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[^0]
# THE NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS* 

ву<br>HOWARD W. EMMONS<br>Harvard University

1. Introduction. Through a consideration of the fundamental aspects of the universe Newton was led to the invention of fluxions. The physical idea of "rate of change," the geometric idea of "slope of a curve" together with his newly invented mathematics proved to be very powerful in describing and predicting a wide range of phenomena of nature. Since Newton's day, an enormous number of physical phenomena have been described in terms of a few "laws of nature." Very often these laws make use of the calculus, especially when applied to a specific problem. Thus large sections of the phenomena of the physical universe are described by the solutions of differential equations for the appropriate boundary conditions.

The engineer in his attempt to make nature work his way is continually presented with problems to be solved. These problems, even when very technical in nature, usually contain an element not present in problems considered by mathematicians or physicists. A "solution" of an engincer's problem is often a numerical answer or a graph obtained in a specified time. A poor answer which meets the deadline date is far superior to a precise answer a week later. Thus the engineer, or any applied scientist, should, when choosing the method of attack on a problem, keep in mind the time when the answer is due.

The following methods of solution are available:

1) the answer can be guessed;
2) some experiments can be run;
3) the result can be computed from the basic laws of nature, by use of whatever mathematical methods are needed.
Obviously the first method has one certain and everlasting superiority over the second and third; it is quick. It can meet any deadline set at a future time. It, of course, has one big disadvantage. The solution is always of doubtful quantitative value even though it is of immense qualitative value. In fact, the guessing process should always be used as a guide to correct results by methods 2 or 3 .

If time permits, the answer may be sought by experiment or computation, or both. At the present time computational methods, when applied to real physical systems to obtain solutions of sufficient accuracy, are often so cumbersome that the vast majority of engineering problems are solved primarily by experiment. Usually com-

[^1]putations are confined to one dimensional approximations to the real problem, or to certain simple two or three dimensional approximations.

In the field of partial differential equations, which includes the key to a vast number of practical problems, the present day need is for methods of solution speedier and more general than the majority of analytical methods thus far discovered. Numerical methods of solution aim at securing an approximate quantitative answer to a given problem as directly as possible from the statement of the problem, so as to reduce the time required for solution. An advantage of this more direct approach is the possibility of injecting into the solution any facts known (or supposed known) by the computer. Thus, physical "facts" of any nature can be made use of, and if some of the "facts" are wrong the solution will indicate this. Obviously, this has an enormous advantage over analytical methods where generally little use can be made of detailed physical observations, except as a check on the final result.
2. An elementary problem. By way of illustrating the methods involved, let us consider the transient flow of heat in a two-dimensional homogeneous, isotropic solid for which the physical law of the conservation of energy yields the familiar differential equation

$$
\begin{equation*}
\frac{\partial T}{\partial t}=a\left(\frac{\partial^{2} T}{\partial x^{2}}+\frac{\partial^{2} T}{\partial y^{2}}\right) \tag{1}
\end{equation*}
$$

where $T$ denotes the temperature, $x$ and $y$ space coordinates, $t$ the time, $a$ the thermal diffusivity (a property of the material, assumed constant). The numerical methods considered here are based upon a finite difference approximation to the differential equation. The simplest of these may be obtained as follows.

By definition

$$
\begin{equation*}
\frac{\partial T(x, y, t)}{\partial t}=\lim _{\delta t \rightarrow 0} \frac{T(x, y, t+\delta t)-T(x, y, t)}{\delta t} \tag{2}
\end{equation*}
$$

and similar expressions hold for the derivatives with respect to $x$ and $y$. As an approximation, the limit operation may be omitted. If this is done the following expressions are obtained

$$
\begin{equation*}
\frac{\partial^{2} T}{\partial x^{2}}=\frac{T(x-\delta x, y, t)-2 T(x, y, t)+T(x+\delta x, y, t)}{\delta x^{2}}=\frac{T_{3}-2 T_{0}+T_{1}}{\delta x^{2}} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial^{2} T}{\partial y^{2}}=\frac{T(x, y-\delta y, t)-2 T(x, y, t)+T(x, y+\delta y, t)}{\delta y^{2}}=\frac{T_{2}-2 T_{0}+T_{4}}{\delta y^{2}} \tag{4}
\end{equation*}
$$

where the last expressions on the right follow the notation of Fig. 1. Substitution of these approximations into Eq. (1) gives, after some rearrangement,

$$
\begin{equation*}
T(x, y, t+\delta t)=\frac{a \delta t}{\delta x^{2}}\left(T_{3}+T_{1}\right)+\frac{a \delta t}{\delta y^{2}}\left(T_{2}+T_{4}\right)+\left(1-\frac{2 a \delta t}{\delta x^{2}}-\frac{2 a \delta t}{\delta y^{2}}\right) T_{0} \tag{5}
\end{equation*}
$$

which relates the temperature at a given point 0 at time $t+\delta t$ to the temperatures which existed in the neighborhood of 0 at time $t$. Since the only restriction on $\delta t, \delta x, \delta y$
is that they be small enough to render the finite difference approximation sufficiently accurate, we set

$$
\begin{equation*}
4 a \delta t=\delta x^{2}=\delta y^{2}=\delta^{2} \tag{6}
\end{equation*}
$$

whence

$$
\begin{equation*}
T(x, y, t+\delta t)=\frac{1}{4}\left(T_{1}+T_{2}+T_{3}+T_{4}\right) \tag{7}
\end{equation*}
$$

This same equation can be derived directly from the physical problem by making physical assumptions only. Such an approach is very valuable, since many more or less vague physical assumptions are always made before any engineering problem can


Fig. 1.
be set up mathematically. Let us assume that the material of the solid body is divided up into three sets of overlapping squares (or rectangles if desired). One set represents all heat conduction of the body in the $x$-direction, as indicated by the dotted square between points 0 and 1 of Fig. 1. The second set of squares represents the conduction in the $y$-direction and finally the third set, surrounding each point, represents all the material as far as heat capacity is concerned. The thermal energy $Q_{1-0}$ which is conducted in unit time to point 0 along the rod $1-0$ is obtained from Fourier's heat conduction equation

$$
\begin{equation*}
q=-k \frac{d T}{d x} \tag{8}
\end{equation*}
$$

in the form

$$
\begin{equation*}
Q_{1-0}=k b\left(T_{1}-T_{0}\right) \tag{9}
\end{equation*}
$$

where $b$ is the thickness of the two dimensional body considered. The energy arriving at the point 0 from all the surrounding points is thus

$$
\begin{equation*}
Q_{0}=k b\left(T_{1}+T_{2}+T_{3}+T_{4}-4 T_{0}\right) \tag{10}
\end{equation*}
$$

This heat will result in an increase of temperature of the material associated with point 0 , whence

$$
\begin{equation*}
Q_{0}=c \rho b \delta^{2} \frac{T(x, y, t+\delta t)-T_{0}}{\delta t} \tag{11}
\end{equation*}
$$

We equate (10) and (11) and rearrange terms. Thus,

$$
\begin{equation*}
T(x, y, t+\delta t)=\frac{k \delta t}{c \rho \delta^{2}}\left(T_{1}+T_{2}+T_{3}+T_{4}\right)+\left(1-\frac{4 k \delta t}{c \rho \delta^{2}}\right) T_{0}, \tag{12}
\end{equation*}
$$

which reduces to (7), since the thermal diffusivity $a=k / c \rho$, and $\delta t, \delta$ are related by (6).
The method of use of Eq. (7) for the solution of a transient heat flow problem expressed by the differential Eq. (1) is direct. The space domain is divided into a square net of points. (The exact relationship between the required net spacing and the desired accuracy of solution has not yet been studied, to the author's knowledge.) The initial values (at $t=0$ ) of temperature are attached to each point and the values at successive times are computed by the averaging process. Figs. 2 illustrates such a solution.


Fig. 2a. (See legend below Fig. 2c.)
By the nature of the process it is clear that the shape of the domain and the boundary conditions (generally some relation between the boundary value and the normal derivative) cause no special difficulty, such as occurs in the analytical approach, since they can be treated numerically as required for the points nearest (or on) the boundary. Boundary conditions and net spacing should be chosen of comparable accuracy as judged physically (in the absence of rigorous methods). Experience indicates that it is seldom worth the trouble to derive special boundary formulas, since generally linear extrapolation or a simple plot on graph paper will give the same accuracy much more speedily.

Let us return to Eq. (1) and the attendant physical problem. It is known that if the thermal conditions at the physical boundaries are held fixed (in time), the interior temperature will (after a sufficient period) reach a steady value (to any given
degree of approximation). In the present problem, this means that with a sufficiently large number of applications of Eq. (7) the solution of Laplace's equation will be obtained for the given domain and boundary conditions. The process proposed above, the one closely followed by nature, is numerically very slow unless an elaborate computing machine is used [1].*

Several schemes have been proposed for speeding up this process. By starting with Laplace's equation and making the finite difference approximations, by starting physically with Eq. (10) and $Q_{0}=0$, or by observing that for steady conditions $T(x, y, t+\delta t)=T_{0}$ in Eq. (7), one obtains

$$
\begin{equation*}
T_{0}=\frac{1}{4}\left(T_{1}+T_{2}+T_{3}+T_{4}\right) \tag{13}
\end{equation*}
$$

When Eq. (13) is written for each of $n$ points of a square net covering a given domain, $n$ linear equations result. Several iteration processes have been devised for solving such a system $[2,5,6]$, and the convergence of some of these has been


Fig. 2b.
discussed $[3,4,7,8]$. All of these methods propose computations on the values of $T$ by a specifically stated iteration process which can be shown rigorously to converge and can be performed by a completely automatic computing machine. In no case is it possible to add physical information after the values guessed initially are attached to each net point, without upsetting the scheme of solution.

A new scheme for solving Eq. (13) for the $n$ points of a two-dimensional domain is given by the relaxation method [9]. This method is so superior to others in point of the time required to reach a solution of given accuracy that it will be discussed in

[^2]detail. Again, the physical problem will be considered as one of heat conduction, although any other phenomena leading to Laplace's equation would serve as well. R. V. Southwell [10], who developed the relaxation method from a consideration of problems of statics, generally speaks in terms of a tension net as an approximation to a soap film or membrane.


Fig. 2c.
Figs. 2a-c. Transient heating of furnace wall (Fig. 5). At time 0 wall at uniform temperature $T=100^{\circ} \mathrm{F}$ has inner surface temperature raised to $500^{\circ} \mathrm{F}$. Wall thermal diffusivity $a=.01 \mathrm{ft}^{2} / \mathrm{hr}$. $\delta_{x}=.208 \mathrm{ft}$. Therefore $\delta_{t}=1.083 \mathrm{hrs}$. The 6 temperatures shown at each point are at time intervals of 1.083 hrs . Each temperature is the average of the temperatures at the 4 surrounding points at the preceding time. Heat loss is $\sum Q^{\prime}$ along inner surface.
(From The numerical solution of heat conduction problems. By H. W. Emmons, Trans. A.S.M.E., 65, 607-615 (1943)).

Instead of focusing attention on the values of $T$ and the averaging process of Eq. (13), let us return to Eq. (10). To solve a problem the domain is drawn and the net points chosen. Values of $T$ are then attached (by guess or any information available from experiment, special solutions, prior work, field mapping, etc.) to each point. From these values the residuals

$$
\begin{equation*}
Q=Q_{0} / k b=T_{1}+T_{2}+T_{3}+T_{4}-4 T_{0} \tag{14}
\end{equation*}
$$

are computed and recorded. The $Q$, thus computed, can be thought of as interior heat sinks which must be removed. Now, instead of setting up a specific iteration process, we merely observe that if the temperature at one point ( 0 ) is altered, all others remaining fixed, the residuals will change according to the pattern in Fig. 3, the "relaxation pattern." Each change of $T$, at any point, effects a redistribution of the residuals,
$Q$, among the net points, and such changes of $T$ are desired which will move all the sinks to the boundary.

These "operating instructions" may appear vague. Indeed they are vague. Their vagueness is the source of their great power because the computer may without any effort alter the procedure to attain more rapid approach to the final answer (of no residuals). There is only one way to appreciate fully the meaning of these remarks and that is to do a problem.

Let us consider the extremely simple problem of Fig. 4, where


Fig. 3. the solution of Laplace's equation is desired with zero boundary values. For illustrations of the various methods, only three interior points are used and the ridiculous trial solution indicated in Fig. 4 is assumed. In this simple problem all the methods are equally easy. The real time-saving advantages


Fig. 4.

0 of the relaxation process only appear with more net points. In Table IA the transient solution [differential equation (1)] is carried out by use 0 of the difference equation (7). In Table IB Liebman's method [2] is used, Eq. (13). In both cases, each value after the first at any point is obtained by adding four numbers and dividing by 4 . This process cannot (for say 4 digit numbers) be carried out mentally. Fifteen and ten changes respectively were needed to make the error less than unity.

Tables IC, D, E show applications of the relaxation method and these will be followed in detail to illustrate various "tricks" which serve to speed up the elimination of residuals. In Tables IC, D, E the initial value of $T$ and the subsequent corrections are shown at the right of each field, the values of the residuals (heat sinks) are shown to the left. The largest heat sink occurs in the vicinity of the greatest deviation of the assumed values from the correct solution, so changes are first made at this point. For purposes of illustration, each time any change is made in Tables IC, D, E a space is left at all unaffected points, so that the work can be followed. (Generally this is not done.)

Let us consider Table IC. To eliminate exactly a residual of -420 at point 3 would require a change in $T_{3}$ of -105 . The making of this change is equivalent exactly to the averaging process. But why do we bother with three digit numbers? Since our residuals are very large, let us make simple large changes of about the right size. Therefore, let us change $T_{3}$ by -100 . By the relaxation pattern of Fig. 3 the residuals become $Q_{3}=-420+4(100)=-20, Q_{2}=-80-100=-180$. Now $Q_{2}$ is largest. Accordingly, we change $T_{2}$; a change of -50 was chosen and the third residuals at each point computed. We notice that the residual at the point 3 , much improved at the first step, has been spoiled again by the following change at point 2 . This always happens when a point is surrounded by other points with residuals of the same sign. We would have done better to overshoot zero at point 3 on the first change. On the next change, at point 1 , we overshoot the zero and make the residual positive. The residual at point 2 changes from +20 to -20 and the previous change might well
have been larger. By not overshooting enough or by overshooting too much we do no harm except that some time is lost. A little practice improves one's guessing.

In Table ID the block relaxation

## Table I

A) Solution by Eq. (17); 15 changes.
B) Solution by Eq. (13) (Liebman's method); 10 changes.
C) Illustrates overshooting; 5changes. ( $\dagger=$ overshot, * $=$ should have overshot).
D) Illustrates block relaxation; 6 changes made, first 3 at once.
E) Illustrates use of prior knowledge; 3 changes.

| Point | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: |
| A | $\begin{aligned} & 40 \\ & 15 \\ & 10 \\ & 1.4 \\ & 1.2 \\ & .2 \end{aligned}$ | $\begin{aligned} & 60 \\ & 40 \\ & 7.5 \\ & 5 \\ & .7 \\ & .6 \end{aligned}$ | $\begin{array}{r} 120 \\ 15 \\ 10 \\ 1.4 \\ 1.2 \\ .2 \end{array}$ |
| B | $\begin{aligned} & 40 \\ & 15 \\ & 8.4 \\ & 1.0 \\ & .1 \end{aligned}$ | 60 33.8 4.2 .5 | $\begin{array}{r} 120 \\ 8.4 \\ 1.0 \\ .1 \end{array}$ |
| C | $\begin{array}{rr} -100 & 40 \\ -150 & \\ 10 & -40 \dagger \\ \hline 0 & \end{array}$ | $\begin{array}{rc} -80 & 60 \\ -180 & \\ 20 & -50 \\ -20 & * \\ -40 & \\ 0 & -10 \end{array}$ | $\begin{array}{rr} -420 & 120 \\ -20 & -100 \\ -70 & \\ -10 & -20 \\ 0 & \end{array}$ |
| D | $\begin{array}{rr} -100 & 40 \\ 50 & -50 \\ -40 & \\ 0 & +10 \end{array}$ | $\begin{array}{rr} -80 & 60 \\ 20 & -50 \\ -50 & \\ -10 & -10 \\ 0 & \end{array}$ | $\begin{array}{rr} -420 & 120 \\ -270 & -50 \\ 10 & -70 \\ 0 & \\ - & \end{array}$ |
| E | $\begin{array}{rc} -100 & 40 \\ -160 & \\ 0 & -40 \end{array}$ | $\begin{array}{rr} -80 & 60 \\ -200 & \\ 60 & -60 \\ 0 & \end{array}$ | $\begin{array}{rr} -420 & 120 \\ 60 & -120 \\ 0 & \\ - & \end{array}$ | process is used, i.e., a block of points is changed simultancously. This is desirable since all the residuals are negative. So large a change is made that their average resultant residual is about zero (or overshot if desired). A change of 50 was chosen. If one computes by the relaxation pattern [Eq. (14)] directly there is no gain by block relaxation. Instead let us consider the physical arrangement. If the temperatures at two adjacent points are both changed by the same amount there will be no change of heat flow along the connecting rod. Thus the relaxation pattern $-1,1$ as given by Eq. (9) is used for each rod independently, and at the points of Table ID, the residual change by $Q_{1}=$ $-100+3(50)=50, Q_{2}=-80+2(50)=20$, $Q_{3}=-420+3(50)=-270$. From this point the solution is continued as before.

Of course we know the solution in the present case is zero everywhere. This knowledge is used in Table IE immediately and the solution obtained in three steps. Naturally one would not have started this problem with so poor an assumed set of initial values. Also the prior knowledge is never as extensive as in the present trivial problem, but observation of trends in the solution often gives a clue to the way in which values should be changed.

It is not possible to judge the speed and ease of the relaxation method compared to the averaging methods by comparison of the various procedures illustrated in Table I. The relaxation method is far superior when a large number of points are used because of its flexibility (possibility of overshooting), its use of simple large numbers when the residuals are large and simple small numbers when they have become small, and the fact that the most difficult operation is to multiply a simple number by 4 and add the result to or subtract it from another, all of which can easily be done mentally.

Figs. 5 illustrate the solution of a somewhat more difficult problem. It should be
noted that the solution is started with a few points only. Then, for greater accuracy more points are added. These need not be added everywhere but only where greater accuracy is desired, or where some variable changes rapidly. In the illustration the points were doubled in number by adding one point at the center of each four points. The resultant net is square but diagonal to the first. Since Laplace's equation is invariant on rotation of the axes, the same average formula applies to values on the new net. Guessed values for starting the finer net are obtained by averaging the surrounding four values. If a fine net is used locally, as might have been done near the reentrant corner in the illustration, it can always be connected with the coarser net through use of the diagonal formula where the two nets meet.


Fig. 5a. Furnace, showing section under consideration.
The results of computation of the furnace wall conduction problem are summarized in Table II. We note particularly the time required for a solution and the

Table II
Numerical Solution of Heat Conduction
Problem of Fig. 5

| Number of <br> points <br> used | Calculated <br> thermal <br> resistance | Deviation from <br> experimental <br> solution | Hours <br> required for <br> calculation |
| :---: | :---: | :---: | :---: |
| by arithmetic mean area | $\frac{.0735}{k b}$ | $15.3 \%$ | .05 |
| 12 | $\frac{.0806}{k b}$ | $5 \%$ | .75 |
| 19 | $\frac{.0825}{k b}$ | $2.8 \%$ | 1.75 |



Fig. 5b. Start solution with linear temperature distribution, relax $Q^{\prime}$ to get final temperature. Heat transferred is $Q=k b[232+208+(4.5 / 5) 202]=622 \mathrm{~kb}$. Thermal resistance $R=\Delta T / 8 Q=400 /(8 \times 622 k b)$ $=.0806 / k b$. Superscripts indicate step of calculation. ${ }^{0}=$ original values, ${ }^{1,2, \text { ete }}=$ successive steps.


Fig. 5c. Start with solution of coarse net Fig. 3a. Relax $Q^{\prime}$ to get final temperature. Heat transferred $Q=k b[195 / 2+2 \times 111+2 \times 102+(4 / 5) 102]=606 \mathrm{~kb}$. Thermal Resistance $R=\Delta T / 8 Q=400 /(8 \times 606 \mathrm{~kb})$ $=.0825 / \mathrm{kb}$.
(From The numerical solution of heat conduction problems. By H. W. Emmons, Trans. A.S.M.E., 65, 607-615 (1943)).
corresponding error. This same problem took 11 hours when solved by the averaging process.

Another problem, still more difficult, is illustrated in Fig. 6. This is the problem of the water tube of a boiler, and the boundary conditions are varied. The water inside the tube is assumed to maintain a constant wall temperature $T=0$. The lower half of the tube is embedded in insulating material, so the normal gradient of tempera-


Fig. 6. Heat transfer by radiation through furnace water wall tube.
ture is there assumed to be zero. On the upper half of the tube heat is transferred by radiation in such a way that the heat input is constant per unit projected area. The normal gradient of temperature is thus proportional to $\sin \theta$. This problem was easiest to solve by a transformation from the $x, y$-plane to the $z, \theta$-plane, where $\theta=\tan ^{-1} y / x$, $z=\log r=\frac{1}{2} \log \left(x^{2}+y^{2}\right)$. The net points used are indicated in the $x, y$-plane by the intersection of the radial lines and circles. The numerical solution checked "exactly" with the analytical solution, as observed by superposing the graphed results. From an engineering point of view this solution is "exact," since it deviates much less from the analytical solution than the uncertainty of the boundary conditions.

So far we have dealt at length with the relation between the well known averaging process and the relaxation process. Indeed, it may appear that we have wasted time with trivialities of arithmetic but the author's conversations with many have indicated that it is just these minor details in point of view that make the relaxation process about five times as rapid as the iteration processes. To appreciate fully the power of the flexibility of the relaxation method, one must take pencil and paper and carry out the numerical process in all its uninteresting details. In fact, for the compputer (as opposed to those who think only about the logic behind the computation methods) the relaxation method has a spirit lacking entirely from the iteration processes. The former challenges one's intellect at each step to make the best possible guess, while the latter reduces one to the status of an automatic computing machine (without the advantage of no computational errors). It should not be inferred that the relaxation process requires high intellectual powers. If changes are chosen in a specifiable way it reduces exactly to the iteration process. The computer can then vary from this completely specified process by whatever amount fits his own skill.
3. Other types of equations solved by the relaxation method. After the essential idea of the relaxation method is grasped, other problems may be solved by rather obvious steps. The differential equation is converted into a difference equation, some quantity (to be zero for the solution) is chosen as the residual, and the relaxation pattern is set up. Changes which make the residuals smaller are then made. We notice that no question of convergence can occur in the general relaxation process, since no specific instructions are given. If, after some steps, the residuals get worse, the intelligent computer goes back and makes changes in the opposite direction. These remarks oversimplify the problem somewhat because of two facts; first, the computer may become confused as to whether or not the residuals are really better, and secondly there is always a question of whether or not a solution with zero residuals exists (see [8]).

The following is a partial list of types of problems solved, with some brief details of their solution.
A. Poisson's equation,

$$
\begin{equation*}
\Delta \varphi=\varphi_{x x}+\varphi_{y y}=f(x, y) . \tag{15}
\end{equation*}
$$

(Subscripts denote partial differentiation.) In finite difference form, for a square net of points the approximating difference equation is (see Fig. 1 for notation)

$$
\begin{equation*}
\varphi_{1}+\varphi_{2}+\dot{\varphi}_{3}+\varphi_{4}-4 \varphi_{0}-f(x, y) \delta x^{2}=Q \tag{16}
\end{equation*}
$$

where the residual $Q$ is to be zero at each net point. Since $f(x, y)$ has a known numerical value at each net point, it enters into the value of the residuals at the start of the computation, and in no way affects the relaxation pattern used for Laplace's equation, Fig. 3. Thus the solution of Poisson's equation is precisely as easy as that of Laplace's.
B. Biharmonic equation,

$$
\begin{equation*}
\Delta \Delta w=\frac{\partial^{4} w}{\partial x^{4}}+2 \frac{\partial^{4} w}{\partial^{2} x \partial^{2} y}+\frac{\partial^{4} w}{\partial y^{4}}=0 . \tag{17}
\end{equation*}
$$

The finite difference equation for a square net of points is most simply derived by converting the two $\Delta$ forms separately. Thus

$$
\begin{equation*}
\Delta w_{2}+\Delta w_{2}+\Delta w_{3}+\Delta w_{4}-4 \Delta w_{0}=0 \tag{18}
\end{equation*}
$$

from which, by expansion of the remaining $\Delta$ 's, one obtains the relaxation pattern shown on Fig. 7, where the solution of a problem of the deflection of a plate with clamped deflected edges is also shown. The accuracy of the present solution cannot be


Fig. 7. Deflection of a plate with clamped deflected edges, No load. $\Delta \Delta w=0$, $w$ given at edge, $w_{n}=0$ at edge. Recorded: $\Delta \Delta w \cdot w$. Solution time: 12 hours. 1 st solution: 6 hours. Checking and eliminating mistakes: 6 hours.
computed, since the exact solution of the differential equation is not known for these boundary conditions. However, one's confidence in its precision is increased by observing that the finite difference solution of a square plate under uniform load is in error less than $1 \%$ in maximum deflection when only 9 interior points are used [11], rather than the present 22.
C. The equation of natural modes of a membrane,

$$
\begin{equation*}
\Delta w+\lambda w=0 . \tag{19}
\end{equation*}
$$

In problems of the vibration of two-dimensional systems, the information sought concerns the natural frequencies and the characteristic functions. In the equation, it is required to find the permissible values of $\lambda$, which is the square of the frequency times certain physical constants. The relaxation method can be applied to this problem in several ways. For example, a value of $\lambda$ could be estimated by Rayleigh's principle from an assumed deflection $z$,

$$
\begin{equation*}
\lambda=\frac{\sum\left(w_{x}^{2}+w_{y}^{2}\right)}{\sum w^{2}}, \tag{20}
\end{equation*}
$$

where the summation extends over all the net points. This value could then be inserted into Eq. (19) in the finite difference form,

$$
\begin{equation*}
w_{1}+w_{2}+w_{3}+w_{4}-\left(4-\lambda \delta^{2}\right) w_{0}=Q, \tag{21}
\end{equation*}
$$

and the relaxation process used to reduce the $Q$. Periodically during the solution $\lambda$ would be reevaluated and reinserted into Eq. (21). Thus new residuals would appear, which could again be reduced, the process being repeated until Eqs. (20) and (21) are satisfied with sufficient accuracy. This method of solution, as used by Southwell [12], imagines that at each net point there is a force $Q$ applied and that these forces are to be removed by changes of the amplitudes $w$, and the "frequency" $\lambda$.

Another method which seems to be superior in some respects has been worked out by Dr. A. Vazsonyi, and will be published shortly.* Equation (19) is written in the form

$$
\begin{equation*}
\lambda=-\frac{\Delta w}{w} . \tag{22}
\end{equation*}
$$

At each net point a value of the amplitude $w$ is assumed and the corresponding values

| 0 | 0 | 0 | 0 | 0 |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 0 | 454 | 761 | 815 | 568 | 0 |  |
|  | 900 | 896 | 894 | 902 |  |  |
| 0 | 646 | 1092 | 1200 | 945 | 440 | 0 |
|  | 895 | 898 | 897 | 903 | 904 |  |
| 0 | 460 | 780 | 870 | 720 | 418 | 135 |
|  | 899 | 895 | 899 | 898 | 901 | 905 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 |  |  |  |  |  |  |



Fig. 8. Lowest frequency and natural mode of quadrangular membranc. Second approximation.

$$
\begin{aligned}
& \Delta v+\frac{\lambda}{a} v=0, \quad v=0 \text { on edges, } \\
& \delta=1, \quad a=24, \quad \lambda_{m}^{*}=\frac{\Sigma v \Delta v}{\Sigma v^{2}}=.8984, \quad \lambda=\frac{a}{\delta^{2}} \lambda_{m}^{*} .
\end{aligned}
$$

First characteristic value: $\lambda=21.562$.
Recorded:
$\left\{\frac{v}{-1000 \frac{\Delta v}{v}\left(=1000 \lambda^{*}\right)}\right\}$.

