## QUARTERLY

## OF

## APPLIED MATHEMATICS

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# QUARTERLY OF APPLIED MATHEMATICS 

# THE TRANSFORMATION OF PARTIAL DIF FERENTIAL EQUATIONS* 

BY

H. BATEMAN<br>California Institute of Technology

1. Introduction. In the early stages of the use of partial differential equations for the solution of problems of mechanics and physics the separation of variables and construction of simple solutions was the primary aim. The introduction of the idea of an exact differential by Fontaine and Euler led to the idea of associated differential equations such as those for the velocity potential and stream function in hydrodynamics, the adjoint equations of Lagrange and Riemann, the contact transformations of Legendre and Ampère, the transformations of Euler and Laplace for the solution of differential equations by definite integrals and other transformations too numerous to mention. Another aim which led to the study of transformations was that of reducing an equation to a canonical form. Laplace's reduction of a linear partial differential equation of the second order to a form in which only one partial derivative of the second order occurs led to the study of transformations which preserve this form and of quantities which have a property of invariance. Conditions were then found that an equation may be reducible by means of a specified type of change of variables to some particular equations which had been fully studied. The conditions found by Campbell [1]† (constancy of his two invariants) that Laplace's canonical equation may be reducible to the equation of Euler and Poisson, may be cited as an example.

A classification of transformations may be made by including in group A all transformations which arise from the condition or conditions that a linear differential form may be of a specified type (for example an exact differential). Transformations arising from the study of a number of linear differential forms may be included in this group. Transformations associated with the Calculus of Variations are also included because the equations of Euler and Lagrange are closely associated with the conditions for an exact differential. The extension of Legendre's transformation found by Carathéodory [2] may be mentioned here. In this article attention will be devoted almost entirely to transformations of group $A$.

Transformations of group B include all those which arise from the conditions that a quadratic differential form may be of a specified type. The transformation of a

[^0]linear differential equation to a form in which the variables are separated is thus a Btransformation. The transformations of group B are not necessarily point transformations, for instance, if $Q(a, b, c)$ is a non-negative quadratic form in the real variables $a, b, c$ transformations from ( $x, y, z, t, u, v, w)$ to ( $X, Y, Z, T, U, V, W$ ) may be considered in which $Q(d x-u d t, d y-v d t, d z-w d t)$ goes over into $Q(d X-U d t, d Y-V d t$, $d Z-W d t)$ the coefficients of $Q$ in the first case being functions of $x, y, z, t, u, v, w$ and in the second case functions of $X, Y, Z, T, U, V, W$. Since the equation $Q=0$ implies that $d x=u d t, d y=v d t, d z=w d t$ it also implies that $d X=U d T, d Y=V d T, d Z=W d T$. In other words if $u, v, w$ can be regarded as the component velocities of a recognizable moving particle of fluid then $U, V, W$ can be regarded as component velocities of a recognizable particle of a corresponding fluid. Such a transformation is of interest because the density of each fluid can be defined in such a way that the equation of continuity is invariant under the transformation.

Group C may be regarded as including all other transformations and some transformation of the other group which arise in the reduction of an equation to a canonical form.
2. Associated equations of the types of Monge and Legendre. In his work on partial differential equations of the second order in two variables $x, y$ which can be regarded as independent, Monge [3] used $z$ as dependent variable, $p$ and $q$ as the first derivatives $z_{x}, z_{y}$ respectively and $r, s, t$ as the second derivatives $z_{x x}, z_{x y}, z_{y y}$. As we shall have applications to fluid dynamics in mind, we shall deviate slightly from the notation of Monge and use $u, v$ in place of $p$ and $q$ so that when $z$ is the velocity potential $u$ and $v$ represent the component velocities as usual. This plan also allows us to use the symbol $p$ to denote the pressure and $q$ to denote the resultant velocity.

The equations of steady motion of a compressible fluid under no body forces when the flow is irrotational and the fluid barotropic (density a function of pressure only) can, as we know, be derived from a variational principle

$$
\begin{equation*}
\delta \iint p(u, v) d x d y=0, \quad u=z_{x}, v=z_{u} \tag{1}
\end{equation*}
$$

in which $p$ is a specified function of $q$. For greater generality at the outset we shall suppose, however, that $p$ is a specified function of $u$ and $v$. Lagrange's partial differential equation for this variational problem is then derivable from Haar's condition [4] that $p_{v} d x-p_{2} d y$ should be an exact differential. When the differentiations are made, the equation has the form

$$
\begin{equation*}
p_{u u} r+2 p_{u v} s+p_{v v} t=0 . \tag{2}
\end{equation*}
$$

When Legendre's transformation is applied to this differential equation the new dependent variable is

$$
\begin{equation*}
w=u x+v y-z \tag{3}
\end{equation*}
$$

and since $d z=u d x+v d y$,

$$
\begin{equation*}
d w=x d u+y d \tau . \tag{4}
\end{equation*}
$$

When $u$ and $v$ can be regarded as independent this equation gives the relations

$$
\begin{equation*}
x=w_{u}, \quad y=w_{v}, \tag{5}
\end{equation*}
$$

and the equation for $w$ is

$$
\begin{equation*}
p_{u u} w_{v v}-2 p_{u v} w_{u v}+p_{v v} w_{u k u}=0 . \tag{6}
\end{equation*}
$$

When, however, $u$ and $v$ are related so that they can be regarded as functions of a single variable $\tau$ the equation (4) indicates that $w$ is then also a function of $\tau$ and we have the equations

$$
\begin{equation*}
w(\tau)=x u(\tau)+y v(\tau)-z, \quad w^{\prime}(\tau)=x u^{\prime}(\tau)+y v^{\prime}(\tau), \tag{i}
\end{equation*}
$$

which furnish a solution of (2) if

$$
\begin{equation*}
p_{u u} u^{\prime 2}(\tau)+2 p_{u \mathrm{r}} u^{\prime}(\tau) v^{\prime}(\tau)+p_{v \mathrm{v} v^{\prime 2}}(\tau)=0 . \tag{8}
\end{equation*}
$$

Let us now seek the conditions that 3 quantities

$$
\begin{equation*}
R=R(u, v), \quad S=S(u, v), \quad T=T(u, v) \tag{9}
\end{equation*}
$$

may be the second derivatives $Z_{x x}, Z_{x y}, Z_{z x}$ of a single function $Z(x, y)$. The required conditions $R_{y}=S_{x}, S_{y}=T_{x}$ may be written in the form

$$
\begin{equation*}
R_{u} s+R_{\mathrm{r}} t=S_{u} r+S_{v} s, \quad S_{u} s+S_{v} t=T_{u} r+T_{v} s \tag{10}
\end{equation*}
$$

We now seek the conditions that these two equations are both satisfied in virtue of equation (2). This will be the case when

$$
\left.\begin{array}{lll}
R_{v}=w p_{r v}, & R_{u}=w p_{u v}+h, & S_{u}=-w p_{u u},  \tag{11}\\
T_{u}=w^{\prime} p_{u u}, & T_{v}=w_{v} p_{u v}+h^{\prime}, & S_{u}=h^{\prime}-w_{u v} \\
w_{u r}, & S_{r}=-w^{\prime} p_{v x}
\end{array}\right\}
$$

Equating the different expressions for $R_{u v}, S_{u v}, T_{u v}$ we obtain the equations

$$
\left.\begin{array}{ll}
h_{u}=w_{u} p_{u r}-w_{r} p_{u u} & h_{r}=w_{u} p_{v u}-w_{v} p_{u r}  \tag{12}\\
h_{u}^{\prime}=w_{r}^{\prime} p_{u u}-w_{u}^{\prime} p_{u v}, & h_{v}^{\prime}=w_{r}^{\prime} p_{u v}-w_{u}^{\prime} p_{r v}
\end{array}\right\}
$$

The elimination of $h$ and $h^{\prime}$ yields the two equations

$$
\left.\begin{array}{l}
w_{u u} p_{r v}-2 w_{u r} p_{u r}+w_{r \cdot} p_{u u}=0  \tag{13}\\
w_{u u}^{\prime} p_{r r}-2 w_{u r}^{\prime} p_{u r}+w_{v r}^{\prime} p_{u u}=0
\end{array}\right\}
$$

which show that $w$ and $w^{\prime}$ are solutions of equation (6). The case of chief hydrodynamical interest is that in which the second derivatives of $w$ and $w$ ' are all zero. We shall, however, look first at a possible alternative case.
Equating the different expressions for $S_{u}$ and $S_{v}$ we obtain the equations

$$
\begin{align*}
& w p_{u u}=w^{\prime} p_{u r}+\int\left[w_{u}^{\prime} d\left(p_{r}\right)-w_{v}^{\prime} d\left(p_{u}\right)\right] \\
& w^{\prime} p_{v r}=w_{u_{u r}}+\int\left[w_{u} d\left(p_{r}\right)-w_{r} d\left(p_{u}\right)\right] \tag{14}
\end{align*}
$$

Hence

$$
\begin{align*}
& w_{\tau}^{\prime} p_{r v}+w^{\prime} p_{u r r}=2 w_{r} p_{u r}+w_{u r r}-w_{u} p_{r \tau}, \\
& 2 w_{u}^{\prime} p_{u r}-w_{r}^{\prime} p_{u u}+w_{r u \tau}^{\prime} p_{u u \tau}=w_{u} p_{u u}+w_{u u u}  \tag{15}\\
& w_{u}^{\prime} p_{r r}+w_{u}^{\prime} p_{u r r}=w_{r} p_{u u}+w_{u u r} p_{u r}
\end{align*}
$$

These equations lead to the relation

$$
\begin{equation*}
w^{\prime} D_{v}=w D_{u}, \quad \text { where } \quad D=p_{u u} p_{r v}-p_{u r}^{2} \tag{16}
\end{equation*}
$$

This equation is satisfied identically when $D$ is constant but it may also be satisfied if $w=E_{v}, w^{\prime}=E_{u}$ where $D$ and $E$ are related. An important case of this second type occurs when $D$ is a function of $q$ only and $w=v, w^{\prime}=u$. In this case
$h=-p_{u}, \quad h^{\prime}=-p_{v}, \quad R=v p_{v}-p_{,} \quad T=u p_{u}-p_{,} \quad S=-u p_{v}=-v p_{u}$,
and $p$ is a function of $q$ only. If $p_{u}=-u \rho, p_{v}=-v \rho$, where $\rho$ is the density of the fluid we have

$$
\begin{equation*}
R=-p-\rho v^{2}, \quad S=\rho u v, \quad T=-p-\rho u u^{2} \tag{18}
\end{equation*}
$$

and so $Z$ is a kind of stress function which satisfies the equation

$$
\begin{equation*}
R T-S^{2}=p\left(p+\rho u^{2}+\rho v^{2}\right)=p\left(p+\rho q^{2}\right)=F(R+T) \tag{19}
\end{equation*}
$$

since $R+T=-2 p-\rho q^{2}$. The partial differential equation for $Z$ is thus of Legendre's type [5]

$$
\begin{equation*}
\mathfrak{H}(R, S, T)=0 \tag{20}
\end{equation*}
$$

In the special case in which $-F(R+T)=K^{2}-\frac{1}{4}(R+T)^{2}$ the equation reduces to one which occurs in Saint Venant's theory of plastic bodies. This equation has been discussed by Hencky [6], Prandtl [7] and Carathéodory [8]. Oseen [9] uses the method of Legendre in which the equation is first solved for $R$, differentiated with respect to $y$ and so reduced to an equation

$$
\begin{equation*}
\left(K^{2}-V_{x}^{2}\right)^{1 / 2}\left(V_{x x}-V_{y v}\right)+2 V_{x} V_{x y}=0 \tag{21}
\end{equation*}
$$

in which

$$
S=V_{x}, \quad T=V_{y}
$$

It should be remarked that if $p_{v}=\bar{u}=\bar{z}_{x}, p_{u}=-\bar{v}=-\bar{z}_{y}$,

$$
\begin{equation*}
p+\rho u^{2}+\rho v^{2}=\bar{p}, \quad 1 / \rho=-\bar{\rho}, \tag{22}
\end{equation*}
$$

we may write

$$
\begin{equation*}
R=-\bar{p}-\bar{\rho} \bar{v}^{2}, \quad S=\bar{\rho} \bar{u} \bar{v}, \quad T=-\bar{p}-\bar{\rho} \bar{u} \bar{u}^{2}, \tag{23}
\end{equation*}
$$

and the equations $R_{y}=S_{x}, S_{y}=T_{z}$ lead to the partial differential equation for the stream-function $\bar{z}$.

In the theory of plane waves of finite amplitude equations of Legendre's type occur in at least two ways one of which is discussed by J. R. Wilton [10]. In the other way use is made of the equations

$$
\begin{equation*}
R=\rho, \quad S=-\rho u, \quad T=p+\rho u u^{2}, \tag{24}
\end{equation*}
$$

where now $y$ denotes the time and $x$ a co-ordinate in the direction in which the waves are travelling. The quantities $u, v$ are again the derivatives of a velocity potential $z, \rho$ is the density, $p$ the pressure and $u$ the velocity of the fluid. The quantities $R, S, T$ are the second derivatives of a stress-function $Z$. The additional equations from which the relation between $R, S$ and $T$ may be derived, are

$$
\begin{equation*}
v+\frac{1}{2} u^{2}=f^{\prime}(\rho), \quad p=f(\rho)-\rho f^{\prime}(\rho) \tag{25}
\end{equation*}
$$

The desired relation is thus

$$
\begin{equation*}
T-R^{-1} S^{2}=f(R)-R f^{\prime}(R) \tag{26}
\end{equation*}
$$

This equation, like that considered by Wilton, may be solved by the method of Legendre in which the equation is differentiated with respect to one of the independent variables (in this case $x$ ) so as to reduce it to an equation of the Monge-Ampère type. The transformation to the new equation can be regarded as a special Bäcklund transformation [11] as Oscen [9] observes. If $U=Z_{x}$, the new equation is

$$
\begin{equation*}
U_{y y}-2\left(U_{y} / U_{x}\right) U_{x y}+\left(U_{y}^{2} / U_{x}^{2}\right) U_{x x}=-R f^{\prime \prime}(R) U_{x x}=c^{2} U_{x x} \tag{27}
\end{equation*}
$$

or

$$
\Pi U_{x x}+2 K U_{x y}+L U_{y y}=0
$$

where $H=\left(U_{y}^{2} / U_{x}^{2}\right)-c^{2}, K=-\left(U_{y} / U_{x}\right), L=1$. The invariant $G$ is

$$
\begin{equation*}
G=K^{2}-H L=c^{2} \tag{28}
\end{equation*}
$$

and the condition $G \neq 0$ is satisfied so long as $c^{2} \neq 0$.
In the present case

$$
\begin{align*}
& R_{u}=-\rho u / c^{2}, R_{v}=-\rho / c^{2}, S_{u}=\rho\left(u^{2} / c^{2}-1\right) S_{v}=\rho u / c^{2} \\
& T_{u}=\rho u\left(1-u u^{2} / c^{2}\right), \quad T_{v}=-\rho\left(1+u^{2} / c^{2}\right) \tag{29}
\end{align*}
$$

The two equations (10) are both equivalent to

$$
\begin{equation*}
\left(u^{2}-c^{2}\right) r+2 u s+t=0 \tag{30}
\end{equation*}
$$

and it is readily seen that $w=-1, w^{\prime}=-u, g=k^{2}-h l=c^{2} \neq 0$.
It should be noticed that if we solve equations (9) for $u$ and $v$ in the form

$$
\begin{equation*}
u=F(R, S), \quad v=G(S, T) \tag{31}
\end{equation*}
$$

the equation $u_{y}=v_{x}$ is satisfied on account of $R_{y}=S_{x}, S_{y}=T_{x}$ if

$$
\begin{equation*}
F_{R}=G_{S}, \quad F_{S}=G_{T} \tag{32}
\end{equation*}
$$

These two equations then are consequences of the single equation $\mathcal{H}(R, S, T)=0$. The expression of such an equation in the two forms (32) may be regarded as a problem of some interest.

In the case when $D$ is a constant and $p$ is a function of $q$ only

$$
\begin{equation*}
D=p_{q} p_{q q} / q=p^{2}\left[1-(q / c)^{2}\right] \tag{33}
\end{equation*}
$$

and the flow is either entirely subsonic $(D>0)$ or entirely supersonic $(D<0)$. In many cases in which $p$ is a function of $q$ only, $D$ can have either sign and so the flow is partly subsonic and partly supersonic. It is then of some interest to seek the condition satisfied by the function $\mathscr{H}(R, S, T)$ when $D>0$. For this purpose we write the equation in the form

$$
\begin{equation*}
0=\mathfrak{H}(R, S, T) \equiv\left[(R-T)^{2}+4 S^{2}\right]^{1 / 2}-J(R+T) \tag{34}
\end{equation*}
$$

where $J$ is a function which is such that

$$
\begin{equation*}
2 \rho f^{\prime}(\rho)=-J[-2 f(\rho)] \tag{35}
\end{equation*}
$$

Here $p=f(\rho)-\rho f^{\prime}(\rho)$ is the relation between the pressure $p$ and density $\rho$. Now we find by differentiation that

$$
\begin{equation*}
\mathscr{C}_{R}+\mathscr{C}_{T}-2=-2-2 J^{\prime}(R+T)=-4-2 \rho f^{\prime \prime}(\rho) / f^{\prime}(\rho)=4\left(c^{2} / q^{2}-1\right) \tag{36}
\end{equation*}
$$

Hence $\mathfrak{K}_{R}+\mathfrak{K}_{T}-2>0$ when $c^{2}>q^{2}$ and $\mathfrak{C}_{R}+\mathfrak{K}_{T}-2<0$ when $c^{2}<q^{2}$. In the case of the plastic equation $J$ is a constant and so

$$
\begin{equation*}
\mathfrak{K}_{R}+\mathfrak{C}_{T}=0 \tag{37}
\end{equation*}
$$

The corresponding flow is characterized by the relation $q^{2}=2 c^{2}$ and is consequently supersonic.

A simple case in which $D$ is constant is obtained by writing

$$
\begin{equation*}
p=a u^{2}+2 c u v+b u^{2} \tag{38}
\end{equation*}
$$

where $a, b$ and $c$ are constants. The functions $w, w^{\prime}$ both satisfy

$$
\begin{equation*}
b w_{u u}+a w_{v v}-2 c w_{u v}=0 \tag{39}
\end{equation*}
$$

and we may write $w=b V_{u}, w^{\prime}=a V_{v}$, where $V$ is a solution of this equation. If a function $W$ is defined by the equations

$$
\begin{equation*}
W_{u}=c V_{u}-a V_{v}, \quad W_{v}=b V_{u}-c V_{v} \tag{40}
\end{equation*}
$$

we may write $h=2 b W_{u}$ and it is readily found that we can write

$$
\begin{equation*}
R=2 b(W+c V), \quad S=-2 a b V, \quad T=2 a(W+c V) \tag{41}
\end{equation*}
$$

where $V$ and $W$ are connected by the foregoing equations. In this case the relation between $R, S$ and $T$ is simply

$$
\begin{equation*}
a \mathcal{F}=a R-b T=0 \tag{42}
\end{equation*}
$$

The quantity $\mathfrak{C}_{R}+\mathfrak{K}_{T}-2$ is now simply $-(a+b) / a$, a constant. There is no change in sign of the expression. It will be noticed that the equations $a R-b T=0$ and

$$
\begin{equation*}
a r+2 c s+b l=0 \tag{43}
\end{equation*}
$$

satisfy the condition of apolarity

$$
\begin{equation*}
A b+B a-2 C c=0 \tag{44}
\end{equation*}
$$

when the first equation is written in the form $A R+2 C S+B T=0$.
3. The transformation of the Monge-Ampère equation. If for the equation

$$
\begin{equation*}
h r+2 k s+l t+m+n\left(r t-s^{2}\right)=0 \tag{45}
\end{equation*}
$$

the expression

$$
\begin{equation*}
g=k^{2}-h l+m n \tag{46}
\end{equation*}
$$

is not zero and so the two systems in the methods of Monge and Boole are distinct, the equation is transformed by a contact transformation

$$
\left.\begin{array}{c}
X=X(x, y, z, u, v), \quad Y=Y(x, y, z, u, v), \quad Z=Z(x, y, z, u, v) \\
U=U(x, y, z, u, v), V=V(x, y, z, u, v), d Z=U d X-V d Y=\sigma(d z-u d x-v d y) \tag{47}
\end{array}\right\}
$$

into an equation

$$
\begin{equation*}
H R+2 K S+L T+M+N\left(R T-S^{2}\right)=0 \tag{48}
\end{equation*}
$$

for which the quantity $G=K^{2}-H L+M N=0$.
In a paper published in 1904 Sophus Lic [12] remarked that it would be desirable to have a direct proof of this theorem and Kürschák [13] gave one based upon a representation of the equation in the form of a Jacobian

$$
\begin{equation*}
d(a, b) / d(x, y) \tag{49}
\end{equation*}
$$

where $a$ and $b$ are functions of $x, y, z, u, v$ and $d / d x=\partial / \partial x+u(\partial / \partial z)+r(\partial / \partial u)$ $+s(\partial / \partial v), d / d y=\partial / \partial y+v(\partial / \partial z)+s(\partial / \partial u)+t(\partial / \partial v)$. When this representation is not used the proof is algebraically more difficult but the analysis is worth giving on account of the numerous relations to which it leads. Reference for this type of proof may be made to a paper by R. Garnier, Sur la transformation des dérivées secondes dans les transformations de contact et les transformations ponctuelles, Bull. des Sci. Math. (2), 64, 13-32 (1940).

We shall suppose that $d z=u d x+v d y$ and that consequently $d Z=U d X+V d Y$. To make $d U=R d X+S d Y, d V=S d X+T d Y$ consequences of $d u=r d x+s d y, d v=s d x$ $+t d y$ we shall require that

$$
\begin{align*}
d U-R d X- & S d Y=\left(U_{u}-R X_{u}-S Y_{u}\right)(d u-r d x-s d y) \\
& +\left(U_{v}-R X_{v}-S Y_{u}\right)(d v-s d x-t d y) \\
d V-S d X-T d Y= & \left(V_{u}-S X_{u}-T Y_{u}\right)(d u-r d x-s d y)  \tag{50}\\
& +\left(V_{v}-S X_{v}-T Y_{v}\right)(d v-s d x-t d y)
\end{align*}
$$

With the notation

$$
\left.\begin{array}{rlrl}
Z_{1} & =Z_{x}+u Z_{z}, & Z_{2} & =Z_{v}+v Z_{z}, \text { etc. } \\
(u u) & =U_{u}-R X_{u}-S Y_{u}, & (u v) & =U_{v}-R X_{v}-S Y_{v},  \tag{51}\\
(v u) & =V_{u}-S X_{u}-T Y_{u}, & (v v) & =V_{v}-S X_{v}-T Y_{v}
\end{array}\right\}
$$

the equations to be satisfied are

$$
\begin{align*}
& r(u u)+s(u v)+U_{1}-R X_{1}-S Y_{1}=0 ; r(u v)+s(v v)+V_{1}-S X_{1}-T Y_{1}=0 \\
& s(u u)+t(u v)+U_{2}-R X_{2}-T Y_{2}=0, \quad s(v u)+t(v v)+V_{2}-S X_{2}-T Y_{2}=0 \tag{52}
\end{align*}
$$

Hence

$$
\begin{align*}
r \Delta= & U_{v} V_{1}-U_{1} V_{v}+R\left(X_{1} V_{v}-V_{1} X_{v}\right)+S\left(Y_{1} V_{v}+U_{1} X_{v}-X_{1} U_{v}-V_{1} Y_{v}\right) \\
& +T\left(U_{1} Y_{v}-Y_{1} U_{v}\right)+\left(R T-S^{2}\right)\left(Y_{1} X_{v}-X_{1} Y_{v}\right), \\
t \Delta= & U_{2} V_{u}-V_{2} U_{u}+R\left(X_{u} V_{2}-X_{2} V_{u}\right)+S\left(X_{2} U_{u}+Y_{u} V_{2}-X_{u} U_{2}-Y_{2} V_{u}\right) \\
& +T\left(Y_{2} U_{u}-Y_{u} U_{2}\right)+\left(R T-S^{2}\right)\left(X_{2} Y_{u}-X_{u} Y_{2}\right), \\
s \Delta= & U_{1} V_{u}-V_{1} U_{u}+R\left(X_{u} V_{1}-X_{1} V_{u}\right)+S\left(X_{1} U_{u}+Y_{u} V_{1}-X_{u} U_{1}-Y_{1} V_{u}\right) \\
& +T\left(U_{u} Y_{1}-U_{1} Y_{u}\right)+\left(R T-S^{2}\right)\left(X_{1} Y_{u}-X_{u} Y_{1}\right), \\
s \Delta= & V_{2} U_{v}-U_{2} V_{v}+R\left(X_{2} V_{v}-X_{v} V_{2}\right)+S\left(U_{2} X_{v}+Y_{2} V_{v}-X_{2} U_{v}-V_{2} Y_{v}\right)  \tag{53}\\
& +T\left(U_{2} Y_{v}-Y_{2} U_{v}\right)+\left(R T-S^{2}\right)\left(Y_{2} X_{v}-X_{2} Y_{v}\right), \\
\Delta\left(r t-s^{2}\right)= & U_{1} V_{2}-U_{2} V_{1}+R\left(V_{1} X_{2}-V_{2} X_{1}\right)+T\left(U_{2} Y_{1}-U_{1} Y_{2}\right) \\
& +S\left(U_{2} X_{1}-U_{1} X_{2}+V_{1} Y_{2}-V_{2} Y_{1}\right)+\left(R T-S^{2}\right)\left(X_{1} Y_{2}-X_{2} Y_{1}\right) \\
\Delta= & U_{u} V_{v}-U_{v} V_{u}+R\left(X_{v} V_{u}-X_{u} V_{v}\right)+T\left(U_{v} Y_{u}-U_{u} Y_{v}\right) \\
& +S\left(Y_{v} V_{u}-Y_{u} V_{v}+U_{v} X_{u}-U_{u} X_{v}\right)+\left(R T-S^{2}\right)\left(X_{u} Y_{v}-X_{v} Y_{u}\right)
\end{align*}
$$

The two expressions for $s$ are equivalent on account of the relations

$$
\begin{equation*}
[U V]=[X V]=[Y U]=[X Y]=0, \quad[X U]=[Y V]=\sigma \tag{54}
\end{equation*}
$$

which, in addition to the relations $[Y Z]=[Z X]=0,[U Z]=\sigma U[V Z]=\sigma V$ are satisfied because the transformation is a contact transformation. In these relations $[A B]$ is the Poisson bracket

$$
\begin{equation*}
[A B]=A_{u} B_{1}-A_{1} B_{u}+A_{v} B_{2}-A_{2} B_{v} \tag{55}
\end{equation*}
$$

The relations are derived by Lie [12] by a clever device. In the book of Cerf [14] the relations are derived from the equation

$$
\begin{equation*}
\sigma[A B]=[a b] \tag{56}
\end{equation*}
$$

where $A, B$ are the expressions for $a(x, y, z, p, q), b(x, y, z, p, q)$ in the new co-ordinates [ $X, Y, Z, P, Q$ ], while F . Engel [15] obtained them with the aid of the bilinear covariant by a development of a method used by G. Darboux.
It is readily seen that the equation $k r+2 k s+l l+m+n\left(r t-s^{2}\right)=0$ becomes $I I R+2 K S$ $+L T+M+N(R T-S)=0$, where

$$
\begin{align*}
H= & h P_{p}+k\left(P_{b}+P_{q}\right)+l P_{a}+m P_{r}+n P_{c}, \\
2 K= & h\left(R_{p}+C_{p}\right)+k\left(C_{b}+R_{b}+C_{q}+R_{q}\right)+l\left(R_{a}+C_{a}\right) \\
& \quad+m\left(R_{r}+C_{r}\right)+n\left(R_{c}+C_{c}\right), \\
L= & h A_{p}+k\left(A_{b}+A_{q}\right)+l A_{a}+m A_{r}+n A_{c},  \tag{57}\\
M= & h Q_{p}+k\left(Q_{b}+Q_{q}\right)+l Q_{a}+m Q_{r}+n Q_{c}, \\
N= & h B_{p}+k\left(B_{b}+B_{q}\right)+l B_{a}+m B_{r}+n B_{c},
\end{align*}
$$

where

$$
\begin{array}{lll}
A_{a}=U_{u} Y_{2}-U_{2} Y_{u}, & A_{b}=U_{u} Y_{1}-U_{1} Y_{u}, & A_{c}=U_{2} Y_{1}-U_{1} Y_{2}, \\
A_{p}=U_{1} Y_{v}-U_{v} Y_{1}, & A_{q}=U_{2} Y_{v}-U_{v} Y_{2}, & A_{r}=U_{v} Y_{u}-U_{u} Y_{v}, \\
B_{a}=X_{2} Y_{u}-X_{u} Y_{2}, & B_{b}=X_{1} Y_{u}-X_{u} Y_{1}, & B_{c}=X_{1} Y_{2}-X_{2} Y_{1}, \\
B_{p}=X_{v} Y_{1}-X_{1} Y_{v}, & B_{q}=X_{v} Y_{2}-X_{2} Y_{v}, & B_{r}=X_{u} Y_{v}-X_{v} Y_{u}, \\
C_{a}=X_{2} U_{u}-X_{u} U_{2}, & C_{b}=X_{1} U_{u}-X_{u} U_{1}, & C_{c}=X_{1} U_{2}-X_{2} U_{1}, \\
C_{p}=X_{v} U_{1}-X_{1} U_{v}, & C_{q}=X_{v} U_{2}-X_{2} U_{v}, & C_{r}=X_{u} U_{v}-X_{v} U_{u}, \\
P_{a}=X_{u} V_{2}-X_{2} V_{u}, & P_{b}=X_{u} V_{1}-X_{1} V_{u}, & P_{c}=X_{2} V_{1}-X_{1} V_{2}, \\
P_{p}=X_{1} V_{v}-X_{v} V_{u}, & P_{q}=X_{2} V_{v}-X_{v} V_{2}, & P_{r}=X_{v} V_{u}-X_{u} V_{v}, \\
Q_{a}=U_{2} V_{u}-U_{u} V_{2}, & Q_{b}=U_{1} V_{u}-U_{u} V_{1}, & Q_{c}=U_{1} V_{2}-U_{2} V_{1}, \\
Q_{p}=U_{v} V_{1}-U_{1} V_{v}, & Q_{q}=U_{v} V_{2}-U_{2} V_{v}, & Q_{r}=U_{u} V_{v}-U_{v} V_{u} \\
R_{a}=Y_{u} V_{2}-Y_{2} V_{u}, & R_{b}=Y_{u} V_{1}-Y_{1} V_{u}, & R_{c}=Y_{2} V_{1}-Y_{1} V_{2}, \\
R_{p}=Y_{1} V_{v}-Y_{\tau} V_{1}, & R_{q}=Y_{2} V_{v}-Y_{v} V_{2}, & R_{r}=Y_{v} V_{u}-Y_{u} V_{v} .
\end{array}
$$

These equations give the relation

$$
\begin{align*}
& K^{2}-H L+M N=J\left(k^{2}-h l+m n\right) \text {, if } \\
& \frac{1}{4}\left(C_{b}+R_{b}+C_{q}+R_{q}\right)^{2}-\left(P_{b}+P_{q}\right)\left(A_{b}+A_{q}\right)+\left(Q_{b}+Q_{q}\right)\left(B_{b}+B_{q}\right)=J \text {, } \\
& \frac{1}{2}\left(R_{r}+C_{r}\right)\left(R_{c}+C_{c}\right)+\left(Q_{r} B_{c}+Q_{c} B_{r}\right)-\left(P_{r} A_{c}+P_{c} A_{r}\right)=J \text {, } \\
& \frac{1}{2}\left(R_{p}+C_{p}\right)\left(R_{a}+C_{a}\right)+\left(Q_{p} B_{a}+Q_{a} B_{p}\right)-\left(P_{p} A_{a}+P_{a} A_{p}\right)=-J, \\
& \frac{1}{4}\left(R_{p}+C_{p}\right)^{2}+Q_{p} B_{p}-P_{p} A_{p}=0, \quad{ }_{1}^{1}\left(R_{a}+C_{a}\right)^{2}+Q_{a} B_{a}-P_{a} A_{a}=0, \\
& \frac{1}{1}\left(R_{r}+C_{r}\right)^{2}+Q_{r} B_{r}-P_{r} A_{r}=0, \quad \frac{1}{1}\left(R_{c}+C_{c}\right)^{2}+Q_{c} B_{c}-P_{c} A_{c}=0, \\
& \frac{1}{2}\left(R_{p}+C_{p}\right)\left(C_{b}+R_{b}+C_{q}+R_{q}\right)+Q_{p}\left(B_{b}+B_{q}\right)+B_{p}\left(Q_{b}+Q_{q}\right) \\
& =P_{p}\left(A_{b}+A_{q}\right)+A_{p}\left(P_{b}+P_{q}\right), \\
& \frac{1}{2}\left(R_{a}+C_{a}\right)\left(C_{b}+R_{b}+C_{q}+R_{q}\right)+Q_{a}\left(B_{b}+B_{q}\right)+B_{a}\left(Q_{b}+Q_{q}\right) \\
& =P_{a}\left(A_{b}+A_{q}\right)+A_{a}\left(P_{b}+P_{q}\right),  \tag{58}\\
& \frac{1}{2}\left(R_{r}+C_{r}\right)\left(C_{b}+R_{b}+C_{q}+R_{q}\right)+Q_{r}\left(B_{b}+B_{q}\right)+B_{r}\left(Q_{b}+Q_{q}\right) \\
& =P_{r}\left(A_{b}+A_{q}\right)+A_{r}\left(P_{b}+P_{q}\right), \\
& \frac{1}{2}\left(R_{c}+C_{c}\right)\left(C_{b}+R_{b}+C_{q}+R_{q}\right)+Q_{c}\left(B_{b}+B_{q}\right)+B_{c}\left(Q_{b}+Q_{q}\right) \\
& =P_{c}\left(A_{b}+A_{q}\right)+A_{c}\left(P_{b}+P_{q}\right), \\
& \frac{1}{2}\left(R_{p}+C_{p}\right)\left(R_{r}+C_{r}\right)+Q_{p} B_{r}+Q_{r} B_{p}-P_{p} A_{r}-P_{r} A_{p}=0, \\
& \frac{1}{2}\left(R_{p}+C_{p}\right)\left(R_{c}+C_{c}\right)+Q_{p} B_{c}+Q_{c} B_{p}-P_{p} A_{c}-P_{c} A_{p}=0, \\
& \frac{1}{2}\left(R_{a}+C_{a}\right)\left(R_{r}+C_{r}\right)+Q_{a} B_{r}+Q_{r} B_{a}-P_{a} A_{r}-P_{r} A_{a}=0, \\
& \frac{1}{2}\left(R_{a}+C_{a}\right)\left(R_{c}+C_{c}\right)+Q_{a} B_{c}+Q_{c} B_{a}-P_{a} A_{c}-P_{c} A_{a}=0 .
\end{align*}
$$

These relations may be established by using a parametric representation of the quantities satisfying Lie's conditions for a contact transformation, we therefore write

$$
\begin{array}{rlll}
X_{1}=a_{1} e+a_{2} e^{\prime}, & X_{2}=b_{1} e+b_{2} e^{\prime}, & X_{u}=c_{1} e+c_{2} e^{\prime}, & X_{v}=d_{1} e+d_{2} e^{\prime} \\
U_{1}=a_{1} f+a_{2} f^{\prime}, & U_{2}=b_{1} f+b_{2} f^{\prime}, & U_{u}=c_{1} f+c_{2} f^{\prime}, & U_{v}=d_{1} f+d_{2} f^{\prime} \\
-Y_{u}=a_{3} p+a_{4} p^{\prime}, & -Y_{v}=b_{3} p+b_{4} p^{\prime}, & Y_{1}=c_{3} p+c_{4} p^{\prime}, & Y_{2}=d_{3} p+d_{4} p^{\prime}  \tag{59}\\
-V_{u}=a_{3} q+a_{4} q^{\prime}, & -V_{v}=b_{3} q+b_{4} q^{\prime}, & V_{1}=c_{3} q+c_{4} q^{\prime}, & V_{2}=d_{3} q+d_{4} q^{\prime}
\end{array}
$$

where the quantities

$$
\begin{array}{llll}
a_{1} & a_{2} & a_{3} & a_{4} \\
b_{1} & b_{2} & b_{3} & b_{4} \\
c_{1} & c_{2} & c_{3} & c_{4} \\
d_{1} & d_{2} & d_{3} & d_{4}
\end{array}
$$

form an orthogonal matrix and the quantities $e, e^{\prime}, f, f^{\prime}, p, p^{\prime}, q, q^{\prime}$ are such that $\left(c_{1} a_{2}-c_{2} a_{1}+d_{1} b_{2}-d_{2} b_{1}\right)\left(e f^{\prime}+e^{\prime} f\right)=\left(a_{3} a_{4}-a_{4} c_{3}+b_{3} d_{4}-b_{4} d_{3}\right)\left(p q^{\prime}-p^{\prime} q\right)=\sigma$.
It is then found that

$$
\left.\begin{array}{rl}
-H & =e q(1,3)+e^{\prime} q(2,3)+e q^{\prime}(1,4)+e^{\prime} q^{\prime}(2,4) \\
-L & =f p(1,3)+f^{\prime} p(2,3)+f p^{\prime}(1,4)+f^{\prime} p^{\prime}(2,4), \\
2 K & =\left(e f^{\prime}-e^{\prime} f\right)(1,2)+\left(p q^{\prime}-p^{\prime} q\right)(3,4)  \tag{61}\\
M & =f q(1,3)+f^{\prime} q(1,3)+f q^{\prime}(1,4)+f^{\prime} q^{\prime}(2,4), \\
N & =e p(1,3)+e^{\prime} p(2,3)+e p^{\prime}(1,4)+e^{\prime} p^{\prime}(2,4),
\end{array}\right\}
$$

where

$$
\begin{align*}
(1,3)= & h\left(d_{1} c_{3}+a_{1} b_{3}\right)-l\left(b_{1} a_{3}+c_{1} d_{3}\right)+k\left(d_{1} d_{3}+b_{1} b_{3}-a_{1} a_{3}-c_{1} c_{3}\right) \\
& +n\left(a_{1} d_{3}-b_{1} c_{3}\right)+m\left(d_{1} a_{3}-c_{1} b_{3}\right), \\
(2,3)= & h\left(d_{2} c_{3}+a_{2} b_{3}\right)-l\left(b_{2} a_{3}+c_{2} d_{8}\right)+k\left(d_{2} d_{3}+b_{2} b_{3}-a_{2} a_{3}-c_{2} c_{3}\right) \\
& +n\left(a_{2} d_{3}-b_{2} c_{3}\right)+m\left(d_{2} a_{3}-c_{2} b_{3}\right), \\
(1,4)= & h\left(d_{1} c_{4}+a_{1} b_{4}\right)-l\left(b_{1} a_{4}+c_{1} d_{4}\right)+k\left(d_{1} d_{4}-b_{1} b_{4}-a_{1} a_{4}-c_{1} c_{4}\right) \\
& +n\left(a_{1} d_{4}-b_{1} c_{4}\right)+m\left(d_{1} a_{4}-c_{1} b_{4}\right), \\
(2,4)= & h\left(d_{2} a_{4}+a_{2} b_{4}\right)-l\left(b_{2} a_{4}+c_{2} d_{4}\right)+k\left(d_{2} d_{4}+b_{2} b_{4}-a_{2} a_{4}-c_{2} c_{4}\right)  \tag{62}\\
& +n\left(a_{2} d_{4}-b_{2} c_{4}\right)+m\left(d_{2} a_{4}-c_{2} b_{4}\right), \\
(1,2)= & h\left(d_{1} a_{2}-d_{2} a_{1}\right)+l\left(b_{1} c_{2}-b_{2} c_{1}\right)+k\left(a_{1} c_{2}-a_{2} c_{1}+d_{1} b_{2}-d_{2} b_{1}\right) \\
& +n\left(a_{1} b_{2}+a_{2} b_{1}\right)+m\left(c_{1} d_{2}-d_{1} c_{2}\right), \\
(3,4)= & h\left(b_{3} c_{4}-b_{4} c_{3}\right)+l\left(a_{4} d_{3}-a_{3} d_{4}\right)+k\left(c_{3} a_{4}-c_{4} a_{3}+b_{3} d_{4}-b_{4} d_{3}\right) \\
& +n\left(d_{3} c_{4}-d_{4} c_{3}\right)+m\left(b_{3} a_{4}-b_{4} a_{3}\right) .
\end{align*}
$$

It is readily seen that

$$
\begin{equation*}
M N-H L=[(1,3)(2,4)-(2,3)(1,4)]\left(e^{\prime} f-e f^{\prime}\right)\left(p^{\prime} q-p q^{\prime}\right) \tag{63}
\end{equation*}
$$

and that, on account of the properties of an orthogonal matrix

$$
\begin{equation*}
\left(e^{\prime} f-e f^{\prime}\right)^{2}=\left(p^{\prime} q-p q^{\prime}\right)^{2} \tag{64}
\end{equation*}
$$

The expression for $K$ also simplifies considerably and the proof may be readily completed. The quantity $J$ as in Kürschák's analysis, is equal to $\sigma^{2}$ and so is not zero.

