\begin{aligned} V_c(z) &= \frac{i}{\epsilon_0\omega}\int_0^l\int_0^{2\pi}\frac{\partial I(z')}{\partial z'}\frac{e^{-ikr}}{r}\frac{d\varphi}{2\pi}dz', \\ F_{cz}(z) &= \mu_0\int_0^l\int_0^{2\pi} I(z')\frac{e^{-ikr}}{r}\frac{d\varphi}{2\pi}dz'. \end{aligned} \tag{10}$$

The currents along the cylinder flow vertically; hence, there are no horizontal components F_{cx}, F_{cy} of the vector potential. The distance r is shown in Fig. 1.

On the flat top of the cylinder, the current flows radially in the horizontal plane; hence, the F_{lz} component is zero, but we find horizontal components, F_{lx} and F_{ly}, of the vector potential:

$$
\begin{aligned}
F_{lx}(z) &= \mu_0 \int_0^a \int_0^{2\pi} I_l(\rho) \frac{e^{-ikr}}{r} \cos\varphi \frac{d\varphi}{2\pi} d\rho, \\
F_{ly}(z) &= \mu_0 \int_0^a \int_0^{2\pi} I_l(\rho) \frac{e^{-ikr}}{r} \sin\varphi \frac{d\varphi}{2\pi} d\rho = 0.
\end{aligned}
\tag{11}
$$

The transverse components F_{ly}, for a point P in the x-z-plane, is obviously zero by symmetry:

$$
V_l(z) = \frac{i}{\epsilon_0 \omega} \int_0^a \int_0^{2\pi} \frac{\partial I_l}{\partial \rho} \frac{e^{-ikr}}{r} \frac{d\varphi}{2\pi} d\rho \tag{12}
$$

and similar formulae for the potentials F_{0x} and V_0 due to currents and charges on the bottom of the cylinder.

The φ integrals are of two fundamental types which will now be explained in connection with Fig. 2.

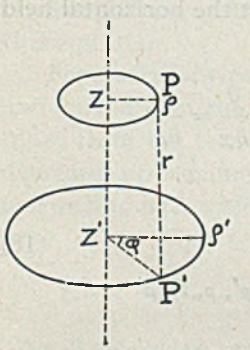

FIG. 2.

$$
r^2 = (z - z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos\varphi, \tag{13}
$$

$$
G_k(\rho, \rho', z - z') = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ikr}}{r} d\varphi, \tag{14}
$$

$$
C_k(\rho, \rho', z - z') = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ikr}}{r} \cos\varphi d\varphi. \tag{15}
$$

G_k and C_k are two functions which will be discussed more fully in section 5. They are symmetrical in ρ, ρ' and even functions of $z - z'$. With these functions, our formulae (10, 11, 12) read

$$
\begin{aligned}
V_c(z) &= \frac{i}{\epsilon_0 \omega} \int_0^l \frac{\partial I}{\partial z'} G_k(a, \rho, z - z') dz', \\
F_{cz}(z) &= \mu_0 \int_0^l I(z') G_k(a, \rho, z - z') dz', \\
V_l(z) &= \frac{i}{\epsilon_0 \omega} \int_0^a \frac{\partial I_l}{\partial \rho'} G_k(\rho', \rho, z - l) d\rho', \\
F_{lx}(z) &= \mu_0 \int_0^a I_l C_k(\rho', \rho, z - l) d\rho',
\end{aligned}
\tag{16}
$$

where $\rho=a$ for the point $P(z)$ on the cylinder.

We now are in a position to compute the longitudinal field $h_z(z)$ at the point P, according to Eqs. (4) and (16).

$$h_z = -\frac{\partial V_c}{\partial z} - \frac{\partial F_{cz}}{\partial t} - \frac{\partial V_0}{\partial z} - \frac{\partial V_l}{\partial z},$$

$$\begin{aligned} -\frac{\epsilon_0\omega}{i} h_z(z) = \int_0^l \left[\frac{\partial I}{\partial z'}\frac{\partial}{\partial z} G_k(a, a, z - z') + k^2 I(z') G_k(a, a, z - z')\right] dz' \\ + \int_0^a \frac{\partial I_l}{\partial \rho'}\frac{\partial}{\partial z} G_k(\rho', a, z - l) d\rho' + \int_0^a \frac{\partial I_0}{\partial \rho'}\frac{\partial}{\partial z} G_k(\rho', a, z) d\rho' = 0. \end{aligned} \tag{17}$$

This is the first integral equation of the problem which corresponds to Eq. (5) for the simplified example of a thin wire. It should be noticed immediately that in the first integral

$$\frac{\partial}{\partial z} G_k(a, a, z - z') = -\frac{\partial}{\partial z'} G_k(a, a, z - z'). \tag{17a}$$

This transformation will be very useful, afterwards, in applying integration by parts.

Another integral equation is obtained by writing the fact that the horizontal field component is zero at a point $P(\rho)$ on the top of the cylinder:

$$h_x(\rho) = -\frac{\partial V_c}{\partial x} - \frac{\partial V_l}{\partial x} - \frac{\partial V_0}{\partial x} - \frac{\partial F_{lx}}{\partial t} - \frac{\partial F_{0x}}{\partial t}, \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \rho},$$

$$\begin{aligned} -\frac{\epsilon_0\omega}{i} h_x(\rho) = \int_0^l \frac{\partial I}{\partial z'}\frac{\partial}{\partial \rho} G_k(a, \rho, l - z') dz' \\ + \int_0^a \frac{\partial I_l}{\partial \rho'}\frac{\partial}{\partial \rho} G_k(\rho', \rho, 0) d\rho' + \int_0^a \frac{\partial I_0}{\partial \rho'}\frac{\partial}{\partial \rho} G_k(\rho', \rho, l) d\rho' \\ + k^2 \int_0^a [I_l(\rho') C_k(\rho', \rho, 0) + I_0(\rho') C_k(\rho', \rho, l)] d\rho' = 0. \end{aligned} \tag{18}$$

A similar equation could be written for the bottom of the cylinder; but this is actually not needed, since it reduces to (18) by reason of symmetry.

The proper oscillations of the cylinder can be divided into two groups:

$$\begin{aligned} &\text{symmetrical oscillations } I_l(\rho') = I_0(\rho'),\ I(l-z) = -I(z), \\ &\qquad \frac{\partial I(l-z)}{\partial z} = \frac{\partial I(z)}{\partial z}; \\ &\text{antisymmetrical oscillations } I_l(\rho') = -I_0(\rho'),\ I(l-z) = I(z), \\ &\qquad \frac{\partial I(l-z)}{\partial z} = -\frac{\partial I(z)}{\partial z}. \end{aligned} \tag{19}$$

These two types will be discussed together in the following formulae. The upper sign corresponds to symmetrical and the lower sign to antisymmetrical vibrations.

3. Discussion of the first integral equation (17). Wave propagation along the cylinder. Equation (17) can now be written in the following way:

$$\int_0^l \left[-\frac{\partial I}{\partial z'} \frac{\partial}{\partial z'} G_k(a, a, z - z') + k^2 I(z') G_k(a, a, z - z') \right] dz'$$

$$= -\int_0^a \frac{\partial I_l}{\partial \rho'} \frac{\partial}{\partial z} \left[G_k(\rho', a, z - l) \pm G_k(\rho', a, z) \right] d\rho' = R(I_l, z). \tag{20}$$

The left hand integral contains only vertical currents, $I(z')$, along the cylindrical boundary; while the right hand terms, R, show the coupling between these vertical currents and the currents or charges on both flat ends of the cylinder.

Let us integrate the left hand integral by parts, starting from $\partial G_k/\partial z'$:

$$\int_0^l \left[\frac{\partial^2 I}{\partial z'^2} + k^2 I(z') \right] G_k(a, a, z - z') dz' = \left. \frac{\partial I}{\partial z'} G_k(a, a, z - z') \right|_{z'=0}^{z'=l} + R(I_l, z)$$

$$= \left(\frac{\partial I}{\partial z'} \right)_{z'=l} \left[G_k(a, a, z - l) \mp G_k(a, a, z) \right] + R(I_l, z). \tag{21}$$

This new formula has been obtained without any approximations. Let us now make a few simplifying assumptions, in order to get a better understanding of the meaning of this equation.

For a very thin and long wire, $l \gg a$, we may neglect the $R(I_l, z)$ term, as both charges and currents on the flat terminals become very small. Furthermore, at a certain distance from the terminals, $G_k(a, a, z-l)$ and $G_k(a, a, z)$ are also very small, since G_k decreases approximately like $1/r$ for large distances. The only important term is the one on the left, which has the obvious solution

$$\frac{\partial^2 I}{\partial z'^2} + k^2 I(z') = 0, \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda} \cdot \tag{22}$$

This shows wave propagation with the velocity of light along the major part of the wire. This result is obtained under the assumption $l \gg a$ and without any restriction about the wave length λ, which can be of the order a or even smaller; but it holds only for the medium part of the wire, far away from both ends.[8]

This shows the connection with the usual elementary theory of antennas. The classical discussion[9] starts from the assumption of sinusoidal standing waves along the wire, which cancels out completely the left hand integral in equation (21). Then, using this current distribution, the longitudinal field along the wire may be computed; and according to (17) and (21) it comes out as

$$h_z(z) = \frac{i}{\epsilon_0 \omega} \left\{ \left(\frac{\partial I}{\partial z'} \right)_{z'=l} \left[G_k(a, a, z - l) \mp G_k(a, a, z) \right] + R(I_l, z) \right\}. \tag{23}$$

---

[8] It should be emphasized, here, that our discussion is limited to the case of oscillations with cylindrical symmetry (see beginning of Section 2). Vibrations with nodal lines parallel to the axis are not included.

[9] L. Brillouin, Radio-électricité, loc. cit.

J. A. Stratton, *Electromagnetic theory*, McGraw-Hill, New York, 1941, pp. 455–460. Stratton uses rational units, hence a $1/4\pi$ factor before the integrals, and he uses the opposite sign before i.

This plays the role of a small additional average impedance Z along the antenna, which can be defined by

$$Z\,\overline{I^2} = \int_0^l h_z(z)I(z)dz. \tag{23a}$$

The real part of Z is called the *radiation resistance*, Z_r, and the expression $Z_r\overline{I^2}$ represents the energy, W, radiated at large distance (see Stratton, p. 458), from which the damping of the antenna oscillations may be computed. For a very thin wire, one may neglect the term $R(I_l, z)$, which represents the role played by the currents and charges on both flat terminals of the wire; and one may take for G_k the expression $(1/r)e^{-ikr}$ as in Eq. (5). With these approximations, our equation (23) becomes identical with Stratton's Eq. (76a), p. 457.

It should be noticed that Eq. (23) is physically wrong, as we know in advance that the longitudinal electric field along the wire is zero. These equations (23) and (23a) merely represent a second approximation in a system of successive approximations starting from (22). An attempt will be made, in the next section, to build up a consistent system of approximations of similar structure.

Returning now to Eq. (20), we may try another integration by parts, starting from $\partial I/\partial z'$, which yields

$$\int_0^l I(z')\left[\frac{\partial^2 G_k}{\partial z'^2} + k^2 G_k(a, a, z - z')\right]dz'$$
$$= I(z')\frac{\partial}{\partial z'}G_k(a, a, z - z')\Bigg|_{z'=0}^{z'=l} + R(I_l, z). \tag{24}$$

Let us again discuss this equation for a very thin wire. The term $R(I_l, z)$ represents the role of both terminals and may be neglected, $I(z')$ is zero at both ends ($z'=0$, $z'=l$), and consequently all the right hand terms are zero. This transformation is very closely connected with the one used by Schelkunoff and Feldman[10] in a recent paper. These authors discuss the problem of forced vibrations in a transmission antenna, instead of the free vibrations which we have in mind. They use both approximations (7A) and (7B) of Oseen and Hallen and take for G the simplified expression $(1/r)e^{-ikr}$, Eq. (5). These approximations may apply for a very thin wire. Furthermore, they split the $(1/r)e^{-ikr}$ function into its real and imaginary parts before performing the integration by parts. Their final result is actually identical with the one derived from the elementary theory and Eq. (23). This is not surprising, as both methods are very closely connected.

4. Principle of a method of successive approximations. As stated in the preceding section, it seems possible to build up a method of successive approximations in order to solve Eq. (21) along a way rather similar to the one followed in the classical elementary discussion.

First of all, we may split the integro-differential equation (21) into an integral equation and a differential equation, by writing:

---

[10] S. A. Schelkunoff and C. B. Feldman, Proc. I.R.E., 30, 511 (1942).

$$\int_0^l F(z')G_k(a, a, z - z')dz' = R'(z), \tag{25}$$

where $R'(z) = (\partial I/\partial z')_{z'=l}[G_k(a, a, z-l) \mp G_k(a, a, z)] + R(I_l, z)$,

$$\frac{\partial^2 I}{\partial z'^2} + k^2 I(z') = F(z'). \tag{26}$$

The first equation is an integral equation of the first kind, with the kernel $G_k(z-z')$. Its solution can be written with the help of the resolving kernel $H_k(z'-z'')$, which satisfies the following conditions

$$\int_0^l G_k(z - z')H_k(z' - z'')dz' = \delta(z - z''), \tag{27}$$

$$F(z') = \int_0^l R'(z'')H_k(z' - z'')dz'', \tag{28}$$

where δ means a delta function. Hence, the first question is to build up the resolving kernel H_k, a problem for which some general methods have been developed. This being done, we are left with Eq. (26) to which we apply the usual Rayleigh-Schrödinger method of successive approximations. Let us first notice that the G_k function becomes very large for $z = z'$ which, according to (27), means that H_k is small. Thus we may rewrite (26) and state explicitly by an ϵ coefficient the smallness of the right hand term:

$$\frac{\partial^2 I}{\partial z'^2} + k^2 I(z') = \epsilon\varphi(z'), \qquad F = \epsilon\varphi. \tag{26a}$$

Then we use the following expansions:

$$\begin{aligned} I(z') &= I_0(z') + \epsilon I_1(z') + \epsilon^2 I_2(z') \cdots, \\ k^2 &= k_0^2 + \epsilon\chi_1 + \epsilon^2\chi_2 \cdots \end{aligned} \tag{29}$$

and obtain the successive approximations:

$$\begin{aligned} &\frac{\partial^2 I_0}{\partial z'^2} + k_0^2 I_0 = 0, \\ &\frac{\partial^2 I_1}{\partial z'^2} + k_0^2 I_1 = -\chi_1 I_0 + \varphi, \\ &\frac{\partial^2 I_2}{\partial z'^2} + k_0^2 I_2 = -\chi_2 I_0 - \chi_1 I_1 \cdots \end{aligned} \tag{30}$$

I_0 is a sinusoidal function, as in the elementary treatment,

$$I_0 = A \sin k_0(z' + \zeta)$$

where the ζ constant is necessary in order to give a small but finite value for the current I_0 at the bottom of the cylinder ($z' = 0$). This is needed for the junction with the currents on the lower flat end of the cylinder. By symmetry, the correction at the upper end must also be ζ; hence,

$$k_0(l + 2\zeta) = n\pi, \qquad l + 2\zeta = n\,\frac{\lambda_0}{2}, \qquad k_0 = \frac{2\pi}{\lambda_0}\cdot \tag{31}$$

The constant ζ will be determined by means of the second integral equation (18) for the flat terminals. Now let us turn to the second equation (30). As is well known, it is necessary for the right hand term to be orthogonal to the solution of the homogeneous equation, which means

$$\int_0^l \sin k_0(z' + \zeta)[-\chi_1 I_0 + \varphi]dz' = 0$$

or

$$A\chi_1 \int_0^l \sin^2 k_0(z' + \zeta)dz' = \int_0^l \varphi \sin k_0(z' + \zeta)dz'. \tag{32}$$

This yields the correction χ_1 to the proper value k_0^2. It is readily seen that equation (32) is very similar to the relation (23a) used in the elementary theory to obtain the average "radiation resistance" of the antenna and thence the damping coefficient in the proper oscillations. The important point, however, is that equation (32) contains φ, which is not R' but is computed from R' by means of (28)–(26a).

Once χ_1 is obtained, the second equation (30) can be solved; then χ_2 is first computed by a similar orthogonality condition, and so on. Hence, the whole procedure should yield a solution along lines parallel to the elementary treatment and show how far the usual formulae can be trusted.

We may already go one step further and write the general expression of the function $F(z')$ on the basis of Eqs. (25) and (27):

$$\begin{aligned}\epsilon\varphi(z') = F(z') &= \int_0^l R'(z'')H_k(z' - z'')dz'' \\ &= \left(\frac{\partial I_0}{\partial z'}\right)_{z'=l}[\delta(z' - l) \mp \delta(z')] + \int_0^l R(I_l, z'')H_k(z' - z'')dz''.\end{aligned} \tag{33}$$

The δ functions appear here automatically, because G_k is an even function of $(z-z')$, and so is H_k for $z'-z''$; hence, the integral in (28) comes out as

$$\int G_k(z'' - l)H_k(z' - z'')dz'' = \int G_k(l - z'')H_k(z'' - z')dz'' = \delta(z' - l)$$

according to (27).

We can use the new expression (28) for the discussion of some simplified examples. Let us start with the *wire of vanishing radius*. The whole $R(I_l, z'')$ term, which represents the terminal effect, drops out; and we are left with an equation

$$\frac{\partial^2 I}{\partial z'^2} + k^2 I(z') = F(z') = \left(\frac{\partial I}{\partial z'}\right)_{z'=l}[\delta(z' - l) \mp \delta(z')]. \tag{34}$$

from (26) and (28). The condition on both terminals is obviously $I(0) = I(l) = 0$; hence $\zeta = 0$ in (31), which results in the following equation:

$$I_0 = A \sin k_0 z', \qquad k_0 = n\pi/l. \tag{35}$$

$n=2m+1$: symmetrical oscillation, sign $- = (-1)^n$ in bracket,
$n=2m$: antisymmetrical oscillation, sign $+ = (-1)^n$ in bracket.

$\epsilon\varphi(z') = F(z') = Ak_0 \,[(-1)^n\delta(z'-l)+\delta(z')]$ as $\cos k_0 l = (-1)^n$ and Eq. (32) reduces to

$$\begin{aligned} \epsilon\chi_1 \frac{l}{2} &= k_0 \int_0^l [(-1)^n\delta(z'-l)+\delta(z')] \sin k_0 z' dz' \\ &= k_0[(-1)^n \sin(k_0 l) + \sin(k_0 0)] = 0 \end{aligned} \tag{36}$$

which gives no damping at all. The physical explanation is the following: a finite amount of energy is radiated per second; but this does not mean any damping of the oscillations, because the energy accumulated in the field around the wire is infinite. As a matter of fact, both electric and magnetic fields are infinite as $1/r$ near the wire of infinitely small radius. The square of the field is of the order $1/r^2$; and the energy is obtained by multiplying by $2\pi r\, dr$ and integrating with respect to r, which gives logarithmic infinite terms. The situation is similar to the one obtained in a circuit with infinite L, zero capacity, and finite resistance R, which yields a negligible damping coefficient $R/2L$.

This shows the difficulties involved in the assumption (7A), as put forth by Oseen and Hallen. When such a condition is used in the rigorous Eqs. (25), (26), it leads directly to (36) and yields practically no damping.

Such is also the case for a *hollow cylinder*. Here again, there is no end effect, no terminals, no R term, and condition (7A) holds good. The whole procedure from (34) to (36) repeats itself and shows again no damping. Of course, the G_k and H_k functions would differ materially in both cases; but these functions have been eliminated from Eq. (34) and finally drop out.

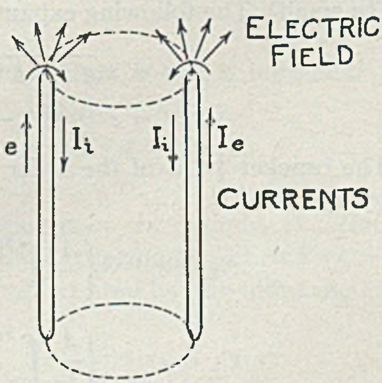

FIG. 3.

The explanation is similar to the one given for the thin wire, but not quite so obvious. The problem of a hollow cylinder of indefinitely small thickness must be considered as the limit of a cylinder of finite wall thickness, as represented in Fig. 3. On such a cylinder, one should take into account, separately, a current I_l flowing along the external surface of the cylinder and another current I_i along the internal surface. At the limit, these two currents merge into a single one, for which the theory indicates a sinusoidal distribution. Hence, for a cylinder of finite thickness, there certainly is a current flowing around the edge of the cylinder, as shown in Fig. 3. On this edge, one must also consider the electric charge; and this results in an accumulation of electric fields and of electro-magnetic energy near the cylinder, while the energy radiated per second at large distance remains finite. Hence the damping becomes negligible.

The result is general and applies for any hollow cylinder of indefinitely small thickness, whatever the shape of the cross-section might be. The field distribution inside the cylinder should correspond to a superposition of E_0 waves (transverse magnetic) and should show a strong decay from both ends down to the middle part of the cylinder, especially when the diameter of the cylinder is small compared to the wave length.

These two simple examples show the importance of the role played by the shape of both terminals and the danger of using assumptions like (7A) or (7B).

5. **Some important formulae.** We have introduced in (14), (15) two fundamental functions:

$$G_k(\rho, \rho', \zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ikr}}{r} d\varphi, \qquad \zeta = z - z',$$

$$C_k(\rho, \rho', \zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ikr}}{r} \cos \varphi d\varphi, \tag{37}$$

$$r^2 = \zeta^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos \varphi = q - 2p \cos \varphi, \qquad q \geqq 2p,$$

$$q = \zeta^2 + \rho^2 + \rho'^2, \qquad p = \rho\rho'.$$

From these relations, we see that G_k and C_k depend upon ρ, ρ', ζ only through the two combinations p and q. Furthermore, it is easily proved that

$$\frac{\partial G_k}{\partial p} = -2\frac{\partial C_k}{\partial q} = -\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{e^{-ikr}}{r}\right) \cos \varphi d\varphi. \tag{38}$$

C_k and G_k being both zero at infinity, this can be written as

$$C_k = \frac{1}{2}\int_q^\infty \frac{\partial G_k}{\partial p} dq. \tag{39}$$

These integrals are closely connected with the complete elliptic integrals K and D,[11] as is seen for a thin wire when the radius a is small compared with the wave length (ka small). The following expansions can be used:

$$\begin{aligned} r &= \sqrt{q - 2p \cos \varphi} = \sqrt{q} + [\sqrt{q - 2p \cos \varphi} - \sqrt{q}] \\ e^{-ikr} &= e^{-ik\sqrt{q}}\{1 - ik[\sqrt{q - 2p \cos \varphi} - \sqrt{q}] \cdots \}. \end{aligned} \tag{40}$$

The bracket [ ] is of the order of magnitude of a, and its product when multiplied by k is small:

$$G_k = e^{-ik\sqrt{q}}\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ik\sqrt{q}}{[q - 2p \cos \varphi]^{1/2}} d\varphi - ik \cdots\right\}$$

$$C_k = e^{-ik\sqrt{q}}\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ik\sqrt{q}}{[q - 2p \cos \varphi]^{1/2}} \cos \varphi d\varphi - 0 + \cdots\right\}. \tag{41}$$

We may write

$$q - 2p \cos \varphi = (q + 2p)(1 - \kappa^2 \sin^2 \psi)$$

$$\kappa^2 = \frac{4p}{q + 2p}, \qquad \psi = \frac{\varphi - \pi}{2}. \tag{42}$$

Hence

[11] E. Jahnke and F. Emde, *Tables of functions*, 2nd ed., Springer, Berlin, 1933, pp. 127–145.

$$\int_0^{2\pi} \frac{d\varphi}{[q - 2p\cos\varphi]^{1/2}} = \frac{2}{[q+2p]^{1/2}} \int_{-\pi/2}^{\pi/2} \frac{d\psi}{[1 - \kappa^2 \sin^2\psi]^{1/2}} = \frac{4}{[q+2p]^{1/2}} K(\kappa)$$

and

$$G_k = e^{-ik\sqrt{q}} \left[\frac{2}{\pi} \cdot \frac{1 + ik\sqrt{q}}{[q+2p]^{1/2}} K(\kappa) - ik \cdots \right]. \tag{43}$$

When $\zeta \to 0$, the variables q and p retain finite values; but when at the same time $\rho = \rho'$, $q = 2p$, then κ is 1 and K is logarithmically infinite. This could easily be foreseen and does not make any special trouble in the integrations. The second integral C_k is transformed in a similar way:

$$\int_0^{2\pi} \frac{\cos\varphi d\varphi}{[q - 2p\cos\varphi]^{1/2}} = \frac{2}{[q+2p]^{1/2}} \int_{-\pi/2}^{\pi/2} \frac{2\sin^2\psi - 1}{[1 - \kappa^2\sin^2\psi]^{1/2}} d\psi$$

$$= \frac{4}{[q+2p]^{1/2}} [2D(\kappa) - K(\kappa)], \tag{44}$$

$$C_k = e^{-ik\sqrt{q}} \left\{ \frac{2}{\pi} \cdot \frac{1 + ik\sqrt{q}}{[q+2p]^{1/2}} [2D(\kappa) - K(\kappa)] \cdots \right\}.$$

These approximate formulae should be used for a thin wire and represent the first two terms in an expansion when a/λ is small but not negligible. For the fundamental vibration, λ is of the order of $2l$ (twice the length of the antenna). Hence using the expansions (43), (44), one should be able to go one step further than Oseen or Hallen, who completely neglected a/l and were satisfied with keeping terms in Ω^{-1}, Ω^{-2}, where

$$\Omega = 2 \log \frac{l}{a} \cdot \tag{6a}$$

This parameter comes in, when integrations are performed on D and K for κ near 1,

$$\kappa'^2 = 1 - \kappa^2 = \frac{q - 2p}{q + 2p} = \frac{\zeta^2 + (\rho - \rho')^2}{\zeta^2 + (\rho + \rho')^2}, \quad \text{small}; \quad \zeta = z - z'.$$

This happens when z and z' are nearly equal for two points on the cylindrical surface $\rho = \rho' = a$. It happens again for two points on one of the flat terminals, when $z = z' = 0$ or l, and ρ is nearly ρ'. In such cases, K and D are represented by the following expansions (Jahnke-Emde, p. 145)

$$K = \Lambda + \frac{\Lambda - 1}{4} \kappa'^2 \cdots, \qquad D = \Lambda - 1 + \tfrac{3}{4}(\Lambda - \tfrac{4}{3})\kappa'^2 \cdots,$$

$$\Lambda = \log \frac{4}{\kappa'} = \log 4 - \tfrac{1}{2} \log \kappa'^2 = \log 4 - \tfrac{1}{2} \log \frac{(z - z')^2 + (\rho - \rho')^2}{(z - z')^2 + (\rho + \rho')^2} \cdot \tag{45}$$

Integration and averaging process carried out on Λ will introduce the parameter Ω.