[^3]of $\Delta w$ and $\lambda$ recorded. These assumed amplitudes constitute the solution to the problem of the vibration of a membrane of non uniform mass distribution. For a uniform mass distribution the value of $\lambda$ should be the same at each point. The values of $w$ are changed to equalize the values of $\lambda$. Whenever a $w$ is changed the changes of $\Delta w$ are immediately computed by the Laplace relaxation pattern, Fig. 3. New values of $\lambda$ need not be computed every time. At any stage of the solution the correct $\lambda$ lies between the highest and iowest value at the net points. Finally a value of $\lambda$ is computed as an average of those at all the net points. The best theoretical way to compute this "average" is by the use of Eq. (20) in the form
\[

$$
\begin{equation*}
\lambda=\frac{\sum w \Delta w}{\sum w^{2}} . \tag{23}
\end{equation*}
$$

\]

Fig. 8 shows the solution for the lowest frequency of a quadrangular membrane.



Fig. 9. Second frequency and natural mode of quadrangular membrane. Second approximation.

$$
\begin{gathered}
\quad \Delta v+\frac{\lambda}{a} v=0, \quad \nabla=0 \text { on edges } \\
\delta=1, \quad a=24, \quad \lambda_{m 1}^{*}=\frac{\Sigma v \Delta v}{\Sigma v^{2}}=1.6716, \quad \lambda=\frac{a}{\delta^{2}} \lambda_{m}^{*} .
\end{gathered}
$$

Second characteristic value $\lambda=40.118$.
Recorded:

$$
\left\{\frac{v}{-100 \frac{\Delta v}{v}\left(=100 \lambda^{*}\right)}\right\}
$$

To compute higher modes, the procedure is exactly the same as outlined above for the lowest mode. The assumed amplitudes should of course include one or more nodal lines, depending upon the mode sought. All of one's information about the vibration of membranes should be used in selecting the amplitudes, and generally it is convenient to make use of the orthogonal properties of the characteristic functions. Thus, it is always possible to use any arbitrary amplitudes $w$ and remove the portion of these arising from the first mode $w_{1}$ (obtained at the end of the first mode computation), to obtain the amplitudes $w_{n}$ of the higher modes only;

$$
\begin{equation*}
w_{n}=w-a w_{1}, \tag{24}
\end{equation*}
$$

where

$$
\begin{equation*}
a=\frac{\sum w_{1} w}{\sum w_{1}^{2}} . \tag{25}
\end{equation*}
$$

The second mode of the quadrangular membrane is shown in Fig. 9, while Table III compares the results. We note the good accuracy obtained with a few points in the square. By the use of symmetry, the 9 point approximation for the square required only 10 minutes.

Table III
Characteristic values of membranes

| Characterization of problem | Square membrane; lowest mode |  |  | Membrane accord. to Figs. 8, 9 ; lowest mode second mode |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Order of approximation | first | second | exact | first | second | first | second |
| Degrees of freedom of approx. system | 9 | 49 | - | 7 | 15 | 7 | 15 |
| Characteristic value | 18.75 | 19.508 | $\begin{gathered} 2 \pi^{2}= \\ 19.739 \end{gathered}$ | 20.61 | 21.562 | 37.10 | 40.118 |
| Number of modifications used | 5 | 35 | - | 12 | 35 | 16 | 65 |
| Accuracy | 5\% | 1.2\% |  |  |  |  |  |

It should be noted that the solution of a forced vibration problem is best carried out by the first method outlined above.
D. Equations of the type,

$$
\begin{equation*}
\varphi_{x x}+f\left(x, y, \varphi, \varphi_{x}, \varphi_{y}\right) \varphi_{y y}=g\left(x, y, \varphi_{,}, \varphi_{x}, \varphi_{y}\right) . \tag{26}
\end{equation*}
$$

Equations of this type are of frequent occurrence in engineering problems, but because of their non-linear aspects only limited assistance is offered by conventional mathematical methods. For so complicated an equation, there are many possible ways of applying the relaxation process. Only one will be mentioned here. We obtain a finite difference equation by transforming the second derivatives only;

$$
\begin{equation*}
\varphi_{1}+\varphi_{2}+f_{0} \varphi_{3}+f_{0} \varphi_{4}-2\left(1+f_{0}\right) \varphi_{0}-g_{0} \delta^{2}=Q, \tag{27}
\end{equation*}
$$

for which the relaxation pattern is shown in Fig. 10. Thus the pattern is different
for each net point and varies during the course of solution. The luxury of an investigation of the classification of equations of the type of (26) (as to whether they are elliptic, parabolic, or hyperbolic, if such a classification is possible), of the nature of solutions, of the permissible boundary conditions, etc., is denied during war time by the urgency to get numerical results. Only casual observations have been made to date, and will not be discussed. However, it is certain that for $f>0$ [Eq. (26) of elliptic type] the process of solution described is quite easy and quick to carry out.

The domain of the problem to be solved is drawn and a solution is guessed at a net of points. From this set of values of $\phi, f_{0}$ and $g_{0}$ and then the $Q$ are computed at each point. By the relaxation process carried out exactly as described for Laplace's equation, except that the influence coefficients of Fig. 10 are used instead of $(-4,1,1,1,1)$, the residuals


Fig. 10. $Q$ are reduced somewhat. Before bothering to eliminate the $Q$ completely, we compute new values of $f_{0}$ and $g_{0}$ and hence corrected values of the residuals attached to each net point. This process of reduction and correction is continued until sufficient accuracy has been attained. Just as in previous problems, a finer net can be added for greater accuracy.

As an illustration, let us consider the distribution of electric potential in the space between two parallel planes, one of which has, standing normal to it, a right circular, cylindrical post with a hemispherical top. Fig. 11 shows the problem and the solution.


Fig. 11. Axially symmetric electric potential distribution.

$$
\frac{\partial^{2} \varphi}{\partial s^{2}}+\frac{1}{r^{2}} \frac{\partial^{2} \varphi}{\partial y^{2}}=0, \text { where } y=\log r, \varphi \text { constant on boundaries. }
$$

If cylindrical coordinates are used with the $z$-axis along the axis of the post and the origin at the base of the post, the potential is described by the equation

$$
\begin{equation*}
\frac{\partial^{2} \phi}{\partial z^{2}}+\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right)=0 \tag{28}
\end{equation*}
$$

with the boundary conditions $\varphi=0$ on the base plane and post, and $\varphi=100$ on the parallel plane. Again, several alternative procedures are possible. For example, Eq. (28) can be written

$$
\begin{equation*}
\varphi_{z z}+\varphi_{r r}=-\frac{1}{r} \varphi_{r} \tag{29}
\end{equation*}
$$

and the solution can be carried out as described above for the general case.
In the case of the present problem it was decided to make the substitution

$$
\begin{equation*}
y=\log r \tag{30}
\end{equation*}
$$

thus converting Eq. (28) into

$$
\begin{equation*}
\varphi_{z z}+\frac{1}{r^{2}} \varphi_{y v}=0 \tag{31}
\end{equation*}
$$

Hence the finite difference equation becomes

$$
\begin{equation*}
\varphi_{1}+\varphi_{3}+B\left(\varphi_{2}+\varphi_{4}\right)-2(1+B) \varphi_{0}=Q, \tag{32}
\end{equation*}
$$

where $B=\delta z^{2} / r^{2} \delta y^{2}, \delta y$ and $\delta z$ being the net spacing in the $y$ and $z$ directions, respectively. The net spacing was chosen so that $\delta z=4 \delta y$. In this way sufficient points appear where they are needed, i.e., near the post. In Fig. 12 the transformed plane is shown, together with the influence coefficients at the top of each column of points.
E. Equations of the type,

$$
\begin{equation*}
\frac{\partial}{\partial x}\left(\mu \varphi_{x}\right)+\frac{\partial}{\partial y}\left(\mu \varphi_{y}\right)=g\left(x, y, \varphi_{,} \varphi_{x}, \varphi_{v}\right), \tag{33}
\end{equation*}
$$

where $\mu=\mu\left(x, y, \varphi, \varphi_{x}, \varphi_{y}\right)$. This equation is of very general occurrence in physical problems. For example, in the case of a soap film with large deflections $w$ under constant excess pressure $p$, we have [14]

$$
\begin{equation*}
\varphi=w, \quad \mu=\frac{1}{\left\{1+w_{x}^{2}+w_{v}^{2}\right\}^{1 / 2}}, \quad g=\frac{p}{2}=\text { constant } ; \tag{34}
\end{equation*}
$$

in the case of the plane irrotational flow of a compressible fluid with velocity potential $\varphi$, we have

$$
\begin{equation*}
\mu=\rho=\operatorname{density}=\rho_{0}\left\{1-\frac{\gamma-1}{1} \frac{v^{2}}{a_{0}^{2}}\right\} \frac{1}{\gamma-1}, \quad g=0 ; \tag{35}
\end{equation*}
$$

in the case of a steady magnetic field ( $\varphi=$ magnetic potential) in a non current carrying medium, we have

$$
\begin{equation*}
\mu=\text { magnetic permeability }=f\left(\varphi_{x}^{2}+\varphi_{v}^{2}\right), \tag{36}
\end{equation*}
$$

where the function $f$ is given by experimental data on the material.

The flow of oil in the film lubrication of a bearing, when the viscosity is assumed independent of temperature, is described by an equation of this type [see Eq. (39)]. In general, physical problems which lead to Laplace's equation when physical parameters are constant, give equations similar to (33) when the physical properties vary. The properties may be known functions of $x, y$ because the material is non-homogeneous, or of $\varphi, \varphi_{x}, \varphi_{y}$, etc., because of the nature of the material itself.


FIG. 12. Transformed $y$, $z$-plane of axially symmetric electric potential problem. Recorded: $\varphi$.

For the numerical solution of these problems it has been found convenient to use the form

$$
\begin{equation*}
\varphi_{x x}+\varphi_{y y}=g-(\log \mu)_{x} \varphi_{z}-(\log \mu)_{\nu} \varphi_{y} \tag{37}
\end{equation*}
$$

In finite differences this becomes

$$
\begin{align*}
\varphi_{1}+ & \varphi_{2} \\
& +\varphi_{3}+\varphi_{4}-4 \varphi_{0}  \tag{38}\\
& =g_{0} \delta^{2}+\frac{1}{4}\left\{\left(\varphi_{1}-\varphi_{3}\right)\left(\log \mu_{3}-\log \mu_{1}\right)+\left(\varphi_{4}-\varphi_{2}\right)\left(\log \mu_{2}-\log \mu_{4}\right)\right\}+Q
\end{align*}
$$

The relaxation pattern used is that used for Laplace's equation. The $Q$ is to be eliminated. The variable terms on the right of Eq. (38) are computed periodically as corrections. This method is well adapted to the equation as long as Eq. (33) remains of elliptic type. The "correction" terms on the right contain $\varphi_{x x}$ and $\varphi_{y y}$ through the relation (34). When these variations are such that Eq. (33) becomes hyperbolic, the
relaxation process becomes confusing unless a great deal of physical knowledge is available to assist the computer.

The solution of a problen of this kind is shown on Fig. 13, which shows the shape of a soap film with large deflections (Eq. 34). The maximum deflection is 100 units,


Fig. 13. Soap film with large deflections.

$$
\frac{\partial}{\partial x}\left(\mu u w_{x}\right)+\frac{\partial}{\partial y}\left(\mu w_{y}\right)=0, \text { where } \mu=\frac{1}{\left(1+w_{x}^{2}+w_{y}^{2}\right)^{1 / 2}}
$$

Recorded: deflection $w$. Time to correct small deflection solution (shown by broken lines): 3 hrs .
compared with the largest dimension of 160 units for the $x y$ projection of the boundary. We note the deflection as given by Laplace's equation and show it for comparison by dotted contours. In solving this problem, the Laplace equation solution was taken as the first approximation. Correction terms had to be computed only twice. This solution required 3 hours.
F. A more general type of equation which arises in the flow of oil in a bearing has been solved by Christopherson [13]. He solves the system of equations

$$
\begin{gather*}
\frac{\partial}{\partial \xi}\left[\frac{H^{3}}{M} \frac{\partial P}{\partial \xi}\right]+\frac{\partial}{\partial \eta}\left[\frac{H^{3}}{M} \frac{\partial P}{\partial \eta}\right]=\frac{\partial H}{\partial \xi},  \tag{39}\\
\frac{\partial T}{\partial \xi}\left[1-\frac{H^{2}}{M} \frac{\partial P}{\partial \xi}\right]+\frac{\partial T}{\partial \eta}\left[1-\frac{H^{2}}{M} \frac{\partial P}{\partial \eta}\right]=A\left[\frac{M}{H^{2}}+3 \frac{\partial P}{\partial \xi}\right], \tag{40}
\end{gather*}
$$

where $H$ is a given function of $\xi, \eta$ and $M$ is a given function of $T, P ; P$ is the pressure in the lubricant, $T$ is the temperature of the lubricant.
$G$. Another very general system of non linear, integral, differential, difference equations arises in the problem of the thermal equilibrium of a nest of $N$ coaxial, conducting cylinders of length $2 b$, between which a hot gas flows. The temperature $T$ of the cylinders varies with the number $n$ of the cylinder and with the axial position $z$ of the point under consideration. The temperature of the gas varies with the number $n$ of the stream, and the position $x$. The equations to be solved are

$$
\begin{align*}
& f_{1}(x, n) T^{4}(x, n)+a \int_{-b}^{b}\left\{T^{4}(\gamma, n+1) f_{2}(x, n, \gamma)+T^{4}(\gamma, n-1) f_{3}(x, n, \gamma)\right\} d \gamma \\
&+b\left\{T_{s}(x, n)+T_{s}(x, n-1)-2 T(x, n)\right\}+c \frac{\partial^{2} T(x, n)}{\partial x^{2}}=0  \tag{41}\\
& \frac{\partial T_{s}(x, n)}{\partial x}+\left\{T(x, n)+\frac{n+1}{n} T(x, n+1)-\frac{2 n+1}{n} T_{s}(x, n)\right\} d=0 \tag{42}
\end{align*}
$$

where $a, b, c, d$ are constants
$f_{1}, f_{2}, f_{3}$ are known functions of the variables indicated. The solution of these equations, which required about 30 hours, is shown in Fig. 14. The temperature contours show only the distribution of cylinder temperatures. This analysis was actually used as the basis for redesign of the instrument involved.


Fig. 14. Thermal equilibrium of a nest of cylinders in a hot gas stream including radiation, conduction and convection. Gas temperature: $1700^{\circ} \mathrm{F}$ abs. Surrounding duct temperature: $1500^{\circ} \mathrm{F}$ abs. Section of nest of cylinders showing cylinder temperature distribution.
4. General remarks on finite difference approximations. In this paper only the simplest approximations for rectangular nets of points have been mentioned. There is no reason why higher order approximations cannot be used, or why triangular or other point arrangements should not be considered. Indeed both of these possibilities have been used $[9,10,13]$. There is no sure way of deciding at the present time just
what net should be used with a given problem. The ultimate requirement is a sufficiently accurate answer in the shortest possible time. The author's experience to date favors the simplest possible formula, the simplest possible relaxation pattern, with accuracy obtained by extra points on a finer net where needed. Southwell and Christopherson $[9,10,13]$ have occasionally found other formulas to be advantageous. Until a person is thoroughly acquainted with the details of the several processes of solution, the "bookkeeping," it is impossible to judge the relative merits in the matter of time required to reach a given accuracy.

This is particularly true in the matter of boundary conditions. Every new computer "discovers" the possibility of deriving special formulas to apply to points near (or on) the boundary of the domain, especially when the boundary itself wanders among the net points. In many cases such formulas are admirable from the point of view of accuracy for a given net size, but fail miserably when compared with linear interpolation (or extrapolation) together with a somewhat finer net (used locally). By far the best general procedure for fitting "queer" boundaries or boundary conditions is to sketch a graph for the last few points approaching the boundary and thus compute the boundary point graphically.
5. Conclusions. This paper stresses particularly the practical aspects. The finite difference approximations to partial differential equations are well known; only the detailed steps in carrying out the solutions are not yet general knowledge, and these are discussed step by step. The relaxation method, first conceived by R. V. Southwell, is the underlying process in the solution of most of the problems discussed. Once the basic idea is grasped by actually solving a problem, it is capable of enormous extension to a great many kinds of problems with only the most meagre knowledge of the current methods (if any) for solving them. From an engineering point of view this is of enormous importance, because the practical man would like to do something better than guess, and yet he cannot afford the time required to become versed in analytical procedures, procedures which too often cannot supply a numerical answer to the real physical problem with reasonable accuracy and speed.

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## STUDIES IN OPTICS

## I. General Coordinates for Optical Systems with Central or Axial Symmetry*

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In previous papers ${ }^{1,2}$ the author has proposed an approach to geometrical optics different from that developed by Hamilton and his successors.

The purpose of the present paper is to generalize the formulas in these papers, and to find the most general treatment of systems with central (point) symmetry and with axial symmetry. By leaving the coordinates general, subject only to the symmetry conditions of the problem, we retain the symmetry in the formulas up to the point where we desire to draw conclusions for a special problem. We can then introduce special coordinates adapted to the problem in question, and find the particular answers.

The fundamental invariants of geometrical optics show no preference for either object or image side, nor for point or angle coordinates as variables. The different approaches suggested by Hamilton, as well as the direct approach just mentioned, are special cases of the methods developed here corresponding to special choices of coordinates. Several different choices of coordinates will be given as examples.

The fundamental formulas ( $\mathrm{A}, \mathrm{B}, \mathrm{B}^{\prime}, \mathrm{C}$ below) are based only on symmetry conditions and on the validity of the Lagrange invariant (A). They are therefore not restricted to optical problems, ${ }^{3}$ but are also valid for problems in mechanics, hydrodynamics, and electron optics.

1. Ray tracing formulas, the Lagrangian invariant. Let us assume a ray traversing a number of optical media with refractive indexes $n, n_{12}, n_{23}, \cdots, n^{\prime}$. Let $\mathrm{a}(x, y, z)$, $\mathbf{a}^{\prime}\left(x^{\prime}, y^{\prime}, z^{\prime}\right)$ be a vector from an arbitrary origin to a point on the object and image rays, respectively. Let $a_{k}\left(x_{k}, y_{k}, z_{k}\right)$ be the vector from the same origin to the intersection point of the ray with the $k$ th surface. Let $\mathrm{s}_{k, k+1}\left(\xi_{k, k+1}, \eta_{k, k+1}, \zeta_{k, k+1}\right)$ be a vector along the ray in the medium between $k$ th and $(k+1)$ th surface, a vector of length equal to the refractive index $n_{k, k+1}$ of the medium.

Let $o_{k}$ be a vector perpendicular to the $k$ th surface at the intersection point. Its length may remain arbitrary, for the moment. The refraction law then reads:

$$
\begin{equation*}
\mathbf{s}_{k, k+1} \times \mathbf{o}_{k}=\mathbf{s}_{k-1, k} \times \mathbf{o}_{k}, \tag{1}
\end{equation*}
$$

where the multiplication sign indicates vector multiplication. Equation (1) shows that $\mathbf{s}_{k, k+1}-\mathbf{s}_{k-1, k}$ has the direction of the surface normal $\mathbf{o}_{k}$, or

[^4]\[

$$
\begin{equation*}
\mathbf{s}_{k, k+1}-\mathbf{s}_{k-1, k}=\phi_{k} \mathbf{o}_{k} . \tag{2}
\end{equation*}
$$

\]

We now can describe the path of the ray through the system by means of the vector equations

$$
\begin{align*}
& a_{1}=a+\lambda s, \quad s_{12}=s+\phi_{1} \mathbf{o}_{1}, \quad a_{2}=a_{1}+\lambda_{12} s_{12}, \\
& \cdots \cdots  \tag{3}\\
& s^{\prime}=s_{v-1, \nu}+\phi_{v} o_{\nu}, \quad a^{\prime}=a_{\nu}+\lambda^{\prime} s^{\prime} .
\end{align*}
$$

The geometrical significance of $\lambda$ and $\phi$ can be seen by multiplying (3) scalarly by $\mathbf{s}_{k, k+1}$ and $\mathbf{o}_{k}$, respectively, keeping in mind that, by definition,

$$
\begin{equation*}
s_{k, k+1}^{2}=n_{k, k+1}^{2} . \tag{3a}
\end{equation*}
$$

We then find that

$$
\begin{align*}
\lambda_{k-1, k} & =\left(\mathbf{a}_{k}-\mathbf{a}_{k-1}\right) \cdot \mathbf{s}_{k-1, k} / n_{k-1, k}  \tag{3b}\\
\phi_{k} & =\left(\mathbf{s}_{k, k+1}-\mathbf{s}_{k-1, k}\right) \cdot o_{k} / o_{k}^{2}, \tag{3c}
\end{align*}
$$

i.e., the $\lambda$ 's are proportional to the distance between the two surfaces along the ray, and the $\phi$ 's are proportional to what might be called the power of the surface for the individual ray.

Since Eqs. (3) are valid for every ray, we now consider a two-dimensional manifold of rays, i.e., we assume the a's, o's, and s's to be vector functions of two variables $t_{1}$ and $t_{2}$. From the definition of $\mathbf{s}_{k, k+1}$ and $\mathbf{o}_{k}$, we find

$$
\begin{equation*}
\mathbf{s}_{k, k+1} \cdot\left(\partial \mathbf{s}_{k, k+1} / \partial t_{\mu}\right)=0, \quad \mathbf{o}_{k} \cdot\left(\partial \mathbf{a}_{k} / \partial t_{\mu}\right)=0, \quad(\mu=1,2) \tag{4}
\end{equation*}
$$

We now differentiate (3) with respect to $t_{1}$ and multiply scalarly by $\partial s_{k, k+1} / \partial t_{2}$ and $\partial \mathrm{a}_{k} / \partial t_{2}$, respectively. Then we differentiate with respect to $t_{2}$ and multiply scalarly by $\partial s_{k, k+1} / \partial t_{1}$ and $\partial \mathrm{a}_{k} / \partial t_{1}$, respectively. Subtraction of the two sets of equations yields the "Lagrangian invariant":

$$
\left|\begin{array}{ll}
\frac{\partial \mathrm{a}}{\partial t_{1}} & \frac{\partial \mathrm{~s}}{\partial t_{1}}  \tag{A}\\
\frac{\partial \mathrm{a}}{\partial t_{2}} & \frac{\partial \mathrm{~s}}{\partial t_{2}}
\end{array}\right|=\left|\begin{array}{ll}
\frac{\partial \mathrm{a}_{1}}{\partial t_{1}} & \frac{\partial \mathrm{~s}_{1_{, 2}}}{\partial t_{1}} \\
\frac{\partial \mathrm{a}_{1}}{\partial t_{2}} & \frac{\partial \mathrm{~s}_{1,2}}{\partial t_{2}}
\end{array}\right|=\left|\begin{array}{ll}
\frac{\partial \mathrm{a}^{\prime}}{\partial t_{1}} & \frac{\partial \mathrm{~s}^{\prime}}{\partial t_{1}} \\
\frac{\partial \mathrm{a}^{\prime}}{\partial t_{2}} & \frac{\partial \mathrm{~s}^{\prime}}{\partial t_{2}}
\end{array}\right| .
$$

This formula was introduced by Lagrange in his astronomical investigations. It is known by the name of the Lagrangian bracket in the theory of partial differential equations. Herzberger ${ }^{3}$ used it in his theory of transversal curves, as the starting point. Let us now see what conclusions can be drawn if the system in question fulfills certain conditions of symmetry.
2. Centrally symmetric systems. In this case all refracting surfaces are concentric spheres with radii $r_{1}, \cdots, r_{n}$. It is therefore appropriate to consider the common center as the coordinate origin, and to choose concentric spheres as the object surface a and the image surface $a^{\prime}$. All the surface normals pass through the common center. We shall give them the length $r_{k}$ from center to surface, with a positive sign if the surface is convex towards the incident light. In other words, we make $\mathbf{o}_{k}=\mathbf{a}_{k}$. Under these conditions, Eqs. (3) become

$$
\begin{array}{ll}
\mathbf{a}_{1}=\mathbf{a}+\lambda \mathbf{s}, & s_{12}=s+\phi_{1} \mathbf{a}_{1},  \tag{5}\\
\cdots & \cdots \\
\mathbf{s}^{\prime}=s_{n-1, n}+\phi_{n} a_{n}, & \mathbf{a}^{\prime}=a_{n}+\lambda^{\prime} s^{\prime}
\end{array}
$$

From Eqs. (3) we can find an invariant vector, namely,

$$
\begin{equation*}
a \times s=a_{1} \times s=a_{1} \times s_{12}=\cdots=a^{\prime} \times s^{\prime}=p \tag{6}
\end{equation*}
$$

Therefore, in a concentric system, both object and image rays lie in a plane through the center, and the optical length $p$ of the perpendicular from the center (the length of the invariant vector) remains constant.

Equations (4) can be combined into

$$
\begin{equation*}
\mathrm{a}^{\prime}=a \mathrm{a}+b \mathrm{~s}, \quad \mathrm{~s}^{\prime}=c \mathrm{a}+d \mathrm{~s}, \quad \text { where } \quad a d-b c=1 \tag{B}
\end{equation*}
$$

The invariant relation, $a d-b c=1$, is found by substituting (5) in (6). It is possible to calculate $a, b, c$, and $d$ as functions of the $\lambda$ 's and $\phi$ 's, if we use "Gaussian brackets." ${ }^{4}$ We find

$$
\begin{array}{rlrl}
a & =\left[\phi_{1}, \lambda_{12}, \cdots, \lambda^{\prime}\right], & b & =\left[\lambda, \phi_{1}, \cdots, \lambda^{\prime}\right] \\
c & =\left[\phi_{1}, \lambda_{12}, \cdots, \phi_{n}\right], & d=\left[\lambda, \phi_{1}, \cdots, \phi_{n}\right] . \tag{7}
\end{array}
$$

In the case of central symmetry, the $\phi$ 's and $\lambda$ 's, and therefore $a, b, c$, and $d$, can be considered as functions of a single variable, and $p$ can be taken as this variable, as shown in (9) and (10). Now

$$
\begin{equation*}
\left(a_{k} \cdot s_{k, k+1}\right)^{2}+\left(a_{k} \times s_{k, k+1}\right)^{2}=a_{k}^{2} s_{k, k+1}^{2} \tag{8}
\end{equation*}
$$

or

$$
\begin{equation*}
a_{k} \cdot \mathbf{s}_{k, k+1}=\sqrt{r_{k}^{2} n_{k, k+1}^{2}-p^{2}} \tag{8a}
\end{equation*}
$$

Thus we find from (3) that

$$
\begin{align*}
\lambda_{k, k+1}= & \frac{1}{n_{k, k+1}^{2}}\left[\sqrt{r_{k+1}^{2} n_{k, k+1}^{2}-p^{2}}-\sqrt{r_{k}^{2} n_{k, k+1}^{2}-p^{2}}\right] \\
& =\frac{1}{n_{k, k+1}}\left[r_{k+1} \sqrt{1-\left(\frac{p}{r_{k+1} n_{k, k+1}}\right)^{2}}-r_{k} \sqrt{1-\left(\frac{p}{r_{k} n_{k, k+1}}\right)^{2}}\right] \tag{9}
\end{align*}
$$

and that

$$
\begin{equation*}
\phi_{k}=\frac{1}{r_{k}^{2}}\left[\sqrt{r_{k}^{2} n_{k, k+1}^{2}-p^{2}}-\sqrt{r_{k}^{2} n_{k-1, k}^{2}-p^{2}}\right] \tag{10}
\end{equation*}
$$

Equations (B) correspond to the direct equations of our theory. A more general representation is given by choosing two arbitrary vector functions, 1 and m , in terms of which the object and image vectors can be expressed. Equations (B) may then be written,

$$
\begin{gather*}
\mathbf{a}=a l+b \mathrm{~m}, \quad \mathbf{a}^{\prime}=a^{\prime} 1+b^{\prime} \mathrm{m} \\
\mathrm{~s}=c \mathrm{l}+d \mathrm{~m}, \quad \mathbf{s}^{\prime}=c^{\prime} 1+d^{\prime} \mathrm{m} \\
a d-b c=a^{\prime} d^{\prime}-b^{\prime} c^{\prime}
\end{gather*}
$$

[^5]$a, b, c, d$, and $a^{\prime}, b^{\prime}, c^{\prime}, d^{\prime}$ are still functions of a single variable, which can be taken as
\[

$$
\begin{equation*}
\pi=(1 \times m)^{2}=1^{2} m^{2}-(1 \cdot m)^{2} \tag{11}
\end{equation*}
$$

\]

It is obvious that under these conditions the image formation described by (11) fulfills all the conditions mentioned, and that the last condition in (B) is equivalent to the validity of the invariant (6).