Contact transformations are not the only ones in which the condition $g \neq 0$ is invariant. In the theory of the steady two-dimensional motion of an inviscid elastic fluid the equations satisfied by the velocity potential $z$ and stream-function $\bar{z}$ are respectively

$$
\begin{equation*}
p_{u u} r+2 p_{u v} s+p_{v v} t=0, \quad p_{u k} \bar{r}+2 p_{u} \bar{s}+p_{v v} \bar{l}=0 . \tag{65}
\end{equation*}
$$

In this case $d z=u d x+v d y, d \bar{z}=\bar{u} d x+\bar{v} d y=p_{v} d x-p_{u} d y$ and so

$$
\begin{equation*}
\bar{u}=p_{v}, \quad \bar{v}=-p_{u}, \quad g=\bar{g}=p_{u v}^{2}-p_{u u} p_{v v}=c^{2}\left(u^{2}+v^{2}-c^{2}\right) \tag{66}
\end{equation*}
$$

Thus $g=0$ either when $c=0$ or when $q=c$. The supersonic region is characterized by the condition $g>0$ and the subsonic region by the condition $g<0$. The curve for which $c=0$ is a boundary for the flow just as in the case of the Prandtl-Meyer flow round a corner. The transformation under consideration is a special Bäcklund transformation and is included in the group of Bäcklund transformations

$$
\begin{equation*}
\bar{x}=X(x, y, u, v), \quad \bar{y}=Y(x, y, u, v), \quad \bar{u}=U(x, y, u, v), \quad \bar{v}=V(x, y, u, v), \tag{67}
\end{equation*}
$$

for which the Jacobian $\partial(X, Y, U, V) / \partial(x, y, u, v)$ is not zero. These transformations have been studied carefully by Goursat [16]. The requirement that

$$
\bar{u} d \bar{x}+\bar{v} d \bar{y}
$$

should be exact leads to an equation of the Monge-Ampère type in which $z$ does not
occur explicitly. It is shown, however, that the general Monge-Ampère equation of this type cannot be obtained in this way and a similar result has been found by J. Clairin [17] in his studies of more general Bäcklund transformations. Clairin has studied in particular transformations of type

$$
\left.\begin{array}{rl}
x^{\prime}=f_{1}\left(x, y, z, u, v, z^{\prime}\right), & y^{\prime}=f_{2}\left(x, y, z, u, v, z^{\prime}\right) \\
u^{\prime} & =f_{3}\left(x, y, z, u, v, z^{\prime}\right), \tag{68}
\end{array} v^{\prime}=f_{4}\left(x, y, z, u, v, z^{\prime}\right) .\right\}
$$

Some of Clairin's work is summarized in the book of Forsyth [18] and illustrated by means of examples.

Another transformation of type (67), which preserves the condition $g \neq 0$ is obtained by writing

$$
d U=R d x+S d y, \quad d V=S d x+T d y
$$

where $R, S, T$ are the functions of $u$ and $v$ used in section 2. Making use of the equations

$$
\begin{equation*}
\left(u p_{u}+v p_{v}\right) d x=p_{u} d z+v d \bar{z}, \quad\left(u p_{u}+v p_{v}\right) d y=p_{v} d z-u d \bar{z} \tag{69}
\end{equation*}
$$

we find that

$$
\left.\begin{array}{r}
(T u-S v) d U+(R v-S u) d V=\left(R T-S^{2}\right) d z  \tag{70}\\
\left(S p_{u}+T p_{v}\right) d U-\left(R p_{u}+S p_{v}\right) d V=\left(R T-S^{2}\right) d \bar{z}
\end{array}\right\}
$$

Hence, if

$$
\begin{equation*}
x^{\prime}=U, \quad y^{\prime}=V, \quad u^{\prime}=\partial z / \partial U, \quad v^{\prime}=\partial z / \partial V, \quad \bar{u}^{\prime}=\partial \bar{z} / \partial U, \quad \bar{v}^{\prime}=\partial \bar{z} / \partial V, \tag{71}
\end{equation*}
$$

we have the relations

$$
\begin{array}{ll}
u^{\prime}=(T u-S v) /\left(R T-S^{2}\right), & v^{\prime}=(R v-S u) /\left(R T-S^{2}\right) \\
\bar{u}^{\prime}=\left(S p_{u}+T p_{v}\right) /\left(R T-S^{2}\right), & \bar{v}^{\prime}=-\left(R p_{u}+S p_{v}\right) /\left(R T-S^{2}\right)
\end{array}
$$

which, in conjunction with the preceding relations define two Bäcklund transformations. The transformations considered in my paper on the lift and drag functions are of this type [19] and are not generally contact transformations as is apparently implied by a statement relating to the correspondence of the supersonic regions in the two associated types of flow.

In the case in which $p$ is a function of $q$ only the relations between $u^{\prime}, v^{\prime}, u, v$ are

$$
u^{\prime}=-u / p, \quad v^{\prime}=-v / p, \quad q^{\prime}=q / p
$$

and, if $p^{\prime}=-1 / p, p^{\prime}+\rho^{\prime} q^{\prime 2}=-1 /\left(p+\rho q^{2}\right)$ we have

$$
\begin{gathered}
\rho^{\prime}=\frac{\rho p}{p+\rho q^{2}} \\
d p^{\prime} / d q^{\prime}=-q \rho /\left(p+\rho q^{2}\right)=-\rho^{\prime} q^{\prime} \\
1-q^{\prime 2} / c^{\prime 2}=\left(1-q^{2} / c^{2}\right)\left[p /\left(p+\rho q^{2}\right)\right]^{2}, \text { where } c^{\prime 2}=d p^{\prime} / d \rho^{\prime}
\end{gathered}
$$

This transformation may be compared with that obtained by means of Haar's adjoint variation problems [20]. In this case

$$
u^{*}=p_{u} /\left(p-u p_{u}-v p_{v}\right), \quad v^{*}=p_{v} /\left(p-u p_{u}-v p_{v}\right), \quad p^{*}=1 /\left(p-u p_{u}-v p_{v}\right),
$$

and, when $p$ depends only on $q$,

$$
\begin{array}{ll}
u^{*}=-\rho u /\left(p+\rho q^{2}\right), & q^{*}=-\rho v /\left(p+\rho q^{2}\right) \\
p^{*}=+1 /\left(p+q^{2}\right), & q^{*}=q /\left(p+q^{2}\right)
\end{array}
$$

Defining $\rho^{*}$ by the equation $d p^{*}=-q^{*} d q^{*}$, we have

$$
\begin{aligned}
\rho^{*} & =\left(p+\rho q^{2}\right) / p \rho, \quad p^{*}+q^{* 2}=1 / p \\
c^{* 2} & =\rho^{2} p^{2} q^{2}\left(q^{2}-c^{2}\right) /\left(p+\rho q^{2}\right)^{2}\left[c^{2}\left(p+\rho q^{2}\right)^{2}+p^{2}\left(q^{2}-c^{2}\right)\right] \\
c^{* 2}-q^{* 2} & =-\rho^{2} q^{2} c^{2} /\left[c^{2}\left(p+\rho q^{2}\right)^{2}+p^{2}\left(q^{2}-c^{2}\right)\right]
\end{aligned}
$$

Hence $c^{* 2}=q^{* 2}$ when $c^{2}=0$ and $c^{* 2}=0$ when $q^{2}=c^{2}$. It should be noticed, however, that

$$
c^{* 2}\left(c^{* 2}-q^{2}\right)=c^{2}\left(c^{2}-q^{2}\right) p^{2} \rho^{4} q^{4} /\left(p+\rho q^{2}\right)^{2}\left[c^{2}\left(p+\rho q^{2}\right)^{2}+p^{2}\left(q^{2}-c^{2}\right)\right]^{2}
$$

This may be compared with Haar's general relation*

$$
\left(p_{u_{v} * p^{*} p_{v v^{*}}^{*}}^{*}-p_{u^{*} v^{*}}^{* 2}\right)\left(p_{u u} p_{v v}-p_{u v}^{2}\right)=p^{-4} p^{*-4}
$$

Transformations more general than those of Bäcklund have been considered by Gau [21] but so far no hydrodynamical applications have been found for these so far as I know. Mention should be made, however, of the equiareal transformations from the Eulerian to Lagrangian co-ordinates in the two-dimensional flow of an incompressible fluid. These transformations have been much used in mapping but the hydrodynamical applications are beset with formidable difficulties.

No mention has been made of the use of transformations in the theory of surfaces, congruences, etc. This is a subject which is well treated in the books of Darboux [22], Forsyth [18], Goursat [16], Bianchi [23] and Eisenhart [24].
4. Transformation of the linear equation. In the special case in which $h, k$ and $l$ are functions of $x$ and $y$ only, $n=0$ and $m$ is a linear homogeneous function of $u, v$ and $z$ with coefficients depending only on $x$ and $y$, the Monge-Ampère equation reduces to a linear equation. The behavior of this equation in transformations of type

$$
\begin{equation*}
X=X(x, y), \quad Y=Y(x, y), \quad Z=z F(x, y) \tag{73}
\end{equation*}
$$

has been studied by Darboux [22], Cotton [25], Rivereau [26], J. E. Campbell after the case $F=1$ had been discussed by Laplace [27], Chini [28] and others [16]. Campbell uses the equation in a form in which $g=1$, a form to which the general equation can be reduced by multiplying it by a suitable factor. He then shows that there are two invariants $I, J$ and an absolute invariant $J / I$ where if suffixes denote partial derivatives

$$
\begin{align*}
I= & h a_{x}+k\left(a_{y}+b_{x}\right)+l b_{y}+\left(h_{x}+k_{y}\right) a+\left(k_{x}+l_{y}\right) b \\
& +h a^{2}+2 k a b+l b^{2}-m_{x}, \quad J=a_{y}-b_{x} \\
2 a= & l\left(h_{x}+k_{y}-m_{u}\right)-k\left(k_{x}+l_{y}-m_{x}\right)  \tag{74}\\
2 b= & h\left(k_{x}+l_{y}-m_{x}\right)-k\left(h_{x}+k_{y}-m_{u}\right) .
\end{align*}
$$

* This is a consequence of the relations

$$
\frac{\partial\left(u^{*}, v^{*}\right)}{\partial(u, v)}=p p^{* 3}\left(p_{u u} p_{v v}-p_{u v}\right), \quad \frac{\partial(u, v)}{\partial\left(u^{*}, v^{*}\right)}=p^{*} p^{3}\left(p_{v^{*} \cdot u^{*}}^{*} p_{v^{*} v^{*}}-p_{u^{*} \cdot v^{*}}^{*}\right) .
$$

In the correspondence between the two hodograph planes complications arise on account of the relation of $p_{w u} p_{u v}-p_{u v}^{2}$ to these Jacobians.

Laplace's invariants are $\frac{1}{2}(I-J), \frac{1}{2}(I+J)$. These are used with a different notation, in Darboux's Théorie des Surfaces, t.2. Campbell shows that in the case of the equation of Euler and Poisson the invariants $I$ and $J$ are constant. This may be compared with Cotton's result. The harmonic equations belong to the group characterized by the relation $J=0$. The equations considered by Burgatti [29] are such that $I=0$.

In mathematical physics the simple solutions of linear equations play an important part and the primary problem is that of separability. Even in the case of the equation with two independent variables there are some unsolved problems. A good idea of the progress which has been made may be derived from Darboux's book [22]. The method of Laplace provides an important way of reducing equations by a cascade process which is particularly useful in the treatment of equations arising in the theory of plane waves of finite amplitude. Reference may be made to a paper of Love and Pidduck [30], an article by Platrier [31], some papers by Bechert [32] and to two papers by Oseen [9] in which the transformation and reduction is given for equations occurring in the theory of earth pressure and in the theory of plasticity.

In the theory of the steady motion of an inviscid compressible fluid the equations in the hodograph plane are linear. These equations are

$$
\left.\begin{array}{ll}
p_{v v} w_{u u}-2 p_{u v} w_{u v}+p_{u u} w_{v v}=0, & p_{v v} \bar{w}_{u \bar{u}}-2 p_{u v} \bar{w}_{\overline{u v}}+p_{u u} \bar{w}_{\overline{\bar{v}}}=0, \\
z=u w_{u}+v w_{v}-w=q w_{q}-w, & \bar{z}=\bar{u} \overline{w_{\bar{u}}}+\bar{v} \bar{w}_{\bar{v}}-\bar{w}=\bar{q} \bar{w}_{q}-\bar{w} . \tag{75}
\end{array}\right\}
$$

When $p$ is a function of $q$ only the equations $\bar{u}=p_{v}, \dot{v}=-p_{u}$ take the form

$$
\begin{equation*}
\bar{u}=-\rho v=-\bar{q} \sin \tau, \quad \bar{v}=\rho u=\bar{q} \cos \tau, \quad \bar{q}=\rho q, \tag{76}
\end{equation*}
$$

and the equations become

$$
\begin{equation*}
w_{r \tau}+q\left(\bar{q} w_{q}\right)_{\bar{q}}=0, \quad \tilde{w}_{r r}+q\left(q \bar{w}_{\bar{q}}\right)_{q}=0 \quad(\bar{q} \text { a function of } q) . \tag{77}
\end{equation*}
$$

These are consequences of simple relations between $w$ and $\bar{w}$

$$
\begin{equation*}
\bar{w}_{r}+\bar{q} w_{\bar{q}}=0, \quad w_{r}-q \bar{w}_{\bar{q}}=0 . \tag{78}
\end{equation*}
$$

The corresponding relations between $z$ and $\bar{z}$ are found to be

$$
\begin{equation*}
q^{2} \bar{z}_{q}=\bar{q} \bar{q}_{\tau}, \quad \bar{q}^{2} z_{q}=-q \bar{z}_{\tau} \tag{79}
\end{equation*}
$$

and so the equations for $z$ and $\bar{z}$ are

$$
\left.\begin{array}{l}
\left.\left.\left(q^{2} / \bar{q}\right)\right]\left(\bar{q}^{2} / q\right) z_{\bar{q}}\right]_{q}+z_{r r}=0,  \tag{80}\\
\left(\bar{q}^{2} / q\right)\left[\left(q^{2} / \bar{q}\right) \bar{z}_{q}\right]_{\bar{q}}+z_{r r}=0 .
\end{array}\right\}
$$

These are equivalent to the equations obtained by Molenbroek [33] and Tschaplygin [34] for the case in which the relation between $p$ and $\rho$ is of the polytopic or adiabatic type. The symmetrical forms of the equations are easy to remember.

It is sometimes useful to introduce other quantities which satisfy linear relations. Thus we may obtain the desired relations between $z, \bar{z}, w, w$ by writing

$$
\begin{array}{ll}
w=\bar{e}_{T}=q e_{\bar{q}}, & \bar{w}=e_{T}=-\bar{q} e_{q}, \\
z=-\bar{e}_{r}-(q / \bar{q}) e_{T r}, & \bar{z}=(\bar{q} / q) e_{T r}-e_{r},
\end{array}
$$

where

$$
e_{T \tau}+\bar{q}\left(q e_{\bar{q}}\right)_{q}=0, \quad \bar{e}_{T \tau}+q\left(\bar{q} \bar{e}_{q}\right)_{q}=0 .
$$

The literature dealing with the transformation of linear equations in several variables is very extensive and only a brief summary can be attempted here. Beltrami's work on differential parameters [35] was extended by Ricci and Levi-Civita [36], Cotton [25], Levi-Civita [37] and many other writers. The development of general relativity, electrodynamics and the theory of elasticity has made this work more or less known. The work of Lamé on simple solutions of the potential equation [38] was much developed by later writers and a good summary of results up to 1893 is given in the book of Bôcher [39]. The use of a variational principle for obtaining the transformation of the equation was recommended by Larmor [40], Volterra and others [41]. Since the advent of the new quantum theory the interest in separable equations and separable systems has much increased. Mention may be made of the work of Staeckel [42], Eisenhart [43] and Robertson [44].

In addition to the simple solutions of partial differential equations there are solutions having the form of products in which one or more or the factors satisfies a partial differential equation instead of an ordinary differential equation. Comparatively little work has been done on this problem. In the case of Laplace's equation $V_{x x}+V_{y y}$ $+V_{32}=0$, the aim is to find a solution of form [45]

$$
V=Z F(X, Y), \quad \text { (generalized binary potential) }
$$

where $F$ satisfies a partial differential equation of the second order in the variables $X$ and $Y$. The problem seems to depend on the formation of a relation of type

$$
\left(p^{2}+q^{2}+r^{2}\right)\left(d x^{2}+d y^{2}+d z^{2}\right)-(p d x+q d y+r d z)^{2}=a d X^{2}+2 h d X d Y+b d Y^{2}
$$

in which $a, b$ and $h$ are functions of $X$ and $Y$ only. There is a similar relation for the corresponding problem in any number of variables.
5. The transformation of Legendre's equation. Legendre's equation

$$
\mathfrak{K}(R, S, T)=0
$$

is unaltered in form by a Legendre contact transformation

$$
X^{\prime}=U, \quad Y^{\prime}=V, \quad U^{\prime}=X, \quad V=Y^{\prime}, \quad Z^{\prime}=U X+V Y-Z,
$$

which makes

$$
R^{\prime}=T /\left(R T-S^{2}\right), \quad S^{\prime}=-S /\left(R T-S^{2}\right), \quad T^{\prime}=R /\left(R T-S^{2}\right)
$$

In particular, the equation

$$
R^{\prime} T^{\prime}-S^{\prime 2}=F\left(R^{\prime}+T^{\prime}\right)
$$

becomes

$$
1 /\left(R T-S^{2}\right)=F\left(\frac{R+T}{R T-S}\right)
$$

an equation of the same general type. Again, if $a, b$ and $h$ are constants the contact transformation

$$
\begin{aligned}
& Z^{\prime}=\frac{1}{2}\left(a X^{2}+2 h X Y+b Y^{2}\right)+Z, \quad X^{\prime}=X, \quad Y^{\prime}=Y, \\
& U^{\prime}=a X+h Y+U, \quad V^{\prime}=h X+b Y+V
\end{aligned}
$$

makes $R^{\prime}=a+R, S^{\prime}=h+S, T^{\prime}=b+T$ and so transforms an equation of Legendre's type into another equation of the same type. Equations of the preceding type usually go into Legendre equations of a slightly different type.

Other transformations may be found by first transforming the equation to the Monge-Ampère form by Legendre's device. If, for instance, the equation is

$$
T=F(R, S)
$$

and we differentiate with respect to $x$ using then the new notation $z=U, R=U_{x}=u$, $S=U_{y}=v, T_{x}=S_{y}=t, R_{x}=r, S_{x}=s$, the new equation is

$$
t=F_{u}(u, v) r+F_{v}(u, v) s .
$$

Comparing this with the equation $p_{u u} r+2 p_{u v} s+p_{v v} t=0$ we find that

$$
F_{u}=-p_{u u} / p_{v v}, \quad F_{v}=-2 p_{u v} / p_{v v} .
$$

Eliminating $F$ we find that the equation for $z$ is not a general equation of the type considered in $\$ 2$ because the function $p(u, v)$ satisfies the condition

$$
\partial D / \partial v=0 \text { where } D=p_{u u} p_{v v}-p_{u v}^{2} .
$$

An equation for which $D$ is constant satisfies this condition and the equation of type

$$
\overline{\mathfrak{K}}(R, S, T)=0
$$

associated with it by the analysis of section 2 may be regarded as a transform of the original equation $\mathscr{C}(R, S, T)=0$. In the case when $D=1$, we may write

$$
p_{u u}=e^{a} \sec b, \quad p_{v u}=e^{-a} \sec b, \quad p_{u v}=\tan b,
$$

where $a$ and $b$ are functions of $u$ and $v$ which must be chosen so that

$$
\begin{aligned}
& \sec ^{2} b b_{u}=e^{2} \sec b\left(a_{v}\right)+e^{a} \sec b \tan b\left(b_{v}\right), \\
& \sec ^{2} b b_{v}=-e^{-a} \sec b\left(a_{u}\right)+e^{-a} \sec b \tan b\left(b_{u}\right) .
\end{aligned}
$$

Also, since $F_{u}=-e^{2 a}, F_{v}=-2 e^{a} \sin v$, we must have the additional equation

$$
2 e^{a} \cos b\left(b_{u}\right)-2 e^{2 a}\left(a_{r}\right)+2 e^{a} \sin v\left(a_{u}\right)=0
$$

which is seen, however, to be a consequence of the other two. Elimination of the derivatives of $a$ gives the equation

$$
e^{a} b_{v v}+e^{-a} b_{u v}-2 \sin b b_{u v}=0 \quad \text { or } \quad p_{u u} b_{v v}+p_{v v} b_{u u}-2 p_{u v} b_{u v}=0
$$

and it is readily seen that $a$ satisfies the same equation. The equation $D=1$ is given as an example in Forsyth's book, p. 220, Ex. 11.

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## THE INTRINSIC THEORY OF THIN SHELLS AND PLATES*

## PART I.-GENERAL THEORY

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1. Introduction. The development of the theory of thin shells and plates by many authors $[1,2,3,4]$ can be summarized under the following three main heads:
(1) All theories are based upon certain simplifying unproved assumptions. For example: (a) the thickness remains unchanged during the deformation, (b) the middle surface in the unstrained state deforms into the middle surface in the strained state, (c) the normals of the unstrained middle surface deform into the normals of the strained middle surface.
(2) All theories involve the use of displacement to describe the state of deformation. This plan works well in the theory of small deflection, but presents considerable difficulty in the case of large deflection.
(3) The various approximations used in the theory of thin shells and plates are confusing. If one attempts to give a complete picture of the theory, one must be able to introduce a systematic method of approximation, which not only clears away the confusion of various approximations, but also leads to a complete classification of all thin shell and plate problems.

The purpose of this paper is to give a systematic treatment of the general problem of the thin shell, which includes the problem of the thin plate as a special case. The work is based on the usual equations of elasticity for a finite body, supposed to be homogeneous and isotropic. The final equations of Part I are the three equations of equilibrium, ( 6.34 ), ( 6.35 ), and the three equations of compatibility, (6.43), (6.44), for the six unknowns, $p_{\alpha \beta}, \boldsymbol{q}_{\alpha \beta}$, which represent extension and change of curvature of the middle surface. When these quantities are found, the strain and stress throughout the shell or plate can be calculated. The displacement does not appear explicitly in the argument. Since we deal rather with stress, strain and curvature (all tensors), the tensor notation proves much more convenient than any other.

In Part II and III, we shall discuss the various approximate forms of the equations arising from consideration of the thinness of the shell or plate and the smallness (or vanishing) of its curvature. The strain is, of course, always supposed to be small. We obtain a complete classification of all shell and plate problems. There are found to be twelve types of plate problems, and thirty-five types of shell problems. To each type, there corresponds a set of six equations which are simplifications of the equations (6.34), (6.35), (6.43), (6.44) of Part I, with certain terms dropped on account of smallness and the uncalculated residual terms omitted. The equations obtained include all the familiar equations in the field of small deflection and the few equations

[^1]already known for large deflection. The new results for finite deflection may prove particularly interesting.

The author is much indebted to Prof. J. L. Synge who not merely has directed this work but has actively participated in it.
2. Reduction of force system to reference surface; macroscopic equations of equilibrium. We shall start in this section by reviewing some main results in a previous paper [5].

All theories of thin shells and plates involve the use of a reference surface. Usually the middle surface is taken without explicit distinction between the middle surface in the unstrained state and the middle surface in the strained state. In the present methodical treatment we shall use a general reference surface in the material in sections $2-5$. In section 6 and later parts we shall use the surface in the strained state formed by the particles of the middle surface in the unstrained state. To the order of approximation used there, this is indistinguishable from the middle surface in the strained state. (This is generally assumed to be the case; cf. E. Reissner [4].)

The following coordinate system will be used: $x^{0}$ at any point $A$ inside the material of the shell or plate is the perpendicular distance of $A$ from the reference surface $S_{0}$, and $x^{\alpha}$ at $A$ are the values of any Gaussian coordinates on $S_{0}$ corresponding to the foot of the perpendicular dropped from $A$. (Throughout the paper, Latin indices have the range $0,1,2$ and Greek indices the range 1,2 ; summation over either of these ranges will be signified by the repetition of an index.) This may be called a normal space coordinate system with respect to $S_{0}$.

Let us denote the line element in space by $d s^{2}=g_{i j} d x^{i} d x^{j}$, where $g_{i j}$ is the fundamental tensor. Furthermore let $g_{[0] i} ;$ be the values of $g_{i j}$ at $S_{0}$, then we have in usual tensor notation

$$
\begin{align*}
g_{00} & =g_{[0] 00}=g^{00}=g_{[0]}^{00}=1, \quad g_{0 \alpha}=g_{[0] 0 \alpha}=g^{0 \beta}=g_{[0]}^{08}=0,  \tag{2.1}\\
g_{[0] \alpha \beta} & =a_{\alpha \beta}, \quad g_{[0]}^{\alpha \beta}=a^{\alpha \beta}, \quad \operatorname{det} .\left(g_{[0] i ;}\right)=g_{[0]}=\operatorname{det} .\left(a_{\alpha \beta}\right)=a, \tag{2.2}
\end{align*}
$$

where $a_{\alpha \beta} d x^{\alpha} d x^{\beta}$ is the metric on $S_{0}$.
We shall now consider forces in the shell or plate. Let $C$ be a curve on $S_{0}$ and $A_{0}$ a point on $C$. Let $n_{[0]}^{\alpha}$ be the unit vector in $S_{0}$ normal to $C$ at $A_{0}$, indicating the positive side; let $\Lambda_{[0]}^{\alpha}$ be an arbitrary unit vector in $S_{0}$ at $A_{0}$. We consider the system of forces acting across an element of area standing on the element $d s_{0}$ of $C$ at $A_{0}$, and terminated by the surfaces of the shell or plate. We replace the forces acting on the element by a statically equipollent system acting at $A_{0}$. This leads to the following invariants, which in fact define the macroscopic tensors, $T^{\alpha 0}$ (shearing stress tensor), $T^{\alpha \beta}$ (membrane stress tensor), $L^{\alpha \beta}$ (bending moment tensor):
$T^{\alpha 0} n_{001} d s_{0}=$ shearing force normal to $S_{0}$ across the element $d s_{0}$,
$T{ }^{\alpha \beta} n_{[0] \alpha} \Lambda_{[0 \mid \beta} d s_{0}=$ component in the direction of $\Lambda_{[0]}^{a}$ of membrane stress across the element $d s_{0}$,
$L^{\alpha \beta} n_{[0] \alpha} \Lambda_{[0] \beta} d s_{0}=$ component in the direction of $\Lambda_{[0]}^{\alpha}$ of the bending moment across the element $d s_{0}$.

Similarly, let $d S_{0}$ be an element of area of $S_{0}$ at $A_{0}$. Consider the external forces acting on the volume element consisting the normals to $S_{0}$ standing on the element
$d S_{0}$ and terminated by the surfaces of the shell or plate. We replace them by a statically equipollent system acting at $A_{0}$. This-leads to the following invariants, which give the definitions of the external force and external moment tensors, $F^{i}, M^{\alpha}$ :

$$
\begin{equation*}
F^{0} d S_{0}=\text { normal component of the external force on } d S_{0} \tag{2.4}
\end{equation*}
$$

$F^{\alpha} \Lambda_{[0] a} d S_{0}=$ component in the direction of $\Lambda_{[0]}^{\alpha}$ of the external force on $d S_{0}$,
$M^{\alpha} \Lambda_{[0] \alpha} d S_{0}=$ component in the direction of $\Lambda_{[0]}^{\alpha}$ of the external moment on $d S_{0}$.
Then from purely statical considerations, we obtain the following six equations of statical equilibrium ([5], p. 110):
(a)

$$
T^{\alpha 0}{ }_{\mid \alpha}-\frac{1}{2} b_{\alpha \beta} T^{\alpha \beta}+F^{0}=0
$$

(b)

$$
T^{\beta \alpha}{ }_{\mid \beta}+\frac{1}{2} a^{\alpha \beta} b_{\beta \gamma} T^{\gamma} 0+F^{\alpha}=0
$$

(c)

$$
\begin{equation*}
L^{\beta \alpha}{ }_{1 \beta}+a^{\alpha \beta} \eta[0] \beta \gamma T^{\gamma 0}+M^{\alpha}=0, \tag{2.5}
\end{equation*}
$$

$$
\begin{equation*}
\eta_{(0) \alpha \beta} T^{\alpha \beta}-\frac{1}{2} b_{\alpha \beta} L^{\alpha \beta}=0 \tag{d}
\end{equation*}
$$

The symbols have the following meanings:

$$
\begin{align*}
& T^{\alpha 0}{ }_{\mid \alpha}=T^{\alpha 0}, \alpha \\
&  \tag{2.6}\\
& T^{\alpha \beta}{ }_{1 \alpha}=T^{\alpha \beta}{ }_{, \alpha}+\left\{\begin{array}{c}
\alpha \\
\pi \alpha
\end{array}\right\}_{a}^{\alpha} T^{* 0}, \\
& b_{\alpha \beta}=\left(g_{\alpha \beta, 0}\right)_{a}{ }_{x}=0, \\
& T^{\pi \beta}+\left\{\begin{array}{c}
\beta \\
\pi \alpha
\end{array}\right\}_{a} T^{\alpha \pi}, \\
& \eta_{[0] \alpha \beta}=a^{1 / 2} \epsilon_{\alpha \beta}, \quad \epsilon_{11}=\epsilon_{22}=0, \quad \epsilon_{12}=-\epsilon_{21}=1 .
\end{align*}
$$

The Christoffel symbols are calculated for $a_{\alpha \beta}$. The quantities $(1 / 2) b_{\alpha \beta}$ are the coefficients of the second fundamental form of $S_{0}$; they vanish if $S_{0}$ is a plane. The radius of curvature $R$ in the direction of a unit vector $\mu_{[0]}^{\alpha}$ (counted positive when $S_{0}$ is convex in the sense of $x^{0}$ increasing) is given by

$$
\begin{equation*}
2 / R=b_{\alpha \beta} \mu_{[0]}^{\alpha} \mu_{[0]}^{\beta} . \tag{2.7}
\end{equation*}
$$

By eliminating $T^{\alpha 0}$ from (2.5) we get a set of three equations,

$$
\begin{gather*}
\left.T^{\pi \alpha}\right|_{\mid \pi}+(1 / 2) a^{\alpha \beta} b_{\beta \gamma} \eta_{[0]}^{\gamma \lambda} a_{\lambda \delta} L^{\pi \delta}{ }_{\mid \pi}+F^{\alpha}+\frac{1}{2} a^{\alpha \beta} b_{\beta \gamma} \eta_{[0]}^{\gamma \delta} a_{\delta \lambda} M^{\lambda}=0,  \tag{2.8a}\\
\eta_{[0]}^{\delta \pi} a_{\pi \gamma} L^{\lambda \gamma}{ }_{\mid \lambda \delta}-\frac{1}{2} b_{\pi \lambda} T^{\pi \lambda}+F^{0}+a_{\pi \lambda} \eta_{[0]}^{\gamma \pi} M_{\mid \gamma}^{\lambda}=0 . \tag{2.8b}
\end{gather*}
$$

These equations, rather than the equations (2.5), are fundamental in the later theory. It should be noted that in the case of repeated covariant differentiation with respect to $a_{\alpha \beta}$, the order of the operations cannot be changed unless the total curvature $K$ of the reference surface $S_{0}$ (cf. Eq. (3.13)) is equal to zero.

The above equations are valid for shells and plates of finite thickness. When we come to deal with approximations based on the smallness of certain quantities, we must of course consider only the magnitudes of dimensionless quantities. It is best therefore to work with dimensionless quantities throughout. Let us introduce a standard length $L$, a lateral dimension of the shell or plate (e.g. the diameter in the case of a circular plate). By $d s$ we shall understand the distance between two adjacent
points, divided by $L$; this dimensionless $d s$ may be called the reduced distance. Similarly all coordinates $x^{i}$ are supposed to be in reduced or dimensionless form. Consequently the fundamental tensor $g_{i j}$ is dimensionless, and all tensor operations (such as raising or lowering suffixes or covariant differentiation) are dimensionless operations. We also reduce stress to dimensionless form by dividing by Young's modulus $E$, and body forces by dividing by $E / L$.

All the relations written above hold equally well for reduced or dimensionless quantities. We shall carry through the work with these quantities; if we wish to translate conclusions into ordinary dimensional form, we have simply to multiply by that combination of the form $L^{m} E^{n}$ which restores the required dimensionality. Young's modulus will not appear explicitly in our work, since it is eliminated from the stressstrain relations by the process of reduction.

We also note that a thin shell or plate is defined as one whose thickness is small in comparison with a lateral dimension $L$. Customarily, a thin shell is defined as one whose thickness is small compared with the radius of curvature; this is unsatisfactory in the limiting case of a plate, and also in the case of a shell whose thickness is small in comparison with the radius of curvature but of the same order as the lateral dimension $L$.
3. Representation of $T^{a 0}, T^{\alpha \beta}, L^{\alpha \beta}, F^{i}, M^{\alpha}$ as power series in the thickness. Let $C$ be a curve on the reference surface $S_{0}$, and $\Sigma$ the surface formed by erecting normals to $S_{0}$ along $C$. Let $d s_{0}$ be an element of $C$, and $d \Sigma$ the strip formed by the normals on $d s_{0}$ (Fig. 1). Let $d s$ be the length of the arc of intersection of $d \Sigma$ and the surface $x^{0}=$ constant, passing through any point $A$ in $d \Sigma$. Let $n_{\alpha}$ be the unit vector normal to $d \Sigma$ at $A$, and $n_{[0] \alpha}$ the unit vector normal to $d \Sigma$ at the reference surface $S_{0}$.

Here we note that there are two distinct classes of quantities: (1) those defined only on the reference surface, such as $a_{\alpha \beta}, b_{\alpha \beta}, T^{\alpha 0}, T^{\alpha \beta}, L^{\alpha \beta}, M^{\alpha}$, $F^{i}, E_{[m]}^{[j]}$ (normal derivatives of the stress tensor
Fig. 1. on $S_{0}$ ); (2) those defined at any point in the material of the shell, such as $g_{i j}, E_{i j}$ (stress tensor), $\lambda_{i}$ (parallel vector field), $n_{\alpha}$. For all space tensors, the change of indices will be effected by applying $g_{i j}$ or $g^{i i}$; and for all surface tensors, the change of indices will be effected by applying $g_{[01 i ;}$ and $g_{101}^{i / j}$ [defined as in (2.1), (2.2)].

By the definition (2.3), the shearing stress tensor and the membrane stress tensor are calculated by the invariant formula:

$$
\begin{equation*}
T^{\alpha i} n_{[01 \alpha} \lambda_{(0] i} d s_{0}=\int_{h}\left(E^{\alpha i} n_{\alpha} \lambda_{i} d s\right) d x^{0} \tag{3.1}
\end{equation*}
$$

where $\lambda_{i}$ is any parallel field of unit vectors, $\lambda_{\text {[0] }}$ the same vector field at the reference surface $S_{0}$, and $E^{\alpha i}$ part of the stress tensor $E^{i j}$. The symbol $h$ under the sign of integration indicates here (and throughout the paper) that the integral runs from $x^{0}=-h_{(-)}$to $x^{0}=+h_{(+)}$, where both $h_{(+)}$and $h_{(-)}$are positive functions of $x^{\alpha}$ (for thin shells or plates, they are assumed to be small). Furthermore, the bending moment tensor is caculated from the invariant formula:

$$
\begin{equation*}
L^{\alpha \beta} n_{[0] \alpha} \lambda_{[0] \beta} d s_{0}=\int_{h}\left(g_{\pi \gamma} \eta^{\gamma \beta} E^{\alpha \pi} \lambda_{\beta} v_{\alpha} d s\right) x^{0} d x^{0} \tag{3.2}
\end{equation*}
$$

where

$$
\begin{array}{lc}
\eta^{\alpha \beta}=g^{-1 / 2} \epsilon^{\alpha \beta} & g=\operatorname{det}\left(g_{\alpha \beta}\right) \\
\epsilon^{11}=\epsilon^{22}=0, & \epsilon^{12}=-\epsilon^{21}=1 . \tag{3.3}
\end{array}
$$

For the external force system, assume that $X^{i} \lambda_{i}$ is the component of the body for per unit volume in the arbitrary direction of $\lambda_{i}$ at any point in the shell, and $Z_{(+)}^{\mathrm{t}} \lambda_{(+) i}$ and $Z_{(-)}^{t} \lambda_{(-) i}$ the components of the given loads per unit are applied to the upper and lower surfaces respectively. Let us consider (Fig. 2) a portion of the shell or plate obtained by drawing normals to its reference surface over a surface element $d S_{0}$. The portions of the boundary surfaces of the shell or plate cut out by these normals are $d \sigma_{(+)}$and $d \sigma_{(-)}$. Let $d S$ be the


Fig. 2. corresponding element drawn at constant normal distance from the reference surface $S_{0}$. Then the external force and moment components are calculated from the invariant formulae:

$$
\begin{align*}
F^{i} \lambda_{[0] i} d S_{0}= & \int_{h}\left(X^{i} \lambda_{i} d S\right) d x^{0}+Z_{(+)}^{i} \lambda_{(+) i} d \sigma_{(+)}+Z_{(-)}^{i} \lambda_{(-) i} d \sigma_{(-)}  \tag{3.4}\\
M_{\alpha} \lambda_{(0]}^{\alpha} d S_{0}= & \int_{h}\left(\eta_{\beta \alpha} \lambda^{\alpha} X^{\beta} d S\right) x^{0} d x^{0}+\eta_{(+) \beta \alpha} \lambda_{(+)}^{\alpha} Z_{(+)}^{\beta} h_{(+)} d \sigma_{(+)} \\
& -\eta_{(-) \beta \alpha} \lambda_{(-)}^{\alpha} Z_{(-)}^{\beta} h_{(-)} d \sigma_{(-)} \tag{3.5}
\end{align*}
$$

where $\eta_{(+) \alpha \beta}, \eta_{(-) \alpha \beta}$ are the values of $\eta_{\alpha \beta}$ at the upper and lower surfaces of the shell or plate respectively.