Finally, let us discuss the dependence on k of both functions G_k and C_k. From the definition itself (37), it is seen that both functions can be expressed in terms of G_1, C_1 corresponding to $k = 1$,

$$G_k = \frac{k}{2\pi} \int \frac{e^{-ikr}}{kr} d\varphi = kG_1(kr) \text{ hence:}$$

$$\begin{aligned} G_k(\rho, \rho', \zeta) &= kG_1(k\rho, k\rho', k\zeta) = kG_1(k^2q, k^2p), \\ C_k(\rho, \rho', \zeta) &= kC_1(k\rho, k\rho', k\zeta) = kC_1(k^2q, k^2p). \end{aligned} \tag{46}$$

The same decomposition can be seen from the expansions (41).

6. Conclusions. The preceding sections show clearly the importance of the rôle played by both end-surfaces, whose exact shape should be taken into consideration very carefully. We have shown, on the example of plane terminations, that the problem consists in finding *two* unknown current distributions, one for the cylindrical surface and one for the (symmetrical) terminal surfaces, and this requires solving *two* integral equations. This is the essential difference from the problems of the proper oscillations of *one* closed algebraic surface, such as an ellipsoid. For plane terminations, a complete study of equations (17) and (18) should be affected, and the successive approximations should be worked out simultaneously on both equations. Other shapes of end-surfaces, like half spherical or half ellipsoidal terminals, would certainly yield quite different results. A discussion of this problem is not attempted in the present paper, the aim of which was merely to offer a precise statement of the mathematical theory of antennas and to emphasize some difficulties which seemed to have been overlooked by previous authors.

STABILITY OF COLUMNS AND STRINGS UNDER PERIODICALLY VARYING FORCES*

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1. Introduction. It is a well known fact that a rigid body hinged at one end and standing vertically can be put into stable equilibrium by applying a vertical periodic force of proper frequency and amplitude at the lower end. The differential equation for small oscillations of the rod is a linear homogeneous equation with a periodic coefficient—it is a Mathieu equation if the applied force is a simple sine or cosine function of the time. Stability of the rod would require that all solutions of this equation be bounded; it is found that this is the case if the frequency and amplitude of the applied force are properly chosen. A more complicated problem of the same general type in a system with more than one degree of freedom has been considered by G. Hamel [4][1]; linear differential equations with periodic coefficients play the essential role in this case also.

We shall be interested here in analogous problems in elastic systems with infinitely many degrees of freedom. One of these is the problem of the column under periodic compressive forces $F(t)$ applied at the ends of the column.[2] The analogue of the problems mentioned above would be as follows: the force $F(t)$ consists of a constant part P plus a periodic part $H \cos \omega t$. Suppose that P were a compressive force larger than the lowest compressive load (the Euler load) for which the column in the original unbent position is instable. The question is, then, whether or not H and ω can be chosen in such a way that small motions in the neighborhood of the undeflected position are stable ones. We shall see that this can always be done, though, as one would expect, the quantity H must be chosen so that the total force $F(t)$ falls below the Euler value during at least part of the time. However, the time average of F (over a cycle) may be very much larger than the Euler load. On the other hand, it is quite possible that the column may be *instable* when P is a compressive force smaller than the Euler load or when P is a tension rather than a compression, if H and ω are properly chosen.[3] From the point of view of the practical applications these latter possibilities are certainly the more important ones. For the case of the column with pinned ends we give diagrams which make it possible to decide whether the column is stable or not under any of these circumstances. The stability of the stretched string under a tension which varies periodically in time is also considered.

In all of these problems the Mathieu equation[4] (more properly, a *sequence* of

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* Received April 9, 1943.

[1] Numbers in square brackets refer to the bibliography at the end.

[2] A special case of this problem has been treated by I. Utida and K. Sezawa [16].

[3] Analogous problems for plates under loads in the plane of the plate have been considered by R. Einaudi [1].

[4] We consider always that the applied forces are simple harmonic functions of the time—otherwise we should have to deal with the more general Hill's equation.

Mathieu equations in the continuous systems) plays a central rôle, since the decision as to stability depends upon the character of the solutions of such equations. For this reason a brief summary of the main facts concerning the solutions of the Mathieu equation is included here. A brief treatment of the Mathieu equation with a viscous damping term added is also included because of its importance for the stability problem.

2. The column under periodic axial forces at its ends. We make the assumptions that are customary in dealing with the transverse oscillations of thin rods. Of these, the principal ones are: 1) the rod is an initially straight uniform cylinder, 2) the lateral deflection w (Fig. 1) and the cross sectional dimensions of the beam are small in comparison with the length l, 3) all stresses remain below the proportional limit,

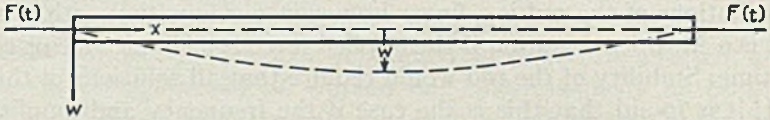

FIG. 1.

4) the effects of shear and rotary inertia are negligible.[5] In addition, we assume that the column is subjected to axial forces F depending on the time t and applied at the ends of the column; these forces are counted positive when they are tensions. With these assumptions the differential equation for the lateral deflection $w(x, t)$ is well known to be as follows:

$$EI\frac{\partial^4 w}{\partial x^4} - F(t)\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} = 0. \tag{2.1}$$

In this equation E and I are Young's modulus of the column and the moment of inertia of its cross section, and m is the mass per unit length. In what follows we assume always that $F(t)$ is given by

$$F(t) = P + H\cos 2\pi ft; \tag{2.2}$$

i.e., it consists of a constant part plus a harmonic component of amplitude H and frequency f.

It should be pointed out that the derivation of (2.1) involved a tacit assumption not included among those enumerated above. This was that the forces $F(t)$ applied at the ends of the column result in forces throughout the column which are, to a sufficiently close approximation, independent of x. We proceed to show that this assumption is warranted under the circumstances normally encountered in practice. The differential equation for the longitudinal displacement $u(x, t)$ of the rod is

$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{2.3}$$

in which ρ is the density of the rod. The total force F transmitted through any cross section of the rod of area A is given by

[5] These effects could be taken into account without difficulty, but nothing new in principle would result.

$$F = AE\frac{\partial u}{\partial x}\cdot \tag{2.4}$$

We assume as boundary conditions

$$u = 0 \quad \text{at} \quad x = 0, \tag{2.5}$$

and

$$F = AE\frac{\partial u}{\partial x} = P + H \cos 2\pi f t \quad \text{at} \quad x = l/2, \tag{2.6}$$

the origin of coordinates being taken at the midpoint of the rod in order to take advantage of symmetry. We seek the forced oscillation and neglect the free oscillation. The result for the quantity F is readily found to be

$$F(x, t) = P + H\frac{\cos \lambda x}{\cos (\lambda l/2)} \cos 2\pi f t, \tag{2.7}$$

with

$$\lambda = 2\pi f(\rho/E)^{1/2}. \tag{2.8}$$

It is convenient to introduce the fundamental frequency f_0 of the free longitudinal vibration of the rod which has a single node at the center. This is given by

$$f_0 = (1/2l)(E/\rho)^{1/2}. \tag{2.9}$$

Upon introducing this into (2.7) we obtain

$$F(x, t) = P + H\frac{\cos (\pi f x/f_0 l)}{\cos (\pi f/2f_0)} \cos 2\pi f t. \tag{2.10}$$

If f is small compared with f_0 it is clear that F will be nearly independent of x. For steel or aluminum $(E/\rho)^{1/2}=17000$ ft./sec., while for brass, concrete, stone, or wood this quantity is about 12000 ft./sec. For any column of usual length f_0 will therefore be of the order of 500 cycles/sec. or more. Hence if the applied axial force $F(t)$ is one of frequency below say 50 cycles/sec. it is reasonable to assume that the variation of the axial force with x may be neglected.

We introduce new independent variables replacing t and x in (2.1) by the equations

$$\vartheta = 2\pi f t \quad \text{and} \quad \xi = \pi x/l. \tag{2.11}$$

In addition, it is convenient to introduce new parameters as follows:

$$P_E = \pi^2 EI/l^2, \qquad \epsilon_0 = P_E/EA, \tag{2.12}$$

$$p = P/P_E, \qquad h = H/P_E. \tag{2.13}$$

The quantity P_E is the negative of the Euler load for the column and ϵ_0 is the tensile strain due to that load. The quantities p and h are the ratios of the constant part and of the amplitude of the oscillating part of the applied load to the negative Euler load. With these new quantities the differential equation (2.1) becomes

$$\frac{\partial^4 w}{\partial \xi^4} - (p + h\cos\vartheta)\frac{\partial^2 w}{\partial \xi^2} + (f^2/f_0^2\epsilon_0)\frac{\partial^2 w}{\partial \vartheta^2} = 0. \tag{2.14}$$

The quantity f_0 is the fundamental frequency of longitudinal vibration of the column given by (2.9).

The general problem which we wish to investigate can now be stated: for given boundary conditions there are certain values of p, h, and f for which all solutions $w(\xi,\vartheta)$ of (2.14) remain bounded when arbitrary initial conditions are prescribed and other values of these quantities for which unbounded solutions exist. In the former case we say that the column is stable and refer to p, h, and f in this case as stable values. Our problem is to separate the stable from the instable values of p, h, and f.

We do not solve the problem in this generality; we choose rather a special case with regard to the boundary conditions to be imposed.

3. Formulation of the stability problem for the column with pinned ends. The boundary conditions we choose are those corresponding to the case of a column with pinned ends; that is, we assume that the deflection w and bending moment $M=EI(\partial^2 w/\partial x^2)$ are both zero at $x=0$ and $x=l$. We have, therefore, as boundary conditions for (2.14):

$$w = \frac{\partial^2 w}{\partial \xi^2} = 0 \quad \text{for} \quad \xi = 0 \quad \text{and} \quad \xi = \pi. \tag{3.1}$$

These boundary conditions can be satisfied by taking for w a solution in the form of a Fourier sine series:

$$w = \sum_{n=1}^{\infty} F_n(\vartheta) \sin n\xi. \tag{3.2}$$

The series (assuming that it converges properly) is a solution of (2.14) provided that the function $F_n(\vartheta)$ satisfies the differential equation

$$\frac{d^2 F_n}{d\vartheta^2} + (\alpha_n + \beta_n \cos \vartheta) F_n = 0, \qquad n = 1, 2, 3, \cdots, \tag{3.3}$$

in which

$$\alpha_n = n^2(f_0^2/\epsilon_0/f^2)(n^2 + p) \tag{3.4}$$

and

$$\beta_n = n^2(f_0^2 \epsilon_0/f^2)(h). \tag{3.5}$$

The quantities f, f_0, ϵ_0, p, and h have been defined by equations (2.2), (2.9), (2.12), and (2.13) respectively. The differential equation (3.3) is, of course, a Mathieu equation.

We can now see why the choice of the boundary conditions (3.1) brings with it essential simplifications. To begin with, it is not possible to separate the variables in (2.14) in the usual way: if we insert for w in (2.14) an expression of the form $w=f(\xi)F(\vartheta)$ we do not obtain a pair of ordinary differential equations for f and F alone. By assuming for w the *special* form given in (3.2) we are able to satisfy (2.14) by virtue of the fact that only even ordered derivatives of w with respect to ξ occur in it. This form of solution is, however, not useful for boundary conditions other than those given by (3.1).[6] The reason for this is as follows: since w satisfies (2.14) we

[6] The problem can be solved for other boundary conditions, but with much more difficulty. It is not possible, for example, to make use of the theory of the Mathieu equation in other cases. For a possible approach, see R. Einaudi [1], and S. Lubkin [8].

must require that $\partial^4 w/\partial\xi^4$ be continuous, since w and $\partial^2 w/\partial\xi^2$ (the bending moment within a constant factor) should be assumed continuous on physical grounds. But the sine series (3.2) can be differentiated four times with respect to ξ if, and only if, w and $\partial^2 w/\partial\xi^2$ vanish at $\xi=0$ and $\xi=\pi$, the end points of the column.[7]

Our definition of stability requires that $w(\xi, \vartheta)$ be bounded for $0 \leqq \vartheta < \infty$ when arbitrary initial conditions are prescribed. Hence we must require for stability that all solutions $F_n(\vartheta)$ of (3.3) for $n=1, 2, 3, \cdots$ and $0 \leqq \vartheta < \infty$ remain bounded when arbitrary initial conditions are prescribed. This is, of course, only a necessary condition for stability. However, we show in an appendix that the Fourier series (3.2) will, roughly speaking, converge for all ϑ if it converges for $\vartheta=0$ and if each $F_n(\vartheta)$ is a stable solution of the Mathieu equation. Such a question does not arise in the more usual type of initial value problem, since the functions analogous to $F_n(\vartheta)$ are generally of the form $e^{-r_n\vartheta}(A_n \cos n\vartheta + B_n \sin n\vartheta)$, $r_n \geqq 0$.

4. The Mathieu equation. The problem of the stability of the column with pinned ends has been reduced to that of determining whether all solutions of the Mathieu equation

$$\frac{d^2F}{d\vartheta^2} + (\alpha + \beta \cos \vartheta)F = 0, \tag{4.1}$$

i.e., of Eq. (3.3) without subscripts, are bounded for given values of α and β or not.

We summarize briefly the known theory of this equation in so far as it is needed for our purposes; more extended discussions and proofs can be found in the pamphlets of M. J. O. Strutt [15] and P. Humbert [5], and in the books of E. L. Ince [7] and Whittaker and Watson [17]. We have also made use of papers of S. Goldstein [2], E. L. Ince [6], and M. J. O. Strutt [14]. The notation we have chosen for the Mathieu equation has been taken to fit our problem; we compare it with the notation used by others:

Strutt	Goldstein	Ince and Whittaker and Watson	Here
u	y	y	F
$2x$	$2x$	$2x$	ϑ
$\lambda/4$	α	$a/4$	α
$-h^2/2$	$-4q$	$4q$	β

It can be shown (theorem of Floquet) that there exist in general two linearly independent solutions F_1 and F_2 of (4.1) which satisfy the relations

$$\begin{aligned} F_1(\vartheta + 2\pi) &= K_1F_1(\vartheta), \\ F_2(\vartheta + 2\pi) &= K_2F_2(\vartheta). \end{aligned} \tag{4.2}$$

The quantities K_1 and K_2 are either conjugate complex or real constants which satisfy the relation

[7] The analogous problem of the rectangular plate with simply supported edges can be treated in the same way as the column with pinned ends. The only essential difference would be that the relations corresponding to (3.4) and (3.5) would contain more parameters.

$$K_1 \cdot K_2 = 1. \tag{4.3}$$

Hence all solutions of (4.1) will be bounded only if

$$|K_1| = |K_2| = 1. \tag{4.4}$$

In case (4.4) is not satisfied, it follows from (4.3) that K_1 and K_2 are both real—a fact of which we make use later on. For certain values of α and β there exist solutions for which the values of K are $+1$ or -1; such solutions are therefore periodic of period 2π or 4π respectively.[8] The pairs of values (α, β) for which such periodic solutions of (4.1) exist can be shown to fill out curves in an α, β-plane which divide that plane into "stable" regions in which (4.4) holds and "instable" regions in which it does not hold. The boundary curves themselves belong to the instable region, the general solution of (4.1) corresponding to (α, β) on such a curve consisting of the sum of a periodic function plus ϑ times a periodic function. Fig. 2 indicates these regions, the stable ones being shaded.

It is of some interest to note that in the stable regions the relation

$$\alpha + |\beta| > 0 \tag{4.5}$$

must hold since otherwise $d^2F/d\vartheta^2$ would always have the sign of F and a solution not identically zero could not remain bounded for $\vartheta \to +\infty$ as well as for $\vartheta \to -\infty$; this would mean instability since $F(-\vartheta)$ is evidently a solution of (4.1) if $F(\vartheta)$ is.

The stable regions are connected at the points $\alpha = k^2/4$, $\beta = 0$, $k = 1, 2, 3, \cdots$, for which the solutions of (4.1) are evidently bounded. As indicated earlier, the boundary curves separating stable and instable regions are characterized by the fact that a periodic solution of period 2π or 4π exists for any pair of values (α, β) on such a curve. This can be made the basis of a method (due to Ince [6]) for determining these curves, as follows: a Fourier series with undetermined coefficients is assumed as a solution of (4.1). Upon substitution in (4.1) an infinite set of linear equations in the coefficients is obtained, each of which involves only three successive coefficients. Each equation may then be solved for the ratio of two successive coefficients in terms of the next higher or of the next lower coefficients. By successive substitution in these relations one is in this way led to two expressions for any such ratio, one of which is a finite and the other an infinite continued fraction. By equating the two, a relation between α and β is obtained which holds at the boundary points separating the stable and instable regions. For a given value of β and with α ranging from $-\infty$ to $+\infty$ one comes first upon the boundary curve C_0 which begins at $\alpha = 0$, $\beta = 0$ (cf. Fig. 2)[9]; the periodic solutions corresponding to points on this curve are of period 2π. Following this, the next two curves, C_1 and S_1, starting at $\alpha = 1/4$, $\beta = 0$ correspond to solutions of period 4π, followed by two, S_2 and C_2, starting at $\alpha = 1$, $\beta = 0$ corresponding to solutions of period 2π, etc. The letters C and S refer to developments in cosine series (for the even solutions) and in sine series (for the odd solutions). The points between two successive curves for which the periods of the corresponding solutions are different are stable points. For small β the boundary curves are given by the following expressions, solutions of type C_{2k} and S_{2k} having the period 2π, while those of type C_{2k+1}, S_{2k+1} have period 4π:

[8] For a given value of β, say, the problem of determining values of α for which such solutions exist is obviously a linear eigenvalue problem.

[9] Essentially the same figure appears in the book of Strutt [15].

$$
\left.\begin{array}{ll}
C_0: & \alpha = -\beta^2/2 + O(\beta^4),\\
C_1: & \alpha = 1/4 - \beta/2 - \beta^2/8 + \beta^3/32 + O(\beta^4),\\
S_1: & \alpha = 1/4 + \beta/2 - \beta^2/8 - \beta^3/32 + O(\beta^4),\\
S_2: & \alpha = 1 - \beta^2/12 + O(\beta^4),\\
C_2: & \alpha = 1 + 5\beta^2/12 + O(\beta^4),\\
C_3: & \alpha = 9/4 + \beta^2/16 - \beta^3/32 + O(\beta^4),\\
S_3: & \alpha = 9/4 + \beta^2/16 + \beta^3/32 + O(\beta^4),\\
\multicolumn{2}{l}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,}\\
S_k, C_k: & \alpha = k^2/4 + \beta^2/2(k^2-1) + O(\beta^4), \qquad k = 4, 5, 6, \cdots
\end{array}\right\} \quad (4.6)
$$

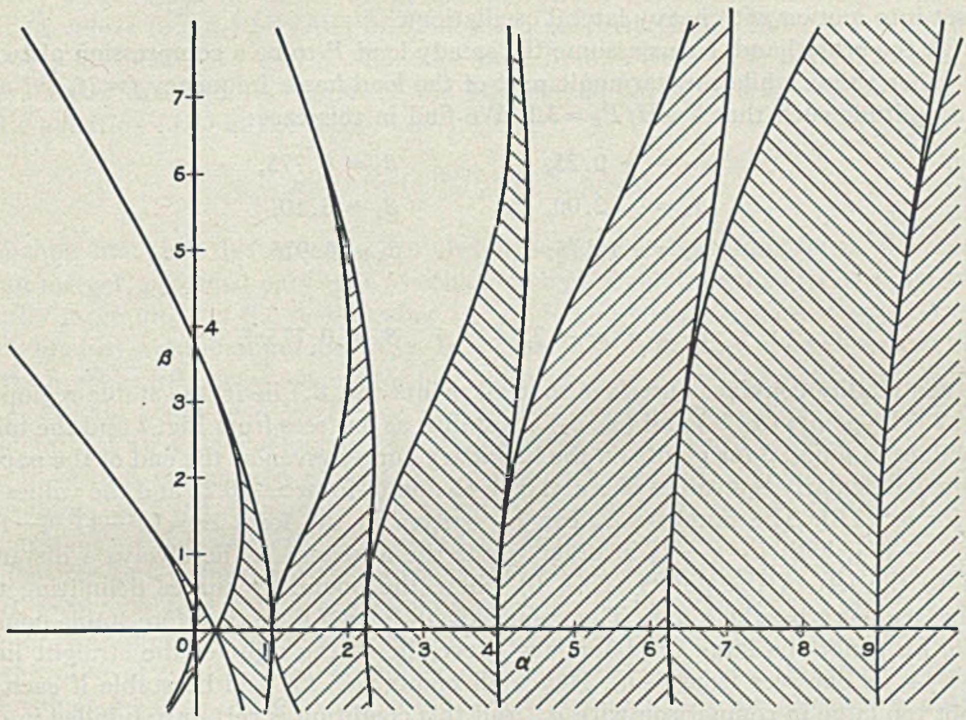

Fig. 2.

Curves of type C_{2k} and S_{2k} have contact of order $2k$ at the points $\alpha = k^2$, $\beta = 0$, while curves of type C_{2k+1} and S_{2k+1} have contact of order $2k+1$ at the points $\alpha = (2k+1)^2/4$, $\beta = 0$. This behavior is clearly indicated in Fig. 2. A table of values of α and β for points on these curves is given at the end of the paper. These values were calculated by means of the procedure outlined above and were checked against values given by S. Goldstein [2] and E. L. Ince [6] where possible.

For large positive values of α the points for which $|\beta| < \alpha$ are stable except for very narrow strips which lie near the lines $\alpha = k^2/4$. For large values of β it has been shown that all boundary curves tend to have the slope -1 (for $\beta > 0$). The stable regions are in general very narrow for $\alpha<0$ and grow narrower as $|\beta|$ increases. These observations are all borne out by Fig. 2.

5. The stability of the column with pinned ends. We may now conclude that the column with pinned ends will be stable only if the applied force $F=P+H\cos\omega t$ is such that all points (α_n, β_n) given by (3.4) and (3.5) fall within the shaded region of Fig. 2. In other words, a set of values (p, h, f) is stable only if every point of the sequence (α_n, β_n) determined by (p, h, f) is stable.

Suppose, for example, that $P=P_E$ (i.e., the steady part of the load is a tension equal in value to that of the Euler load) and that the harmonic part of the load has a frequency $f=f_0(\epsilon_0/2)^{1/2}$. We find that $\alpha_1=1$ and that the column (it is, rather, a tensile member in this case) is instable even for small amplitudes H of the oscillatory part of the load (i.e., for $|\beta_1|$ small), since the points $(1, \beta_1)$ are clearly seen with reference to Fig. 2 to be instable if $|\beta_1|\neq 0$ is small. We could expect the column to be set into motion with heavy lateral oscillations.

On the other hand, let us assume the steady load P to be a compression of twice the Euler value, while the harmonic part of the load has a frequency $f=2f_0\epsilon_0^{1/2}$ and an amplitude such that $h=H/P_E=3.1$. We find in this case:

$$
\begin{array}{ll}
\alpha_1 = -0.25, & \beta_1 = 0.775, \\
\alpha_2 = 2.00, & \beta_2 = 3.10, \\
\alpha_3 = 15.75, & \beta_3 = 6.975, \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\alpha_n = n^2(n^2-2)/4, & \beta_n = 0.775n^2.
\end{array}
$$

We can readily convince ourselves that all points (α_n, β_n) lie in the stable region of Fig. 2. The points (α_1, β_1) and (α_2, β_2) are stable, as one sees from Fig. 2 and the table of values of α and β for points on the boundary curves given at the end of the paper. (Note particularly the values of α and β on C_0 and C_1 for $\alpha\simeq-0.25$ and the values on C_2 for $\alpha\simeq 2.0$). The numbers α_n can be written in the form $\alpha_n=(n^2-1)^2/4-1/4$ $=k^2/4-1/4$, with $k=n^2-1$; in other words the abscissae α_n lie always a distance 1/4 to the left of the points $(k^2/4, 0)$ where the boundary curves delimiting the stable regions cross the α-axis. The points $(\alpha_n, 0)$ for $n>1$ are therefore stable points. Also, for β not too large the boundary curves lie to the right of the straight lines $\alpha=k^2/4$, as one sees from (4.6). Hence all points (α_n, β_n) will be stable if each β_n is not too large in comparison with α_n, and this condition is certainly fulfilled in our case for $n\geqq 2$. Note, for example, that β must be taken larger than 8 for a point of instability when $\alpha=8.75$ (that is, a value 1/4 less than 9). For $\alpha_3=15.75$ we have β_3 only 6.975 in value so that (α_3, β_3) is certainly stable. Since the α_n increase like n^4 while the β_n increase only like n^2, it becomes obvious that all (α_n, β_n) are stable. *The column is therefore stable even though the steady value of the load is twice that of the Euler load.*[10] However, the total compressive load always, as in this case, drops below P_E in value during at least part of the cycle if the column is stable: we have seen (cf. 4.5)) that the inequality $\alpha_n+|\beta_n|>0$ holds for stable solutions; in particular, for $n=1$ this leads to

[10] A. Stephenson [13] appears to have been the first to point out the possibility of such phenomena in general. This paper appeared in 1908.

$$p + |h| > -1, \tag{5.1}$$

as one sees from (3.4) and (3.5), and our statement follows from (2.13).

Thus there exist both stable and instable sets of values (p, h, f). However, our definition of stability leaves out of account a possibility which is always inherent in any physical problem, i.e., that slight changes in the parameters of the problem (p, h, and f in our case) may be sufficient to cause a stable motion to become an instable one. *A set of values (p, h, f) should be considered stable in any proper physical sense only if a complete neighborhood of these values exists which is made up entirely of what we have defined as stable sets of values.*

We proceed to show that the problem of the column never has a stable solution in this more restricted sense; i.e., we show that arbitrarily small changes δf in f and δp in p, for example, can always be found such that $(p+\delta p, h, f+\delta f)$ is instable no matter what values are chosen for p, h, and f. This is done by showing that a certain pair of values (α_n, β_n) becomes instable when properly chosen but arbitrarily small changes are made in f and p. Our statement follows from (3.4) and (3.5) and the character of the instable regions of the Mathieu equation for high values of α. We write equation (3.4) in the form

$$\alpha_n^{1/4}/n = (\epsilon_0 f_0^2/f^2)^{1/4}(1 + p/n^2)^{1/4}, \tag{5.2}$$

and show first that this equation can always be satisfied by taking for α_n the square of an integer, provided only that f is changed by a small amount δf and n is a sufficiently large integer: the real number $(\epsilon_0 f_0^2/f^2)^{1/4}$ can be approximated as accurately as desired by a rational number N/n. It is clear that n can always be chosen so large that an arbitrarily small change δf in f will suffice to make the right hand side of (5.2) exactly equal to N/n. Hence $\alpha_n = N^4$ and our statement is proved. It is also evident that an α_n of the form $n^2/4$ could have been determined in the same manner. We have thus determined a point (α, β) for which $\alpha = n^2/4$, n and $f+\delta f$ being now considered as fixed. We recall the fact that the instable regions of the Mathieu equation cross the α-axis at right angles at the points where $\alpha = n^2/4$ and that these regions for high values of n are narrow strips which remain (for not too large values of β) very near to the vertical straight lines $\alpha = n^2/4$. Since the values of β_n increase like n^2, while those of α_n increase like n^4 it becomes evident that a small change δp in the value of p in (3.4) will be sufficient to cause the point (α', β) corresponding to the values $p+\delta p$, h, $f+\delta f$ to fall inside an instable region of the Mathieu equation. We repeat: no values of p, h, and f $(h, f \neq 0)$ can be found such that the column is stable when small variations in these quantities are permitted.