Formulas (B) are a special case of Eqs. ( $B^{\prime}$ ) if we choose $I=a, m=s$, which corresponds to $a=d=1, b=c=0$.

The connection between $a, b, c, d$ and $a^{\prime}, b^{\prime}, c^{\prime}, d^{\prime}$ becomes clearer if we introduce some auxiliary angles. Let us write

$$
\begin{gather*}
\Varangle(\mathrm{a}, \mathrm{~m})=\alpha, \quad \Varangle(1, \mathrm{a})=\beta, \quad \Varangle(\mathrm{s}, \mathrm{~m})=\gamma, \quad \Varangle(\mathrm{l}, \mathrm{~s})=\delta,  \tag{12}\\
\Varangle(\mathrm{l}, \mathrm{~m})=\psi, \quad \Varangle(\mathrm{a}, \mathrm{~s})=\sigma, \quad \Varangle\left(\mathrm{a}^{\prime}, \mathrm{s}^{\prime}\right)=\sigma^{\prime} .
\end{gather*}
$$

From (B), if we write $r$ for the absolute value of vector a, we find that

$$
\begin{array}{ll}
a=(r m / \pi) \sin \alpha, & b=(r l / \pi) \sin \beta  \tag{13}\\
c=(n m / \pi) \sin \gamma, & d=(n l / \pi) \sin \delta
\end{array}
$$

with analogous expressions for the primed quantities. Moreover, we find from (12) that

$$
\begin{gather*}
\alpha+\beta=\gamma+\delta=\psi=\alpha^{\prime}+\beta^{\prime}=\gamma^{\prime}+\delta^{\prime} \\
\alpha-\gamma=\delta-\beta=\sigma, \quad \alpha^{\prime}-\gamma^{\prime}=\delta^{\prime}-\beta^{\prime}=\sigma^{\prime} \tag{14}
\end{gather*}
$$

where $\sigma$ and $\sigma^{\prime}$, according to (9), are connected by

$$
\begin{equation*}
p=u r \sin \sigma=n^{\prime} r^{\prime} \sin \sigma^{\prime}, \quad \pi=l m \sin \phi \tag{15}
\end{equation*}
$$

If we write

$$
\begin{equation*}
a d-b c=a^{\prime} d^{\prime}-b^{\prime} c^{\prime}=D \tag{16}
\end{equation*}
$$

we have finally

$$
p=D \pi
$$

Thus, for a given system of coordinates $l, m$, and $\phi$, and given object and image spheres (radii $r$ and $r^{\prime}$ ), we can calculate all the functions $a, b, c, d, a^{\prime}, b^{\prime}, c^{\prime}, d^{\prime}$, if only one of them is given on each side. For instance, let us assume $\gamma$ and $\gamma^{\prime}$ to be given. We then find $\sigma$ and $\sigma^{\prime}$ from (15), and obtain

$$
\begin{array}{rlrrr}
\alpha=\sigma+\gamma, & \beta=\psi-\sigma-\gamma, & \gamma=\gamma, & \delta=\psi-\gamma, \\
\alpha^{\prime}=\sigma^{\prime}+\gamma^{\prime}, & \beta^{\prime}=\psi-\sigma^{\prime}-\gamma^{\prime}, & \gamma^{\prime}=\gamma^{\prime}, & \delta^{\prime}=\psi-\gamma^{\prime} . \tag{17}
\end{array}
$$

Thus, according to (13), we determine $a, b, c, d, a^{\prime}, b^{\prime}, c^{\prime}$, and $d^{\prime}$.
Let us now consider some special cases.
a) The direct method. We choose $1=\dot{\mathbf{a}}, \mathrm{m}=\mathrm{s}$. This means that

$$
\begin{gather*}
\psi=\sigma, \quad \beta=\gamma=0, \quad \alpha=\delta=\sigma \\
a=d=1, \quad b=c=0  \tag{18}\\
\alpha^{\prime}=\sigma^{\prime}+\gamma^{\prime}, \quad \beta^{\prime}=\sigma-\sigma^{\prime}-\gamma^{\prime}, \quad \gamma^{\prime}=\gamma^{\prime}, \quad \delta^{\prime}=\sigma-\gamma^{\prime},
\end{gather*}
$$

where $n r \sin \sigma=n^{\prime} r^{\prime} \sin \sigma^{\prime}$.
b) Hamilton's point coordinates. We choose $\mathrm{I}=\mathrm{a}, \mathrm{m}=\mathrm{a}^{\prime}$, or $a=1, b=0, a^{\prime}=0$, $b^{\prime}=1$. This means that

$$
\begin{align*}
& \alpha=\psi, \quad \beta=0, \quad \gamma=\psi-\sigma, \quad \delta=\sigma,  \tag{16'}\\
& \alpha^{\prime}=0, \quad \beta^{\prime}=\psi, \quad \gamma^{\prime}=-\sigma^{\prime}, \quad \quad \delta^{\prime}=\psi+\sigma^{\prime},
\end{align*}
$$

or, since $1=r, m=r^{\prime}$,

$$
\begin{array}{rlrlrl}
a & =1, & b & =0, & c & =\left[n r^{\prime} \sin (\psi-\sigma)\right] / \pi, \\
a^{\prime} & =0, & b^{\prime} & =1, & c^{\prime} & =p / \pi, \\
& d^{\prime} & =\left[n^{\prime} r \sin \left(\psi+\sigma^{\prime}\right)\right] / \pi,
\end{array}
$$

where

$$
n r \sin \sigma=n^{\prime} r^{\prime} \sin \sigma^{\prime}=p, \quad r r^{\prime} \sin \psi=\pi .
$$

We note especially that $c^{\prime}-d=0$.
c) Hamilton's angle characteristic. We choose $\mathbf{1}=\mathbf{s}, \mathrm{m}=\mathbf{s}^{\prime}$, or $c=1, d=0, c^{\prime}=0$, $d=1$. We find from ( $\mathrm{B}^{\prime}$ ) that

$$
\begin{equation*}
a d-b c=a^{\prime} d^{\prime}-b^{\prime} c^{\prime} \tag{19}
\end{equation*}
$$

or $-b=a^{\prime}=D=p / \pi$. For the auxiliary angles, we obtain

$$
\begin{align*}
& \alpha=\sigma+\psi, \quad \beta=-\sigma, \quad \gamma=\psi, \quad \delta=\sigma,  \tag{20}\\
& \alpha^{\prime}=\sigma^{\prime}, \quad \beta^{\prime}=\psi-\sigma^{\prime}, \quad \gamma^{\prime}=0, \quad \delta^{\prime}=\psi,
\end{align*}
$$

or

$$
\begin{equation*}
a=\left(n^{\prime} r / \pi\right) \sin (\psi+\sigma), \quad b=-a^{\prime}=-p / \pi, \quad b^{\prime}=\left(n^{2} / \pi\right) \sin \left(\psi-\sigma^{\prime}\right), \tag{21}
\end{equation*}
$$

where

$$
\pi=n n^{\prime} \sin \psi, \quad p=n r \sin \sigma=n^{\prime} r^{\prime} \sin \sigma^{\prime} .
$$

d) We take as coordinates the intersection points of the ray with two spheres in the object space: the object sphere (radius $r$ ), and a second sphere (radius $R$, and A, the vector to the intersection point on this sphere). We define
and

$$
\mathrm{l}=\mathrm{a}, \quad \mathrm{~m}=\mathrm{A},
$$

$$
\begin{array}{ll}
\mathbf{a}=\mathbf{a}, & \mathbf{a}^{\prime}=a^{\prime} \mathbf{a}+b^{\prime} \mathbf{A},  \tag{22}\\
\mathbf{s}=-c \mathbf{a}+d \mathbf{A}, & \mathbf{s}^{\prime}=c^{\prime} \mathbf{a}+d^{\prime} \mathbf{A},
\end{array}
$$

where $a^{\prime} d^{\prime}-b^{\prime} c^{\prime}=c$ and

$$
\begin{equation*}
c=[n R \sin (\psi-\sigma)] / \pi=-[n r \sin \sigma] / \pi=-p / \pi . \tag{23}
\end{equation*}
$$

We find that

$$
\sin \sigma^{\prime}=[n r \sin \sigma] / n^{\prime} r^{\prime}, \quad \sin (\psi-\sigma)=[n \sin \sigma] / n R,
$$

and, finally that

$$
\begin{array}{ll}
a^{\prime}=\left[r^{\prime} R \sin \left(\sigma^{\prime}+\gamma^{\prime}\right)\right] / \pi, & c^{\prime}=\left[n^{\prime} R \sin \gamma^{\prime}\right] / \pi  \tag{24}\\
b^{\prime}=\left[r r^{\prime} \sin \left(\psi-\sigma^{\prime}-\gamma^{\prime}\right)\right] / \pi, & d^{\prime}=\left[n^{\prime} r \sin \left(\psi-\gamma^{\prime}\right)\right] / \pi
\end{array}
$$

These are the equations for the image formation.
3. Systems with rotational symmetry. Let us now assume that the system has symmetry only around an axis, the unit vector along which we shall designate by $k$. In this case, all the surface normals intersect the axis, and we shall give to the vector $\mathbf{o}_{k}$ in (3) the length $r_{k}$, which is the distance along the normal between its intersection points with the axis and the surface.

We now project all the vectors on a plane perpendicular to the axis, and define the projected vectors $\mathrm{b}_{k}$ and $\mathrm{t}_{k, k+1}$ by the equations

$$
\begin{equation*}
\mathbf{a}_{k}=\mathbf{b}_{k}+z_{k} \mathbf{k}, \quad \mathbf{s}_{k, k+1}=\mathbf{t}_{k, k+1}+\zeta_{k, k+1} \mathbf{k}, \quad \mathbf{o}_{k}=\mathbf{b}_{k}+\left(z_{k}-z_{N k}\right) \mathbf{k} \tag{25}
\end{equation*}
$$

where $z_{N}$ is the quantity known in geometry as the subnormal, and

$$
\zeta_{k, k+1}=\sqrt{n_{k, k+1}^{2}-\xi_{k, k+1}^{2}-\eta_{k, k+1}^{2}}
$$

$\zeta_{k, k+1}$ is the (optical) cosine of the angle between the ray and the axis.
Let us now assume that object and image origins lie on two planes perpendicular to the axis. We can then replace all the vectors in (3) and (A) by their projections in these planes, and find, instead of (A), for a two-dimensional manifold of rays (parameters $t_{1}, t_{2}$ ),

$$
\left|\begin{array}{ll}
\frac{\partial \mathrm{b}}{\partial t_{1}} & \frac{\partial \mathrm{t}}{\partial t_{1}}  \tag{26}\\
\frac{\partial \mathrm{~b}}{\partial t_{2}} & \frac{\partial \mathrm{t}}{\partial t_{2}}
\end{array}\right|=\left|\begin{array}{ll}
\frac{\partial \mathrm{b}^{\prime}}{\partial t_{1}} & \frac{\partial \mathrm{t}^{\prime}}{\partial t_{1}} \\
\frac{\partial \mathrm{~b}^{\prime}}{\partial t_{2}} & \frac{\partial \mathrm{t}^{\prime}}{\partial t_{2}}
\end{array}\right|
$$

and instead of (B),

$$
\begin{equation*}
\mathbf{b}^{\prime}=a^{\prime} \mathbf{b}+b^{\prime} \mathbf{t}, \quad \mathbf{t}^{\prime}=c^{\prime} \mathbf{b}+d^{\prime} \mathbf{t} \tag{27}
\end{equation*}
$$

where $\mathrm{b}^{\prime} \times \mathrm{t}^{\prime}=\mathrm{b} \times \mathrm{t}$ and therefore $a^{\prime} d^{\prime}-b^{\prime} c^{\prime}=1$. The functions $a^{\prime}, b^{\prime}, c^{\prime}, d^{\prime}$ are given by formula (7), where $\phi$ and $\lambda$ have the same meaning as before.

Moreover, $a^{\prime}, b^{\prime}, c^{\prime}, d^{\prime}$ are no longer functions of a single variable, but are functions of the three symmetric functions and $b$ and $t$, namely,

$$
\begin{equation*}
u=\mathrm{b}^{2}, \quad v=\mathrm{b} \cdot \mathrm{t}, \quad w=\mathrm{t}^{2} \tag{28}
\end{equation*}
$$

Equation (27) corresponds to the formulas of the direct image error theory. The most general choice of coordinates might be described as follows ( $t$ and $m$, as well as the other vectors, lie in a plane perpendicular to the axis) : let

$$
\begin{array}{ll}
\mathbf{b}=a \mathbf{l}+b \mathrm{~m}, & \mathbf{b}^{\prime}=a^{\prime} 1+b^{\prime} \mathrm{m}  \tag{29}\\
\mathrm{t}=\mathrm{l}+d \mathrm{~m}, & \mathbf{t}^{\prime}=c^{\prime} 1+d^{\prime} \mathrm{m}
\end{array}
$$

where

$$
b \times t=b^{\prime} \times t^{\prime}
$$

or

$$
\begin{equation*}
a d-b c=a^{\prime} d^{\prime}-b^{\prime} c^{\prime} \tag{30}
\end{equation*}
$$

Let us assume that $a, b, c$, and $d$ are functions of the symmetric functions of 1 and $m$; that is,

$$
\begin{equation*}
u=1^{2}, \quad v=1 \cdot m, \quad w=m^{2} \tag{31}
\end{equation*}
$$

$\mathrm{b}, \mathrm{t}, \mathrm{b}^{\prime}, \mathrm{t}^{\prime}$ must fulfill Eq. (26), if we set $t_{1}$ and $t_{2}$ alternatively equal to $u, v$, and $w$. Thus, we find the following equations for $a, b, c$, and $d$. Let us write

$$
\begin{array}{ll}
A=a u+b v, & C=c u+d v \\
B=a v+b w, & D=c v+d w \tag{32}
\end{array}
$$

and introduce the abbreviation $\Delta$ for the difference of an expression in the object and image spaces. An easy computation then gives

$$
\Delta\left\{\left|\begin{array}{ll}
\frac{\partial A}{\partial t_{1}} & \frac{\partial A}{\partial t_{2}}  \tag{C}\\
\frac{\partial c}{\partial t_{1}} & \frac{\partial c}{\partial t_{2}}
\end{array}\right|+\left|\begin{array}{ll}
\frac{\partial B}{\partial t_{1}} & \frac{\partial B}{\partial t_{2}} \\
\frac{\partial d}{\partial t_{1}} & \frac{\partial d}{\partial t_{2}}
\end{array}\right|+\left|\begin{array}{ll}
\frac{\partial a}{\partial t_{1}} & \frac{\partial a}{\partial t_{2}} \\
\frac{\partial C}{\partial t_{1}} & \frac{\partial C}{\partial t_{2}}
\end{array}\right|+\left|\begin{array}{ll}
\frac{\partial b}{\partial t_{1}} & \frac{\partial b}{\partial t_{2}} \\
\frac{\partial D}{\partial t_{1}} & \frac{\partial D}{\partial t_{2}}
\end{array}\right|\right\}=0,
$$

These are the necessary and sufficient conditions that (29) describe an axially symmetric image formation. We repeat that Eqs. (29) and (C) demand only the validity of Lagrange's invariant (26), and axial symmetry. Their application is therefore not restricted to optical problems.

Let us now again investigate what forms the fundamental formulas assume for special choices of coordinates.
a) Hamilton's (Bruns') point characteristic. Hamilton (Bruns) chose as variables the coordinates of a point in the object and image spaces. That corresponds to taking $\mathrm{l}=\mathrm{b}, \mathrm{m}=\mathrm{b}^{\prime}$. Equations (29) become

$$
\begin{array}{ll}
\mathbf{b}=\mathrm{b} & \mathbf{b}^{\prime}=\mathrm{b}^{\prime}  \tag{33}\\
\mathrm{t}=\mathrm{c} \mathbf{b}+d \mathrm{~b}^{\prime}, & \mathrm{t}^{\prime}=c^{\prime} \mathbf{b}+d^{\prime} \mathbf{b}^{\prime}
\end{array}
$$

or

$$
a=b^{\prime}=1, \quad b=a^{\prime}=0
$$

These are conditions for the coefficients. Equations (32) now become

$$
\begin{array}{llll}
A=u, & C=c u+d v, & A^{\prime}=v, & C^{\prime}=c^{\prime} u+d^{\prime} v  \tag{34}\\
B=v, & D=c v+d w, & B^{\prime}=w, & D^{\prime}=c^{\prime} v+d^{\prime} w,
\end{array}
$$

and we find instead of (C) that

$$
\begin{equation*}
d=-c^{\prime}, \tag{35a}
\end{equation*}
$$

$$
\left|\begin{array}{ll}
\frac{\partial u}{\partial t_{1}} & \frac{\partial u}{\partial t_{2}}  \tag{35b}\\
\frac{\partial c}{\partial t_{1}} & \frac{\partial c}{\partial t_{2}}
\end{array}\right|+\left|\begin{array}{ll}
\frac{\partial v}{\partial t_{1}} & \frac{\partial v}{\partial t_{2}} \\
\frac{\partial d}{\partial t_{1}} & \frac{\partial d}{\partial t_{2}}
\end{array}\right|+\left|\begin{array}{ll}
\frac{\partial c^{\prime}}{\partial t_{1}} & \frac{\partial c^{\prime}}{\partial t_{2}} \\
\frac{\partial v}{\partial t_{1}} & \frac{\partial v}{\partial t_{2}}
\end{array}\right|+\left|\begin{array}{ll}
\frac{\partial d^{\prime}}{\partial t_{1}} & \frac{\partial d^{\prime}}{\partial t_{2}} \\
\frac{\partial w}{\partial t_{1}} & \frac{\partial w}{\partial t_{2}}
\end{array}\right|=0 .
$$

Equation (35b) stands for three equations, which we can obtain by replacing $t_{1}$ and $t_{2}$ in (35) by $u$ and $v, u$ and $w, v$ and $w$, respectively. This yields

$$
\begin{equation*}
\frac{\partial c}{\partial v}-\frac{\partial d}{\partial u}=-\frac{\partial c^{\prime}}{\partial u}, \quad \frac{\partial c}{\partial w}=-\frac{\partial d^{\prime}}{\partial u}, \quad \frac{\partial d}{\partial w}=\frac{\partial c^{\prime}}{\partial w}-\frac{\partial d^{\prime}}{\partial v} . \tag{35c}
\end{equation*}
$$

Equations (35a) and (C), when integrated, lead to a function $V(u, v, w)$ such that

$$
\begin{equation*}
c=2 \frac{\partial V}{\partial u}, \quad c^{\prime}=-\frac{\partial V}{\partial v}, \quad d=\frac{\partial V}{\partial v}, \quad d^{\prime}=-2 \frac{\partial V}{\partial w} \tag{36}
\end{equation*}
$$

$V$ is the characteristic function of Hamilton, the "eiconal" of Bruns. Formulas (36) agree with Hamilton's formulas, except that he used $l^{2} / 2$ and $\mathrm{m}^{2} / 2$ as variables. Our choice of variables simplifies the form of the general formulas (C).
b) The angle characteristic. Hamilton chose as coordinates the direction cosines of the object and image rays. This means that $I=t, m=t$, or

$$
\begin{array}{ll}
\mathrm{b}=a \mathrm{t}+b \mathrm{t}, & \mathrm{~b}^{\prime}=a^{\prime} \mathrm{t}+b^{\prime} \mathrm{t}^{\prime} \\
\mathrm{t}=\mathrm{t}, & \mathrm{t}^{\prime}=\mathrm{t}^{\prime}, \tag{37}
\end{array}
$$

or

$$
\begin{equation*}
c=1, \quad d=0, \quad c^{\prime}=0, \quad d^{\prime}=1 \tag{38}
\end{equation*}
$$

Equations (29) now give

$$
\begin{equation*}
C=u, \quad C^{\prime}=v, \quad D=v, \quad D^{\prime}=w \tag{39}
\end{equation*}
$$

and (C) becomes

$$
\begin{gather*}
b+a^{\prime}=0 \\
\frac{\partial b}{\partial u}-\frac{\partial a}{\partial v}=\frac{\partial a^{\prime}}{\partial u}, \quad-\frac{\partial a}{\partial w}=\frac{\partial b^{\prime}}{\partial u}, \quad-\frac{\partial b}{\partial w}=-\frac{\partial a^{\prime}}{\partial w}+\frac{\partial b^{\prime}}{\partial v} \tag{40}
\end{gather*}
$$

Equation (4) is solved if we introduce the angle characteristic $T(u, v, w)$, and set

$$
\begin{equation*}
a=\frac{1}{2} \frac{\partial T}{\partial u}, \quad a^{\prime}=-\frac{\partial T}{\partial v}, \quad b=\frac{\partial T}{\partial v}, \quad b^{\prime}=-\frac{1}{2} \frac{\partial T}{\partial w} \tag{41}
\end{equation*}
$$

We see that this also agrees with Hamilton's theory.
c) The direct method. In the papers mentioned, ${ }^{1,2}$ we took as variables the object point and the direction of the object ray, i.e., we chose $l=b$ and $m=t$. This gives

$$
\begin{array}{ll}
\mathrm{b}=\mathrm{b}, & \mathrm{~b}^{\prime}=a^{\prime} \mathrm{b}+b^{\prime} \mathrm{t}  \tag{42}\\
\mathrm{t}=\mathrm{t}, & \mathrm{t}^{\prime}=c^{\prime} \mathrm{b}+d^{\prime} \mathrm{t}
\end{array}
$$

That is, we put

$$
\begin{equation*}
a=d=1, \quad b=c=0 \tag{43}
\end{equation*}
$$

Equations (27) then give

$$
\begin{equation*}
A=u, \quad C=v, \quad B=v, \quad D=w, \tag{44}
\end{equation*}
$$

and Eqs. (C) give

$$
\left|\begin{array}{cc}
\frac{a^{\prime} d^{\prime}-b^{\prime} c^{\prime}=1}{\partial t_{1}} & \frac{\partial A^{\prime}}{\partial t_{2}} \\
\frac{\partial c^{\prime}}{\partial t_{1}} & \frac{\partial c^{\prime}}{\partial t_{2}}
\end{array}\right|+\left|\begin{array}{ll}
\frac{\partial B^{\prime}}{\partial t_{1}} & \frac{\partial B^{\prime}}{\partial t_{2}}  \tag{45}\\
\frac{\partial d^{\prime}}{\partial t_{1}} & \frac{\partial d^{\prime}}{\partial t_{2}}
\end{array}\right|+\left|\begin{array}{ll}
\frac{\partial a^{\prime}}{\partial t_{1}} & \frac{\partial a^{\prime}}{\partial t_{2}} \\
\frac{\partial C^{\prime}}{\partial t_{1}} & \frac{\partial C^{\prime}}{\partial t_{2}}
\end{array}\right|+\left|\begin{array}{cc}
\frac{\partial b^{\prime}}{\partial t_{1}} & \frac{\partial b^{\prime}}{\partial t_{2}} \\
\frac{\partial D^{\prime}}{\partial t_{1}} & \frac{\partial D^{\prime}}{\partial t_{2}}
\end{array}\right|=0 .
$$

If we denote the sum of the four determinants in (45) by $I^{\prime}$ when $t_{1}=u, t_{2}=v$, by $I I^{\prime}$ when $t_{1}=u, t_{2}=w$, and by $I I I^{\prime}$ when $t_{1}=v, t_{2}=w$, we can write (45) in the form

$$
\begin{equation*}
a^{\prime} d^{\prime}-b^{\prime} c^{\prime}=1, \quad I^{\prime}=I I^{\prime}=I I I^{\prime}=0 \tag{46}
\end{equation*}
$$

Again, if we disregard the fact that the variables $u$ and $w$ differ by a factor of two from those used previously ${ }^{1,2}$ we find Eqs. (46) to be identical with those given before.
d) Object and stop coordinates. To analyze image errors, we investigate the manner in which the image ray changes with the position of the object point and the position of the intersection with the diaphragm plane, for which we frequently substitute the entrance pupil of the system. If we choose these as the coordinates of the ray, assuming that the distance between object and entrance pupils is equal to $k$, we find that

$$
\begin{array}{ll}
\mathbf{b}=\mathbf{b}, & \mathbf{b}^{\prime}=a^{\prime} \mathbf{b}+b^{\prime} \mathbf{b}_{p} \\
\mathbf{t}=\gamma\left(\mathrm{b}-\mathrm{b}_{p}\right), & \mathbf{t}^{\prime}=c^{\prime} \mathbf{b}+d^{\prime} \mathbf{b}_{p} \tag{47}
\end{array}
$$

where

$$
\begin{equation*}
\gamma=\frac{n}{\sqrt{k^{2}+\left(\mathrm{b}-\mathrm{b}_{p}\right)^{2}}}=\frac{n}{\sqrt{k^{2}+u-2 v+w}} \tag{48}
\end{equation*}
$$

From (48) we conclude that

$$
\begin{equation*}
\gamma_{u}=-\frac{1}{2} \gamma_{v}=\gamma_{v}=\frac{1}{2} \frac{n}{\sqrt{\left(k^{2}+u-2 v+w\right)^{3}}} \tag{49}
\end{equation*}
$$

Thus Eq. (29) gives

$$
\begin{equation*}
A=u, \quad B=v, \quad C=\gamma(u-v), \quad D=\gamma(v-w) . \tag{50}
\end{equation*}
$$

The fundamental equations (C) become

$$
\begin{gather*}
a^{\prime} d^{\prime}-b^{\prime} c^{\prime}=\frac{n}{\sqrt{k^{2}+u-2 v+w}}  \tag{51}\\
I^{\prime}=-I I^{\prime}=I I I^{\prime}=\frac{1}{2} \frac{n}{\left(k^{2}+u-2 v+w\right)^{3 / 2}}
\end{gather*}
$$

# LATERAL BENDING OF SYMMETRICALLY LOADED CONICAL DISCS* 

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1. Introduction. The general theory of lateral bending for thin circular plates of variable thickness is given in Timoshenko's book, "Theory of plates and shells," where also may be found numerous references to the literature of the subject. Of particular interest is a reference to Föppl who indicated the analogy existing between the rotating disc problem and that of lateral bending in a circular plate of variable thickness. Comparison of the solution for the rotating conical disc problem with the corresponding one for lateral bending shows that the basic differential equations involved, and the expressions for the stresses, are analogous. Therefore, previously described methods ${ }^{2}$ for obtaining solutions of the former problem in terms of hypergeometric functions are applicable to the latter problem. It will appear later that the special type of hypergeometric differential equation associated with the lateral bending problem has solutions which give the stress coefficients with less labor than in the case of the rotating conical disc.