In order to carry out the integrations in (3.1)-(3.5), we must express all the quantities in the integrands as functions of $x^{0}$. These can be written as follows:

$$
\begin{align*}
g_{\alpha \beta} & =a_{\alpha \beta}+b_{\alpha \beta} x^{0}+\frac{1}{2} c_{\alpha \beta}\left(x^{0}\right)^{2},  \tag{3.6}\\
g & =a\left\{1+2 H x^{0}+K\left(x^{0}\right)^{2}\right\}^{2},  \tag{3.8}\\
n_{\alpha} \frac{d s}{d s_{0}} & =n_{[0] \alpha}(g / a)^{1 / 2}=n_{[01 \alpha}\left\{1+2 H x^{0}+K\left(x^{0}\right)^{2}\right\},  \tag{3.7}\\
\lambda_{0} & =\lambda_{[010}, \quad \lambda_{\alpha}=\lambda_{[0] \alpha}+\frac{1}{2} b^{\beta}{ }_{\alpha} \lambda_{[0] \beta} x^{0} . \tag{3.9}
\end{align*}
$$

The first relation is well known; $a_{\alpha \beta},(1 / 2) b_{\alpha \beta},(1 / 2) c_{\alpha \beta}$ are the first, second and third fundamental tensors of the reference surface $S_{0}$ respectively. These tensors are not independent, but connected by six geometrical conditions of flat space ([5], p. 112). Three of these read

$$
\begin{equation*}
c_{\alpha \beta}=\frac{1}{2} a^{\pi \lambda} b_{\alpha \approx} b_{\beta \lambda}, \tag{3.10}
\end{equation*}
$$

and the other three are the well known equations of Codazzi and Gauss,

$$
\begin{gather*}
b_{\alpha \beta \mid \gamma}-b_{\alpha \gamma \mid \beta}=0  \tag{3.11}\\
4 R_{\rho \alpha \beta \gamma}=b_{p, 3} b_{\alpha \gamma}-b_{\rho \gamma} b_{\alpha \beta} . \tag{3.12}
\end{gather*}
$$

Here the single stroke indicates covariant differentiation with respect to $x^{\alpha}$ and the tensor $a_{\alpha \beta} ; R_{\rho \alpha \beta \gamma}$ is the two dimensional curvature tensor formed from the tensor $a_{\alpha \beta}$ (sometimes called the Riemann Christoffel tensor of the surface $S_{0}$, see [6], p. 182, Eq. (50)).

The relation (3.7) can be obtained by direct calculation from the definition of $g$. Here $K$ is the total curvature and $H$ the mean curvature of the reference surface $S_{0}$. If $R_{1}, R_{2}$ are the principal radii of curvature, then

$$
\begin{equation*}
H=\frac{1}{2}\left(\frac{1}{R_{1}}+\frac{1}{R_{2}}\right), \quad K=\frac{1}{R_{1} R_{2}} ; \tag{3.13a}
\end{equation*}
$$

therefore, in tensor notation,

$$
\begin{equation*}
8 K=a^{\pi \gamma} b_{\pi \gamma} a^{\lambda \delta} b_{\lambda \delta}-b^{\pi \lambda} b_{\pi \lambda}, \quad 4 H=a^{\pi \lambda} b_{\pi \lambda} . \tag{3.13b}
\end{equation*}
$$

We also note the following relations, which are often used in later calculations,

$$
\begin{equation*}
\epsilon^{\alpha \pi} \epsilon^{\beta \lambda} a_{\pi \lambda}=a a^{\alpha \beta}, \tag{a}
\end{equation*}
$$

$$
\begin{align*}
& \epsilon^{\alpha \pi} \epsilon^{\beta \lambda} b_{\pi \lambda}=a\left(4 H a^{\alpha \beta}-b^{\alpha \beta}\right)  \tag{b}\\
& \epsilon^{\alpha \pi} \epsilon^{\beta \lambda} c_{\pi \lambda}=2 a\left\{\left(4 I^{2}-K\right) a^{\alpha \beta}-H b^{\alpha \beta}\right\} \tag{c}
\end{align*}
$$

The proof of the geometrical relation (3.8) is long, but not difficult. The relation (3.9) is obvious; since $\lambda_{i}$ is a parallel vector field, we have

$$
\begin{equation*}
\lambda_{i \| i}=0 . \tag{3.15}
\end{equation*}
$$

Here the double stroke indicates covariant differentiation with respect to the space coordinates $x^{i}$ and the tensor $g_{i j}$. Putting $i=0, i=\alpha, j=0$ in (3.15), the relation (3.9) follows at once.

Besides all the relations (3.6)-(3.9), we also need the following geometrical results:

$$
\begin{equation*}
\frac{d S}{d S_{0}}=\left(\frac{g}{a}\right)^{1 / 2}, \quad \frac{d \sigma_{(+)}}{d S_{0}}=\frac{1}{\left|N_{(+)}^{0}\right|}\left(\frac{g_{(+)}}{a}\right)^{1 / 2}, \quad \frac{d \sigma_{(-)}}{d S_{0}}=\frac{1}{\left|N_{(-)}^{0}\right|}\left(\frac{g_{(-)}}{a}\right)^{1 / 2}, \tag{3.16}
\end{equation*}
$$

the positive root being understood. Here $N_{(+) \text {i }}, N_{(-) i}$ are unit normal vectors, drawn out from the upper and lower boundary surfaces respectively, and $g_{(+)}, g_{(-)}$are the values of $g$ on the boundary surfaces.

Substituting (3.8), (3.9) in (3.1), we have

$$
\begin{align*}
n_{[0] \alpha} \lambda_{[0] \beta}\left\{T^{\alpha \beta}\right. & \left.-\int_{A} E^{\alpha \gamma}\left(a_{\gamma}^{\beta}+\frac{1}{2} b_{\gamma}^{\beta} x^{0}\right)\left(\frac{g}{a}\right)^{1 / 2} d x^{0}\right\} \\
& +n_{[0] \alpha} \lambda_{[0] 0}\left\{T^{\alpha 0}-\int_{A} E^{\alpha 0}\left(\frac{g}{a}\right)^{1 / 2} d x^{0}\right\}=0 \tag{3.17}
\end{align*}
$$

But $n_{[0 ; \alpha}$ and $\lambda_{[0] ;}$ are arbitrary, and consequently we have for the shearing stress tensor

$$
\begin{equation*}
T^{\alpha 0}=\int_{A} E^{\alpha 0}\left(\frac{g}{a}\right)^{1 / 2} d x^{0} \tag{3.18}
\end{equation*}
$$

and for the membrane stress tensor

$$
\begin{equation*}
T^{\alpha \beta}=\int_{\hbar} E^{\alpha \gamma}\left(a_{\gamma}^{\beta}+\frac{1}{2} b_{\gamma}^{\beta} x^{0}\right)\left(\frac{g}{a}\right)^{1 / 2} d x^{0} \tag{3.19}
\end{equation*}
$$

Similarly, substituting (3.6)-(3.9) in (3.2), we obtain the bending moment tensor

$$
\begin{equation*}
L^{\alpha \beta}=\eta_{[0]}^{\lambda \beta} a_{\pi \lambda} \int_{\star} E^{\alpha \gamma}\left(a_{\gamma}^{\pi}+\frac{1}{2} b_{\gamma}^{\pi} x^{0}\right) x^{0}\left(\frac{g}{a}\right)^{1 / 2} d x^{0} \tag{3.20}
\end{equation*}
$$

Furthermore, substituting (3.9), (3.16) in (3.3), (3.4), we have for the external force system

$$
\begin{align*}
F^{0}= & \int_{h} X^{0}\left(\frac{g}{a}\right)^{1 / 2} d x_{0}+\frac{Z_{(+)}^{0}}{\left|N_{(+)}^{0}\right|}\left(\frac{g_{(+)}}{a}\right)^{1 / 2}+\frac{Z_{(-)}^{0}}{\left|N_{(-)}^{0}\right|}\left(\frac{g_{(-)}}{a}\right)^{1 / 2},  \tag{3.21}\\
F^{\alpha}= & \int_{h} X^{\gamma}\left(a_{\gamma}^{\alpha}+\frac{1}{2} b_{\gamma}^{\alpha} x_{0}^{0}\right)\left(\frac{g}{a}\right)^{1 / 2} d x^{0}+\frac{Z_{(+)}^{\gamma}}{\left|N_{(+)}^{0}\right|}\left(a_{\gamma}^{\alpha}+\frac{1}{2} b_{\gamma}^{\alpha} h_{(+)}\right)\left(\frac{g_{(+1}}{a}\right)^{1 / 2} \\
& +\frac{Z_{(-)}^{\gamma}}{\left|N_{(-)}^{0}\right|}\left(a_{\gamma}^{\alpha}-\frac{1}{2} b_{\gamma}^{\alpha} h_{(-)}\right)\left(\frac{g_{(-)}}{a}\right)^{1 / 2}, \tag{3.22}
\end{align*}
$$

and for the external moment system

$$
\begin{align*}
M^{\alpha}= & \eta_{[01}^{\lambda \alpha} a_{\tau \lambda}\left\{\int_{A} X^{\gamma}\left(a_{\gamma}^{\tau}+\frac{1}{2} b_{\gamma}^{\tau} x^{0}\right)\left(\frac{g}{a}\right)^{1 / 2} x^{0} d x^{\top}\right. \\
& +\frac{Z \gamma_{+)}}{\left|N_{(+)}^{0}\right|}\left(a_{\gamma}^{\pi}+\frac{1}{2} b_{\gamma}^{\tau} h_{(+)}\right) h_{(+)}\left(\frac{g_{(+)}}{a}\right)^{1 / 2} \\
& -\frac{Z_{(-)}^{\gamma}}{\left|N_{(-)}^{0}\right|}\left(a_{\gamma}^{\tau}-\frac{1}{2} b_{\gamma}^{\tau} h_{(-)}\right) h_{(-)}\left(\frac{g_{(-)}}{a}\right)^{1 / 2} \tag{3.23}
\end{align*}
$$

It should be emphasized that the above expressions are exact, no approximation or assumption based on the thickness of the shell or plate being involved.

Now we assume that $E^{i j}$ and $X^{i}$ can be expanded in power series in $x^{0}$ :

$$
\begin{equation*}
E^{i j}=\sum_{0}^{\infty} \frac{1}{m!} E_{\{m]}^{i j}\left(x^{0}\right)^{m}, \quad X^{i}=\sum_{0}^{\infty} \frac{1}{m!} X_{[m]}^{i}\left(x^{0}\right)^{m} . \tag{3.24}
\end{equation*}
$$

We also introduce the abbreviations

$$
\begin{align*}
B_{\gamma}^{\beta} & =\frac{1}{2}\left(4 H a_{\gamma}^{\beta}+b_{\gamma}^{\beta}\right), \quad C_{\gamma}^{\beta}=K a_{\gamma}^{\beta}+H b_{\gamma}^{\beta}, \quad D_{\gamma}^{\beta}=\frac{1}{2} K b_{\gamma}^{\beta},  \tag{3.25}\\
d^{(n)} & =h_{(+)}^{n}-h_{(-)}^{n}, \quad t^{(n)}=h_{(+)}^{n}+h_{(-)}^{n}, \quad d^{(1)}=d, \quad t^{(1)}=t,  \tag{3.26}\\
Q^{(n) i} & =\frac{Z_{(+)}^{i}}{\left|N_{(+)}^{0}\right|} h_{(+)}^{n}-\frac{Z_{(-)}^{i}}{\left|N_{(-)}^{0}\right|} h_{(-)}^{n}, \quad Q^{(0) i}=Q^{i},  \tag{3.27}\\
P_{(n) i} & =\frac{Z_{(+)}^{i}}{\left|N_{(+)}^{0}\right|} h_{(+)}^{n}+\frac{Z_{(-)}^{i}}{\left|N_{(-)}^{0}\right|} h_{(-),}^{n}, \quad P^{(0) i}=P^{i} . \tag{3.28}
\end{align*}
$$

Here $H$ and $K$ are mean and total curvature, as in (3.13), and $t$ is the thickness of
the shell or plate, measured normally to $S_{0}$. We substitute (3.7), (3.24) into (3.18)(3.23), and carry out the integrations. This leads to series in $t, d^{(2)}, t^{(3)}, d^{(4)}, \cdots$; namely,

$$
\begin{align*}
& T^{\alpha 0}=E_{[01}^{\alpha 0} t+\left(2 H E_{[01}^{\alpha 0}+E_{[1]}^{\alpha 0}\right) \frac{d^{(2)}}{2!}+\left(2 K E_{[0]}^{\alpha 0}+4 H E_{(1)}^{\alpha 0}+E_{[2]}^{\alpha 0}\right) \frac{t^{(3)}}{3!} \\
& +R_{(1)}\left(E^{\alpha 0}\right) \text {, }  \tag{3.29}\\
& T^{\alpha \beta}=E_{[01}^{\alpha \beta} t+\left(B_{\gamma}^{\beta} E_{[0]}^{\alpha \gamma}+E_{[1]}^{\alpha \beta}\right) \frac{d^{(2)}}{2!}+\left(2 C_{\gamma}^{\beta} E_{[0]}^{\alpha \gamma}+2 B_{\gamma}^{\beta} E_{[1]}^{\alpha \gamma}+E_{[2]}^{\alpha \beta}\right) \frac{t^{(3)}}{3!} \\
& +R_{(2)}\left(E^{\alpha \beta}\right) \text {, }  \tag{3.30}\\
& L^{\alpha r}=\eta_{[0]}^{\lambda \alpha} a_{\beta \lambda}\left\{E_{[0]}^{\alpha \beta} \frac{d^{(2)}}{2!}+2\left(B_{\gamma}^{\beta} E_{[0]}^{\alpha \gamma}+E_{[1]}^{\alpha \beta}\right) \frac{t^{(3)}}{3!}+R_{(3)}\left(E^{\alpha \beta}\right)\right\},  \tag{3.31}\\
& F^{0}=X_{[0]}^{0} t+\left(2 H X_{[01}^{0}+X_{[1]}^{0}\right) \frac{d^{(2)}}{2!}+\left(2 K X_{[01}^{0}+4 H X_{[1]}^{0}+X_{[2]}^{0}\right) \frac{t^{(3)}}{3!} \\
& +P^{0}+2 H Q^{(1) 0}+K P^{(2) 0}+R_{(1)}\left(X^{0}\right),  \tag{3.32}\\
& F^{\alpha}=X_{[00}^{\alpha} t+\left(B_{\gamma}^{\alpha} X_{\{0\}}^{\gamma}+X_{[1]}^{\alpha}\right) \frac{d^{(2)}}{2!}+\left(2 C_{\gamma}^{\alpha} X_{[0]}^{\gamma}+2 B_{\gamma}^{\alpha} X_{[1]}^{\gamma}+X_{[2]}^{\alpha}\right) \frac{t^{(3)}}{3!} \\
& +P^{\alpha}+B_{\gamma}^{\alpha} Q^{(1) \gamma}+C_{\gamma}^{\alpha} P^{(2) \gamma}+D_{\gamma}^{\alpha} Q^{(3) \gamma}+R_{(2)}\left(X^{\alpha}\right),  \tag{3.33}\\
& M^{\alpha}=\eta_{[0]}^{\lambda \alpha} a_{\beta \lambda}\left\{X_{[0]}^{\beta} \frac{d^{(2)}}{2!}+2\left(B_{r}^{\beta} X_{[0]}^{\gamma}+X_{[11}^{\beta}\right) \frac{t^{(3)}}{3!}+R_{(3)}\left(X^{\beta}\right)\right. \\
& \left.+Q^{(1) \beta}+B_{\gamma}^{\beta} P^{(2) \gamma}+C_{\gamma}^{\beta} Q^{(3) \gamma}+D_{\gamma}^{\beta} P^{(4) \gamma}\right\} . \tag{3.34}
\end{align*}
$$

In these series the remainders are as follows:

$$
\begin{align*}
R_{(1)}\left(E^{\alpha 0}\right) & =\sum \frac{1}{m+5}\left\{\frac{E_{[m+4]}^{\alpha 0}}{(m+4)!}+\frac{2 H E_{[m+3]}^{\alpha 0}}{(m+3)!}+\frac{K E_{[m+2]}^{\alpha 0}}{(m+2)!}\right\} i^{(m+5)} \\
& +\sum \frac{1}{m+4}\left\{\frac{E_{[m+3]}^{\alpha 0}}{(m+3)!}+\frac{2 H E_{[m+2]}^{\alpha 0}}{(m+2)!}+\frac{K E_{[m+1]}^{\alpha 0}}{(m+1)!}\right\} d^{(m+4)},  \tag{3.35}\\
R_{(2)}\left(E^{\alpha \beta}\right) & =\sum \frac{1}{m+5}\left\{\frac{a_{\gamma}^{\beta} E_{[m+4]}^{\alpha \gamma}}{(m+4)!}+\frac{B_{\gamma}^{\beta} E_{[m+3]}^{\alpha \gamma}}{(m+3)!}+\frac{C_{\gamma}^{\beta} E_{[m+2]}^{\alpha \gamma}}{(m+2)!}+\frac{D_{\gamma}^{\beta} E_{[m+1]}^{\alpha \gamma}}{(m+1)!}\right\} t^{(m+5)} \\
& +\sum \frac{1}{m+4}\left\{\frac{a_{\gamma}^{\beta} E_{[m+3]}^{\alpha \gamma}}{(m+3)!}+\frac{B_{\gamma}^{\beta} E_{[m+2]}^{\alpha \gamma}}{(m+2)!}+\frac{C_{\gamma}^{\beta} E_{[m+1]}^{\alpha \gamma}}{(m+1)!}+\frac{D_{\gamma}^{\beta} E_{(m)}^{\alpha \gamma}}{m!}\right\} d^{(m+4)} \tag{3.36}
\end{align*}
$$

$R_{(3)}\left(E^{\alpha \beta}\right)=\sum \frac{1}{m+5}\left\{\frac{a_{\gamma}^{\beta} E_{[m+3]}^{\alpha \gamma}}{(m+3)!}+\frac{B_{\gamma}^{\beta} E_{[m+2]}^{\alpha \gamma}}{(m+2)!}+\frac{C_{\gamma}^{\beta} E_{[m+1]}^{\alpha \gamma}}{(m+1)!}+\frac{D_{\gamma}^{\beta} E_{[m]}^{\alpha \gamma}}{m!}\right\} t^{(m+5)}$

$$
\begin{align*}
& +\sum \frac{1}{m+4}\left\{\frac{a_{\gamma}^{\beta} E_{[m+2]}^{\alpha \gamma}}{(m+2)!}+\frac{B_{\gamma}^{\beta} E_{[m+1]}^{\alpha \gamma}}{(m+1)!}+\frac{C_{\gamma}^{\beta} E_{[m]}^{\tilde{\alpha} \gamma}}{m!}\right\} d^{(m+4)} \\
& +\sum \frac{1}{m+6} \frac{D_{\gamma}^{\beta} E_{[m+1]}^{\alpha \gamma}}{(m+1)!} d^{(m+6)} \tag{3.37}
\end{align*}
$$

These are accurate expressions, the summations being all for $m=0,2,4, \ldots$ The remainders $R_{(1)}\left(X^{0}\right), R_{(2)}\left(X^{\alpha}\right), R_{(3)}\left(X^{\beta}\right)$ are obtained simply by replacing $E_{[m]}^{\alpha 0}$ with $X_{[m]}^{0}$ in $R_{(1)}\left(E^{\alpha 0}\right), E_{[m]}^{\beta \gamma}$ with $X_{[m]}^{\gamma}$ in $R_{(2)}\left(E^{\beta \alpha}\right), E_{[m]}^{\alpha \gamma}$ with $X_{[m]}$ in $R_{(3)}\left(E^{\alpha \beta}\right)$ respectively.

In the ordinary case where the body force is the force of gravity, we can regard $X^{i}$ as a parallel vector field of constant magnitude. Then by (3.9), we have the following relations:

$$
\begin{gather*}
X_{[m]}^{0}=0  \tag{3.38}\\
\text { for } m \geqq 1 \\
X_{[1]}^{\alpha}=-\frac{1}{2} b_{\gamma}^{\alpha} X_{[0]}^{\gamma}, \quad X_{[m]}^{\alpha}=0 \text { for } m \geqq 2
\end{gather*}
$$

And consequently the tensor $F^{i}, M^{\alpha}$ in (3.32)-(3.34) can be simplified to the ( $l^{(n)}, d^{(n)}$ ) polynomial of few terms. The most important equations of this section, for future use, are (3.29)-(3.34)
4. The tensors $T^{a i}, L^{\alpha \beta}$ in terms of the six quantities $p_{, ~}, q_{\alpha \beta}$. We shall devote the present section to finding expressions for the tensors, (3.29)-(3.31), in terms of the six quantities $p_{\alpha \beta}$ and $q_{\alpha \beta}$. Here $p_{\alpha \beta}$ represents the extension of the reference surface $S_{0}$ and $q_{\alpha \beta}$ is closely connected to the change of curvature of the reference surface during the deformation; both were introduced in the paper [5] (p. 114, Eq. (44)), but will be formally defined below in (4.4).

We shall now proceed to determine $E_{[m],}^{[0]} E_{[m]}^{\alpha \beta}$ defined in (3.24) in terms of $p_{\alpha \beta}$ and $q_{\alpha \beta}$. This is accomplished by means of (i) the equations of microscopic equilibrium, (ii) the stress-strain relation, (iii) some geometrical relations, (iv) the conditions on the upper and lower boundary surfaces. The successive steps are as follows:
(I) By means of (i), we express $E_{[m]}^{i 0}$ successively for $m=1,2, \ldots$ in terms of $E_{[0]}^{i 0}$ and $E_{[n]}^{\alpha \beta}$, where $n=0,1,2, \cdots,(m-1)$.
(II) With the aid of (ii), (iii) and the results of step (I), we determine $E_{[0]}^{\alpha \beta}, E_{[1]}^{40}$, $E_{[1]}^{\alpha \beta}, E_{[2]}^{10}, E_{[2]}^{\alpha \beta}, \cdots$ successively in terms of $E_{[0]}^{00}, p_{\alpha \beta}, q_{\alpha \beta}$.
(III) Then using (iv) and the results in step (II), we determine $E_{[0 \mid}^{00}$, in terms of $p_{\alpha \beta}, q_{\alpha \beta}$. The surface force system $\left(P^{i}, Q^{i}\right)$ is supposed to be given. Thus we have at once $E_{[m]}^{00}$ and $E_{[m]}^{\alpha \beta}$ for all $m$ in terms of $p_{\alpha \beta}$ and $q_{\alpha \beta}$.
(IV) Substituting $E_{[m]}^{\alpha \beta}$ from step (III) into (3.30), (3.31), we obtain the required expressions for $T^{\alpha \beta}$ and $L^{\alpha \beta}$. The expression for $T^{\alpha 0}$ can be found either by substituting $E_{[m]}^{\infty 0}$ from step (III) into (3.29), or by using the equation of macroscopic equilibrium ( 2.5 c ).

The geometry of strain and the definitions of $p_{\alpha \beta}$ and $q_{\alpha \beta}$. Let us introduce comoving coordinates $[7,8]$. The same coordinates are attached to each particle during deformation, and the coordinates form a normal system in the strained state. The fundamental tensor in the strained state is $g_{i j}$ (satisfying (2.1), (2.2)), and in the unstrained state it is $g_{j}^{\prime}$. Let $S_{0}^{\prime}$ be the surface in the unstrained state which is carried over into the reference surface $S_{0}$ in the strained state (after we reach section 6 , we shall define $S_{0}^{\prime}$ to be the middle surface in the unstrained state). The parametric lines of $x^{0}$ are not in general normal to the reference surface $S_{0}^{\prime}$ in the unstrained state.

The strain tensor $e_{i j}$ is defined as

$$
\begin{equation*}
e_{i j}=\frac{1}{2}\left(g_{i j}-g_{i j}^{\prime}\right) ; \tag{4.1}
\end{equation*}
$$

this is the definition usually adopted (cf. $[7,8,9]$ ). For small deformation, the principal part of the extension of an element in the direction $\lambda^{i}$ is $e=e_{i j} \lambda^{i} \lambda^{j}$.

We shall throughout raise suffixes by means of $g^{i j}$; thus

$$
\begin{align*}
g^{\prime m n}=g^{m i} g^{n j} g_{i j}^{\prime}, & g_{m}^{\prime i}=g^{i j} g_{j m}^{\prime},  \tag{4.2a}\\
e^{m n}=g^{m i} g^{n i} e_{i j}, & e_{m}^{i}=g^{i j} e_{j m} . \tag{4.2b}
\end{align*}
$$

Now we assume that $e_{i j}$ is expansible in power series in $x^{0}$, so that

$$
\begin{equation*}
e_{i j}=\sum_{m=0}^{\infty} \frac{1}{m!} e_{[m] i j}\left(x^{0}\right)^{m} . \tag{4.3}
\end{equation*}
$$

Let us define

$$
\begin{equation*}
p_{i j}=e_{[0] i j}, \quad q_{i j}=e_{[1] i j}, \quad r_{i j}=e_{[2] i j} ; \tag{4.4}
\end{equation*}
$$

obviously these tensors are symmetric. The six components $p_{a \beta}, q_{\alpha \beta}$ are the basic dependent variables of our theory.

For small strain $p_{i j}$ must be small, yet $q_{i j_{1}} r_{i j}$, etc. may be finite in a thin shell or plate. These quantities are not independent but are connected by certain geometrical relations. Since space is flat, $g_{i j}$ and $g_{j j}^{\prime}$ are not arbitrary functions of the coordinates. They must satisfy the equations

$$
\begin{align*}
& \widehat{R}_{i j k l}=0  \tag{4.5a}\\
& \widehat{R}_{i j k l}^{\prime}=0 \tag{4.5b}
\end{align*}
$$

Here $\widehat{R}_{i j k l}$ is the curvature tensor for $g_{i j}$,

$$
\begin{align*}
\widehat{R}_{i j k l}= & \frac{1}{2}\left(g_{i l, j k}+g_{j k, i l}-g_{i k, j l}-g_{j l, i k}\right) \\
& +g^{m n}\left\{[i l, n l]_{\sigma}[j k, n]_{\vartheta}-[i k, m]_{\sigma}[j l, n]_{\sigma}\right\}, \tag{4.6}
\end{align*}
$$

while $\widehat{R}_{i j k l}$ is the corresponding curvature tensor for $g_{l j}^{\prime}$,

$$
\begin{align*}
\widehat{R}_{i j k l}^{\prime}= & \frac{1}{2}\left(g_{i l, j k}^{\prime}+g_{i k, i l}^{\prime}-g_{i k, i l}^{\prime}-g_{i l, i k}^{\prime}\right) \\
& +\bar{g}^{\prime m n}\left\{[i l, m]_{o^{\prime}}[j k, n]_{o^{\prime}}-[i k, m]_{o^{\prime}}[j l, n]_{o^{\prime}}\right\} . \tag{4.7}
\end{align*}
$$

It should be noted that $\widehat{R}_{i j k l}$ is the curvature tensor in space, while $R_{\rho \alpha \beta \gamma}$ is the two dimensional curvature tensor of $S_{0}$ (cf. (3.11)). In (4.7), $\dot{g}^{\prime_{m n}}$ by definition denotes the cofactor of $g_{m n}^{\prime}$ in $g^{\prime}$, divided by $g^{\prime}$; namely,

$$
\begin{equation*}
\tilde{g}^{\prime m n}=\frac{1}{2!g^{\prime}} \epsilon^{m r s} \epsilon^{n k l} g_{r k}^{\prime} g_{s l}^{\prime}, \tag{4.8}
\end{equation*}
$$

where $\epsilon^{m k l}$ is the usual permutation symbol, and $g^{\prime}$ is the determinant of $g_{i j}^{\prime}$

$$
\begin{equation*}
g^{\prime}=\frac{1}{3!} \epsilon^{r t t_{\epsilon}^{m n p}} g_{r m}^{\prime} g_{z n}^{\prime} g_{t p}^{\prime} \tag{4.9}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\tilde{g}^{\prime m m}=\frac{3!\eta^{m r t} \eta^{n k l} g_{r k}^{\prime} g_{t l}^{\prime}}{2!\eta^{i j b} \eta^{u v v}} g_{i u}^{\prime} g_{i v}^{\prime} g_{s w}^{\prime}, \tag{4.10}
\end{equation*}
$$

where

$$
\begin{equation*}
\eta^{r s t}=\epsilon^{r s t} g^{-1 / 2} \tag{4.11}
\end{equation*}
$$

this is a contravariant tensor and satisfies the relation

$$
\begin{equation*}
\eta^{r a t} \eta_{r l k}=\delta_{l}^{s} \delta_{k}^{t}-\delta_{k}^{s} \delta_{l}^{t} . \tag{4.12}
\end{equation*}
$$

The equations ( $4.5 \mathrm{a}, \mathrm{b}$ ) form a set of twelve independent geometrical conditions for $g_{i j}$ and $g_{i j}^{\prime}$. We shall now regard $g_{i j}$ as given, and develop (4.5b) explicitly in terms of $e_{i j}$ by using (4.5a), (4.1). The resulting expressions represent the six independent geometrical conditions for $e_{i j}$ :
$-\left(e_{i l, j k}+e_{j k, i l}-e_{i k, j l}-e_{j l, i k}\right)$
$+2 e^{m n}\left\{[i l, m]_{0}[j k, n]_{0}-[i k, m]_{0}[j l, n]_{0}\right\}$
$-2 g^{m n}\left\{[i l, m]_{0}[j k, n]_{\theta}+[i l, m]_{0}[j k, n]_{\theta}-[i k, m]_{0}[j l, n]_{0}-[i k, m]_{0}[j l, n]_{0}\right\}$
$+\Phi_{i j k l}^{(2)}+\Phi_{i j k l}^{(3)}+\Phi_{i j k l}^{(4)}=0$.
This is a polynomial in $e_{i j}$, with linear terms explicitly stated. The other terms proceed with increasing degree in $e_{i j}$, and have the following exact expressions:

$$
\begin{align*}
& \Phi_{i j k l}^{(2)}=2 e_{t}^{t}\left(e_{i l, j k}+e_{j k, i l}-e_{i k, j l}-e_{j l, i k}\right) \\
& +4 g^{m n}\left\{[i l, m]_{e}[j k, n]_{\theta}-[i k, m]_{e}[j l, n]_{0}\right\} \\
& +4\left(e^{n s} e_{s}^{m}-e^{m n} e_{s}^{z}\right)\left\{[i l, m]_{0}[j k, n]_{0}-[i k, m]_{\rho}[j l, n]_{0}\right\} \\
& +4\left(g^{m n} e_{s}^{*}-e^{m n}\right)\left\{[i l, m]_{0}[j k, n]_{\theta}+[i l, m]_{0}[j k, n]_{0}\right. \\
& \left.-[i k, m]_{0}[j l, n]_{0}-[i k, m]_{\theta}[j l, n]_{e}\right\},  \tag{4.13a}\\
& \dot{\Phi}_{i j k l}^{(3)}=-2\left(e_{s}^{s} e_{l}^{i}-e^{s t} e_{a}^{i}\right)\left(e_{i l, j k}+e_{j k, i l}-e_{i k, j l}-e_{j l, i k}\right) \\
& -\frac{2}{3} \eta^{s t p} \eta^{u v w} e_{s u} e_{t v} e_{p v}\left(g_{i l, j k}+g_{j k, i l}-g_{i k, j l}-g_{j l, i k}\right) \\
& -8\left(g^{m n} e_{s}^{*}-e^{m n}\right)\left\{[i l, m]_{0}[j k, n]_{e}-[i k, m]_{e}[j l, n]_{0}\right\} \\
& -4 \eta^{m r s} \eta^{n t p_{e}} e_{r l} e_{s p}\left\{[i l, m]_{0}[j k, n]_{0}-[i k, m]_{0}[j l, n]_{0}\right. \\
& \left.+[i l, m]_{0}[j k, n]_{s}-[i k, m]_{0}[j l, n]_{0}\right\},  \tag{4.13b}\\
& \Phi_{i j k l}^{(4)}=\frac{4}{8} \eta^{\Delta t p} \eta^{u \tau w} e_{u u} e_{t v} e_{p w}\left(\epsilon_{i l, j k}+e_{j k, i l}-e_{i k, j l}-e_{i l, i k}\right),  \tag{4.13c}\\
& {[i j, k]_{0}=\frac{1}{2}\left(g_{i k, j}+g_{j k, i}-g_{i j, k}\right),}  \tag{4.13~d}\\
& {[i j, k]_{0}=\frac{1}{2}\left(e_{i k, j}+e_{j k, i}-e_{i j, k}\right) .} \tag{4.13e}
\end{align*}
$$

Three of (4.13) involve only $e_{i j}, e_{i j, 0}$ and their derivatives with respect to $x^{\alpha}$. When $x^{0}=0$, these three conditions become the equations of compatibility of the twelve quantities $p_{i j}$ and $q_{i j}$; a detailed formulation of these equations will be given in section 5 .

The other three of (4.13). involve not only $e_{i j}, e_{i j, 0}$ and their derivatives with respect to $x^{\alpha}$, but also $e_{\alpha \beta, 00}$. In fact, by these equations, we express $e_{\alpha \beta, 00}$ in terms of $e_{i j}$, $e_{i j, 0}$ and their derivatives with respect to $x^{\alpha}$. When $x^{0}=0$, these equations give us the expression of $\gamma_{\alpha \beta}$ in terms of $p_{i j}, q_{i j}$ and their derivatives with respect to $x^{\alpha}$. Putting $i=\alpha, j=0, k=\beta, l=0$ and $x^{0}=0$ in (4.13), and solving for $r_{\alpha \beta}$, we obtain

$$
\begin{align*}
r_{\alpha \beta}= & e_{[2] \alpha \beta}=q_{\alpha 0 \mid \beta}+q_{\beta 0 \mid \alpha}-p_{00 \mid \alpha \beta}+\frac{1}{2}\left(b_{\alpha}^{\pi} a_{\beta}^{\lambda}+b_{\beta}^{\lambda} a_{\alpha}^{\pi}\right) q_{\lambda \pi}+\frac{1}{2} b_{\alpha \beta} q_{00} \\
& -\frac{1}{2} b_{\beta}^{\lambda} b_{\alpha}^{\pi} p_{\lambda \pi}+\frac{1}{2}\left(b_{\beta}^{\pi} a_{\alpha}^{\lambda}+b_{\alpha}^{\pi} a_{\beta}^{\lambda}-b_{\beta}^{\lambda} a_{\alpha}^{\pi}-b_{\alpha}^{\lambda} a_{\beta}^{\pi}\right) p_{=0 \mid \lambda} \\
& -q_{00} q_{\alpha \beta}-q_{\alpha}^{\pi} q_{\beta \pi}+O_{(2) \alpha \beta}\left(\dot{p}^{2}, \hat{p} \hat{q}\right) \tag{4.14}
\end{align*}
$$

where $O_{(2) \alpha \beta}$ is a residual term, to be explained below.
To find $e_{[3] \alpha \beta}$, we have to differentiate $e_{\alpha \beta, 00}$ with respect to $x^{0}$, and put $x^{0}=0$ in the resulting equations. By (4.14), this gives $e_{[3] \alpha \beta}$ in terms of $p_{i j}, q_{i j}, r_{i 0}$ and their derivatives with respect to $x^{\alpha}$. To find the other $e_{[m] \alpha \beta}$, we merely repeat the process over and over again. Thus we can express $e_{[m] \alpha \beta}$ in terms of $p_{i j}, q_{i j}, e_{[n] i 0}$ and their derivatives with respect to $x^{\alpha}$, where $n<m$.

Actually, to obtain the principal parts of the final equations, we only require explicit calculation for $m=2$, as in (4.14).

All the above results are of a purely geometrical nature.
The symbol $O_{(2)}\left(\hat{p}^{2}, p \hat{q}\right)$ in (4.14) represents the terms which are not explicitly calculated. The quantities in parentheses indicate order of magnitude of these terms for small $\hat{p}$ and $\hat{q}$, which denote the magnitudes of the tensors $p_{i j}, q_{i j}$ respectively. If $\dot{p}, \hat{q}$ approach zero simultaneously and independently, these terms converge to zero at least as fast as $\hat{p} \hat{q}$ or $\dot{p}^{2}$. Symbolically, we may write

$$
\begin{equation*}
O_{\alpha \beta}\left(\hat{p}^{2}, \hat{p} \hat{q}\right)=O_{\alpha \beta}\left(\dot{p}^{2}\right)+O_{\alpha \beta}(\hat{p} \hat{q}) \tag{4.14a}
\end{equation*}
$$

The label "(2)" is to distinguish this $O$-symbol from later symbols of the same type. The indices attached to $O_{(2)}$ are the tensorial indices of every terms involved. We assume throughout that differentiation with respect to the coordinates $x^{\alpha}$ does not change the order of magnitude; i.e., $p_{\alpha \beta}, p_{a \beta, \gamma}$ are of the same order. On the other hand, we never make any assumption regarding the effect of differentiation with respect to $x^{0}$.

This $O$-symbol notation will be used extensively throughout the paper. The notations used inside the parentheses of $O$-symbols are given in the following table:

| Tensors | $b_{\alpha \beta}$ | $b_{a \beta}$ | $e_{\alpha \beta}$ | $e_{i j}$ | $E^{i 0}$ | $E_{01}^{i 0}$ | $p_{i j}$ | $p_{\alpha \beta}$ | $p_{\alpha \beta}$ | $P^{\alpha}$ | $P^{i}$ | $q_{i j}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Symbols for magnitudes | $b$ | $b$ | $e$ | $\hat{e}$ | $\hat{E}$ | $E_{0}$ | $\hat{p}$ | $p$ | $p$ | $P$ | $\hat{P}$ |  |


| Tensors | $q_{\alpha \beta}$ | $q_{\alpha \beta}$ | $Q^{i}$ | $Q^{\alpha}$ | $X_{[\mathrm{m}]}^{\prime}$ | $X_{[m \mathrm{~m}}^{\alpha}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Symbols for magnitudes | $q$ | $q$ | $\hat{Q}$ | $Q$ | $X$ | $X$ |

The expressions of $E_{[m]}^{[0}$ in terms of $E_{[0]}^{00}$ and $E_{[n]}^{\alpha \beta}(n=0,1, \cdots,(m-1))$.
We start the process outlined in the beginning of this section by writing down the microscopic equations of equilibrium under the body force $X^{i}$ :

$$
\begin{equation*}
E^{k i}{ }_{\| k}+X^{i}=0 \tag{4.15}
\end{equation*}
$$

the double stroke indicating covariant differentiation using $g_{i j}$. Putting in turn $i=0$ and $i=\alpha$, we get three equations which may be written as

$$
\begin{align*}
& E^{00}{ }_{.0}=\frac{1}{2} g_{\pi \lambda, 0} E^{\pi \lambda}-\frac{1}{2} g^{\pi \lambda} g_{\pi \lambda, 0} E^{00}-\left.E^{\pi 0}\right|_{0}-X^{0}  \tag{4.16}\\
& E^{\alpha 0} .0=-\frac{1}{2} g^{\pi \lambda} g_{\pi \lambda, 0} E^{\alpha 0}-g^{\alpha \pi} g_{\pi \lambda, 0} E^{\lambda 0}-\left.E^{\alpha \pi}\right|_{0}-X^{\alpha} \tag{4.17}
\end{align*}
$$

where $g$ under the stroke indicates covariant differentiation using $g_{\alpha \beta}$,

$$
\begin{align*}
& E_{0}^{\star 0}{ }_{0}=E^{\star 0}, \pi+\left\{\begin{array}{c}
\pi \\
\lambda \pi
\end{array}\right\}_{\theta} E^{\lambda 0}  \tag{4.18}\\
& E^{\star \alpha}{ }_{0}{ }_{0}=E^{\star \alpha} \cdot \pi+\left\{\begin{array}{c}
\pi \\
\lambda \pi
\end{array}\right\}_{0} E^{\lambda \alpha}+\left\{\begin{array}{c}
\alpha \\
\lambda \pi
\end{array}\right\}_{0} E^{\lambda \pi} \tag{4.19}
\end{align*}
$$

This operation should be clearly distinguished from that indicated in (2.6). Putting $x^{0}=0$ in (4.16), (4.17), we get

$$
\begin{align*}
& E_{[1]}^{00}=-X_{[0]}^{0}-E_{[0] \mid x}^{00}+\frac{1}{2} b_{\tau \lambda} E_{[0]}^{x \lambda}-2 H E_{[0]}^{00},  \tag{4.19a}\\
& E_{[1]}^{a 0}=-X_{[0]}^{\alpha}-E_{[0] \mid x}^{\alpha a}-b_{\pi}^{\alpha} E_{[0]}^{00}-2 H E_{\{00}^{\alpha 0} . \tag{4.19b}
\end{align*}
$$

These are the expressions of $E_{[1]}^{[0}$ in terms of $E_{[0]}^{10}$ and $E_{[0]}^{\alpha \beta}$, the body forces being supposed given. To find $E_{[m]}^{[0]}$, we have to differentiate (4.16), (4.17) with respect to $x^{0}$ over and over again, and put $x^{0}=0$ in each of the resulting equations. We shall only write the expressions of $E_{[2]}^{10}$ explicitly:

$$
\begin{align*}
& E_{[2]}^{00}=-X_{[1]}^{0}+2 H X_{[0]}^{0}+X_{[01 \mid \pi}^{\tau}+E_{[0| | \pi \lambda}^{\pi \lambda}+\frac{1}{2} b_{\pi \lambda} E_{[1]}^{\pi \lambda}-a_{\pi \lambda} K E_{[0]}^{\pi \lambda} \\
& +\left(8 H^{2}-2 K\right) E_{[01}^{00}+b_{\lambda}^{\pi} E_{[01 \mid \pi}^{\lambda 0}+4\left(H E_{[0]}^{\lambda 0}\right)_{\mid \lambda,}  \tag{4.20a}\\
& E_{[2]}^{0 \alpha}=-X_{[1]}^{\alpha}+\left(b_{\pi}^{\alpha}+2 H a_{\pi}^{\alpha}\right) X_{[0 \mid}^{\pi}-a_{\pi}^{\alpha} E_{[1] \mid \lambda}^{\pi \lambda}+\left(b_{\pi}^{\alpha}+2 H a_{\pi}^{\alpha}\right) E_{[0| | \lambda}^{\pi \lambda} \\
& -\frac{1}{2}\left(4 H a_{\pi}^{\alpha}+b_{\pi}^{\alpha}\right)_{\{\lambda} E_{[0]}^{\pi \lambda}+\left(10 H b_{\approx}^{\alpha}+8 H^{2} a_{\pi}^{\alpha}-8 K a_{\pi}^{\alpha}\right) E_{[0]}^{\approx 0} . \tag{4.20b}
\end{align*}
$$

This completes Step (I).
Stress-strain relations and expressions for $E_{[m]}^{[0]}, E_{[m]}^{\alpha \beta}$ in terms of $p_{\alpha \beta}, q_{\alpha \beta}, E_{[0]}^{00}$.
We shall accept, as the stress-strain relation for an isotropic body, the equation

$$
\begin{equation*}
E^{i j}=\frac{1}{(1+\sigma)(1-2 \sigma)}\left\{\sigma g^{i j} g^{k l}+(1-\sigma) g^{i k} g^{i l}\right\} e_{k l}+O_{(\sigma)}^{i j}\left(\hat{e}^{2}\right), \tag{4.21}
\end{equation*}
$$

where $\sigma$ is Poisson's ratio, and $E^{i j}$ is of course the reduced stress (see section 2); therefore Young's modulus does not appear. For small strain problems, $\hat{e}$ is small, and the terms in $O^{i i}\left(\hat{e}^{2}\right)$ are negligible in comparison with the terms linear in $e^{i j}$ in (4.21); in that case, (4.21) becomes the usual linear stress-strain relation for an isotropic body (in rectangular Cartesians, see [1], p. 102, Eq. (18)). Any modification, such as replacement of $g^{i j}$ by $g^{\prime i j}$, leads to no real change, because the difference is taken care of by the $O$-symbol.