In the actual physical problem, however, there is an important element present, i.e., viscous damping, which has been neglected so far. In a later section we shall show that the presence of even the slightest amount of viscous damping will suffice to make all values (α, β) stable for which $\alpha \geqq \alpha_0 > 0$, and $|\beta| < \alpha$, when α_0 is a certain constant which may be large. In other words, damping acts in such a way as to cut out the narrow instable strips which occur for large α in the regions for which $|\beta| < \alpha$. Under these circumstances it becomes sufficient to test only a certain *finite* number of the points (α_n, β_n) for stability. Thus the column may be stable if viscous damping is present even when small variations in the quantities p, h, and f take place, though, as we have seen, this is not the case without damping.

Figures 3, 4, and 5 show the stable values of f and h (frequency and relative amplitude $h=H/P_E$ of the vibratory part of the load) for the values $p=P/P_E = -1.5$, -1.0, and 1.0 respectively. The stable regions are shaded.[11] These diagrams have been constructed on the assumption that the amount of viscous damping is large enough that values of α larger than 10 can be ignored. In other words, Figs. 3, 4 and 5 were constructed by combining the stability regions of Fig. 2, which includes values of α up to 10 only, for a suitable number of values of n.

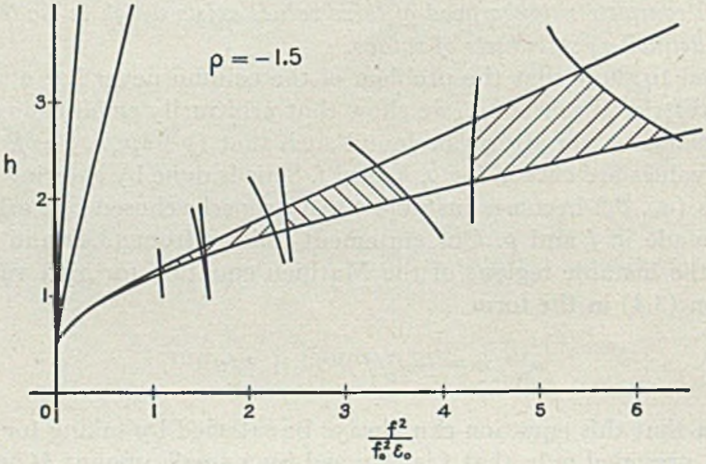

FIG. 3.

The general character of Figs. 3 and 4 is typical for the cases in which $p<-1$, i.e., in which the steady part of the load is a compression larger than the Euler load. We note that the shaded stable regions for $p=-1.5$ are much smaller than

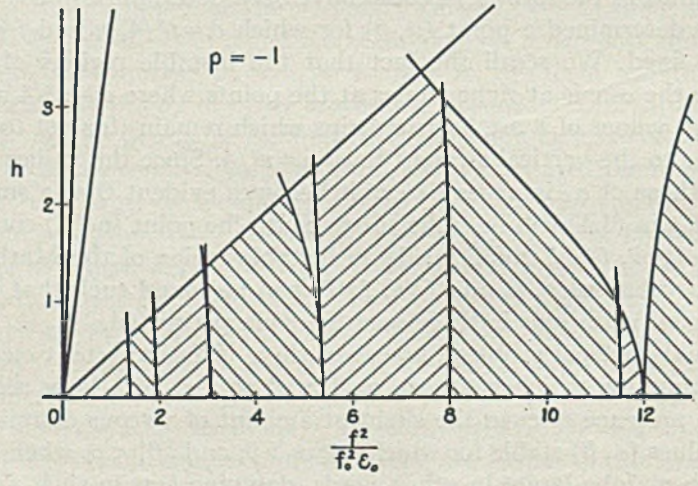

FIG. 4.

those for $p=-1.0$, as was to be expected; for the higher values of the steady compressive load beyond the Euler load it is necessary to make more accurate

[11] Without damping, as we have seen, there could be no stable *regions* though there are stable points. It would have a certain mathematical interest to investigate the set of stable points in detail in this case.

adjustments in the frequency and amplitude of the oscillatory part of the load in order to obtain stability. The full lines which cut through the shaded regions in the figures are not really curves; they represent, rather, narrow instable regions. However, the two curves in Figs. 3 and 4 which appear to be straight lines running near the h-axis indicate narrow stable regions. Fig. 5 is typical for all cases in which $p > -1.0$, i.e., for cases in which the steady part of the load is either a tension or a compression less in value than the Euler load. In these cases the column is stable for all frequencies when $h=0$; it is in fact stable almost everywhere in the neighborhood of the axis $h=0$.

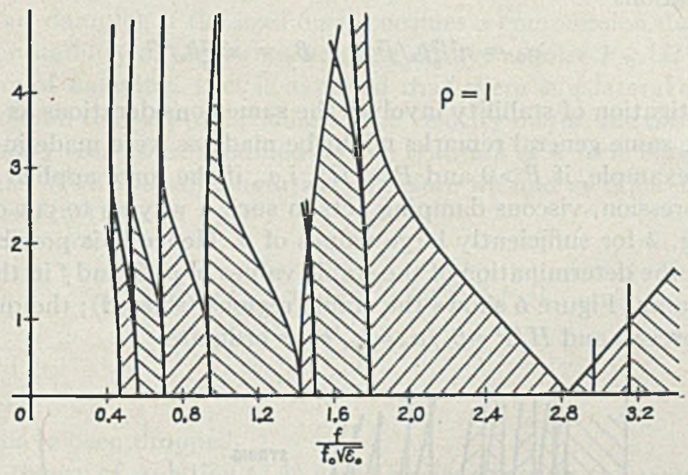

FIG. 5.

It is of some interest to consider the special case in which the amplitude H of the oscillatory part of the applied load is very small so that the values of β_n are small (for n not too large). We note that the natural frequencies f_n of the free lateral oscillations of the rod under steady load (that is, in this case, for $H=0$) are given by $f_n = f\alpha_n^{1/2}$ as one can readily verify. From Fig. 2 we observe that the rod is instable for small values of β when $\alpha_n = k^2/4$, k being any integer. Hence instability occurs for small amplitudes of the oscillatory part of the load whenever

$$f = 2f_n/k, \qquad k = 1, 2, 3, \cdots, \tag{5.3}$$

that is, whenever the load frequency is twice any integral submultiple of a natural frequency of oscillation. At such frequencies one could expect that heavy oscillations would be built up.[12] However, the most favorable case for the production of oscillations is, in general, that for which $n=k=1$. Consider, for example, the case $p=1$. For $n=k=1$ we find readily that $f/f_0\epsilon_0^{1/2} = 8^{1/2} = 2.83$, and one readily sees from Fig. 5 that this furnishes the most favorable frequency for instability at small amplitudes of the oscillatory force.

6. The flexible string under harmonically varying tension. With only slight modifications our preceding results can be used to discuss the problem of the vibrating

[12] This problem has been considered both experimentally and theoretically by I. Utida and K. Sezawa [16].

string subjected to a harmonically varying tension.[13] We have only to set $I=0$ in (2.1) to obtain the fundamental differential equation. The tension $F(t)$ in the string is assumed given by (2.2) and the same independent variables as before are introduced. However, the parameters p and h in (2.3) can obviously not be used here. Instead, we introduce the quantities

$$p_s = P/EA, \qquad h_s = H/EA. \tag{6.1}$$

We may assume for w the expansion (3.2) for a string with fixed ends and will obtain (3.3) as differential equation for the quantities $F_n(\vartheta)$ if we now define α_n and β_n by the equations

$$\alpha_n = n^2 f_0^2 p_s/f^2, \qquad \beta_n = n^2 f_0^2 h_s/f^2. \tag{6.2}$$

The investigation of stability involves the same considerations as for the column, and much the same general remarks might be made as were made in the case of the column. For example, if $P>0$ and $P>|H|$, i.e., if the force applied to the string is never a compression, viscous damping acts in such a way as to cut out the instable regions of Fig. 2 for sufficiently large values of α. Hence it is possible to construct a diagram for the determination of the stable values of p, h, and f in the same manner as for the column. Figure 6 shows the stable regions (shaded); the quantity $f/f_0 p_2^{1/2}$ is taken as abscissa and $H/P=\beta_n/\alpha_n=h_s/p_s$ as ordinate.

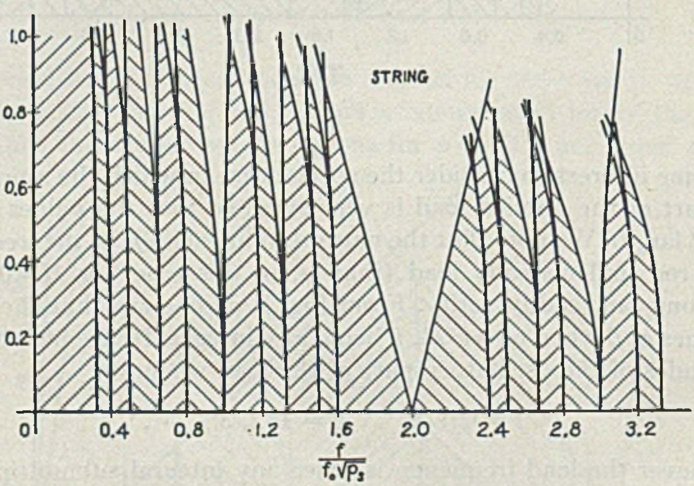

FIG. 6.

It is readily seen that the natural frequencies f_n for the free lateral oscillation of the string (under constant tension) are given by $f_n = f\alpha_n^{1/2}$, just as in the case of the column. The string is instable for low amplitudes of the oscillatory part of the tension when $\alpha_n = k^2/4$, $k=1, 2, 3, \cdots$. In this case we know in addition that $f_n = nf_1$, in which f_1 is the fundamental frequency of the string. Hence "resonance," that is, heavy oscillations for low amplitudes of the applied oscillatory force, will occur when

[13] This problem was first discussed by Lord Rayleigh [11]. The problem was discussed later by A. Stephenson [12], and [13], and by C. V. Raman [10].

$$f = 2nf_1/k, \qquad n,\ k = 1, 2, 3, \cdots, \tag{6.3}$$

that is, at twice any rational multiple of the fundamental frequency of free lateral oscillation of the string. However, the most favorable case for the production of oscillations is readily seen to be that for which $n=k=1$ (i.e., that corresponding to $f/f_0p_s^{1/2}=2.0$). In Melde's experiment lateral oscillations of a string are produced in accordance with (6.3) by attaching one end of the string to the prong of a tuning fork.

There is one marked (though not unexpected) difference between the behavior of the column and that of the string: it could be shown that the string is never stable even with viscous damping if the load on it becomes a compression during any part of the cycle. For stability of the string we must always require $P \geqq |H|$.

7. The effect of damping. If it is assumed that there is a lateral damping force acting on the column that is proportional to the velocity $\partial w/\partial t$, the differential equation (2.1) is readily seen to be modified by the addition of a term $\delta(\partial w/\partial t)$, $\delta>0$, to its left hand side. With the same notation as before we find as differential equation for the functions $F_n(\vartheta)$:

$$\frac{d^2F}{d\vartheta^2} + 2\nu\frac{dF}{d\vartheta} + (\alpha + \beta\cos\vartheta)F = 0, \tag{7.1}$$

where

$$\nu = \delta/4\pi mf, \tag{7.2}$$

and subscripts have been dropped.

The general theory of equation (7.1) could be developed in the same way as that for the Mathieu equation without damping (for a treatment which includes a damping term, see the papers of G. Gorelik [3]). In particular, the α, β-plane could be divided into stable and instable regions. We confine ourselves here to one special problem, i.e., to a discussion of the behavior of the solutions of (7.1) for a given value of ν and large positive values of α. We assume also that $|\beta|<\alpha$.

Upon making the substitutions

$$F = e^{-\nu\vartheta}G, \qquad \alpha' = \alpha - \nu^2 \tag{7.3}$$

Eq. (7.1) becomes

$$\frac{d^2G}{d\vartheta^2} + (\alpha' + \beta\cos\vartheta)G = 0. \tag{7.4}$$

Obviously, if G is bounded, F is not only bounded but approaches zero as ϑ increases. Also, even at boundary points (α', β) separating stable and instable regions of (7.4), the corresponding solutions F tend to zero since no solution G of (7.4) increases faster than ϑ in this case. If the amount of damping is slight (that is, if ν is small), the boundary curves for (7.1) would lie near those for $\nu=0$, but they would not intersect the α-axis except at the origin since all solutions F of (7.1) are clearly bounded for $\beta=0$, $\alpha>0$. This reasoning makes it seem rather evident that the narrow instable regions which occur for large positive values of α when $|\beta|<\alpha$ are cut out when a damping term is added.

We proceed to give a proof of the following statement: *if* $\nu>0$ *and* $|\beta|<\alpha$, *all* *solutions of (7.1) are stable for all values of α which exceed a certain value $\alpha_0>0$.* It was pointed out earlier that there exist two linearly independent solutions G_1 and G_2 of (7.4) such that $G_1(\vartheta+2\pi)=KG_1(\vartheta)$ and $KG_2(\vartheta+2\pi)=G_2(\vartheta)$ with $|K|>1$ in case (α', β) is in an instable region for (7.4). We know also that K is a real number in this case. The solutions of (7.1) will, however, remain bounded even in such an instable case for (7.4) provided that

$$e^{2\pi\nu} > |K|, \tag{7.5}$$

as one sees from (7.3). Consequently our statement will be proved if we can show that $|K|\to 1$ as $\alpha'\to\infty$. This we prove through the use of the following asymptotic formula for K, valid under our assumptions, which has been given by Strutt [14]:

$$K + 1/K = 2 \cosh \lambda \cos \zeta + O(1/\sqrt{\alpha'}) \tag{7.6}$$

in which

$$\zeta + \lambda\sqrt{-1} = \int_0^{2\pi} (\alpha' + \beta \cos \vartheta)d\vartheta, \tag{7.7}$$

and $O(1/\sqrt{\alpha'})$ means that all terms neglected are of order $1/\sqrt{\alpha'}$ or higher. Since we assume that $|\beta| < \alpha'$ the integral in (7.7) is real and $\lambda=0$. We have, therefore:

$$| K + 1/K | < | 2 \cos \zeta | + O(1/\sqrt{\alpha'}) < 2 + O(1/\sqrt{\alpha'}). \tag{7.8}$$

Since K is real it is readily seen that

$$2 \leqq | K + 1/K |, \tag{7.9}$$

equality holding only for $|K|=1$. From this and inequality (7.8) it follows at once that

$$| K | \to 1 \quad \text{when} \quad \alpha' \to \infty. \tag{7.10}$$

In the case of the column we note from Eqs. (3.4) and (3.5) that $|\beta_n| < \alpha_n$ for sufficiently large n and that $\alpha_n\to\infty$ with n. The assumptions under which (7.10) was derived are thus fulfilled in this case. When damping is present we are therefore justified in neglecting all values of α larger than a certain positive value α_0 in discussing the stable values for the column. Our diagrams were drawn under the assumption that $\alpha_0=10$. In the case of the string, α_n and β_n increase at the same rate with increase of n; consequently our conclusions regarding the effect of damping in this case are valid only when $P>|H|$ (which ensures that $|\beta_n| < \alpha_n$,) i.e., when P is a tension and H is such that the total force in the string is always a tension.

Appendix

Sufficient conditions for stability. For stability we required always that the solution

$$w = \sum_{n=1}^{\infty} F_n(\vartheta) \sin n\xi \tag{A1}$$

of our problems be bounded for arbitrary initial conditions; it is thus necessary to assume for stability that each $F_n(\vartheta)$ be bounded for $0 \leqq \vartheta < \infty$ (ϑ is essentially the time variable). In this appendix we prove a statement made at the end of section (3) to the effect that the series will converge for all ϑ if it converges for $\vartheta=0$ and if the $F_n(\vartheta)$ are all stable solutions of the Mathieu equation.

In order to state our theorem precisely we introduce the series

$$\sum_n D_n \sin n\xi, \qquad \sum_n V_n \sin n\xi \tag{A2}$$

in which D_n and V_n are defined by

$$D_n = F_n(0), \qquad V_n = \left.\frac{dF_n}{d\vartheta}\right|_{\vartheta=0}. \tag{A3}$$

We assume that the series (A2) are such that

$$\sum_n \left\{ \rho_n \left| D_n \right| + \rho_n \alpha_n^{-1/2} \left| V_n \right| \right\} < \infty, \tag{A4}$$

in which ρ_n is a certain positive quantity and α_n is one of the two parameters in the Mathieu equation for the functions $F_n(\vartheta)$:

$$\frac{d^2F_n}{d\vartheta^2} + (\alpha_n + \beta_n \cos \vartheta)F_n = 0. \tag{A5}$$

We assume in addition that the $F_n(\vartheta)$ are stable solutions of (A5) for which

$$\left| \beta_n \right| < k\alpha_n, \qquad 0 \leqq k < 1, \tag{A6}$$

at least for all $n>N$, say.[14] Under these assumptions we show that: *the series*

$$\sum_n F_n(\vartheta) \sin n\xi \quad \textit{and} \quad \sum_n \frac{dF_n(\vartheta)}{d\vartheta} \sin n\xi$$

converge for $0 \leqq \vartheta < \infty$ *in the same sense as the series* (*A*2), *i.e., the convergence relation*

$$\sum_n \left\{ \rho_n \left| F_n(\vartheta) \right| + \rho_n \alpha_n^{-1/2} \left| \frac{dF_n(\vartheta)}{d\vartheta} \right| \right\} < \infty \tag{A7}$$

holds for $0 \leqq \vartheta < \infty$.

If it were assumed that $\rho_n = 1$ in (A4) then $\sum_n F_n(\vartheta) \sin n\xi$ would converge, but its derivative with respect to ϑ would not necessarily converge. If ρ_n were assumed to be $\alpha_n^{1/2}$, the differentiated series would converge. In our cases $\alpha_n^{1/2}$ is of order n for the string and of order n^2 for the column. To assume $\rho_n = \alpha_n^{1/2}$ in (A4) would therefore not seem unduly restrictive when it is considered that the series (A1) should be assumed to converge when it is differentiated twice with respect to ξ in the case of the string and four times with respect to ξ in the case of the column.

We prove our theorem by showing that every stable solution of the Mathieu equation

$$\frac{d^2F}{d\vartheta^2} + (\alpha + \beta \cos \vartheta)F = 0 \tag{A8}$$

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[14] These latter conditions are fulfilled in the stable cases for both column and string. This follows from (3.4) and (3.5) for the column, and from (6.2) and the fact that $\left| h_s \right| < p_s$ in the case of the string.

for which

$$|\beta| < k\alpha, \qquad 0 \leqq k < 1, \tag{A9}$$

and

$$F(0) = D, \qquad \left.\frac{dF(\vartheta)}{d\vartheta}\right|_{\vartheta=0} = V \tag{A10}$$

satisfies the inequality

$$|F(\vartheta)| + \alpha^{-1/2}\left|\frac{dF(\vartheta)}{d\vartheta}\right| \leqq C\{|D| + \alpha^{-1/2}|V|\}, \tag{A11}$$

for $0 \leqq \vartheta < \infty$, C being a constant which depends only upon k. Upon reintroduction of the subscript n in (A11) followed by multiplication with $\rho_n > 0$ and a summation with respect to n, it is clear that (A7) would result from (A4), since C is independent of n.

We proceed to establish the validity of the inequality (A11). For this purpose it is convenient to introduce a new independent variable φ as well as a new dependent variable f in (A8) as follows:[15]

$$\phi(\vartheta) = \int_0^\vartheta \chi^{1/2} d\vartheta, \qquad \chi = \alpha + \beta\cos\vartheta, \tag{A12}$$

$$f = \chi^{1/4} F. \tag{A13}$$

In these variables the differential equation (A8) becomes

$$\frac{d^2 f}{d\varphi^2} + \left(1 + \frac{\beta\cos\vartheta}{4\chi^2} + \frac{5\beta^2\sin^2\vartheta}{16\chi^3}\right) f = 0, \tag{A14}$$

or, as we prefer to write it

$$\frac{d^2 f}{d\varphi^2} + f = \alpha^{-1}\gamma f, \tag{A15}$$

with

$$\gamma = -\alpha\left(\frac{\beta\cos\vartheta}{4\chi^2} + \frac{5\beta^2\sin^2\vartheta}{16\chi^3}\right). \tag{A16}$$

From now on we consider $f(\varphi)$ to be the solution of (A15) which satisfies the initial conditions

$$f(0) = 1, \qquad \frac{df}{d\varphi} = i, \qquad i = \sqrt{-1}. \tag{A17}$$

It is then readily verified that $f(\varphi)$ and its derivative satisfy the integral equations

$$f(\varphi) = e^{i\varphi} - \frac{i}{2\alpha}\left(e^{i\varphi}\int_0^\varphi \gamma(\tau)f(\tau)e^{-i\tau}d\tau - e^{-i\varphi}\int_0^\varphi \gamma(\tau)f(\tau)e^{i\tau}d\tau\right), \tag{A18}$$

$$\frac{df(\varphi)}{d\varphi} = ie^{i\varphi} + \frac{1}{2\alpha}\left(e^{i\varphi}\int_0^\varphi \gamma(\tau)f(\tau)e^{-i\tau}d\tau - e^{-i\varphi}\int_0^\varphi \gamma(\tau)f(\tau)e^{i\tau}d\tau\right). \tag{A19}$$

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[15] This transformation is frequently used in the treatment of various questions relating to the asymptotic behavior of the solutions of certain types of second order ordinary differential equations.

From the general theory of the Mathieu equation it is known that every stable solution $F(\vartheta)$ of (A8) can be expressed in the form $H(\vartheta)e^{ia\vartheta}$, in which $H(\vartheta)$ is a periodic function of period 2π and a is a real constant. It follows from (A13) that $f(\varphi(\vartheta))$ can be expressed in the form $h(\vartheta)e^{ia\vartheta}$ with $h = H\chi^{1/4}$; $h(\vartheta)$ is thus also periodic of period 2π in ϑ. Consequently we may write

$$G = \max \left| f(\vartheta) \right| = \max \left| h(\vartheta) \right| = \max_{|\vartheta| \leqq \pi} \left| h(\vartheta) \right| = \max_{|\vartheta| \leqq \pi} \left| f(\vartheta) \right|. \tag{A20}$$

The validity of (A20) is the essential point in our proof; because of it, bounds for our quantities in the interval $-\pi \leqq \vartheta \leqq \pi$ hold also for $0 \leqq \vartheta < \infty$.

We find from (A16), the definition of χ in (A12), and (A9) that

$$|\gamma| \leqq k/4(1-k^2) + 5k^2/16(1-k^2) = \Gamma. \tag{A21}$$

We note also that

$$\varphi(\pi) \leqq \pi\sqrt{\alpha+\beta} \leqq \pi\sqrt{\alpha}\sqrt{1+k}, \tag{A22}$$

as one sees from (A12). Finally we obtain from (A18) the following inequality for $G = \max |f(\vartheta)|$:

$$G \leqq 1 + \frac{\Gamma\pi\sqrt{1+k}}{\sqrt{\alpha}}G. \tag{A23}$$

In view of our purpose it is permissible to assume from now on that

$$\alpha \geqq \alpha_0 > \Gamma^2\pi^2(1+k) = \alpha_1; \tag{A24}$$

once this is done (A23) may be written in the form

$$G \leqq 1/(1 - \sqrt{\alpha_1/\alpha_0}) = G_0. \tag{A25}$$

In a similar fashion we can show that

$$\max \left| \frac{df(\varphi)}{d\varphi} \right| \leqq G_0; \tag{A26}$$

since $df/d\varphi$ satisfies (A19) and, like $f(\varphi)$ itself, can be written in the form $h(\vartheta)e^{ia\vartheta}$ with h of period 2π in ϑ.

Since the function $f(\varphi(\vartheta))$ given by (A18) and its complex conjugate are linearly independent solutions of (A15) it follows that we may write the general real solution F of (A8) in the form

$$F(\vartheta) = \operatorname{Re} C\chi^{-1/4}f(\vartheta), \qquad C = A - iB, \tag{A27}$$

in which Re means that the real part of what follows is to be taken, and A and B are real but otherwise arbitrary constants. The quantity $dF/d\vartheta$ is then given by the expression

$$\frac{dF}{d\vartheta} = \operatorname{Re}\left\{(\alpha+\beta\cos\vartheta)(A-iB)\frac{df}{d\varphi} + 1/4\,\frac{(A-iB)f(\varphi(\vartheta))\beta\sin\vartheta}{(\alpha+\beta\cos\vartheta)^{5/4}}\right\}. \tag{A28}$$

We find at once, since

$$\varphi(0) = 0 \quad \text{and} \quad \left.\frac{df}{d\varphi}\right|_{\varphi=0} = i,$$

$$D = F(0) = (\alpha + \beta)^{-1/4}A, \qquad V = \left.\frac{dF}{d\vartheta}\right|_{\vartheta=0} = (\alpha+\beta)^{1/4}B, \tag{A29}$$

from which we obtain

$$A = (\alpha+\beta)^{1/4}D, \qquad B = (\alpha+\beta)^{-1/4}V. \tag{A30}$$

For $|F(\vartheta)|$ we then have the inequality

$$\begin{aligned} |F(\vartheta)| &\leqq \left\{\left(\frac{1+k}{1-k}\right)^{1/4} |D| + \frac{1}{\alpha^{1/2}(1-k^2)^{1/4}}|V|\right\}G_0 \\ &< p_0|D| + q_0\alpha^{-1/2}|V|, \end{aligned} \tag{A31}$$

in which p_0 and q_0 depend only upon the constant k introduced in (A9), and G_0 is the bound for max $|f(\vartheta)|$ given in (A25). From (A28) we find

$$\begin{aligned} \left|\frac{dF}{d\vartheta}\right| &< 1/4\,\frac{k}{1-k}(\alpha-\beta)^{-1/4}|C|G_0 + (\alpha+\beta)^{1/4}|C|G_0 \\ &\leqq \alpha^{1/2}p_1|D| + q_1|V|, \end{aligned} \tag{A32}$$

where

$$p_1 = \left\{\frac{1}{4\alpha_0^{1/2}}\,\frac{k}{1-k}\left(\frac{1+k}{1-k}\right)^{1/4} + (1+k)^{1/2}\right\}G_0 \tag{A33}$$

and q_1 is of similar nature. The quantities p_1 and q_1, like p_0 and q_0 in (A31), depend only upon k. Division of both sides of (A32) by $\sqrt{\alpha}$, followed by addition to (A31) yields

$$|F| + \alpha^{-1/2}\left|\frac{dF}{d\vartheta}\right| \leqq p|D| + \alpha^{-1/2}q|V|, \tag{A34}$$

which establishes the validity of (A11) and thus completes the proof of our theorem.