The stress coefficients have been arranged conveniently for numerical calculation of conical discs, which are component parts of a wide variety of engineering structures. The head of a large poppet valve provides a particular example where the principal stress member can be approximated by a system of incomplete conical discs. In order to illustrate an application of the theory, stress coefficients for conical discs subject to lateral bending as well as for rotating conical discs will be used to estimate stress distributions in a steel valve head of constant weight and various proportions. Since the coefficients are obtained from solutions of differential equations for thin discs, the approximate method breaks down in the neighborhood of the valve stem. These limitations have little effect near the periphery, which makes it possible to calculate valve proportions corresponding to approximately uniform stress distribution throughout the head. The description of the illustrative example at the end of the paper explains the method of calculation in detail.
2. Derivation of differential equation. Let $M_{r}$ and $M_{l}$ denote radial and tangential bending moments per unit length acting on an element of a circular plate at distance $r$ from the center; then if $Q$ is the corresponding circumferential shearing force per unit length, the equation of equilibrium is

$$
\begin{equation*}
M_{r}+r d M_{r} / d r-M_{t}=-Q r \tag{1}
\end{equation*}
$$

If $w$ denotes downward deflection of the middle surface, then

[^6]\[

$$
\begin{align*}
& M_{r}=-D\left(\frac{d^{2} w}{d r^{2}}+\frac{\sigma}{r} \frac{d w}{d r}\right)=D\left(\frac{d \varphi}{d r}+\frac{\sigma}{r} \varphi\right) \\
& M_{i}=-D\left(\frac{1}{r} \frac{d w}{d r}+\sigma \frac{d^{2} w}{d r^{2}}\right)=D\left(\frac{\varphi}{r}+\sigma \frac{d \varphi}{d r}\right) \tag{2}
\end{align*}
$$
\]

where $\sigma$ is Poisson's ratio, $\varphi=-d w / d r$ and $D=E h^{3} / 12\left(1-\sigma^{2}\right)$, $E$ being Young's modulus and $h$ the thickness of the plate; $D$ is called the flexural rigidity.

In the case of a conical profile, $h$ is a linear function of $r$, so that substitution from Eqs. (2) in Eq. (1) gives

$$
\begin{equation*}
D \frac{d}{d r}\left(\frac{d \varphi}{d r}+\frac{\varphi}{r}\right)+\frac{d D}{d r}\left(\frac{d \varphi}{d r}+\sigma \frac{\varphi}{r}\right)=-Q . \tag{3}
\end{equation*}
$$

In order to reduce this equation to non-dimensional form, we introduce the radius $R$ to the knife edge of the disc and the thickness $h_{0}$ at the center. If $r / R=x$, then $h=h_{0}(1-x)$, and Eq. (3) becomes

$$
\begin{equation*}
\frac{d^{2} \varphi}{d x^{2}}+\left(\frac{1}{x}-\frac{3}{1-x}\right) \frac{d \varphi}{d x}-\left(\frac{1}{x^{2}}+\frac{3 \sigma}{x(1-x)}\right) \varphi=-\frac{12 Q R^{2}\left(1-\sigma^{2}\right)}{E h_{0}^{3}(1-x)^{3}} \tag{4}
\end{equation*}
$$

For any particular type of symmetrical loading the shearing force $Q$ is a function of $x$ alone. The maximum radial and tangential bending stresses $S_{r}$ and $S_{t}$ are obtained from the general solution of Eq. (4) with the aid of Eqs. (2) and the relations

$$
\begin{equation*}
S_{r}=6 M_{r} / h^{2}, \quad S_{t}=6 M_{t} / h^{2} . \tag{4a}
\end{equation*}
$$

The problem of a conical disc supporting a concentrated vertical load $P$ at the center has some interesting practical applications. In this case $Q=P / 2 \pi r=P / 2 \pi R x$, and Eq. (4) becomes

$$
\begin{equation*}
x(1-x) \frac{d^{2} \varphi}{d x^{2}}+(1-4 x) \frac{d \varphi}{d x}-\left(\frac{1-x}{x}+3 \sigma\right) \varphi=-\frac{6 P R\left(1-\sigma^{2}\right)}{\pi E h_{0}^{3}(1-x)^{2}} \tag{5}
\end{equation*}
$$

It can be verified by substitution that a particular integral of Eq. (5) is

$$
\begin{equation*}
\varphi_{3}(x)=-\frac{2 P R(1+\sigma)}{\pi E h_{0}^{3}(1-3 \sigma)}\left(\frac{2-3 \sigma}{(1-x)^{2}}+\frac{1}{x}+\frac{1}{1-x}\right) . \tag{6}
\end{equation*}
$$

The auxiliary equation, the solutions of which are independent of the type of loading, is obtained by setting the right hand side of Eq. (5) equal to zero. After making the substitution $\varphi=x F$, we obtain

$$
\begin{equation*}
x(1-x) \frac{d^{2} F}{d x^{2}}+3(1-2 x) \frac{d F}{d x}-3(1+\sigma) F=0 \tag{7}
\end{equation*}
$$

which is recognized to be of hypergeometric type.
3. Complementary functions. Equation (7) is of the form

$$
x(1-x) \frac{d^{2} F}{d x^{2}}+[c-(a+b+1) x] \frac{d F}{d x}-a b F=0
$$

where $c=3, a+b=5$, and $a b=3(1+\sigma)$. The first solution can be represented by a power series; the integral exponent difference ${ }^{3} 1-c=-2$ shows that the second solution contains a logarithm.* In the notation of the hypergeometric function, the first solution is

$$
\begin{equation*}
F_{1}(x)=F(a, b, c, x)=1+\frac{a b}{1 \cdot c} x+\frac{a(a+1) b(b+1)}{1 \cdot 2 \cdot c(c+1)} x^{2}+\cdots \tag{8}
\end{equation*}
$$

which converges absolutely and uniformly when $|x|<1$. The asymptotic behavior of the hypergeometric function in the neigborhood of its poles is given by ${ }^{4}$

$$
\begin{equation*}
F(a, b, c, x) \sim \frac{\Gamma(c) \Gamma(a+b-c)}{\Gamma(a) \Gamma(b)}(1-x)^{c-a-b} \tag{9}
\end{equation*}
$$

whenever $c-a-b$ is an integer less than zero, $\Gamma(z)$ being the well known gamma function. Thus $F_{1}(x)$ has a second order singularity at $x=1$ such that,

$$
\begin{equation*}
F_{x \rightarrow 1}(x) \sim \frac{\Gamma(3) \Gamma(2)}{\Gamma(a) \Gamma(b)}(1-x)^{-2}=\frac{2(1-x)^{-2} \sin a \pi}{\pi(a-1)(a-2)(a-3)(a-4)} \tag{10}
\end{equation*}
$$

which may be used to approximate the function for values of $x$ near unity. The presence of singularities of lower order in the remainder term for $F_{1}(x)$ makes this method unsuitable for accurate numerical work. Better approximations for similar functions with second and third order singularities are given in Ref. (2).

The logarithmic solution ${ }^{5}$ of Eq. (7) is $\dagger$

$$
\begin{equation*}
F_{2}(x)=-\frac{(a b-4)(a b-6)}{2} F_{1}(x) \log _{0} x+\frac{1}{x^{2}}-\frac{a b-6}{x}-g(x) \tag{11}
\end{equation*}
$$

where

$$
\begin{equation*}
g(x)=\sum_{n=2}^{\infty} \frac{(n+a-3) \cdots(a-2)(n+b-3) \cdots(b-2)}{n!(n-2)!} x^{n-2} \Phi_{n} \tag{12}
\end{equation*}
$$

and

$$
\begin{align*}
\Phi_{n}= & \psi(n-3+a)+\psi(n-3+b)-\psi(n)-\psi(n-2) \\
= & \frac{1}{a-2}+\cdots+\frac{1}{a+n-3}+\frac{1}{b-2}+\cdots+\frac{1}{b+n-3} \\
& -1-\frac{1}{2}-\cdots-\frac{1}{n}-1-\frac{1}{2}-\cdots-\frac{1}{n-2} \tag{13}
\end{align*}
$$

The principal part of expansion (11) shows that $F_{2}(x)$ has a second order singularity at the origin. The nature of the singularity at $x=1$ can be recognized by observing

[^7]the limiting form of the $n$th term of $g(x)$ which is proportional to that of $F_{1}(x)$ provided $\lim _{n \rightarrow \infty} \Phi_{n}$ remains finite. That this is the case can be shown with the aid of the logarithmic derivative ${ }^{6}$ of the Gamma function, which gives
\[

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \Phi_{n}=2\left(\frac{1}{a-2}+\frac{1}{a-1}+\frac{1}{a}\right)-2 \gamma-2 \Psi(a)-\pi \cot a \pi \tag{14}
\end{equation*}
$$

\]

where $\Psi(a)=\Gamma^{\prime}(a+1) / \Gamma(a+1)$ and $\gamma$ is Euler's constant. Thus $F_{2}(x)$ has a second order singularity at $x=1$ of magnitude

$$
\begin{equation*}
\underset{x \rightarrow 1}{F_{2}(x) \sim-\lim _{n \rightarrow \infty} \Phi_{n} \frac{\sin a \pi}{\pi} /(1-x)^{2} . . . . ~} \tag{15}
\end{equation*}
$$

The slow convergence of the power series near the singularities of $F_{1}(x)$ and $F_{2}(x)$ makes numerical evaluation of the stress coefficients for all values of $x$ between zero and unity exceedingly difficult, in spite of available asymptotic approximations. One scheme for removing this difficulty would be to construct from the transformed differential equation (7) two new solutions of argument $1-x$ and combine them linearly with $F_{1}(x)$ and $F_{2}(x)$, as described in Ref. (2). This differential equation is invariant under transformation by $1-x$, which brings about added convenience of calculation; however, considerable further reduction in computation can be accomplished by expressing $F_{1}(x)$ and $F_{2}(x)$ in terms of symmetrical hypergeometric functions.
4. Solutions in terms of even and odd functions. Whenever $2 c=a+b+1$, which condition is satisfied by Eq. (7), the transformation $(1-2 x)^{2}=t$ reduces the standard form of the hypergeometric equation to

$$
\begin{equation*}
t(1-t) \frac{d^{2} F}{d t^{2}}+\left[\frac{1}{2}-\left(\frac{a}{2}+\frac{b}{2}+1\right) t\right] \frac{d F}{d t}-\frac{a b}{4} F=0 \tag{16}
\end{equation*}
$$

The solutions of this equation as functions of $x$ are $^{7}$
$F\left\{\frac{1}{2} a, \frac{1}{2} b, \frac{1}{2},(1-2 x)^{2}\right\} \equiv G_{1}(x), \quad(1-2 x) F\left\{\frac{1}{2}(a+1), \frac{1}{2}(b+1), \frac{3}{2},(1-2 x)^{2}\right\} \equiv G_{2}(x)$.
This shows that $G_{1}(x)=G_{1}(1-x)$ and $G_{2}(x)=-G_{2}(1-x)$. Since only functions of $x$ are involved,

$$
\begin{align*}
& G_{1}(x)=C_{1} F_{1}(x)+C_{2} F_{2}(x)  \tag{17}\\
& G_{2}(x)=D_{1} F_{1}(x)+D_{2} F_{2}(x)
\end{align*}
$$

where $C_{1}, C_{2}, D_{1}, D_{2}$ are constants; $G_{1}(x)$ and $G_{2}(x)$ are respectively even and odd relative to the point $x=\frac{1}{2}$. The series for $G_{1}(x)$ and $G_{2}(x)$ are very convenient for computation when $.25 \leqq x \leqq .50$, while those for $F_{1}(x)$ and $F_{2}(x)$ are equally so when $0 \leqq x \leqq .25$. Since the $G$ 's are symmetrical it is necessary to compute only one half as many fundamental values for constructing tables of stress coefficients as would be required with the $F$ 's. From this point on therefore, the $F$ 's are subordinated to the role of "helping functions," while the $G$ 's form the basis of all subsequent calculations.

Returning to Eq. (17), we employ the familiar method of comparison of singularities for evaluation of the linear factors. It is apparent from the character of the F's that the $G$ 's have second order singularities at zero and unity whose values may be deduced from Eq. (9). After some reduction, we obtain

[^8]\[

$$
\begin{align*}
& C_{1}=\frac{\sqrt{\pi} \Gamma(a) \Gamma(b)}{32 \Gamma\left(\frac{1}{2} a\right) \Gamma\left(\frac{1}{2} b\right)}\left[1+\lim _{n \rightarrow \infty} \Phi_{n} \frac{\sin a \pi}{\pi}\right] \\
& C_{2}=\sqrt{\pi} /\left[16 \Gamma\left(\frac{1}{2} a\right) \Gamma\left(\frac{1}{2} b\right)\right] \\
& D_{1}=-\frac{\sqrt{\pi} \Gamma(a) \Gamma(b)}{64 \Gamma\left\{\frac{1}{2}(a+1)\right\} \Gamma\left\{\frac{1}{2}(b+1)\right\}}\left[1-\lim _{n \rightarrow \infty} \Phi_{n} \frac{\sin a \pi^{-}}{\pi}\right]  \tag{18}\\
& D_{2}=\sqrt{\pi} /\left[32 \Gamma\left\{\frac{1}{2}(a+1)\right\} \Gamma\left\{\frac{1}{2}(b+1)\right\}\right] .
\end{align*}
$$
\]

The functions $G_{1}(x), G_{2}(x)$, and their derivatives are tabulated in Table 1.
Since the $F$ 's and $G^{\prime}$ s are linearly dependent, $x G_{1}(x)$ and $x G_{2}(x)$ are fundamental solutions of Eq. (5), from which, by use of Eqs. (2) and (4a), the stress coefficients follow immediately.
5. Determination of the deflection functions. The deflection $w(x)$ can be expressed in the form

$$
\begin{equation*}
w(x)=R\left[w_{1}(x)+w_{2}(x)+\bar{w}_{3}(x)\right], \tag{18a}
\end{equation*}
$$

where $w_{1}(x)$ and $w_{2}(x)$ arise from the complementary functions respectively, and $\bar{w}_{3}(x)$ arises from the particular integral. The calculation of $\bar{w}_{3}(x)$ presents no difficulty, since only elementary functions with known integrals are involved. Direct in tegration of Eq. (6) gives

$$
\begin{equation*}
\bar{w}_{3}(x)=-\int \varphi_{3}(x) d x=\frac{2 P R(1+\sigma)}{\pi E h_{0}^{3}(1-3 \sigma)}\left[\frac{2-3 \sigma}{1-x}+\log _{e} \frac{x}{1-x}\right] \tag{19}
\end{equation*}
$$

The construction of the deflection functions $w_{1}(x)$ and $w_{2}(x)$ is considerably more difficult, since it is necessary to evaluate integrals of the type $\int x G_{i}(x) d x(i=1,2)$, which involves additional infinite series. For purposes of computation, a convenient procedure, that also has the advantage of being easy to check, is to use a combination of analytical and numerical methods. A prerequisite for this calculation is a fairly extensive and accurate tabulation of the $G$ 's.

A straightforward step by step numerical integration process is seen to fail near the poles of the $G$ 's, due to the presence of ordinary singularities in the integrands. The procedure for constructing the functions $w_{1}(x)$ and $w_{2}(x)$ in tabular form consists of removing these singularities analytically and integrating the resulting functions numerically.
6. Removal of singularities from the integrands of $\int x G_{1}(x) d x$ and $\int x G_{2}(x) d x$. Let us consider a "substracting off" function $H_{1}(x)$ which has the property that $G_{1}(x)-H_{1}(x)$ is bounded uniformly, i.e., without finite jumps, throughout the interval of existence of $G_{1}(x)$. It is necessary that $H_{1}(x)$ be continuous except for poles which are of the same order as, and coincide with, those of $G_{1}(x)$. This specification is not sufficient however, since at every point of the interval the difference $G_{1}(x)-H_{1}(x)$ is finite, which requires the principal parts of $G_{1}(x)$ and $H_{1}(x)$ to be identical. The principal parts of $G_{1}(x)$ at zero and unity are readily obtainable from Eqs. (11) and (17) together with the relation $G_{1}(x)=G_{1}(1-x)$. Since $G_{2}(x)$ $=-G_{2}(1-x)$ and the $F$ 's and $G^{\prime}$ s are linearly dependent, the corresponding principal parts of $G_{2}(x)$ may be found by the same process.

The integral parts of $G_{1}(x)$ and $G_{2}(x)$ can be approximated by polynomials of low degree, which makes it possible to reduce the differences $G_{1}(x)-H_{1}(x)$ and $G_{2}(x)-H_{2}(x)$ to uniformly small values throughout the interval by an intelligent choice of the "subtracting off" functions. Incidentally this process provides a convenient check on the accuracy of the tabulated values of the $G^{\prime}$ s. After some manipulation a pair of suitable "subtracting off" functions were found to be

$$
\begin{align*}
H_{1}(x)= & C_{2}\left[-7+a b-g(0)+C_{1} / C_{2}-\frac{1}{2}(a b-4)(a b-6) \log _{\varepsilon} x(1-x)\right. \\
& \left.+1 / x^{2}+1 /(1-x)^{2}-(a b-6) / x-(a b-6) /(1-x)\right]  \tag{20}\\
H_{2}(x)= & D_{2}\left[\left(7-a b-g(0)+D_{1} / D_{2}\right)(1-2 x)-\frac{1}{2}(a b-4)(a b-6) \log \cdot\{x /(1-x)\}\right. \\
& \left.+1 / x^{2}-1 /(1-x)^{2}-(a b-6) / x+(a b-6) /(1-x)\right] \tag{21}
\end{align*}
$$

These functions have the added property that

$$
\lim _{x \rightarrow 0, x \rightarrow 1}\left\{G_{1}(x)-H_{1}(x)\right\}=\lim _{x \rightarrow 0, x \rightarrow 1}\left\{G_{2}(x)-H_{2}(x)\right\}=0
$$

Integrals of the type $\int_{x} G(x) d x$ now can be evaluated directly from the identity

$$
\begin{equation*}
-w_{i}(x)=\int x G_{i}(x) d x \equiv \int x\left[G_{i}(x)-H_{i}(x)\right] d x+\int x H_{i}(x) d x, \quad(i=1,2) . \tag{22}
\end{equation*}
$$

The second integral on the right-hand side is expressible in terms of elementary functions, while the first one behaves like a polynomial which can be computed easily with any numerical integration formula having a suitably small remainder depending on the magnitude of the differences of $x\left[G_{i}(x)-H_{i}(x)\right]$. Evaluation of the second integral of Eq. (22) with $\sigma=.3$ gives, with the constant of integration chosen so that $w_{1}\left(\frac{1}{2}\right)=w_{2}\left(\frac{1}{2}\right)=0$,

$$
\begin{align*}
& \int x H_{1}(x) d x=.060,042,74\left\{.082,589,7 x^{2}+.052,500,0 x-2.103,471\right. \\
& \left.\quad+\frac{1}{1-x}+\log _{e} x-1.047,500,0 \log _{e}(1-x)-.052,500,0 x^{2} \log _{e} x(1-x)\right\},  \tag{23}\\
& \int x H_{2}(x) d x=.040,784,50\left\{.080,308,8 x^{3}-.060,231,6 x^{2}+4.147,500 x+1.349,471,6\right. \\
& \left.\quad-\frac{1}{1-x}+\log _{\curvearrowleft} x+1.0475,000,0 \log _{\curvearrowleft}(1-x)+.052,500,0 x^{2} \log _{e} \frac{1-x}{x}\right\} . \tag{24}
\end{align*}
$$

7. Deflection and stress coefficients. It is convenient to state the actual deflection in the form

$$
\begin{equation*}
w=\frac{2 R^{2}\left(1-\sigma^{2}\right)}{E h_{0}}\left[A w_{1}+B w_{2}+\left(P / h_{0}^{2}\right) w_{3}+C\right], \tag{25}
\end{equation*}
$$

where, from Eqs. (18a) and (19),

$$
w_{3}=\frac{1}{\pi(1-3 \sigma)(1-\sigma)}\left[\frac{2-3 \sigma}{1-x}+\log _{a} \frac{x}{1-x}\right]
$$

and $w_{1}$ and $w_{2}$ are non-dimensional functions of $x$ defined by Eq. (22). $w_{1}, w_{2}, w_{3}$
are deflection coefficients, and are tabulated in Table 2. The constants $A, B$, and $C$ are seen to have the dimensions of stress.

The bending stresses can be stated in a form entirely analogous to that obtained for the rotating disc problem. With the aid of Eqs. (4a), (2), and (25), we have

$$
\begin{align*}
& S_{r}=A p_{1}+B p_{2}+\left(P / h_{0}^{2}\right) p_{3},  \tag{26}\\
& S_{\iota}=A q_{1}+B q_{2}+\left(P / h_{0}^{2}\right) q_{3},
\end{align*}
$$

where

$$
\begin{aligned}
p_{1} & =(1-x)\left[x d G_{1} / d x+(1+\sigma) G_{1}\right], \\
q_{1} & =(1-x)\left[\sigma x d G_{1} / d x+(1+\sigma) G_{1}\right] ;
\end{aligned}
$$

a similar pair of relations apply to $p_{2}$ and $q_{2}$; when $\sigma=.3, p_{3}$ and $q_{3}$ can be computed directly from the formulas

$$
\begin{align*}
& p_{3}=-4.547284 \frac{1-x}{x}\left[\frac{.63+2.27 x-.7 x^{2}}{(1-x)^{3}}-\frac{.7}{x}\right] \\
& q_{3}=-4.547284 \frac{1-x}{x}\left[\frac{2.1-2.14 x+.7 x^{2}}{(1-x)^{3}}+\frac{.7}{x}\right] \tag{27}
\end{align*}
$$

$p_{1}, p_{2}, p_{3}, q_{1}, q_{2}, q_{3}$ are the stress coefficients, and are tabulated in Tables 3 and 4.
The tables of coefficients are especially convenient for approximating a plate of variable thickness with a system of conical profiles. Calculations in this type of problem show that it is necessary that the coefficients be accurate to six significant figures in order to obtain four significant figures in the final results. Consequently the tables have been calculated accurately to five parts in two million. Their general usefulness can be extended considerably with the aid of an auxiliary table of interpolation coefficients. It was found that such a table based on Bessel's central difference formula ${ }^{8}$ for six ordinates gives interpolated values of the coefficients as accurately as the tabulated ones, except near the ends of the table where the values are seldom used. In such cases a knowledge of the singularities of the tabulated functions indicates the necessary procedure for applying an interpolation formula.

## Illustrative Example

Stress distributions in a steel valve of constant weight and various proportions were estimated by an approximate method based on thin conical disc stress coefficients tabulated for both the lateral bending and rotating cases. The valve head is represented by a system of truncated conical shells of variable thickness, whose apex angles are nearly $180^{\circ}$ as shown in Fig. 1. The angle of the seat determines the direction of the reaction which imposes two independent stress systems on the valve head. An approximation to these stresses can be made on the assumption that the membrane and bending stresses correspond to those in an equivalent system of conical discs. This assumption is admissible, since it has been demonstrated ${ }^{9}$ for conical shells of constant thickness, that the stress distribution has the same character as that in a

[^9]circular plate whenever the apex angle of the shell is between $168^{\circ}$ and $180^{\circ}$. The loads on the composite disc shown in Fig. 2 are determined by resolving the valve seat reaction (of which the axial force is a component) into two perpendicular components, one of which produces pure compression on a section normal to the middle


Fig. 1. Half section of valve.
surface at the periphery, and the other of which produces pure bending.* The peripheral forces per unit length are proportional to the resultant vertical force acting on the valve, so that the force resolution in Figs. 1 and 2 has been made in terms of the axial force $P$, which is considered as a concentrated load, such as would


FIG. 2. Half section of composite disc.
be imposed on the valve by impact against its seat. It is safe to assume a concentrated axial load since the impact forces are proportional to the total valve weight, of which approximately $50 \%$ is in the stem.

The next step in the calculation is to represent the valve head by a system of equivalent conical discs in the usual manner. The tabular solution for the bending stresses is obtained from the calculation procedure described in Ref. (2), except that

[^10]the $p$ 's and $q$ 's now refer to lateral bending coefficients and $\rho R_{n}^{2} \omega^{2}$ is replaced by $P / h_{r 0}^{2}$. The solution for the membrane stresses is unchanged with the exception that the coefficients of $p_{3}$ and $q_{3}$ are zero, which corresponds to a static stress distribution


Fig. 3.
in a rotating disc. If $\bar{S}_{r}$ and $\bar{S}_{\iota}$ refer to the corresponding membrane stresses respectively, then the appropriate boundary conditions are: at the boundary between valve stem and head, $S_{t}=\sigma S_{r}, \bar{S}_{t}=\bar{S}_{r}$; at the periphery $S_{r}=0, \bar{S}_{r}$ assigned.


Fig. 4.

The dimensions of the valve head and the results of the stress calculations are shown in Figs. 3 and 4 respectively.