From (4.21), we have

$$
\begin{align*}
& E^{00}=\frac{1}{(1+\sigma)(1-2 \sigma)}\left\{\sigma g^{\lambda \lambda} e_{\lambda \pi}+(1-\sigma) e_{00}\right\}+O_{(3)}^{00}\left(\hat{e}^{2}\right),  \tag{4.22}\\
& E^{\alpha \beta}=\frac{1}{(1+\sigma)(1-2 \sigma)}\left\{\sigma g^{\alpha \beta} \varepsilon_{00}+\left[\sigma g^{\alpha \beta} g^{\pi \lambda}+(1-2 \sigma) g^{\alpha \pi g^{g \lambda \lambda}} e_{\pi \lambda}\right\}+O_{(3)}^{\alpha \beta}\left(\hat{e}^{2}\right),\right.  \tag{4.23}\\
& E^{\alpha 0}=\frac{1}{(1+\sigma)} g^{\alpha x} e_{\pi 0}+O_{(3)}^{\alpha 0}\left(\hat{e}^{2}\right) . \tag{4.24}
\end{align*}
$$

It is evident from (4.21) that all the stress components are small, of at least as high an order as the largest strain components. The elimination of $e_{0 i}$ from the last three equations gives

$$
\begin{equation*}
E^{\alpha \beta}=\frac{\sigma}{1-\sigma} g^{\alpha \beta} E^{00}+\frac{1}{1-\sigma^{2}}\left\{\sigma g^{\alpha \beta} E^{\pi \lambda}+(1-\sigma) g^{\alpha \pi} g^{\beta \lambda}\right\} e_{\pi \lambda}+O_{(4)}^{\alpha \beta}\left(e^{2}, \widehat{E}^{2}, \widehat{E} e\right) . \tag{4.25}
\end{equation*}
$$

When $x^{0}=0$, the equation (4.25) becomes

$$
\begin{equation*}
E_{[0]}^{\alpha \beta}=\frac{\sigma}{1-\sigma} a^{\alpha \beta} E_{[0]}^{00}+A_{(1)}^{\alpha \beta \lambda \pi} p_{\pi \lambda}+O_{(4)}^{\alpha \beta}\left(p^{2}, \widehat{E}_{0}^{2}, \widehat{E}_{0} p\right), \tag{4.26}
\end{equation*}
$$

where the abbreviation $A_{(1)}^{\alpha \beta \pi \lambda}$ means

$$
\begin{equation*}
A_{(1)}^{\alpha \beta \pi \lambda}=\frac{1}{1-\sigma^{2}}\left\{\sigma a^{\alpha \beta} a^{\pi \lambda}+(1-\sigma) a^{\alpha \pi} a^{\lambda \beta}\right\} \tag{4.27}
\end{equation*}
$$

Equation (4.26) is the required expression for $E_{[0]}^{\alpha 8}$.
The substitution of $E_{[0]}^{\alpha \beta}$ from (4.26) into (4.19a, b) gives

$$
\begin{align*}
E_{[1]}^{00}= & -X_{[0]}^{0}-E_{[0] \mid \pi}^{\pi 0}-\frac{2(1-2 \sigma)}{(1-\sigma)} H E_{[0]}^{00}+\frac{1}{2} A_{(1)}^{\delta \gamma \pi \lambda} b_{\delta \gamma} p_{\pi \lambda} \\
& +O_{(5)}^{00}\left(b p^{2}, b \widehat{E}_{0} p, b \widehat{E}_{0}^{2}\right)  \tag{4.28}\\
E_{[1]}^{\alpha 0}= & -X_{[0]}^{\alpha}-b_{\pi}^{\alpha} E_{[0]}^{\pi 0}-2 H E_{[0]}^{\alpha 0}-\frac{\sigma}{1-\sigma} a^{\alpha \pi} E_{[0] \mid \pi}^{00}-A_{(1)}^{\lambda \alpha \pi \delta} p_{\pi \delta \mid \lambda} \\
& +O_{(6)}^{\alpha 0}\left(p^{2}, p \widehat{E}_{0}, \widehat{E}_{0}^{2}\right) \tag{4.29}
\end{align*}
$$

Here $X_{[0]}^{1}$ are supposed to be given. These are the required expressions of $E_{[1]}^{10}$.
Now, differentiating (4.25) with respect to $x^{0}$ and putting $x^{0}=0$ in the resulting equations, we obtain in consequence of $(4.26),(4.28)$

$$
\begin{align*}
E_{[11]}^{\alpha \beta}= & -\frac{\sigma}{1-\sigma} a^{\alpha \lambda} X_{[0]}^{0}-\frac{\sigma}{1-\sigma} a^{\alpha \beta} E_{[011 \pi}^{r 0}-\frac{\sigma}{1-\sigma}\left\{b^{\alpha \beta}+\frac{2(1-2 \sigma)}{(1-\sigma)} H a^{\alpha \beta}\right\} E_{[0]}^{00} \\
& +\left\{\frac{\sigma}{2(1-\sigma)} a^{\alpha \beta} A_{(1)}^{\lambda \delta \pi \gamma}-a^{\beta \delta} A_{(1)}^{\alpha \lambda \pi \gamma}-a^{\alpha \beta} A_{(1)}^{\alpha \beta \delta \gamma}\right\} b_{\lambda \delta} p_{\pi \gamma} \\
& +A_{(1)}^{\alpha \beta \pi \lambda} q_{\pi \lambda}+O_{(7)}^{\alpha \beta}, \tag{4.30}
\end{align*}
$$

where

$$
\begin{equation*}
O_{(7)}^{\alpha \beta}=O_{(7)}^{\alpha \beta}\left(b p^{2}, b p \widehat{E}_{0}, b \widehat{E}_{0, p q}^{2} p q, q \widehat{E}_{0}, p \widehat{X}, \widehat{X} \widehat{E}_{0}\right) \tag{4.31}
\end{equation*}
$$

Equation (4.30) is the required expression of $E_{[1]}^{\alpha \beta}$.
The substitution of $E_{[0]}^{\alpha \beta}, E_{[1]}^{\alpha \beta}$ from (4.26), (4.30) into (4.20a, b) gives the required expressions for $E_{[2]}^{\ell 0}$ :

$$
\begin{align*}
& E_{[2]}^{00}=\frac{2(1-2 \sigma)}{(1-\sigma)} H X_{[0]}^{0}-X_{[1]}^{0}+X_{[0] \mid x}^{\pi} \\
& +\frac{2(1-2 \sigma)}{(1-\sigma)^{2}}\left\{2(2-3 \sigma) \Pi^{2}-(1-\sigma) K\right\} E_{[0]}^{00} \\
& +\frac{\sigma}{1-\sigma} a^{\pi \lambda} E_{[0] \mid \times \lambda}^{00}+\frac{2(2-3 \sigma)}{(1-\sigma)} H E_{[0] \mid \pi}^{\pi 0}+\left(b_{\pi}^{\lambda} E_{[0]}^{x 0}\right)_{\mid \lambda}+A_{(1)}^{\lambda \delta \pi \gamma} p_{\pi \gamma \mid \lambda \delta} \\
& -\frac{1}{1-\sigma^{2}}\left\{(4-3 \sigma) H b^{\pi \lambda}+\frac{4 \sigma(2-3 \sigma)}{1-\sigma} H^{2} a^{\star \lambda}-(3-\sigma) K a^{\pi \lambda}\right\} p_{\pi \lambda} \\
& +\frac{1}{2\left(1-\sigma^{2}\right)}\left\{4 \sigma H a^{\pi \lambda}+(1-\sigma) b^{\pi \lambda}\right\} q_{\pi \lambda}+O_{(8)}^{00} .  \tag{4.32}\\
& E_{(2)}^{\alpha 0}=\left(b_{\pi}^{\alpha}+2 H a_{\pi}^{\alpha}\right) X_{[01}^{\pi}-X_{[1]}^{\alpha}+\frac{\sigma}{1-\sigma} a^{\alpha x} X_{[0] \mid x}^{0} \\
& +\frac{\sigma}{1-\sigma} a^{\alpha \lambda} E_{[0] \mid \pi \lambda}^{r 0}+\frac{2 \sigma}{1-\sigma} b^{\alpha x} E_{[011 \pi}^{00} \\
& +\frac{2 \sigma(2-3 \sigma)}{(1-\sigma)^{2}} a^{\alpha \pi}\left(H E_{[0]}^{00}\right)_{1 \pi}+\left(10 H b_{\pi}^{\alpha}+8 H^{2} a_{\pi}^{\alpha}-8 K a_{\pi}^{\alpha}\right) E_{[0]}^{\pi 0}-A_{(1)}^{\alpha \delta \pi \lambda} q_{\pi \lambda 1 \delta} \\
& +\frac{1}{1-\sigma^{2}}\left\{\frac{1}{2} \sigma b^{\pi \lambda} \left\lvert\, \delta a^{\alpha \delta}+\frac{1-\sigma}{2} b^{\alpha \lambda}{ }_{1 \delta} a^{\delta \pi}\right.\right. \\
& \left.-\frac{2 \sigma^{2}}{1-\sigma} H_{1 \delta} a^{\alpha \delta} a^{\pi \lambda}+3(1-\sigma) H_{1 \delta} a^{\alpha \pi} a^{\delta \lambda}\right\} p_{\pi \lambda} \\
& +\frac{1}{1-\sigma^{2}}\left\{\sigma\left(2 b^{\alpha \delta} a^{\pi \lambda}+\frac{1}{2} b^{\pi \lambda} a^{\alpha \delta}\right)+\frac{2 \sigma(1-2 \sigma)}{(1-\sigma)} H a^{\alpha \delta} a^{\pi \lambda}\right. \\
& \left.+(1-\sigma)\left(2 b^{\alpha \pi} a^{\delta \lambda}+a^{\alpha \pi} b^{\delta \lambda}+2 H a^{\alpha \pi} a^{\delta \lambda}\right)\right\} p_{\pi \lambda \mid \delta}+O_{(9)}^{\alpha 0} . \tag{4.33}
\end{align*}
$$

Here

$$
\begin{align*}
& O_{(8)}^{00}=O_{(8)}^{00}\left(p^{2}, p \widehat{E}_{0}, \widehat{E}_{0}^{2}, b p q, b q \widehat{E}_{0}, b p \widehat{X}, b \widehat{X} \widehat{E}_{0}\right)  \tag{4.34}\\
& O_{(9)}^{\alpha 0}=O_{(9)}^{\alpha 0}\left(b p^{2}, b p \widehat{E}_{0}, b \widehat{E}_{0}^{2}, p q, q \widehat{E}_{0}, p \widehat{X}, \widehat{X} \widehat{E}_{0}\right) \tag{4.35}
\end{align*}
$$

To find $E_{[2]}^{\alpha \beta}$, we have to determine $p_{i 0}, q_{i 0}$ and then $r_{\sigma \beta}$ in terms of $p_{\alpha \beta}, q_{\alpha \beta}, E_{[0]}^{i 0}$. Eliminating $e_{00}$ from (4.22), (4.24) and solving the resulting expression for $e_{\alpha 0}$, we obtain

$$
\begin{equation*}
e_{\alpha 0}=(1+\sigma) g_{\alpha \beta} E^{0 \beta}+O_{(10) \alpha}\left(e^{2}, e \widehat{E}, \widehat{E}^{2}\right) \tag{4.36}
\end{equation*}
$$

when $x^{0}=0$, this becomes

$$
\begin{equation*}
p_{\alpha 0}=(1+\sigma) a_{a \beta} E_{[0]}^{\beta 0}+O_{(10) \alpha}\left(p^{2}, p \widehat{E}_{0}, \widehat{E}_{0}^{2}\right) \tag{4.37}
\end{equation*}
$$

We now differentiate (4.36) with respect to $x^{0}$ and put $x^{0}=0$. In consequence of (4.26), (4.28), (4.29), this gives

$$
\begin{align*}
q_{\alpha 0}= & -\frac{1+\sigma}{1-\sigma}\left\{(1-\sigma) X_{[0] \alpha}+\sigma E_{[0] \mid \alpha}^{00}+2(1-\sigma) H a_{\alpha \beta} E_{[0]}^{\beta 0}\right\}-\frac{\sigma}{1-\sigma} a^{\pi \lambda} p_{\pi \lambda \mid \alpha} \\
& -a^{\lambda \pi} p_{\alpha \pi \mid \lambda}+O_{(11) \alpha 0}\left(p^{2}, p \widehat{E}_{0}, \widehat{E}_{\left.0, p q, q \widehat{E}_{0}, p \widehat{X}, \widehat{X} \widehat{E}_{0}\right)} .\right. \tag{4.38}
\end{align*}
$$

Similarly, from (4.22), the values of $e_{00}$ and $e_{00,0}$ on $x^{0}=0$ are, by (4.28), (4.29)

$$
\begin{align*}
p_{00}= & \frac{(1+\sigma)(1-2 \sigma)}{(1-\sigma)} E_{[0]}^{00}-\frac{\sigma}{1-\sigma} a^{\pi \lambda} p_{\pi \lambda}+O_{(12) 00}\left(p^{2}, p \widehat{E}_{0}, \widehat{E}_{0}^{2}\right)  \tag{4.39}\\
q_{00}= & -\frac{(1+\sigma)(1-2 \sigma)}{(1-\sigma)}\left\{X_{[0]}^{0}+E_{[0] \mid \pi}^{\pi 0}+\frac{2(1-2 \sigma)}{1-\sigma} H E_{[0]}^{00}\right\}-\frac{\sigma}{1-\sigma} a^{\lambda \pi} q_{\pi \lambda} \\
& +\frac{1}{2(1-\sigma)}\left\{4 \sigma(1-2 \sigma) H a^{\pi \lambda}+(1-\sigma) b^{\pi \lambda}\right\} p_{\pi \lambda}+O_{(13) 00} \tag{4.40}
\end{align*}
$$

where

$$
\begin{equation*}
O_{(13) 00}=O_{(13) 00}\left(b p^{2}, b p \widehat{E}_{0}, b \widehat{E}_{0}^{2}, p q, q \widehat{E}_{0}, p \widehat{X}, \widehat{X} \widehat{E}_{0}\right) \tag{4.41}
\end{equation*}
$$

Substituting $p_{i 0}, q_{i 0}$ from (4.37)-(4.40) into (4.41), we obtain

$$
\begin{equation*}
r_{\alpha \beta}=r_{\alpha \beta}^{(1)}(q)+r_{\alpha \beta}^{(2)}(p)+r_{\alpha \beta}^{(3)}(X)+r_{\alpha \beta}^{(4)}\left(\widehat{E}_{0}\right)+r_{\alpha \beta}^{(5)}\left(q^{2}\right)+O_{(14) \alpha \beta}, \tag{4.42}
\end{equation*}
$$

where the abbreviations represent

$$
\begin{align*}
& r_{\alpha \beta}^{(1)}(q)=\frac{1}{2}\left(b_{\alpha}^{\pi} a_{\beta}^{\lambda}+b_{\beta}^{\lambda} a_{\alpha}^{\pi}-\frac{\sigma}{1-\sigma} a^{\lambda \tau} b_{\alpha \beta}\right) q_{\lambda \pi},  \tag{4.42a}\\
& r_{\alpha \beta}^{(2)}(p)=-\left(a^{\pi \lambda} a_{\alpha}^{\gamma} a_{\beta}^{\delta}+a^{\pi \lambda} a_{\beta}^{\gamma} a_{\alpha}^{\delta}+\frac{\sigma}{1-\sigma} a^{\tau \gamma} a_{\alpha}^{\lambda} a_{\beta}^{\delta}\right) p_{\gamma \pi \mid \lambda \delta} \\
& +\frac{1}{4(1-\sigma)}\left\{\frac{4 \sigma(1-2 \sigma)}{(1-\sigma)} H b_{\alpha \beta} a^{\pi \lambda}+b_{\alpha \beta} b^{\lambda \pi}-2(1-\sigma) b_{\alpha}^{\lambda} b_{\beta}^{\pi}\right\} p_{\lambda \pi},  \tag{4.42b}\\
& r_{\alpha \beta}^{(3)}(\hat{X})=-\frac{(1+\sigma)}{2(1-\sigma)}\left\{2(1-\sigma)\left(X_{(0] \alpha \mid \beta}+X_{[0|\beta| \alpha}\right)+(1-2 \sigma) b_{\alpha \beta} X_{\{0 \mid}^{0}\right\},  \tag{4.42c}\\
& r_{\alpha \beta}^{(4)}\left(\widehat{E}_{0}\right)=-\frac{1+\sigma}{1-\sigma}\left\{E_{[0| | \alpha \beta}^{00}+\frac{(1-2 \sigma)^{2}}{(1-\sigma)} H b_{\alpha \beta} E_{[01}^{00}\right\} \\
& -2(1+\sigma)\left(a_{\alpha \pi} a_{\beta}^{\lambda}+a_{\beta \pi} a_{\alpha}^{\lambda}\right)\left(H E_{[01}^{\pi 0}\right)_{\mid \lambda}-\frac{1+\sigma}{2}\left\{b_{\beta}^{\pi} a_{\alpha \lambda}+b_{\alpha}^{\pi} a_{\beta \lambda}\right. \\
& \left.-b_{\beta \lambda} a_{\alpha}^{\pi}-b_{\alpha \lambda} a_{\beta}^{\pi}+\frac{(1-2 \sigma)}{1-\sigma} b_{\alpha \beta} a_{\lambda}^{\pi}\right\} E_{[0] \mid \pi}^{\lambda 0},  \tag{4.42~d}\\
& r_{\alpha \beta}^{(5)}\left(q^{2}\right)=\left\{\frac{\sigma}{1-\sigma} a^{\gamma \pi} a_{\alpha}^{\lambda} a_{\beta}^{\delta}-a_{\alpha}^{\lambda} a^{\pi \delta} a_{\beta}^{\gamma}\right\} q_{\lambda \delta} q_{\gamma \pi},  \tag{4.42e}\\
& O_{(14) \alpha \beta}=O_{(14) \alpha \beta}\left(p^{2}, p \widehat{E}_{0}, \widehat{E}_{0}^{2}, p q, q \widehat{E}_{0}, p \widehat{X}, \widehat{X} \widehat{E}_{0}, q \widehat{X}\right) \text {. } \tag{4.42f}
\end{align*}
$$

It will be noted that $r_{\alpha \beta}^{(1)}(q)$ is linear in the $q$ 's, $r_{\alpha \beta}^{(2)}(p)$ is linear in the $p$ 's, and so on.
We now differentiate (4.25) with respect to $x^{0}$ twice, and put $x^{0}=0$ in the resulting equation. In consequence of (4.28), (4.32), (4.42), this gives

$$
\begin{equation*}
E_{[2]}^{\alpha \beta}=E_{[2]}^{(1) \alpha \beta}(q)+E_{[2]}^{(2) \alpha \beta}(p)+E_{[2]}^{(3) \alpha \beta}(\widehat{X})+E_{[2]}^{(4) \alpha \beta}\left(\widehat{E}_{0}\right)+\cdot E_{[2]}^{(5) \alpha \beta}\left(q^{2}\right)+O_{(15)}^{\alpha \beta}, \tag{4.43}
\end{equation*}
$$

where

$$
\begin{equation*}
O_{(15)}^{\alpha \beta}=O_{(16)}^{\alpha \beta}\left(p^{2}, p \widehat{E}_{0}, \widehat{E}_{0}^{2}, p q, q \widehat{E}_{0}, p \widehat{X}, \widehat{X} \widehat{E}_{0}, q \widehat{X}\right) ; \tag{4.43a}
\end{equation*}
$$

the other abbreviations are

$$
\begin{align*}
& E_{[2]}^{(1) \alpha \beta}(q)=-\frac{1}{2\left(1-\sigma^{2}\right)}\left\{\sigma a^{\alpha \beta} b^{\pi \lambda}+5 \sigma b^{\alpha \beta} a^{\pi \lambda}+3(1-\sigma)\left(b^{\alpha \pi} a^{\beta \lambda}+b^{\beta \lambda} a^{\alpha \pi}\right)\right\} q_{\lambda \pi}  \tag{4.43b}\\
& E_{[2]}^{(2) a \beta}(p)=-\frac{1}{1-\sigma^{2}}\left\{\sigma\left(a^{\alpha \lambda} a^{\beta \delta} a^{\pi \gamma}+a^{\alpha \beta} a^{\delta \gamma} a^{\pi \lambda}\right)\right. \\
& \left.+(1-\sigma)\left(a^{\alpha \gamma} a^{\beta \delta}+a^{\alpha \delta} a^{\beta \gamma}\right) a^{\pi \lambda}\right\} p_{\pi \gamma \mid \lambda \delta} \\
& +\frac{1}{1-\sigma^{2}}\left\{\sigma a^{\alpha \beta} b^{\pi \lambda} H-\frac{4 \sigma^{2}}{1-\sigma} a^{\alpha \beta} a^{\tau \gamma} H^{2}-\frac{\sigma(7-9 \sigma)}{(1-\sigma)} a^{\alpha \beta} K a^{\pi \gamma}\right. \\
& +\frac{\sigma(7-12 \sigma)}{(1-\sigma)} H a^{\pi \gamma} b^{\alpha \beta}+\frac{(1-4 \sigma)}{4} b^{\alpha \beta} b^{\sigma \gamma}-\frac{1}{2}(1-\sigma) b^{\alpha \pi} b^{\beta \gamma} \\
& \left.+6(1-\sigma) H\left(a^{\alpha \pi} b^{\beta \gamma}+b^{\alpha \pi} a^{\beta \gamma}\right)-(1-\sigma) 12 K a^{\alpha \pi} a^{\beta \gamma}\right\} p_{\pi \gamma},  \tag{4.43c}\\
& E_{[2]}^{(3) \alpha \beta}(\widehat{X})=\frac{1}{1-\sigma}\left\{\frac{1}{2}(6 \sigma-1) b^{\alpha \beta} X_{[0]}^{0}-\sigma a^{\alpha \beta} X_{[1]}^{0}-\sigma a^{\alpha \beta} X_{[01 \mid \gamma}^{\gamma}\right\} \\
& -a^{\alpha \pi} a^{\beta \gamma}\left(X_{[01 \pi \mid \gamma}+X_{\{01 \gamma \mid \pi}\right) \text {, }  \tag{4.43d}\\
& E_{[2]}^{(4) \alpha \beta}\left(\widehat{E}_{0}\right)=-\frac{1}{1-\sigma^{2}}\left\{\left(18 \sigma^{2}-14 \sigma+1\right) H b^{\alpha \beta}\right. \\
& \left.-4 \sigma(1-2 \sigma) H^{2} a^{\alpha \beta}+2 \sigma(4-5 \sigma) K a^{\alpha \beta}\right\} E_{[0]}^{00} \\
& -\frac{1}{1-\sigma}\left\{a^{\alpha \pi} a^{\beta \gamma}+\sigma a^{\alpha \beta} a^{\pi \gamma}\right\} E_{[01 \mid \tau \gamma}^{00}+2\left(a^{\beta \gamma} a_{\pi}^{\alpha}+a^{\alpha \gamma} a_{\pi}^{\beta}\right)\left(H E_{[0]}^{\pi 0}\right)_{1 \gamma} \\
& +\frac{\sigma}{1-\sigma} a^{\alpha \beta} b_{\gamma}^{\pi} E_{[01 \mid \mathrm{x}}^{\gamma 0}+\frac{1}{2(1-\sigma)}\left\{(6 \sigma-1) b^{\alpha \beta}-4 \sigma H a^{\alpha \beta}\right\} E_{[01 \mid x}^{\mp 0} \\
& +\frac{1}{2}\left\{a^{\alpha x} b_{\gamma}^{\beta}+a^{\beta x} b_{\gamma}^{\alpha}-a_{\gamma}^{\beta} b^{\alpha \tau}-a_{\gamma}^{\alpha} b^{\beta x}\right\} E_{\{01 \mid x}^{\gamma 0},  \tag{4.43e}\\
& E_{[2]}^{(5) \alpha \beta}\left(q^{2}\right)=A_{(1)}^{\alpha \beta \pi \lambda}\left(\frac{\sigma}{1-\sigma} a^{\left.\xi \pi a_{\pi}^{\gamma} a_{\lambda}^{\delta}-a_{\pi}^{\xi} a_{\lambda}^{\gamma} a^{\eta \delta}\right) q_{\gamma \delta q_{\xi \eta}} . . . . ~}\right. \tag{4.43f}
\end{align*}
$$

Equation (4.43) is the required expression of $E_{[2]}^{\alpha \beta}$.
By similar steps, we can express all the other $E_{[m]}^{\{0}$ and $E_{[m]}^{\alpha \beta}$ in terms of $p_{\alpha \beta}, q_{a \beta}$ and $E_{[0]}^{40}$. However, for the purpose of the first approximation in the equations of equilibrium, the knowledge of the required expressions of $E_{[0]}^{[j}, E_{[1]}^{\ell j}, E_{[2]}^{[j]}$ is sufficient. This completes Step (II).

Conditions of the boundary surfaces, and the expressions of $E_{[0]}^{00}$ in terms of $p_{a \beta}$ and $q_{\alpha \beta}$. We have now succeeded in expressing $E_{[m]}^{4}$ in terms of $p_{\alpha \beta}, q_{\alpha \beta}, E_{[0]}^{\{0}$. Our next task is to express $E_{[0]}^{00}$ in terms of $p_{\alpha \beta}, q_{\alpha \beta}$ by applying the following six surface conditions:

$$
\begin{align*}
& N_{(+) k} E_{(+)}^{h i}=Z_{(+)}^{i} \quad \text { for } \quad x^{0}=+h_{(+)}  \tag{4.44a}\\
& N_{(-) k} E_{(-)}^{k_{i}}=Z_{(-)}^{i} \quad \text { for } \quad x^{0}=-h_{(-)} . \tag{4.44b}
\end{align*}
$$

The unit normal vectors $N_{(+) i}, N_{(-) i}$, drawn out from the shell or plate, are determined as functions of $x^{\alpha}$ by the equations

$$
\begin{array}{ll}
N_{(+) \alpha}=-N_{(+) 0} h_{(+), \alpha}, & N_{(-) \alpha}=N_{(-) 0} h_{(-), \alpha} \\
N_{(+) 0}=\left(1-N_{(+) \alpha} N_{(+)}^{\alpha}\right)^{1 / 2}, & N_{(-) 0}=\left(1-N_{(-) \alpha} N_{(-)}^{\alpha}\right)^{1 / 2} \tag{4.45b}
\end{array}
$$

the positive roots being understood. $Z_{(+) i}, Z_{(-) ;}$are tensor components of the given loads per unit area applied to the upper and lower surfaces.

Making use of $(4.45 \mathrm{a}, \mathrm{b})$, we find that $(4.44 \mathrm{a}, \mathrm{b})$ take the form

$$
\begin{equation*}
E_{(+)}^{i 0}-E_{(+)}^{\pi i} h_{(+), \pi}=\frac{Z_{(+)}^{i}}{\left|N_{(+)}^{0}\right|}, \quad E_{(-)}^{i 0}+E_{(-)}^{\alpha i} h_{(-), \alpha}=-\frac{Z_{(-)}^{i}}{\left|N_{(-)}^{0}\right|} \tag{4.46}
\end{equation*}
$$

Substituting (3.24) into (4.46), and adding the resulting expressions, we obtain in consequence of (3.26), (3.28)

$$
\begin{align*}
2 E_{[0]}^{00}+E_{[1]}^{00} d+\frac{1}{2!} E_{[2]}^{00} t^{(2)} & +\cdots \\
& -\left[E_{[0]}^{\pi 0} d_{1}+\frac{1}{2!} E_{[1]}^{\pi 0} t_{,}^{(2)}+\frac{1}{3!} E_{[2]}^{\pi 0} d_{, \pi}^{(3)}+\cdots\right]=Q^{0} \tag{4.47a}
\end{align*}
$$

$2 E_{[0]}^{\alpha 0}+E_{[1]}^{\alpha 0} d+\frac{1}{2!} E_{\left[21 l^{(2)}\right.}^{\alpha 0}+\cdots$

$$
\begin{equation*}
-\left[E_{[0]}^{\pi \alpha} d_{, \pi}+\frac{1}{2!} E_{[1]}^{\pi \alpha} t_{\pi}^{(2)}+\frac{1}{3!} E_{[2]}^{\pi \alpha} d_{\pi}^{(3)}+\cdots\right]=Q^{\alpha} . \tag{4.47b}
\end{equation*}
$$

Substituting $E_{[m]}^{y}$ from the results of Step (II) into (4.47a, b), we obtain a set of three equations in terms of nine quantities $p_{\alpha \beta}, q_{\alpha \beta}, E_{[0]}^{i 0}$. Solving these equations for $E_{[0]}^{00}$, we have

$$
\begin{align*}
& E_{[01}^{00}=\frac{1}{2} Q^{0}+\frac{(1-2 \sigma)}{2(1-\sigma)} H Q^{0} d+\frac{1}{4}\left(Q^{\pi} d\right)_{\mid x}-\frac{\sigma}{8(1-\sigma)} a^{\lambda \pi}\left(Q_{\mid \lambda}^{0} t^{(2)}\right)_{\left.\right|_{r}} \\
& -\frac{1-2 \sigma}{4(1-\sigma)^{2}}\left\{(2-3 \sigma) 2 H^{2}-(1-\sigma) K\right\} Q^{0} t^{(2)}-\frac{1}{8}\left\{\left(b^{\pi}+2 H a_{\lambda}^{\pi}\right) Q^{\lambda} t^{(2)}\right\}_{1 \pi} \\
& -\frac{1}{1}\left\{\left(H Q^{\pi}\right)_{\mid x}-\frac{\sigma}{1-\sigma} H Q_{1 \pi}^{\pi}\right\} t^{(2)}+\frac{1}{2} X_{[0]}^{0} d \\
& +\frac{1}{4}\left\{X_{[1]}^{0}-\frac{2(1-2 \sigma)}{1-\sigma} H X_{[0]}^{0}\right\} t^{(2)}-\frac{1}{4}\left(X_{[0]}^{\mathrm{T}} t^{(2)}\right)_{\mid \pi} \\
& -\frac{1}{2\left(1-\sigma^{2}\right)}\left\{2 \sigma H a^{\pi \lambda}+(1-\sigma) b^{\pi \lambda}\right\} p_{\pi \lambda} d-\frac{1}{4} A_{(1)}^{\lambda)^{\delta \gamma}}\left(p_{\delta \gamma \mid \lambda} t^{(2)}\right)_{\mid \sigma} \\
& +\frac{1}{4\left(1-\sigma^{2}\right)}\left\{(4-3 \sigma) H b^{\pi \lambda}+\frac{4 \sigma(2-3 \sigma)}{(1-\sigma)} H^{2} a^{\pi \lambda}-(3-\sigma) K a^{\pi \lambda}\right\} p_{\pi \lambda} t^{(2)} \\
& -\frac{1}{8\left(1-\sigma^{2}\right)}\left\{4 H \sigma a^{\pi \lambda}+(1-\sigma) b^{\pi \lambda}\right\} q_{\pi \lambda} l^{(2)}+O_{(16)}^{00}, \tag{4.48a}
\end{align*}
$$

$$
\begin{align*}
E_{[0]}^{\alpha 0}= & \frac{1}{2} Q^{\alpha}+\frac{\sigma}{4(1-\sigma)} a^{\alpha \pi}\left(Q^{0} d\right)_{\mid \pi}+\frac{1}{4}\left(b_{\pi}^{\alpha}+2 H a_{\pi}^{\alpha}\right) Q^{\pi} d \\
& -\frac{\sigma}{8(1-\sigma)} a^{\alpha \pi}\left(Q_{\mid \lambda}^{\lambda} l^{(2)}\right)_{\mid \pi}-\frac{\sigma}{8(1-\sigma)}\left\{\left(b^{\alpha \gamma}+\frac{2(1-2 \sigma)}{(1-\sigma)} H a^{\alpha \gamma}\right) Q^{0} l^{(2)}\right\}_{\mid \gamma} \\
& +\frac{1}{2} X_{[0]}^{\alpha} d-\frac{\sigma}{4(1-\sigma)} a^{\alpha \lambda}\left(X_{\left.[0]^{0} l^{(2)}\right)_{\mid \lambda}}\right. \\
& +\frac{\sigma}{16(1-\sigma)}\left\{\left(4 H a^{\alpha \lambda}+b^{\alpha \lambda}\right)_{\mid \lambda} Q^{0}-\left(b^{\alpha \lambda}+2 H a^{\alpha \lambda}\right) Q_{\mid \lambda}^{0}\right\} l^{(2)} \\
& +\frac{1}{4}\left(5 H b_{\lambda}^{\alpha}+4 H H^{2} a_{\lambda}^{\alpha}-4 K a_{\lambda}^{\alpha}\right) Q^{\alpha} t^{(2)}+\frac{1}{4}\left\{X_{[1]}^{\alpha}-\left(b_{\pi}^{\alpha}+2 H a_{\pi}^{\alpha}\right) X_{[0]}^{\pi}\right\} l^{(2)} \\
& +\frac{1}{4} A_{(1)}^{\alpha \beta \pi \lambda}\left(\left.q_{\pi \lambda} t^{(2)}\right|_{\mid \beta}+\frac{1}{2} A_{(1)}^{\alpha \pi \lambda \delta}\left(p_{\lambda \delta} d\right)_{\mid \pi}+\frac{1}{4}\left(4 H a_{\pi}^{\alpha}+b_{\pi}^{\alpha}\right)_{\mid \lambda} A_{(1)}^{\gamma \delta \pi} p_{\gamma \delta} t^{(2)}\right. \\
& -\frac{1}{4(1+\sigma)}\left\{\left[\frac{\sigma}{1-\sigma}\left(b^{\alpha \beta} a^{\pi \lambda}+\frac{1}{2} a^{\alpha \beta} b^{\pi \lambda}\right)-\frac{2 \sigma^{2}}{(1-\sigma)^{2}} H a^{\alpha \beta} a^{\pi \lambda}\right.\right. \\
& \left.+\left(b^{\alpha \pi} a^{\beta \lambda}+a^{\alpha \pi} b^{\beta \lambda)}\right] p_{\pi \lambda} l^{(2,}\right\}_{\mid \beta} \\
& -\frac{1}{2}\left(b_{\pi}^{\alpha}+2 H a_{\pi}^{\alpha}\right) A_{(1)}^{\pi \gamma \delta} p_{\lambda \delta \mid \gamma} t^{(2)}+O_{(17)}^{\alpha 0} . \tag{4.48b}
\end{align*}
$$

Here $O_{(16)}^{00}, O_{(17)}^{c 0}$ stand as usual for terms not explicitly calculated. It is possible to exhibit their orders of magnitude as in (4.43a), for example, but the expressions are very long and will therefore be omitted here; the full expressions may be found in the author's Ph.D. Thesis, "The intrinsic theory of elastic shells and plates" (University of Toronto Library). The important fact about these residual terms is that, in the case of small strain, they are small compared with the terms shown explicitly.

The substitution of $E_{[0]}^{i 0}$ from (4.48a, b) into (4.26), (4.28), (4.29), (4.30), (4.32), (4.33), (4.43) etc., gives the expressions for $E_{[m]}^{[y}$ in terms of $p_{\alpha \beta}$ and $q_{\alpha \beta}$. This completes Step (III).