Coordinates of Points on the Boundary Curves of Fig. 2.

β	$\alpha(C_0)$	$\alpha(C_1)$	$\alpha(S_1)$	$\alpha(S_2)$
0.0	0.00000	0.25000	0.25000	1.00000
0.2	−0.01966	0.14525	0.34475	0.99667
0.4	−0.07510	0.03191	0.42796	0.98670
0.6	−0.15836	−0.08872	0.49816	0.97018
0.8	−0.26148	−0.21555	0.55906	0.94724
1.0	−0.37849	−0.34767	0.59480	0.91806
1.2	−0.50535	−0.48430	0.62006	0.88284
1.4	−0.63942	−0.62480	0.63015	0.84183
1.6	−0.77898	−0.76867	0.62592	0.79529
1.8	−0.92281	−0.91545	0.60857	0.74349
2.0	−1.07013	−1.06480	0.57950	0.68672

β	$\alpha(C_0)$	$\alpha(C_1)$	$\alpha(S_1)$	$\alpha(S_2)$
2.2	−1.22031	−1.21640	0.54012	0.62526
2.4	−1.37291	−1.37002	0.49174	0.55938
2.6	−1.52760	−1.52544	0.43554	0.48935
2.8	−1.68410	−1.68248	0.37253	0.41542
3.0	−1.84221	−1.84098	0.30357	0.33785
3.2	−2.00175	−2.00081	0.22938	0.25684
3.4	−2.16258	−2.16185	0.15057	0.17263
3.6	−2.32457	−2.32402	0.06763	0.08541
3.8	−2.48764	−2.48720	−0.01901	−0.00468
4.0	−2.65168	−2.65134	−0.10899	−0.09734
4.4	−2.98242	−2.98220	−0.29781	−0.29009
4.8	−3.31627	−3.31614	−0.49688	−0.49171
5.2	−3.65286	−3.65277	−0.70474	−0.70124
5.6	−3.99186	−3.99180	−0.92026	−0.91787
6.0	−4.33302	−4.33298	−1.14253	−1.14088
6.4	−4.67611	−4.67609	−1.37085	−1.36970
6.8	−5.02097	−5.02096	−1.60460	−1.60383
7.2	−5.36744	−5.36743	−1.84328	−1.84271
7.6	−5.71537	−5.71537	−2.08644	−2.08607
8.0	−6.06467	−6.06466	−2.33382	−2.33353
8.4	−6.41522	−6.41522	−2.58498	−2.58478
8.8	−6.76694	−6.76694	−2.83970	−2.83955
9.2	−7.11974	−7.11974	−3.09772	−3.09761
9.6	−7.47357	−7.47357	−3.35883	−3.35875
10.0	−7.82835	−7.82835	−3.62283	−3.62277
11.0	−8.71911	−8.71911	−4.29436	−4.29434
12.0	−9.61474	−9.61474	−4.98065	−4.98064
13.0	−10.51465	−10.51465	−5.67983	−5.67982
14.0	−11.41834	−11.41834	−6.39044	−6.39043
15.0	−12.32542	−12.32542	−7.11126	−7.11126
16.0	−13.23556	−13.23556	−7.84129	−7.84129
18.0	−15.06389	−15.06389	−9.32566	−9.32566
20.0	−16.90154	−16.90154	−10.83807	−10.83807

β	$\alpha(C_2)$	$\alpha(C_3)$	$\alpha(S_3)$	$\alpha(S_4)$
0.0	1.00000	2.25000	2.25000	4.00000
0.2	1.01633	2.25225	2.25275	4.00133
0.4	1.06171	2.25808	2.26203	4.00530
0.6	1.12806	2.26622	2.27933	4.01181
0.8	1.20733	2.27554	2.30589	4.02075
1.0	1.29317	2.28515	2.34258	4.03192

β	$\alpha(C_2)$	$\alpha(C_3)$	$\alpha(S_3)$	$\alpha(S_4)$
1.2	1.38126	2.29429	2.38967	4.04512
1.4	1.46860	2.30233	2.44680	4.06010
1.6	1.55305	2.30878	2.51308	4.07660
1.8	1.63302	2.31323	2.58723	4.09433
2.0	1.70727	2.31536	2.66777	4.11301
2.2	1.77487	2.31495	2.75314	4.13236
2.4	1.83509	2.31175	2.84194	4.15212
2.6	1.88745	2.30568	2.93284	4.17199
2.8	1.93163	2.29660	3.02467	4.19175
3.0	1.96752	2.28448	3.11640	4.21115
3.2	1.99517	2.26925	3.20712	4.22997
3.4	2.01478	2.25092	3.29604	4.24800
3.6	2.02665	2.22950	3.38247	4.26507
3.8	2.03118	2.20500	3.46578	4.28099
4.0	2.02881	2.17748	3.54547	4.29563
4.4	2.00521	2.11356	3.69216	4.32053
4.8	1.95947	2.03826	3.81969	4.33886
5.2	1.89487	1.95216	3.92636	4.34996
5.6	1.81419	1.85589	4.01149	4.35338
6.0	1.71968	1.75014	4.07538	4.34881
8.0	1.09281	1.09947	4.12172	4.20467
10.0	0.28857	0.29018	3.84895	3.87349
12.0	−0.63494	−0.63452	3.38071	3.38817
14.0	−1.64702	−1.64690	2.77777	2.78016
16.0	−2.72859	−2.72855	2.07287	2.07367
18.0	−3.86669	−3.86668	1.28641	1.28668
20.0	−5.05198	−5.05198	0.43241	0.43251

β	$\alpha(C_4)$	$\alpha(C_5)$	$\alpha(S_5)$	$\alpha(S_6)$
0.0	4.00000	6.25000	6.25000	9.00000
0.2	4.00134	6.25083	6.25083	9.00057
0.4	4.00538	6.25333	6.25333	9.00229
0.6	4.01226	6.25750	6.25751	9.00515
0.8	4.02215	6.26334	6.26337	9.00915
1.0	4.03530	6.27084	6.27094	9.01430
1.2	4.05204	6.27999	6.28025	9.02060
1.4	4.07273	6.29077	6.29134	9.02806
1.6	4.09776	6.30317	6.30427	9.03667
1.8	4.12755	6.31714	6.31911	9.04643
2.0	4.16245	6.33264	6.33594	9.05735
2.2	4.20283	6.34961	6.35487	9.06943
2.4	4.24889	6.36800	6.37604	9.08267

β	$\alpha(C_4)$	$\alpha(C_5)$	$\alpha(S_5)$	$\alpha(S_6)$
2.6	4.30085	6.38773	6.39956	9.09705
2.8	4.35867	6.40871	6.42560	9.11259
3.0	4.42220	6.43085	6.45432	9.12927
3.2	4.49121	6.45406	6.48591	9.14707
3.4	4.56533	6.47821	6.52052	9.16600
3.6	4.64406	6.50321	6.55837	9.18603
3.8	4.72688	6.52893	6.59962	9.20714
4.0	4.81318	6.55525	6.64444	9.22930
4.4	4.99383	6.60921	6.74533	9.27671
4.8	5.18127	6.66411	6.86185	9.32798
5.2	5.37113	6.71898	6.99394	9.38281
5.6	5.55951	6.77289	7.14093	9.44078
6.0	5.74803	6.82500	7.30201	9.50150
8.0	6.50217	7.03409	8.23272	9.82875
10.0	6.89864	7.11706	9.16125	10.14742
12.0	6.97136	7.05384	9.87814	10.40143
14.0	6.82083	6.85144	10.30874	10.55621
16.0	6.51561	6.52721	10.48838	10.59848
18.0	6.09463	6.09902	10.48167	10.52959
20.0	5.58132	5.58302	10.33749	10.35813

BIBLIOGRAPHY

[1] R. EINAUDI, *Sulle configurazioni di equilibrio instabile di una piastra sollecitata da sforzi tangentiali pulsante*, Atti. Accad. Gioenia Catania, in two parts: I) mem. **20**, 1–5 (1936); II) mem. **5**, 1–20 (1937).

Part I) deals with the rectangular plate having simply supported edges, II) with the rectangular plate assuming other boundary conditions. The theory is applied only to cases in which the steady part of the applied load is less than the lowest Euler load and the harmonic component is small in amplitude.

[2] S. GOLDSTEIN, *Mathieu functions*, Trans. Cambridge Phil. Soc., **23**, 303–336 (1937).

[3] G. GORELIK, *Resonanzerscheinungen in linearen Systemen mit periodisch veränderlichen Parametern*, (in Russian), Zeit. tech. Phys., Leningrad, in three parts: I) **4**, 1783–1817 (1934); II) **5**, 195–215 (1935); III) **5**, 489–517 (1935).

Treats the nonhomogeneous "Mathieu equation" with a viscous damping term added. Among other things it is proved that the solutions of the homogeneous Mathieu equation with a damping term all tend to zero with increase of the independent variable if the solutions without damping are stable. (These observations were taken from the Zentralblatt f. Mech., **3**, 302 (1935); **4**, 18 (1936).)

[4] G. HAMEL, *Über lineare homogene Differentialgleichungen zweiter Ordnung mit periodischen Koeffizienten*, Math. Ann., **73**, 371 (1912).

[5] P. HUMBERT, *Fonctions de Lamé et fonctions de Mathieu*, Mém. des Sci. Math. X, Gauthier-Villars, Paris, 1926.

[6] E. L. INCE, *Researches into the characteristic numbers of the Mathieu equation*, Proc. Royal Soc. of Edinburgh, **46**, 20–29 (1925); **46**, 316–322 (1926); **47**, 294–301 (1927).

[7] E. L. INCE, *Ordinary differential equations*, Longmans, Green and Co., London, 1927.

[8] S. LUBKIN, *Stability of columns under periodically varying loads*, 1939, manuscript in the library of New York University.

A thesis for the degree of Doctor of Philosophy. Outlines a treatment of the problem for any boundary conditions.

[9] S. LUBKIN, *Stability of columns under periodically varying loads*, 1943, privately printed. 11 pages.

A brief summary of the above thesis.

[10] C. V. Raman, *Experimental investigations on the maintenance of vibrations*, Proc. Indian Assoc. for the Cultivation of Sci. Bulletin 6, 1912.

A lengthy experimental investigation, beautifully and profusely illustrated with photographs of vibrating strings. Some anomalies are explained by considering a nonlinearity in the phenomena; for this purpose the differential equation

$$\ddot{u} + k\dot{u} + (\eta^2 - 2\alpha \sin 2pt + \beta u^2)u = 0$$

is introduced.

[11] Lord Rayleigh, *On the maintenance of vibrations by forces of double frequency and on the propagation of waves through a medium endowed with a periodic structure*, Phil. Mag. (5) 24, 145–159 (1887).

Considers only the possibility of excitation of the lowest mode of vibration of the string. The effect of damping is considered.

[12] A. Stephenson, *On a class of forced oscillations*, Quart. Journ. of Math. 37, 353–360 (1906).

Amplifies the results of Rayleigh noted above on the vibration of strings. Observes the possibility of exciting vibrations when the frequency of the applied force is a rational multiple of the fundamental frequency of lateral oscillation of the string.

[13] A. Stephenson, *On a new type of dynamical stability*, Mem. and Proc. Manchester Literary and Phil. Soc. 52, No. 8 (1908).

The theoretical possibility of converting the instable equilibrium position of a rigid rod standing on end by applying a vertical periodic force at the bottom seems to have been pointed out for the first time in this paper.

[14] M. J. O. Strutt, *Der characteristische Exponent der Hillschen Differentialgleichung*, Math. Ann., 101, 559–569 (1929).

A study of the asymptotic character of the stable regions of the general Hill's equation.

[15] M. J. O. Strutt, *Lamésche, Mathieusche u. verwandte Funktionen in Physik u. Technik*, Ergeb. der Math., Springer, Berlin, 1932.

A very complete summary of both theory and applications, with an extensive bibliography.

[16] I. Utida and K. Sezawa, *Dynamical stability of a column under periodic longitudinal forces*, Report of the Aero. Res. Inst., Tokyo Imp. Univ., 15, 193 (1940).

It is assumed that the constant part of the applied load is zero (that is, $P=0$ in our notation). The paper is both experimental and theoretical in character. In the experiments in some cases jump phenomena similar to those observed in working with forced oscillations of systems with one degree of freedom and a nonlinear restoring force were noted when the amplitude of the oscillations was large. The boundary conditions in the experiments were nearly those for clamped ends rather than pinned ends. In this paper a reference is given to a paper by K. Nisino: Journ. Aero. Res. Inst., Tokyo, No. 176, May 1939, which apparently deals with the same problem; the writers were not able to find this paper.

[17] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th Ed., Cambridge University Press, 1927.

[18] N. Krylov and N. Bogoliuboff, *Investigation of the influence of resonance in transverse vibration of rods caused by periodic normal forces at one end*, (in Russian), Ukrainian Scientific Research Institute of Armament, Recueil, Kiev, 1935. The writers were not able to locate this paper.

ON MOMENT BALANCING IN STRUCTURAL DYNAMICS*

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1. The method of moment balancing. In recent years several writers in this country have developed the method of moment balancing in the analysis of continuous beams and frameworks. Mention should be made especially of the basic paper by Hardy Cross.[1,2] One could also classify as related procedures the method of balancing angle changes given in a paper by L. E. Grinter,[3] and the whole field of relaxation methods being investigated by R. V. Southwell.[4] That such interest is taken in these methods would seem to indicate that their extension to the dynamics of beams and frameworks might be desirable, and it is the purpose of this article to provide at least the beginning of this extension.

We assume that we are dealing with plane structures on which loads are acting in the plane of the structure. Members of the structure consist of uniform straight beams; and they meet in stiff joints, which are assumed to be fixed against translation. All connections to a foundation are either built-in or hinged.

The method of moment balancing depends upon three very simple ideas, namely, fixed-end moment, stiffness and carry-over factor. We give their definitions here:

> The "fixed-end moment" at the end of a member is the moment which would exist at that end if all joints to which it is connected were fixed against rotation.
>
> If one end of a member is simply-supported, its "stiffness" is the moment required to produce unit rotation of that end. The other end may be built-in, simply-supported or free.
>
> The "carry-over factor" is the numerical value of the moment induced at one end of a member by a unit moment acting at the other end.

Methods of finding these characteristics of beams and other components of a structure are numerous and well-known. Having determined them for all components

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[1] H. Cross, *Analysis of continuous frames by distributing fixed-end moments*, Trans. A.S.C.E. 96, 1 (1932). This paper is followed by discussions, that by L. E. Grinter, pp. 11–20, being particularly informative.

[2] See also: Hardy Cross and N. D. Morgan, *Continuous frames of reinforced concrete*, John Wiley and Sons, 1932, Chapter IV, pp. 81–125; *Moment distribution applied to continuous concrete structures*, Portland Cement Association, Second Edition, 1942.

[3] L. E. Grinter, *Analysis of continuous beams by balancing angle changes*, Trans. A.S.C.E. 102, 1020 (1937).

[4] R. V. Southwell, *Relaxation methods in engineering science*, Oxford University Press, 1940.

of a framework, we assume that all joints of the framework (in Fig. 1, for example) are fixed against rotation; and determine the resulting fixed-end moments acting at the ends of each member. Built-in, simply-supported or free ends are not considered as joints. Then at any joint, say D, a moment equal but opposite in sign to the sum of its fixed-end moments is applied, representing the effect of releasing the joint. This moment is distributed to the members AD, CD, BD, ED, meeting at D, in proportion to their stiffnesses, since all members meeting at D rotate through the same angle. The share falling to each member is called the "balancing moment" acting at the end D of this member. The joint D is now balanced, but the new balancing moment M_{DA} acting at the end D of AD will induce an additional moment

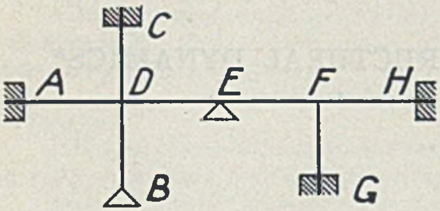

FIG. 1.

$$M_{AD} = C_{AD}M_{DA}$$

at the opposite end A. C_{AD} is the carry-over factor for the member AD, and the moment M_{DA} is said to be "carried over." Likewise, moments are carried over to C and E, but none to B since $C_{DB}=0$. The joint D is again locked—this time in its balanced position—and the process repeated for all joints of the framework until the balancing moments are negligible. The order of choosing unbalanced joints for balancing is not obligatory, but usually the joint with the largest total unbalanced moment at any given stage is balanced. Signs of the moments are chosen so that a positive moment acting on the end of the beam tends to rotate it in a clockwise direction. Likewise, a rotation in the clockwise direction is considered positive.

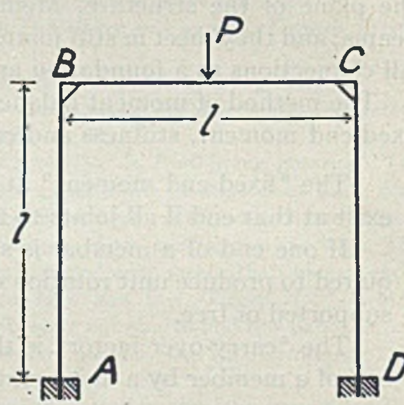

FIG. 2.

Example 1. As a simple example consider the rectangular bent formed of uniform and equal bars, illustrated in Fig. 2. All of the bars are of equal stiffness and the carry-over factor in each case is 1/2. The only non-vanishing fixed-end moments are $-.125Pl$ and $.125Pl$ at the left and right ends of the horizontal bar. The calculations used in the method of moment balancing are shown in Table I. In a given column, say that headed M_{CB}/Pl, we find recorded successively the fixed-end moment and the balancing moment. These are added, and since at this stage $M_{CB}+M_{CD}=0$, the joint C is balanced. The balancing moment has been carried over to column M_{BC}/Pl, and the joint B is balanced next. The same steps are followed until after five balancings the moments to be carried over are negligible. The results obtained agree with those computed by other methods.

2. Dynamics of a simple beam. It is clear that if we can set up analogous definitions for fixed-end moment, carry-over factor and stiffness for a beam on which an oscillating force is acting, and if we can find these characteristics for the oscillating beam, it may be possible to use the method of balancing moments just as it is in the dynamic case. A procedure adapted to this purpose can be found in an article of W. Prager's,[5] the essentials of which will be given here.

TABLE I.

	B		C		
M_{AB}/Pl	M_{BA}/Pl	M_{BC}/Pl	M_{CB}/Pl	M_{CD}/Pl	M_{DC}/Pl
.000	.000	−.125	.125	.000	.000
			−.063	−.062	
		−.032			−.031
—	—	—	—	—	—
.000	.000	−.157	.062	−.062	−.031
	.078	.079			
.039			.040		
—	—	—	—	—	—
.039	.078	−.078	.102	−.062	−.031
			−.020	−.020	
		−.010			−.010
—	—	—	—	—	—
.039	.078	−.088	.082	−.082	−.041
	.005	.005			
.002			.002		
—	—	—	—	—	—
.041	.083	−.083	.084	−.082	−.041
			−.001	−.001	
—	—	—	—	—	—
.041	.083	−.083	.083	−.083	−.041

The differential equation for the deflection, $y(x, t)$, of a uniform beam with no external load is taken as

$$\mu \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0,$$

where μ is the mass per unit length of the beam and EI is its flexural rigidity (Fig. 3). Following a well-known procedure we write $y(x, t) = u(x) \cos \omega t$, $u(x)$ being the amplitude of the assumed harmonic motion and ω its circular frequency. Hence

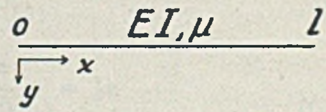

FIG. 3.

$$\frac{d^4 u}{dx^4} + n^4 u = 0,$$

where $n^4 = \omega^2 \mu / EI$; and from this equation

$$u(x) = A \cosh nx + B \sinh nx + C \cos nx + D \sin nx.$$

It is convenient to express the four constants of integration in terms of four quantities of immediate physical importance: the *amplitudes* of the moments acting on the ends of the beam, and of the displacements at the ends of the beam. This can be done by use of the relations

[5] W. Prager, *Die Beanspruchung von Tragwerken durch schwingende Lasten*, Ingenieur-Archiv 1, 527 (1930).

$$u_0 = A + C$$

$$u_1 = A \cosh \lambda + B \sinh \lambda + C \cos \lambda + D \sin \lambda,$$

$$M_0 = -EI \left.\frac{d^2u}{dx^2}\right]_{x=0} = -A - C\}EIn^2,\{$$

$$M_1 = EI \left.\frac{d^2u}{dx^2}\right]_{x=l} = \{A \cosh \lambda + B \sinh \lambda - C \cos \lambda - D \sin \lambda\}EIn^2,$$

where $\lambda = nl$. It can be seen that the amplitudes of the deflection, angle of rotation, bending moment and shear for any value of x will involve linearly the amplitudes of the end deflections and end moments. These quantities can be expressed in much simpler form if the following functions and abbreviations are introduced:

$$\phi(\lambda) = (\coth \lambda - \cot \lambda)/2\lambda, \qquad \bar{\phi}(\lambda) = \frac{\lambda}{2}(\coth \lambda + \cot \lambda),$$

$$\psi(\lambda) = -(\operatorname{csch} \lambda - \csc \lambda)/2\lambda, \qquad \bar{\psi}(\lambda) = \frac{\lambda}{2}(\operatorname{csch} \lambda + \csc \lambda),$$

$$l' = l/EI.$$

Then we find the following expressions for the amplitudes of the angles of rotation at the end points (Fig. 4):

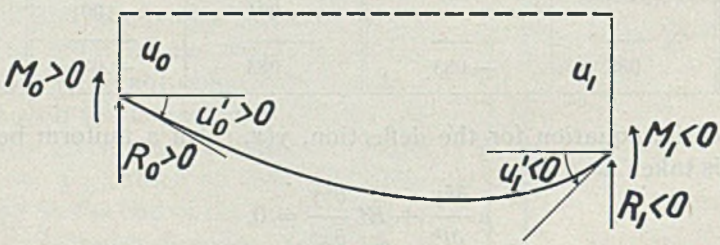

FIG. 4.

$$u_0' = -\frac{u_0\bar{\phi}(\lambda)}{l} + \frac{u_1\bar{\psi}(\lambda)}{l} + M_0 l'\phi(\lambda) - M_1 l'\psi(\lambda), \tag{1}$$

$$u_1' = -\frac{u_0\bar{\psi}(\lambda)}{l} + \frac{u_1\bar{\phi}(\lambda)}{l} - M_0 l'\psi(\lambda) + M_1 l'\phi(\lambda); \tag{2}$$

and for the amplitudes of the reactions:

$$R_0 = -EIu_0''' = u_0 \frac{\lambda^4}{l^2 l'}\phi(\lambda) + u_1 \frac{\lambda^4}{l^2 l'}\psi(\lambda) - \frac{M_0}{l}\bar{\phi}(\lambda) - \frac{M_1}{l}\bar{\psi}(\lambda), \tag{3}$$

$$R_1 = -EIu_1''' = -u_0 \frac{\lambda^4}{l^2 l'}\psi(\lambda) - u_1 \frac{\lambda^4}{l^2 l'}\phi(\lambda) - \frac{M_0}{l}\bar{\psi}(\lambda) - \frac{M_1}{l}\bar{\phi}(\lambda). \tag{4}$$

If the beam, simply supported at both ends, is loaded at its center by an oscillating load, $P \cos \omega t$ (Fig. 5),

$$M_0 = u_0 = u'\left(\frac{l}{2}\right) = 0, \qquad R\left(\frac{l}{2}\right) = P/2.$$

Then, from (2) and (4),

$$\frac{2}{l}\, u\left(\frac{l}{2}\right)\bar{\phi}\left(\frac{\lambda}{2}\right) + \frac{l'}{2}\, M\left(\frac{l}{2}\right)\phi\left(\frac{\lambda}{2}\right) = 0,$$

$$-\frac{1}{2}\, u\left(\frac{l}{2}\right)\frac{\lambda^4}{l^2 l'}\, \phi\left(\frac{\lambda}{2}\right) - \frac{2}{l}\, M\left(\frac{l}{2}\right)\bar{\phi}\left(\frac{\lambda}{2}\right) = \frac{P}{2};$$

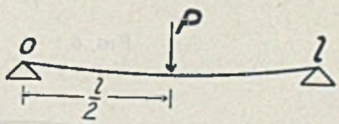

FIG. 5.

so that

$$u\left(\frac{l}{2}\right) = Pl^2 l'\Phi(\lambda), \tag{5}$$

$$M\left(\frac{l}{2}\right) = -\,Pl\bar{\Phi}(\lambda), \tag{6}$$

where

$$\Phi(\lambda) = -\,(\tanh \tfrac{1}{2}\lambda - \tan \tfrac{1}{2}\lambda)/4\lambda^3, \qquad \bar{\Phi}(\lambda) = (\tanh \tfrac{1}{2}\lambda + \tan \tfrac{1}{2}\lambda)/4\lambda.$$

Also, from formulas (5) and (6), and with (1)–(4), we find that

$$u_0' = Pll'\Psi(\lambda) = -\,u_1', \tag{7}$$

and

$$R_0 = P\bar{\Psi}(\lambda) = -\,R_1, \tag{8}$$

where

$$\Psi(\lambda) = -\,(\operatorname{sech} \tfrac{1}{2}\lambda - \sec \tfrac{1}{2}\lambda)4\lambda^2, \qquad \bar{\Psi}(\lambda) = (\operatorname{sech} \tfrac{1}{2}\lambda + \sec \tfrac{1}{2}\lambda)/4.$$

Now, if the beam in question is on unyielding supports but has moments acting on its ends in addition to the load acting at its center, we find by addition of (1) and (7) that

$$u_0' = M_0 l'\phi(\lambda) - M_1 l'\psi(\lambda) + Pll'\Psi(\lambda); \tag{9}$$

and, similarly

$$u_1' = -\,M_0 l'\psi(\lambda) + M_1 l'\phi(\lambda) - Pll'\Psi(\lambda). \tag{10}$$

Obviously, with the equations derived, problems in dynamics of frameworks are reduced to problems in statics of frameworks. To facilitate this work, tables of the functions $\phi(\lambda)$, $\bar{\phi}(\lambda)$, $\psi(\lambda)$, $\bar{\psi}(\lambda)$, $\Phi(\lambda)$, $\bar{\Phi}(\lambda)$, $\Psi(\lambda)$ and $\bar{\Psi}(\lambda)$ are available.[6]

3. Dynamic moment balancing. By substituting "moment-amplitude" and "rotation-amplitude" for "moment" and "rotation," respectively, wherever they occur in the definitions of fixed-end moment, stiffness and carry-over factor, we arrive at suitable definitions for the corresponding dynamic quantities. These three quantities will give us a basis for the application of the moment balancing method to problems in dynamics of frameworks.