Table 1.*-Fundamental solutions of hypergeometric equation, $\sigma=.30$.

| $x$ | $G_{1}(x)=G_{1}(1-x)$ | $G_{1}^{\prime}(x)=-G_{1}^{\prime}(1-x)$ | $G_{2}(x)=-G_{2}(1-x)$ | $G_{2}^{\prime}(x)=G_{2}^{\prime}(1-x)$ |
| :---: | :---: | :---: | :---: | :---: |
| . 00 | 613 | -121 | $11{ }^{\infty}$ | - $\sim^{-\infty}$ |
| . 01 | 613.258 | -121,346.7 | 416.297 | $-82,426.1$ |
| . 02 | $156.631,5$ $71.137,6$ | $-15,325.94$ $-4,587.63$ | 106.126,0 | $-10,410.62$ |
| . 04 | $40.900,5$ | -1,955.005 | 48.050, ${ }^{47.507,5}$ | $\begin{aligned} & -3,116.55 \\ & -1,328.327 \end{aligned}$ |
| 05 | 26.762,0 | -1,010.946 | 17.900,04 | -687.077 |
| . 06 | 19.005,02 | - 590.775 | 12.627,16 | -401.683 |
| . 07 | 14.282,02 | - 375.613 | 9.415,02 | -255.543 |
| . 08 | 11.187,24 | -254.002 | 7.308,76 | -172.948,9 |
| . 09 | 9.045,62 | $-180.034,7$ | 5.849,83 | -122.717,7 |
| . 10 | 7.499,77 | -132.421,0 | 4.795,46 | -90.387,7 |
| .11 | 6.345,89 | -100.354,8 | 4.007,22 | -68.619,0 |
| . 12 | 5.460,74 | -77.948,5 | 3.401,39 | -53.412,3 |
| . 14 | $4.766,18$ $4.210,70$ | $-61.804,5$ $-49.867,4$ | $2.924,89$ $2.542,72$ | $-42.459,9$ $-34.365,6$ |
| . 15 | 3.759,18 | -40.842,7 | 2.231,01 | -28.250,3 |
| . 16 | 3.386,98 | -33.887, 3 | 1.973,046 | -23.541,2 |
| . 17 | 3.076,44 | -28.435, 8 | 1.756,797 | -19.854,16 |
| . 18 | 2.814,56 | -24.098,7 | 1.573,446 | -16.924,95 |
| . 19 | 2.591,67 | -20.602,3 | 1.416,397 | -14.567,50 |
| . 20 | $2.400,38$ | -17.749,86 | $1.280,639$ | -12.648,25 |
| . 21 | $2.235,01$ | -15.397,53 | 1.162,300 | -11.069,57 |
| . 22 | $2.091,12$ | -13.438,55 | 1.058,356 | -9.758,98 |
| 24 | $1.965,196$ $1.854,433$ | $-11.792,44$ $-10.397,74$ | . 9686,410 | -8.661,87 |
|  | 1.756,562 | -9.206,96 | .811,215 | -6.950,83. |
| . 26 | 1.669,738 | -8.182,97 | . 745,149 | $\begin{aligned} & -6.950,83 \\ & -6.279,56 \end{aligned}$ |
| . 27 | 1.592,445 | -7.296,48 | . 685,309 | -5.702,90 |
| . 28 | 1.523,429 | -6.524,11 | . 630,829 | $-5.205,08$ |
| . 29 | 1.461,645 | -5.847,10 | .580,987 | -4.773,43 |
| . 30 | 1.406,220 | -5.250,21 | 535,174 | -4.397,71 |
|  | 1.356,416 |  |  |  |
| . 32 | 1.311, 608 | -4.249,27 | 453,647 | -3.782,13 |
| . 33 | 1.271,267 | -3.826,49 | .417,115 | -3.529, 80 |
| . 34 | 1.234,939 | -3.445,58 | 382,950 | -3.307,88 |
| . 35 | 1.202,236 | -3.100,60 | . 350,869 | -3.112,48 |
| . 37 | 1.172,824 | -2.786,52 | . 320,623 | $-2.940,33$ |
| . 37 | 1.146,417 | -2.499,05 | . 291,994 | -2.788,70 |
| . 38 | $1.122,767$ $1.101,661$ | $-2.234,53$ $-1.989,781$ | .264,788 | $-2.655,29$ $-2.538,15$ |
| . 40 | 1.082,915 | -1.762,067 | 213,976 | -2.435,65 |
| . 41 | 1.066,371 | -1.548,980 | .190,076,6 |  |
| . 42 | 1.051,893 | -1.348,397 | .167,007,9 | -2.269,25 |
| . 43 | 1.039,367 | -1.158,434 | 144,654,6 | -2.203,20 |
| . 44 | 1.028,695 | -.977,394 | .122,909,6 | -2.147,46 |
| . 45 | 1.019,795 | -.803,742 | .101,673,3 | -2.101,34 |
| . 46 | 1.012,600 | $-.636,068$ | .080, 852,4 | -2.064,31 |
| . 47 | 1.007,058 | $-.473,063$ | $.060,358,2$ | $-2.035,94$ |
| . 48 | $1.003,128$ $1.000,780$ | $\begin{aligned} & =.313,494 \\ & -.156,186,3 \end{aligned}$ | $.040,105,8$ | $-2.015,90$ |
| . 49 | 1.000,780 | -.156,186,3 | .020,013,2 | -2.003,96 |
| . 50 | 1.000,000 | 0 | 0 | -2.000,00 |

[^11]TAble 2.-Deflection coefficients for lateral bending of conical discs, $\sigma=.30$.

| $r / R$ | $w_{1}$ | $w_{2}$ | $w_{3}$ | $r / R$ | $w_{1}$ | $w_{2}$ | $w_{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 00 | $\infty$ | $\infty$ | - | . 50 | 0 | 0 | 10.00402 |
| . 01 | . 341490 | . 1727156 | -15.84278 | . 51 | $-.00505132$ | . 0000506834 | 10.39010 |
| . 02 | . 298578 | . 1436068 | $-12.59312$ | . 52 | -. 01021073 | . 000205606 | 10.78484 |
| . 03 | . 272917 | . 1262435 | -10.65009 | . 53 | -. 01548680 | . 000469405 | 11.18891 |
| . 04 | . 254305 | . 1136970 | -9.24108 | . 54 | -. 0208887 | . 000847188 | 11.60306 |
| . 05 | . 239546 | . 1037962 | -8.12392 | . 55 | -. 0264263 | . 001344576 | 12.02809 |
| . 06 | . 227215 | . 0955742 | -7.19072 | . 56 | -. 0321101 | . 001967757 | 12.46484 |
| . 07 | 216551 | . 0885157 | -6.38390 | . 57 | -. 0379516 | . 00272355 | 12.91425 |
| . 08 | . 207102 | . 0823131 | -5.66908 | . 58 | -. 0439632 | . 00361945 | 13.37730 |
| . 09 | . 1985728 | . 0767682 | -5.02404 | . 59 | -. 0501582 | . 00466373 | 13.85508 |
| . 10 | . 1907641 | . 0717460 | -4.43361 | . 60 | -. 0565514 | . 00586555 | 14.34880 |
| . 11 | . 1835327 | . 0671501 | -3.88695 | . 61 | -. 0631588 | . 00723498 | 14.85973 |
| . 12 | . 1767727 | . 0629098 | -3.37604 | . 62 | -. 0699978 | . 00878323 | 15.38931 |
| .13 | . 1704034 | . 0589712 | -2.89476 | . 63 | -. 0770879 | . 01052271 | 15.93909 |
| . 14 | . 1643621 | . 0552928 | -2.43835 | . 64 | -. 0844501 | . 01246720 | 16.51082 |
| . 15 | . 1585986 | . 0518418 | -2.00300 | . 65 | -. 0921083 | . 01463208 | 17.10641 |
| . 16 | . 1530724 | . 0485919 | -1.585657 | . 66 | -. 1000884 | . 01703454 | 17.72799 |
| . 17 | . 1477501 | . 0455217 | $-1.183777$ | . 67 | -. 1084197 | . 01969381 | 18.37793 |
| . 18 | . 1426040 | . 0426135 | -. 795248 | . 68 | -. 1171349 | . 0226315 | 19.05890 |
| . 19 | . 1376105 | . 0398528 | -. 418284 | . 69 | -. 1262706 | . 0258720 | 19.77389 |
| 20 | . 1327494 | . 0372275 | -. 0513587 | . 70 | -. 1358681 | . 0294430 | 20.5263 |
| . 21 | . 1280035 | . 0347271 | . 306849 | . 71 | $-.1459743$ | . 0333756 | 21.3199 |
| . 22 | . 1233577 | . 0323431 | . 657492 | . 72 | -. 1566424 | . 0377057 | 22.1591 |
| . 23 | . 1187984 | . 0300681 | 1.001586 | . 73 | -. 1679333 | . 0424741 | 23.0488 |
| . 24 | .1143140 | . 0278957 | 1.340034 | . 74 | -. 1799171 | . 0477280 | 23.9948 |
| 25 | . 1098938 | . 0258206 | 1.673648 | . 75 | $-.1926744$ | . 0535219 | 25.0038 |
| . 26 | . 1055282 | . 0238382 | 2.00316 | . 76 | -. 206299 | . 0599190 | 26.0833 |
| . 27 | . 1012084 | . 0219446 | 2.32924 | . 77 | -. 220901 | . 0669937 | 27.2424 |
| . 28 | . 0969264 | . 0201366 | 2.65250 | . 78 | -. 236610 | . 0748334 | 28.4918 |
| . 29 | . 0926748 | . 01841116 | 2.97352 | . 79 | -. 253578 | . 0835418 | 29.8439 |
| . 30 | . 0884466 | . 01676613 | 3.29283 | . 80 | -. 271990 | . 0932430 | 31.3139 |
| . 31 | . 0842354 | . 01519954 | 3.61092 | . 81 | -. 292065 | . 1040868 | 32.9200 |
| . 32 | . 0800348 | . 01370985 | 3.92829 | . 82 | -. 314074 | . 1162557 | 34.6842 |
| . 33 | . 0758391 | . 01229585 | 4.24537 | . 83 | -. 338347 | . 1299744 | 36.6339 |
| . 34 | . 0716426 | . 01095665 | 4.56262 | . 84 | -. 365296 | . 1455227 | 38.8030 |
| . 35 | . 0674398 | . 00969164 | 4.88046 | . 85 | -. 395443 | . 1632539 | 41.2345 |
| . 36 | . 0632252 | . 00850050 | 5.19930 | . 86 | -. 429454 | . 1836207 | 43.9833 |
| . 37 | . 0589937 | . 00738318 | 5.51956 | . 87 | -. 468197 | . 207213 | 47.1212 |
| . 38 | . 0547400 | . 00633986 | 5.84165 | . 88 | -. 512828 | . 234815 | 50.7436 |
| . 39 | . 0504589 | . 00537098 | 6.16596 | . 89 | -. 564915 | . 267491 | 54.9800 |
| . 40 | . 0461453 | . 00447722 | 6.49292 | . 90 | -. 626646 | . 306728 | 60.0115 |
| . 41 | . 0417938 | . 00365951 | 6.82293 | . 91 | $-.701169$ | . 354663 | 66.0987 |
| . 42 | . 0373993 | . 00291902 | 7.15642 | . 92 | -. 793190 | . 414495 | 73.6312 |
| . 43 | . 0329561 | . 00225716 | 7.49380 | . 93 | -. 910079 | . 491227 | 83.2197 |
| . 44 | . 0284588 | . 001675592 | 7.83553 | . 94 | -1.064072 | . 593171 | 95.8789 |
| . 45 | . 0239016 | . 001176243 | 8.18206 | . 95 | $-1.277112$ | . 735232 | 113.4294 |
| . 46 | . 01927865 | . 0000761311 | 8.53386 | . 96 | -1.592926 | . 947112 | 139.5018 |
| . 47 | . 01458362 | . 0000433277 | 8.89143 | . 97 | -2.11315 | 1.297850 | 182.5406 |
| . 48 | . 00981010 | . 000194922 | + 9.25528 | . 98 | -3.14154 | 1.993762 | 267.798 |
| . 49 | . 00495128 | . 000049349 | 9.62595 | . 99 | -6.19036 | 4.06209 | 521.097 |
| . 50 | 0 | 0 | 10.00402 | 1.00 | - $\infty$ | $\infty$ | $\infty$ |

Table 3.-Stress coefficients for lateral bending of conical discs, $\sigma=.30$.

| $r / R$ | $p_{1}$ | $-p_{2}$ | $p_{3}$ | $r / R$ | $p_{1}$ | - $p_{2}$ | $p_{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 00 | - | $\infty$ | $\infty$ | . 50 | . 650000 | . 500000 | -51.4753 |
| . 01 | -412.069 | 280.244 | 31209.9 | . 51 | . 676528 | . 513539 | -53.6293 |
| . 02 | $-100.8399$ | 68.8436 | 7638.76 | . 52 | . 704200 | . 528194 | -55.8788 |
| . 03 | -43.7956 | 30.1006 | 3318.31 | . 53 | . 733152 | 544032 | -58.2347 |
| . 04 | -24.0284 | 16.67837 | 1821.085 | . 54 | . 763534 | . 561125 | -60.7094 |
| . 05 | -14.96886 | 10.52962 | 1134.831 | . 55 | . 795506 | . 579561 | -63.3159 |
| . 06 | -10.09557 | 7.22454 | 765.640 | . 56 | . 829243 | . 599438 | -66.0685 |
| . 07 | -7.18545 | 5.25310 | 545.144 | . 57 | . 864938 | . 620867 | -68.9831 |
| . 08 | -5.31459 | 3.98776 | 403.367 | . 58 | . 902803 | . 643975 | -72.0770 |
| . 09 | -4.04388 | 3.13023 | 307.050 | . 59 | . 943074 | . 668906 | -75.3697 |
| . 10 | -3.14316 | 2.52420 | 238.760 | . 60 | . 986012 | . 695824 | -77.8828 |
| . 11 | -2.48255 | 2.08145 | 188.6601 | . 61 | 1.031911 | . 724915 | -82.6402 |
| . 12 | -1.984277 | 1.749150 | 150.8584 | . 62 | 1.081101 | . 756391 | -86.6694 |
| . 13 | -1.599538 | 1.494168 | 121.6581 | . 63 | 1.133955 | . 790495 | -91.0010 |
| . 14 | -1.296465 | 1.294864 | 98.6451 | . 64 | 1.190896 | . 827503 | -95.6698 |
| . 15 | $-1.053553$ | 1.136640 | 80.1902 | . 65 | 1.252404 | . 867734 | $-100.7157$ |
| . 16 | -. 855874 | 1.009365 | 65.1626 | . 66 | 1.319031 | . 911553 | -106.1840 |
| . 17 | -. 692812 | . 905838 | 52.7577 | . 67 | 1.391409 | . 959381 | -112.1271 |
| . 18 | -. 556649 | . 820830 | 42.3909 | . 68 | 1.470270 | 1.011709 | -118.6053 |
| . 19 | -. 441674 | . 750472 | 33.6293 | . 69 | 1.556461 | 1.069105 | -125.6887 |
| . 20 | -. 343586 | . 691855 | 26.1469 | 70 | 1.650971 | 1.132236 | -133.4592 |
| . 21 | -. 259098 | . 642759 | 19.69457 | . 71 | 1.754958 | 1. 201882 | -142.0124 |
| 22 | -. 1856618 | . 601468 | 14.07923 | . 72 | 1.869789 | 1.278967 | -151.4616 |
| . 23 | -. 1212805 | . 566640 | 9.14940 | . 73 | 1.997083 | 1.364585 | -161.9406 |
| . 24 | -. 0643692 | . 537215 | 4.78487 | . 74 | 2.13878 | 1.460047 | -173.6096 |
| . 25 | -. 01365723 | . 512346 | . 889247 | . 75 | 2.29719 | 1.566925 | -186.6610 |
| . 26 | . 0318840 | . 491353 | -2.61552 | . 76 | 2.47513 | 1.687126 | -201.327 |
| . 27 | . 0730951 | . 473684 | -5.79325 | . 77 | 2.67604 | 1.822974 | -217.893 |
| . 28 | .1106690 | . 458889 | -8.69658 | . 78 | 2.90412 | 1.977330 | -236.707 |
| . 29 | . 1451814 | . 446599 | -11.36928 | . 79 | 3.16461 | 2.15375 | $-258.203$ |
| . 30 | . 1771155 | . 436510 | -13.84808 | . 80 | 3.46408 | 2.35669 | -282.926 |
| . 31 | . 206879 | . 428369 | -16.16404 | . 81 | 3.81084 | 2.59179 | -311.566 |
| . 32 | . 234821 | . 421968 | -18.34365 | . 82 | 4.21558 | 2.86631 | -345.007 |
| . 33 | . 261238 | . 417132 | -20.4097 | . 83 | 4.69218 | 3.18967 | -384.402 |
| . 34 | . 286390 | . 413718 | -22.3820 | . 84 | 5.25895 | 3.57432 | -431.270 |
| . 35 | . 310503 | . 411605 | -24.2778 | . 85 | 5.94048 | 4.03696 | -487.652 |
| . 36 | . 333776 | . 410694 | -26.1125 | . 86 | 6.77038 | 4.60040 | $-556.337$ |
| . 37 | . 356387 | . 410903 | -27.8996 | . 87 | 7.79558 | 5.29653 | -641.223 |
| . 38 | . 378496 | . 412167 | -29.6516 | . 88 | 9.08323 | 6.17096 | -747.885 |
| . 39 | . 400248 | . 414431 | -31.3796 | . 89 | 10.73220 | 7.29083 | $-884.537$ |
| . 40 | . 421777 | . 417655 | -33.0941 | . 90 | 12.89286 | 8.75831 | $-1063.672$ |
| . 41 | . 443208 | . 421806 | -34.8048 | . 91 | 15.80318 | 10.73501 | -1305.066 |
| . 42 | . 464658 | . 426865 | -36.5208 | . 92 | 19.85801 | 13.48915 | -1641.541 |
| . 43 | . 486239 | . 432816 | -38.2510 | . 93 | 25.7521 | 17.49263 | -2130.86 |
| . 44 | . 508060 | . 439656 | -40.0040 | . 94 | 34.8021 | 23.6398 | -2882.53 |
| . 45 | . 530227 | . 447386 | -41.7881 | . 95 | 49.7595 | 33.7997 | -4125.41 |
| . 46 | . 552846 | . 456017 | -43.6117 | . 96 | 77.1990 | 52.4381 | -6406.56 |
| . 47 | . 576023 | . 465566 | -45.4835 | . 97 | 136.2744 | 92.5655 | -11320.03 |
| . 48 | . 599866 | . 476057 | -47.4119 | . 98 | 304.461 | 206.807 | -25315.3 |
| . 49 | . 624487 | . 487522 | -49.4061 | . 99 | 1209.305 | 821.430 | -100647.9 |
| . 50 | . 650000 | . 500000 | -51.4753 | 1.00 | $\infty$ | $\infty$ | $-\infty$ |

Table 4.-Stress coefficients for lateral bending of conical discs, $\sigma=.30$.

| $r / R$ | $q 1$ | $q_{2}$ | $-q_{3}$ | $r / R$ | $q{ }_{1}$ | $q:$ | $-g_{s}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 00 | ${ }^{\infty}$ | ${ }^{\infty}$ | ${ }^{\infty}$ | . 50 | . 650000 | -. 1500000 | 50.2020 |
| . 01 | 428.863 | 290.968 | 32477.1 | . 51 | . 649206 | -. 1629858 | 50.2128 |
| . 02 | 109.4320 | 73.9901 | 8285.68 | . 52 | . 649426 | -. 1759765 | 50.3035 |
| . 03 | 49.6545 | 33.3836 | 3758.74 | . 53 | . 650664 | -. 1890247 | 50.4748 |
| 1.04 | 28.5222 | 19.02706 | 2158.48 | . 54 | . 652935 | -. 202182 | 50.7282 |
| . 05 | 18.64509 | 12.31570 | 1410.590 | . 55 | . 656258 | -. 215504 | 51.0654 |
| . 06 | 13.22822 | 8.63391 | 1000.466 | . 56 | . 660662 | -. 229044 | 51.4891 |
| . 07 | 9.93124 | 6.39200 | 750.873 | . 57 | . 666186 | -. 242863 | 52.0026 |
| . 08 | 7.77158 | 4.92257 | 587.402 | . 58 | . 672875 | -. 257023 | 52.6097 |
| . 09 | 6.27752 | 3.90518 | 474.331 | . 59 | . 680785 | -. 271589 | 53.3154 |
| . 10 | 5.19936 | 3.17022 | 392.751 | . 60 | . 689984 | $-.286635$ | 54.1253 |
| . 11 | 4.39477 | 2.62101 | 331.883 | . 61 | . 700553 | -. 302237 | 55.0462 |
| . 12 | 3.77767 | 2.19909 | 285.210 | . 62 | . 712583 | -. 318481 | 56.0859 |
| . 13 | 3.29352 | 1.867385 | 248.602 | . 63 | . 726185 | -. 335463 | 57.2535 |
| . 14 | 2.90636 | 1.601472 | 219.336 | . 64 | .741486 | -. 353287 | 58.5598 |
| . 15 | 2.59166 | 1.384697 | 195.5559 | . 65 | . 758634 | -. 372072 | 60.0169 |
| . 16 | 2.33225 | 1.205387 | 175.9608 | . 66 | . 777800 | -. 391950 | 61.6393 |
| . 17 | 2.11579 | 1.055157 | 159.6167 | . 67 | . 799185 | -. 413074 | 63.4435 |
| . 18 | 1.933233 | . 927856 | 145.8386 | . 68 | . 823021 | -. 435615 | 65.4490 |
| . 19 | 1.777816 | . 818885 | 134.1146 | . 69 | . 849583 | -. 459771 | 67.6783 |
| . 20 | 1.644399 | . 724748 | 124.0556 | . 70 | . 879189 | -. 485773 | 70.1581 |
| . 21 | 1.529018 | . 642750 | 115.3617 | . 71 | . 912216 | -. 513887 | 72.9195 |
| . 22 | 1.428577 | . 570780 | 107.7985 | . 72 | . 949106 | -. 544425 | 75.9993 |
| . 23 | 1.340629 | . 507172 | 101.1808 | . 73 | . 990389 | -. 577756 | 79.4413 |
| . 24 | 1.263215 | . 450588 | 95.3605 | . 74 | 1.036693 | -. 614316 | 83.2977 |
| . 25 | 1. 194757 | . 399950 | 90.2181 | . 75 | 1.088774 | -. 654629 | 87.6312 |
| . 26 | 1.133967 | . 354377 | 85.6563 | . 76 | 1.147548 | -. 699322 | 92.5176 |
| . 27 | 1.079790 | . 313145 | 81.5951 | . 77 | 1.214126 | -. 749161 | 98.0492 |
| . 28 | 1.031351 | . 275652 | 77.9685 | . 78 | 1.289876 | -. 805081 | 104.3394 |
| . 29 | . 987924 | . 241396 | 74.7215 | . 79 | 1.376492 | -. 868241 | 111.5287 |
| . 30 | . 948897 | . 209953 | 71.8079 | 80 | 1.476092 | -. 940082 | 119.7926 |
| . 31 | . 913757 | . 1809649 | 69.1890 | . 81 | 1.591352 | -1.022432 | 129.3533 |
| . 32 | . 882070 | . 1541266 | 66.8319 | . 82 | 1.725701 | -1.117624 | 140.4952 |
| . 33 | . 853463 | . 1291751 | 64.7086 | . 83 | 1.883578 | -1.228709 | 153.5868 |
| . 34 | . 827622 | . 1058843 | 62.7951 | . 84 | 2.07083 | -1.359573 | 169.1134 |
| . 35 | . 804274 | . 0840574 | 61.0712 | . 85 | 2.29527 | -1.515621 | 187.7236 |
| . 36 | . 783186 | . 0635226 | 59.5192 | . 86 | 2.56756 | -1.704062 | 210.302 |
| . 37 | . 764157 | . 0441290 | 58.1239 | . 87 | 2.90251 | $-1.934972$ | 238.080 |
| . 38 | . 747014 | . 0257434 | 56.8723 | . 88 | 3.32128 | $-2.22272$ | 272.814 |
| . 39 | . 731606 | . 00824728 | 55.7532 | . 89 | 3.85488 | -2.58838 | 317.080 |
| . 40 | 717805 | -. 00846535 | 54.7569 | . 90 | 4.55034 | $-3.06387$ | 374.786 |
| . 41 | . 705497 | -. 0244898 | 53.8749 | . 91 | 5.48179 | -3.69960 | 452.095 |
| . 42 | . 694587 | -. 0399126 | 53.1001 | . 92 | 6.77183 | -4.57882 | 559.197 |
| . 43 | . 684992 | -. 0548125 | 52.4264 | . 93 | 8.63538 | $-5.84753$ | 713.963 |
| . 44 | . 676641 | -. 0692620 | 51.8486 | . 94 | 11.47831 | -7.78139 | 950.147 |
| . 45 | . 669475 | -. 0833284 | 51.3624 | . 95 | 16.14551 | -10.95435 | 1338.035 |
| . 46 | . 663446 | -. 0970743 | 50.9640 | . 96 | 24.6485 | -16.73272 | 2045.00 |
| . 47 | . 658511 | -. 1105590 | 50.6507 | . 97 | 42.8244 | -29.0814 | 3556.83 |
| . 48 | . 654640 | -. 1238390 | 50.4203 | . 98 | 94.1889 | -63.9738 | 7831.14 |
| . 49 | . 651808 | -. 1369684 | 50.2710 | . 99 | 368.372 | -250.217 | 30658.4 |
| . 50 | . 650000 | -. 1500000 | 50.2020 | 1.00 | $\infty$ | $-\infty$ | $\infty$ |

# KRON'S METHOD OF SUBSPACES* 

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Introduction. Gabriel Kron has introduced new and powerful methods of applying tensor analysis to complicated engineering problems, presenting his major contributions in the field of clectrical engineering. The manner of presentation, and the rarity of a simultaneous knowledge of the hitherto almost unrelated subjects of electrical engineering and tensor analysis, have unfortunately served to limit his audience. Recent experimental confirmation of some of his investigations dealing with equivalent circuits, however, has attracted the serious attention of a wider engineering following.

In view of the growing importance of the whole subject, and of the controversy which has surrounded it, it has seemed desirable to present some particular aspect of Kron's work in a form which may appeal to a less highly specialized audience. To avoid complications as far as possible, the present paper must ignore such important topics as electrical networks, electrical machines, and equivalent circuits. It confines itself to purely dynamical problems, and to that particular idea of Kron's which may be called the method of subspaces. $\dagger$

Much discussion has arisen over Kron's claim that he uses tensor analysis. It is the considered opinion of the present writer that Kron does indeed make a full and proper use of tensor analysis. Possibly the belief that Kron employs only matrices may have arisen from the fact that, in order to present his actual mathematical procedure in a form that may be understood and used by those not familiar with the intricacies of the tensor calculus, he often presents this procedure in matrix form. However, he is always careful to point out that the underlying concepts are wholly tensorial in character. In the present paper the method of subspaces will first be presented in terms of a simple example, and in purely matrix form, merely as a set of rules of procedure, the essentially tensorial significance of the procedure being discussed only after the actual procedure has been brought before the reader. The theoretical discussion will then be followed by three simple, related examples illustrative of various aspects of the method.

Scope of the method of subspaces. Let us consider a system, which may be dynamical, electrodynamical, or otherwise, containing several standard parts, such as a fly-wheel, a governor, a pair of synchronous machines, a system of levers, etc. The equations of performance of the individual parts are usually well known, but the equations of performance of the complex whole will depend on the manner in which they are interconnected. Usually it is extremely difficult to trace out the full influence of each interconnection in setting up the equations of performance. The method

[^12]of subspaces suggested by Kron yields the equations of performance by a routine and quite straightforward manipulation of the known equations of performance of the constituent parts of the system, a brief inspection sufficing to yield all needed information as to the manner of interconnection.

Simple dynamical example. To illustrate the actual mathematical procedure in its simplest form, we shall first consider a quite trivial dynamical problem. Naturally it will not reveal the power and economy of the method any more than it would the peculiar virtues of, say, the Hamilton-Jacobi equation, were that applied to it. But it will serve to bring the routine mathematical procedure before us without unnecessary distraction from complexities which are merely incidental.

Let us consider the dynamical system $S$ consisting of three particles free to move on a line, the masses being $m_{1}, m_{2}, m_{3}$, the coordinates ${ }^{1} x^{1}, x^{2}, x^{3}$, and the forces $f_{1}, f_{2}, f_{3}$. The equations of motion are

$$
\begin{equation*}
f_{1}=m_{1} \ddot{x}^{1}, \quad f_{2}=m_{2} \dot{x}^{2}, \quad f_{3}=m_{3} \dot{x}^{3} . \tag{1}
\end{equation*}
$$

Let us consider now the new dynamical system $\bar{S}$ which arises when the particles 2 and 3 are made to coalesce. Its equations of motion may be written down at once:

$$
\begin{equation*}
f_{1}=m_{1} \bar{x}^{1}, \quad f_{2}+f_{3}=\left(m_{2}+m_{3}\right) \dot{x}^{2} . \tag{2}
\end{equation*}
$$

Let us suppose, though, that it had been a highly complex system of interconnected simple parts. We would then welcome a routine method of obtaining (2) from (1) which required no detailed thought and avoided constant preoccupation with the effects of the interconnections. The method of subspaces would be applied to the present problem in the following routine manner:

The first step is to write equations (1) of system $S$ in matrix form,

$$
\begin{equation*}
F=M \ddot{X} \tag{3}
\end{equation*}
$$

i.e.,

$$
\left[\begin{array}{ccc}
m_{1} & 0 & 0  \tag{4}\\
0 & m_{2} & 0 \\
0 & 0 & m_{3}
\end{array}\right]\left[\begin{array}{l}
\vec{x}^{1} \\
\ddot{x}^{2} \\
\ddot{x}^{3}
\end{array}\right]=\left[\begin{array}{l}
f_{1} \\
f_{2} \\
f_{3}
\end{array}\right],
$$

where, it will be noted, the masses form a square array rather than a single row or column such as one might at first expect.

Next, a relationship is set up between the coordinates $x^{1}, x^{2}, x^{3}$ of $S$ and the coordinates $\bar{x}^{1}, \bar{x}^{2}$ of $\bar{S}$. This relationship can be taken to be

$$
\begin{equation*}
x^{1}=\bar{x}^{1}, \quad x^{2}=\bar{x}^{2}, \quad x^{3}=\bar{x}^{2}\left(\operatorname{not} \bar{x}^{3}\right) . \tag{5}
\end{equation*}
$$

From this is obtained the matrix $C$ defined by

$$
\begin{equation*}
X=C \bar{X} \tag{6}
\end{equation*}
$$

It is

$$
C=\left[\begin{array}{ll}
1 & 0  \tag{7}\\
0 & 1 \\
0 & 1
\end{array}\right]
$$

[^13]If we denote the transposed matrix by $C^{\text {tr }}$, then the new forces $\bar{f}_{1}, \bar{f}_{2}$ are given by the routine matrix multiplication

$$
\bar{F}=C^{t r} F=\left[\begin{array}{lll}
1 & 0 & 0  \tag{8}\\
0 & 1 & 1
\end{array}\right]\left[\begin{array}{l}
f_{1} \\
f_{2} \\
f_{3}
\end{array}\right]=\left[\begin{array}{c}
f_{1} \\
f_{2}+f_{3}
\end{array}\right]
$$

The new masses are given by routine matrix multiplications as follows:

$$
\begin{align*}
\bar{M}=C^{t r} M C & =\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 1
\end{array}\right]\left[\begin{array}{ccc}
m_{1} & 0 & 0 \\
0 & m_{2} & 0 \\
0 & 0 & m_{3}
\end{array}\right]\left[\begin{array}{ll}
1 & 0 \\
0 & 1 \\
0 & 1
\end{array}\right] \\
& =\left[\begin{array}{ccc}
m_{1} & 0 & 0 \\
0 & m_{2} & m_{3}
\end{array}\right]\left[\begin{array}{ll}
1 & 0 \\
0 & 1 \\
0 & 1
\end{array}\right]=\left[\begin{array}{cc}
m_{1} & 0 \\
0 & m_{2}+m_{3}
\end{array}\right] \tag{9}
\end{align*}
$$

Finally the new equations of motion are

$$
\begin{equation*}
\bar{F}=\bar{M} \ddot{\bar{X}} \tag{10}
\end{equation*}
$$

i.e.,

$$
\left[\begin{array}{cc}
m_{1} & 0 \\
0 & m_{2}+m_{3}
\end{array}\right]\left[\begin{array}{l}
\ddot{x}^{1} \\
\ddot{\tilde{x}}^{2}
\end{array}\right]=\left[\begin{array}{c}
f_{1} \\
f_{2}+f_{3}
\end{array}\right] .
$$

This yields the two equations

$$
m_{1} \ddot{\vec{x}}^{1}=f_{1}, \quad\left(m_{2}+m_{3}\right) \ddot{\bar{x}}^{2}=f_{2}+f_{3}
$$

which are equivalent to (2) above.
The tensor form of the problem. Using Latin indices for the range $1,2,3$, and Greek for the range 1,2 , we may write the various expressions and equations above in the familiar index notation of the tensor calculus.