Expressions of $T^{\alpha 0}, T^{\alpha \beta}, L^{\alpha \beta}$ in terms of $p_{\alpha \beta}, q_{\alpha \beta}$. If we substitute the expressions of $E_{m m}^{a \beta}$ from the results of Step (III) into (3.30), (3.31), we immediately obtain the set of eight quantities $T^{\alpha \beta}$ and $L^{\alpha \beta}$ in terms of the six quantities $p_{\alpha \beta}$ and $q_{\alpha \beta}$; the quantities $X_{[m]}^{i}, Q^{i}$ are supposed to be given. Therefore for the membrane stress tensor $T^{\alpha \beta}$ and the bending moment tensor $L^{\alpha \beta}$, we have

$$
\begin{align*}
T^{\alpha \beta}= & A_{(1)}^{\alpha \beta \pi \lambda} p_{\pi \lambda} t-B_{(2)}^{\alpha \beta \pi \lambda} q_{\pi \lambda} t t^{(2)}+B_{(3)}^{\alpha \beta \pi \lambda} q_{\pi \lambda} l^{(3)}+\frac{1}{2} A_{(1)}^{\alpha \beta \pi \lambda} q_{\pi \lambda} d^{(2)} \\
& -A_{(2)}^{\alpha \beta \pi \gamma \lambda \delta} q_{\pi \gamma} q_{\lambda \delta} t^{(3)}+B_{(4)}^{\alpha \beta \pi \lambda} p_{\pi \lambda} d^{(2)}-\left.\frac{\sigma}{1-\sigma} a^{\alpha \beta} A_{(1)}^{\lambda \pi \gamma}\left(p_{\delta \gamma \mid \lambda} t^{(2)}\right)\right|_{\pi} l \\
& +C_{(1)}^{\alpha \beta \pi \lambda} p_{\pi \lambda} t^{(2)}-\left(a^{\beta \delta} A_{(1)}^{\alpha \lambda \lambda \pi}+a^{\alpha \lambda} A_{(1)}^{\alpha \beta \delta \gamma}\right) p_{\pi \gamma\left(\lambda \delta t^{(3)}\right.}+C_{(2)}^{\alpha \beta \pi \lambda} p_{\pi \lambda} l^{(3)} \\
& +\frac{\sigma}{2(1-\sigma)} a^{\alpha \beta} Q^{0} t+T_{(1)}^{\alpha \beta}\left\{X^{0}\right\}+T_{(2)}^{\alpha \beta}\{X\}+T_{(3)}^{\alpha \beta}\{Q\}+O_{(18)}^{\alpha \beta},  \tag{4.49}\\
L^{\alpha \beta}= & \eta_{[10}^{\gamma \beta} a_{\pi \gamma}\left\{\frac{1}{2} A_{(1)}^{\alpha \gamma \lambda \delta} p_{\lambda \delta} d^{(2)}+B_{(5)}^{\alpha \pi \lambda \delta} p_{\lambda \delta} t^{(3)}+\frac{1}{3} A_{(1)}^{\alpha \pi \lambda \delta} q_{\lambda \delta} t^{(3)}\right\} \\
& +\frac{\sigma}{12(1-\sigma)} \eta_{10\}}^{\gamma \beta}\left\{a_{\gamma}^{\alpha}\left(3 Q^{0} d^{(2)}+\frac{4 \sigma}{1-\sigma} H Q^{0} t^{(3)}-4 X_{(01}^{0} t^{(3)}-2 Q_{1 \lambda}^{\lambda} t^{(3)}\right)\right. \\
& \left.-b_{\gamma}^{\alpha} Q^{0} t^{(3)}\right\}+O_{(19)}^{\alpha \beta}, \tag{4.50}
\end{align*}
$$

where $O_{(18)}^{\alpha \beta}, O_{(10)}^{a \beta}$ stand for the residual terms. The other abbreviations are

$$
\begin{align*}
& A_{(1)}^{\alpha \beta \pi \lambda}=\frac{1}{1-\sigma^{2}}\left\{\sigma a^{\alpha \beta} a^{\pi \lambda}+(1-\sigma) a^{\alpha \pi} a^{\beta \lambda}\right\}, \\
& A_{(2)}^{\alpha \beta \pi \gamma \lambda \delta}=\frac{1}{6\left(1-\sigma^{2}\right)}\left\{\sigma a^{\alpha \beta} a^{\pi \delta} a^{\gamma \lambda}+(1-\sigma) a^{\alpha \pi} a^{\beta \delta} a^{\gamma \lambda}-\frac{\sigma^{2}}{1-\sigma^{2}} a^{\alpha \beta} a^{\pi \gamma} a^{\delta \lambda}\right. \\
& \left.-\sigma a^{\alpha \pi} a^{\beta \gamma} a^{\delta \lambda}\right\}=\frac{1}{6}\left\{A_{(1)}^{\alpha \beta \delta \delta} a^{\gamma \lambda}-\frac{\sigma}{1-\sigma} A_{(1)}^{\alpha \beta \pi \gamma} a^{\delta \lambda}\right\}, \\
& B_{(2)}^{\alpha \beta \pi \lambda}=\frac{\sigma}{8(1-\sigma)\left(1-\sigma^{2}\right)}\left\{4 \sigma H a^{\pi \lambda}+(1-\sigma) b^{\pi \lambda}\right\} a^{\alpha \beta}, \\
& B_{(3)}^{a \beta \pi \lambda}=\frac{1}{12\left(1-\sigma^{2}\right)}\left\{\sigma\left(8 H a^{\alpha \beta} a^{\pi \lambda}-a^{\alpha \beta} b^{\pi \lambda}-3 b^{\alpha \beta} a^{\pi \lambda}\right)\right. \\
& \left.+(1-\sigma)\left(8 H a^{\alpha \pi} a^{\beta \lambda}-3 b^{\alpha x} a^{\beta \lambda}-b^{\beta \lambda} a^{\alpha x}\right)\right\} \text {, }  \tag{4.51~d}\\
& B_{(4)}^{\alpha \beta \pi \lambda}=\frac{1}{4\left(1-\sigma^{2}\right)}\left\{\sigma\left(4 H a^{\alpha \beta} a^{\pi \lambda}-3 a^{\alpha \beta} b^{\pi \lambda}-a^{\pi \lambda} b^{\alpha \beta}\right)\right. \\
& \left.+(1-\sigma)\left(4 H a^{\alpha r} a^{\beta \lambda}-2 b^{\alpha x} a^{\beta \lambda}-a^{\alpha \pi} b^{\beta \lambda}\right)\right\},  \tag{4.51e}\\
& B_{(5)}^{\alpha \pi \lambda \delta}=\frac{1}{6\left(1-\sigma^{2}\right)}\left\{\frac{4 \sigma}{1-\sigma} H a^{\alpha \tau} a^{\lambda \delta}+(1-\sigma) 4 H a^{\alpha \lambda} a^{\tau \delta}-\sigma\left(a^{\lambda \delta} b^{\alpha \pi}+b^{\lambda \delta} a^{\alpha \pi}\right)\right. \\
& \left.-2(1-\sigma)\left(b^{\alpha \lambda} a^{\star \delta}+\frac{1}{2} a^{\alpha \lambda} b^{\star \delta}\right)\right\} \text {, }  \tag{4.51f}\\
& C_{(1)}^{\alpha \beta \pi \lambda}=\frac{\sigma}{4\left(1-\sigma^{2}\right)}\left\{(4-3 \sigma) H b^{\approx \lambda}+\frac{4 \sigma(2-3 \sigma)}{(1-\sigma)} H^{2} a^{\pi \lambda}\right. \\
& \left.-(3-\sigma) K a^{\pi \lambda}\right\} a^{\alpha \beta},  \tag{4.51~g}\\
& C_{(2)}^{\alpha \beta \pi \lambda}=\frac{1}{6\left(1-\sigma^{2}\right)}\left\{2(1-\sigma) H\left(a^{\alpha \pi} b^{\beta \lambda}+b^{\alpha \pi} a^{\beta \lambda}\right)-\frac{1-2 \sigma}{4} b^{\alpha \beta} b^{\pi \lambda}\right. \\
& -\frac{3}{2}(1-\sigma) b^{\alpha \pi} b^{\beta \lambda}+\frac{4 \sigma^{2}}{1-\sigma} H^{2} a^{\alpha \beta} a^{\pi \lambda}-\sigma H a^{\alpha \beta} b^{\pi \lambda}+\frac{\sigma(1-4 \sigma)}{1-\sigma} H b^{\alpha \beta} a^{\pi \lambda} \\
& \left.-8(1-\sigma) K a^{\alpha \pi} a^{\beta \lambda}-\frac{\sigma(1-3 \sigma)}{1-\sigma} K a^{\alpha \beta} a^{\pi \lambda}\right\} \text {, } \tag{4.51h}
\end{align*}
$$

$T_{(1)}^{\alpha \beta}\left\{X^{0}\right\}=\frac{\dot{\sigma}}{4(1-\sigma)} a^{\alpha \beta}\left\{X_{[1]}^{0}-\frac{2(1-2 \sigma)}{1-\sigma} H X_{[0]}^{0}\right\} t^{(2)}-\frac{1}{2} \frac{\sigma}{1-\sigma} a^{\alpha \beta} X_{[0]}^{0} d^{(2)}$

$$
\begin{equation*}
-\frac{1}{6(1-\sigma)}\left\{\frac{1}{2}(1-4 \sigma) b^{\alpha \beta} X_{[01}^{0}+4 \sigma H a^{\alpha \beta} X_{[0]}^{0}+\sigma a^{\alpha \beta} X_{[1]}^{0}\right\} t^{(3)} \tag{4.51i}
\end{equation*}
$$

$$
\begin{align*}
& T_{(2)}^{\alpha \beta}\{X\}=-\frac{\sigma}{4(1-\sigma)} a^{\alpha \beta}\left(X_{\left[0, t^{(2)}\right)_{\mid \pi} t-\frac{\sigma}{\sigma(1-\sigma)} a^{\alpha \beta} X_{[0] \mid \pi}^{\pi} t^{(3)}}\right. \\
&-\frac{1}{6} a^{\alpha \pi} a^{\beta \lambda}\left(X_{[0] \pi \mid \lambda}+X_{[0] \lambda \mid \pi}\right) t^{(3)} \tag{4.51j}
\end{align*}
$$

$$
\begin{align*}
T_{(3)}^{\alpha \beta}\{Q\}= & -\frac{\sigma}{8(1-\sigma)} a^{\alpha \beta}\left\{\left(b_{\lambda}^{\pi}+2 H a_{\lambda}^{\pi}\right) Q^{\lambda} t^{(2)}\right\}_{\mid \pi} t-\frac{\sigma}{4(1-\sigma)} a^{\alpha \beta} Q_{\left.\right|_{\pi} d^{(2)}} \\
& -\frac{1}{6}\left\{\left.a^{\beta \pi}\left(H Q^{\alpha}\right)\right|_{\mid \pi}+\left.a^{\alpha \pi}\left(H Q^{\beta}\right)\right|_{\pi}\right\} t^{(3)}+\frac{\sigma}{4(1-\sigma)} a^{\alpha \beta}\left(Q^{\pi} d\right)_{\mid \pi} t \\
& -\frac{\sigma}{4(1-\sigma)} a^{\alpha \beta}\left(Q^{\pi} H\right)_{\mid \pi} t l^{(2)}+\frac{\sigma}{4(1-\sigma)^{2}} a^{\alpha \beta} H Q_{\mid \pi}^{\pi} t t^{(2)} \\
& +\frac{1}{24}\left\{a^{\alpha \lambda} b^{\beta \pi}+a^{\beta \lambda} b^{\alpha \pi}-b^{\alpha \lambda} a^{\beta \pi}-b^{\beta \lambda} a^{\alpha \pi}+\frac{2 \sigma}{1-\sigma} a^{\alpha \beta} b^{\pi \lambda}\right. \\
& \left.+\frac{4 \sigma-1}{1-\sigma} a^{\lambda \pi} b^{\alpha \beta}-\frac{12 \sigma}{1-\sigma} H a^{\alpha \beta} a^{\lambda \pi}\right\} Q_{\pi \mid \lambda} t^{(3)} \tag{4.51k}
\end{align*}
$$

Furthermore, by solving (2.5c), we have

$$
\begin{equation*}
T^{\alpha 0}=\eta_{[0]}^{\alpha \lambda} a_{\lambda \delta}\left(L_{\mid r}^{\pi \delta}+M^{\delta}\right), \tag{4.52}
\end{equation*}
$$

in which $L^{\alpha \beta}$ is given by (4.50) and $M^{\delta}$ by (3.34).
Equations (4.49), (4.50) express $T^{\alpha \beta}$ and $L^{\alpha \beta}$ in terms of $p_{\alpha \beta}$ and $q_{\alpha \beta}$. When $L^{\alpha \beta}$ is known, $T^{\alpha 0}$ is calculated from (4.52). This completes the last step of the procedure outlined at the beginning of this section.

It should be noted that $p_{\alpha \beta}$ and $q_{\alpha \beta}$ correspond respectively to the extension and change of curvature of the reference surface $S_{0} ; X_{[m]}^{t}$ is the normal derivatives of the $m$ th order on $S_{0}$ of the body force, supposed to be given. If the form of the reference surface $S_{0}$ is given (in the strained state), the following quantities are known: $a_{\alpha \beta}$, $(1 / 2) b_{\alpha \beta},(1 / 2) c_{\alpha \beta}$ are the first, second and third fundamental tensors, $H$ the mean curvature and $K$ the total curvature. Furthermore, if we know the positions of the boundary surfaces of the shell in the strained state and also the surface loads on the boundary surfaces, then the quantities $t^{(m)}, d^{(m)}, Q^{(m) i}, P^{(m) i}$ can be determined by using (3.26)-(3.28).
5. Equations of equilibrium and compatibility in terms of the six unknowns $p_{\alpha \beta}$ and $q_{\alpha \beta}$. Having now expressed the macroscopic stress tensors in terms of $p_{\alpha \beta}$ and $q_{\alpha \beta}$ we shall substitute these expressions into the macroscopic equations of equilibrium, ( $2.8 \mathrm{a}, \mathrm{b}$ ). In consequence of (3.32)-(3.34), this gives the following three equations in terms of the six unknowns $p_{\alpha \beta}$ and $a_{\alpha \beta}$ :

$$
\begin{align*}
-\frac{1}{2} b_{\rho \gamma} A_{(1)}^{\rho \gamma \pi \lambda} & p_{\pi \lambda} t+\left.\frac{1}{3} A_{(1)}^{\rho \gamma \pi \lambda}\left(q_{\pi \lambda} t^{(3)}\right)\right|_{\rho \gamma}+\frac{1}{2} b_{\rho \gamma} B_{(2)}^{\rho \gamma \pi \lambda} q_{\pi \lambda} t t^{(2)} \\
& -\frac{1}{2} b_{\rho \gamma} B_{(3)}^{\rho \gamma \pi \lambda} q_{\pi \lambda} t^{(3)}+\frac{1}{2} b_{\rho \gamma} A_{(2)}^{\rho \gamma \pi \beta \lambda \delta} q_{\pi \beta} q_{\lambda \delta \lambda} t^{(3)}-\frac{1}{4} A_{(1)}^{\rho \gamma \pi \lambda} b_{\rho \gamma} q_{\pi \lambda} d^{(2)} \\
& -\frac{1}{2} b_{\rho \gamma} B_{(4)}^{\rho \gamma \pi \lambda} p_{\pi \lambda} d^{(2)}+\frac{\sigma}{2(1-\sigma)} a^{\rho \gamma} b_{\rho \gamma} A_{(1)}^{\lambda \pi \delta \beta}\left(p_{\partial \beta|\lambda| \lambda} l^{(2)}\right)_{\mid \pi} t \\
& -\frac{1}{2} C_{(1)}^{\rho \gamma \pi \lambda} b_{\rho \gamma} p_{\pi \lambda} t t^{(2)}+\frac{1}{2} b_{\rho \gamma}\left(a^{\gamma \delta} A_{(1)}^{\rho \lambda \beta \pi}+a^{\pi \lambda} A_{(1)}^{\rho \gamma} \delta \beta\right) p_{\pi \beta \mid \lambda \delta \delta} l^{(3)} \\
& -\frac{1}{2} b_{\rho \gamma} C_{(2)}^{\rho \gamma \pi \lambda} p_{\pi \lambda} t^{(3)}+\frac{1}{2} A_{(1)}^{\rho \gamma \pi \lambda}\left(p_{\pi \lambda} d^{(2)}\right)_{\mid \rho \gamma}+\left(B_{(5)}^{\rho \gamma \pi \lambda} p_{\pi \lambda} l^{(3)}\right)_{\mid \rho \gamma}+P^{0} \\
& -\frac{\sigma}{1-\sigma} H Q^{0} t+2 H Q^{(1) 0}+K P^{(2) 0}+I_{(1)}^{0}\left\{Q^{\pi}\right\}+I_{(2)}^{0}\left\{p_{\pi}\right\}+I_{(3)}^{0}\left\{X^{0}\right\} \\
& +I_{(4)}^{0}\left\{X^{\pi}\right\}=O_{(20)}^{0}, \tag{5.1}
\end{align*}
$$

$$
\begin{align*}
A_{(1)}^{\pi \alpha \delta \lambda}\left(p_{\lambda \delta} t\right)_{\mid \pi} & +\left\{B_{(3)}^{\pi \alpha \lambda \delta} q_{\lambda \delta} t^{(3)}-B_{(2)}^{\pi \alpha \lambda \delta} q_{\lambda \delta} t t^{(2)}\right\}_{\mid \pi}-A_{(2)}^{\pi \alpha \gamma \rho \lambda \delta}\left(q_{\gamma \rho} q_{\lambda \delta} t^{(3)}\right)_{\mid \pi} \\
& +\frac{1}{6} a^{\alpha \gamma} b_{\gamma \rho} A_{(1)}^{\pi \rho \lambda \delta}\left(q_{\lambda \delta} t^{(3)}\right)_{\mid \pi}+\left(B_{(\lambda)}^{\rho \alpha \pi \gamma} p_{\pi \gamma} d^{(2)}\right)_{\mid \rho}+\left(C_{(1)}^{\rho \alpha \pi \gamma} p_{\pi \gamma} t t^{(2)}\right)_{\mid \rho} \\
& -\frac{\sigma}{1-\sigma} a^{\rho \alpha} A_{(1)}^{\lambda \pi \delta \gamma}\left[\left.\left(p_{\delta \gamma \mid \lambda} t^{(2)}\right)\right|_{\mid \pi} t\right]_{\mid \rho}-\left(a^{\alpha \delta} A_{(1)}^{\rho \gamma \lambda \pi}+a^{\pi \lambda} A_{(1)}^{\rho \alpha \delta \gamma}\right)\left(p_{\left.\left.\pi \gamma \mid \lambda t^{\prime}\right)^{(3)}\right)_{\mid \rho}}\right. \\
& +\left(C_{(2)}^{\rho \alpha \pi \gamma} p_{\pi \gamma} t^{(3)}\right)_{\mid \rho}-\frac{1}{2} A_{(1)}^{\rho \alpha \pi \gamma}\left(q_{\tau \gamma} d^{(2)}\right)_{\mid \rho}+P^{\alpha} \\
& +\frac{\sigma}{2(1-\sigma)}\left(Q^{0} t\right)_{\mid \pi} a^{\alpha \pi}+3 H b_{\gamma}^{\alpha} P^{(2) \gamma}+K\left(H b_{\gamma}^{\alpha}-K a_{\gamma}^{\alpha}\right) P^{(4) \gamma}+I_{(1)}^{\alpha}\left\{Q^{\tau}\right\} \\
& +I_{(2)}^{\alpha}\left\{X^{0}\right\}+I_{(3)}^{\alpha}\left\{X^{\pi}\right\}=O_{(21) .}^{\alpha} . \tag{5.2}
\end{align*}
$$

The symbols $A_{(1)}^{a \beta \pi \lambda}, A_{(2)}^{\alpha \beta \pi \lambda \lambda \delta}$, given by (4.51a, b), are functions of $a_{\alpha \beta} ; B_{(2)}^{\alpha \beta \pi \lambda}, B_{(3)}^{\alpha \beta \pi \lambda}$, $B_{(4)}^{\alpha \beta \pi \lambda}, B_{(5)}^{\alpha \beta \pi \lambda}$, given by ( $4.51 \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}$ ), are linear functions of $b_{\alpha \beta}$; and $C_{(1)}^{\alpha \beta \pi \lambda}, C_{(5)}^{\alpha \beta \pi \lambda}$, given by $(4.51 \mathrm{~g}, \mathrm{~h})$, are quadratic functions of $b_{a \beta}$. The other abbreviations in (5.1), (5.2) are listed as follows:

$$
\begin{align*}
& I_{(1)}^{0}\left\{Q^{x}\right\}=\frac{\sigma}{4(1-\sigma)} H\left\{\left(b_{\lambda}^{\pi}+2 H a_{\lambda}^{\pi}\right) Q^{\lambda} t^{(2)}\right\}_{{ }_{\mid x} t}-\frac{\sigma}{2(1-\sigma)} H Q^{\pi} d_{\mid \pi} t \\
& +\frac{\sigma}{2(1-\sigma)} H\left\{\left(Q^{\pi} H\right)_{\mid \pi}-\frac{\sigma}{1-\sigma} H Q_{\left.\right|^{\pi}}^{\pi}\right\} l^{(2)}-\frac{\sigma}{\sigma(1-\sigma)} a^{\gamma \lambda}\left(Q_{\mid \pi}^{\pi} t^{(3)}\right)_{\mid \gamma \lambda} \\
& -\frac{1}{6}\left\{\frac{\sigma}{1-\sigma} H b^{\lambda x}-2 H^{2} a^{\lambda x}-\frac{4 \sigma-1}{1-\sigma} K a^{\lambda \pi}\right\} Q_{\lambda \mid x} t^{(3)} \\
& +\frac{1}{6} b_{\lambda}^{\pi}\left(H Q^{\lambda}\right)_{{ }_{1 \times} \iota^{(3)}}+\left\{Q^{(1) \pi}+\left(K a_{\lambda}^{\pi}+H b_{\lambda}^{\pi}\right) Q^{(3) \lambda}\right\}_{|\pi|}  \tag{5.3a}\\
& I_{(2)}^{0}\left\{P^{\pi}\right\}=\left\{\frac{1}{2}\left(4 H a_{\lambda}^{\pi}+b_{\lambda}^{\pi}\right) P^{(2) \lambda}+\frac{1}{2} b_{\lambda}^{\pi} K P^{(4) \lambda}\right\}_{1 \pi},  \tag{5.3b}\\
& I_{(3)}^{0}\left\{X^{0}\right\}=X_{[0]}^{0} t+\left(2 H X_{[01}^{0}+X_{[1]}^{0}\right) \frac{d^{(2)}}{2}-\frac{\sigma}{3(1-\sigma)} a^{\lambda x}\left(X_{\left[01 t^{(3)}\right.}^{0}\right)_{\mid \lambda x} \\
& +\frac{1}{6(1-\sigma)}\left\{\left(4 H^{2}+6 \sigma K\right) X_{[0]}^{0}+2(1-\sigma) H X_{[1]}^{0}+X_{[2]}^{0}\right\} t^{(3)} \\
& -\frac{\sigma}{2(1-\sigma)} H\left\{X_{[1]}^{0}-\frac{2(1-2 \sigma)}{1-\sigma} H X_{[0]}^{0}\right\} u^{(2)} \text {, }  \tag{5.3c}\\
& I_{(4)}^{0}\left\{X^{\pi}\right\}=\frac{1}{2}\left\{X_{[0]}^{x} d^{(2)}\right\}_{1 \pi}+\frac{1}{3}\left\{\frac{1}{2}\left(4 H a_{\lambda}^{x}+b_{\lambda}^{\pi}\right) X_{[0]}^{\lambda} \iota^{(3)}+X_{(1)}^{\pi} t^{(3)}\right\}_{1 \pi} \\
& +\frac{\sigma}{2(1-\sigma)} H\left(X_{[0]}^{\pi} t^{(2)}\right)_{\mid \pi} t+\frac{\sigma}{3(1-\sigma)} H X_{[0| | \pi}^{\pi} t^{(3)}+\frac{1}{6} b^{\pi \lambda} X_{[0] \pi \mid \lambda} t^{(3)}, \tag{5.3d}
\end{align*}
$$

$I_{(1)}^{\alpha}\left\{Q^{\pi}\right\}=\frac{\sigma}{8(1-\sigma)} a^{\alpha \beta}\left\{\left[\left(b_{\lambda}^{\pi}+2 H a_{\lambda}^{\pi}\right) Q^{\lambda} t^{(2)}\right]_{\mid x_{x}} t+2 Q^{\pi} d_{\mid \pi} \ell\right.$
$\left.-2\left[\left(Q^{\pi} H\right)_{1_{\pi}}-\frac{\sigma}{1-\sigma} H Q_{\left.\right|^{\pi}}^{\pi}\right] t^{(2)}\right\}_{\mid \beta}$
$-\frac{1}{6}\left\{\left[a^{\beta \pi}\left(Q^{\alpha} H\right)_{\left.\right|_{\pi}}+\left.a^{\alpha \pi}\left(Q^{\beta} H\right)\right|_{\mid \pi}\right] t^{(3)}\right\}_{1 \beta}$

$$
\begin{align*}
& +\frac{1}{24}\left\{\left[a^{\alpha \lambda} b^{\beta \pi}+a^{\beta \lambda} b^{\alpha \pi}-b^{\alpha \lambda} a^{\beta \pi}-b^{\beta \lambda} a^{\alpha \pi}+\frac{2 \sigma}{1-\sigma} a^{\alpha \beta} b^{\lambda \pi}\right.\right. \\
& \left.\left.+\frac{4 \sigma-1}{1-\sigma} b^{\alpha \beta} a^{\lambda \pi}-\frac{12 \sigma}{1-\sigma} H a^{\alpha \beta} a^{\lambda \pi}\right] Q_{\pi \mid \lambda} t^{(3)}\right\}_{\mid \beta} \\
& +\left(2 H a_{\pi}^{\alpha}+b_{\pi}^{\alpha}\right) Q^{(1) \pi}+\left(K b_{\pi}^{\alpha}+2 H^{2} b_{\pi}^{\alpha}-2 K H a_{\pi}^{\alpha}\right) Q^{(3) \pi},  \tag{5.4a}\\
& I_{(2)}^{\alpha}\left\{X^{0}\right\}=\frac{\sigma}{4(1-\sigma)} a^{\alpha \beta}\left\{\left[X_{(1)}^{0}-\frac{2(1-2 \sigma)}{1-\sigma} H X_{[01}^{0}\right] \not U^{(2)}\right\}_{\mid \beta} \\
& -\frac{\sigma}{6(1-\sigma)}\left\{\left[\frac{1}{2}(1-4 \sigma) b^{\alpha \beta} X_{[0]}^{0}+4 \sigma H a^{\alpha \beta} X_{[0]}^{0}+\sigma a^{\alpha \beta} X_{[1]}^{0}\right] t^{(3)}\right\}_{\mid \beta} \\
& -\frac{\sigma}{6(1-\sigma)} b^{\alpha \beta}\left(X_{\text {O01 }}^{0} t^{(3)}\right)_{\mid \beta,}  \tag{5.4b}\\
& I_{(3)}^{\alpha}\left\{X^{\pi}\right\}=X_{[01}^{\alpha} t+\frac{1}{2} X_{[1]}^{\alpha} d^{(2)}-\frac{1}{6}\left(2 H a_{x}^{\alpha}+b_{\pi}^{\alpha}\right) X_{[0]}^{\pi} d^{(2)}+\frac{1}{6} X_{[2]}^{\alpha} t^{(3)}
\end{align*}
$$

$$
\begin{align*}
& +H b_{x}^{\alpha} X_{[01}^{\pi} t^{(3)}-\frac{1}{6} a^{\alpha \pi} a^{\beta \lambda}\left\{\left(X_{[0] x \mid \lambda}+X_{[0] \lambda \mid x}\right)^{(3)}\right\}_{[\beta} \\
& +\frac{1}{3}\left(2 H a_{x}^{\alpha}+b_{\pi}^{\alpha}\right) X_{[1]^{( } t^{(3)}} . \tag{5.4c}
\end{align*}
$$

Equations (5.1), (5.2) are the three equations of equilibrium in terms of the six unknowns $p_{\alpha \beta}$ and $q_{\alpha \beta}$. All the other quantities are supposed to be given.

The basic unknowns are the six quantities $p_{\alpha \beta}$ and $q_{\alpha \beta}$. To find them, we have to solve the three equations of equilibrium (5.1), (5.2), together with the three equations of compatibility, funished by the geometrical conditions.

We shall now convert (4.13) into a form in which $p_{\alpha \beta}$ and $q_{\alpha \beta}$ are the only unknowns. Let us put in turn $i=0, j=\alpha, k=\beta, l=\gamma$, and $i=\rho, j=\alpha, k=\beta, l=\gamma$ in (4.13), and then put $x^{0}=0$; we thus obtain threc equations in $p_{i j}$ and $q_{i j}$. We now substitute (4.37)-(4.40) for $p_{00}, p_{\alpha 0}, q_{00}, q_{\alpha 0}$, and (4.48a, b) for $E_{[0 \mid}^{0}$; this gives three equations in $p_{\alpha \beta}$ and $q_{\alpha \beta}$. Two of these, after being multiplied by $\eta_{[0]}^{\gamma \beta}$ and simplified by using (3.13), (3.14), can be written as follows:

$$
\begin{align*}
\eta_{[01}^{\gamma \beta}\left\{2 q_{\alpha \gamma \mid \beta}-\right. & \frac{\sigma}{1-\sigma} a^{\rho \gamma} b_{\alpha \beta} p_{\nabla \rho \mid \gamma}+b_{\beta}^{\gamma}\left(p_{\alpha \pi \mid \gamma}+p_{\gamma \gamma \mid \alpha}-p_{\alpha \gamma \mid \pi}\right) \\
& \left.+(1+\sigma)\left[Q_{\beta \mid \alpha \gamma}-a_{\alpha \beta} K Q_{\gamma}+\frac{1-2 \sigma}{2(1-\sigma)} b_{\alpha \beta} Q_{i \gamma}^{\gamma}\right]\right\}=O_{(22) \alpha .} \tag{5.5a}
\end{align*}
$$

The third equation, after being multiplied by $\eta_{[0]}^{\alpha \beta} \eta_{00}^{\text {of }}$, becomes

$$
\begin{align*}
& 2 \eta_{[01}^{\alpha \beta} \eta_{[0] 1}^{p \gamma} p_{\rho \beta \mid \alpha \gamma}-\eta_{[0 \beta}^{\alpha \beta} \eta_{[0]}^{\rho \gamma} q_{\rho \beta} q_{\alpha \gamma}+\frac{2(1-3 \sigma)}{1-\sigma} a^{\pi \lambda} K p_{\pi \lambda}+\left(b^{\pi \lambda}-4 H a^{\pi \lambda}\right) q_{\pi \lambda} \\
&  \tag{5.5b}\\
& +\frac{2(1+\sigma)(1-2 \sigma)}{1-\sigma} Q^{\alpha} K-(1+\sigma)\left(b^{\pi \lambda}-4 H a^{\pi \lambda}\right) Q_{\approx \mid \lambda}=O_{(23) .} .
\end{align*}
$$

Equations (5.5a, b) contain the three equations of compatibility in $p_{\alpha \beta}$ and $q_{\alpha \beta}$; these are to be associated with the three equations of equilibrium (5.1), (5.2) for the solution of plate and shell problems.
6. The equations of equilibrium and compatibility referred to the middle surface in the unstrained state. The quantities $a_{a \beta}, b_{\alpha \beta}$, which occur in (5.1), (5.2), (5.5a, b), refer to the strained state. But it is usual in elasticity to regard the unstrained state as given, rather than the strained state, and from this point of view $a_{\alpha \beta}, b_{\alpha \beta}$ are unknown. We should use, instead of them, the fundamental tensors of that surface $S_{o}^{\prime}$ in the unstrained state, which passes over into the reference surface $S_{0}$ in the strained state. So far $S_{0}$ has been quite general. Now we shall follow the usual method by choosing $S_{0}$ so that $S_{0}^{\prime}$ is the middle surface of the shell or plate in the unstrained state. This means that $S_{0}$ is not accurately the middle surface in the strained state.

For the metric in the unstrained state, we have by (4.1)

$$
\begin{equation*}
g_{i j}^{\prime}=g_{i j}-2 e_{i j} ; \tag{6.1}
\end{equation*}
$$

and so, if we define

$$
\begin{equation*}
a_{i j}^{\prime}=\left(g_{i j}^{\prime}\right)_{2_{0}=0}, \quad b_{i j}^{\prime}=\left(g_{i j, 0}^{\prime}\right)_{x=0}=0, \tag{6.2}
\end{equation*}
$$

we have

$$
\begin{array}{lll}
a_{\alpha \beta}^{\prime}=a_{\alpha \beta}-2 p_{\alpha \beta}, & a_{\alpha 0}^{\prime}=-2 p_{\alpha 0}, & a_{00}^{\prime}=1-2 p_{00}, \\
b_{\alpha \beta}^{\prime}=b_{\alpha \beta}-2 q_{\alpha \beta}, & b_{\alpha 0}^{\prime}=-2 q_{\alpha 0}, & b_{00}^{\prime}=-2 q_{00}, \tag{6.3}
\end{array}
$$

where $p_{i j}$ and $q_{i j}$ are defined as in (4.4). $a_{\alpha \beta}^{\prime}$ is the fundamental tensor of $S_{o}^{\prime}$; we regard it as given. We may substitute in the proceding theory

$$
\begin{equation*}
a_{\alpha \beta}=a_{\alpha \beta}^{\prime}+2 p_{\alpha \beta} . \tag{6.4}
\end{equation*}
$$

The tensor $b_{\alpha \beta}^{\prime}$ does not represent the curvature of the middle surface $S_{o}^{\prime}$ in the natural state; since in general $g_{a 0}^{\prime} \neq 0$ on $S_{0}^{\prime}$, the parametric lines of $x^{0}$ cuts $S_{0}^{\prime}$ obliquely. Let us introduce the normal coordinates $x^{i}$ based on $S_{0}^{\prime}$, choosing $x^{\alpha}=x^{\alpha}$ on $S_{0}^{\prime}$, and $\mathbf{x}^{0}$ normal to $S_{0}^{\prime}$. The metric $g_{i j}$ corresponding to the coordinates $\mathbf{x}^{i}$ satisfies

$$
\begin{equation*}
g_{00}=1, \quad g_{a 0}=0 . \tag{6.5}
\end{equation*}
$$

Let us put

$$
\begin{equation*}
a_{\alpha \beta}=\left(\mathcal{g}_{\alpha \beta}\right)_{x=0}, \quad b_{\alpha \beta}=\left(\frac{\partial \hat{g}_{\alpha \beta}}{\partial x^{0}}\right)_{x^{0}=0} . \tag{6.6}
\end{equation*}
$$

Now $\delta_{i j}$ and $g_{u}^{\prime}$ are metrics corresponding to the coordinate systems $x^{i}$ and $x^{i}$ respectively, for the description of the geometry of the unstrained state. Hence the tensor $\dot{g}_{i j}$ determines $g_{v}^{\prime}$, and vice versa.

It should be noted that the quantities $(1 / 2) b_{\alpha \beta}$ are the coefficients of the second fundamental form of $S_{0}^{\prime}$; they vanish if $S_{0}^{\prime}$ is a plane. The radius of curvature $R$ in the direction of a unit vector $\psi_{[0]}^{\alpha}\left(R\right.$ counted positive when $S_{0}^{\prime}$ is convex in the sense of $x^{0}$ increasing) is"given by

$$
\begin{equation*}
\frac{1}{R}=\frac{1}{2} b_{a \beta} \mathbb{U}_{[0]}^{\alpha} \mathbb{w}_{[0]}^{s} . \tag{6.7}
\end{equation*}
$$

We proceed to find in particular the quantities (6.6). In the first place, since $\mathrm{x}^{\alpha}=x^{\alpha}$ on $S_{0}^{\prime}$, we have

$$
\begin{equation*}
a_{\alpha \beta}=a_{\alpha \beta}^{\prime}, \quad a_{\alpha \beta}=a_{\alpha \beta}+2 p_{\alpha \beta} \tag{6.8}
\end{equation*}
$$

We now follow a straight line normal to $S_{0}^{\prime}$, starting at the point $x^{\alpha}$. Since it is a geodesic, we have (cf. 6, p. 301)

$$
\frac{d^{2} x^{i}}{\left(d x^{0}\right)^{2}}+\left\{\begin{array}{c}
i  \tag{6.9}\\
j k
\end{array}\right\}_{a^{\prime}} \frac{d x^{j}}{d x^{0}} \frac{d x^{k}}{d x^{0}}=0
$$

and so we can develop $x^{i}$ as power series in $x^{0}$. For $x^{0}=0$, we have

$$
\begin{equation*}
x^{\alpha}=\mathrm{x}^{\alpha}, \quad x^{0}=0, \quad \frac{d x^{i}}{d \mathrm{x}^{0}}=\frac{\tilde{g}^{\prime 20}}{\left(\tilde{g}^{\prime 00}\right)^{1 / 2}} \tag{6.10}
\end{equation*}
$$

where $\bar{g}^{\prime i j}$ is the conjugate of $g_{i j}^{\prime}$. The last follows from the fact that the line is normal to $S_{0}^{\prime}$, and so

$$
\begin{equation*}
g_{\alpha j}^{\prime} \frac{d x^{i}}{d \mathrm{x}^{0}}=0, \quad g_{i j}^{\prime} \frac{d x^{i}}{d \mathrm{x}^{0}} \frac{d x^{i}}{d \mathrm{x}^{0}}=1 \tag{6.11}
\end{equation*}
$$

On carrying out the development in power series, and using the transformation

$$
\begin{equation*}
g_{i i}=g_{m n}^{\prime} \frac{d x^{m}}{d x^{i}} \frac{d x^{n}}{d x^{i}} \tag{6.12}
\end{equation*}
$$

we obtain after a little calculation

$$
\begin{equation*}
b_{\alpha \beta}-2 q_{\alpha \beta}=b_{\alpha \beta}^{\prime}=b_{\alpha \beta}\left(\tilde{a}^{\prime 00}\right)^{1 / 2}+a_{\alpha 0, \beta}^{\prime}+a_{\beta 0, \alpha}^{\prime}+2[\alpha \beta, \gamma]_{a^{\prime}} \bar{a}^{\prime} \gamma 0 / \tilde{a}^{\prime 00} \tag{6.13}
\end{equation*}
$$

where $\bar{a}^{\prime i j}=\bar{g}^{\prime i j}$ for $x^{0}=0$. The Christoffel symbol is calculated for $a_{\alpha \beta}^{\prime}$. By (4.10) and (6.3), we have

$$
\begin{equation*}
\tilde{a}^{\prime 00}=1+2 p_{00}+O_{(24)}^{00}\left(\hat{p}^{2}\right), \quad \tilde{a}^{\prime 0 \alpha}=2 p^{0 \alpha}+O_{(25)}^{0 \alpha}\left(\hat{p}^{2}\right) \tag{6.14}
\end{equation*}
$$

Thus (6.13) becomes

$$
\begin{equation*}
b_{\alpha \beta}-2 q_{\alpha \beta}^{\prime}=b_{\alpha \beta}^{\prime}=b_{\alpha \beta}-b_{\alpha \beta} p_{00}-2 p_{0 \alpha \mid \beta}-2 p_{0 \beta \mid \alpha}+O_{(26) \alpha \beta}\left(\dot{p}^{2}\right) \tag{6.15}
\end{equation*}
$$

where a under the stroke indicates the covariant differentiation with respect to $x^{\alpha}$ and $\boldsymbol{a}_{\alpha \beta}$.