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[6] K. Hohenemser and W. Prager, *Dynamik der Stabwerke*, Julius Springer, Berlin, 1933.

First let us consider the amplitudes of the moments acting at the ends of a centrally loaded built-in beam (Fig. 6). Equations (9) and (10) can be applied, and we find that

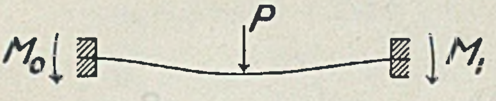

FIG. 6.

$$M_0\phi(\lambda) - M_1\psi(\lambda) = -Pl\Psi(\lambda),$$

and

$$M_0\psi(\lambda) - M_1\phi(\lambda) = -Pl\Psi(\lambda).$$

Then, since $\phi(\lambda)+\psi(\lambda)=2\overline{\Phi}(\lambda)$, the amplitudes of the moment are given by the relations

$$M_0 = -M_1 = -\frac{Pl\Psi(\lambda)}{2\overline{\Phi}(\lambda)}\cdot \tag{11}$$

These quantities give the amplitudes of the fixed-end moments for a beam loaded at its center with a load of amplitude P.

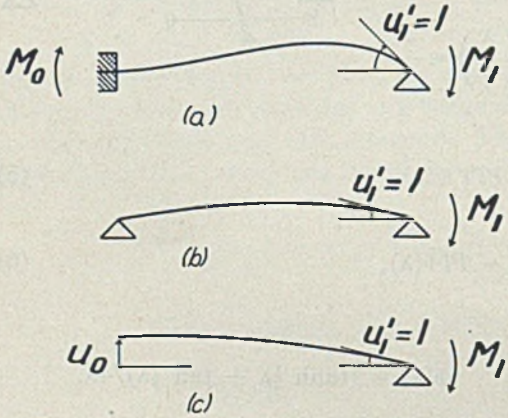

FIG. 7.

The problem of finding the dynamic stiffness is illustrated in Fig. 7. If the far end of the beam is built-in (Fig. 7*a*), we find from (2) that

$$-M_0l'\psi(\lambda) + M_1l'\phi(\lambda) = 1,$$

and from (9)

$$M_0\phi(\lambda) - M_1\psi(\lambda) = 0.$$

Since M_1 is by definition the stiffness, K,

$$K = \frac{\phi(\lambda)}{l'\{[\phi(\lambda)]^2 - [\psi(\lambda)]^2\}} = -\frac{\lambda B(\lambda)}{l'D(\lambda)}, \tag{12}$$

where

$$B(\lambda) = \cosh\lambda\sin\lambda - \cos\lambda\sinh\lambda,$$

$$D(\lambda) = \cosh\lambda\cos\lambda - 1.$$

Tables exist for these functions and for the quotient $B(\lambda)/D(\lambda)$.[6]

For a beam on two simple supports (Fig. 7*b*), equation (2) gives

$$1 = M_1l'\phi(\lambda), \quad \text{so} \quad K = 1/l'\phi(\lambda).$$

To find the stiffness of the cantilever beam (Fig. 7*c*), we find from (2) and (3) that

$$-\frac{u_0}{l}\bar{\psi}(\lambda) + M_1l'\phi(\lambda) = 1,$$

$$u_0\frac{\lambda^4}{l^2l'}\phi(\lambda) - \frac{M_1}{l}\bar{\psi}(\lambda) = 0.$$

Hence,

$$K = \frac{\lambda^4\phi(\lambda)}{l'\{\lambda^4[\phi(\lambda)]^2 - [\bar{\psi}(\lambda)]^2\}}\cdot \tag{13}$$

The carry-over factor needs to be found only for the case illustrated in Fig. 7*a*, since it is zero in the other two cases.

From equation (1)

$$M_0\phi(\lambda) - M_1\psi(\lambda) = 0,$$

so that the carry-over factor is defined by

$$C = \frac{M_0}{M_1} = \frac{\psi(\lambda)}{\phi(\lambda)}\cdot \tag{14}$$

We are now equipped to apply the moment balancing method to problems in dynamics of frameworks.

Example 2. Consider again the bent illustrated in Fig. 2, but now suppose that the frequency ω has a value such that $\lambda = 3.30$ for each bar. Then we have fixed-end moment-amplitudes of $-.169Pl$ and $.169Pl$ at the left and right ends of the horizontal bar, equal carry-over factors of 1.22, and equal stiffnesses for each bar. Table II gives the calculations involved in solving this problem. The values obtained from the 12 balancings are correct to two significant figures.

4. Dynamic balancing of angle changes. The application of the results of Section 2 to L. E. Grinter's method of balancing angle changes[3] is not difficult. In balancing a given joint, the members of the framework meeting at the joint are assumed to be simply-supported and disconnected there. Then rotations are forced by means of applied moments until the angular discontinuities between the members are negligible. To work with rotations rather than moments we require two more definitions.

By "angle-change" will be meant the change in slope produced at the end of a member either by loads or by an applied end moment.

The "angle carry-over factor" is the numerical value of the angle change induced at one end of a member by a unit angle-change imposed upon the other end.

The amplitudes of the angle changes, at the ends of a simply supported beam, due to a central load of amplitude P are seen from (7) to be

$$u_0' = -u_1' = Pll'\Psi(\lambda).$$

The angle carry-over factor can be found by consideration of a simply supported beam, one of whose ends is rotated by means of an applied moment-amplitude (Fig. 8). From equations (1) and (2), $u_1' = -u_0'\ \psi(\lambda)/\phi(\lambda)$ so that the angle carry-over factor is $\psi(\lambda)/\phi(\lambda)$.

Similarly, by consideration of equations (1), (2) and (4) we can arrive at an angle carry-over factor for a cantilever beam:

Fig. 8.

$$C = \frac{\bar{\phi}(\lambda)\bar{\psi}(\lambda) + \lambda^4\phi(\lambda)\psi(\lambda)}{[\bar{\psi}(\lambda)]^2 - \lambda^4[\phi(\lambda)]^2}\cdot$$

Since $\bar{\phi}(0) = \bar{\psi}(0) = 1$, it is seen that this reduces to unity in the static case.

Continuity is established between a member and a joint by giving the joint a rotation-amplitude $K_i\theta_i/\Sigma K$, where θ_i is the amplitude of the angle change in the ith

TABLE II.

	B		C		
M_{AB}/Pl	M_{BA}/Pl	M_{BC}/Pl	M_{CB}/Pl	M_{CD}/Pl	M_{DC}/Pl
.000	.000	−.169	.169	.000	.000
			−.084	−.085	
		−.103			−.103
———	———	———	———	———	———
.000	.000	−.272	.085	−.085	−.103
	.136	.136			
.166			.166		
———	———	———	———	———	———
.166	.136	−.136	.251	−.085	−.103
			−.083	−.083	
		−.101			−.101
———	———	———	———	———	———
.166	.136	−.237	.168	−.168	−.204
	.051	.050			
.062			.062		
———	———	———	———	———	———
.228	.187	−.187	.230	−.168	−.204
			−.031	−.031	
		−.038			−.038
———	———	———	———	———	———
.228	.187	−.225	.199	−.199	−.242
	.019	.019			
.023			.023		
———	———	———	———	———	———
.251	.206	−.206	.222	−.199	−.242
			−.011	−.012	
		−.014			−.014
———	———	———	———	———	———
.251	.206	−.220	.211	−.211	−.256
	.007	.007			
.009			.009		
———	———	———	———	———	———
.260	.213	−.213	.220	−.211	−.256
			−.005	−.004	
		−.005			−.005
———	———	———	———	———	———
.260	.213	−.218	.215	−.215	−.261
	.002	.003			
.003			.003		
———	———	———	———	———	———
.263	.215	−.215	.218	−.215	−.261
			−.001	−.002	
		−.002			−.002
———	———	———	———	———	———
.263	.215	−.217	.217	−.217	−.263
	.001	.001			
———	———	———	———	———	———
.263	.216	−.216	.217	−.217	−.263

member, K_i its stiffness, and the summation extends over all bars meeting at the joint; and at the same time the end of the member itself is given a rotation-amplitude $-[\theta_i - K_i\theta_i/\Sigma K]$. This is done for each member meeting at the joint, thereby balancing that joint; then the assigned rotation-amplitudes are carried over, and the balancing process continues.

After the rotation-amplitudes for all joints have been found with the desired accuracy, the moment-amplitudes can be found from a combination of (9) and (10):

$$M_0 = \frac{u_0' \phi(\lambda) + u_1' \psi(\lambda)}{l'\{[\phi(\lambda)]^2 - [\psi(\lambda)]^2\}} - \frac{Pl\Psi(\lambda)}{2\bar{\Phi}(\lambda)}, \tag{15}$$

$$M_1 = \frac{u_0' \psi(\lambda) + u_1' \phi(\lambda)}{l'\{[\phi(\lambda)]^2 - [\psi(\lambda)]^2\}} + \frac{Pl\Psi(\lambda)}{2\bar{\Phi}(\lambda)}. \tag{16}$$

TABLE III.

B		C	
θ_{BA}/Pll'	θ_{BC}/Pll'	θ_{CB}/Pll'	θ_{CD}/Pll'
.0000	.0625	−.0625	.0000
		.0357	−.0268
	−.0178		
———	———	———	———
.0000	.0447	−.0268	−.0268
.0192	−.0225		
		.0128	
———	———	———	———
.0192	.0192	−.0140	−.0268
		−.0073	.0055
	.0036		
———	———	———	———
.0192	.0228	−.0213	−.0213
.0016	−.0020		
		.0010	
———	———	———	———
.0208	.0208	−.0203	−.0213
		−.0006	.0004
	.0003		
———	———	———	———
.0208	.0211	−.0209	−.0209
.0001	−.0002		
		.0001	
———	———	———	———
.0209	.0209	−.0208	−.0209
		−.0001	
———	———	———	———
.0209	.0209	−.0209	−.0209

$M_{AB} = .0418\ Pl$	$M_{BA} = .0836\ Pl$	$M_{CB} = .0832\ Pl$
$M_{DC} = -.0419\ Pl$	$M_{BC} = -.0832\ Pl$	$M_{CD} = -.0836\ Pl$

It is interesting to observe that when $\lambda=0$ equations (15) and (16) reduce to the slope-deflection equations for a centrally loaded beam.[7]

Example 3. As an illustration let us solve Example 1 by the method of balancing angle changes. For this method, the stiffness of the horizontal bar will be the moment required to produce unit rotation of one end while the other end is simply-supported; while the stiffness of a vertical bar requires the other end to be built-in. Hence the ratio of the stiffness of the horizontal bar to the stiffness of a vertical bar is 3/4. If all joints are assumed to be pin-connected, we have angle changes $\theta_{BC}=.0625Pll'$ and $\theta_{CB}=-.0625Pll'$ due to the load P. For simplicity, u_i' is replaced by θ_i.

In Table III we find the computation used in solving this example. Joint C is balanced first by rotating the member CB through the angle $-(1-\frac{3}{7})(-.0625)Pll'$ $=.0357Pll'$ and the other members of the joint (that is, CD) through the angle $\frac{3}{7}(-.0625)Pll'=-.0268Pll'$. Continuity at that joint is then established, but the rotation of CD induces a rotation at the other end of the beam, $\theta_{BC}=-.178Pll'$. This leaves a total unbalance of $.0447Pll'$ at joint B, which is balanced next. These balancings continue until the angle changes to be carried over are negligible. The resulting moments, computed from (15) and (16) are also listed, and compare favorably with the results obtained for Example 1 by moment balancing.

Example 4. If, now, ω has a value such that $\lambda=3.00$, we find angle changes $\theta_{BC}=.381Pll'=-\theta_{CB}$ due to the load $P\cos\omega t$. Furthermore the angle carry-over factor for the horizontal bar is $-.872$, and as to stiffnesses, $K_{BC}=.549\,l'$, $K_{AB}=K_{CD}$ $=3.102\,l'$. Table IV gives the computation involved in 12 balancings of angle changes in this case. The values of the moment-amplitudes obtained are compared with those obtained by moment balancing.

5. Convergence of the moment balancing process. Convergence of the process of moment balancing can be assured if the frequency of the forced vibration is smaller than the first natural frequency of the structure. The first step of the method of moment balancing leads to the determination of the amplitudes of the unbalanced moments. For the following steps these unbalanced moments are considered as exterior couples acting on the joints of the structure. In the type of structure considered here (joints fixed against translation) the amplitudes of displacement and bending moment of any member are completely determined by the frequency ω and the rotation-amplitudes at the two ends of the member. If a set of values of the rotation-amplitudes at the n joints of the structure is assumed, it is therefore possible to compute the amplitudes of the periodic couples which must be applied to the joints in order to produce the assumed rotation-amplitudes. Let $\theta_i=u_i'$, $(i=1, 2, \cdots, n)$, be the rotation-amplitudes and A_i the corresponding amplitudes of the couples. Furthermore, let B_i be the amplitudes of the exterior couples obtained by the first step of the method of moment balancing. Then, if the assumed θ_i represent the actual configuration enforced by the loads B_i, $A_i-B_i=0$; but in general

$$A_i - B_i = C_i, \tag{17}$$

where C_i is the residual moment-amplitude.

Amongst all possible systems θ_i the actual one minimizes the energy function

[7] See, for example, J. I. Parcel and G. A. Maney, *Statically indeterminate stresses*, John Wiley and Sons, 1936, p. 149.

TABLE IV.

θ_{BA}/Pll'	θ_{BC}/Pll'	θ_{CB}/Pll'	θ_{CD}/Pll'
.000	.381	−.381	.000
		.307	−.074
	−.268		
.000	.113	−.074	−.074
.022	−.091		
		.079	
.022	.022	.005	−.074
		−.064	.015
	.055		
.022	.077	−.059	−.059
.011	−.044		
		.038	
.033	.033	−.021	−.059
		−.030	.008
	.027		
.033	.060	−.051	−.051
.005	−.022		
		.019	
.038	.038	−.032	−.051
		−.015	.004
	.013		
.038	.051	−.047	−.047
.003	−.010		
		.008	
.041	.041	−.039	−.047
		−.006	.002
	.006		
.041	.047	−.045	−.045
.001	−.005		
		.005	
.042	.042	−.040	−.045
		−.004	.001
	.003		
.042	.045	−.044	−.044
.001	−.002		
.043	.043	−.044	−.044

$\theta_{AB}=\theta_{DC}=0.$

	Balancing angle changes	Balancing moments
M_{AB}/Pl	.116	.118
M_{BA}/Pl	.134	.135
M_{BC}/Pl	−.138	−.135
M_{CB}/Pl	.132	.135
M_{CD}/Pl	−.136	−.135
M_{DC}/Pl	−.119	−.118

$$H = \frac{1}{2}\sum_{i,k=1}^{n} \alpha_{ik}\theta_i\theta_k - \sum_{k=1}^{n} B_k\theta_k. \tag{18}$$

The first term on the right side represents the internal energy,

$$\frac{1}{2}\sum_{i,k=1}^{n} \alpha_{ik}\theta_i\theta_k = \sum\left\{EI\int_0^l (u'')^2 dx - \mu\omega^2\int_0^l u^2 dx\right\}, \tag{19}$$

where the right hand sum is to be taken over all members of the structure. The relation (19) arises from the fact that, for any member, u'' and u can be expressed linearly in terms of the rotation-amplitudes at the ends of this member. Note that $\alpha_{ik} = \alpha_{ki}$ and $A_i = \sum_{k=1}^{n} \alpha_{ik}\theta_k$.

Let us denote the first natural frequency of the structure by ω_1. The values $\alpha_{11}, \alpha_{22}, \cdots, \alpha_{nn}$ then can be shown to be positive as long as $\omega < \omega_1$. Indeed, by Rayleigh's principle

$$\omega_1^2 \leqq \left[\sum EI\int_0^l (u'')^2\, dx\right] \Big/ \left[\sum \mu\int_0^l u^2 dx\right], \tag{20}$$

where the sums are to be extended over all members of the structure. As the function u in (20) let us take the displacements corresponding to $\theta_1 = 1$, $\theta_2 = \theta_3 = \cdots = \theta_n = 0$. From (19) and (20) together with the condition $\omega < \omega_1$ it is then clear that $\alpha_{11} > 0$. Similarly $\alpha_{22} > 0, \alpha_{33} > 0, \cdots, \alpha_{nn} > 0$.

Let a first set of values $\theta_i = \theta_i^{(1)}$ be given and compute the corresponding residual moment-amplitudes $C_i^{(1)}$. Suppose the subscripts $1, 2, \cdots, n$ to be arranged in such a manner that $|C_1^{(1)}| \geqq |C_i^{(1)}|$, $(i = 2, 3, \cdots, n)$. We now define a second set of values $\theta_i^{(2)}$ which differs from the first one only in so far as the value of θ_1 is concerned:

$$\theta_1^{(2)} = \theta_1^{(1)} + \phi, \qquad \theta_i^{(2)} = \theta_i^{(1)}, \qquad (i = 2, 3, \cdots, n).$$

We propose to determine ϕ in such a manner that the value of H is decreased.[8] We have

$$H(\theta^{(2)}) - H(\theta^{(1)}) = \left[\sum_{k=1}^{n} \alpha_{1k}\theta_k^{(1)} - B_1\right]\phi + \frac{\alpha_{11}}{2}\phi^2 = C_1^{(1)}\phi + \frac{\alpha_{11}}{2}\phi^2. \tag{21}$$

Taking

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[8] See G. Temple, *The General theory of relaxation methods applied to linear systems*, Proc. R. Soc. of London, Ser. A, 169, 476, (1938–1939).

$$\phi = -\frac{C_1^{(1)}}{\alpha_{11}} \tag{22}$$

we obtain

$$H(\theta^{(2)}) - H(\theta^{(1)}) = -\tfrac{1}{2}[C_1^{(1)}]^2/\alpha_{11} \tag{23}$$

which is certainly negative as long as $\omega < \omega_1$. The residual moment-amplitude $C_1^{(2)}$ corresponding to the new values $\theta_i^{(2)}$ equals

$$C_1^{(2)} = \sum_{k=1}^{n} \alpha_{1k}\theta_k^{(2)} - B_1 = C_1^{(1)} + \alpha_{11}\phi = 0.$$

This shows that the choice of ϕ according to (22) corresponds precisely to the process of moment balancing where in each step the greatest absolute residual moment is "liquidated." For the next step the subscripts i have to be rearranged, so that $C_1^{(2)}$ is the greatest absolute residual moment. Continuing in this way we obtain a decreasing sequence of values of H. If we simplify our notation by writing $H^{(p)}$ instead of $H(\theta^{(p)})$, this sequence becomes

$$H^{(1)} > H^{(2)} > \cdots > H^{(p)} > \cdots > H_{\min},$$

with

$$H^{(p+1)} - H^{(p)} = -\tfrac{1}{2}[C_1^{(p)}]^2/\alpha_{11}^{(p)} < 0.$$

Here $\alpha_{11}^{(p)}$ has been written instead of the α_{11} of (23), since as a consequence of the rearrangement of the subscripts the value of this quantity changes from step to step. Now $\alpha_{11}^{(p)}$ is positive and can assume only a finite number of different values (n at the most). Furthermore, the sequence $H^{(p)}$ is decreasing monotonically and is bounded from below by $H_{\min}$. Therefore

$$\lim_{p\to\infty} [C_1^{(p)}]^2 = 0.$$

Since $C_1^{(p)}$ is the greatest absolute residual moment in the pth step, this means that ultimately all residual moments will disappear. The structure is then completely balanced.

This convergence may be rather slow, especially if ω is near ω_1. For example, compare the 12 balancings used in Example 2, when $\lambda = 3.30$, to the 5 needed in Example 1, for the same structure when $\lambda = 0$. For this structure $\lambda_1 \doteq 3.55$.

The method of balancing angle changes may not always converge when $\omega < \omega_1$, as will be seen if Example 4 is attempted when $\lambda = 3.30$. Usually the method of balancing moments converges more rapidly than the method of balancing angle changes.

AN APPLICATION OF THE METHOD OF THE ACCELERATION POTENTIAL*

BY

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In this paper Prandtl's theory of the acceleration potential is used, in conjunction with conformal mapping, in order to determine the pressure distribution on a symmetrical control surface consisting of a fin and flap separated by a gap of finite width, under the assumption of a steady irrotational flow of an incompressible perfect fluid. The method used is essentially that by which M. A. Biot[1] recently derived the well-known formulae for lift and moment of a symmetrical airfoil with flap in a remarkably simple manner. The present problem is considerably more complicated than that of a single airfoil; but it is still possible to obtain formulae in closed form. In the case where the gap between fin and flap is large in proportion to the chord of either, the formulae obtained here for the pressure distribution on the fin, or on the flap, do not differ materially from those used for a single symmetrical airfoil.

For treatments of this or related problems by the classical velocity-potential method, see I. Flügge-Lotz and I. Ginzel, *Die ebene Strömung um ein geknicktes Profil mit Spalt*, Ingenieur-Archiv 11, 268–292 (1940), which also contains references to earlier studies. Flügge-Lotz and Ginzel do not restrict themselves to the symmetrical case (fin and flap of equal length) as is done in this paper. The method by which they obtain the complex velocity potential is essentially the same as the one we use to derive the acceleration potential. They compute the pressure distribution for an unsymmetrical split wing rather than the total lift and moment, so that a comparison of their numerical results with ours is not practicable.

A paper by Kutta in the Sitzber. Bayerische Akad. of 1911 considers the special case in which the two airfoils have the same angle of attack. Our results agree with his if we make the identification $\sin 2\alpha = 2\alpha$, where α is the angle of attack.

The first part of the paper gives a description of the methods and results; while some details of the mathematical methods used are given in the second part.

I. General Description of Methods and Results

1. The acceleration potential. The equation of motion of an incompressible perfect fluid of density ρ is

$$\rho \bar{a} = - \operatorname{grad} p, \tag{1.1}$$

where $\bar{a}$ denotes the acceleration vector, and p the pressure. According to Prandtl,[2]

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[1] M. A. Biot, *Some Simplified Methods in Airfoil Theory*, Journal of the Aeronautical Sciences, 9, 185–190 (1942).

[2] L. Prandtl, *Beitrag zur Theorie der tragenden Fläche*, Zeitschrift f. angew. Math. u. Mech., 16, 360–361 (1936).

the function

$$\varphi = \frac{-p}{\rho} \tag{1.2}$$

may be called the acceleration potential, since $\bar{a} = \text{grad } \varphi$.

We shall consider the steady irrotational plane flow around a symmetrical control surface consisting of a fin and flap separated by a finite gap. Taking the axis of x parallel to the velocity U which prevails at an infinite distance from the control surface, we write the components of the velocity vector $\bar{q}$ as

$$q_x = U + u, \qquad q_y = v. \tag{1.3}$$

In the case of a thin profile with a small angle of attack, the terms u and v can be assumed to be small compared with the velocity U of the undisturbed stream.

For a steady flow, the components of the acceleration vector are

$$a_x = q_x \frac{\partial q_x}{\partial x} + q_y \frac{\partial q_x}{\partial y}, \qquad a_y = q_x \frac{\partial q_y}{\partial x} + q_y \frac{\partial q_y}{\partial y}.$$

Introducing the acceleration potential φ on the left sides and the expressions (1.3) on the right sides of these relations, and neglecting terms of the second order in u and v, we obtain

$$\frac{\partial \varphi}{\partial x} = U \frac{\partial u}{\partial x}, \qquad \frac{\partial \varphi}{\partial y} = U \frac{\partial v}{\partial x}.$$

According to the condition of incompressibility, $\partial u/\partial x + \partial v/\partial y = 0$, we have $\partial u/\partial x = -\partial v/\partial y$. Substituting this in the first of our equations gives

$$\frac{\partial \varphi}{\partial x} = -U \frac{\partial v}{\partial y}, \qquad \frac{\partial \varphi}{\partial y} = U \frac{\partial v}{\partial x}. \tag{1.4}$$

Elimination of v between these two equations leads to

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0. \tag{1.5}$$

The acceleration potential φ is thus seen to satisfy the Laplace equation and, consequently, can be taken as the real part of an analytic function, $f(x+iy) = \varphi(x, y) + i\psi(x, y)$, of the complex variable $x+iy$. M. A. Biot has shown that the conjugate function of the acceleration potential, $\psi(x, y)$, also has an immediate physical significance. Indeed, from (1.4) and the well-known Cauchy-Riemann relations, $\partial\varphi/\partial x = \partial\psi/\partial y$, $\partial\varphi/\partial y = -\partial\psi/\partial x$, it follows that the function ψ may be defined so as to equal $-Uv$.

In the rest of this section we shall speak of only one airfoil on the understanding that what is said applies equally to the fin and the flap.

The undisturbed flow U along the x-axis involves a normal velocity $+U\alpha$ at the surface of a thin straight airfoil having the small angle of attack α. The boundary condition of tangential flow on the surface will be satisfied by a function giving the velocity $-U\alpha$ normal to the surface. As is usual in the linear approximation, this boundary condition is applied at the x-axis instead of at the surface; so that we now take a part of the x-axis to represent the airfoil (see Fig. 1). We have, then, $v = -U\alpha$; and hence,

$$\psi = -Uv = U^2\alpha; \tag{1.6}$$

that is, ψ is constant on the surface of the airfoil.

The problem of finding the pressures, and hence the lift on the airfoil, may now be restated. One seeks an analytic function of the complex variable $x+iy$ the imaginary part of which satisfies the condition (1.6) and, since $\psi=-Uv$, vanishes at infinity. The real part of this analytic function may then be used as the acceleration potential from which the pressures may be obtained by (1.2). Taking into account the properties of harmonic functions, the fact that ψ must vanish at infinity in the z-plane $(z=x+iy)$ makes it necessary that ψ have at least one singularity. Aerodynamical considerations indicate the leading edge of the airfoil as the obvious location of this singularity, partly by the analogy with the classical thin-wing theory in which the velocity turned out to be infinite at the leading edge. As in the case studied by Biot, we assume this singularity to be a source-sink doublet with axis parallel to the y-axis.

To simplify further the determination of ψ, the segment of the x-axis now used to represent the airfoil is mapped conformally into a circle in the w-plane.