The equations of motion of $S$ may be written ${ }^{2}$

$$
f_{a}=m_{a b} \dot{x}^{b}
$$

the matrix $C$ may be written

$$
\begin{equation*}
C_{\alpha}^{a}=\frac{\partial x^{\alpha}}{\partial \bar{x}^{\alpha}} \tag{7}
\end{equation*}
$$

and the relations between the forces, etc., in $S$ and $\bar{S}$ may be put in the form

$$
\bar{f}_{\alpha}=\frac{\partial x^{a}}{\partial \bar{x}^{\alpha}} f_{a,}, \quad\left(8^{\prime}\right) ; \quad \bar{m}_{a \beta}=\frac{\partial x^{\alpha}}{\partial \bar{x}^{\alpha}} \frac{\partial x^{b}}{\partial \bar{x}^{\beta}} m_{a b}, \quad\left(9^{\prime}\right) ; \quad x^{a}=\frac{\partial x^{a}}{\partial \bar{x}^{\alpha}} \bar{x}^{\alpha} .
$$

Also the equations of motion of $\bar{S}$ are

$$
\bar{f}_{\alpha}=\bar{m}_{\alpha \beta} \ddot{x^{\beta}}
$$

Provided we interpret the time derivatives as absolute derivatives, or alterna-

[^14]tively, as is permissible in the present simple case, provided we avoid non-linear coordinate transformations, the above have the form of tensor equations and tensor transformations. The mere fact that the work can be expressed in the tensor notation does not, in itself, imply tensorial character. The essential criterion of tensor character is the tensor law of transformation. In view of $\left(7^{\prime}\right)$, equations $\left(8^{\prime}\right),\left(9^{\prime}\right)$, and ( $6^{\prime}$ ) are tensor transformations, and equation ( $10^{\prime}$ ) is the result of transforming the tensor equation $\left(1^{\prime}\right)$. The fact that $C$ is singular does not destroy the tensor character of the transformations; its significance will appear shortly.

The tensor theory. Since tensor equations have objective significance we may look for a geometrical picture of the process described above. Naturally this will be sought in configuration space. The basic tensorial and geometrical significance of Lagrangean dynamics being quite familiar, the corresponding significance of Kron's method may be explained here quite briefly.

As is well known, in Lagrangean dynamics the motion of the dynamical system $S$ is represented by the motion of a point in a three dimensional configuration space, $K$, having $x^{d}$ as coordinates and $m_{a b}$ as metrical tensor. When particles 2 and 3 coalesce, the system $S$ loses one degree of frecdom and becomes the system $\bar{S}$. Thus the trajectory of $\bar{S}$ belongs to a two dimensional configuration space, $\bar{K}$, which is in fact a subspace of $K$, for it is defined by a relation (in more general cases, by a set of relations) between the coordinates of $K$. Specifically, the subspace here is defined by the relation

$$
x^{2}=x^{3}
$$

In parametric form this subspace is given by the relations (5) above, which represent the three coordinates $x^{a}$ as functions of the two variables $\hat{x}^{\alpha}$. (Compare with the relations $x=\cos \theta \cos \varphi, y=\cos \theta \sin \varphi, z=\sin \theta$ which express the Cartesian coordinates $x, y, z$ as functions of the two parameters $\theta, \varphi$. This defines the two dimensional subspace of ordinary three dimensional space constituting the surface of a unit sphere.)

By the well known theory of subspaces, the projections of the covariant tensors $f_{a}, m_{a b}$, are given by the singular transformations $\left(8^{\prime}\right),\left(9^{\prime}\right)$. Thus the equations of motion of $\bar{S}$ are the projection on $\bar{K}$ of the equations of motion of $S$. And, since the initial conditions of $S$ and $\bar{S}$ coincide in $\bar{K}$, the trajectory of $\bar{S}$ is the projection on $\bar{K}$ of the trajectory of $S$.

There are two ways of viewing the relationship between the systems $S$ and $\bar{S}$ : ( $\alpha$ ). We may regard $S$ as the same physical system as $\bar{S}$, the forces between particles 2 and 3 which keep them together being included explicitly in the force vector $f_{a}$. The trajectory of $S$ in $K$ is then identical with that of $\bar{S}$ in $\bar{K}$.
$(\beta)$. We may regard $S$ as a different physical system from $\bar{S}$ inasmuch as particles 2 and 3 are not united in $S$. The forces in $\bar{S}$ are the same as those in $S$ except for those forces in $S$ which tend to separate particles 2 and 3. These latter forces have components in $K$ which are normal to the subspace $\bar{K}$, and thus have zero projection on $\bar{K}$.
Both viewpoints are of significance, and more will be said about them later.
A formal proof. A formal proof will now be given that the transition to a subspace and the tensor transformations that go with it, are justified in the general case. This proof will thus also justify the general procedure given by Kron, of which the above example was a particular illustration.

The proof will be made brief by basing it on certain standard results in dynamics and tensor analysis.

Let us consider a rigorous proof of the validity of the Lagrangean equations, such as is given, for instance, in Whittaker's Analytical Dynamics, third edition, starting on page 34 . We are concerned here with the case in which $t$ does not enter explicitly, and we shall use the tensor notation. The dynamical system under consideration in Whittaker has $n$ degrees of freedom, and $n$ generalized coordinates. Let us denote the latter by $\bar{x}^{\alpha}$, using the Greek indices $\alpha, \beta, \gamma$ for the range 1 to $n$. We denote the number of individual particles in the system by $N / 3$, so that their combined coordinates number $N$ and we denote these $N$ coordinates by $\tilde{x}^{\lambda}$, using $\lambda, \mu, \nu$ for the range 1 to $N$. In this notation, the proof of the Lagrangian equations has the following outline:

The equations of motion of the $N / 3$ individual particles, in a self-explanatory notation, have the form

$$
\begin{equation*}
\tilde{m}_{\lambda_{\mu}} \ddot{\tilde{x}}^{4}=\tilde{f}_{\lambda} . \tag{11}
\end{equation*}
$$

These $N$ equations are not independent, since the $N$ coordinates $\tilde{x}^{\lambda}$ are related, as are the $N$ forces $\tilde{f}_{\lambda}$. The relations between the coordinates $\tilde{x}^{\lambda}$ are defined by equations

$$
\begin{equation*}
\tilde{x}^{\tilde{x}}=\tilde{x}^{x}\left(\tilde{x}^{\alpha}\right) \tag{12}
\end{equation*}
$$

which express them in terms of the $n$ generalized coordinates of the system. (These correspond to Whittaker's equations $x_{i}=f_{i}\left(q_{1}, q_{2}, \cdots, q_{n}, t\right)$, etc., with $t$ omitted.) From (12) may be computed the quantities $\partial \tilde{x}^{\lambda} / \partial \bar{x}^{\alpha}$. The equations of motion (11) of the individual particles are multiplied individually by such quantities and then added in groups, the process being precisely that described by the equation

$$
\begin{equation*}
\frac{\partial \tilde{x}^{\lambda}}{\partial \tilde{x}^{\alpha}} \tilde{m}_{\lambda_{\lambda}} \ddot{x}^{\alpha}=\frac{\partial \tilde{x}^{\lambda}}{\partial \bar{x}^{\alpha}} \tilde{f}_{\lambda} \tag{13}
\end{equation*}
$$

where the summation convention is employed, as usual. After some manipulation, the left-hand side is then reduced to the standard Lagrangean form, and the righthand side is interpreted as a set of generalized forces in the familiar manner.

Later, on page 39, Whittaker gives an explicit form of the Lagrangean equations for the case in which $t$ does not enter explicitly. This reveals that the left-hand side has the form of a covariant derivative with respect to $\bar{m}_{\alpha \beta}$ as metrical tensor. The equations, in fact, may be written

$$
\begin{equation*}
\bar{m}_{\alpha \beta} \dot{\bar{x}}^{\theta}, \gamma \dot{x}^{r}=\bar{f}_{\alpha} . \tag{14}
\end{equation*}
$$

Since, in the original form (11), the coordinates were Cartesian for each individual particle, the ordinary derivatives there coincided with the covariant derivatives; thus (11) may be written as

$$
\begin{equation*}
\tilde{m}_{\lambda \mu} \dot{x}^{\mu}, \dot{x}^{\prime}=\tilde{f_{\lambda}}, \tag{15}
\end{equation*}
$$

the subscript preceded by a comma here denoting the covariant derivative with respect to $\tilde{m}_{\lambda_{\mu}}$ as metrical tensor.

It will be observed that the initial step, represented by the equation (13), is a transition to a subspace, complete with a singular transformation matrix $\partial \tilde{x}^{\lambda} / \partial \bar{x}^{\alpha}$
of the type which, for some reason, excites the critics of Kron when it is used by him. The relations between $\tilde{m}_{\lambda \mu}$ and $\bar{m}_{\alpha \beta}$, and between $\tilde{f}_{\lambda}$ and $\bar{f}_{\alpha}$, follow from the analysis and are the usual tensor transformations, with singular transformation matrix.

Now Lagrange goes from an initial configuration space of $N$ dimensions to a final subspace of $n$ dimensions in one step. Kron's idea is, in theory, simply to make this transition in more than one step, using subsidiary subspaces as resting places when the mathematics tends to become too complicated. For example, he would, in theory, go from the first space having coordinates $\tilde{x}^{\lambda}$ to an intermediary subspace having coordinates $x^{a}$ and then to the final subspace having coordinates $\bar{x}^{\alpha}$ which is also a subspace of the intermediary subspace. In practice, of course, as is also the case in the Lagrangean method, the initial space having coordinates $\tilde{x}^{\lambda}$ is entirely neglected, having served its sole purpose in providing the theoretical basis for the equations and procedures actually used.

Since ${ }^{3}$

$$
\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\alpha}}=\frac{\partial \tilde{x}^{\lambda}}{\partial \tilde{x}^{\alpha}}
$$

the transition in several steps will yield the same result as the transition in one step, the various tensors involved being transformed according to the standard tensor law.

Thus the proof of Kron's theory and procedure is a direct corollary of the proof of Lagrange's equations and the tensor theory of subspaces.

The proof given in Whittaker is based on viewpoint $(\alpha)$, since the forces $\tilde{f}_{\lambda}$ between the individual particles are regarded as the forces actually existing between them in the ultimate system. The trajectory in the $N$ dimensional space is actually confined to the $n$ dimensional subspace.

It is important to note, though, that the proof is equally valid for viewpoint ( $\beta$ ). For those forces which do no work do not contribute to the values of the generalized forces of the ultimate system. Since they do not affect the ultimate system, it is clear that, for the purpose of setting up the equations of that system, they may be omitted from the $N$ basic equations of the individual particles. When these forces are ignored, however, the forces between the individual particles are very much changed, especially in the case of inelastic bodies. The system of individual particles is then no longer physically equivalent to the ultimate system. It has a quite different motion (for instance, the particles of a rigid body here move in divergent directions) and its trajectory, in general, spans the whole $N$ dimensional space. Nevertheless, according to the above reasoning, the projection of its trajectory on the $n$ dimensional subspace coincides with the trajectory of the ultimate system.

Non-linear transformations. Since Kron has made the widest application of his method of subspaces to electrical networks and other electrodynamical problems in which the interconnection transformation is very of ten linear, the impression has sometimes arisen that the method is applicable only to situations in which this linearity is present. The following examples of the method involve non-linear transformations. They are simple enough so that the more usual methods of solution are

[^15]hardly more complicated than those by Kron's method, but this is inevitable in simple problems. The power of Kron's method begins to be felt when the systems in question involve a larger number of interconnections, and of more complicated mechanisms than the simple rods of the examples below.


Fig. 1. System I.

Illustrative example I. We begin with a system consisting of two rods hinged together without friction, one rod being suspended by its free end from a fixed point $O$ (Fig. 1). We denote the masses of the rods by $m_{1}, m_{2}$, their lengths by $2 a_{1}, 2 a_{2}$, and their moments of inertia about their centers of gravity, which we shall assume to coincide with their midpoints, by $I_{1}, I_{2}$. In addition to gravity, let us consider a force $F$, not necessarily conservative, which acts horizontally and to the right at the mid-point of the lower rod. The system has two degrees of freedom, and we may take as the generalized coordinates the angles $\theta, \varphi$ which the rods make with the vertical.

The problem is to set up the equations of motion. From previous experience with Lagrangean dynamics, we may regard a single rod as a known system, in the sense that we can instantly write down its equations of motion, or have already tabulated them for quick reference. The present system consists of two of these known systems interconnected. We therefore begin by considering the system consisting of the two rods not interconnected, the forces being the same as those acting externally on the original system. ${ }^{4}$ Kron calls this system the primitive system. It is shown in Fig. 2, and has here four degrees of freedom. We may take the four generalized coordinates to be the angles $\theta, \varphi$ above together with the coordinates $y, z$ of the center of gravity of the lower rod. We let Latin indices refer to the primitive system, and Greek to the actual system under discussion. For the primitive system the


Fig. 2. Primitive system of system I. metrical tensor and the force vector can be written down at once. They are

[^16]|  | $\theta$ | $\varphi$ | $y$ | $z$ |
| :---: | :---: | :---: | :---: | :---: |
| $\theta$ | $I_{1}+m_{1} a_{1}^{2}$ | 0 | 0 | 0 |
| $\varphi$ | 0 | $I_{2}$ | 0 | 0 |
| $y$ | 0 | 0 | $m_{2}$ | 0 |
| $z$ | 0 | 0 | 0 | $m_{2}$ |

and

$$
f_{a}=\begin{array}{c|c|}
\hline \theta & -m_{1} g a_{1} \sin \theta  \tag{17}\\
\hline \varphi & 0 \\
\hline y & F \\
\hline z & -m_{2} g \\
\hline
\end{array}
$$

The restraint arising from the interconnection of the two rods imposes the following two conditions on the four coordinates $x^{a}$ of the primitive system:

$$
\begin{equation*}
y=2 a_{1} \sin \theta+a_{2} \sin \varphi, \quad z=-2 a_{1} \cos \theta-a_{2} \cos \varphi \tag{18}
\end{equation*}
$$

These two equations define the subspace of the configuration space of the primitive system to which the given system is confined. We may express them in the form of a transformation, that is to say, in parametric form, by writing

$$
\begin{equation*}
\theta=\bar{\theta}, \quad \varphi=\bar{\varphi}, \quad y=2 a_{1} \sin \bar{\theta}+a_{2} \sin \bar{\varphi}, \quad z=-2 a_{1} \cos \bar{\theta}-a_{2} \cos \bar{\varphi} \tag{19}
\end{equation*}
$$

which is of the form

$$
\begin{equation*}
x^{a}=x^{a}\left(\bar{x}^{\alpha}\right) . \tag{20}
\end{equation*}
$$

The transformation matrix $C_{\alpha}^{a}$, or $\partial x^{a} / \partial \bar{x}^{a}$, is (in the form $C^{t r}$ )

$\frac{\partial x^{a}}{\partial \bar{x}^{\alpha}}=$| $\bar{x}^{\alpha}$ | $\theta$ | $\varphi$ | $y$ | $z$ |
| :---: | :---: | :---: | :---: | :---: |
| $\bar{\theta}$ | 1 | 0 | $\frac{2 a_{1} \cos \bar{\theta}}{\bar{\varphi}}$ | 0 |
|  | 1 | $a_{2} \cos \bar{\varphi}$ | $\frac{2 a_{1} \sin \bar{\theta}}{a_{2} \sin \bar{\varphi}}$ |  |.

Thus the metfical tensor for the given system, namely the projection $\bar{m}_{a \beta}$ of $m_{a b}$, is given by

$$
\bar{m}_{a \beta}=\frac{\partial x^{a}}{\partial \bar{x}^{a}} \frac{\partial x^{b}}{\partial \bar{x}^{\beta}} m_{a b}
$$

i.e.,
$\bar{m}_{a \beta}=\left[\begin{array}{cccc}1 & 0 & 2 a_{1} \cos \bar{\theta} & 2 a_{1} \sin \bar{\theta} \\ 0 & 1 & a_{2} \cos \bar{\varphi} & a_{2} \sin \bar{\varphi}\end{array}\right]\left[\begin{array}{cccc}I_{1}+m_{1} a_{1}^{2} & 0 & 0 & 0 \\ 0 & I_{2} & 0 & 0 \\ 0 & 0 & m_{2} & 0 \\ 0 & 0 & 0 & m_{2}\end{array}\right]\left[\begin{array}{cc}1 & 0 \\ 0 & 1 \\ 2 a_{1} \cos \bar{\theta} & a_{2} \cos \bar{\varphi} \\ 2 a_{1} \sin \bar{\theta} & a_{2} \sin \bar{\varphi}\end{array}\right]$

$$
\begin{align*}
& =\left[\begin{array}{cccc}
I_{1}+m_{1} a_{1}^{2} & 0 & 2 a_{1} m_{2} \cos \bar{\theta} & 2 a_{1} m_{2} \sin \bar{\theta} \\
0 & I_{2} & a_{2} m_{2} \cos \bar{\varphi} & a_{2} m_{2} \sin \bar{\varphi}
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & 1 \\
2 a_{1} \cos \bar{\theta} & a_{2} \cos \bar{\varphi} \\
2 a_{1} \sin \bar{\theta} & a_{2} \sin \bar{\varphi}
\end{array}\right] \\
& =\left[\begin{array}{cc}
I_{1}+m_{1} a_{1}^{2}+4 m_{2} a_{1}^{2} & 2 m_{2} a_{1} a_{2} \cos (\bar{\theta}-\bar{\varphi}) \\
2 m_{2} a_{1} a_{2} \cos (\bar{\theta}-\bar{\varphi}) & I_{2}+m_{2} a_{2}^{2}
\end{array}\right] . \tag{22}
\end{align*}
$$

Likewise, the force vector is given by

$$
\bar{f}_{a}=\frac{\partial x^{a}}{\partial \bar{x}^{\alpha}} f_{a}
$$

or

$$
\begin{align*}
\bar{f}_{\alpha} & =\left[\begin{array}{cccc}
1 & 0 & 2 a_{1} \cos \bar{\theta} & 2 a_{1} \sin \bar{\theta} \\
0 & 1 & a_{2} \cos \varphi & a_{2} \sin \bar{\varphi}
\end{array}\right]\left[\begin{array}{c}
-m_{1} g a_{1} \sin \bar{\theta} \\
0 \\
F \\
-m_{2} g
\end{array}\right] \\
& =\left[\begin{array}{c}
-\left(m_{1}+2 m_{2}\right) g a_{1} \sin \bar{\theta}+2 a_{1} F \cos \bar{\theta} \\
a_{2} F \cos \bar{\varphi}-a_{2} m_{2} g \sin \bar{\varphi}
\end{array}\right] \tag{23}
\end{align*}
$$

The kinetic energy function of the given system is

$$
\begin{align*}
T & =\frac{1}{2} \bar{m}_{\alpha \beta} \dot{\bar{x}} \dot{\bar{x}}^{\beta} \\
& =\frac{1}{2}\left(I_{1}+m_{1} a_{1}^{2}+4 m_{2} a_{1}^{2}\right) \dot{\theta}^{2}+2 m_{2} a_{1} a_{2} \cos (\bar{\theta}-\bar{\varphi}) \dot{\bar{\theta}} \dot{\bar{\varphi}}+\frac{1}{2}\left(I_{2}+m_{2} a_{2}^{2}\right) \dot{\varphi}^{2} . \tag{24}
\end{align*}
$$

The Lagrangean equations for the given system may now be written in the usual form. ${ }^{5}$ They are, on dropping the bars over $\theta$ and $\varphi$, but without simplification,

$$
\begin{array}{r}
\frac{d}{d t}\left\{\left(I_{1}+m_{1} a_{1}^{2}+4 m_{2} a_{1}^{2}\right) \dot{\theta}+2 m_{2} a_{1} a_{2} \cos (\theta-\varphi) \dot{\varphi}\right\}-\left\{-2 m_{2} a_{1} a_{2} \sin (\theta-\varphi) \dot{\theta} \dot{\varphi}\right\} \\
\\
=-\left(m_{1}+2 m_{2}\right) g a_{1} \sin \theta+2 a_{1} F \cos \theta \\
\begin{aligned}
\frac{d}{d t}\left\{2 m_{2} a_{1} a_{2} \cos (\theta-\varphi) \dot{\theta}+\left(I_{2}+m_{2} a_{2}^{2}\right) \dot{\varphi}\right\}-\left\{2 m_{2} a_{1} a_{2} \sin (\theta-\varphi) \dot{\theta} \dot{\varphi}\right\}
\end{aligned} \\
=a_{2} F \cos \varphi-a_{2} m_{2} g \sin \varphi
\end{array}
$$

Since no forces were introduced at the points $A, A^{\prime}$ of the primitive system, the latter was physically different from the given system, for in the given system opposite forces acted at the hinge $A$. Thus we have been using viewpoint ( $\beta$ ). To make the two systems physically equivalent, it would be necessary to impose appropriate initial conditions on the primitive system and to introduce the proper opposite forces at $A$ and $A^{\prime}$ corresponding to the reactions at the hinge in the given system. This,

[^17]however, would entail a knowledge of the reactions at $A$, and the advantage of the $(\beta)$ viewpoint, which makes it the more appropriate viewpoint for the Kron method here, is that it enables one to proceed without bringing in the reactions at all, except indirectly insofar as they imply the equations of constraint. It is possible to use viewpoint ( $\alpha$ ) by introducing unknown opposite reactions at $A$ and $A^{\prime}$ in the primitive, denoting them by some symbol, say $R$ and $-R$; they will automatically cancel when the transition is made to the subspace.

To give some indication of the flexibility of the Kron method and its ability to extract cumulative dividends from such calculations as may previously have been performed, we conclude with a brief and sketchy discussion of two further systems.

Illustrative example II. Let us consider the system illustrated in Fig. 3. It is the same as system I above except that the end of the lower rod is constrained to move without friction on a fixed vertical line distant $2 c$ from 0 .

System I has already been investigated. It is now a known system. Instead, therefore, of taking the primitive


Fig. 3. System II. of system II to be the same as the primitive of system I, we may take it to be system I itself.

The new constraint reduces the number of degrees of freedom to one and may be represented mathematically by the condition

$$
\begin{equation*}
a_{1} \sin \theta+a_{2} \sin \varphi=c . \tag{25}
\end{equation*}
$$

The subspace now can be written parametrically as

$$
\begin{equation*}
\theta=\bar{\theta}, \quad \varphi=\sin ^{-1}\left\{\frac{c-a_{1} \sin \bar{\theta}}{a_{2}}\right\} \tag{26}
\end{equation*}
$$

The transformation matrix $C$ is given by

$$
C=\left[\begin{array}{l}
1  \tag{27}\\
\lambda
\end{array}\right], \quad \lambda=\frac{\partial \varphi}{\partial \bar{\theta}}=\frac{-a_{1} \cos \bar{\theta}}{\sqrt{a_{2}^{2}-\left(c-a_{1} \sin \bar{\theta}\right)^{2}}} .
$$

By implicit differentiation of (25) we may also obtain the useful relation

$$
\begin{equation*}
a_{1} \cos \theta+a_{2} \lambda \cos \varphi=0 \tag{28}
\end{equation*}
$$

The new metrical tensor $\bar{m}_{\text {or }}$ (which here has only one component) may be obtained by the usual transformation formula, or the new expression for $T$ may be obtained directly from (24) by substituting for $\varphi$ and $\dot{\varphi}$ in terms of $\bar{\theta}$ and $\bar{\theta}$ by means of (26),
the latter method being the simpler in this particular problem. The generalized force is

$$
\begin{align*}
\bar{f}_{\sigma} & =\left[\begin{array}{ll}
1 & \lambda
\end{array}\right]\left[\begin{array}{c}
-\left(m_{1}+2 m m_{2}\right) g a_{1} \sin \bar{\theta}+2 a F \cos \bar{\theta} \\
a_{2} F \cos \varphi-a_{2} m_{2} g \sin \varphi
\end{array}\right] \\
& =-\left(m_{1}+2 m_{2}\right) g a_{1} \sin \bar{\theta}+2 a_{1} F \cos \bar{\theta}+\lambda a_{2} F \cos \varphi-\lambda a_{2} m_{2} g \sin \varphi \\
& =-\left(m_{1}+2 m_{2}\right) g a_{1} \sin \bar{\theta}+a_{1} F \cos \bar{\theta}+\frac{a_{1} m_{2} g\left(c-a_{1} \sin \bar{\theta}\right) \cos \bar{\theta}}{\sqrt{a-\left(c-a_{1} \sin \bar{\theta}\right)^{2}}} \tag{29}
\end{align*}
$$

the terms in $F$ partially cancelling in view of (28). The equation of motion may now be written down in the usual manner. Solving it is another matter!

Illustrative example III. The preceding example made use of system I in the role of a known system, and essentially dealt with the imposition of a constraint on that system. One may, however, join several known systems together by the Kron method, and this is, in fact, the procedure of principal importance in practical problems. To illustrate the idea, let us outline the method of attack on the system shown in Fig. 4,


Fig. 4. System III.
the points $O, D$ being fixed, and motion being confined to a vertical plane. This system may be regarded as system I interconnected with another system of the same type. Thus the primitive may be taken to consist of two systems of the type I, as shown in Fig. 5. For each system of type I the metrical tensor and force vector are known, being of the form (22) and (23). For brevity, we denote them in shape only by the following symbols:

$$
\left[\begin{array}{c:c}
(1) & \\
\hdashline & {\left[1^{\prime}\right)} \\
\hdashline
\end{array}\right] ; \quad\left[\begin{array}{l:l}
(2) & \cdots
\end{array}\right], \quad\left[\begin{array}{l}
\left(2^{\prime}\right) \\
\hdashline
\end{array}\right] .
$$

Then for the whole primitive system the corresponding quantities are

which, of course, may be written down at once. The configuration space of the present primitive system is the direct product of the configuration spaces of the two systems of type I.


Fig. 5. Primitive system of system III.

The interconnection of the two systems at $B$ introduces a single constraint, and by the method of subspaces the equations of motion of system III may be obtained in a routine manner.

The problems discussed above involved only very simple interconnections. When the interconnections are numerous and complicated, and the elements interconnected are themselves known complex dynamical, electrodynamical, or hydrodynamical systems, Kron's method of subspaces assumes the highest practical importance. In concluding the author wishes to thank Mr. Kron for many stimulating discussions of his work extending over several years.

Added April 12, 1944. While the general mathematical theory underlying Kron's method of subspaces as applied to dynamical systems is implied in the "Formal Proof" of the present paper, it is not there given in explicit detail. Professor Synge of the Ohio State University has suggested that a more explicit proof be included which goes directly to the mathematical basis of the method, and has kindly communicated the following outline of a method of proof from a different point of view which will be of interest to mathematicians wishing to see clearly what is fundamentally involved mathematically.