Let us define $p_{\alpha \beta}, \boldsymbol{q}_{\alpha \beta}$ so that

$$
\begin{equation*}
2 \bar{q}=b_{\alpha \beta}-b_{\alpha \beta}, \quad 2 p_{\alpha \beta}=2 p_{\alpha \beta}=a_{\alpha \beta}-a_{\alpha \beta} \tag{6.16}
\end{equation*}
$$

then the extension and change of curvature of the middle surface $S_{0}^{\prime}$ along the direction of a unit vector $\mathcal{u}_{[0]}^{\alpha}$ are given by

$$
\begin{equation*}
p_{\alpha \beta} \mathcal{U}_{[01}^{\alpha} \mathcal{U}_{[0]}^{\beta}, \quad q_{a \beta} \mu_{[0]}^{\alpha} \mathcal{U}_{[0]}^{\beta} . \tag{6.17}
\end{equation*}
$$

From (6.15), (6.16), we have in consequence of (4.37), (4.39), (4.48a, b) and (6.8)

$$
\begin{align*}
& 2 q_{\alpha \beta}=2 q_{\alpha \beta}-\frac{\sigma}{1-\sigma} a^{\pi \lambda} b_{\alpha \beta} P_{\pi \lambda}+\frac{(1+\sigma)(1-2 \sigma)}{2(1-\sigma)} b_{\alpha \beta} Q^{0}+(1+\sigma) Q_{\alpha \mid \beta} \\
& \quad+(1+\sigma) Q_{\beta \mid \alpha}+O_{(27) \alpha \beta}  \tag{6.18}\\
& b_{\alpha \beta}=2 q_{\alpha \beta}+b_{\alpha \beta}, \quad a_{\alpha \beta}=2 p_{\alpha \beta}+a_{\alpha \beta}, \quad p_{\alpha \beta}=p_{a \beta} . \tag{6.19}
\end{align*}
$$

The symbol $O_{(27) \alpha \beta}$ represents the residual terms with the order of magnitude shown symbolically by

$$
\begin{array}{r}
O_{(27) \alpha \beta}=O_{(27) \alpha \beta}\left(p^{2}, \widehat{Q}_{p}, \widehat{Q}^{2}, p d, b p t^{(2)}, q t^{(2)}, Q^{0} d, b Q^{0} t^{(2)}, b Q d, Q t^{(2)}, q Q d,\right. \\
\left.b X^{0} d, X^{0} t^{(2)}, X d, X t^{(2)}, q X^{0} d\right) . \tag{6.20}
\end{array}
$$

Let us now denote the thickness of the shell or plate in the natural state by $2 h$. Then by ( 6.10 ), (4.10), (6.1), we have

$$
\begin{align*}
& h=\int_{0}^{h_{(+)}} \frac{d x^{0}}{\left(\tilde{g}^{\prime 00}\right)^{1 / 2}}=\int_{0}^{h_{(+)}}\left[1-e^{00}+O_{(28)}^{00}\left(e^{2}\right)\right] d x^{0},  \tag{6.21a}\\
& h=\int_{-h_{(-)}}^{0} \frac{d x^{0}}{\left(\hat{g}^{\prime 00}\right)^{1 / 2}}=\int_{-h_{(-)}}^{0}\left[1-e^{00}+O_{(28)}^{00}\left(\hat{e}^{2}\right)\right] d x^{0} . \tag{6.21b}
\end{align*}
$$

By (4.3), these become

$$
\begin{align*}
& h=h_{(+)}-p^{00} h_{(+)}+O_{(29)}\left(\dot{p}^{2} h_{(+)}, \hat{q} h_{(+)}^{2}, \hat{q} \hat{p} h_{(+)}^{2}\right)  \tag{6.22a}\\
& h=h_{(-)}-p^{00} h_{(-)}+O_{(30)}\left(\hat{p}^{2} h_{(-)}, \hat{q} h_{(-)}^{2}, \hat{q} \tilde{p} h_{(-)}^{2}\right) . \tag{6.22b}
\end{align*}
$$

Substituting (4.37)-(4.40) for $p_{i 0}, q_{i 0},(4.48 a, b)$ for $E_{[0]}^{i 0}$ and (6.18), (6.19) for $q_{\alpha \beta}, p_{\alpha \beta}$, $a_{\alpha \beta}, b_{\alpha \beta}$ in to ( $6.22 \mathrm{a}, \mathrm{b}$ ), we have two equations for the determination of $h_{(+)}$and $h_{(-)}$. We now solve these equations for $h_{(+)}$and $h_{(\rightarrow)}$. In deciding what terms to retain explicitly, we note that, for a thin or plate undergoing small strain, the quantities $h$, $h_{(t)}, h_{(-)}, p_{\alpha \beta}, Q^{i}, X_{[m]}^{1}$ are small. We obtain

$$
\begin{align*}
& h_{(+)}=h-\frac{\sigma}{1-\sigma} a^{\pi \lambda} p_{\pi \lambda} h+\frac{(1+\sigma)(1-2 \sigma)}{2(1-\sigma)} Q^{0} h+O_{(31)}  \tag{6.23a}\\
& h_{(-)}=h-\frac{\sigma}{1-\sigma} a^{\pi \lambda} p_{\pi \lambda} h+\frac{(1+\sigma)(1-2 \sigma)}{2(1-\sigma)} Q^{0} h+O_{(32)} . \tag{6.23b}
\end{align*}
$$

The $O$-symbols represent the residual terms with orders of magnitude shown symbolically by

$$
\begin{equation*}
p^{2} h, \widehat{Q}^{2} h, p \widehat{Q h}, p h^{2}, \widehat{Q} h^{2}, \widehat{X} h^{2}, q h^{2} . \tag{6.24}
\end{equation*}
$$

Hence from $(6.23 \mathrm{a}, \mathrm{b})$, we have immediately, in the notation of (3.26),

$$
\begin{align*}
t^{(n)} & =2\left(1-\frac{n \sigma}{1-\sigma} a^{\pi \lambda} p_{\pi \lambda}+\frac{n(1+\sigma)(1-2 \sigma)}{2(1-\sigma)} Q^{0}\right)+O_{(33)}  \tag{6.25a}\\
d^{(n)} & =O_{(34)} \tag{6.25b}
\end{align*}
$$

where

$$
\begin{align*}
& O_{(33)}=O_{(33)}\left(p^{2} h^{n}, \widehat{Q}^{2} h^{n}, p \widehat{Q} h^{n}, \widehat{Q} h^{n+1}, \widehat{X} h^{n+1}, p h^{n+1}, q h^{n+1}\right),  \tag{6.26a}\\
& O_{(34)}=O_{(3+)}\left(\widehat{X} h^{n+1}, \widehat{Q} h^{n+1}, q h^{n+1}, p h^{n+1}\right) . \tag{6.26b}
\end{align*}
$$

Similarly, substituting (6.23a, b) into (3.27), (3.28), we get

$$
\begin{align*}
& P^{(n) i}=P^{i} h^{n}+O_{(35)}^{i},  \tag{6.27a}\\
& Q^{(n) i}=Q^{i} h^{n}+O_{(36)}^{i}, \tag{6.27b}
\end{align*}
$$

where

$$
\begin{align*}
& O_{(35)}^{i}=O_{(35)}^{i}\left(\widehat{P} P h^{n}, \widehat{P Q} h^{n}, \widehat{P} \widehat{X} h^{n+1}, \widehat{P} q h^{n+1}\right),  \tag{6.28a}\\
& O_{(36)}^{i}=O_{(36)}^{i}\left(\widehat{Q} p h^{n}, \widehat{Q}^{2} h^{n}, \widehat{Q} \widehat{X} h^{n+1}, \widehat{Q} q h^{n+1}\right), \tag{6.28b}
\end{align*}
$$

where $P^{i}$ and $Q^{i}$ defined by (3.28), represent the sum and difference of the components of the surface loads with respect to the surface $S_{0}$ in the strained state.

With (6.18), (6.19), (6.25), (6.27) established, the expression for $T^{\alpha \beta}, L^{\alpha \beta}, T^{a 0}$ in (4.49), (4.50), (4.52) will now be reduced to forms involving $p_{\alpha \beta}, q_{\alpha \beta}$ instead of $p_{\alpha \beta}$, $q_{\alpha \beta}$. The results are as follows:
(i) The membrane stress tensor $T^{\alpha \beta}$,
$T^{\alpha \beta}=2 A_{(1)}^{\alpha \beta \pi \lambda} p_{x \lambda} h+A_{(\lambda)}^{\alpha \beta \pi \gamma \lambda \delta} b_{\lambda \delta} q_{\pi \gamma} h^{3}-A_{(3)}^{\alpha \beta \pi \gamma \lambda \delta} q_{\pi \gamma} q_{\lambda \delta} h^{3}+\frac{\sigma}{1-\sigma} a^{\alpha \beta} Q^{0} h+O_{(\eta 7)}^{\alpha \beta}$. (6.29)
(ii) The bending moment tensor $L^{\alpha \beta}$,

$$
\begin{align*}
L^{\alpha \beta}= & n_{(0)}^{\gamma \beta} a_{\pi \gamma}\left\{\frac{2}{3} A_{(1)}^{\alpha \pi \lambda \delta} q_{\lambda \delta}^{\alpha}+2 A_{(5)}^{\alpha \pi \lambda \delta \omega} b_{\left.\lambda \delta p_{\rho \omega}\right\}}\right\} h^{3} \\
& +\frac{\sigma}{\sigma(1-\sigma)} n_{[01}^{\gamma \beta}\left\{a_{\gamma}^{\alpha}\left(\frac{4 \sigma}{1-\sigma} H Q^{0}-4 X_{(01}^{0}-2 Q_{a}^{\lambda}\right)-b_{\gamma}^{\alpha} Q^{0}\right\} h^{3}+O_{(38)}^{\alpha \beta} . \tag{6.30}
\end{align*}
$$

(iii) The shearing stress tensor $T^{a 0}$,

$$
\begin{align*}
& T^{\alpha 0}=2\left\{A_{(6)}^{\pi \alpha \delta \delta \gamma} b_{\lambda \delta} P_{\rho \gamma} h^{3}+\frac{1}{3} A_{(1)}^{T \alpha \lambda \delta} q_{\lambda \delta} h^{3}\right\}_{a}+Q^{\alpha} h \\
& +\frac{1}{2}\left(4 H P^{\alpha}+a^{\alpha \pi} b_{\pi \gamma} P^{\gamma}\right) h^{2}+\left(a^{\pi \gamma} P^{\alpha}+a^{\alpha \pi} P^{\gamma}\right) q_{\pi \gamma} h^{2} \\
& +\frac{\sigma}{\sigma(1-\sigma)}\left\{\left[\mathbf{a}^{\alpha \tau}\left(\frac{4 \sigma}{1-\sigma} H Q^{0}-4 X_{[01}^{0}+\frac{2 \sigma}{1-\sigma} a^{\lambda \delta} q_{\lambda \delta} Q^{0}\right)\right.\right. \\
& \left.\left.-b^{\alpha \sigma} Q^{0}-2 q^{\alpha \sigma} Q^{0}\right] h^{3}\right\}_{a}+\frac{1}{3}\left\{\left(4 H a_{\pi}^{\alpha}+b_{\pi}^{\alpha}\right) X_{[0]}^{\pi}+X_{[11}^{\alpha}\right\}^{3} \\
& +\frac{2}{3}\left(a^{\pi \gamma} q_{\tau \gamma} a_{\lambda}^{\alpha}+a^{\alpha \tau} q_{\tau \lambda}\right) X_{[0]}^{\lambda} h^{3}+O_{(39)}^{\alpha 0} . \tag{6.31}
\end{align*}
$$

The residual terms in (6.29)-(6.31) are

$$
\begin{align*}
& O_{(38)}^{\alpha \beta}=O_{(38)}^{a \beta}\left(b p^{2} h^{3}, \widehat{Q} \widehat{Q}^{2} h^{3}, p \widehat{Q} h^{3}, \widehat{X Q} h^{3}, q p h^{3}, \widehat{X} p h^{3}, \widehat{Q} q h^{3}, b p h^{5}, q h^{5},\right. \\
& \left.b Q^{0} h^{5}, Q h^{5}, \widehat{X} h^{5}\right),  \tag{6.32a}\\
& O_{(37)}^{\alpha \beta}=O_{(37)}^{\alpha \beta}\left(p^{2} h, \widehat{\left.Q^{2} h, \widehat{Q} p h, p h^{3}, Q Q^{0} h^{3}, b Q h^{3}, q Q h^{3}, \widehat{X} h^{3}, b q p h^{3}, q^{2} p h^{3}, q h^{5}\right),}\right.  \tag{6.32b}\\
& O_{(39)}^{\alpha o}=O_{(39)}^{\alpha 0}\left(Q p h, Q \widehat{Q} h, Q \widehat{X} h^{2}, Q q h^{2}, b P p h^{2}, q P p h^{2}, b P \widehat{Q} h^{2}, q P \widehat{Q} h^{2}, p q h^{3},\right. \\
& \left.\quad b P q h^{3}, P q^{2} h^{3}, b P h^{4}, b p h^{5}, q h^{5}, b^{3} Q Q^{0} h^{5}, b^{2} Q h^{5}, \widehat{X} h^{5}\right) . \tag{6.32c}
\end{align*}
$$

The abbreviations are as follows:

$$
\begin{align*}
A_{(1)}^{\alpha \beta \pi \lambda} & =\frac{1}{1-\sigma^{2}}\left\{\sigma a^{\alpha \beta} a^{\pi \lambda}+(1-\sigma) a^{\alpha \pi} a^{\beta \lambda}\right\},  \tag{6.33a}\\
A_{(3)}^{\alpha \beta \pi \gamma \lambda \delta} & =\frac{2(2 \sigma-1)}{3(1-\sigma)} a^{\delta \lambda} A_{(1)}^{\alpha \beta \pi \gamma}+\frac{5}{3} a^{\delta \pi} A_{(1)}^{\alpha \beta \lambda \gamma},  \tag{6.33b}\\
A_{(4)}^{\alpha \beta \pi \gamma \lambda \delta} & =\frac{1}{3} a^{\delta \lambda} A_{(1)}^{\alpha \beta \pi \gamma}-\frac{\sigma}{2(1-\sigma)} a^{\pi \gamma} A_{(1)}^{\alpha \beta \delta \lambda}-\frac{1}{6} a^{\delta \gamma} A_{(1)}^{\alpha \beta \pi \lambda}-\frac{1}{2} a^{\pi \lambda} A_{(1)}^{\alpha \beta \delta \gamma},  \tag{6.33c}\\
A_{(6)}^{\alpha \pi \lambda \delta \rho \gamma} & =\frac{1}{6}\left\{\frac{\sigma}{1-\sigma} a^{\alpha \pi} A_{(1)}^{\lambda \delta \gamma \gamma}-2 a^{\rho \delta} A_{(1)}^{\alpha \pi \lambda \gamma}-a^{\pi \delta} A_{(1)}^{\alpha \lambda \rho \gamma}+a^{\lambda \delta} A_{(1)}^{\alpha \pi \rho \gamma}\right\},  \tag{6.33d}\\
n_{101}^{\alpha \beta} & =(a)^{-1 / 2} \epsilon^{\alpha \beta}, \quad a=\operatorname{det} .\left(a_{\alpha \beta}\right), \quad \epsilon^{11}=\epsilon^{22}=0,  \tag{6.33e}\\
\epsilon^{12} & =-\epsilon^{21}=1, \\
H & =\frac{1}{4} a^{\pi \lambda} b_{\pi \lambda}, \tag{6.33f}
\end{align*}
$$

where $H$ is the mean curvature of the middle surface $S_{0}^{\prime}$ in the unstrained state. All these quantities are determined by the geometry of the middle surface (in the unstrained state) of the shell or plate.

With (6.18), (6.19), (6.25), (6.27) established, (5.1), (5.2) can be reduced by direct substitution to the form involving $p_{\alpha \beta}$ and $q_{\alpha \beta}$ instead of $p_{\alpha \beta}$ and $q_{\alpha \beta}$. Thus we have the following three equations of equilibrium in terms of $p_{\alpha \beta}$ and $q_{\alpha \beta}$ :

$$
\begin{align*}
& -A_{(1)}^{\rho \gamma \pi \lambda} b_{\rho \gamma} p_{\pi \lambda} h-2 A_{(1)}^{\rho \gamma \pi \lambda} q_{\rho \gamma} p_{\pi \lambda} h+\frac{2}{3} A_{(1)}^{\rho \gamma \pi \lambda}\left(q_{\pi \lambda} h^{3}\right)_{l_{\rho \gamma}} \\
& -\frac{1}{2} A_{(\lambda)}^{\rho \gamma \pi \omega \lambda \delta} b_{\rho \gamma} b_{\lambda \delta} q_{\pi \omega} h^{3}+A_{(6)}^{\rho \gamma \pi \omega \lambda \delta} q_{\pi \omega} q_{\lambda \delta} b_{\rho \gamma} h^{3}+A_{(3)}^{p \gamma \pi \omega \lambda \delta} q_{\pi \omega} q_{\lambda \delta} q_{\rho \gamma} h^{3} \\
& +P^{0}+2 X_{[0]}^{0} h+\left(Q^{\pi} h\right)_{\left.\right|_{a}}+\frac{2(1-2 \sigma)}{1-\sigma} H Q^{0} h \\
& +\frac{1-2 \sigma}{1-\sigma} q_{\pi \lambda} a^{\pi \lambda} Q^{0} h=O_{(40)}^{0},  \tag{6.34}\\
& 2 A_{(1)}^{\rho \alpha \lambda \lambda}\left(p_{\pi \lambda} h\right)_{a}+A_{(\lambda)}^{\rho \alpha \pi \gamma \lambda \delta}\left(b_{\lambda \delta} q_{\pi \gamma} h^{3}\right)_{\left.\right|_{a}}+\frac{1}{3} A_{(1)}^{\gamma \beta \lambda \delta} a^{\alpha \pi} b_{\gamma \gamma}\left(q_{\lambda \delta} h^{3}\right)_{a} \\
& -A_{(3)}^{\rho \alpha \pi \gamma \delta \lambda}\left(q_{\pi \gamma} q_{\lambda \delta} h^{3}\right)_{a}+\frac{2}{3} a^{\alpha \pi} q_{\pi \gamma} A_{(1)}^{\gamma \beta \lambda \delta}\left(q_{\lambda \delta} h^{3}\right)_{a}+\frac{\sigma}{1-\sigma} a^{\alpha \beta}\left(Q^{0} h\right)_{a} \\
& +P^{\alpha}+2 X_{[0]}^{\alpha} h+\left(2 H a_{\pi}^{\alpha}+b_{\pi}^{\alpha}\right) Q^{\pi} h \\
& +\left(a^{\pi \lambda} q_{\pi \lambda} a_{\delta}^{\alpha}+2 a^{\alpha \pi} q_{\pi \delta}\right) Q^{\delta} h=O_{(41)}^{\alpha}, \tag{6.35}
\end{align*}
$$

where

$$
\begin{gather*}
O_{(40)}^{0}=O_{(40)}^{0}\left(Q p h, Q \widehat{Q} h, Q \widehat{X} h^{2}, Q q h^{2}, b P h^{2}, q P h^{2}, b^{2} P^{0} h^{2}, q^{2} P^{0} h^{2}, q b P^{0} h^{2},\right. \\
\left.b p h^{3}, b Q^{0} h^{3}, q Q^{0} h^{3}, Q h^{3}, \widehat{X} h^{3}, q p h^{3}, q h^{5}\right)  \tag{6.36a}\\
O_{(41)}^{\alpha}= \tag{6.36b}
\end{gather*}
$$

The abbreviations $A_{(1)}^{\rho \gamma \lambda \lambda}, A_{(3)}^{\rho \gamma \pi \omega \lambda \delta}, A_{(4)}^{p \gamma \pi \omega \lambda \delta}$ are given by $(6.33 \mathrm{a}, \mathrm{b}, \mathrm{c})$, while $A_{(0)}^{p \gamma \pi \omega \lambda \delta}$ is given by

$$
\begin{align*}
& A_{(6)}^{\rho \gamma \pi \omega \lambda \delta}=\frac{1}{\sigma\left(1-\sigma^{2}\right)}\left\{\frac{(9 \sigma-4) \sigma}{1-\sigma} a^{\rho \gamma} a^{\pi \omega} a^{\lambda \delta}-(2-7 \sigma) a^{\delta \pi} a^{\lambda \omega} a^{\rho \gamma}\right. \\
&\left.+9(1-\sigma) a^{\omega \delta} a^{\pi \rho} a^{\lambda \gamma}+(11 \sigma-2) a^{\pi \omega} a^{\lambda \rho} a^{\delta \gamma}\right\} \tag{6,37}
\end{align*}
$$

All these tensors are determined by the geometry of the unstrained state.
If we were to substitute $(6.18),(6.19),(6.25),(6.27)$ into $(5.5 \mathrm{a}, \mathrm{b})$, we would immediately obtain three equations of compatibility for $p_{\alpha \beta}, q_{\alpha \beta}$, with certain terms not explicitly calculated. However it is wiser to adopt an entirely different method, for the required equations of compatibility can be obtained in an exact form by a purely geometrical method.

We first introduce the equations of Codazzi and Gauss for the reference surface $S_{0}$ in the strained state,

$$
\begin{gather*}
b_{\alpha \beta \mid \gamma}-b_{\alpha \gamma \mid \beta}=0  \tag{6.38a}\\
R_{\rho \alpha \beta \gamma}=\frac{1}{4}\left(b_{\rho \beta} b_{\alpha \gamma}-b_{p \gamma} b_{\beta \alpha}\right), \tag{6.38b}
\end{gather*}
$$

and the corresponding equations for the middle surface $S_{0}^{\prime}$ in the unstrained state,

$$
\begin{gather*}
b_{\alpha \beta \mid \gamma}-b_{\alpha \gamma \mid \beta}=0,  \tag{6.39a}\\
R_{\rho \alpha \beta \gamma}=\frac{1}{4}\left(b_{\rho \beta} b_{\alpha \gamma}-b_{\rho \gamma} b_{\beta \alpha}\right) . \tag{6.39b}
\end{gather*}
$$

Here we recall that

$$
\begin{align*}
b_{\alpha \beta \mid \gamma}= & b_{a \beta, \gamma}-a^{\pi \rho}[\alpha \gamma, \rho]_{a} b_{\beta \pi}-a^{\pi \rho}[\beta \gamma, \rho]_{a} b_{\alpha \pi},  \tag{6.40a}\\
b_{\alpha \beta \mid \gamma}= & b_{a \beta, \gamma}-a^{\pi \rho}[\alpha \gamma, \rho]_{a} b_{\beta \pi}-a^{\pi \rho}[\beta \gamma, \rho]_{a} b_{\alpha \pi},  \tag{6.40b}\\
R_{\rho \alpha \beta \gamma}= & \frac{1}{2}\left(a_{\rho \gamma, \alpha \beta}+a_{\alpha \beta, \rho \gamma}-a_{\rho \beta, \alpha \gamma}-a_{\alpha \gamma, \rho \beta}\right) \\
& +a^{\pi \lambda}\left\{[\rho \gamma, \pi]_{a}[\alpha \beta, \lambda]_{a}-[\rho \beta, \pi]_{a}[\alpha \gamma, \lambda]_{a}\right\},  \tag{6.40c}\\
R_{\rho \alpha \beta \gamma}= & \frac{1}{2}\left(a_{\rho \gamma, \alpha \beta}+a_{\alpha \beta, \rho \gamma}-a_{\rho \beta, \alpha \gamma}-a_{\alpha \gamma, \rho \beta}\right) \\
& +a^{\pi \lambda}\left\{[\rho \gamma, \pi]_{a}[\alpha \beta, \lambda]_{a}-[\rho \beta, \pi]_{a}[\alpha \gamma, \lambda]_{a}\right\} . \tag{6.40d}
\end{align*}
$$

Furthermore by definition, we have

$$
\begin{align*}
a^{\alpha \omega} & =\frac{1}{2} \epsilon^{\pi \lambda} \epsilon^{\omega \delta} a_{\lambda \delta}, & a & =\frac{1}{2} \epsilon^{\pi \lambda} \epsilon^{\omega \delta} a_{\lambda \delta} a_{\pi \omega},  \tag{6.41a}\\
a^{\pi \omega} & =\frac{1}{2} \epsilon^{\pi \lambda} \epsilon^{\omega \delta} a_{\lambda \delta}, & a & =\frac{1}{2} \epsilon^{\pi \lambda} \epsilon^{\omega \delta} a_{\lambda \delta} a_{\pi \omega},  \tag{6.41b}\\
a_{\gamma}^{\pi} & =\delta_{\gamma}^{\pi}=\frac{1}{2} \epsilon^{\pi \lambda} \epsilon^{\rho \delta} a_{\lambda \delta} a_{\rho \gamma}, & a_{\gamma}^{\pi} & =\delta_{\gamma}^{\pi}=\frac{1}{2} \epsilon^{\pi \lambda} \epsilon^{\rho \delta} a_{\lambda \delta} a_{\rho \gamma}, \tag{6.41c}
\end{align*}
$$

where $\delta_{\gamma}^{\pi}$ is the Kronecker delta. Substitution of $a_{\lambda \delta}$ from ( 6.19 ) into ( $6.41 \mathrm{a}, \mathrm{c}$ ) gives, with (6.41b)

$$
\begin{align*}
a^{\pi \omega} & =\frac{a}{a}\left(2 \eta_{[0]}^{\pi \lambda} \eta_{[0]}^{\omega \delta} p_{\lambda \delta}+a^{\pi \omega}\right)  \tag{6.42a}\\
\frac{a}{a} & =1+2 \eta_{[01}^{\pi \lambda} \eta_{[0] 1}^{\gamma \delta} p_{\lambda \delta} p_{\pi \gamma}+2 p_{\pi \lambda} a^{\pi \delta} \tag{6.42b}
\end{align*}
$$

$$
\begin{equation*}
\delta_{\gamma}^{\pi}=\frac{a}{a}\left\{\delta_{\gamma}^{\pi}+2 p_{\lambda \gamma} a^{\lambda \pi}+2 \eta_{[0]}^{\pi \lambda} \eta_{[0]}^{\rho \delta} p_{\lambda j} a_{\rho \gamma}+4 \eta_{[01}^{\pi \lambda} \eta_{[01}^{\eta^{\delta}} p_{\lambda \delta} p_{\rho \gamma}\right\} \tag{6.42c}
\end{equation*}
$$

where $\mathbf{n}_{[0]}^{\text {rid }}$ is given as in (6.33f).
We now multiply (6.38a), (6.38b) respectively by (a/a) $n_{[0]}^{\beta \gamma},(a / a) n_{[0]}^{\beta \gamma} n_{[0]}^{p \alpha}$, and substitute $a_{\alpha \beta}, b_{\alpha \beta}$ from (6.19) into the resulting equations. This gives, in consequence of ( $6.42 \mathrm{a}, \mathrm{b}, \mathrm{c}$ ) and ( $6.39 \mathrm{a}, \mathrm{b}$ ), the following three equations in $p_{\alpha \beta}$ and $q_{\alpha \beta}$ :

$$
\begin{align*}
& 2 \mathbf{n}_{[01}^{\beta \gamma} q_{\alpha \beta \mid \gamma}\left(1+2 \eta_{[0]}^{\pi \lambda} \eta_{[0]}^{\omega \delta} p_{\pi \omega} p_{\lambda \delta}+2 a^{\pi \omega} p_{\pi \omega}\right) \\
& -\mathbf{n}_{[0]}^{\beta \gamma}\left(2 q_{\beta \pi}+b_{\beta \pi}\right)\left(a^{\pi \omega}+2 \mathbf{n}_{[0]}^{\beta \lambda} n_{[0]}^{\omega \delta} p_{\lambda \delta}\right)\left(p_{\alpha \omega \mid \gamma}^{a}+p_{\gamma \omega \mid \alpha}-p_{\alpha \gamma \mid \omega}\right)=0,  \tag{6.43}\\
& \left(1+2 \mathbf{n}_{00]}^{\pi \lambda} n_{[0]}^{\gamma \delta} p_{\lambda \delta} p_{\pi \gamma}+2 a^{\pi \gamma} p_{\pi \gamma}\right)\left\{2 \mathfrak{n}_{[0]}^{\rho \alpha} n_{[0]}^{\beta \omega} p_{\rho \omega \mid \alpha \beta}^{a}\right. \\
& \left.+n_{[0]}^{\rho \alpha} n_{[0]}^{\beta \omega} \boldsymbol{q}_{\rho \omega} \boldsymbol{q}_{a \beta}+2 \boldsymbol{a}^{\alpha \beta} p_{\alpha \beta} K-\left(4 H a^{\alpha \beta}-b^{\alpha \beta}\right) \boldsymbol{q}_{a \beta}\right\} \\
& +\mathbf{n}_{[0]}^{\rho \alpha} n_{[0]}^{\beta \gamma}\left(a^{\pi \omega}+2 n_{[0]}^{\pi \lambda} n_{[0 \mid}^{\mu \delta \delta} p_{\lambda \delta}\right)\left(p_{x \rho \mid \gamma}+p_{\gamma \pi \mid \rho}-p_{\gamma \rho \mid \pi}\right)\left(p_{\alpha \omega \mid \beta}^{a}+p_{\beta \omega \mid \alpha}-p_{\alpha \beta \mid \omega}\right)=0 . \tag{6.44}
\end{align*}
$$

Equations (6.43), (6.44) are the three equations of compatibility for $p_{\alpha \beta}$ and $\boldsymbol{q}_{\alpha \beta}$; these are to be associated with the three equations of equilibrium, (6.34), (6.35) for the solutions of plate or shell problems. We note that $K$ is the total curvature of the middle surface $S_{0}^{\prime}$ in the unstrained state, and satisfies

$$
\begin{equation*}
K=\frac{1}{\delta}\left(a^{\pi \gamma} b_{\pi \gamma} a^{\lambda \delta} b_{\lambda \delta}-b^{\pi \lambda} b_{\pi \lambda}\right) \tag{6.45}
\end{equation*}
$$

Conclusion. Equations (6.34), (6.35), (6.43), (6.44) are the final forms of six differential equations in the six unknowns $p_{\alpha \beta}$ and $q_{\alpha \beta}$. Here $p_{\alpha \beta}$ and $q_{\alpha \beta}$, as indicated in (6.17), represent the extension and change of curvature of $S_{0}^{\prime}$, the middle surface in the unstrained state. With $p_{\alpha \beta}$ and $q_{\alpha \beta}$ known, the macroscopic tensors $T^{\alpha \beta}, T^{\alpha 0}$, $L^{\alpha \beta}$ can be calculated from (6.29)-(6.31).

We recall that the tensors $a_{\alpha \beta}$ and $(1 / 2) b_{\alpha \beta}$ are respectively the first and second fundamental tensors of the middle surface $S_{0}^{\prime}$ in the unstrained state; $H$ and $K$ are the mean and total curvature of $S_{0}^{\prime}$ as in (6.33e), (6.45); $\sigma$ is Poisson's ratio; $2 h$ is the thickness of the shell or plate in the unstrained state; $P^{i}$ and $Q^{i}$ represent the sum and difference of the surface forces on the upper and lower surfaces of the shell or plate in the strained state as in (5.28) ; $X_{[m]}^{t}$ are the normal derivatives of the body force on the reference surface $S_{0}$ in the strained state as in (3.24). All these quantities may be regard as given. The covariant differentiations are calculated for $\boldsymbol{a}_{\alpha \beta}$ and $\mathrm{x}^{\alpha}$.

We also note that the six equations (6.34), (6.35), (6.43), (6.44) are exact, in the sense that no terms have been omitted, but of course the residual terms, represented by $O$-symbols, have not been calculated explicitly. However, it will be shown in Parts II and III that in all cases of small thickness and small strain, the residual terms are small compared with those shown explicitly, and it is legitimate to neglect them in a first approximation.

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# ELECTROMAGNETIC WAVES IN A BENT PIPE OF RECTANGULAR CROSS SECTION* 

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The analysis of electromagnetic wave propagation in a bent pipe of rectangular cross section, $(x=R, x=R+a, y=0, y=b)$, is based on the Maxwell field equations, expressed in cylindrical coordinates ( $r, \theta, y$ ) (Fig. 1). As in the case of the straight pipe, ${ }^{1}$ the time variation is given by the exponential $e^{j \omega t}$, where $\omega$ is the angular frequency. The angular variation is given by $e^{-\Sigma \theta}$, where $\Sigma$ is the propagation constant for the bent portion. The equations may be written

$$
\begin{array}{r}
-\Sigma E_{v}-r \partial E_{\theta} / \partial y+r j \omega \mu H_{r}=0, \\
\partial E_{r} / \partial y-\partial E_{y} / \partial r+j \omega \mu H_{\theta}=0, \\
r \partial E_{\theta} / \partial r+E_{\theta}+\Sigma E_{r}+r j \omega \mu H_{y}=0, \\
-\Sigma H_{y}-r \partial H_{\theta} / \partial y-r j \omega \epsilon E_{r}=0,  \tag{1}\\
\partial H_{r} / \partial y-\partial H_{y} / \partial r-j \omega \epsilon E_{\theta}=0, \\
r \partial H_{\theta} / \partial r+H_{\theta}+\Sigma H_{r}-r j \omega \epsilon E_{v}=0, \\
r \partial H_{r} / \partial r+H_{r}-\Sigma H_{\theta}+r \partial H_{\nu} / \partial y=0, \\
r \partial E_{\tau} / \partial r+E_{r}-\Sigma E_{\theta}+r \partial E_{y} / \partial y=0 .
\end{array}
$$

In (1) $H_{\theta}, H_{r}, H_{y}, E_{\theta}, E_{r}$ and $E_{y}$ are the components of magnetic and electric field, $\epsilon$ is the electric inductive capacity, and $\mu$ the magnetic inductive capacity. The electrical conductivity, $\sigma$, and charge density, $\rho$, are assumed to be zero.

The field components $H_{r}, H_{\nu}, E_{r}$ and $E_{y}$ may be expressed in terms of $H_{\theta}$ and $E_{\theta}$ by various combinations of the equations (1). These give

$$
\begin{align*}
H_{r}\left(G r^{2}\right) & =-\Sigma r \partial H_{\theta} / \partial r-\Sigma H_{\theta}-r^{2} j \omega \epsilon \partial E_{\theta} / \partial y,  \tag{2a}\\
H_{\nu}\left(G r^{2}\right) & =-\Sigma r \partial H_{\theta} / \partial y+r j \omega \epsilon E_{\theta}+r^{2} j \omega \epsilon \partial E_{\theta} / \partial r,  \tag{2b}\\
E_{r}\left(G r^{2}\right) & =-\Sigma r \partial E_{\theta} / \partial r-\Sigma E_{\theta}+r^{2} j \omega \mu \partial H_{\theta} / \partial y,  \tag{2c}\\
E_{y}\left(G r^{2}\right) & =-\Sigma r \partial E_{\theta} / \partial y-r j \omega \mu H_{\theta}-r^{2} j \omega \mu \partial H_{\theta} / \partial r, \tag{2d}
\end{align*}
$$

where

$$
\left(G r^{2}\right)=\Sigma^{2}+r^{2} \omega^{2} \mu \epsilon .
$$

Using the last two of Eqs. (1) and Eqs. (2), $H_{r}, H_{y}, E_{r}$ and $E_{y}$ may be eliminated and equations in $H_{\theta}$ and $E_{\theta}$ readily obtained.

$$
\begin{equation*}
\frac{\partial}{\partial r}\left[\frac{1}{G r} \frac{\partial\left(r H_{\theta}\right)}{\partial r}\right]+H_{\theta}+G^{-1} \frac{\partial^{2} H_{\theta}}{\partial y^{2}}+\frac{r j \omega \epsilon}{\Sigma} \frac{\partial E_{\theta}}{\partial y} \frac{\partial G^{-1}}{\partial r}=0, \tag{3a}
\end{equation*}
$$

[^2]\[

$$
\begin{equation*}
\frac{\partial}{\partial r}\left[\frac{1}{G r} \frac{\partial\left(r E_{\theta}\right)}{\partial r}\right]+E_{\theta}+G^{-1} \frac{\partial^{2} E_{\theta}}{\partial y^{2}}-\frac{r j \omega \mu}{\Sigma} \frac{\partial H_{\theta}}{\partial y} \frac{\partial G^{-1}}{\partial r}=0 . \tag{3b}
\end{equation*}
$$

\]

The boundary conditions for this case are

$$
\begin{array}{lll}
y=0, & y=b . & E_{\theta}=0, \quad E_{r}=0 \\
r=R, & r=R+a . & E_{\theta}=0, \quad E_{y}=0 \tag{4}
\end{array}
$$

A more useful form may be obtained from (2d):

$$
\begin{array}{lll}
y=0, & y=b . & E_{\theta}=0, \quad E_{r}=0 . \\
r=R, & r=R+a . & E_{\theta}=0, \quad \partial\left(r H_{\theta}\right) / \partial r=0 \tag{5}
\end{array}
$$

By considering $E_{\theta}$ and $H_{\theta}$ as functions of $y$ and $r$, these conditions establish the dependence of $E_{\theta}$ on $\sin k_{y} y$, and $H_{\theta}$ on $\cos k_{y} y$, where $k_{y}=n \pi / b, n$ being an in teger. By substituting

$$
E_{\theta}(r, y)=E_{\theta}(r) \sin k_{y} y, \quad H_{\theta}(r, y)=H_{\theta}(r) \cos k_{y} y,
$$

in (3) and simplifying, equations in $E_{\theta}$ and $H_{\theta}$ as functions of $r$ alone are obtained.

$$
\begin{align*}
& \frac{\partial^{2} H_{\theta}}{\partial r^{2}}+A(r) \frac{\partial H_{\theta}}{\partial r}+B(r) H_{\theta}+\epsilon C(r) E_{\theta}=0  \tag{6a}\\
& \frac{\partial^{2} E_{\theta}}{\partial r^{2}}+A(r) \frac{\partial E_{\theta}}{\partial r}+B(r) E_{\theta}+\mu C(r) H_{\theta}=0 \tag{6b}
\end{align*}
$$

where the coefficients $A(r), B(r)$ and $C(r)$ have the values

$$
\begin{align*}
& A(r)=\frac{1}{r}+\frac{2 \Sigma^{2}}{G r^{3}}=\frac{1}{r}+\frac{2 \Sigma^{2}}{r\left(\Sigma^{2}+r^{2} \omega^{2} \mu \epsilon\right)} \\
& B(r)=G-k_{y}^{2}-\frac{1}{r^{2}}+\frac{2 \Sigma^{2}}{G r^{4}}=\frac{\Sigma^{2}-1}{r^{2}}+\omega^{2} \mu \epsilon-k_{y}^{2}+\frac{2 \Sigma^{2}}{r^{2}\left(\Sigma^{2}+r^{2} \omega^{2} \mu \epsilon\right)}  \tag{7}\\
& C(r)=\frac{2 j k_{y} \omega \Sigma}{G r^{2}}=\frac{2 j k_{y} \omega \Sigma}{\left(\Sigma^{2}+r^{2} \omega^{2} \mu \epsilon\right)}
\end{align*}
$$



Fig. 1

Equations (6a) and (6b) show that $E_{\theta}$ and $H_{\theta}$ are not independent in the bent pipe. Furthermore, $E_{\theta}$ and $H_{\theta}$ do not vanish in this case, hence the methods of solution used for the straight pipe fail. $E_{\theta}$ and $H_{\theta}$ are not expressible in terms of Bessel functions. One possible method of solution of these equations, namely to substitute

$$
\begin{aligned}
& \phi_{1}=(\mu)^{1 / 2} H_{\theta}+(\epsilon)^{1 / 2} E_{\theta}, \\
& \phi_{2}=(\mu)^{1 / 2} H_{\theta}-(\epsilon)^{1 / 2} E_{\theta},
\end{aligned}
$$

and thus to obtain separated equations - in $\phi_{1}$ and $\phi_{2}$, is incorrect because the boundary conditions (4) are not satisfied.

Equations (6a) and (6b) have been
solved completely by a method of approximation, using the theory of the Schrödinger equation with perturbations. Only the zero order and first order terms are considered. This does not affect the generality of the solution, because in practice the radius of curvature of the pipe, $R$, may be chosen very large compared to the constants of the equations and to the dimension $a$ of the pipe.