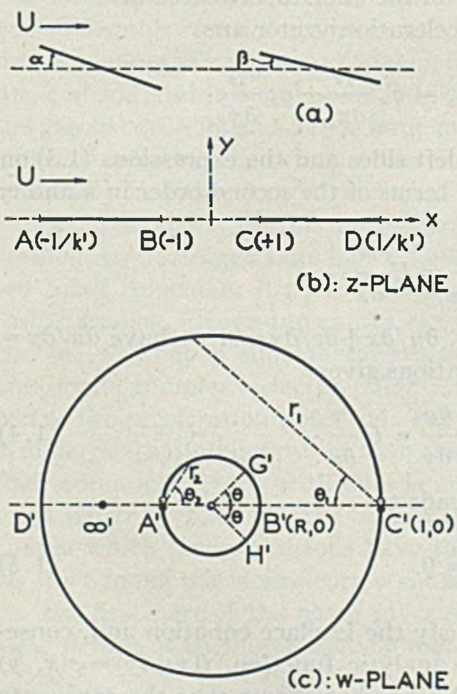

FIG. 1. The actual fin and flap are indicated in (a); the slits used to represent them in (b): it is on these slits that the condition (1.6) is actually met. (c) shows the circles into which (b) is mapped. It is convenient to use temporarily several sets of polar coordinates in the w-plane: (r, θ), with origin at center of circles; (r_1, θ_1), with origin at C'; (r_2, θ_2), with origin at A'.

2. The mapping. We treat only the symmetrical case in which the fin and flap are of equal length.* Let these be represented by slits along the real axis: the fin from $-1/k'$ to -1, the flap from 1 to $1/k'$, $(0<k'<1)$. The z-plane exterior to the two slits can be mapped into the interior of a circular ring in the w-plane, as indicated in Fig. 1. The radius of the outer circle may be taken as unity, the other radius, R, being then fixed. The slits transform into the boundary circles of the ring. The function giving the required mapping can be written in closed form using elliptic functions (see §6).

In the mapping, the upper edge of the slit AB (Fig. 1) goes into the upper semicircle $A'B'$, the lower edge of the slit into the lower semicircle. Moreover, points such as G and H on AB, G being on the upper edge of the slit and H on the lower edge immediately below G, map into points such as $G'(R, \theta)$ and $H'(R, -\theta)$ for which the values of θ are equal but opposite in sign. Similarly for CD. For uniqueness, we define the coordinate θ in the w-plane so that $|\theta| \leqq \pi$. The point $z=\infty$ maps into $\infty'(\sqrt{R}, \pi)$.

* The unsymmetrical case could be handled just as easily at this stage. Any fin-flap arrangement could be mapped into the symmetrical case by a preliminary linear transformation of the type $z'=(az+b)/(cz+d)$; subsequent steps being the same as here. It is in meeting the condition $\psi(\infty)=0$ that the symmetrical case is notably easier to treat; unsymmetrical cases might require numerical handling from that point on.

The radius of the inner circle (transform of the fin) is given by

$$R = e^{-2\pi K'/K}, \tag{2.1}$$

where K and K′ are the complete elliptic integrals of the first kind (modulus k), k being the complementary modulus to k' ($k^2+k'^2=1$). The relation between the coordinate z of a point on one of the slits in Fig. 1(b) and the coordinate θ of the corresponding point in Fig. 1(c) is

$$z = \pm \frac{1}{\mathrm{dn}(K\theta/\pi,\ k)} = \pm \frac{1}{\sqrt{k'}} \frac{\vartheta_4(\frac{1}{2}\theta,\ \sqrt{R})}{\vartheta_3(\frac{1}{2}\theta,\ \sqrt{R})}, \tag{2.2}$$

in which the plus sign holds for z on the flap. The notation of the elliptic and ϑ-functions is that used in Whittaker and Watson, *Modern Analysis* (Cambridge Univ. Press, 4th Ed., 1927; Chap. XXI and XXII).

3. The potential. The problem now is to find a potential which will vanish at infinity and have a singularity of the doublet type at the leading edges of fin and flap (the points A' and C' in the w-plane), and of which the imaginary part ψ will satisfy condition (1.6) separately for each airfoil; namely,

$$\psi = U^2\alpha \text{ on } r = R, \quad \text{and} \quad \psi = U^2\beta \text{ on } r = 1.^* \tag{3.1}$$

In addition, corresponding to the Kutta-Joukowski condition in the classical theory, the acceleration potential must be continuous at the trailing edges of fin and flap.

We place a plane doublet of strength a_1 at C' and one of strength a_2 at A', the axis of each doublet being perpendicular to the real axis. The values of a_1 and a_2 will be determined later by (3.1). The complex potential at the point ξ, in the plane of a complex variable u, due to a doublet of strength m at the point $u=0$ whose axis makes an angle η with the real axis is

$$m\xi^{-1}e^{i\eta}.$$

Thus the potential due to the doublet at A', for example, is given by (see Fig. 1(c))

$$a_2\bar{r_2}^{-1}e^{i(\pi/2-\theta_2)} = a_2\bar{r_2}^{-1}\sin\theta_2 + ia_2\bar{r_2}^{-1}\cos\theta_2.$$

The imaginary part of this potential, $a_2\bar{r_2}^{-1}\cos\theta_2$, has the constant value $a_2/2R$ on the circle $r=R$, but is not constant on $r=1$. Similarly, *mutatis mutandis*, for $a_1\bar{r_1}^{-1}\cos\theta_1$. The function $a_1\bar{r_1}^{-1}\cos\theta_1+a_2\bar{r_2}^{-1}\cos\theta_2$ will not, then, serve as the ψ function, but it may be modified as follows.

We obtain the harmonic function $F(r,\theta)$ which, to within an additive constant, takes the values $-a_1\bar{r_1}^{-1}\cos\theta_1$ on $r=R$, and $-a_2\bar{r_2}^{-1}\cos\theta_2$ on $r=1$. Then the function

$$\psi = a_1\bar{r_1}^{-1}\cos\theta_1 + a_2\bar{r_2}^{-1}\cos\theta_2 + F(r,\theta) + c, \tag{3.2}$$

where a_1, a_2, and the available constant c are chosen so that (3.1) and

$$\psi(\infty') = \psi(\sqrt{R},\ \pi) = 0 \tag{3.3}$$

are satisfied, will be the imaginary part of the potential required.

Equations (3.1) and (3.3) are satisfied (see §7) when

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* When (r,θ) is the map of z we shall use $\psi(z)$ and $\psi(r,\theta)$ indiscriminately where no misunderstanding can arise. The variable indicates the plane (and the point in the plane) at which the ψ function is to be considered. Similarly for other functions with a physical significance.

$$c = -a_1\left(\frac{k'\mathrm{K}}{2\pi} + \frac{3}{4}\right) + \frac{a_2}{R}\left(\frac{\mathrm{K}}{2\pi} - \frac{1}{4}\right),$$

$$a_1 = U^2\left(\frac{2\mathrm{K}(\alpha - \beta) - \pi(\alpha + \beta)}{\mathrm{K}(1 + k')}\right), \tag{3.4}$$

$$\frac{a_2}{R} = U^2\left(\frac{2k'\mathrm{K}(\alpha - \beta) + \pi(\alpha + \beta)}{\mathrm{K}(1 + k')}\right).$$

(The apparent difference between the ways in which a_1 and a_2 appear in (3.4) is due to the fact that a_2 is on a circle of radius R.)

The function $F(r, \theta)$ in (3.2) was obtained in the form of a Fourier series:

$$F(r, \theta) = \sum_1^\infty (A_n r^n + B_n r^{-n}) \cos n\theta.$$

The conjugate harmonic function may then be written immediately as:

$$-\sum_1^\infty (A_n r^n - B_n r^{-n}) \sin n\theta.$$

The conjugate function to

$$a_1 r_1^{-1} \cos \theta_1 + a_2 r_2^{-1} \cos \theta_2 \quad \text{is} \quad -a_1 r_1^{-1} \sin \theta_1 + a_2 r_2^{-1} \sin \theta_2,$$

(the minus sign being due to the fact that θ_1, as defined in Fig. 1(c), is measured in the negative sense). In this way the acceleration potential is obtained to within an additive constant b (which disappears in evaluating the lift). Dropping the auxiliary coordinates, we have finally (see §7)

$$\text{on the fin:} \quad \varphi(R_1\, \theta) = \frac{-a_2}{2R}\frac{\vartheta_2'}{\vartheta_2}(\tfrac{1}{2}\theta, R) - \frac{a_1}{2}\frac{\vartheta_4'}{\vartheta_4}(\tfrac{1}{2}\theta, R) + b,$$

$$\text{on the flap:} \quad \varphi(1, \theta) = -\frac{a_1}{2}\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R) - \frac{a_2}{2R}\frac{\vartheta_3'}{\vartheta_3}(\tfrac{1}{2}\theta, R) + b; \tag{3.5}$$

in which $\vartheta_i'/\vartheta_i(\frac{1}{2}\theta, R)$ is written for

$$\frac{1}{\vartheta_i(\frac{1}{2}\theta, R)}\frac{d}{d(\frac{1}{2}\theta)}(\vartheta_i(\tfrac{1}{2}\theta, R)).$$

4. The lift and moment. From (1.2) we have $p = -\rho\varphi$. The lift on the air-foil at such a point as G is given by

$$l(z) = p(H) - p(G) = p(H') - p(G') = \rho[\varphi(\theta) - \varphi(-\theta)] = 2\rho(\varphi(\theta) - b) = l(\theta).$$

Thus, on the fin,

$$l(R, \theta) = \rho\left(-\frac{a_2}{R}\frac{\vartheta_2'}{\vartheta_2}(\tfrac{1}{2}\theta, R) - a_1\frac{\vartheta_4'}{\vartheta_4}(\tfrac{1}{2}\theta, R)\right), \tag{4.1}$$

on the flap,

$$l(1, \theta) = \rho\left(-a_1\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R) - \frac{a_2}{R}\frac{\vartheta_3'}{\vartheta_3}(\tfrac{1}{2}\theta, R)\right); \tag{4.2}$$

in which a_1 and a_2 are given by (3.4).

The total lift (on the flap, say) would then be given by

$$L_2 = \int_1^{1/k'} l(z)dz = \int_0^{\pi} l(\theta)\,\frac{dz}{d\theta}\,d\theta. \tag{4.3}$$

The moment of the flap about the point $z=0$ is

$$M_2(0) = \int_1^{1/k'} l(z)zdz; \tag{4.4}$$

and about any point z_1, it is

$$M_2(z_1) = \int_1^{1/k'} l(z)(z - z_1)dz = M_2(0) - z_1L_2. \tag{4.5}$$

The functions involved in these integrals are found in (4.2) and (2.2). It will be observed that the ϑ-functions appearing have different parameters.

5. Results. The above integrations can be carried out exactly. The method is described in §8, and the results are:

$$\rho^{-1}U^{-2}L_2 = \frac{\alpha}{k'}\,[\pi(1-k')+2\mathrm{K}k'-\mathrm{E}] + \frac{\delta}{2\pi k'}\,(1-k')(\pi^2+4\mathrm{K}^2k'),$$

$$\rho^{-1}U^{-2}M_2(0) = \frac{\alpha}{k'^2}\,(1-k')\,[2\mathrm{E}-\tfrac{1}{2}\pi\,(1-k')]$$
$$+\frac{\delta}{\pi k'^2}[\pi(1-k')\{\mathrm{K}k'+\mathrm{E}-\tfrac{1}{4}\pi\,(1-k')\}+(\mathrm{K}k'-\mathrm{E})^2], \tag{5.1}$$

$$\rho^{-1}U^{-2}L_1 = \frac{\alpha}{k'}\,[2(\mathrm{E}-\mathrm{K}k'^2)-(1-k')(2\mathrm{K}k'-\pi)]$$
$$+\frac{\delta}{2\pi k'}\,[\pi\{2\mathrm{E}-\mathrm{K}(1+k'^2)\}-\tfrac{1}{2}(1-k')(2\mathrm{K}k'-\pi)(2\mathrm{K}+\pi)],$$

where L_2, $M_2(0)$ have the meanings given in §4, L_1 is the total lift on the fin, E is the complete elliptic integral of the second kind modulus k, and $\delta=\beta-\alpha$.

As a limiting case we consider the situation where the gap between fin and flap is so large in proportion to the chord of the fin (or flap) that they may be expected to act as independent airfoils. In this case $k'\doteq 1$, $k\doteq 0$. Reference to tables of elliptic integrals (e.g., Jahnke, Emde, *Tables of Functions*, Teubner, 2nd Ed., 1938) or direct integration gives $K\doteq E\doteq \pi/2$; so that, from (2.1) we have $R\doteq 0$. As an approximation, we neglect powers of R, and our formulae become (Whittaker, Watson, p. 489)

$$a_1 = -2\beta U^2, \qquad a_2/R = 2\alpha U^2,$$

$$l(1,\theta) = 2\beta\rho U^2 \cot \tfrac{1}{2}\theta. \tag{5.2}$$

The expression in (5.2) is the usual one for the lift on a single straight wing with angle of attack β. For example: when the gap between fin and flap equals 8 times the chord of either, $R\doteq 0.0008$; and evaluating (4.2) we get, in this case,

$$l(1, \theta) \doteq \rho U^2(2.1\,\beta - 0.1\,\alpha) \cot \tfrac{1}{2}\theta, \tag{5.3}$$

which differs only slightly from (5.2) unless $|\beta - \alpha|$ is large; while for $\beta = \alpha$, (5.3) reduces to (5.2):

Some numerical examples are given in the table which follows. C_L, C_H, C_P are the usual lift and moment coefficients: lift coefficient, $C_L = L/\rho(U^2/2)c$, c being the total chord from leading edge of fin to trailing edge of flap; moment coefficient of the flap about its leading edge, $C_H = M_2(1)/\rho(U^2/2)c^2$; moment coefficient of flap about P, $C_P = C_H - C_{L_2}\overline{HP}/c$ where $\overline{HP}$ is the distance from the leading edge of the flap to P (see Fig. 2). The subscripts 1 and 2 indicate fin and flap, respectively, while L indicates lift on either.

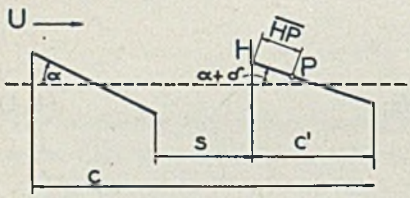

FIG. 2.

In the table $\overline{HP}$ is expressed as a fraction of c', the chord of the flap; and $\delta = \beta - \alpha$ and α are expressed in radians.

TABLE

(α and δ in radians)

s/c'	0.1	0.2	0.3	0.4	0.5
	LIFT COEFFICIENTS				
Fin: C_{L_1}	$4.57\alpha + 2.50\delta$	$4.19\alpha + 2.05\delta$	$3.88\alpha + 1.67\delta$	$3.63\alpha + 1.42\delta$	$3.41\alpha + 1.23\delta$
Flap: C_{L_2}	$1.41\alpha + 2.05\delta$	$1.51\alpha + 2.18\delta$	$1.58\alpha + 2.22\delta$	$1.61\alpha + 2.21\delta$	$1.62\alpha + 2.18\delta$
$\overline{HP}$	MOMENT COEFFICIENTS				
0; C_H	$0.21\alpha + 0.28\delta$	$0.20\alpha + 0.26\delta$	$0.19\alpha + 0.25\delta$	$0.18\alpha + 0.24\delta$	$0.17\alpha + 0.22\delta$
$0.1c'$; $C_{.1}$	$0.15\alpha + 0.18\delta$	$0.13\alpha + 0.16\delta$	$0.12\alpha + 0.15\delta$	$0.12\alpha + 0.14\delta$	$0.11\alpha + 0.14\delta$
$0.2c'$; $C_{.2}$	$0.080\alpha + 0.080\delta$	$0.065\alpha + 0.064\delta$	$0.056\alpha + 0.058\delta$	$0.053\alpha + 0.052\delta$	$0.044\alpha + 0.049\delta$
$0.25c'$; $C_{.25}$	$0.046\alpha + 0.032\delta$	$0.031\alpha + 0.015\delta$	$0.022\alpha + 0.0094\delta$	$0.017\alpha + 0.0060\delta$	$0.012\alpha + 0.0048\delta$

Finally, it might be of interest to note that one can locate a point about which the moment on the flap would be proportional to δ. Although, if hinged at this point, the flap would not remain in the position $\delta = 0$ without some restraint, still, when in this position, the moment on it would be zero. Thus if:

$$s/c' = 0.5, \quad \overline{HP} = 0.270c', \quad C_{.270} = -\,0.014\delta;$$

$$s/c' = 0.1, \quad \overline{HP} = 0.313c', \quad C_{.313} = -\,0.036\delta.$$

II. MATHEMATICAL APPENDIX

The following references will be used:

W = Whittaker and Watson, *Modern Analysis*, Cambridge, University Press, 4 ed., 1927.

J = Jahnke and Emde, *Tables of Functions*, Leipzig, Teubner, 3 ed., 1938.

6. The mapping. (See §2.) The mapping is accomplished in two steps. First, we map the slit z-plane into the interior of a rectangle in the ζ-plane by the function

$$\zeta = \int_0^z [(1 - x^2)(1 - k'^2x^2)]^{-1/2}dx. \tag{6.1}$$

This rectangle has vertices at $\pm K'(k) \pm iK(k)$, the elliptic integrals being expressed as functions of the modulus k complementary to k'. The correspondence is: $z=1 \rightarrow \zeta = K'$, $z=k'^{-1} \rightarrow \zeta = K' \pm iK$, (plus, for the upper boundary of the cut), etc. The integral being multiply valued, the mapping is repeated infinitely often in the ζ-plane, covering the plane without gaps by a net of congruent rectangles.

The second step of the mapping is

$$w = \exp [\pi(\zeta - K')/K]. \tag{6.2}$$

This is periodic with period $2iK$, hence w takes *both* horizontal sides of the rectangle into the part of the real w-axis between -1 and $-R$, $(R = \exp(-2\pi K'/K))$, i.e., into the segment $D'A'$ (Fig. 1(c)). The vertical sides are mapped into the circles $|w| = R$, $|w| = 1$. Because of the periodicity of the exponential function, the full network of rectangles in the ζ-plane is mapped into the ring $R \leqq |w| \leqq 1$, *congruent* points of different rectangles going into the *same* point of the ring. Thus the mapping between the z- and w-planes is one-to-one.

The inverse mapping is, with $w = re^{i\theta}$, (W, p. 492),

$$z = \operatorname{sn}(\zeta, k') = \operatorname{sn}\left(\frac{K}{\pi}\log w + K', k'\right) = \operatorname{sn}\left(\frac{K}{\pi}(i\theta + \log r) + K', k'\right).$$

For the two important cases, $r=1$, $r=R$, the last formula reduces to (2.2) if we transform the elliptic function to one having a real argument (W, pp. 500–506). Note that when the elliptic function is expressed in terms of ϑ-functions, the parameter, q, of the ϑ-functions is $\exp(-\pi K'/K) = R^{1/2}$ (W, p. 479, ex. 3).

7. The potential. (See §3.) The function $F(r, \theta)$, assumed as a Fourier series

$$F(r, \theta) = \sum_1^\infty (A_n r^n + B_n r^{-n}) \cos n\theta, \tag{7.1}$$

must, to within an additive constant, satisfy the boundary conditions

$$\begin{aligned} F(R, \theta) &= -a_1 r_1^{-1} \cos\theta_1 = \frac{-a_1(1 - R\cos\theta)}{1 + R^2 - 2R\cos\theta}, \\ F(1, \theta) &= -a_2 r_2^{-1} \cos\theta_2 = \frac{-a_2(R + \cos\theta)}{1 + R^2 + 2R\cos\theta}, \end{aligned} \tag{7.2}$$

the expressions on the right being obtained from Fig. 1(c). But

$$\frac{1 - R\cos\theta}{1 + R^2 - 2R\cos\theta} = \Re\frac{1}{1 - Re^{i\theta}} = \sum_{n=0}^\infty R^n \cos n\theta,$$

$$\frac{R + \cos\theta}{1 + R^2 + 2R\cos\theta} = \Re\frac{e^{-i\theta}}{1 + Re^{-i\theta}} = \sum_{n=0}^\infty (-1)^n R^n \cos(n+1)\theta,$$

where $\Re$ denotes the real part; and hence, substituting the last two expressions in (7.2) and comparing with (7.1), we get

$$A_nR^n + B_nR^{-n} = -a_1R^n, \quad A_n + B_n = -a_2(-1)^{n-1}R^{n-1}, \quad (n \geqq 1), \tag{7.3}$$

with solutions

$$A_n = \frac{a_1R^{2n} + a_2(-1)^nR^{n-1}}{1 - R^{2n}}, \qquad B_n = -\frac{a_1R^{2n} + a_2(-1)^nR^{3n-1}}{1 - R^{2n}}. \tag{7.4}$$

On substituting these values for A_n, B_n, or, more conveniently Eqs. (7.3), in the definition (7.1) of $F(r, \theta)$, we find

$$F(R, \theta) = -a_1r_1^{-1}\cos\theta_1 + a_1, \qquad F(1, \theta) = -a_2r_2^{-1}\cos\theta_2; \tag{7.5}$$

so that $F(r, \theta)$ satisfies the required conditions. The constant c in (3.2) must be chosen so that ψ vanishes at the infinite point of the z-plane; i.e., (see (3.3)),

$$\psi(R^{1/2}, \pi) = [-a_1r_1^{-1}\cos\theta_1 + a_2r_2^{-1}\cos\theta_2]_{r=R^{1/2},\theta=\pi} + \sum_1^\infty (-1)^n(A_nR^{1/2n} + B_nR^{-1/2n}) + c = 0.$$

The value of the bracket is obtained from Fig. 1(c); and using (7.4) we find

$$c = -\frac{a_1}{1 + R^{1/2}} + \frac{a_2}{R^{1/2}(1 - R^{1/2})} + a_1\sum_1^\infty (-1)^n\frac{R^{3n/2}(1 - R^n)}{1 - R^{2n}} - \frac{a_2}{R}\sum_1^\infty \frac{R^{3n/2}(1 - R^n)}{1 - R^{2n}}. \tag{7.6}$$

For the evaluation of the infinite sums in (7.6) we need certain formulae from the theory of elliptic functions. These are

$$\sum_1^\infty \frac{q^n}{1 + q^{2n}} = \frac{K}{2\pi} - \frac{1}{4} \qquad \text{(W, p. 511, ex. 1)},$$

$$\sum_1^\infty \frac{(-1)^nq^n}{1 + q^{2n}} = \frac{Kk'}{2\pi} - \frac{1}{4} \qquad \text{(ibid., ex. 2)}.$$

Using $R = q^2$ (cf. end of §6), and making some obvious algebraic reductions, we find for c the value given in (3.4).

The values of a_1 and a_2 are found from (3.1) which with (3.2) and (7.5) give the equations

$$\tfrac{1}{2}a_2/R + a_1 + c = U^2\alpha, \qquad \tfrac{1}{2}a_1 + c = U^2\beta.$$

The solutions of these are contained in (3.4); and thus ψ is completely determined.

The acceleration potential φ is now obtained as the harmonic conjugate to the function ψ. We have, (see §3)

$$\varphi = -a_1r_1^{-1}\sin\theta_1 + a_2r_2^{-1}\sin\theta_2 - \sum_1^\infty (A_nr^n - B_nr^{-n})\sin n\theta + b,$$

where b is an arbitrary constant; or, in terms of r and θ,

$$\varphi(r, \theta) = \frac{-a_1r\sin\theta}{1 + r^2 - 2r\cos\theta} + \frac{a_2r\sin\theta}{R^2 + r^2 + 2rR\cos\theta} - \sum_1^\infty (A_nr^n - B_nr^{-n})\sin n\theta + b. \tag{7.7}$$

For the purpose of evaluating the lift we need only $\varphi(R, \theta)$ and $\varphi(1, \theta)$. These are given in (3.5); we establish the expression for $\varphi(1, \theta)$, that for $\varphi(R, \theta)$ being obtained in a similar way.

If, in (7.7), we set $r=1$, use (7.4), and collect terms we find

$$\begin{aligned}\varphi(1, \theta) = & -\frac{a_1}{2}\left\{\cot \tfrac{1}{2}\theta + 4\sum_1^\infty \frac{R^{2n}}{1-R^{2n}} \sin n\theta\right\} \\ & -\frac{a_2}{R}\left\{\frac{-R\sin\theta}{1+R^2+2R\cos\theta} - \sum_1^\infty (-1)^n R^n \sin n\theta \right. \\ & \left. + 2\sum_1^\infty \frac{(-1)^n R^n}{1-R^{2n}} \sin n\theta\right\} + b. \end{aligned} \tag{7.8}$$

But

$$\frac{-R\sin\theta}{1+R^2+2R\cos\theta} = \mathfrak{I}\left(\frac{Re^{-i\theta}}{1+Re^{-i\theta}}\right) = \sum_1^\infty (-1)^n R^n \sin n\theta,$$

where $\mathfrak{I}$ denotes the imaginary part. Using this in (7.8) we obtain

$$\begin{aligned}\varphi(1, \theta) = & -\frac{a_1}{2}\left\{\cot \tfrac{1}{2}\theta + 4\sum_1^\infty \frac{R^{2n}}{1-R^{2n}} \sin n\theta\right\} \\ & -\frac{a_2}{2R}\left\{4\sum_1^\infty \frac{(-1)^n R^n}{1-R^{2n}} \sin n\theta\right\} + b. \end{aligned} \tag{7.9}$$

Finally, the terms in the brackets in (7.9) may be expressed by means of ϑ-functions. Making use of the results of W, p. 489, ex. 12, we get

$$\varphi(1, \theta) = -\frac{a_1}{2}\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R) - \frac{a_2}{2R}\frac{\vartheta_3'}{\vartheta_3}(\tfrac{1}{2}\theta, R) + b, \tag{7.10}$$

as stated in (3.5). It is worth pointing out that the parameter of the ϑ-functions met here is R, whereas in equation (2.2) the parameter is $R^{1/2}$.

8. Some definite integrals. (See §4.) The integrals in §4 are combinations of the following eight integrals:

$$I_m = \int_0^\pi \frac{\vartheta_m'}{\vartheta_m}(\tfrac{1}{2}\theta, R)\frac{dz}{d\theta}\, d\theta, \tag{8.11}$$

$$J_m = \int_0^\pi \frac{\vartheta_m'}{\vartheta_m}(\tfrac{1}{2}\theta, R)\, z\frac{dz}{d\theta}\, d\theta, \qquad (m = 1, 2, 3, 4), \tag{8.12}$$

where z and θ are connected by (2.2). For their evaluation we can make use of the following formulae:

$$\frac{d}{d\theta}\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R) = \frac{Az}{z-1} + B, \tag{8.21}$$

$$\frac{d}{d\theta}\frac{\vartheta_2'}{\vartheta_2}(\tfrac{1}{2}\theta, R) = \frac{A}{1-k'z} + B, \tag{8.22}$$

$$\frac{d}{d\theta}\frac{\vartheta_3'}{\vartheta_3}(\tfrac{1}{2}\theta, R) = \frac{A}{1+k'z} + B, \tag{8.23}$$

$$\frac{d}{d\theta}\frac{\vartheta_4'}{\vartheta_4}(\tfrac{1}{2}\theta, R) = \frac{Az}{z+1} + B, \tag{8.24}$$

where

$$z = \frac{1}{\mathrm{dn}(\mathrm{K}\theta/\pi, k)}, \tag{8.25}$$

$$A = -\mathrm{K}^2k^2/\pi^2, \qquad B = \mathrm{K}(\mathrm{K}-\mathrm{E})/\pi^2. \tag{8.26}$$

We shall prove only (8.21), the others being established in exactly the same way.