Let $(a, b)(e, f)(i, j)$ take three different ranges of values, with the usual summa-
tion convention for each. Let there be two independent holonomic dynamical systems:
I. Coordinates: $x^{n}$,
K. Energy: $T_{(1)}=\frac{1}{2} m_{a b} \hat{x}^{a} \hat{x}^{b}$,

Generalized forces: $X_{a}$.
II. Coordinates: $x$,
K. Energy: $T_{(2)}=\frac{1}{2} m_{\text {ef }} \hat{x}^{-} \hat{x}^{\prime}$,

Generalized forces: $X_{6}$.
Let us clefine, with $D=d / d t$,

$$
\begin{align*}
& S_{a}=D\left(\partial T_{(1)} / \partial \dot{x}^{a}\right)-\partial T_{(1)} / \partial x^{a} \\
& S_{a}=D\left(\partial T_{(2)} / \partial \dot{x}^{c}\right)-\partial T_{(2)} / \partial x^{c} \tag{A}
\end{align*}
$$

Then the equations of motion of I and II are $S_{a}=X_{a}, S_{a}=X_{e}$.
Now establish constraints between I and II, the reactions of constraint being workless. These constraints may be written

$$
x^{a}=x^{a}\left(x^{i}\right), \quad x^{e}=x^{e}\left(x^{i}\right),
$$

where $x^{i}$ are the generalized coordinates of the system III resulting from the combination. Write

$$
C_{i}^{a}=\partial x^{a} / \partial x^{i}, \quad C_{i}^{e}=\partial x^{e} / \partial x^{i} .
$$

We have then

$$
\dot{x}^{a}=C_{1}^{a} \dot{x}^{i}, \quad \dot{x}^{e}=C_{1}^{e} \dot{x}^{i} .
$$

It is easy to prove that

$$
\begin{align*}
D C_{i}^{a} & =\partial \dot{x}^{a} / \partial x^{i}, & D C_{i}^{e} & =\partial \dot{x}^{e} / \partial x^{i}  \tag{B}\\
C_{i}^{a} & =\partial \dot{x}^{a} / \partial \dot{x}^{i}, & C_{i}^{e} & =\partial \dot{x}^{e} / \partial \dot{x}^{i}
\end{align*}
$$

Let $X_{a}^{\prime}, X_{e}^{\prime}$ be the reactions due to the constraint. We have

$$
X_{a}^{\prime} \delta x^{a}+X_{e}^{\prime} \delta x^{e}=0
$$

for any displacement satisfying the constraints, i.e., for

$$
\delta x^{a}=C_{1}^{a} \delta x^{i}, \quad \delta x^{e}=C_{t}^{e} \delta x^{i} .
$$

Hence

$$
X_{a}^{\prime} C_{i}^{a}+X_{e}^{\prime} C_{i}^{e}=0
$$

Now the equations of motion of I and II under the constraint are

$$
S_{a}=X_{a}+X_{a}^{\prime} \quad S_{a}=X_{0}+X_{e}^{\prime}
$$

and hence

$$
\begin{equation*}
S_{a} C_{i}^{a}+S_{e} C_{i}^{e}=X_{a} C_{i}^{a}+X_{e} C_{i}^{e} \tag{C}
\end{equation*}
$$

We have for system III :
Coordinates: $x^{i}$,
K. Energy: $T_{(3)}=\frac{1}{2} m_{i j} \dot{x}^{i} \hat{x}^{i}$, Generalized forces: $X_{i}$.
Let us define

$$
S_{i}=D\left(\partial T_{(3)} / \partial \dot{x}^{i}\right)-\partial T_{(3)} / \partial x^{i}
$$

Our problem is to show how $S_{i}, X_{i}$ are to be computed in terms of the elements of I and II. (We know from dynamical theory that $S_{i}=X_{i}$, but we can forget this knowledge, as we prove it incidentally below.)

We have

$$
T_{(3)}=T_{(1)}+T_{(2)}
$$

and hence

$$
\begin{equation*}
m_{i j}=m_{a b} C_{i}^{a} C_{j}^{b}+m_{o j} C_{i}^{e} C_{j}^{f} \tag{D}
\end{equation*}
$$

It is easily seen by direct transformation that

$$
\begin{equation*}
S_{i}=S_{a} C_{t}^{a}+S_{c} C_{i}^{e} \tag{E}
\end{equation*}
$$

By considerations of work we have

$$
X_{i} \delta x^{i}=X_{a} \delta x^{a}+X_{\Delta} \delta x^{e}
$$

and so

$$
\begin{equation*}
X_{i}=X_{a} C_{i}^{a}+X_{0} C_{i}^{e} \tag{F}
\end{equation*}
$$

Hence by equation (C), we have, as equations of motion of III, $S_{i}=X_{i}$, where $S_{i}$ and $X_{i}$ are given by (E) and (F). The transformation of the metric is given by (D).

We may sum up essentially by saying: Metric and force transform by (D) and (F) when two systems are linked by workless constraints. The extension to the linkage of any number of systems is immediate.

# RIGOROUS SOLUTIONS FOR THE SPANWISE LIFT DISTRIBUTION OF A CERTAIN CLASS OF AIRFOILS* 

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1. Introduction. The problem of the spanwise lift distribution of an airfoil in a uniform stream of air leads, as was shown by Prandtl, ${ }^{1}$ to the following integrodifferential equation for the circulation $\Gamma(y)$ :

$$
\begin{equation*}
\frac{1}{4 \pi V} \int_{-b / 2}^{+b / 2} \frac{d y^{\prime}}{y-y^{\prime}} \frac{d \Gamma}{d y^{\prime}}+\frac{2 \Gamma}{m c(y) V}=\alpha \tag{1}
\end{equation*}
$$

In this equation $y$ is the span coordinate, and $-\frac{1}{2} b \leqq y \leqq \frac{1}{2} b ; V$ is the velocity of the air at infinity, its direction being parallel to the $x$-axis; $c(y)$ is the chord function determining the shape or planform of the airfoil; $m$ is a numerical constant, which the theory of wings of infinite span fixes at $2 \pi$ but which seems to have an experimental value in the vicinity of $5.5 ; \alpha$ is the geometric angle of attack, which for flat wings is a constant but for twisted wings is a given function of $y$. For a derivation, the reader is referred to the papers mentioned in footnote 1 ; we only wish to note here that the downwash velocity $w$ is essentially the first term of (1) and has the form

$$
\begin{equation*}
w(y)=\frac{1}{4 \pi} \frac{2}{b} \int_{-1}^{+1} \frac{d \eta^{\prime}}{\eta-\eta^{\prime}} \frac{d \Gamma}{d \eta^{\prime}}, \tag{2}
\end{equation*}
$$

where $\eta=2 y / b$ is a dimensionless span coordinate. Because of the singular integrand, the Cauchy principal value must be taken both in (1) and (2). Of fundamental importance, furthermore, is the elliptic wing of chord distribution

$$
\begin{equation*}
c(y)=c_{0} \sqrt{1-\eta^{2}} \tag{3}
\end{equation*}
$$

for which $\Gamma(y)$ is also of elliptic shape while $w(y)$ is constant.
In the extensive literature dealing with solutions of (1) for a given planform $c(y)$, the following procedure is used almost always: the substitution $\eta=\sin \theta$ is introduced, and the chord function $c(y)$ is developed in a Fourier series. It is assumed that $\Gamma$ is also of this form, whence equation (1) is satisfied at a discrete number of $y$ values. This leads to a system of linear equations for the Fourier coefficients, the solution of which is usually extremely laborious.

In contrast to this method, the point of view adopted by Trefftz ${ }^{2}$ seems to me more powerful. This author considers the potential flow in the complex $y+i z$ plane

[^18]at a large distance behind the wing. The wake, stretching along the $y$-axis from $-b / 2$ to $+b / 2$ becomes a line of discontinuity along which the velocity potential suffers a jump determined by the circulation. The integro-differential equation (1) reveals itself as a boundary condition which the velocity potential has to obey along the line $-b / 2 \leqq y \leqq b / 2$. "With the aid of conformal transformation the field is brought into relation with the field outside of a circle of unit radius; then the potential is approximated by a trigonometric expression and an approximate fulfilment of the boundary condition is sought. ${ }^{3}$

In the present paper the point of view which was used by Trefftz is extended, but instead of "seeking an approximate fulfillment of the boundary condition" for an arbitrary chord function $c(y)$, a simultancous choice of the function and of a conformal mapping function transforms the problem into a boundary value problem of classical type, which can be solved rigorously and in every detail. The resulting formulae lend themselves readily to numerical computation.

The family of planforms to which one is led in this fashion is represented by the equation

$$
\begin{equation*}
c(y)=c_{0}\left[\left(1-\eta^{2}\right)\left(1-\kappa^{2} \eta^{2}\right)\right]^{1 / 2} . \tag{4}
\end{equation*}
$$

For

$$
0 \leqq \kappa \leqq 1,
$$

this results in airfoils of taper greater than in the elliptic case (3), while for

$$
0 \leqq \kappa \leqq i,
$$

airfoils blunter than the elliptic ones are obtained. Fig. I shows the entire family for a fixed span and aspect ratio. ${ }^{4}$


Fig. 1: $\mathcal{A}=8$.
To be sure, for a given planform elaborate approximate methods, such as the Fourier series method described above, will probably always have to be used. How-

[^19]ever, we shall now show that at least for one planform family, which depends on two parameters, the problem can be completely solved. It is hoped that the method used here will be generalizable so as to furnish perhaps other chord functions in the future. At any rate, for a planform which is close to one of family (4) a method of successive approximations can be readily set up.

It seems that a comparison with the methods used in the theory of wings of infinite span is not superfluous. Here rigorous solutions are available for certain families of simple profiles, the simplest one of which is that of Joukowski. Although the designer will probably not be satisfied with such a simplification and will turn to more elaborate methods, there is an advantage in having easily derived formulae in closed form-both as far as quick estimates and classroom presentation are concerned. In lectures it is standard procedure to present the flow around the Joukowski and perhaps also around the Kármán-Trefftz profiles before turning to more general methods; on the other hand, when dealing with wings of finite span, the derivation of Prandtl's equation (1) is usually followed only by a detailed discussion of the elliptic case and then by a mere description of the Fourier series methods. The formulae given below are intended to fill this gap.
2. Prandtl's equation as a boundary condition in the complex plane $u=y+i z$. A brief recapitulation of Trefftz's work ${ }^{2}$ is necessary following the presentation of Mises and Friedrichs. ${ }^{1}$ In the $y+i z$ plane there exists a flow derivable from a potential $\phi(y, z)$. This potential is everywhere continuous except on the projection of the airfoil

$$
\begin{equation*}
z=0, \quad-b / 2 \leqq y \leqq+b / 2, \tag{5}
\end{equation*}
$$

where, because of the existence of a circulation, $\phi$ is discontinuous. Assuming that the values of $\phi$ at opposite points of the slit (5) are equal and opposite, we have

$$
\begin{equation*}
\Gamma(y)=4 \phi(y, 0) \tag{6}
\end{equation*}
$$

The downwash velocity $w$ becomes

$$
\begin{equation*}
w(y)=\left(\frac{\partial \phi}{\partial z}\right)_{0} \tag{7}
\end{equation*}
$$

The problem is therefore to find a solution of the Laplace equation

$$
\begin{equation*}
\Delta \phi=0 \tag{8}
\end{equation*}
$$

which on the upper side of the slit (5) satisfies the boundary condition

$$
\begin{equation*}
\left(\frac{\partial \phi}{\partial z}\right)_{0}-\frac{8 \phi(y, 0)}{m c(y)}=-V \alpha(y) . \tag{9}
\end{equation*}
$$

This condition results from substituting (2), (6) and (7) into (1). The corresponding condition on the lower side of the slit differs from (9) in that the sign of the second term is opposite.

Condition (9) characterizes the present problem as a boundary value problem of the third kind. As mentioned earlier, Trefftz now maps the region outside slit (5) into the interior of the unit circle by means of

$$
\begin{equation*}
u=\frac{b}{4}\left(t+\frac{1}{t}\right) \tag{10}
\end{equation*}
$$

Through the introduction of polar coordinates $t=r(\cos \theta+i \sin \theta)$ he is then led to approximate solutions having the form of a Fourier series in $\theta$.

It is evident that, because of the occurrence in (9) of the function $c(y)$ in the coefficient of $\phi(y, 0)$, exact solutions of the problem will in general be difficult to obtain. But let us also examine the effect of a conformal mapping upon (8) and (9). While the former is invariant under arbitrary conformal transformations the latter is not, because of the occurrence of the first derivative. One may say that each conformal transformation causes a new variable coefficient to appear both in the $\phi$ term on the left and on the right side of (9).

We propose to let these two effects counteract one another and to choose both appropriate conformal transformations and appropriate chord functions $c(y)$ so as to arrive at a boundary condition with constant coefficients which lends itself to treatment by classical methods. An example will best serve to illustrate this procedure.

Instead of introducing real polar coordinates within the circle into which the slit (5) is mapped by (10), we shall map the interior of the circle onto the strip bounded by the points $0,+\infty,+\infty+2 \pi i, 2 \pi i$ of a $\lambda$ plane by means of

$$
\begin{equation*}
t=e^{\lambda}, \quad u=\frac{b}{2} \operatorname{ch} \lambda, \tag{11}
\end{equation*}
$$

where $\lambda=\mu+i \nu$. The potential must be a solution of the Laplace equation in $\lambda$ and $\mu$, and must satisfy the following boundary condition along the imaginary axis onto which the circle $|t|=1$ is mapped:

$$
\begin{equation*}
\left(\frac{\partial \phi}{\partial \mu}\right)_{0}-\frac{4 b \sin \nu}{m c\left(\frac{1}{2} b \cos \nu\right)} \phi(0, \nu)=-\frac{1}{2} b V \alpha \sin \nu . \tag{12}
\end{equation*}
$$

In order that the coefficient of $\phi$ be a constant we must choose $c\left(\frac{1}{2} b \cos \nu\right)$ proportional to $\sin \nu$, whence, by virtue of (11), we have the elliptic chord distribution (3).
3. A transformation mapping the interior of the circle $|t|=1$ into that of a rectangle. We now replace the function (11) by others which map the interior of the unit circle into various other regions. Each time, in order to have a boundary condition with constant coefficients, an appropriate chord function must be chosen. When mapping the unit circle of the $t$ plane into a rectangle in the usual Schwartzian way such that four arbitrarily chosen points on the periphery correspond to the corners, a chord function results which possesses singularities at those points of the span which are the images of the corners. It is, however, possible to map a rectangle onto the unit circle, such that two opposite sides of the rectangle become two opposite semicircular arcs while each of the two other sides of the rectangle map onto a slit protruding radially part way into the circle. ${ }^{5}$ This is illustrated in Fig. 2. The mapping function which accomplishes this is

$$
\begin{equation*}
t=\sqrt{k} \operatorname{sn} \frac{2 K}{\pi} Z \tag{13}
\end{equation*}
$$

where the circle is in the $t$-plane $(t=r+i s)$, the rectangle is in the $Z$-plane $(Z=X+i Y)$,

[^20]$k$ is the modulus of the elliptic function and is between 0 and $1, K$ is the complete elliptic integral. We now separate real and imaginary parts by means of the addition theorem of the sn function and put
\[

$$
\begin{equation*}
Y= \pm \frac{\pi}{4} \frac{K^{\prime}}{K} \tag{14}
\end{equation*}
$$

\]

In view of the fact that $\operatorname{sn}\left(i K^{\prime} / 2\right)=i k^{-1 / 2}$, we obtain readily

$$
\begin{equation*}
r=(1+k) \frac{\operatorname{sn} 2 K X / \pi}{1+k \operatorname{sn}^{2} 2 K X / \pi} \quad s= \pm \frac{\mathrm{cn} 2 K X / \pi \operatorname{dn} 2 K X / \pi}{1+k \operatorname{sn}^{2} 2 K X / \pi} \tag{15}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
r^{2}+s^{2}=1 \tag{16}
\end{equation*}
$$

Thus the upper straight line (14) maps into the upper semicircle, and lower one into


Fig. 2.
the lower. If in (15) we set $X= \pm \frac{1}{2} \pi$, we obtain $s=0$. The four corners of the rectangle are hereby fixed. To see what corresponds to the vertical sides, we set $Z= \pm \pi / 2+i Y$ in (13), to obtain

$$
\begin{equation*}
r= \pm \sqrt{k} \mathrm{nd}\left(\frac{2 K}{\pi} Y, k^{\prime}\right) \tag{17}
\end{equation*}
$$

where $k^{\prime 2}=1-k^{2}$. The slits protruding into the circle along the real axis therefore terminate at $r= \pm k^{1 / 2}$.
4. The boundary condition in the $Z$-plane. For the upper side of the rectangle this condition becomes, by use of (9), (10) and (13),

$$
\begin{equation*}
\left(\frac{\partial \phi}{\partial Y}\right)_{\pi K^{\prime} / 4 K}+\frac{8}{m c}\left(\frac{d u}{d Z}\right)_{\pi K^{\prime} / 4 K} \phi\left(X, \frac{\pi}{4} \frac{K^{\prime}}{K}\right)=V \alpha\left(\frac{d u}{d Z}\right)_{\pi K^{\prime} / 4 K} \tag{18}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{d u}{d Z}=\frac{b}{2 \pi} \sqrt{k} K\left(1-k^{-1} \operatorname{ns}^{2} \frac{2 K}{\pi} Z\right) \mathrm{cn} \frac{2 K}{\pi} Z \operatorname{dn} \frac{2 K}{\pi} Z \tag{19}
\end{equation*}
$$

and $Z$ must be put equal to $X+i \pi K^{\prime} / 4 K$. The chord $c(y)$, now regarded as a func-
tion of $X$, must be chosen proportional to (19). Therefore it becomes necessary to express (19) as a function of $y$ in the original $u$ plane. We have at first from (13),

$$
\frac{d u}{d Z}=\frac{b \sqrt{\bar{k}} K}{2 \pi}\left(1-\frac{1}{t^{2}}\right)\left[\left(1-\frac{t^{2}}{k}\right)\left(1-k t^{2}\right)\right]^{1 / 2}
$$

where $|t|=1$. Using for the moment the angle $\nu$ as in (12); this may be written in the form

$$
\frac{d u}{d Z}=\frac{i b \sqrt{k} K}{\pi} \sin \nu\left[2 \cos 2 \nu-\left(k+\frac{1}{k}\right)\right]^{1 / 2}
$$

which, since $\cos \nu=\eta$, becomes finally

$$
\begin{equation*}
\frac{d u}{d Z}=\frac{1}{\pi} b K(1+k)\left[\left(1-\eta^{2}\right)\left(1-\kappa^{2} \eta^{2}\right)\right]^{1 / 2} \tag{20}
\end{equation*}
$$

The abbreviation

$$
\begin{equation*}
\frac{1}{\kappa}=\frac{1}{2}\left(\frac{1}{\sqrt{k}}+\sqrt{k}\right) \tag{21}
\end{equation*}
$$

was used, where $\kappa$ is between 0 and 1 . We see thus that we are led to the planforms (4), (4') which represent wings of taper greater than the elliptic wing.

Before (4) and (20) are substituted into (18), it is convenient to calculate the area $S$ and the aspect ratio $A=b^{2} / S$. It stands to reason that the area

$$
S=\int_{-b / 2}^{b / 2} c_{0}\left[\left(1-\eta^{2}\right)\left(1-\kappa^{2} \eta^{2}\right)\right]^{1 / 2} d y
$$

is expressible in terms of the complete elliptic integrals $E$ and $K$ of $\kappa$, but in view of the fact that (21) is the well known Landen transformation, $S$ can also be reduced to the complete elliptic integrals of the modulus $k$. The result is

$$
\begin{equation*}
S=b c_{0} G(k) \tag{22}
\end{equation*}
$$

where

$$
\begin{equation*}
G(k)=\frac{1}{6 k} \cdot\left[\frac{1+6 k+k^{2}}{1+k} E(k)-(1-k)(1-3 k) K(k)\right] \tag{1}
\end{equation*}
$$

$G(k)$ is a purely numerical constant depending on taper. For the aspect ratio we have

$$
\begin{equation*}
A=\frac{b}{G(k) c_{0}} \tag{23}
\end{equation*}
$$

and for the average chord $\bar{\varepsilon}$,

$$
\begin{equation*}
\bar{c}=G(k) c_{0} \tag{1}
\end{equation*}
$$

Substitution of (4), (20) and (23) into (18) gives the final form of the boundary condition,

$$
\begin{align*}
& \left(\frac{\partial \phi}{\partial Y}\right)_{K^{\prime} / 4 K}+\frac{8}{\pi} \frac{\mathcal{A}}{m}(1+k) G K \phi\left(X, \frac{\pi}{4} \frac{K^{\prime}}{K}\right) \\
& =\frac{V b}{2 \pi} \sqrt{k} K\left[\alpha\left(1-k^{-1} n s^{2} \frac{2 K}{\pi} Z\right) \mathrm{cn} \frac{2 K}{\pi} Z \operatorname{dn} \frac{2 K}{\pi} Z\right]_{Y=\pi K^{\prime} / 4 K} \tag{24}
\end{align*}
$$

Variable coefficients now only occur on the right. It should be noted that for $Z=X+i \pi K^{\prime} / 4 K$ the right-hand side represents a real function of $X$.
5. Solution of the boundary value problem for tapered wings. This solution has to satisfy the following requirements:
(1) Inside the rectangle $|X| \leqq \pi / 2,|Y| \leqq \pi K^{\prime} / 4 K$, it must satisfy Laplace's equation.
(2) On the top side $Y=\pi K^{\prime} / 4 K$ it must satisfy (24).
(3) On the lower side it must satisfy a similar condition differing from (24) only in the sign of the first term.

We shall seek a solution of the form

$$
\begin{equation*}
\phi(X, Y)=\sum_{n} \operatorname{sh} n Y\left(A_{n} \cos n X+B_{n} \sin n X\right) . \tag{25}
\end{equation*}
$$

In this series each term is a solution of the Laplace equation. The sh function of $Y$ is alone present because of condition (3), or in other words, because we want $\phi$ to have equal and opposite values at opposite points on slit (5) in the $y+i z$ plane. To determine $A_{n}$ and $B_{n}$ the right-hand side of (24), considered as a function of $X$, has to be expanded in a Fourier series.

At this moment it should be remembered that for twisted airfoils $\alpha$ is a function of $y$ or $\eta$. It will be an even function as long as the ailerons are in their normal position. Otherwise, $\alpha$ may have an odd component or may even be discontinuous. At the present time only the case of constant $\alpha$ will be treated. ${ }^{6}$

To develop the right-hand side of (24) into a Fourier series we begin with the familiar series ${ }^{7}$ for sn :

$$
\begin{equation*}
\text { sn } \frac{2 K}{\pi} Z=\sum_{n=0}^{\infty} a_{n} \sin (2 n+1) Z, \quad a_{n}=\frac{\pi}{K k} \operatorname{cosech} \pi\left(n+\frac{1}{2}\right) \frac{K^{\prime}}{K} . \tag{26}
\end{equation*}
$$

This series converges uniformly as long as $|I(z)|<\pi K^{\prime} / 2 K$, i.e., within a horizontal strip whose median line is the real axis and whose width is $\pi K^{\prime} / K$. But since this is just twice the vertical dimension of the rectangle of the boundary value problem (see Fig. 2), series (26) will certainly converge absolutely anywhere that it is needed at present. We differentiate and put $Z=i \pi K^{\prime} / 4 K+X$, to obtain

$$
\begin{equation*}
\text { cn } \frac{2 K}{\pi} Z \mathrm{dn} \frac{2 K}{\pi} Z=\frac{\pi}{2 K} \sum_{n=0}^{\infty}(2 n+1) a_{n} \cos (2 n+1)\left(i \frac{\pi}{4} \frac{K^{\prime}}{K}+X\right) . \tag{27}
\end{equation*}
$$

To get the other part of the right-hand side of (24), we put $Z=i \pi K^{\prime} / 2 K+Z^{\prime}$, whence

$$
\mathrm{ns} \frac{2 K}{\pi} Z=k \operatorname{sn} \frac{2 K}{\pi} Z^{\prime}
$$

[^21]The application of (26) with $Z^{\prime}$ as the variable gives

$$
k^{-1} \mathrm{~ns} \frac{2 K}{\pi} Z=\sum_{n=0}^{\infty} a_{n} \sin (2 n+1) Z^{\prime}
$$

After differentiating with respect to $Z$ or $Z^{\prime}$, we put $Z^{\prime}=-i \pi K^{\prime} / 4 K+X$ (which is the same as putting $\left.Z=+i \pi K^{\prime} / 4 K+X\right)$, to obtain

$$
\begin{equation*}
-k^{-1} \operatorname{cs} \frac{2 K}{\pi} Z \text { ds } \frac{2 K}{\pi} Z=\frac{\pi}{2 K} \sum_{n=0}^{\infty}(2 n+1) a_{n} \cos (2 n+1)\left(-i \frac{\pi}{4} \frac{K^{\prime}}{K}+X\right) \tag{28}
\end{equation*}
$$

The sum of (27) and (28) furnishes the desired series which, because of the second formula of (26), may be written in the form

$$
\begin{equation*}
\left(1-k^{-1} \mathrm{~ns}^{2} \frac{2 K}{\pi} Z\right) \mathrm{cn} \frac{2 K}{\pi} Z \operatorname{dn} \frac{2 K}{\pi}=\frac{\pi^{2}}{2 K^{2} k} \sum_{n=0}^{\infty}(2 n+1) \frac{\cos (2 n+1) X}{\operatorname{sh}\left(n+\frac{1}{2}\right) \pi K^{\prime} / 2 K}, \tag{29}
\end{equation*}
$$

where it is to be understood that $Z=i \pi K^{\prime} / 4 K+X$. This series converges uniformly for all real values of $X$.

The final solution of the problem can now be written down. Introducing (29) and (25) into (24) one obtains for the velocity potential,

$$
\begin{equation*}
\phi(X, Y)=\frac{\pi}{4} \frac{V \alpha b}{K \sqrt{k}} \sum_{n=0}^{\infty} \frac{\operatorname{sh}(2 n+1) Y \cos (2 n+1) X}{D_{n} \operatorname{sh}(2 n+1) \pi K^{\prime} / 4 K}, \tag{30}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{n}=\operatorname{ch}(2 n+1) \frac{\pi}{4} \frac{K^{\prime}}{K}+\frac{8}{\pi} \frac{\mathcal{A}}{m} \frac{(1+k) G K}{2 n+1} \operatorname{sh}(2 n+1) \frac{\pi}{4} \frac{K^{\prime}}{K} \tag{31}
\end{equation*}
$$

This series converges uniformly as long as $|Y|<\pi K^{\prime} / 2 K$, which is more than we need for numerical calculations, since the rectangle only extends to $Y= \pm \pi K^{\prime} / 4 K$.

Of greater practical importance than $\phi$ is the sectional lift or the sectional lift coefficient $c_{l}$ which may be defined as follows: let $l$ be the lift per unit span,

$$
\begin{equation*}
l=\frac{1}{2} \rho V^{2} \bar{c} c_{l}, \tag{32}
\end{equation*}
$$

where $\bar{c}$ is the average chord (231). Since from (6) $l$ is also $4 \rho V \phi\left(X, \pi K^{\prime} / 4 K\right)$, the following expression results:

$$
\begin{equation*}
c_{l}=2 \pi \alpha \frac{A}{K \sqrt{k}} \sum_{n=0}^{\infty} \frac{1}{D_{n}} \cos (2 n+1) X \tag{33}
\end{equation*}
$$

For this rapidly converging series, numerical tests have shown that three or four terms suffice to give a result correct to one part in ten thousand. To formula (33) should be added the relation between $X$ and the original span coordinate $y$. From (15) and (10) it is found to be

$$
\begin{equation*}
y=\frac{b}{2}(1+k) \frac{\operatorname{sn} 2 K X / \pi}{1+k \operatorname{sn}^{2} 2 K X / \pi} \tag{34}
\end{equation*}
$$

Formulae (33) and (34) give the spanwise lift distribution as a function of $y$.
6. Results for the blunt wing. The method of the previous section may readily be used to get similar results for wings blunter than elliptic ones. The underlying idea is the following: in section 2 we mapped the $t$ circle onto a square such that the semicircle in the upper half plane became the upper side of a rectangle in the $Z$ plane, and correspondingly for the lower semicircle. Now, the interior of the unit circle in the $t$ plane will be mapped so that the right and left semicircles will correspond to the right and left sides of the rectangle, while two slits from $\pm i$ to $\pm i \sqrt{k}$ will correspond to the upper and lower sides. In short, the role on the real and imaginary axes in both planes will be reversed. Results will merely be given and formulas will receive the same numbers as the corresponding formula of the "tapered" case, except that primes will be added.