To rewrite (6a) and (6b) in the familiar Schrödinger form, let.

$$
\begin{align*}
& \quad\left(r H_{\theta}\right) / R=\beta \\
& r=R+s=R(1+s / R), \quad 0<s<a  \tag{8}\\
& \gamma=\Sigma / R .
\end{align*}
$$

Thus

$$
\begin{align*}
\partial^{2} \beta / \partial s^{2}+f_{1}(s) \partial \beta / \partial s+g(s) \beta+h(s) \epsilon E_{\theta} & =0  \tag{9a}\\
\partial^{2} E_{\theta} / \partial s^{2}+f_{2}(s) \partial E_{\theta} / \partial s+g(s) E_{\theta}+h(s)_{\mu} I I_{\theta} & =0 \tag{9b}
\end{align*}
$$

The coefficients, to the first approximation in $R^{-1}$, are given by

$$
\begin{aligned}
f_{1}(s)= & R^{-1}\left(-1+2 \gamma^{2} / K^{2}\right), \quad f_{2}(s)=R^{-1}\left(1+2 \gamma^{2} / K^{2}\right) \\
& g(s)=K^{2}-k_{y}^{2}-2 \gamma^{2} s / R=k_{x}^{2}-2 \gamma^{2} s / R \\
& h(s)=2 j k_{y} \omega \gamma / R K^{2}
\end{aligned}
$$

where

$$
K^{2}=\gamma^{2}+\omega^{2} \mu \epsilon=k_{x}^{2}+k_{y}^{2}
$$

Continuing the approximation, $E_{\theta}$ and $\beta$ may be written as

$$
\begin{equation*}
E_{\theta}=\left(E_{\theta}\right)_{0}+R^{-1}\left(E_{\theta}\right)_{1}+\cdots, \quad \beta=\beta_{0}+R^{-1} \beta_{1}+\cdots, \tag{10}
\end{equation*}
$$

and the perturbation of the angular coefficient for each case as

$$
\begin{equation*}
k_{x}^{2}=k_{s}^{2}+R^{-1} \varepsilon_{1} \quad \text { for } E_{\theta}, \quad k_{x}^{2}=k_{s}^{2}+R^{-1} h_{1} \text { for } \beta \tag{11}
\end{equation*}
$$

By substituting (10) and (11) in (9), the zero order and first order approximations may be written separately. For $E_{\theta}$ these are

$$
\begin{gather*}
\frac{\partial^{2}\left(E_{\theta}\right)_{0}}{\partial s^{2}}+k_{\sim}^{2}\left(E_{\theta}\right)_{0}=0  \tag{12a}\\
\frac{\partial^{2}\left(E_{\theta}\right)_{1}}{\partial s^{2}}+\frac{\partial\left(E_{\theta}\right)_{0}}{\partial s}\left[1+2 \gamma^{2} / K^{2}\right]+\left(E_{\theta}\right)_{I} k_{\theta}^{2} \\
+e_{1}\left(E_{\theta}\right)_{0}-2 \gamma^{2} s\left(E_{\theta}\right)_{0}+\frac{2 j k_{\nu} \gamma \omega \mu\left(H_{\theta}\right)_{0}}{R K^{2}}=0 . \tag{12b}
\end{gather*}
$$

The zero order equation (12a) has the same form as the equation for the straight pipe, with the solution

$$
\begin{equation*}
\left(E_{\theta}\right)_{0}=E_{m, n} \sin k_{n} S, \tag{13}
\end{equation*}
$$

where $m$ and $n$ are integers.
Similar equations may be written for $\beta$, giving

$$
\begin{equation*}
\beta_{0}=\lim \frac{r}{R}\left(H_{\theta}\right)_{0}=H_{m, n} \cos k_{s} s=\left(H_{\theta}\right)_{0} \tag{14}
\end{equation*}
$$

Eq. (12b) may be rewritten as

$$
\begin{align*}
\frac{\partial^{2}\left(E_{\theta}\right)_{1}}{\partial s^{2}}+k_{\Delta}^{2}\left(E_{\theta}\right)_{1}= & -e_{1}\left(E_{\theta}\right)_{0}+2 \gamma^{2} s\left(E_{\theta}\right)_{0} \\
& -\left(1+2 \gamma^{2} / K^{2}\right) \frac{\partial\left(E_{\theta}\right)_{0}}{\partial s}-\frac{2 j k_{\nu} \omega \mu \gamma\left(H_{\theta}\right)_{0}}{K^{2}} \tag{15}
\end{align*}
$$

This is the general form of the Schrödinger equation with perturbations, where the usual perturbation factor $\lambda$ is equal to $1 / R$.

By using the orthogonality condition for the Schrödinger theory, the value of $e_{1}$ may be readily determined:

$$
\begin{aligned}
\int_{0}^{a}\left(-e_{1}+2 \gamma^{2} s\right) E_{m, n}^{2} \sin ^{2} k_{z} s d s & -\int_{0}^{a}\left(1+2 \gamma^{2} / K^{2}\right) k_{s} E_{m, n}^{2} \cos k_{7} s \sin k_{s} s d s \\
& -\int_{0}^{a} 2 j k_{y} \omega \mu \gamma K^{-2} H_{m, n} E_{m, \pi} \cos k_{s} s \sin k_{s} s d s=0
\end{aligned}
$$

Therefore

$$
e_{1}=\gamma^{2} a
$$

By using this value and (13), the first approximation (15) may be solved for $\left(E_{\theta}\right)_{1}$. The solution, satisfying the boundary conditions, is given by

$$
\begin{align*}
\left(E_{\theta}\right)_{1}= & E_{m, n} \cos k_{s} s\left[\left(\gamma^{2} s\right)(a-s)\left(2 k_{s}\right)^{-1}\right] \\
& +E_{m, n} \sin k_{s} s\left[(s)\left(2 k_{s}^{2}\right)^{-1}\left\{\gamma^{2}-\left[1+\frac{2 \gamma^{2}}{K^{2}}\right] k_{s}^{2}\right\}\right] \\
& -H_{m, n} \sin k_{s} s\left[\left(j k_{y} s \omega \mu \gamma\right)\left(K^{2} k_{s}\right)^{-1}\right] . \tag{16a}
\end{align*}
$$

In like manner, from the $\beta$ approximation equations,

$$
h_{1}=\gamma^{2} a
$$

Since $e_{1}=h_{1}$, there is no change in the angle variable during the perturbations. The solution of the $\beta$ equation, satisfying the boundary conditions, and corresponding to (16a) is

$$
\begin{align*}
\beta_{1}= & E_{m, n} \cos k_{8} s\left[\left(j k_{y} s \omega \epsilon \gamma\right)\left(k_{s} K^{2}\right)^{-1}\right]-E_{m, n} \sin k_{s} s\left[\left(j k_{\Downarrow} \omega \in \gamma\right)\left(k_{\varepsilon}^{2} K^{2}\right)^{-1}\right] \\
& +H_{m, n} \cos k_{s} s\left[(s / 2)\left(1-2 \gamma^{2} / K^{2}+\gamma^{2} / k_{s}^{2}\right)\right] \\
& +H_{m, n} \sin k_{s} s\left[\left(2 k_{s}\right)^{-1}\left(-\gamma^{2} a s-1+\gamma^{2} s^{2}+2 \gamma^{2} / K^{2}-\gamma^{2} / k_{z}^{2}\right)\right] \tag{16b}
\end{align*}
$$

The complete solutions of (6a) and (6b), including both the zero order and first order approximations, may be written as

$$
\begin{align*}
E_{\theta}= & \left\{E_{m, n} \sin k_{s} s\left[1+c_{1} s\right]+E_{m, n} \cos k_{s} s\left[s(a-s) c_{2}\right]\right. \\
& \left.-H_{m, n} \sin k_{s} s\left[\mu c_{3} s\right]\right\}\left\{\sin k_{y} y e^{j \omega t \Sigma \theta}\right\}  \tag{17a}\\
H_{\theta}= & \left\{H_{m, n} \cos k_{s} s\left[1+c_{1} s\right]-H_{m, n} \sin k_{s} s\left[s(a-s) c_{2}-c_{4}\right]\right. \\
& \left.-E_{m, n} \sin k_{s} s\left[\epsilon c_{3} / k_{s}\right]+E_{m, n} \cos k_{s} s\left[\epsilon c_{3} s\right]\right\}\left\{\cos k_{y} y e^{j \omega t-\Sigma \theta}\right\}, \tag{17b}
\end{align*}
$$

where

$$
\begin{aligned}
& c_{1}=\frac{\gamma^{2}}{2 R k_{s}^{2}}-\frac{1}{2 R}-\frac{\gamma^{2}}{R K^{2}}, \quad c_{2}=\frac{\gamma^{2}}{2 R k_{s}}, \\
& c_{3}=\frac{j k_{y} \omega \gamma}{R k_{8} K^{2}}, \quad c_{4}=\frac{-\gamma^{2}}{2 R k_{s}^{3}}-\frac{1}{2 R k_{z}}+\frac{\gamma^{2}}{R K^{2} k_{s}} .
\end{aligned}
$$

By using (2) and the approximations (8) and (10), the components $H_{r}, H_{y}, E_{r}$ and $E_{y}$ are seen to become

$$
\begin{align*}
-K^{2} H_{r}= & \left\{H _ { m , n } \operatorname { s i n } k _ { s } s \left[-\gamma k_{s}\left(1+c_{1} s-s / R+2 \gamma^{2} s / R K^{2}\right)\right.\right. \\
& \left.+c_{2}(2 s-a) \gamma-j \omega \mu \epsilon c_{3} k_{y} s\right] \\
& +H_{m, n} \cos k_{s} s\left[\gamma\left\{c_{1}+R^{-1}-c_{2} s(a-s) k_{s}\right\}+\gamma k_{s} c_{4}\right] \\
& +E_{m, n} \sin k_{s} s\left[j \omega \epsilon k_{y}\left(1+c_{1} s+2 \gamma^{2} s / R K^{2}\right)-k_{s} c_{3} \gamma \epsilon s\right] \\
& \left.+E_{m, n} \cos k_{s} s\left[j \omega \epsilon k_{y} s(a-s) c_{2}\right]\right\} \cos k_{y} y e^{j \omega t-\Sigma \theta},  \tag{18a}\\
K^{2} H_{y}= & \left\{E_{m, n} \sin k_{s} s\left[j \omega \epsilon\left\{c_{1}-k_{8} s(a-s) c_{2}+R^{-1}\right\}-\left(c_{3} k_{y} \gamma \epsilon\right)\left(k_{s}\right)^{-1}\right]\right. \\
& +E_{m, n} \cos k_{s} s\left[j \omega \epsilon k_{s}\left(1+c_{1} s+2 \gamma^{2} s / R K^{2}\right)-j \omega \epsilon(2 s-a) c_{2}+k_{y} c_{3} \gamma \epsilon s\right] \\
& -H_{m, n} \sin k_{s} s\left[j \omega \mu \epsilon \epsilon_{3}+\gamma k_{v}\left\{s(a-s) c_{2}-c_{4}\right\}\right] \\
& \left.+H_{m, n} \cos k_{s} s\left[-j \omega \mu \epsilon k_{s} c_{3} s+\gamma k_{y}\left(1+c_{1} s-s / R+2 \gamma^{2} s / R K^{2}\right)\right]\right\} \\
& \cdot \sin k_{y} y e^{j \omega t-\Sigma \theta},  \tag{18b}\\
-K^{2} E_{r}= & \left\{E_{m, n} \sin k_{s} s\left[\gamma\left\{c_{1}-c_{2} s(a-s) k_{s}+R^{-1}\right\}-\left(k_{y} c_{3} j \omega \mu \epsilon\right)\left(k_{s}\right)-1\right]\right. \\
& +E_{m, n} \cos k_{s} s\left[\gamma k_{s}\left(1+c_{1} s-s / R+2 \gamma^{2} s / R K^{2}\right)+j \omega \mu \epsilon k_{y} c_{3} s-c_{2} \gamma(2 s-a)\right] \\
& -H_{m, n} \sin k_{s} s\left[c_{3} \gamma \mu+j \omega \mu k_{y}\left\{s(a-s) c_{2}-c_{4}\right\}\right] \\
& \left.+H_{m, n} \cos k_{s} s\left[j \omega \mu k_{y}\left(1+c_{1} s+2 \gamma^{2} s / R K^{2}\right)-k_{s} c_{3} \gamma \mu s\right]\right\} \\
& \cdot \sin k_{y} y e^{j \omega t-\Sigma \theta} \tag{18c}
\end{align*}
$$

$-K^{2} E_{\nu}=\left\{E_{m, n} \sin k_{s} s\left[\gamma k_{y}\left(1+c_{1} s-s / R+2 \gamma^{2} s / R K^{2}\right)-c_{3} k_{v} j \omega \mu \epsilon s\right]\right.$
$+E_{m, n} \cos k_{s} s\left[\gamma k_{1} s(a-s) c_{2}\right]$
$+H_{m, n} \sin k_{s} s\left[-j \omega \mu k_{s}\left(1+c_{1} s+2 \gamma^{2} s / R K^{2}\right)+j \omega \mu c_{2}(2 s-a)-k_{v} c_{3} s \gamma \mu\right]$
$\left.+H_{m, n} \cos k_{s} s\left[j \omega \mu\left\{c_{1}-k_{3} s(a-s) c_{2}+R^{-1}\right\}+j \omega \mu k_{3} c_{4}\right]\right\}$ - $\cos k_{y} y e^{j \omega t-\Sigma \theta}$.

The solutions for the field components (18) satisfy the Maxwell field equations (1) within the approximation conditions imposed on the solution of the problem.

For the special cases of $H_{m, n}$ and $E_{m, n}$ when one of the integers $m$ or $n$ is zero, the components may be obtained from (18). For $m=0$ and $n$ not equal to zero:

$$
\begin{aligned}
E_{\theta} & =H_{0, n}\left[j \omega \mu \gamma k_{y} s(a-s) / R K_{m}^{2}\right] \sin k_{y} y e^{j \omega t-\Sigma \theta}, \\
H_{\theta} & =H_{0, n}\left[1-s / R-\gamma^{2} a s^{2} / 2 R+\gamma^{2} s^{3} / 3 R\right] \cos k_{y} y e^{j \omega t-\Sigma \theta}, \\
K_{m}^{2} H_{r} & =H_{0, n}\left[\gamma s(a-s) R^{-1}\left(\omega^{2} \mu \epsilon k_{y}^{2} / K_{m}^{2}+\gamma^{2}\right)\right] \cos k_{y} y e^{j \omega t-\Sigma \theta}, \\
K_{m}^{2} H_{y} & =H_{0, n}\left[k_{y} \gamma R^{-1}\left(-\omega^{2} \mu \epsilon a / K_{m}^{2}+R-\gamma^{2} a s^{2} / 2+\gamma^{2} s^{3} / 3\right)\right] \sin k_{y} y e^{j \omega t-\Sigma \theta}, \\
-K_{m}^{2} E_{r} & =H_{0, n}\left[j \omega \mu k_{y} R^{-1}\left(\gamma^{2} a / K_{m}^{2}+R-s-\gamma^{2} a s^{2} / 2+\gamma^{2} s^{3} / 3\right)\right] \sin k_{y} y e^{i \omega t-\Sigma \theta}, \\
E_{y} & =0,
\end{aligned}
$$

where

$$
K_{m}^{2}=\left(K^{2}\right)_{m=0}=k_{y}^{2}+\gamma^{2} a / R
$$

For $m$ not equal to zero, $n=0$ :

$$
\begin{aligned}
E_{\theta}= & 0 \\
H_{\theta}= & \left\{H_{m, 0} \cos k_{s} s\left[1+c_{1} s\right]-H_{m, 0} \sin k_{s} s\left[(a-s) c_{2} s-c_{4}\right]\right\} \cos k_{y} y e^{j \omega t-\Sigma \theta}, \\
-K_{n}^{2} H_{r}= & \left\{H_{m, 0} \sin k_{s} s\left[c_{2} \gamma(2 s-a)-\gamma k_{s}\left(1+c_{1} s-s / R+2 \gamma^{2} s / R K_{n}^{2}\right)\right]\right. \\
& \left.+H_{m, 0} \cos k_{s} s\left[\gamma\left\{c_{1}+R^{-1}-c_{2} s(a-s) k_{s}\right\}+k_{s} c_{4} \gamma\right]\right\} \cos k_{y} y e^{j \omega t-\Sigma \theta}, \\
H_{y}= & 0, \quad E_{r}=0 \\
-K_{n}^{2} E_{y}= & \left\{H_{m, 0} \sin k_{s} s\left[-j \omega \mu k_{s}\left(1+c_{1} s+2 \gamma^{2} s / R K_{n}^{2}\right)+j \omega \mu c_{2}(2 s-a)\right]\right. \\
& +H_{m, 0} \cos k_{s} s\left[j \omega \mu\left\{c_{1}-k_{s} s(a-s) c_{2}+R^{-1}\right\}+j \omega \mu c_{4} k_{s}\right] \cos k_{y} y e^{j \omega t-\Sigma \theta},
\end{aligned}
$$

where the $c_{1}, c_{2}, c_{3}, c_{4}$ are calculated for $n$ vanishing, and

$$
K_{n}^{2}=\left(K^{2}\right)_{n=0}=k_{s}^{2}+\gamma^{2} a / R
$$

It should be noted that both the $E_{0, n}$ and $E_{m, 0}$ are missing.
A consideration of the continued propagation of $E$ and $H$ waves from a straight pipe into a bent pipe yields some interesting results. A pure $E_{m, n}$ or $H_{m, n}$ wave in the straight pipe will be reflected, partially, at the junction with the bent pipe. After reflection the amplitudes are proportional to $a / R$, and intensities to $a^{2} / R^{2}$, hence, for the first approximation, the reflected portion may be neglected. Thus a pure $E_{m, n}$ or a pure $H_{m, n}$ wave in the straight pipe may be traced into the bent pipe, where it will become a mixed $E$ and $H$ wave.

For a mixed $E$ and $H$ wave in the straight portion, the intensities are proportional to $a / R$ and must be considered. A mixed $E$ and $H$ wave in the straight pipe, because of the reflected portion at the junction, sets up an undetermined condition within the pipe, not predictable from the results of this paper.

If the propagation constant is measured along the center line, $a / 2$, of the bent pipe, there is no change in its value from that of the straight pipe.

# AN APPLICATION OF ORTHOGONAL MOMENTS TO PROBLEMS IN STATICALLY INDETERMINATE STRUCTURES* 

BY<br>W. M. KINCAID and V. MORKOVIN<br>Brown University

1. Numerous methods have been devised for determining moment profiles ${ }^{1}$ in statically indeterminate structures. Most of these can be classified as either approximate or exact. The approximate methods are usually simple to apply, but when employing them it is necessary to specify numerical values for the dimensions and stiffnesses (or their ratios) of the structure involved. The exact methods consist in solving systems of linear equations obtained by setting up relations between generalized displacements or by the equivalent means of using Castigliano's Principle. As the degree of indeterminacy of the structure increases, the solution of the equations becomes more laborious. Most of the methods aim at reducing such labor by a suitable choice of unknowns. This paper is an attempt in that direction.
2. Consider a structure whose degree of statical indeterminacy is $N$. Denote by $M$ the true moment profile in the structure under a given load, and by $M_{0}$ the moment profile under the same load when the structure has been made statically determinate by removing $N$ constraints (the so-called basic structure). The effect of the removed constraints may be replaced by the combined effect of $N$ unknown generalized forces (couples and forces) $X_{1}, X_{2}, \cdots, X_{N}$. The generalized displacements of the loaded structure at the points of application and in the directions of $X_{1}, X_{2}, \ldots, X_{N}$, are assumed to be known and will be denoted by $\delta_{1}, \delta_{2}, \cdots, \delta_{N}$. (If $X_{i}$ is a couple, $\delta_{i}$ is a rotation; if $X_{i}$ is an ordinary force, $\delta_{i}$ is an ordinary displacement.) Let $M_{i}$ be the moment profile obtained when the force corresponding to $X_{i}=1, X_{j}=0$ for $j \neq i$, acts on the (unloaded) basic structure. Then the moment profile $X_{i} M_{i}$ represents the effect of the $i$ th constraint. Superposing the moment profiles due to the load and the constraints yields the true moment profile $M$ :

$$
\begin{equation*}
M=M_{0}+\sum_{i=1}^{N} X_{i} M_{i} \tag{1}
\end{equation*}
$$

The unknown quantities $X_{i}$ in (1) are to be determined by means of Castigliano's Principle.

Disregarding as usual the contributions of shearing stresses and axial forces, we get for the total strain energy $U$ the following expression:

$$
\begin{equation*}
U=\frac{1}{2} \int M^{2} d x^{\prime} \tag{2}
\end{equation*}
$$

[^3]where $d x^{\prime}=d x / E I, E$ is Young's modulus, $I$ is the moment of inertia of any crosssection about the neutral axis, and the integral is taken over the entire structure. Castigliano's Principle states that
$$
\frac{\partial U}{\partial \bar{X}_{i}}=\delta_{i} \quad(i=1,2, \cdots, N)
$$
or, by virtue of (1) and (2),
\[

$$
\begin{equation*}
M_{0} M_{i} d x^{\prime}+\sum_{j=i}^{N} X_{j} \int M_{i} M_{i} d x^{\prime}=\delta_{i}(i=1,2, \cdots, N) \tag{3}
\end{equation*}
$$

\]

It will be observed that (1) and (3) are still valid if each $X_{i}$ represents a set of generalized forces (rather than a single force) acting simultaneously at different points. (See for instance Fig. 2d.) In such a case, each $\delta_{i}$ would be made up of the generalized displacements corresponding to the given set of generalized forces.

Previous attacks upon the problem have essentially consisted in choosing the points of application and lines of action of the unknown forces $X_{i}$ so as to make the system of equations (3) as simple as possible. Thus the moment profiles $M_{i}$ were completely determined. We propose to reverse this procedure by specifying the moment profiles $M_{i}$ first. Let us choose these profiles so that

$$
\begin{equation*}
\int M_{i} M_{j} d x^{\prime}=0 \quad(i, j=1,2, \cdots, N ; i \neq j) \tag{4}
\end{equation*}
$$

Then each of the equations (3) will contain only one unknown, and we get at once

$$
\begin{equation*}
X_{i}=\frac{\delta_{i}-\int M_{0} M_{i} d x^{\prime}}{\int M_{i}^{2} d x^{\prime}} \quad(i=1,2, \cdots, N) \tag{5}
\end{equation*}
$$

The true moment profile $M$ may now be obtained by substituting these values $X_{i}$ into (1).

It will be recognized that equations (4) require that the moment profiles $M_{i}$ form an orthogonal system over the structure. Such a system can always be constructed by the standard orthogonalization process from the original set of moment profiles (or any similar set of linearly independent moment profiles). ${ }^{2}$ Therefore, the system of orthogonalized moment profiles and the corresponding generalized forces which appear in (5) will consist of linear combinations of the original sets of $M_{i}$ and $X_{i}$, respectively. In most practical applications, the exact form of these relations, as well as the physical interpretation of the generalized forces $X_{i}$, is immaterial; only the orthogonalized moment profiles $M_{i}$ are needed. The simple example that follows will illustrate the notions introduced in the preceding


Fig. 1 discussion.
3. Let us consider the triply indeterminate bent ABCD (Fig. 1), which has undergone a vertical displacement $d$ and a rotation $\theta$ at D (as a result of a settlement of

[^4]

Fig. 2
the foundation). The member $A B$ is subjected to a uniform horizontal pressure $p$. The stiffnesses $E I$ of the membersAB and $C D$ are assumed to be equal. In this and succeeding examples, the positive direction will be taken as downward in vertical members and toward the right in horizontal members.

The solution will be carried out as follows: (1) the construction of an orthogonal system will be shown, and (2) this system will be used to obtain the desired moment profiles.

Figures 2a, 2b, and 2 c show three self-equilibrating systems of generalized forces and the corresponding moment profiles; it is obvious that any set of reactions (in equilibrium) can be built up by superposing these three in the right proportions.

We note that the moment profiles $M_{2 a}$ and $M_{2 b}$ are symmetric with respect to a vertical axis through the midpoint of $B C$ and would be orthogonal to any antisymmetric profile. Therefore we replace $M_{2 c}$ by $M_{2 d}$ (Fig. 2d). It remains to replace $M_{2 a}$ by a linear combination of $M_{2 a}$ and $M_{2 b}$, say $\alpha M_{2 a}+\beta M_{2 b}$, that will be orthogonal to $M_{2 b}$. The orthogonality condition reads

$$
\begin{equation*}
0=\alpha \int M_{2 a} M_{2 b} d x^{\prime}+\beta \int M_{2 b}^{2} d x^{\prime}=\alpha\left(l^{\prime}+h^{\prime}\right)+\beta\left(l^{\prime}+2 h^{\prime}\right), \tag{6}
\end{equation*}
$$

where $l^{\prime}=l / E I_{B C}, h^{\prime}=h / E I_{A B}$. Eq. (6) will be satisfied if $\alpha=\kappa+2$ and $\beta=-(\kappa+1)$, where $\kappa=l^{\prime} / h^{\prime}$. The resulting moment profile $M_{2 e}$ is shown in


Fig. 3.

$$
\begin{equation*}
M=M_{0}+X_{2 b} M_{2 b}+X_{2 d} M_{2 d}+X_{2 \theta} M_{2 \theta} . \tag{7}
\end{equation*}
$$

Clearly, the quantities $X$ may best be interpreted as mere coefficients by which the generalized forces in Figs. 2b, 2d, and 2e must be multiplied in order to yield, upon superposition, the correct reactions at the supports. ${ }^{3}$ As for the interpretation of $\delta_{2 b}, \delta_{2 d}$, and $\delta_{2 e}$, we recall that these quantities are equal to the partial derivatives, with respect to the corresponding quantities $X$, of the work done by the reactions. For instance,

$$
\delta_{2 d}=\frac{\partial}{\partial X_{2 d}}\left[\left\{X_{2 d}-X_{2 b}+(\kappa+1) X_{2 \theta}\right\} \theta-\frac{2 X_{2 d}}{l} d\right]=\theta-\frac{2 d}{l} .
$$

Similarly $\delta_{2 b}=-\theta$ and $\delta_{2 e}=(\kappa+1) \theta$.
Next, we evaluate the following important quantities:

$$
\begin{gather*}
\int M_{2 b}^{2} d x^{\prime}=h^{\prime}(\kappa+2), \quad \int M_{2 d}^{2} d x^{\prime}=\frac{1}{3} h^{\prime}(\kappa+6), \\
\int M_{2 e}^{2} d x^{\prime}=\frac{1}{3} h^{\prime}(\kappa+2)(2 \kappa+1) ;  \tag{8}\\
\int M_{0} M_{2 b} d x^{\prime}=\int M_{0} M_{2 d} d x^{\prime}=-\frac{1}{6} \phi h^{2} h^{\prime}, \quad \int M_{0} M_{2 d} d x^{\prime}=\frac{1}{24} p h^{2} h^{\prime}(3 \kappa+2) . \tag{9}
\end{gather*}
$$

Thus the equations (5) specialize to

$$
\begin{gather*}
X_{2 b}=\frac{-6 \theta+p h^{2} h^{\prime}}{6 h^{\prime}(\kappa+2)}, \quad X_{2 d}=\frac{6 \theta-12 d / l+p h^{2} h^{\prime}}{2 h^{\prime}(\kappa+6)} \\
X_{2 \theta}=\frac{24(\kappa+1) \theta-(3 \kappa+2) p h^{2} h^{\prime}}{8 h^{\prime}(\kappa+2)(2 \kappa+1)} . \tag{10}
\end{gather*}
$$

To complete the solution we have only to substitute (10) into (7) and tabulate the values of $M$ at $\mathrm{A}, \mathrm{B}, \mathrm{C}$, and D .
4. In the present method, the problem of solving the system (3) is replaced by that of constructing a set of orthogonal moment profiles. Once such a set is known for a given structure, the solution $M$ corresponding to any loading is obtained merely by evaluating the right-hand member of (5) and substituting into (1). ${ }^{4}$ This is an advantage of the present method over others in which the complete system (3) has to be solved anew every time the load is altered. Thus, for important structures, it may be worthwhile to construct an orthogonal set, even in cases where the orthogonalization itself is fairly complicated.

Fortunately, the rather lengthy standard orthogonalization process (see footnote 2 ) seldom needs to be used in its entirety, as the preceding example indicates. Shortcuts involving the use


Fig 4. of symmetry, antisymmetry, and other characteristics of the

[^5]structure are usually available. For instance, in the case of the triply indeterminate frame in Fig. 4 (with $I_{A B}=I_{C D}$ and $I_{B C}=I_{A D}$ ), we obtain the $M_{i}$ 's immediately from symmetry considerations (Fig. 5).


Fig. 5.
The evaluation of integrals $\int M_{0} M_{i} d x^{\prime}$ in (5) presents few difficulties, because the moments $M_{i}$ always vary linearly over any member and $M_{0}$ can be made to vanish over all but a few members by a suitable choice of the basic structure. Furthermore, there exist tables of such integrals for the common forms of moment profiles (trapezoidal, parabolic, etc.). ${ }^{5}$

Similarly, the remaining operations can be systematized so that even a person with little mathematical training can perform them. This is particularly true when numerical values of the stiffnesses $E I$ are known; then substituting (5) into (1) reduces to taking scalar products on a computing machine.
5. We conclude with an example illustrating an efficient arrangement of the work.

Consider the four-legged bent shown in Fig. 6; assume that $I_{A B}=I_{C D}=I_{E F}=I_{G I I}$ and $I_{B C}=I_{C F}=I_{F G}$, and denote by $\kappa$ the ratio $l^{\prime} / h^{\prime}=l I_{A B} / h I_{B C}$, as before.


Fig. 6.


Fig. 7.

Since the structure is statically indeterminate of the ninth degree, we obtain nine orthogonal moment profiles $M_{1}, M_{2}, \cdots, M_{9}$, whose values at different points of the structure are given in the first ten columns of Table I. (We understand by $M_{i}(C B)$ the value of $M_{i}$ at the end of C of the member BC , and give a similar meaning to $M_{i}(A B), M_{i}(B C)$, etc.) It will be observed that $M_{1}, M_{2}, M_{3}, M_{7}, M_{8}$, and $M_{9}$ are essentially reproductions of the moment profiles $M_{2 b}, M_{2 d}$, and $M_{2 e}$ found for the two-legged bent in section 3. Considerations of symmetry are helpful in constructing $M_{4}, M_{5}$, and $M_{6}$. The integrals $\int M_{8}^{2} d x^{\prime}$ are now computed by means of the tables referred to in footnote 5 , and are given in the eleventh column of our table.

The parts of the table so far discussed can be used to find the moment profile due to any loading of the bent. Suppose the member BC is subjected to a set of vertical

[^6]loads and moments. By selecting the basic structure as indicated in Fig. 7, we confine the moment profile $M_{0}$ to BC .
Defining
$$
P_{l}=\frac{1}{l^{2}} \int_{\mathrm{BC}} M_{0} x d x, \quad P_{r}=\frac{1}{l^{2}} \int_{\mathrm{BC}} M_{0}(l-x) d x,
$$
we see at once that
\[

$$
\begin{equation*}
\int M_{0} M_{i} d x^{\prime}=l^{\prime}\left[P_{r} M_{i}(B C)+P_{l} M_{i}(C B)\right] \quad(i=1,2, \cdots, 9) \tag{11}
\end{equation*}
$$

\]

The values $X_{i}$ (twelfth column) are now obtained by dividing the negatives of the right members of (11) by the corresponding values in the eleventh column.

To find the value of the true moment profile at (say) A, we multiply each term $M_{i}$ (AB) (first column) by the corresponding $X_{i}$ and add the results, and similarly for other points. It would ordinarily not be necessary to carry out the algebraic simplification of these sums, but this has been done for the sake of compactness:

$$
\begin{aligned}
& M(A B)=\frac{\kappa}{S T}\left\{\left(4 \kappa^{3}-170 \kappa^{2}-414 \kappa-225\right) P_{r}-2(\kappa+1)\left(10 \kappa^{2}-6 \kappa-3\right) P_{l}\right\}, \\
& M(B C)=\frac{\kappa}{S T}\left\{-\left(64 \kappa^{3}+628 \kappa^{2}+1188 \kappa+585\right) P_{r}+2(\kappa+1)\left(16 \kappa^{2}+102 \kappa+51\right) P_{l}\right\}, \\
& M(C B)=\frac{1}{S T}\left\{2 \kappa\left(16 \kappa^{3}+118 \kappa^{2}+153 \kappa+51\right) P_{r}-4(\kappa+1)\left(16 \kappa^{3}+141 \kappa^{2}+195 \kappa+72\right) P_{l}\right\}, \\
& M(D C)=\frac{\kappa}{S T}\left\{\left(20 \kappa^{3}+96 \kappa^{2}+233 \kappa+135\right) P_{r}+2\left(-2 \kappa^{3}+51 \kappa^{2}+91 \kappa+39\right) P_{l}\right\}, \\
& M(C F)=\frac{1}{S T}\left\{\kappa\left(140 \kappa^{2}+412 \kappa+237\right) P_{r}-2\left(86 \kappa^{3}+355 \kappa^{2}+411 \kappa+144\right) P_{l}\right\}, \\
& M(F C)=\frac{1}{S T}\left\{-\kappa\left(124 \kappa^{2}+248 \kappa+93\right) P_{r}+2\left(70 \kappa^{3}+179 \kappa^{2}+144 \kappa+36\right) P_{l}\right\}, \\
& M(E F)=\frac{1}{S T}\left\{\kappa\left(36 \kappa^{3}+260 \kappa^{2}+377 \kappa+135\right) P_{r}-2 \kappa\left(18 \kappa^{3}+125 \kappa^{2}+176 \kappa+69\right) P_{l}\right\}, \\
& M(G F)=\frac{1}{S T}\left\{-\kappa\left(108 \kappa^{2}+208 \kappa+135\right) P_{r}+2 \kappa\left(54 \kappa^{2}+73 \kappa+21\right) P_{l}\right\}, \\
& M(F G)=\frac{1}{S T}\left\{2 \kappa\left(54 \kappa^{2}+73 \kappa+21\right) P_{r}+2\left(-54 \kappa^{3}-38 \kappa^{2}+51 \kappa+36\right) P_{l}\right\}, \\
& M(H G)=\frac{1}{S T}\left\{\kappa\left(36 \kappa^{3}+198 \kappa^{2}+284 \kappa+135\right) P_{r}-2 \kappa\left(18 \kappa^{3}+90 \kappa^{2}+104 \kappa+33\right) P_{l}\right\} .
\end{aligned}
$$

The arrangement given here is especially adapted to the use of computing machines in case numerical values of $\kappa$ and the other quantities appearing are known. ${ }^{6}$

[^7]Table I

|  | $M_{i}(A B)$ | $M$ ( $B C)$ | $M_{i}(C B)$ | $M_{i}(D C)$ | $M_{i}(C F)$ | $M_{i}(F C)$ | $M_{i}(E F)$ | $M_{i}(F G)$ | $M_{i}(G F)$ | $M_{i}(H G)$ | $\int M_{i}^{2} d x^{\prime}$ | $-X_{i}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $M_{1}$ | $k+1$ | 1 | 1 | $-(x+1)$ | - | - |  | - | - | - | $h^{\prime}(x+2)(2 x+1) / 3$ | $3 x\left(P_{5}+P_{l}\right) /(x+2)(2 x+1)$ |
| $M$ | -1 | 1 | 1 | 1 | - | - | - | - | - | - | $h^{\prime}(x+2)$ | $\kappa\left(P_{r}+P_{l}\right) /(\kappa+2)$ |
| $M$ | -1 | 1 | -1 | -1 | - | - | - | - |  | - | $h^{\prime}(x+6) / 3$ | $3 x\left(P_{r}-P_{l}\right) /(\kappa+6)$ |
| $M$. | $-\mathrm{k}(3 x+7)$ | $-11 \times$ | $13 x+12$ | $-x(x-5)$ | $\lambda$ | $\lambda$ | $x(x-5)$ | $13 x+12$ | $-11 \mathrm{k}$ | $k(3 x+7)$ | म'к入R | $\left[-11 x P_{r}+(13 x+12) P_{l}\right] / \lambda R$ |
| Ms | $-{ }^{-2}(3 x+7)$ | $-11 x$ | $13 \mathrm{k}+12$ | - $\mathrm{K}^{(k-5)}$ | $\lambda$ | - $\lambda$ | $-x(x-5)$ | $-(13 x+12)$ | - $+11 k$ | $-x(3 x+7)$ | $h^{\prime} \times \lambda / 3 / 3$ | $3\left[-11 \kappa P_{r}+(13 x+12) P_{d}\right] / \lambda S$ |
| $M$ | $2 x^{2}+8 x+5$ | $6 x+5$ | $-2(3 x+2)$ | $2 x^{2}+9 x+5$ | $2 x+1$ | $2 x+1$ | $-\left(2 x^{2}+9 x+5\right)$ | $-2(3 x+2)$ | $6 x+5$ | $-\left(2 k^{2}+8 k+5\right)$ | $h^{\prime} R T / 3$ | $3 \mathrm{x}\left[(6 \mathrm{c}+5) P_{r}-2(3 \mathrm{l}+2) P_{l}\right] / R T$ |
| $M_{1}$ | - | - | - | - | - | - | $x+1$ | 1 | 1 | $-(x+1)$ | $h^{\prime}(x+2)(2 x+1) / 3$ | 0 |
| M | - | - | - | - | - | - | -1 | 1 | 1 | 1 | $h^{2}(\kappa+2)$ | 0 |
| $M$, |  | - | - | - | - | - | 1 | -1 | 1 | 1 | $h^{\prime}(x+6) / 3$ | 0 |

## A STRAIN ENERGY DERIVATION OF THE TORSIONALFLEXURAL BUCKLING LOADS OF STRAIGHT COLUMNS OF THIN-WALLED OPEN SECTIONS*

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In the thin-walled open section columns of modern aluminum alloy aircraft torsional buckling and combinations of torsional and flexural buckling are of consider. able importance. The critical loads corresponding to these types of instability have been calculated by Wagner, ${ }^{1}$ Kappus, ${ }^{2}$ Lundquist and Fligg, ${ }^{3}$ and Goodier ${ }^{4}$ through integrating the differential equations of the problem. In the present paper the tor-sional-flexural buckling loads are determined with the aid of the Rayleigh-RitzTimoshenko method. This procedure obviates the derivation and integration of the differential equations as well as the geometric considerations connected with what Goodier termed "Wagner's hypothesis."

The equilibrium of a straight bar of a length $L$ and a cross-sectional area $A$, loaded axially with a compressive force of a magnitude $\sigma A$ distributed uniformly over the end section, can be investigated by assuming that each section of the bar undergoes a virtual displacement. The end sections of the bar are assumed to be restrained in a manner which precludes translations as well as rotations about any axis perpendicular to the end section, but which permits rotations about axes in the plane of the end section and warping of the end section. Barring displacements that would change the shape of the cross section (such displacements lead to plate- or shell-buckling), the most general virtual displacement pattern of the bar can be represented by the following infinite series:

$$
\begin{align*}
& u=\sum_{n=1}^{\infty} a_{n} \sin (n \pi z / L)  \tag{1a}\\
& v=\sum_{n=1}^{\infty} b_{n} \sin (n \pi z / L)  \tag{1b}\\
& \beta=\sum_{n=1}^{\infty} c_{n} \sin (n \pi z / L) \tag{1c}
\end{align*}
$$

[^8]In these equations $u$ is the virtual translation in the $x$-direction of a section of the bar at a distance $z$ from the bottom section, $v$ that in the $y$-direction, and $\beta$ the virtual rotation of the section about its shear center. The $x$ - and $y$-axes are parallel to the principal axes of inertia, and pass through the shear center of each section. The displacements of a section are shown in Fig. 1.


Fig. 1.

The increment $\delta U$ of the strain energy because of the virtual displacements is

$$
\begin{align*}
\delta U= & \frac{1}{2} E I_{y} \int_{0}^{L}\left(d^{2} u / d z^{2}\right)^{2} d z+\frac{1}{2} E I_{x} \int_{0}^{L}\left(d^{2} v / d z^{2}\right)^{2} d z \\
& +\frac{1}{2} G C \int_{0}^{L}(d \beta / d z)^{2} d z+\frac{1}{2} E \Gamma \int_{0}^{L}\left(d^{2} \beta / d z^{2}\right)^{2} d z \tag{2}
\end{align*}
$$

In Eq. (2) $E I_{x}$ and $E I_{y}$ stand for the bending rigidities of the bar when the bending moment vector is parallel to the $x$-axis and the $y$-axis, respectively, and $G C$ is the torsional rigidity calculated from the Saint-Venant theory of uniform torsion. The fourth term is due to the direct stress caused by non-uniform warping, and $\Gamma$ is the warping constant defined in the theory of non-uniform torsion of thin-walled open sections. This theory is discussed in the previously mentioned references and in a paper written by the author. ${ }^{5}$ Eq. (2) follows from the theory of non-uniform torsion of thin-walled open sections and from the Bernoulli-Euler theory of bending, if the strain energy due to shear associated with bending and that associated with nonuniform warping are neglected.