From the quasi-periodic properties of the ϑ-functions (W, p. 465, ex. 4) we see that the left member of (8.21) is a doubly periodic function with periods 2π, $2\pi\tau$, where $\tau = 2i\mathrm{K}'/\mathrm{K}$. We may restrict ourselves to a single period rectangle, say the one having the origin as the southwest vertex. Here the left member of (8.21) is regular except for a pole with principal part $-2/\theta^2$ (W, p. 466, p. 489, ex. 12, and the Laurent series for the cotangent). But, using (8.25) and W, p. 504.

$$\begin{aligned} z &= \left(1 - \frac{k^2}{2}\frac{\mathrm{K}^2\theta^2}{\pi^2} + \cdots\right)^{-1} = 1 + \frac{k^2}{2\pi^2}\mathrm{K}^2\theta^2 + \cdots, \\ \frac{Az}{z-1} &= \frac{2A\pi^2}{k^2\mathrm{K}^2\theta^2}(1 + O(\theta^2)) = -\frac{2}{\theta^2} + \text{const.} + \cdots. \end{aligned} \tag{8.3}$$

Furthermore, $Az/(z-1)$ is regular at points in the period rectangle other than $\theta = 0$, for z is an elliptic function of order 2 and (8.3) shows that it takes the value 1 twice at $\theta = 0$.

We see, therefore, that the difference

$$\frac{d}{d\theta}\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R) - \frac{Az}{z-1}$$

is a doubly periodic function without singularities; thus it is a constant, B, which we evaluate at $\theta = \pi$. Using W, p. 489, ex. 12, we have

$$\frac{d}{d\theta}\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R) = -\tfrac{1}{2}\csc^2\tfrac{1}{2}\theta + 4\sum_1^\infty \frac{nR^{2n}\cos n\theta}{1-R^{2n}};$$

and at $\theta = \pi$,

$$\begin{aligned} \frac{d}{d\theta}\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\pi, R) &= -\tfrac{1}{2} + 4\sum_1^\infty \frac{(-1)^n nR^{2n}}{1-R^{2n}} \\ &= -\tfrac{1}{2} + 2\left\{\sum_1^\infty \frac{(-1)^n nq^{2n}}{1-q^{2n}} - \sum_1^\infty \frac{(-1)^n nq^{2n}}{1+q^{2n}}\right\}, \quad q = R^{1/2}. \end{aligned}$$

The sums are evaluated by reference to W, p. 535, ex. 57, and p. 512, second formula differentiated. We obtain

$$\frac{d}{d\theta}\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\pi, R) = -\frac{\mathrm{K}^2k'}{\pi^2} - \frac{\mathrm{KE}}{\pi^2}. \tag{8.4}$$

Also, by (8.25), $z = k'^{-1}$ when $\theta = \pi$. Hence,

$$B = \frac{\mathrm{K}}{\pi^2}\left[-\mathrm{K}k' - \mathrm{E} + \mathrm{K}k^2k'^{-1}(k'^{-1} - 1)^{-1}\right] = \frac{\mathrm{K}}{\pi^2}(\mathrm{K} - \mathrm{E}), \qquad (k^2 + k'^2 = 1),$$

as in (8.26).

As before, we shall evaluate only one of the integrals, say I_1 in (8.11). This is accomplished by an integration by parts.

$$\begin{aligned} I_1 &= \int_0^\pi \frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R)\frac{dz}{d\theta}\,d\theta \\ &= \frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R)z(\theta)\Big|_0^\pi - \int_0^\pi \frac{d}{d\theta}\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R)z(\theta)d\theta \\ &= -\left[\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R)\right]_{\theta=0} - \int_1^{1/k'} \frac{d}{d\theta}\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R)\,z\,\frac{d\theta}{dz}\,dz. \end{aligned} \tag{8.5}$$

In the extreme right member of (8.5) both the integrated part and the integral are infinite, but their difference, considered as the limit

$$\lim_{\epsilon\to 0}\left(-\left[\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R)\right]_{\theta=\epsilon} - \int_{z(\epsilon)}^{1/k'} \frac{d}{d\theta}\frac{\vartheta_1'}{\vartheta_1}(\tfrac{1}{2}\theta, R)z\frac{d\theta}{dz}\,dz\right),$$

is finite.

The integration can now be carried out without difficulty. In the integral in the right member of (8.5) use is made of (8.21), while $d\theta/dz$ is obtained from (8.25), the result being then expressed as a function of z. It is necessary to show that the singular contributions from the two terms of the right member of (8.5) cancel. This is done conveniently by employing their Laurent expansions. The integrals which present themselves are at worst elliptic integrals of the first two kinds, and can be found in J, pp. 52–56. In this way we obtain the results of §5.

PERIODIC PROPERTIES OF THE SEMI-PERMANENT ATMOSPHERIC PRESSURE SYSTEMS*

BY

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The outstanding features of the general circulation of the atmosphere are the belts of westerly winds and, on the equatorial side of these, the system of semi-permanent sub-tropical high pressure areas. In a previous paper[1] the author has discussed the problem of the formation of such high pressure systems. In particular, it was shown that these systems probably represent dynamically stable concentrations of vorticity similar to the Kármán "vortex street" which is formed behind any two-dimensional bluff body over a wide range of values of the Reynolds number. It now appears that a further examination of the periods of the characteristic oscillations of such systems is of considerable interest. It is seen that the period of these oscillations is of the order of magnitude of years. This indicates that oscillations of this type may be of importance in the calculation of the long period displacements of the Pacific or Azores high pressure systems.

It is believed that this is the first time that atmospheric motions have been discussed which have a period of the order of magnitude of, but different from, a year. Since the weather shows large variations from one year to another, it is apparent that such motions must exist; and, since the non-seasonal variation of the only external parameter, the solar energy input, is very small, these long period motions must be explainable in terms of the free oscillations of the earth's atmosphere.

It seems that the horizontal field of motion is of primary importance in determining the motion of these large scale systems; so it is assumed that the atmosphere can be treated as a single layer of fluid of constant density with the vertical velocities being of small importance so that the pressure can be determined from the hydrostatic equation. It is also assumed that the apparent acceleration is negligible when compared to the Coriolis acceleration. In addition the effects of friction and of the variation of the Coriolis parameter with latitude are neglected. This latter factor means that the fluid motions considered are those taking place on a rotating disc rather than on a rotating sphere.

The notation used in the discussion is as follows:

x, y = Cartesian coordinates on a rotating disc, u = velocity in x direction, v = velocity in y direction, ω = angular velocity of the disc, h = depth of the fluid, g = acceleration due to gravity.

If the motion could have been started from rest with a uniform depth h_0, the principle of conservation of the absolute vorticity states that

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{2\omega}{h_0}(h - h_0). \tag{1}$$

* Received August 4, 1943.

[1] Stewart, H. J., Proc. Nat. Acad. of Sci., 26, 604 (1940).

If the velocity components are eliminated from this equation by means of the geostrophic wind equations,

$$-2\omega v = -g\frac{\partial h}{\partial x}, \qquad 2\omega u = -g\frac{\partial h}{\partial y}, \tag{2}$$

an expression determining the depth of the atmosphere (i.e. the sea-level pressure) is obtained. This is

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} - \frac{4\omega^2}{gh_0}(h - h_0) = 0. \tag{3}$$

This equation can be further simplified by the introduction of dimensionless variables, $X = 2\omega x/\sqrt{gh_0}$, $Y = 2\omega y/\sqrt{gh_0}$ and $\eta = (h - h_0)/h_0$. With these new variables, Eq. (3) becomes

$$\frac{\partial^2 \eta}{\partial X^2} + \frac{\partial^2 \eta}{\partial Y^2} - \eta = 0. \tag{4}$$

In terms of the dimensionless depth and dimensionless velocities defined by $U = u/\sqrt{gh_0}$ and $V = v/\sqrt{gh_0}$, the geostrophic wind equation can be rewritten as

$$\frac{\partial \eta}{\partial X} = V, \qquad \frac{\partial \eta}{\partial Y} = -U. \tag{5}$$

The only solution of Eq. (4) which vanishes at infinity and which represents flow in circles about the origin and thus corresponds to a simple vortex is

$$\eta = \alpha K_0(r) \tag{6}$$

where α is an arbitrary constant, $r = \sqrt{X^2 + Y^2}$ and $K_0(r)$ is a modified Bessel function[2] of the second kind. If α is positive the motion is anticyclonic; if α is negative the motion is cyclonic. All of the motions considered in the present investigation are built up through superposition of vortices of this type. From Eq. (5), this vortex has a dimensionless tangential velocity, u_θ, given by

$$u_\theta = -\alpha K_1(r) = \frac{\partial \eta}{\partial r}\cdot \tag{7}$$

The geostrophic wind equations used in the above development can be shown to be valid unless $r \ll 1$.

Based on a homogeneous atmosphere having a mean sea-level pressure and density of 1.013×10^6 dynes/cm² and 1.22×10^{-3} gm/cm³ respectively, the same as the standard atmosphere, the characteristic velocities and distances used above to produce dimensionless variables are $\sqrt{gh_0} = 2.87\times 10^4$ cm/sec and $\sqrt{gh_0}/2\omega = 1.97\times 10^8$ cm. At a distance of 2000 km. from the center of the Pacific or Azores high pressure systems, the characteristic velocity is of the order of 10 meters/second. Since $K_1(1) = 0.602$, this indicates that these anticyclones have a strength such that α is approximately 0.06.

If the interaction between the northern and southern hemispheres is neglected, the ring of subtropical anticyclones can be roughly represented by N equal anticyclones of the type given by Eq. (5) which are placed on a ring of radius a and spaced

[2] Grey, Mathews and MacRobert, *Bessel Functions*, Macmillan and Co., London, 1931.

at equal angles $\tau = 2\pi/N$ as shown in Fig. 1. In the northern hemisphere, the Pacific and Azores high pressure regions are well defined. They are about 120° of longitude apart. There is some evidence of a third such system over India and equidistant from the other two; however this evidence is far from conclusive due to the low level interference from the monsoon. In the southern hemisphere there are also three such systems. The best model is thus obtained with $N=3$.

The surface deflection for such a system in its equilibrium state is

$$\eta = \alpha \sum_{n=1}^{N} K_0[a^2 + r^2 - 2ar \cos(\theta - n\tau)]^{1/2}. \tag{8}$$

From Eq. 5 the dimensionless velocities in the radial and tangential directions, u_r and u_θ respectively, are given by

$$u_r = -\frac{1}{r}\frac{\partial \eta}{\partial \theta}, \qquad u_\theta = \frac{\partial \eta}{\partial r}. \tag{9}$$

The velocity of any vortex is the velocity at that point due to all of the remaining vortices. From the second of the expressions of Eq. 9, the system shown in Fig. 1 is seen to have a dimensionless angular velocity Ω given by

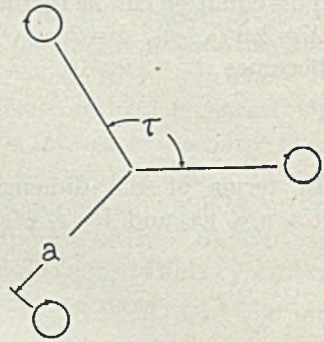

FIG. 1. Ring system of anticyclones.

$$\Omega = -\frac{\alpha}{a}\sum_{n=1}^{N-1} K_1[2a \sin \tfrac{1}{2}n\tau] \sin \tfrac{1}{2}n\tau. \tag{10}$$

In Table 1 are given the values of Ω and of $a\Omega\sqrt{gh_0}$, the linear velocity of the vortices, for $\alpha = 0.06$, $N=3$, and $a=3.0$ and 3.5. The values of $a=3.0$ and 3.5 correspond to the ring of subtropical anticyclones being placed at latitude 37° and 29° respectively. From this result it is seen that such a vortex system would have a slow precession to the west. In the atmosphere, there is also a region of distributed cyclonic vorticity to the north of the westerly winds. It is easily seen that this cyclonic vorticity tends to produce an eastward displacement of the subtropical highs. It appears that these two displacements cancel one another so that the systems are practically stationary, a condition which is undoubtedly also imposed by the thermodynamic and topographical factors. Since the westward drift shown in Table 1 is very small, no attempt will be made to correct the vortex system shown in Fig. 1 in order to take into account the polar cyclonic vorticity.

TABLE 1.

ANGULAR VELOCITIES OF VORTEX SYSTEMS, $\alpha = 0.06$, $N=3$

a	Ω	$a\Omega\sqrt{gh_0}$
3.0	−.000112	−9.6 cm/sec
3.5	−.000037	−3.7 cm/sec

If the nth vortex is displaced by a distance Δr_n in the radial direction and by an angle $\Delta\theta_n$ in the tangential direction, the surface deflection in the displaced condition is

$$\eta = \alpha \sum_{n=1}^{N} K_0[(a + \Delta r_n)^2 + r^2 - 2(a + \Delta r_n) r \cos(\theta - n\tau - \Delta\theta_n)]^{1/2}. \tag{11}$$

The velocities of the vortices may be calculated as before by Eq. (9). If the displacements are small, the changes in velocity of the Nth vortex from the equilibrium value indicated by Eq. (10) may be written as

$$\Delta u_r = \frac{d}{dt}(\Delta r_N) = \frac{\alpha}{2}\sum_{n=1}^{N-1}\Delta r_n K_0(R_n)\sin(n\tau)$$
$$+\alpha\sum_{n=1}^{N-1}a(\Delta\theta_n - \Delta\theta_N)\left\{\frac{K_1(R_n)}{R_n} + K_0(R_n)\cos^2\tfrac{1}{2}n\tau\right\}$$
$$\Delta u_\theta = a\frac{d}{dt}(\Delta\theta_N) + \Omega\Delta r_N = \alpha\Delta r_N\sum_{n=1}^{N-1}\left\{K_0(R_n)\sin^2\tfrac{1}{2}n\tau - \frac{K_1(R_n)}{R_n}\cos(n\tau)\right\} \tag{12}$$
$$+\alpha\sum_{n=1}^{N-1}\left[\Delta r_n\left\{K_0(R_n)\sin^2\tfrac{1}{2}n\tau + \frac{K_1(R_n)}{R_n}\right\} + \tfrac{1}{2}a\Delta\vartheta_n K_0(R_n)\sin(n\tau)\right],$$

where

$$R_n = 2a\sin\tfrac{1}{2}n\tau.$$

It should be noted that the t in Eq. (12) is a dimensionless time. If the actual time is t^*, then

$$t = 2\omega t^*. \tag{13}$$

Expressions similar to Eq. (12) for the velocities of the other vortices could be written from symmetry. These would form a set of simultaneous differential equations for the displacements.

If the N equations in each of the two sets indicated in Eq. (12) are added, it is seen that

$$\frac{d}{dt}\left\{\sum_{n=1}^{N}\Delta r_n\right\} = 0$$
$$\frac{d}{dt}\left\{\sum_{n=1}^{N}\Delta\theta_n\right\} = \alpha\left\{\sum_{n=1}^{N}\Delta r_n\right\}\left\{\sum_{n=1}^{N-1}\left[2K_0(R_n)\sin^2\tfrac{1}{2}n\tau + \frac{1}{a}K_1(R_n)\sin\tfrac{1}{2}n\tau\right]\right\}. \tag{14}$$

From this it is seen that if a mean value of a is chosen so that $\sum_{n=1}^{N}\Delta r_n = 0$ initially, then from Eq. (14) both $\sum_{n=1}^{N}\Delta r_n$ and $\sum_{n=1}^{N}\Delta\theta_n$ will remain constant. These results correspond to similar equations for two dimensional line vortices which state that the impulse of a system having no external forces remains constant.[3]

The disturbed motion of the vortex system as described in Eq. (12) can best be discussed by considering the normal modes of oscillation. From the symmetry conditions, the displacements in each normal mode must be of the form $\Delta r_n = \Delta r_N e^{in\varphi}$ and $\Delta\theta_n = \Delta\theta_N e^{in\varphi}$ where φ characterizes the normal mode and is a member of the series $2\pi/N$, $4\pi/N$, $\cdots$, $2\pi(N-1)/N$, 2π. With this notation, Eq. (12) can be written as

$$\frac{1}{\alpha}\frac{d}{dt}(\Delta r_N) = A\Delta r_N - Ba\Delta\theta_N, \qquad \frac{a}{\alpha}\frac{d}{dt}(\Delta\theta_N) = C\Delta r_N + Aa\Delta\theta_N, \tag{15}$$

where

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[3] Lamb, H., *Hydrodynamics*, 6th edition, Cambridge University Press, London, 1932, 220.

$$A = \frac{1}{2}\sum_{n=1}^{N-1} K_0(R_n)e^{in\varphi}\sin(n\tau)$$

$$B = \sum_{n=1}^{N-1}(1 - e^{in\varphi})\left\{\frac{K_1(R_n)}{R_n} + K_0(R_n)\cos^2 \tfrac{1}{2}n\tau\right\} \tag{16}$$

$$C = \sum_{n=1}^{N-1}\left\{K_0(R_n)(1 + e^{in\varphi})\sin^2 \tfrac{1}{2}n\tau + \frac{K_1(R_n)}{R_n}(1 - 2\cos(n\tau) + e^{in\varphi})\right\}.$$

If the amplitudes Δr_N and $\Delta\theta_N$ are assumed to vary like e^{ipt} and p is the dimensionless normal frequency, then from Eq. (15),

$$p = \alpha\{-iA \pm \sqrt{BC}\}. \tag{17}$$

From Eq. (16), it can be seen that for $\varphi = 2\pi$, $A = B = 0$ and there are thus two zero normal frequencies. These two zero frequencies are those shown in Eq. (14). It can also be seen that for the specified values of φ, A is always a purely imaginary quantity, B is always real and not less than zero. From Eq. (17), the condition that the frequencies be real is that C be real and non-negative. Complex frequencies, of course, characterize systems in which the amplitudes increase with time and are thus unstable. Now C is always real and is always positive for $N < 7$. If $N = 7$, C is always positive if $a > 71$. For $N > 7$, C is negative for one or more of the given values of φ. A value of $a > 71$ corresponds either to disturbances of such great wave lengths or to motions of such a shallow layer of air in the earth's atmosphere that it probably is of no significance. The vortex system is thus stable if N is less than or equal to six.

The frequencies and modes of oscillation will be discussed in some detail for the cases where $N = 3$ and $a = 3.0$ and 3.5. From Eq. (13) it may be seen that the period of an oscillation is given by

$$T = \left|\frac{1}{2p}\right| \text{ sidereal days.} \tag{18}$$

The normal mode of oscillation, from Eq. (15), is given by

$$\frac{a\Delta\theta_n}{\Delta r_n} = \frac{a\Delta\theta_N}{\Delta r_N} = \frac{A - ip/\alpha}{B}. \tag{19}$$

The results for $a = 3.0$ and 3.5 are given in Tables 2 and 3, respectively. The two normal modes thus show a short period (2000 days to 5000 days) and a long period oscillation (70,000 days to 250,000 days). The path of the vortex is in each case an ellipse. For the short period oscillation the vortices are east of their mean position when traveling south and west when traveling north. For the long period oscillation the sense of the rotation is reversed.

Conclusion. The present calculations cannot be considered as a quantitative theory of the oscillations of the semi-permanent high pressure systems; they must be considered rather as an existence proof. Since the essential features of the model, vorticity concentrations at distances of roughly 10,000 km., are also found in the atmosphere, motions of this type must exist in the atmosphere. It might be expected that the effects of coupling between the systems of the Northern and Southern Hemispheres and of any cyclonic vorticity concentrations on the polar sides of the westerly winds

TABLE 2.

NORMAL MODES OF OSCILLATION FOR $N=3$, $a=3.0$, $\alpha=0.06$.

φ	$2\pi/3$		$4\pi/3$	
$-iA$	0.002248		-0.002248	
B	0.00414		0.00414	
C	0.001182		0.001182	
p	0.000268	0.000002	-0.000268	-0.000002
T-days	1,865	2.5×10^5	1,865	2.5×10^5
$\frac{a\Delta\theta_N}{\Delta r_N}$	$-0.53i$	$+0.53i$	$+0.53i$	$-0.53i$

TABLE 3.

NORMAL MODES OF OSCILLATION FOR $N=3$, $a=3.5$, $\alpha=0.06$.

φ	$2\pi/3$		$4\pi/3$	
$-iA$	0.000859		-0.000859	
B	0.001467		0.001467	
C	0.000375		0.000375	
p	0.000096	0.000007	-0.000096	-0.000007
T-days	5,200	7.1×10^4	5,200	7.1×10^4
$\frac{a\Delta\theta_N}{\Delta r_N}$	$-0.51i$	$0.51i$	$0.51i$	$-0.51i$

would be to decrease the period of the shortest oscillation and to introduce additional natural frequencies. No attempt has as yet been made to estimate the magnitude of these effects or of the errors involved in using velocity distributions corresponding to vortices on a rotating disc rather than to vortices on a rotating sphere and in neglecting the seasonal variations in the strength of the semi-permanent high pressure systems. It is suggested that the present calculations may prove useful as a guide for the statistical analysis of empirically obtained data.

—NOTES—

ON HERZBERGER'S DIRECT METHOD IN GEOMETRICAL OPTICS*

By J. L. SYNGE (*Ohio State University*)

1. Introduction. In recent papers M. Herzberger[1,2] has developed a "direct method" for analytical ray-tracing through an instrument of revolution. At the end of the first paper he refers to Hamilton's method, which he says "leads to an elimination problem, hitherto unsolved." Nevertheless the question arises: What is the connection between Herzberger's approach and that of Hamilton? This question is best answered by attacking Herzberger's problem by the method of Hamilton. As we shall see, this is quite feasible. Indeed, if we combine Herzberger's "direct method" with Hamilton's character function we obtain a very powerful technique.

Section 2 contains the formulation of the problem of determining the Herzberger transformation when Hamilton's angle-characteristic is known for the instrument in question. Herzberger's identity ($AD-BC=1$) is obtained immediately.

In Section 3 the case of a single surface (refracting or reflecting) is considered. It is found that the coefficients are connected by a new relation.

In Section 4 I show how the problem of the sphere may be treated, Herzberger's geometrical approach being replaced by a more systematic analytical method.

2. The Herzberger transformation. To facilitate comparison with Herzberger's work, I shall use his notation. The following table shows the correspondence between the notations of Herzberger and Hamilton:

	Herzberger	Hamilton
Coordinates of point on incident ray	x, y, z	x', y', z'
Components of incident ray	ξ, η, ζ	$\sigma', \tau', \upsilon'$
Coordinates of point on final ray	x', y', z'	x, y, z
Component of final ray	ξ', η', ζ'	σ, τ, υ

According to the method of Hamilton there exists an angle-characteristic T, a function of ξ, η, ξ', η', such that the equations of the incident and final rays are[3]

$$\begin{aligned} x - z\xi/\zeta &= T_\xi, & x' - z'\xi'/\zeta' &= -T_{\xi'}, \\ y - z\eta/\zeta &= T_\eta, & y' - z'\eta'/\zeta' &= -T_{\eta'}. \end{aligned} \tag{2.1}$$

The subscripts denote partial derivatives.

Now suppose that the instrument is of revolution and that the axes Oz, $O'z'$ lie along its axis. Then T is a function of the quantities

* Received May 21, 1943.

[1] M. Herzberger, Trans. Amer. Math. Soc. 53, 218–229 (1943).

[2] M. Herzberger, Quarterly of Applied Mathematics, 1, 69–77 (1943).

[3] J. L. Synge, *Geometrical optics*, Cambridge, 1937, p. 31.

$$u_3 = \tfrac{1}{2}(\xi^2 + \eta^2), \qquad u_4 = \xi\xi' + \eta\eta', \qquad u_5 = \tfrac{1}{2}(\xi'^2 + \eta'^2). \tag{2.2}$$

Let us write $\partial T/\partial u_3 = T_3$, etc. Then, by (2.1), the intersections of the rays with the planes $z=0$, $z'=0$, satisfy

$$\begin{aligned} x &= T_3\xi + T_4\xi', & x' &= -T_4\xi - T_5\xi', \\ y &= T_3\eta + T_4\eta', & y' &= -T_4\eta - T_5\eta'. \end{aligned} \tag{2.3}$$

These equations involve the eight quantities

$$x', y', \xi', \eta'; \qquad x, y, \xi, \eta.$$

The basis of the Herzberger method is to express the first set in terms of the second set. To do this, we introduce

$$u_1 = \tfrac{1}{2}(x^2 + y^2), \qquad u_2 = x\xi + y\eta. \tag{2.4}$$

Let us multiply the x, y equations in (2.3) by ξ, η, respectively, and add; this gives

$$u_2 = 2T_3u_3 + T_4u_4. \tag{2.5}$$

Rearranging the x, y equations in (2.3), squaring and adding, we get

$$T_4^2u_5 = u_1 - T_3u_2 + T_3^2u_3. \tag{2.6}$$

Supposing T known as a function of u_3, u_4, u_5, we have in (2.5), (2.6) two equations to determine u_4, u_5 in terms of u_1, u_2, u_3; suppose the solutions are

$$u_4 = f(u_1, u_2, u_3), \qquad u_5 = g(u_1, u_2, u_3). \tag{2.7}$$

Making this substitution, we may express T_3, T_4, T_5 as functions of u_1, u_2, u_3.