[^9]Because of the orthogonal properties of the trigonometric functions the integrals indicated in Eq. (2) can be easily calculated:

$$
\begin{align*}
\delta U= & \left(\pi^{4} E I_{y} / 4 L^{3}\right) \sum_{\pi=1}^{\infty} n^{4} a_{n}^{2}+\left(\pi^{4} E I_{x} / 4 L^{3}\right) \sum_{n=1}^{\infty} n^{4} b_{n}^{2} \\
& +\left(\pi^{4} E \Gamma / 4 L^{3}\right) \sum_{n=1}^{\infty} n^{4} c_{n}^{2}+\left(\pi^{2} G C / 4 L\right) \sum_{n=1}^{\infty} n^{4} c_{n}^{2} \tag{3}
\end{align*}
$$

The decrease $-\delta V$ of the potential of the external forces is equal to the work $\delta W$ done by them during the virtual displacements. The work $d \delta W$ done by the infinitesimal force $\sigma d A$ is equal to the force times the shortening of the distance between the end points of the fiber upon which it is acting. The shortening $\Delta L_{x, y}$ of the distance between the end points of the fiber passing through a point $x, y$ can be calculated from the equation

$$
\begin{equation*}
\Delta L_{x, y}=\frac{1}{2} \int_{0}^{L}\left[\left(d u_{x, v} / d z\right)^{2}+\left(d v_{x, v} / d z\right)^{2}\right] d z \tag{4}
\end{equation*}
$$

It may be seen from Fig. 1 that for small displacements

$$
\begin{align*}
& u_{x, y}=u-y \beta  \tag{5a}\\
& v_{x, y}=v+x \beta . \tag{5b}
\end{align*}
$$

Upon substitution of the expressions of equations (1) and (5) into Eq. (4), integration yields

$$
\begin{align*}
\Delta L_{x, y}= & \left(\pi^{2} / 4 L\right)\left\{\sum_{n=1}^{\infty} n^{2} a_{n}^{2}+\sum_{n=1}^{\infty} n^{2} b_{n}^{2}-2 y \sum_{n=1}^{\infty} n^{2} a_{n} c_{n}\right. \\
& \left.+2 x \sum_{n=1}^{\infty} n^{2} b_{n} c_{n}+x^{2} \sum_{n=1}^{\infty} n^{2} c_{n}^{2}+y^{2} \sum_{n=1}^{\infty} n^{2} c_{n}^{2}\right\} . \tag{6}
\end{align*}
$$

The sum $\delta W$ of the work done by all the infinitesimal forces $\sigma d A$ is

$$
\begin{equation*}
\delta W=\int_{A} \sigma \Delta L_{x, y} d A \tag{7}
\end{equation*}
$$

where the integral is extended over the total cross-sectional area. With
$\int_{A} \sigma d A=P$, the total compressive force,
$\int_{\Lambda} y d A=y_{0} A$, the static moment of the section with respect to the $x$-axis passing through the shear center,
$\int_{A} x d A=x_{0} A$, the static moment of the section with respect to the $y$-axis passing through the shear center,
$\int_{A}\left(x^{2}+y^{2}\right) d A=\int_{A} r^{2} d A=I_{p}$, the polar moment of inertia of the section with respect to the shear center, and

$$
\rho^{2}=I_{p} / A
$$

equation (7) can be written as

$$
\begin{align*}
\delta W= & \left(\pi^{2} P / 4 L\right)\left\{\sum_{n=1}^{\infty} n^{2} a_{n}^{2}+\sum_{n=1}^{\infty} n^{2} b_{n}^{2}-2 y_{0} \sum_{n=1}^{\infty} n^{2} a_{n} c_{n}\right. \\
& \left.+2 x_{0} \sum_{n=1}^{\infty} n^{2} b_{n} c_{n}+\rho^{2} \sum_{n=1}^{\infty} n^{2} c^{2}\right\} . \tag{8}
\end{align*}
$$

According to the principle of virtual displacements the bar is in equilibrium in its original straight-line form if the change of the total potential $\delta(U+V)=\delta U-\delta W$ is zero for any virtual displacement provided that first order small terms alone are considered. Since both $\delta U$ (Eq. (3)) and $\delta W$ (Eq. (8)) contain only second order terms in the $a_{n}, b_{n}, c_{n}$, the original straight-line form is a configuration of equilibrium. This equilibrium is stable only if the total potential increases, that is $\delta(U+V)$ is positive, for any virtual displacement. With the notation

$$
\begin{align*}
& N=\pi^{2} E I_{y} / L^{2}  \tag{9a}\\
& Q=\pi^{2} E I_{x} / L^{2}  \tag{9b}\\
& R=\pi^{2} E \Gamma / L^{2} \tag{9c}
\end{align*}
$$

the increment of the total potential can be written in the form

$$
\begin{align*}
\delta(U+V)= & \left(\pi^{2} P / 4 L\right) \sum_{n=1}^{\infty} n^{2}\left\{\left[n^{2}(N / P)-1\right] a_{n}^{2}+\left[n^{2}(Q / P)-1\right] b_{n}^{2}\right. \\
& \left.+\left[n^{2}(R / P)+(G C / P)-\rho^{2}\right] c_{n}^{2}+2 y_{0} a_{n} c_{n}-2 x_{0} b_{n} c_{n}\right\} \tag{10}
\end{align*}
$$

With the notation

$$
\begin{align*}
& A_{n}=n^{2}(N / P)-1, \quad B_{n}=n^{2}(Q / P)-1 \\
& C_{n}=n^{2}(R / P)+(G C / P)-\rho^{2}  \tag{11}\\
& F_{n}=-x_{0}, \quad G_{n}=y_{0} \\
& X_{n}=A_{n} a_{n}^{2}+B_{n} b_{n}^{2}+C_{n} c_{n}^{2}+2 F_{n} b_{n} c_{n}+2 G_{n} c_{n} a_{n} \tag{12}
\end{align*}
$$

the infinite sum on the right hand side can be written as $\sum_{n=1}^{\infty} X_{n}$. The necessary and sufficient conditions for its positive definite character are that all $X_{n}$ for $n=1,2, \ldots$ must be positive definite. Necessary and sufficient conditions for this are

$$
\begin{equation*}
A_{n}>0, \quad B_{n}>0, \quad A_{n} B_{n} C_{n}-A_{n} F_{n}^{2}-B G_{n}^{2}>0 \tag{13}
\end{equation*}
$$

or,

$$
\begin{gather*}
n^{2}(N / P-1)>0  \tag{14}\\
n^{2}(Q / P)-1>0  \tag{15}\\
\rho^{2}\left[n^{2}(N / P)-1\right]\left[n^{2}(Q / P)-1\right][(T / P)-1] \\
-x_{0}^{2}\left[n^{2}(N / P)-1\right]-y_{0}^{2}\left[n^{2}(Q / P)-1\right]>0 \tag{16}
\end{gather*}
$$

Since inequalities (14)-(16) are necessary as well as sufficient conditions of stability, the bar may buckle if any one of them is not satisfied. Neutral equilibrium prevails, therefore, if any one of the following "buckling conditions" is fulfilled:

$$
\begin{gather*}
n^{2}(N / P)-1=0  \tag{17}\\
n^{2}(Q / P)-1=0  \tag{18}\\
\rho^{2}\left[n^{2}(N / P)-1\right]\left[n^{2}(Q / P)-1\right][(T / P)-1] \\
-x_{0}^{2}\left[n^{2}(N / P)-1\right]-y_{0}^{2}\left[n^{2}(Q / P)-1\right]=0 \tag{19}
\end{gather*}
$$

It is easy to prove that the original straight-line form of the bar corresponds to stable equilibrium if the compressive force $P$ is sufficiently small, since by decreasing
$P$ the first three terms on the right Kând side of Eq. (12) can be made positive and large as compared to the last two terms. In the investigation of the stability of the bar under increasing values of $P$ it is to be noted that if any of the inequalities (14)(16) is satisfied for $n=k$, it is also satisfied for $n=k+p$, where $k$ and $p$ are arbitrary positive integers. Consequently the bar is stable if, and only if, inequalities (14)-(16) are satisfied when $n=1$. The smallest critical load can be calculated from Eqs. (17)(19) if 1 is substituted for $n$. This load alone is of practical importance unless geometric constraints (for instance rigid end fixation or additional supports between the ends of the bar) prevent displacement corresponding to the first terms of the Fourier series in Eqs. (1a)-(1c). In such a case the smallest value of $n$ that is compatible with the restraints must be used in Eqs. (17)-(19) for the calculation of the buckling load, for instance $n=2$ when the ends of the bar are prevented from rotating and warping.

Eqs. (17)-(19) permit a discussion of the various types of buckling of bars of different cross section. With an asymmetric section $x_{0} \neq 0$ and $y_{0} \neq 0$. In this case with increasing $P$ a value is reached at which Eq. (19) is fulfilled while the left sides of Eqs. (17) and (18) are still greater than zero. The buckling load $P$ can be calculated from Eq. (19) which is a cubic in ( $1 / P$ ). The deflection pattern is flexural-torsional since it contains the non-vanishing coefficients $a_{n}, b_{n}$, and $c_{n}$ simultaneously.

If the section has one plane of symmetry, one of the coordinates of the centroid, say $x_{0}$, vanishes. Then Eq. (19) reduces to

$$
\begin{equation*}
\left[n^{2}(Q / P)-1\right]\left\{\rho^{2}\left[n^{2}(N / P)-1\right][(T / P)-1]-y_{0}^{2}\right\}=0 . \tag{20}
\end{equation*}
$$

Consequently two distinct types of buckling are possible. One is purely flexural and symmetric. It corresponds to a buckling load which is the solution of Eq. (18). The other is flexural-torsional since it simultaneously contains displacement components corresponding to the non-vanishing coefficients $a_{n}$ and $c_{n}$; it is antisymmetric; its buckling load $P$ is the (smaller) root of the quadratic in ( $1 / P$ ) that can be obtained by dividing Eq. (20) by $\left[n^{2}(Q / P)-1\right]$. The smaller of the two distinct buckling loads is of practical importance. The buckling load according to Eq. (17) is always greater than the smaller root of the quadratic.

Finally, if the section is doubly symmetric or point symmetric, $x_{0}=y_{0}=0$. Then Eq. (19) reduces to

$$
\begin{equation*}
(T / P)-1=0 . \tag{21}
\end{equation*}
$$

Buckling is either purely flexural or purely torsional. Of practical importance is the smallest of the solutions for $P$ of the three Eqs. (17), (18), and (21).

## -NOTES-

## USE OF SINE TRANSFORM FOR NON-SIMPLY SUPPORTED BEAMS*

By A. G. STRANDHAGEN (Carnegic Institute of Technology)

The problem of non-simply supported beams is approached by various mathematical procedures. In certain applications several of the common methods are long and tedious. By employing the sine transform a certain ease can be claimed for most cases.

The definition of the sine transform of a function $y(x)$ in the interval $(0, l)$ is

$$
\begin{equation*}
S[y(x)]=\int_{0}^{l} y(x) \sin (n \pi x / l) d x=v(n) . \quad(0<x<l ; n=1,2, \cdots) \tag{1}
\end{equation*}
$$

Recalling that the expression of a function $y(x)$ in a Fourier sine series is

$$
\begin{equation*}
y(x)=\sum_{n=1}^{\infty} b_{n} \sin n \pi x / l \tag{2}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{n}=(2 / l) \int_{0}^{l} y(x) \sin (n \pi x / l) d x, \quad(0<x<l ; n=1,2, \cdots) \tag{3}
\end{equation*}
$$

it becomes evident that the connection between the sine transform and the coefficients of the Fourier sine series is

$$
\begin{equation*}
S[y(x)]=(l / 2) b_{n} . \tag{4}
\end{equation*}
$$

Forms given by Eq. (2) and Eq. (3) are altered for the sake of convenience as follows:

$$
\begin{equation*}
y(x)=(2 / l) \sum_{n=1}^{\infty} v(n) \sin (n \pi x / l) \tag{5}
\end{equation*}
$$

where

$$
\begin{equation*}
v(n)=S[y(x)]=\int_{0}^{l} y(x) \sin (n \pi x / l) d x \tag{6}
\end{equation*}
$$

For example, consider the sine transform of $\left(d^{2} y / d x^{2}\right)$ in the interval $(0, l)$; by definition

$$
S\left[d^{2} y / d x^{2}\right]=\int_{0}^{l}\left(d^{2} y / d x^{2}\right) \sin (n \pi x / l) d x . \quad(n=1,2, \cdots)
$$

Integrating formally by parts gives

$$
\begin{equation*}
S\left[d^{2} y / d x^{2}\right]=-\frac{n \pi}{l}\left[(-1)^{n} y(l)-y(0)\right]-\left(\frac{n \pi}{l}\right)^{2} v(n) . \quad(n=1,2, \cdots) \tag{7}
\end{equation*}
$$

Likewise the sine transform of $\left(d^{4} y / d x^{4}\right)$ in $(0, l)$ is:

[^10]$S\left[d^{4} y / d x^{4}\right]=-\frac{n \pi}{i}\left[(-1)^{n} y^{\prime \prime}(l)-y^{\prime \prime}(0)\right]$
\[

$$
\begin{equation*}
+\left(\frac{n \pi}{l}\right)^{3}\left[(-1)^{n} y(l)-y(0)\right]+\left(\frac{n \pi}{l}\right)^{4} v(n), \quad(n=1,2, \ldots) \tag{8}
\end{equation*}
$$

\]

where $v(n)$ in (7) and (8) is defined by Eq. (1).
Consider a beam fixed at $x=0$ with axial loads $P$. The intensity of transverse loading is $q(x)$, Fig. 1. The differential equation and boundary conditions are as follows:

$$
\text { 1. } d^{4} y / d x^{4}+k^{2}\left(d^{2} y / d x^{2}\right)=q(x) / E I, \quad(0<x<l)
$$

2. $y(0)=y(l)=0$,
3. $y^{\prime \prime}(l)=0, \quad y^{\prime}(0)=0$,
where

$$
\begin{array}{rlrl}
q(x) & =0 \quad \text { when } \quad 0<x<b \\
& =\theta(x) & & \text { when } \quad b<x<c \\
& =0 \quad & \text { when } \quad c<x<l
\end{array}
$$

and $c>b$. Let $k^{2}=P / E I$, and primes indicate differentiation with respect to $x$. Let $S[y(x)]=v(n)$. Transforming $d^{4} y / d x^{4}$ and $d^{2} y / d x^{2}$ and $q(x)$ and substituting $y(0)$ $=y(l)=y^{\prime \prime}(l)=0$, there results


Fig. 1.

$$
(n \pi / l) y^{\prime \prime}(0)+(n \pi / l)^{4} v(n)-k^{2}(n \pi / l)^{2} v(n)=(1 / E I) \int_{0}^{c} \theta(x) \sin (n \pi x / l) d x .
$$

Solving for $v(n)$, where $\alpha^{2}=(k l / \pi)^{2}$,

$$
\begin{equation*}
v(n)=-(l / \pi)^{3} y^{\prime \prime}(0) \frac{1}{n\left(n^{2}-\alpha^{2}\right)}+\frac{l^{4}}{\pi^{4} E I} \frac{1}{n^{2}\left(n^{2}-\alpha^{2}\right)} \int_{b}^{c} \theta(x) \sin (n \pi x / l) d x . \tag{9}
\end{equation*}
$$

Since $y(x)=(2 / l) \sum_{n-1}^{\infty} v(n) \sin n \pi x / l$, then

$$
\begin{align*}
y(x)= & -\left(2 l^{2} / \pi^{3}\right) y^{\prime \prime}(0) \sum_{n=1}^{\infty} \frac{1}{n\left(n^{2}-\alpha^{2}\right)} \sin (n \pi x / l) \\
& +2\left(l^{3} / \pi^{4} E I\right) \sum_{n=1}^{\infty} \frac{\sin (n \pi x / l)}{n^{2}\left(n^{2}-\alpha^{2}\right)} \int_{0}^{c} \theta\left(x^{\prime}\right) \sin \left(n \pi x^{\prime} / l\right) d x^{\prime} . \quad(n \neq \alpha) \tag{10}
\end{align*}
$$

The remaining boundary condition $y^{\prime}(0)=0$ gives the following:

$$
\begin{equation*}
y^{\prime \prime}(0) \sum_{n=1}^{\infty} \frac{1}{\left(n^{2}-\alpha^{2}\right)}=\frac{l}{\pi E I} \sum_{n=1}^{\infty} \frac{1}{n\left(n^{2}-\alpha^{2}\right)} \int_{b}^{c} \theta\left(x^{\prime}\right) \sin \left(n \pi x^{\prime} / l\right) d x^{\prime} . \tag{11}
\end{equation*}
$$

Since $\sum_{n-1}^{\infty} 1 /\left(n^{2}-\alpha^{2}\right)=\left(1 / 2 \alpha^{2}\right)(1-\pi \alpha \cot \pi \alpha)$, then $y^{\prime \prime}(0)$ becomes

$$
\begin{equation*}
y^{\prime \prime}(0)=\frac{\left(2 \alpha^{2} l / \pi E I\right)}{(1-\pi \alpha \cot \pi \alpha)} \sum_{n=1}^{\infty} \frac{1}{n\left(n^{2}-\alpha^{2}\right)} \int_{b}^{c} \theta\left(x^{\prime}\right) \sin \left(n \pi x^{\prime} / l\right) d x^{\prime} . \quad(n \neq \alpha) \tag{12}
\end{equation*}
$$

Further simplifications are possible in Eq. (12). Interchanging formally the integral and summation sign and summing, the following is obtained for $y^{\prime \prime}(0)$ :

$$
y^{\prime \prime}(0)=\frac{(l / E I)}{(1-\pi \alpha \cot \pi \alpha)} \int_{0}^{c} \theta(x)\left\{\frac{\sin k(l-x)}{\sin k l}-\frac{l-x}{l}\right\} d x,
$$

where

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{\sin (n \pi x / l)}{n\left(n^{2}-\alpha^{2}\right)}=\frac{\pi}{2 \alpha^{2}}\left\{\frac{\sin k(l-x)}{\sin k l}-\frac{l-x}{l}\right\}=\phi(x) . \tag{13}
\end{equation*}
$$

Thus by substitution of (13) in (10),

$$
\begin{align*}
y(x)= & -\frac{4 \alpha^{2} l^{3}}{\pi^{4} E I} \frac{\phi(x)}{(1-\pi \alpha \cot \pi \alpha)} \int_{b}^{c} \theta\left(x^{\prime}\right) \phi\left(x^{\prime}\right) d x^{\prime} \\
& +\frac{2 l^{3}}{\pi^{4} E I} \sum_{n=1}^{\infty} \frac{\sin (n \pi x / l)}{n^{2}\left(n^{2}-\alpha^{2}\right)} \int_{b}^{c} \theta\left(x^{\prime}\right) \sin \left(n \pi x^{\prime} / l\right) d x^{\prime} . \quad(n \neq \alpha, 0<x<l) \tag{14}
\end{align*}
$$

Knowing the variation of $\theta(x)$ it is a matter of integration to obtain the required results. Now suppose that $P=0$, i.e., the beam is under no axial loads, and subject to the same boundary conditions. Thus $k=\alpha=0$ in equations (9), (10), and (11) and then

$$
y^{\prime \prime}(0)=\frac{6 l}{\pi^{3} E I} \sum_{n=1}^{\infty} \frac{1}{n^{3}} \int_{b}^{c} \theta\left(x^{\prime}\right) \sin \left(n \pi x^{\prime} / l\right) d x^{\prime}
$$

Again interchanging formally the integral and summation sign,

$$
y^{\prime \prime}(0)=\frac{1}{2 l^{2} E I} \int_{0}^{c} \theta\left(x^{\prime}\right) x^{\prime}\left(x^{\prime}-l\right)\left(x^{\prime}-2 l\right) d x^{\prime}
$$

where

$$
\sum_{n=1}^{\infty}\left(1 / n^{3}\right) \sin (n \pi x / l)=\frac{\pi^{3}}{12}\left\{2(x / l)-3(x / l)^{2}+(x / l)^{3}\right\} . \quad(0<x / l<2 .)
$$

The equation for the elastic line becomes

$$
\begin{aligned}
y(x)= & -\frac{1}{12 E I}\left[2(x / l)-3(x / l)^{2}+(x / l)^{3}\right] \int_{b}^{c} \theta\left(x^{\prime}\right)\left(x^{\prime}-l\right)\left(x^{\prime}-2 l\right) d x^{\prime} \\
& +\frac{2 l^{3}}{\pi^{4} E I} \sum_{n=1}^{\infty} \frac{\sin (n \pi x / l)}{n^{4}} \int_{b}^{c} \theta\left(x^{\prime}\right) \sin \left(n \pi x^{\prime} / l\right) d x^{\prime} . \quad(0<x<l)
\end{aligned}
$$

To be sure, further summation in finite terms is possible, but this will lead to $y(x)$ being defined in distinct intervals in $(0, l)$, as in the solution furnished by the classical methods of differential equations; unquestionably, this is a disadvantage in engineering computations. The above results, however, remain in the desired form, with one function $y(x)$ in $(0, l)$ regardless of the discontinuities of transverse loading.

In like manner other boundary conditions may be imposed, and other beam problems, such as beams on elastic foundations, can be solved.

# THE TREATMENT OF DISCONTINUITIES IN BEAM DEFLECTION PROBLEMS* 

By C. L. BROWN (Purdue University)

The multitude of methods of determining the deflections of beams all stem from the fundamental differential equation

$$
\begin{equation*}
\frac{d^{2} y}{d x^{2}}=\frac{M}{E I}, \tag{1}
\end{equation*}
$$

where the abscissa $x$ is measured along the axis of the beam, and $y$ denotes the deflection, $M$ the bending moment, and $E I$ the bending stiffness. The most obvious method of determining. $y$ is direct integration of (1). However, in most cases the right hand side of (1) is but sectionally analytical. A differential equation of the form (1) is then written for each section of the beam. When these equations are integrated, two constants of integration appear for each section. The evaluation of these constants of integration, though elementary, is extremely cumbersome.

It is possible to avoid this sectionalizing treatment through the use of Heaviside's unit step function, well known from operational calculus. This function is defined as follows:

$$
H_{a}(x)= \begin{cases}0 & \text { for } x<a  \tag{2}\\ 1 & \text { for } x>a\end{cases}
$$

It can readily be seen that

$$
H_{a} H_{b}=\left\{\begin{array}{lll}
H_{a} & \text { if } & a>b,  \tag{3}\\
H_{b} & \text { if } & a<b .
\end{array}\right.
$$

Furthermore, if $u(x)$ and $v(x)$ are analytic, the continuous solution of

$$
\frac{d y}{d x}=u(x)+H_{a^{2}} v(x-a)
$$

is given by

$$
\begin{equation*}
y=\int_{0}^{x} u(\xi) d \xi+H_{u} \int_{0}^{x-a} v(\xi) d \xi+C \tag{4}
\end{equation*}
$$

where $C$ denotes a constant of integration.

The use of the unit step function in the analysis of beams with concentrated loads is best shown by an example. The bending moment of the beam in Fig. 1 can be written as


Fig. 1.

$$
E I \frac{d^{2} y}{d x^{2}}=M=-M_{1}+R_{1} x-H_{a} P(x-a) .
$$

[^11]Integrating according to (4) and taking account of the fact that slope and deflection vanish for $x=0$, we find

$$
\begin{equation*}
E I \frac{d y}{d x}=-M_{1} x+\frac{1}{2} R_{1} x^{2}-\frac{1}{2} H_{a} P(x-a)^{2} \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
E I y=-\frac{1}{2} M_{1} x^{2}+\frac{1}{6} R_{1} x^{3}-\frac{1}{6} H_{a} P(x-a)^{3} . \tag{6}
\end{equation*}
$$

Both slope and deflection are zero at $x=1$. Thus, from (5) and (6)

$$
-M_{1} l+\frac{1}{2} R_{1} l^{2}-\frac{1}{2} P b^{2}=0, \quad-\frac{1}{2} M_{1} l^{2}+\frac{1}{6} R_{1} l^{3}-\frac{1}{6} P b^{3}=0 .
$$

Solving for $M_{1}$ and $R_{1}$,

$$
M_{1}=P a b^{2} / l^{2}, \quad R_{1}=P b^{2}(3 a+b) / l^{3} .
$$

The deflection is then given by

$$
E I y=-\frac{P a b^{2}}{2 l^{2}} x^{2}+\frac{P b^{2}(3 a+b)}{6 l^{3}} x^{3}-H_{a} \frac{P(x-a)^{3}}{6} .
$$



Fig. 2.

In order to illustrate the use of the unit step function in the analysis of beams with sectionally constant moment of inertia, we consider the beam shown in Fig. 2. The reciprocal of the bending stiffness can be written as

$$
\frac{1}{E I}=\frac{1}{E I_{1}}\left[1+H_{a}\right]
$$

and the bending moment as

$$
M=\frac{P x}{2}-P H_{b}(x-b) .
$$

Thus,

$$
E I_{1} \frac{d^{2} y}{d x^{2}}=\left[1+H_{a}\right]\left[\frac{P x}{2}-P H_{b}(x-b)\right]
$$

or, considering (3)

$$
\begin{equation*}
E I_{1} \frac{d^{2} y}{d x^{2}}=\frac{P x}{2}+\frac{P}{2}\left[H_{a}(x-a)+a H_{a}\right]-P H_{b}(x-b)-P H_{a}(x-b), \tag{7}
\end{equation*}
$$

in which the second term on the right hand side has been written in this particular form in order to facilitate the application of (4). Integrating (7) we obtain

$$
\begin{align*}
E I_{1} y=\frac{P x^{3}}{12} & +\frac{P}{2}\left[H_{a} \frac{(x-a)^{3}}{6}+a H_{a} \frac{(x-a)^{2}}{2}\right] \\
& -P H_{b} \frac{(x-b)^{3}}{6}-P H_{a} \frac{(x-b)^{3}}{6}+C_{1} x+C_{2} \tag{8}
\end{align*}
$$

The deflection is zero at $x=0$ and $x=l$. The application of these conditions to (8) yields

$$
C_{1}=-\frac{P}{12 b}\left(6 b^{3}-3 a^{2} b+a^{3}\right), \quad C_{2}=0
$$

If, in addition to concentrated loads, the beam carries loads which are uniformly distributed over certain sections, the treatment is similar.

## A VARIATIONAL PRINCIPLE FOR A STATE OF COMBINED PLASTIC STRESS*

## BX G. H. HANDELMAN $\dagger$ (Brown University)

In a recent paper ${ }^{1}$ M. A. Sadowsky has stated a heuristic principle of maximum plastic resistance which he has applied to several states of combined plastic stress. The principle states that "among all statically possible stress distributions (satisfying all three equations of equilibrium, the condition of plasticity, and boundary conditions), the actual stress distribution in plastic flow requires a maximum value of the external effort necessary to maintain the flow." W. Prager, in a contribution to the discussion of this paper ${ }^{2}$, has shown that the principle can be so interpreted as to lead to the correct differential equation for a beam under combined torsion and tension. This note is concerned with an accurate statement of the principle together with a proof of its validity for the case of a beam in a perfectly plastic state under combined torsion and bending by couples, the cross-section of the beam having an axis of symmetry. Specifically, we shall prove the following variational principle for such a system.

Among all statically possible stress distributions in a beam under a given torque (satisfying the equations of equilibrium, the condition of plasticity, and boundary conditions), the actual stress distribution when plastic flow occurs is the one for which the bending moment is stationary.

Let us choose the coordinate axes in the following fashion. $y$ lies along the axis of symmetry of the cross-section, $z$ passes through the center of gravity of the crosssection and is parallel to the generators of the cylindrical beam, and $x$ is perpendicular to $y$ and $z$. We assume that the strain velocities, $v_{x}, v_{y}, v_{z}$, are given by the same expressions as in the case of an incompressible elastic material; i.e.,

$$
\begin{aligned}
& v_{x}=-\omega y z+\frac{1}{2} \theta x y \\
& v_{y}=\omega x z-\frac{1}{4} \theta\left(x^{2}-y^{2}-2 z^{2}\right) \\
& v_{z}=\omega w(x, y)-\theta y z
\end{aligned}
$$

[^12]$\omega$ and $\theta$ are constants ( $\omega=$ angle of twist per unit length per unit time) and $w(x, y)$ is an unknown function. The components of the strain velocity tensor, which are
$$
\epsilon_{x y}=\frac{1}{2}\left(\frac{\partial v_{x}}{\partial y}+\frac{\partial v_{y}}{\partial x}\right),
$$
etc., will then be
\[

$$
\begin{array}{lll}
\epsilon_{x x}=\frac{\theta}{2} y, & \epsilon_{y y}=\frac{\theta}{2} y, & \epsilon_{z z}=-\theta y \\
\epsilon_{x y}=0, & \epsilon_{x z}=\frac{\omega}{2}\left(\frac{\partial w}{\partial x}-y\right), & \epsilon_{y z}=\frac{\omega}{2}\left(\frac{\partial w}{\partial y}+x\right) .
\end{array}
$$
\]

The components of the stress tensor will be denoted by $\sigma_{2 x}, \sigma_{x y}$, etc. The material is assumed to be incompressible and perfectly plastic. The stress-strain relations can therefore be written in the form

$$
\begin{align*}
\left(\sigma_{x x}-\sigma_{y y}\right):\left(\sigma_{y y}-\sigma_{z z}\right):\left(\sigma_{x z}\right. & \left.-\sigma_{x x}\right): \sigma_{x y}: \sigma_{y z}: \sigma_{z x} \\
& =\left(\epsilon_{x x}-\epsilon_{y y}\right):\left(\epsilon_{y y}-\epsilon_{z z}\right):\left(\epsilon_{z z}-\epsilon_{x x}\right): \epsilon_{x y}: \epsilon_{y z}: \epsilon_{z x} \tag{1}
\end{align*}
$$

In addition the following yield condition must be fulfilled:

$$
\begin{equation*}
\left(\sigma_{x x}-\sigma_{y y}\right)^{2}+\left(\sigma_{y y}-\sigma_{z z}\right)^{2}+\left(\sigma_{z z}-\sigma_{x x}\right)^{2}+6\left(\sigma_{x y}^{2}+\sigma_{y z}^{2}+\sigma_{x z}^{2}\right)=6 k^{2}, \tag{2}
\end{equation*}
$$

where $k$ is a given constant. Eq. (1) will be satisfied if the stress tensor has the form

$$
\begin{align*}
\sigma_{x x}=\sigma_{y y}=\sigma_{x y}=0, & \sigma_{x z}=\mu \omega\left(\frac{\partial w}{\partial x}-y\right) \\
\sigma_{y z}=\mu \omega\left(\frac{\partial w}{\partial y}+x\right), & \sigma_{z z}=-3 \theta \mu y \tag{3}
\end{align*}
$$

$\mu=\mu(x, y)$ being an unknown function of $x$ and $y$. The equations of equilibrium for such a stress system reduce to

$$
\frac{\partial \sigma_{x z}}{\partial x}+\frac{\partial \sigma_{y z}}{\partial y}=0
$$

Consequently we may introduce a stress function $k u(x, y)$ such that

$$
\begin{equation*}
\sigma_{x z}=k \frac{\partial u}{\partial y}, \quad \sigma_{y z}=-k \frac{\partial u}{\partial x} \tag{4}
\end{equation*}
$$

Since the surface of the beam is not stressed, $u=$ const. on the surface. For convenience, this constant may be taken as zero. Combining Eq. (4) and the yield condition, Eq. (2), we have

$$
\sigma_{x z}=k \sqrt{ } 3\left(1-u_{x}^{2}-u_{y}^{2}\right)^{1 / 2},
$$

where $u_{x}=\partial u / \partial x, u_{y}=\partial u / \partial y$. Therefore the function $\mu$ must be given by

$$
\begin{equation*}
\mu=\frac{-k}{y \theta \sqrt{3}}\left(1-u_{x}^{2}-u_{y}^{2}\right)^{1 / 2} \tag{5}
\end{equation*}
$$

By means of Eq. (3) and Eq. (4), we can compute $\partial w / \partial x$ and $\partial w / \partial y$ in terms of the stress function $u$. This will yield

$$
\frac{\partial w}{\partial x}=\frac{k}{\omega \mu} u_{y}+y, \quad \frac{\partial w}{\partial y}=\frac{-k}{\omega \mu} u_{x}-x .
$$

Taking cross derivatives and subtracting one finds

$$
\frac{\partial}{\partial y}\left(\frac{1}{\mu} u_{y}\right)+\frac{\partial}{\partial x}\left(\frac{1}{\mu} u_{x}\right)+\frac{2 \omega}{k}=0
$$

which becomes by virtue of Eq. (5)

$$
\begin{equation*}
\frac{\partial}{\partial x}\left[y u_{x}\left(1-u_{x}^{2}-u_{y}^{2}\right)^{-1 / 2}\right]+\frac{\partial}{\partial y}\left[y u_{y}\left(1-u_{x}^{2}-u_{y}^{2}\right)^{-1 / 2}\right]-\frac{2 \omega}{\theta \sqrt{3}}=0 \tag{6}
\end{equation*}
$$

We shall now determine the Euler equation for the variational principle stated previously. The bending moment and torque are given by

$$
\begin{aligned}
\text { Bending Moment } & =\iint \sigma_{z z} y d x d y=k \sqrt{3} \iint y\left(1-u_{x}^{2}-u_{\psi}^{2}\right)^{1 / 2} d x d y \\
\text { Torque } & =2 k \iint u d x d y .
\end{aligned}
$$

The domain of integration is the cross section of the beam. We note that the symmetry assumption has been used in writing the bending moment in the form above. According to the usual procedure, let us form the function

$$
\varphi\left(u, u_{x}, u_{\nu}\right)=y\left(1-u_{x}^{2}-u_{v}^{2}\right)^{1 / 2}+\lambda \mu,
$$

where $\lambda$ is an unknown constant. The Euler equation can then be written as

$$
\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial u_{x}}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial u_{\nu}}\right)-\frac{\partial \varphi}{\partial u}=0 .
$$

Substituting for $\varphi$ from Eq. (7), one finds

$$
\begin{equation*}
\frac{\partial}{\partial x}\left[y u_{x}\left(1-u_{x}^{2}-u_{y}^{2}\right)^{-1 / 2}\right]+\frac{\partial}{\partial y}\left[y u_{y}\left(1-u_{x}^{2}-u_{y}^{2}\right)^{-1 / 2}\right]+\lambda=0 \tag{8}
\end{equation*}
$$

The unknown constant $\lambda$ is determined by the fact that the torque is prescribed. On the other hand, $\omega$ and $\theta$ can not be given arbitrarily but must be found from exactly the same condition. Consequently Eq. (8) is the same as Eq. (6), which proves the variational principle.

## ON THE ANTENNA PROBLEM*

## By S. A. SCHELKUNOFF (Bell Telephone Laboratories)

The following remarks are made apropos to Brillouin's recent discussion ${ }^{1}$ of mathematical difficulties involved in the retarded potential method of solving the antenna problem. The approximations involved in the actual solution of the final integral equation may be a source of far greater errors than the approximations in the equation itself (such as 7A and 7B of Brillouin's paper). Fig. 1 shows the first maximum


Fig. 1. The first maximum input resistance of center fed cylindrical antennas in free space as a function of the average characteristic impedance: (1) is Hallen's approximation to the solution of the integral equation for infinitely thin cylindrical shells, (2) is Schelkunoff's approximation to the solution of Maxwell's equations for solid cylinders, (3) is Schelkunoff's approximation to the solution of Maxwell's equations for infinitely thin cylindrical shells, (4) is Gray's recent approximation to the solution of the integral equation for infinitely thin cylindrical shells. $K_{a}=120(\log 2 l / a-1)=60 \Omega-120$ ohms, where $l$ is the length of each half of the antenna, $\Omega$ is the parameter in Brillouin's paper. The range of $K_{a}$ in the figure corresponds to $38<l / a<1080$ and $8.66<\Omega<15.33$.

[^13]resistance of center fed cylindrical antennas in free space. Curves (1), (3) and (4) refer to infinitely thin cylindrical shells (for which the antenna current vanishes at the ends); (1) is Hallén's approximate solution of the integral equation, (3) is Schelkunoff's approximate solution of Maxwell's equations, ${ }^{2}$ and (4) is Gray's recent approximation to the solution of the integral equation. ${ }^{3}$ The difference between (1) and (4) is due solely to the difference in the methods of successive approximations. L. V. King's approximation to the integral equation gives $R_{\max , 1}=4000$ for $K_{a}=420$; this is considerably higher than even Hallén's approximation. Curve (2) was calculated by the same method as (3) but for solid cylinders. The "end" or "cap" capacitance was estimated as explained elsewhere; ${ }^{4}$ this estimated capacitance is probably higher than the actual capacitance and in that region where (2) and (3) diverge the true curve is likely to be somewhat lower. In the case of hemispherical ends, curve (2) is raised still further. In all these curves it is assumed that there is no excessive localized capacitance in the vicinity of the input terminals; the effect of such capacitance is to lower these curves. Incidentally it means that one should never assume a "point generator" except when the input terminals are tapered to mere points.

[^14]

## SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The Editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

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[^0]:    * Received March 2, 1943.
    $\dagger$ Numbers in square brackets refer to the list of references at the end of the paper.

[^1]:    * Received May 13,1943 . This paper is part of a thesis submitted in conformity with the requirements for the degree of doctor of philosophy in the University of Toronto, 1942.

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[^2]:    * Received June 29, 1943. The paper constitutes part of a thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Brown University, May, 1943. The author wishes to thank Professor L. Brillouin of Brown University for his many helpful suggestions.
    ${ }^{1}$ This case has been discussed by Lord Rayleigh, Phil. Mag. 43, 125-132 (1897); Brillouin, Rev. Gén. de l'Élec. 40, 227-239 (1936); Schelkunoff, Proc. Inst. Radio Eng. 25, 1457-1492 (1937); Chu and Barrow, Proc. Inst. Radio Eng. 26, 1520-1555 (1938); Slater, Microwave transmission, McGraw-Hill, 1942, pp. 124-150.

[^3]:    * Received July 1, 1943.
    ${ }^{1}$ By the moment profile of a structure under a given load we understand the magnitude of the bending moment as a function of position; the graph of this function is commonly called the bending moment diagram.

[^4]:    ${ }^{2}$ A discussion of orthogonal systems, explaining this process, will be found in R. Courant and D. Hilbert, Methoden der Mathematischen Physik (Berlin, 1931), vol. 1, p. 40, or in E. T. Whittaker and G. N. Watson, A Course of Modern Analysis (Cambridge, 1927), p. 224.

[^5]:    ${ }^{3}$ In most problems displacements at the constraints are zero. In such cases it is altogether unnecessary to visualize what particular generalized forces generate the orthogonal moment profiles because that physical notion is needed only for evaluating the $\delta_{i}$ 's.
    ${ }^{4}$ The integrals $\int M_{i}^{2} d x^{\prime}$ which appear in the denominator of (5) are also independent of the loading and therefore can be computed once for all.

[^6]:    ${ }^{5}$ See, e.g., H. F. B. Müller-Breslau, Die Graphische Statik der Baukonstruktionen (Leipzig, 1925), vol. 2, part 2, p. 56.

[^7]:    ${ }^{6}$ After the manuscript of this paper had been completed, our attention was drawn to a work along similar lines by S. Müller (S. Müller, Zur Berechnung mehrfach statisch unbestimmter Tragwerke, Zentralblatt der Bauverw. 1907, p. 23). Müller introduces a "system of forces $X_{i}{ }^{n}$ instead of single forces, and reduces the system of equations to the diagonal form. However, his point of view and emphasis are quite different from ours, and his procedure is more lengthy than that presented here.

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[^9]:    ${ }^{5}$ Hoff, N. J., Stresses in space-curved rings reinforcing the edges of cut-outs in monocoque fuselages, Journal Roy. Aeron. Soc., 47, 64 (1943).

[^10]:    * Received August 9, 1943.

[^11]:    * Received Aug. 26, 1943.

[^12]:    * Received Scpt. 9, 1943.
    $\dagger$ This note was prepared at the suggestion of Professor W. Prager while the author was a participant in the Program of Advanced Instruction and Research in Mechanics at Brown University and was presented to the American Mathematical Society on Sept. 12, 1943 under the title of On a principle of M. A. Sadowsky.
    ${ }^{1}$ M. A. Sadowsky, Journal of Applied Mechanics 10, A-65 (1943).
    ${ }^{2}$ Journal of Applied Mechanics 10, A-238 (1943).

[^13]:    * Received Nov. 20, 1943.
    ${ }^{1}$ Leon Brillouin, The Antenna Problem, Quarterly of Applied Mathematics, 1, 201 (1943).

[^14]:    ${ }^{2}$ S. A. Schelkunoff, Electromagnetic Waves, D. Van Nostrand Co., New York, 1943, Chapter 11.
    ${ }^{3}$ Marion C. Gray, A modification of Hallen's solution of the antenna problem, Journal of Applied Physics, 15, No. 1, Jan. 1944.

    - Reference 2, page 465.