Now let us rearrange (2.3) into the Herzberger form:

$$\begin{aligned} x' &= Ax + B\xi, & \xi' &= Cx + D\xi, \\ y' &= Ay + B\eta, & \eta' &= Cy + D\eta. \end{aligned} \tag{2.8}$$

The coefficients are as follows:

$$A = -T_5T_4^{-1}, \qquad B = T_3T_5T_4^{-1} - T_4, \qquad C = T_4^{-1}, \qquad D = -T_3T_4^{-1}. \tag{2.9}$$

We immediately deduce Herzberger's identity

$$AD - BC = 1. \tag{2.10}$$

To sum up: *Given the angle-characteristic $T(u_3, u_4, u_5)$ of an instrument of revolution, we obtain the coefficients A, B, C, D of the Herzberger transformation in two steps*:

(i) *We solve* (2.5), (2.6) *for u_4, u_5 in terms of u_1, u_2, u_3.*

(ii) *We substitute these values in* (2.9), *and so obtain A, B, C, D in terms of u_1, u_2, u_3.*

For future reference, let us solve (2.9) for T_3, T_4, T_5:

$$T_3 = -DC^{-1}, \qquad T_4 = C^{-1}, \qquad T_5 = -AC^{-1}. \tag{2.11}$$

3. An identity satisfied by the Herzberger coefficients for a single surface. Consider a surface of revolution

$$z = f(r), \qquad r^2 = x^2 + y^2. \tag{3.1}$$

For refraction or reflection at this surface, the angle-characteristic is[4] (if we take the origins O, O' coincident)

$$T = (\xi - \xi')x + (\eta - \eta')y + (\zeta - \zeta')z, \tag{3.2}$$

from which x, y, z are to be eliminated by the relations

$$\frac{\xi - \xi'}{\zeta - \zeta'} = -\frac{\partial z}{\partial x} = -f'(r)\frac{x}{r}, \qquad \frac{\eta - \eta'}{\zeta - \zeta'} = -f'(r)\frac{y}{r}. \tag{3.3}$$

It is clear that T will be a function of the two quantities

$$\phi = \tfrac{1}{2}[(\xi - \xi')^2 + (\eta - \eta')^2], \qquad \psi = \zeta - \zeta',$$

or, in the notation of (2.2),

$$\phi = u_3 - u_4 + u_5, \qquad \psi = \theta(n^2 - 2u_3)^{1/2} - \theta'(n'^2 - 2u_5)^{1/2}. \tag{3.4}$$

Here n, n' are the refractive indices of the initial and final media, and θ, θ' are ± 1; for refraction we have $\theta\theta' = 1$, and for reflection $\theta\theta' = -1$. If we take refraction with the rays proceeding in the positive sense, we have

$$\theta = \theta' = 1. \tag{3.5}$$

If we take reflection with the incident rays in the positive sense, we have

$$\theta = 1, \qquad \theta' = -1, \qquad n = n'. \tag{3.6}$$

By (3.4) we have

$$\begin{aligned} T_3 &= T_\phi - T_\psi \theta(n^2 - 2u_3)^{-1/2}, \\ T_4 &= -T_\phi, \\ T_5 &= T_\phi + T_\psi \theta'(n'^2 - 2u_5)^{-1/2}. \end{aligned} \tag{3.7}$$

Hence

$$\frac{T_3 + T_4}{T_4 + T_5} = -k\frac{(n'^2 - 2u_5)^{1/2}}{(n^2 - 2u_3)^{1/2}}, \tag{3.8}$$

where

$$\begin{aligned} k &= 1 \text{ for refraction}, \\ k &= -1 \text{ for reflection}. \end{aligned} \tag{3.9}$$

Let us substitute from (2.11) in (3.8); this gives

$$\frac{D - 1}{A - 1} = -k\frac{(n'^2 - 2u_5)^{1/2}}{(n^2 - 2u_3)^{1/2}}, \tag{3.10}$$

and so

$$u_5 = \tfrac{1}{2}n'^2 - \tfrac{1}{2}(n^2 - 2u_3)\left(\frac{D - 1}{A - 1}\right)^2. \tag{3.11}$$

When we substitute this value in (2.6), and at the same time substitute for T_3, T_4 from (2.11), we get

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[4] Synge, op. cit., 33.

$$n'^2 - (n^2 - 2u_3)\left(\frac{D-1}{A-1}\right)^2 = 2(C^2u_1 + CDu_2 + D^2u_3). \tag{3.12}$$

To sum up: *For refraction or reflection at a surface of revolution, the coefficients* A, C, D *are connected by the identity* (3.12).

If $B=0$, then $A=D^{-1}$, and (3.12) simplifies to

$$n'^2 - n^2D^2 = 2C(Cu_1 + Du_2). \tag{3.13}$$

As an alternative procedure we may use the fact that T is of the form

$$T = (\zeta - \zeta')F(\chi), \tag{3.14}$$

where

$$\chi = [(\xi - \xi')^2 + (\eta - \eta')^2]/(\zeta - \zeta')^2. \tag{3.15}$$

This is evident from (3.2) and (3.3); the form of the function F depends on the form of the surface. On differentiating (3.14) we obtain three equations analogous to (3.7), but containing F and its derivative on the right hand sides. If we eliminate these quantities we obtain (3.8) and hence the identity (3.12).

4. The Herzberger transformation for a sphere. Let us take the origins O, O' at the center of a sphere of radius $|r|$. The angle characteristic for refraction or reflection at the sphere is[5]

$$T = \pm|r|\,[(\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2]^{1/2}. \tag{4.1}$$

If we suppose the rays incident in the positive sense, all ambiguities of sign are removed by writing

$$T = r(kn' - n)\rho^{1/2}, \tag{4.2}$$

where

$$\rho = 1 + \frac{2}{(kn' - n)^2}\,[knn' - u_4 - k(n^2 - 2u_3)^{1/2}(n'^2 - 2u_5)^{1/2}]. \tag{4.3}$$

Here r is positive if the rays are incident on the convex side, and negative if they are incident on the concave side; $k=1$ for refraction and $k=-1$ for reflection. All roots are positive.

We have then

$$\begin{aligned} T_3 &= kr(kn' - n)^{-1}\rho^{-1/2}\,\frac{(n'^2 - 2u_5)^{1/2}}{(n^2 - 2u_3)^{1/2}}, \\ T_4 &= -\,r(kn' - n)^{-1}\rho^{-1/2}, \\ T_5 &= kr(kn' - n)^{-1}\rho^{-1/2}\,\frac{(n^2 - 2u_3)^{1/2}}{(n'^2 - 2u_5)^{1/2}}. \end{aligned} \tag{4.4}$$

It is evident that

$$T_4^2 = T_3T_5, \tag{4.5}$$

and so, by (2.9) and (2.10),

$$B = 0, \qquad A = D^{-1}. \tag{4.6}$$

[5] Synge, op. cit., p. 36.

We now solve (4.4) for u_4, u_5 in terms of T_3, T_4, obtaining

$$u_4 = k(n^2 - 2u_3)\frac{T_3}{T_4} - \frac{1}{2}\frac{r^2}{T_4^2} + \frac{1}{2}(n'^2 + n^2),$$

$$u_5 = \frac{1}{2}n'^2 - \frac{1}{2}(n^2 - 2u_3)\frac{T_3^2}{T_4^2}. \tag{4.7}$$

Substitution of these values into (2.5), (2.6) gives

$$\begin{aligned} u_2 T_4 &= \tfrac{1}{2}(n^2 + n'^2)T_4^2 + n^2 T_3 T_4 - \tfrac{1}{2}r^2, \\ u_1 &= u_2 T_3 + \tfrac{1}{2}n'^2 T_4^2 - \tfrac{1}{2}n^2 T_3^2. \end{aligned} \tag{4.8}$$

These are two equations for T_3, T_4; they may be written

$$T_3 = n^{-2}[u_2 - \tfrac{1}{2}(n^2 + n'^2)T_4 + \tfrac{1}{2}r^2 T_4^{-1}], \tag{4.9}$$

$$T_4^4(n^2 - n'^2)^2 + 4T_4^2[p^2 - \tfrac{1}{2}r^2(n^2 + n'^2)] + r^4 = 0, \tag{4.10}$$

where (in Herzberger's notation)

$$p^2 = 2n^2 u_1 - u_2^2. \tag{4.11}$$

Solving (4.10) we get, after some simple reductions,

$$C = T_4^{-1} = r^{-1}[\theta_1(n^2 - p^2/r^2)^{1/2} + \theta_2(n'^2 - p^2/r^2)^{1/2}], \tag{4.12}$$

where θ_1 and θ_2 are each ± 1, for the moment undetermined. We remove the ambiguity of sign by considering the case $\xi = \eta = 0$, so that by (2.8) $\xi' = Cx$, $\eta' = Cy$. It is evident from elementary considerations that C has the same sign as $(n - kn')/r$. Therefore $\theta_1 = 1$, $\theta_2 = -k$, and so in general

$$C = r^{-1}[(n^2 - p^2/r^2)^{1/2} - k(n'^2 - p^2/r^2)^{1/2}]. \tag{4.13}$$

By (2.11) and (4.9) we have

$$D = n^{-2}[p^2/r^2 + k(n^2 - p^2/r^2)^{1/2}(n'^2 - p^2/r^2)^{1/2} - u_2 C]. \tag{4.14}$$

We verify that if $x = y = 0$, then $D = kn'/n$, as it must be by (2.8) from elementary considerations. It is easy to check that (3.13) is satisfied by (4.13), (4.14).

For the case of refraction ($k = 1$) the formula (4.13) agrees with Herzberger's equation (36)[1], except for a reversal of sign.

ON THE FORCE AND MOMENT ACTING ON A BODY IN SHEAR FLOW*

By YUNG-HUAI KUO (*California Institute of Technology*)

Recently, H. S. Tsien solved the problem[1] of a Joukowsky airfoil in a steady, two-dimensional flow of constant vorticity distribution. It is interesting to note that the hydrodynamical forces can be expressed in a form similar to the well known Blasius' theorem, involving contour integration of the complex potential function. The following derivation of the formulae is believed to be simpler than that of Tsien.

1. Equations of motion. Let u and v be the velocity components parallel to the x- and y-axis, respectively. In the case of two-dimensional steady motion, the Eulerian dynamical equations are:

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x},\tag{1.1}$$

$$u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}+u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y},\tag{1.2}$$

where p is the pressure and ρ, the density of the fluid. The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.\tag{1.3}$$

For the type of shear flow considered by Tsien,[1] the vorticity is constant everywhere in the field and equal to $-k$. Thus

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=-k,\qquad k>0.\tag{1.4}$$

At the first sight, it seems that the problem might not be definite as one has four equations for three variables. By eliminating p between Eqs. (1.1) and (1.2), however, the result can be reduced to Eq. (1.3) by means of Eq. (1.4). This shows that any solution which satisfies Eqs. (1.3) and (1.4) is consistent with Eqs. (1.1) and (1.2).

To simplify the problem, the solution is written in the following form:

$$u=ky+u',\tag{1.5}$$

$$v=v'.\tag{1.6}$$

Then Eqs. (1.3) and (1.4) reduce to

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0,\tag{1.7}$$

$$\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}=0.\tag{1.8}$$

* Received June 21, 1943.

[1] H. S. Tsien, *Symmetrical Joukowski airfoils in shear flow*, Quarterly Appl. Math., 1, 129 (1943).

These equations are satisfied by

$$u' = \frac{\partial \psi}{\partial y}, \qquad v' = -\frac{\partial \psi}{\partial x}; \tag{1.9}$$

or

$$u' = \frac{\partial \varphi}{\partial x}, \qquad v' = \frac{\partial \varphi}{\partial y}; \tag{1.10}$$

where ψ and φ are the imaginary and real parts of the complex potential $F(z)$; namely,

$$\varphi + i\psi = F(z), \qquad z = x + iy; \tag{1.11}$$

and

$$u' - iv' = w'(z). \tag{1.12}$$

For a given problem the function $F(z)$ is so determined that the velocity component normal to the contour of the body is zero.

By virtue of Eqs. (1.4), (1.5), and (1.6), Eqs. (1.1) and (1.2) give

$$p = -\frac{\rho}{2}q'^2 - \rho k u' y + \rho k \psi, \tag{1.13}$$

where $q'^2 = u'^2 + v'^2$, and the constant of integration is absorbed in ψ.

2. **Force and moment.** If the motion is two-dimensional and steady, the components of the hydrodynamical force and moment[2] acting on the body are given by

$$X = -\oint p dy - \rho \oint u(udy - vdx), \tag{2.1}$$

$$Y = \oint p dx + \rho \oint v(vdx - udy), \tag{2.2}$$

$$M = \oint p(xdx + ydy) - \rho \oint (-v^2 x dx - u^2 y dy + uvydx + uvxdy), \tag{2.3}$$

where the contour integrals are taken along a closed curve containing the body. Using Eqs. (1.5), (1.6) and (1.13), the above equations can be written as:

$$X = -\frac{\rho}{2}\oint [(u'^2 - v'^2)dy - 2u'v'dx] - \rho k \oint [(\psi + u'y)dy - v'ydx], \tag{2.4}$$

$$Y = -\frac{\rho}{2}\oint [(u'^2 - v'^2)dx + 2u'v'dy] + \rho k \oint [(\psi - u'y)dx - v'ydy], \tag{2.5}$$

$$M = -\operatorname{Re}\left[\frac{\rho}{2}\oint zw'^2 dz\right]$$
$$+ \rho k \oint [(\psi - u'y)(xdx + ydy) - (v'yx - 2u'y^2)dy + v'y^2 dx]. \tag{2.6}$$

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[2] W. F. Durand, *Aerodynamic theory*, vol. 2, Springer, Berlin, 1935, pp. 31–33.

If only bodies with closed boundary are considered, no sources can exist within the field of flow. Then the stream function ψ is single-valued, and

$$\oint \psi dx = \oint x(v'dx - u'dy),$$

$$\oint \psi dy = \oint y(v'dx - u'dy).$$

From these relations, it is not difficult to deduce

$$X = -\frac{\rho}{2}\oint [(u'^2 - v'^2)dy - 2u'v'dx], \tag{2.7}$$

$$Y = -\frac{\rho}{2}\oint [(u'^2 - v'^2)dx + 2u'v'dy] + \rho k\oint [v'(xdx - ydy) - u'(ydx + xdy)], \tag{2.8}$$

$$M = -\operatorname{Re}\left[\frac{\rho}{2}\oint zw'^2dz\right] + \frac{\rho k}{2}\oint [-u'\{(x^2 - y^2)dy + 2xydx\} + v'\{(x^2 - y^2)dx - 2xydy\}]. \tag{2.9}$$

These at once suggest the following alternative expressions:

$$X - iY = \frac{i\rho}{2}\oint w'^2dz + i\operatorname{Im}\left[\rho k\oint w'zdz\right], \tag{2.10}$$

and

$$M = -\operatorname{Re}\left[\frac{\rho}{2}\oint z\left(w' - \frac{ikz}{2}\right)^2 dz\right]. \tag{2.11}$$

Eqs. (2.10) and (2.11) may be regarded as an extension of Blasius' theorem. They can be easily identified with the expressions given by Tsien.[1] The calculation of force and moment, however, can be simplified to a certain extent by using these new expressions.

The writer wishes to thank Dr. H. S. Tsien for the use of his paper before publication and for his helpful discussions.

A CHART FOR PLOTTING RELATIONS BETWEEN VARIABLES OVER THEIR ENTIRE REAL RANGE*

By L. H. DONNELL (*Illinois Institute of Technology*)

The following simple method of graphical respresentation, covering the entire real range, seems rather obvious and the writer has found it useful for many years; however he has never seen it described in the literature. It consists of an ordinary Car-

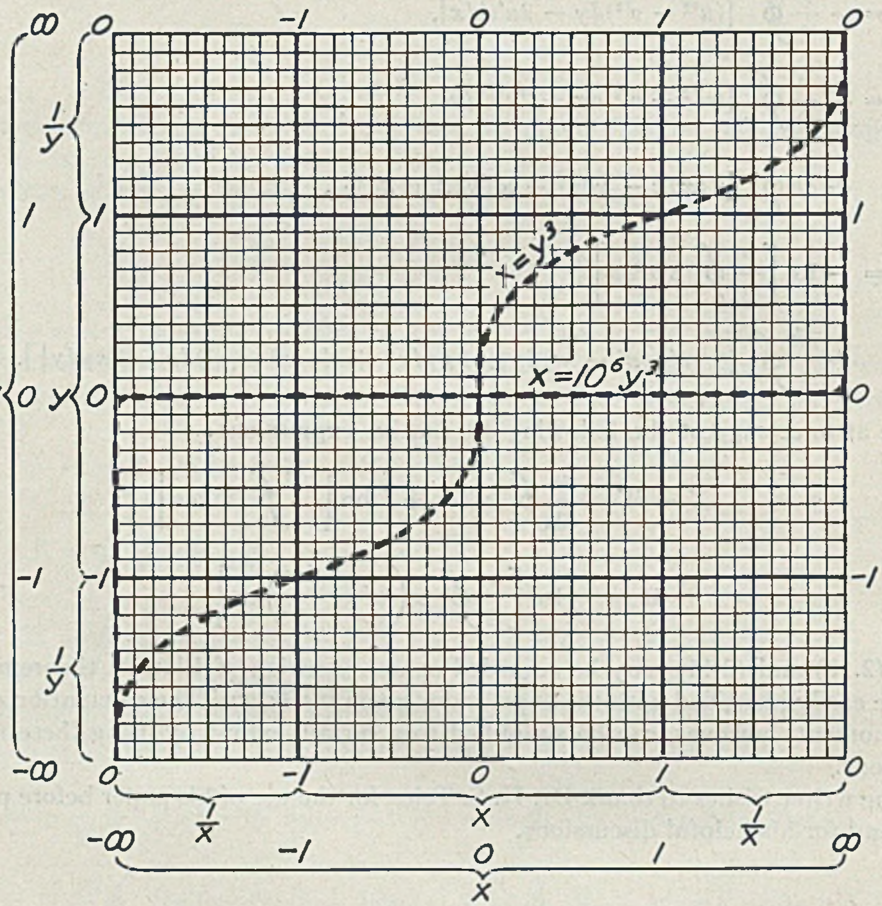

Fig. 1.

tesian plot over the range -1 to $+1$ of each variable, with adjoining Cartesian plots of the reciprocals of the variables from 0 to -1 and from $+1$ to 0, arranged as shown in Fig. 1. This is evidently equivalent to a plot over the range $-\infty$ to $+\infty$ for each variable. It is also evident that curves and their slopes will be continuous over the dividing lines between the two kinds of plots if the functions represented and their first derivatives are continuous at these points.

* Received March 22, 1943.

To illustrate the method, the power relation $x=y^n$ has been plotted in Fig. 2 for values of x and y between $-\infty$ and $+\infty$, and for various values of n.

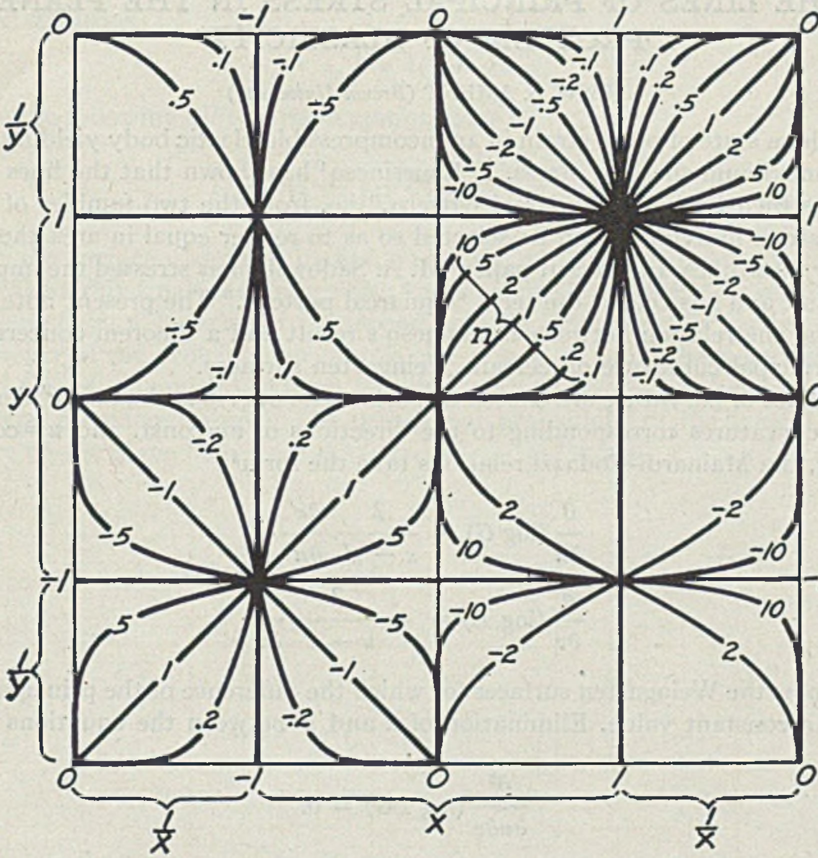

FIG. 2. Plot of $x=y^n$.

In many applications only a part of such a chart would be required. For instance only the upper right hand quarter would be needed to cover all positive real values of the variables.

In some cases a change of variable will make the resulting plot much more useful. For instance a plot of the relation $x=10^6y^3$, (see Fig. 1) follows the coordinate axes closely and would be nearly indistinguishable from many other functions, such as $x=10^{10}y^3$ or $x=10^{10}y^5$. On the other hand, a plot of $x=y'^3$, where $y'=100y$, gives a much more illuminating and characteristic picture of the function. Any change of variable which makes the resulting curve pass through or near such points as (1, 1), (1, -1), (-1, 1) or (-1, -1) would accomplish this purpose in most cases.

THE LINES OF PRINCIPAL STRESS IN THE PLANE PROBLEM OF PLASTICITY*

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Consider a state of plane strain in an incompressible plastic body yielding under a constant maximum shearing stress. J. Boussinesq[1] has shown that the lines of principal stress then form an "equiareal pattern," i.e. from the two families of lines of principal stress individuals can be selected so as to render equal in area the meshes formed by these lines. In a recent paper[2] M. A. Sadowsky has stressed the importance of this result and has coined the term "equiareal pattern." The present note aims at establishing the relation between Boussinesq's result and a theorem concerning the lines of principal curvature on certain Weingarten surfaces.

If the lines of curvature are chosen as parametric curves and κ and κ' denote the principal curvatures corresponding to the directions of $v=$const. and $u=$const. respectively, the Mainardi-Codazzi relations take the form[3]

$$\begin{aligned}\frac{\partial}{\partial u}(\log G) &= \frac{2}{\kappa-\kappa'}\frac{\partial\kappa'}{\partial u},\\ \frac{\partial}{\partial v}(\log E) &= -\frac{2}{\kappa-\kappa'}\frac{\partial\kappa}{\partial v}.\end{aligned}\tag{1}$$

Consider now the Weingarten surfaces for which the difference of the principal curvatures has a constant value. Elimination of κ and κ' between the equations (1) then leads to

$$\frac{\partial^2}{\partial u\partial v}(\log EG) = 0.$$

Hence $EG=f(u)g(v)$. A transformation of the type $\bar{u}=\bar{u}(u)$, $\bar{v}=\bar{v}(v)$ only relabels the parametric curves but does not affect their geometric properties. Define $\bar{u}$ and $\bar{v}$ by

$$\frac{d\bar{u}}{du} = \sqrt{f(u)}, \qquad \frac{d\bar{v}}{dv} = \sqrt{g(v)}.$$

For these new parameters $\overline{EG}=1$, i.e. the meshes formed by the parametric curves corresponding to two sets of equidistant values of $\bar{u}$ and $\bar{v}$ are equal in area. *The lines of principal curvature on a Weingarten surface with $\kappa-\kappa'=$const. therefore form an equiareal pattern.*

The relation between this theorem and Boussinesq's result is immediate. Introduce rectangular Cartesian coordinates O, x, y, z, the plane O, x, y having the orientation of the plane of strain. The normal stresses σ_x, σ_y and the shearing stress τ then can be derived from a stress function F according to

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[1] C. R. Ac. Sci. Paris, **74**, 242 (1872).

[2] Trans. Am. Soc. Mech. Eng. **63**, A-74 (1941).

[3] See for instance C. E. Weatherburn, *Differential geometry*, vol. II, Cambridge 1930, p. 52.

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau = -\frac{\partial^2 F}{\partial x \partial y}.$$

The yield condition

$$(\sigma_x - \sigma_y)^2 + 4\tau^2 = \text{const.}$$

furnishes the following differential equation for F:

$$\left(\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2}\right)^2 + 4\left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 = \text{const.} \tag{2}$$

Consider the surface S defined by $z(x, y) = \alpha F(x, y)$, where α is an arbitrary small constant rendering small the slope which the tangential planes of S have with respect to the plane O, x, y. If terms of the second order in $\partial F/\partial x$ and $\partial F/\partial y$ are neglected, the difference of the principal curvatures of S is seen to equal the square root of the left hand side of (2) multiplied by α. The surface S therefore is a Weingarten surface of the type considered above. The orthogonal projections of its lines of principal curvature on the plane O, x, y are the lines of principal stress.

CONSERVATION OF SCHOLARLY JOURNALS

The American Library Association created in 1941 the Committee on Aid to Libraries in War Areas, headed by John R. Russell, the Librarian of the University of Rochester. The Committee is faced with numerous serious problems and hopes that American scholars and scientists will be of considerable aid in the solution of one of these problems.

One of the most difficult tasks in library reconstruction after the first World War was that of completing foreign institutional sets of American scholarly, scientific, and technical periodicals. The attempt to avoid a duplication of that situation is now the concern of the Committee.

Many sets of journals will be broken by the financial inability of the institutions to renew subscriptions. As far as possible they will be completed from a stock of periodicals being purchased by the Committee. Many more will have been broken through mail difficulties and loss of shipments, while still other sets will have disappeared in the destruction of libraries. The size of the eventual demand is impossible to estimate, but requests, received by the Committee already give evidence that it will be enormous.

With an imminent paper shortage attempts are being made to collect old periodicals for pulp. Fearing this possible reduction in the already limited supply of scholarly and scientific journals, the Committee hopes to enlist the cooperation of subscribers to this journal in preventing the sacrifice of this type of material to the pulp demand. It is scarcely necessary to mention the appreciation of foreign institutions and scholars for this activity.

Questions concerning the project or concerning the Committee's interest in particular periodicals should be directed to Dorothy J. Comins, Executive Assistant to the Committee on Aid to Libraries in War Areas, Library of Congress Annex, Study 251, Washington, 25, D. C.

American Library Association,
Committee on Aid to Libraries in War Areas.

SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The Editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. The papers submitted should be in final form. Only typographical errors may be corrected on proofs; if authors wish to add material, they may do so at their own expense.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical work: Only very simple symbols and formulas should be typewritten. All others should be carefully written by hand in ink. Ample space for marking should be allowed above and below all equations. Greek letters used in formulas should be designated by name in the margin. The difference between capital and lower-case letters should be clearly shown; and care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1) and the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n. All subscripts and exponents should be clearly marked, and dots and bars over letters should be avoided as far as possible. Square roots of complicated expressions should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{\quad}$. Complicated exponents and subscripts should be avoided. Any complicated expression that reoccurs frequently should be represented by a special symbol.

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be given as footnotes. Only in longer expository articles may references be grouped together in a bibliography at the end of the manuscript. The arrangement should be as follows: (for books)—author, title, volume, publisher, place of publication, year, page referred to; (for periodicals)—author, title, name of periodical, volume, page, year. All references should be complete and thoroughly checked.

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