The mapping just described is performed by

$$
t=i^{-1} \sqrt{k} \operatorname{sn} \frac{2 K}{\pi} i Z
$$

Upon putting $X= \pm \pi K^{\prime} / 4 K$ the point in the $t$-plane moves on a circle $r^{2}+s^{2}=1$,
t-PLANE


Z-plane


Fig. 3.
while $Y= \pm \pi / 2$ gives the slits along the imaginary axis of $t$. This is shown in Fig. 3. The interior of the unit circle of the $t$ plane is then mapped by means of (10) onto the exterior of the slit (5) in the $u$ plane.

The boundary condition, now along the two vertical sides, becomes

$$
\left(\frac{\partial \phi}{\partial X}\right)_{\pi K^{\prime} / 4 K}+\frac{8}{i m c}\left(\frac{d u}{d Z}\right)_{\pi K^{\prime} / 4 K} \phi\left(\frac{\pi}{4} \frac{K^{\prime}}{K}, Y\right)=i^{-1 V \alpha}\left(\frac{d u}{d Z}\right)_{\pi K^{\prime} / 4 K}
$$

with

$$
\begin{equation*}
\left(\frac{d u}{d Z}\right)_{\pi K^{\prime} / 4 K}=\frac{b}{2 \pi} \sqrt{k} K\left(1+k^{-1} \mathrm{~ns}^{2} \frac{2 K}{\pi} i Z\right) \operatorname{cn} \frac{2 K}{\pi} i Z \operatorname{dn} \frac{2 K}{\pi} i Z \tag{19'}
\end{equation*}
$$

which is imaginary for $Z=\pi K^{\prime} / 4 K+i Y$. The determination of the planform results from expressing this in terms of $\eta$ :

$$
i^{-1}\left(\frac{d u}{d Z}\right)_{\pi K^{\prime} / 1 K}=\frac{1}{\pi} b K(1-k) \sqrt{\left(1-\eta^{2}\right)\left(1+\kappa^{\prime 2} \eta^{2}\right)},
$$

where

$$
\frac{1}{\kappa^{\prime}}=\frac{1}{2}\left(\frac{1}{\sqrt{k}}-\sqrt{k}\right)
$$

The different signs should be noted. The two functions $\kappa$ of $\kappa^{\prime}$ of $k$ are plotted in Fig. 4. In this way a planform blunter than the elliptic one results, namely (4), ( $4^{\prime \prime}$ ). When $\kappa^{\prime}$ becomes greater than one, the chord function has a maximum between $\eta=0$ and 1 . The bluntest wing shape without such a maximum is attained for $\kappa^{\prime}=1$ or

$$
\begin{equation*}
k_{C R}=3-\sqrt{8} \tag{35}
\end{equation*}
$$

in which case the lemniscate functions result. Thus the formulae of this section, while being valid for any $k$ between 0 and 1 have aerodynamical importance only for $k$ between 0 and $k_{c r}$. ${ }^{8}$


Fig. 4.

The area of the wing becomes

$$
S=b c_{0} G(-k),
$$

where $G(-k)$ is the function in equation (22 ${ }_{1}$ ) for negative $k .{ }^{9}$ In the case of (34), which interests us especially, the elliptic integral becomes reducible to Gamma functions:

$$
\begin{equation*}
G\left(-k_{C R}\right)=\frac{\Gamma(3 / 2) \Gamma(5 / 4)}{\Gamma(7 / 4)}=.87402 \tag{36}
\end{equation*}
$$

Formulae for $\mathcal{A}$ and the mean chord $\bar{c}$ are taken over with $G(-k)$.
For a constant angle of attack, the only case treated below, the function in (19') will be developed in a Fourier series on the vertical sides $X= \pm \pi K^{\prime} / 4 K$ of the rectangle. This is readily accomplished, the difference from (29) being that only terms in $\sin (2 n+1) Y$ occur. The final result for the velocity potential is

$$
\phi(X, Y)=\frac{\pi}{4} \frac{V \alpha b}{K \sqrt{k}} \sum_{n=0}^{\infty} \frac{\operatorname{ch}(2 n+1) X \sin (2 n+1) Y}{\Delta_{n} \operatorname{ch}(2 n+1) \pi K^{\prime} / 4 K},
$$

where

$$
\begin{equation*}
\Delta_{n}=\operatorname{sh}(2 n+1) \frac{\pi}{4} \frac{K^{\prime}}{K}+\frac{8}{\pi} \frac{A}{m} \frac{(1-k) G(-k) K}{2 n+1} \operatorname{ch}(2 n+1) \frac{\pi}{4} \frac{K^{\prime}}{K} . \tag{31'}
\end{equation*}
$$

[^22]Convergence properties are the same as before. The sectional lift coefficient (32) is given by

$$
c_{l}=2 \pi \alpha \frac{A}{K \sqrt{k}} \sum_{n=0}^{\infty} \frac{1}{\Delta_{n}} \sin (2 n+1) Y
$$

while the relation between the variable $Y$ and span coordinate $y$ is now

$$
y=\frac{b}{2} \frac{\operatorname{cn} 2 K Y \operatorname{dn} 2 K Y / \pi}{1+k \operatorname{sn}^{2} 2 K Y / \pi}
$$

7. The elliptic wing as a limiting case. From (20) or (21) it is evident that the chord function represents an ellipse when $k$ goes to zero. For decreasing $k$ the elliptic integrals $K$ and $G$ approach $\pi / 2$ and $\pi / 4$ respectively, while

$$
\lim _{x \rightarrow 0}\left(K^{\prime}-\log \frac{4}{k}\right)=0
$$

The quotient $K^{\prime} / K$ becomes large and the convergence of the series (33) improves. Keeping only the first term we see that

$$
\lim _{k \rightarrow 0}\left\{\binom{\operatorname{ch}}{\operatorname{sh}} \frac{\pi}{4} \frac{K^{\prime}}{K}-k^{-1 / 2}\right\}=0
$$

Thus

$$
\begin{equation*}
c_{l}=\frac{4}{\pi} \alpha m \frac{\cos X}{1+\pi c A / m} \tag{37}
\end{equation*}
$$

which is in complete agreement with Durand II, p. 169 , since now $\cos X=\left(1-\eta^{2}\right)^{1 / 2}$.
8. The limiting case of extreme taper, $k=1$. This case is of more interest than the preceding one, inasmuch as it leads to new and simple formulae in closed form. It follows from (20) that the planform is now given by

$$
\begin{equation*}
c=c_{0}\left(1-\eta^{2}\right) \tag{38}
\end{equation*}
$$

which represents a parabolic arc or two such arcs joined along the $y$-axis (see Fig. 1). Although the lift distribution can be obtained from (33) by letting $k$ tend towards 1 , we prefer to derive it afresh.

The transformation

$$
\begin{equation*}
\iota=\tanh Z \tag{39}
\end{equation*}
$$

maps the interior of the unit circle in the $t$ plane onto a strip parallel to the $X$-axis and bounded by $Y= \pm \pi / 4$. The points $t= \pm 1$, where the $t$ circle crosses the real axis move, respectively, towards $+\infty$ and $-\infty$. Transformations (10) and (39) together map the outside of slit (5) in the $u$ plane on the above mentioned strip, in such a way that the upper and lower sides of the slit become, respectively, the upper and lower sides of the strip, while the tips ( $y= \pm b / 2$ ) move to infinity.

The boundary condition on the upper side is the same as equation (18), except that the subscript on the various derivatives now is $Y=\pi / 4$. Since the transformation from the $u$ to the $Z$ plane is now

$$
\begin{equation*}
u=\frac{1}{2} b \operatorname{coth} 2 Z \tag{40}
\end{equation*}
$$

as is seen from (39) and (10), we have as the analogue of (19) the equation

$$
\begin{equation*}
\left(\frac{d u}{d Z}\right)_{Y=\pi / 4}=b \operatorname{sech}^{2} 2 X \tag{41}
\end{equation*}
$$

which, when expressed as function of $\eta$, becomes $b\left(1-\eta^{2}\right)$. Hence the chord is chosen as in (38). We note the formulae for mean chord $\bar{i}$, area $S$, and aspect ratio R:

$$
\begin{equation*}
\bar{c}=\frac{2}{3} c_{0} ; \quad S=\frac{2}{3} b c_{0} ; \quad A=\frac{3}{2} \frac{b}{c_{0}} . \tag{42}
\end{equation*}
$$

The final form of the boundary condition on the upper side of the infinite strip is

$$
\begin{equation*}
\left(\frac{\partial \phi}{\partial Y}\right)_{\pi / 4}+\frac{16}{3} \frac{A}{m} \phi\left(X, \frac{\pi}{4}\right)=V \alpha b \operatorname{sech}^{2} 2 X . \tag{43}
\end{equation*}
$$

Due to the fact that the range of $X$ is from $-\infty$ to $+\infty$, the right-hand side of (43) must now be expanded in a Fourier integral. This can be done by standard methods, the result being

$$
\begin{equation*}
\operatorname{sech}^{2} 2 X=\frac{1}{4} \int_{0}^{\infty} \frac{\zeta d \zeta \cos \zeta X}{\operatorname{sh} \pi \zeta / 4} \tag{44}
\end{equation*}
$$

For the velocity potential $\phi(X, Y)$ we assume a solution of the form

$$
\begin{equation*}
\phi(X, Y)=\int_{0}^{\infty} d \zeta A(\zeta) \operatorname{sh} \zeta Y \cos \zeta X \tag{45}
\end{equation*}
$$

where the integrand satisfies the Laplace equation. After introducing (45) and (44) into (43), $A(\zeta)$ is easily determined. The final form for $\phi(X, Y)$ is

$$
\begin{equation*}
\phi(X, Y)=\frac{1}{4} V \alpha b \int_{0}^{\infty} \frac{\operatorname{sh} \zeta Y \cos \zeta X}{D(\zeta) \operatorname{sh} \pi \zeta / 4} d \zeta \tag{46}
\end{equation*}
$$

where

$$
\begin{equation*}
D(\zeta)=\operatorname{ch} \frac{\pi}{4} \zeta+\frac{16}{3} \frac{\mathcal{A}}{m} \frac{\operatorname{sh} \pi \zeta / 4}{\zeta} \tag{47}
\end{equation*}
$$

The sectional lift coefficient assumes the form

$$
\begin{equation*}
c_{l}=2 \mathcal{A} \alpha \int_{0}^{\infty} \frac{\cos \zeta X}{D(\zeta)} d \zeta \tag{48}
\end{equation*}
$$

The integral in (46) converges for all $X$ and $Y$ within the strip under consideration. In order to obtain from (48) the lift distribution as a function of $y$, we note here that, due to (40),

$$
\begin{equation*}
y=\frac{1}{2} b \tanh 2 X . \tag{49}
\end{equation*}
$$

The above integral may be brought into a rather different and interesting form which makes the numerical evaluation very much easier. Let us consider the integral

$$
\begin{equation*}
\int_{-\infty}^{+\infty} \frac{\tau d \tau \cos \tau \xi}{\tau \operatorname{ch} \tau+a \operatorname{sh} \tau} \tag{50}
\end{equation*}
$$

which, except for a simpler notation, is the same as that in (48). On the imaginary $\tau$
axis, the integrand has infinitely many poles, which can be computed as the roots of the transcendental equation

$$
\begin{equation*}
\sigma \cos \sigma+a \sin \sigma=0 \tag{51}
\end{equation*}
$$

Denoting the $n$th root by $\sigma_{n}$, we find from this that

$$
\cos \sigma_{n}=(-1)^{n} a\left(a_{n}+\sigma_{n}^{2}\right)^{-1 / 2}, \quad \sin \sigma_{n}=(-1)^{n+1} \sigma_{n}\left(a^{2}+\sigma_{n}^{2}\right)^{-1 / 2} .
$$

For $\tau=0$ the integrand is regular. It now becomes convenient to decompose the cos in the numerator into two exponentials. Two integrals arise in this way, one containing $\exp i \tau \xi$, the other $\exp (-i \tau \xi)$. According to a standard argument, the integral with the positive exponential tends to zero when integrated along the circumference of a large semicircle in the upper half plane, as the radius increases; similarly for the other integral and a semicircle around the origin in the lower half plane. Thus the integral with the positive exponential, taken along the real axis from $-\infty$ to $+\infty$, is equal to the sum of the residues at the poles $\sigma_{n}$ which lie in the upper half plane, and the integral with the negative exponentials is equal to the sum of the residues at the poles of the lower half plane.

Applying these considerations to (48) we see that the quantity $a$, which determines the roots of the transcendental equation, has the value

$$
a=\frac{8}{3} \frac{A}{m},
$$

while the sectional lift coefficient is given by

$$
\begin{equation*}
c_{l}=8 \alpha \mathcal{A} A \sum_{n=0}^{\infty}(-1)^{n+1} \frac{\sigma_{n}\left[a^{2}+\sigma_{n}\right]^{1 / 2}}{a(a+1)+\sigma_{n}^{2}} e^{-4 \sigma_{n} x / \tau} . \tag{52}
\end{equation*}
$$

This is a very rapidly converging series, which, although especially convenient near the wing tips, may be used to advantage for values of $\eta$ as small as .3 or .2. Very near the wing tips $X$ is approaching $\infty$. Hence, using (49) we may write approximately

$$
c_{l}=8 a c A \frac{\sigma_{1}\left[a^{2}+\sigma_{1}^{2}\right]^{1 / 2}}{a(a+1)+\sigma_{1}^{2}}\left(\frac{1-\eta}{2}\right)^{\sigma_{1} / \pi}
$$

which shows that $c_{2}$ has a vertical tangent at the tips since $\sigma_{1}$ is between $\pi / 2$ and $\pi$.
The only inconvenient region is that of small $X$, especially the point $X=0$. It seems that only graphic integration can be used to find $c_{l}$ at the center of the wing from formula (48).
9. Total lift and total induced drag. For these important quantities rapidly convergent expressions are readily derived. The total lift $L$ is obtained by integrating $\rho V \Gamma$ across the span,

$$
L=4 \rho V \int_{-b / 2}^{+b / 2} \phi(y, 0) d y
$$

and transforming to the $Z$ plane, to obtain

$$
L=4 \rho V \int_{-\pi / 2}^{+\pi / 2} \phi\left(X, \frac{\pi}{4} \frac{K^{\prime}}{K}\right)\left(\frac{d u}{d Z}\right)_{\pi K^{\prime} / 4 K} d X .
$$

The two Fourier series for $\phi$ and $d u / d Z$ are then substituted from (30), (19) and (29). From the completeness relation it follows that the integral of the product of the two series is equal to the sum of the products of the coefficients. Instead of the formula for $L$, merely the final formula for the over-all lift coefficient $c_{L}$, defined by $L=\frac{1}{2} \rho V^{2} S c_{L}$, will be given: ${ }^{10}$

$$
\begin{equation*}
c_{L}=\frac{\pi^{3}}{4} \frac{\mathcal{A} \alpha}{K^{2} k} \sum_{n=0}^{\infty} \frac{2 n+1}{D_{n} \operatorname{sh}(2 n+1) \pi K^{!} / 4 K} . \tag{53}
\end{equation*}
$$

The abbreviation $D_{n}$ was defined in (31). In the limit of vanishing $k$ this reduces exactly to formula (2.15), p. 169 of Durand, Vol. II.

For wings blunter than elliptic ones the result is

$$
\begin{equation*}
c_{L}=\frac{\pi^{3}}{4} \frac{C A}{K^{2} k} \sum_{n=0}^{\infty} \frac{2 n+1}{\Delta_{n} \operatorname{ch}(2 n+1) \pi K^{\prime} / 4 K}, \tag{53'}
\end{equation*}
$$

where $\Delta_{n}$ is the coefficient defined in ( $31^{\prime}$ ).
For the parabolic wing treated in section 7 , it is necessary to go through the operation just described for the Fourier integrals (44) and (46). The lift coefficient is now

$$
\begin{equation*}
c_{L}=\frac{\pi}{2} \mathcal{A} \alpha \int_{0}^{\infty} \frac{\zeta d \zeta}{D(\zeta) \operatorname{sh} \pi \zeta / 4} . \tag{54}
\end{equation*}
$$

For the induced drag $D_{i}$ similarly simple formulae may be obtained. Using (7) we have first in the original $u$-plane ( $u=y+i z$ )

$$
D_{i}=4 \rho \int_{-b / 2}^{b / 2} \phi(y, 0)\left(\frac{\partial \phi}{\partial \bar{Z}}\right)_{0} d y .
$$

In the $Z$ plane of the tapered wing this may be written

$$
D_{i}=4 \rho \int_{-\pi / 2}^{\pi / 2} \phi\left(X, \frac{\pi}{4} \frac{K^{\prime}}{K}\right)\left(\frac{\partial \phi}{\partial Y}\right)_{\pi K^{\prime} / 4 K} d X .
$$

Now the completeness relation must be applied to the Fourier series (30) and its own derivative. The result is, expressed in terms of the drag coefficient,

$$
\begin{equation*}
c_{D}=\frac{\pi^{3}}{4} \frac{\alpha^{2} A}{K^{2} k} \sum_{n=0}^{\infty} \frac{2 n+1}{D_{n}^{2}} \operatorname{coth}(2 n+1) \frac{\pi}{4} \frac{K^{\prime}}{K}, \tag{55}
\end{equation*}
$$

while for the blunt wing we have

$$
c_{D}=\frac{\pi^{3}}{4} \frac{\alpha^{2} A}{K^{2} k} \sum_{n=0}^{\infty} \frac{2 n+1}{\Delta_{n}^{2}} \tanh (2 n+1) \frac{\pi}{4} \frac{K^{\prime}}{K},
$$

[^23]and for the parabolic wing
\[

$$
\begin{equation*}
c_{D}=\frac{\pi}{2} \alpha^{2} \mathcal{A} \int_{0}^{\infty} \frac{\zeta d \zeta}{D^{2}(\zeta)} \operatorname{coth} \frac{\pi \zeta}{4} \tag{56}
\end{equation*}
$$

\]

Formulae (53) to (56) give the dependence of lift and induced drag upon aspect ratio $\mathcal{A}$ and taper $k$. However, it is not necessary to compute more than $c_{L}$, for there exists the following simple relation between $c_{L}$ and $C_{D}$ :

$$
\begin{equation*}
c_{D}=\alpha \mathcal{A} A \frac{\partial c_{L}}{\partial \mathcal{C} A} \tag{57}
\end{equation*}
$$

Thus after having plotted a set of curves showing the dependence of $c_{L}$ upon the aspect ratio, one may obtain $c_{D}$ by graphic differentiation.

Further over-all quantities of practical interest are the rolling moment

$$
4 \rho V \int_{-b / 2}^{b / 2} \phi(y, 0) y d y
$$

and the yawing moment

$$
4 \rho \int_{-b / 2}^{b / 2} \phi(y, 0)\left(\frac{\partial \phi}{\partial Z}\right)_{0} y d y
$$

but since in this paper we have confined ourselves through out to constant angles of attack, they will vanish. However, when $\alpha(y)$ is assumed to have an odd component the methods described here still work and the above moments yield formulae of type (53) to (56).
10. Numerical results. All formulae given in the preceding sections for the sectional lift coefficients are readily evaluated, due to the fact that the series converge rapidly. An exception is $c_{l}$ for the parabolic wing shape for small or zero span coordinate, in which case it is necessary to resort to graphic integration. Following v. Kármán and Burgers, ${ }^{3}$ we plot $c_{l} / m \alpha$ as a function of the span coordinate $\eta$ rather than $c_{l}$ itself. In this way it is necessary to vary only the two parameters $A / m$ and the taper constant $k$.

In Tables 1 and 2 values of $c_{l} / m \alpha$ are given which are computed from formulae (33) and (34) for $k=\sqrt{ } .1$ and $k=\sqrt{ } .2$. In the first column are the values of $2 K X / \pi$ which quantity serves as a convenient parameter, in the second $X$, in the third $\eta$, computed by means of the tables by Milne-Thomson ${ }^{11}$. In the last three columns the values of $c_{l} / m \alpha$ are to be found for $A / m$ equal to $1.0,1.5$ and 2.0 , corresponding to aspect ratios of about $5.5,9.25$ and 11 .

Among wings blunter than elliptic only the one with $k_{\mathrm{CR}}$ of (35) has been computed using formulae ( $33^{\prime}$ ) and ( $34^{\prime}$ ). However, since elliptic functions with $k$ values as small as $k_{\text {CR }}$ were not tabulated by Milne-Thomson, equation (34') was expanded in powers of that small constant. Values of $c_{l} / m \alpha$ are given in Table 3.

Finally in Table 4 the same data are given for the parabolic wing of section 7. In column 1 are the values of the parameter $X$, in column 2 the $\eta$ values obtained from it with (49), and in the next three columns the values of $c_{l} / m \alpha$. For $X=0, .1, .2$ they were obtained by graphical integration from (48), and for all greater values of $X$ from

[^24](52). The roots of the transcendental equation (51) do not seem to have been calculated before, at least not for the values ( $51^{\prime \prime}$ ) of the constant $a$. They were determined to five significant figures.

These tables are supplemented by the following figures: Fig. 5 makes possible a


Fig. 5.
quick determination of the span coordinate associated with a certain coordinate value on the side of the rectangle of the boundary value problem. The two curves marked $k=\sqrt{ } .1$ and $k=\sqrt{ } .2$ represent plots of equation (34) and go with Tables 1 and 2.

Table I
Tapered Wing $k=\sqrt{.1}$

|  |  |  | $c_{l} / m \alpha$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $2 K X / \pi$ | $X$ | $\eta$ | $\mathcal{3} / m=1$ | $\mathcal{A} / m=1.5$ | $\mathcal{A} / m=2$ |
| 0 | 0 | 0 | 1.0191 | 1.1207 | 1.1807 |
| .1 | .09742 | .13096 | 1.0070 | 1.1068 | 1.1655 |
| .3 | .29225 | .37801 | .91652 | 1.0026 | 1.0527 |
| .5 | .48709 | .58643 | .76447 | .82908 | .86553 |
| .8 | .77934 | .80778 | .50527 | .53846 | .55574 |
| 1.1 | 1.07159 | .93247 | .28109 | .29429 | .30021 |

The third curve is a plot of equation (34') and goes with Table 3. Figs. 6, 7, 8 show the dependence of lift coefficient upon aspect ratio and taper. They should be compared with Fig. 75 of Durand, vol. II (which was computed by the approximate method de-


FIG. 6. $A / m=1.0$.


Fig. 7. $\mathcal{A} / m=1.5$.


Fig. 8. $\mathcal{A} / m=2.0$.
Table II
Tapered Wing $k=\sqrt{.2}$

|  |  |  | $c_{l} / m \alpha$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $2 K X / \pi$ | $X$ | $\eta$ | $A / m=1$ | $\mathcal{A} / m=1.5$ | $A / m=2$ |
| 0 | 0 | 0 | 1.0306 | 1.1365 | 1.1987 |
| .1 | .09465 | .14380 | 1.0150 | 1.1184 | 1.1792 |
| .3 | .28394 | .41052 | .90134 | .98667 | 1.0370 |
| .5 | .47324 | .62545 | .72028 | .77905 | .81219 |
| .8 | .75718 | .83613 | .44105 | .46565 | .47700 |
| 1.1 | 1.04113 | .94328 | .23139 | .23804 | .24100 |

Table III
Blunt Wing $k=3-\sqrt{8}$

|  |  |  |  |  |  | $c_{l} / m \alpha$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\eta$ | $c A / m=1$ | $c A / m=1.5$ | $c A / m=2$ |  |  |  |  |  |
| $\pi / 2$ | 0 | .91288 | .98248 | 1.0205 |  |  |  |  |  |
| 1.2 | .30920 | .88828 | .96045 | 1.0006 |  |  |  |  |  |
| 1.0 | .47438 | .84591 | .91985 | .96191 |  |  |  |  |  |
| .7 | .70700 | .71481 | .78512 | .82654 |  |  |  |  |  |
| .5 | .83962 | .56705 | .62678 | .66262 |  |  |  |  |  |
| .35 | .91816 | .42178 | .46800 | .49601 |  |  |  |  |  |

Table IV
Parabolic Wing $k=1.0$

|  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $X$ | $\eta$ | $c_{l} / m \alpha$ |  |  |
| 0 | 0 | $A / m=1$ | $\mathcal{A} / m=1.5$ | $\mathcal{A} / m=2$ |
| 1 | .19737 | 1.050 | 1.164 | 1.230 |
| .2 | .37995 | 1.020 | 1.130 | 1.166 |
| .4 | .66404 | .9297 | 1.029 | 1.057 |
| .6 | .83365 | .66568 | .71789 | .74320 |
| .9 | .94681 | .42208 | .43141 | .44003 |

scribed in the introduction). The lift coefficient shows, of course, the expected span dependence, inasmuch as a more highly tapered wing gives rise also to a more highly tapered lift curve. Of special interest is the parabolic case, for which the lift coefficient goes to zero with infinite tangent although the chord function $c(y)$ has a finite tangent at the tips.

The author should like to thank Mr. H. Yoshihara for extensive assistance with the numerical calculations. A grant from the Faculty Research Fund of the University of Michigan is gratefully acknowledged.

# SOLUTION BY RELAXATION METHODS OF PLANE POTENTIAL PROBLEMS WITH MIXED BOUNDARY CONDITIONS* 

BY<br>L. FOX<br>Imperial College of Science and Technology, London

1. Introduction. The method of relaxation, as originally propounded by Southwell [1], was used to calculate the stresses in braced frameworks. A physical picture of the method, as presented by him in the Wright Brothers Memorial Lecture for 1941, is the following. At each joint of the structure constraints are applied which prevent joint displacements and bear all the load. One constraint is then relaxed, thereby transferring some of its load to the members of the framework and some to adjacent constraints. Each constraint is relaxed in turn, and more of the load is imposed on the framework, until the residual loads (still borne by the constraints) may be deemed negligible.

In a series of eight papers [2:I-VIII], Southwell and collaborators have applied relaxation methods to various engineering problems. In some of these the method is applied to two-dimensional problems [2:III], and solutions are obtained of the equation

$$
\begin{equation*}
\nabla^{2} w=Z \tag{1}
\end{equation*}
$$

for any boundary on which $w$ is prescribed, $Z$ being a given function of $x$ and $y$. Here Prandtl's membrane analogy [3] is used, in which $w$ is the displacement of a membrane fastened at its boundary and acted upon by a transverse force $Z$. The membrane is replaced by a mesh of uniformly tensioned strings, the mesh lines forming squares or equilateral triangles, and the tension in the strings being proportional to the surface tension of the membrane. Initially the mesh is flat and the load $Z$ is taken by constraints acting at the mesh points. The constraints are relaxed one by one, just as in the framework, until the loads are all taken by the strings, and the resulting displacements of the mesh points are recorded. Evidently, as the meshlength decreases, the approximation of the mesh to the continuous membrane is improved.

In a recent paper [2:VIII], Southwell and Vaisey extend the membrane-net analogy to obtain solutions of Laplace's equation in the case when the normal gradient $\partial w / \partial \nu$, instead of the function $w$, is given at the boundary. Here $\partial w / \partial \nu$ is regarded as a line intensity of transverse loading applied round the boundary of the membrane. This load is then integrated and distributed statically to the strings which cross the boundary.

There is a mathematical treatment of the above problems, based on finite differences. For Laplace's problem of the first kind, in which the function is specified on the boundary, the finite difference equations have been derived [2:III]. They are in general identical with the equations obtained from the analogy of membranes and tensioned nets.

[^25]The corresponding mathematical treatment of Laplace's problem of the second kind, however, in which the normal gradient of the function is given on the boundary, was not given by Vaisey and Southwell, and it is this treatment with which this paper is concerned. The finite-difference equations are substantially different from those obtained in their paper.

The technigue presented here would seem to be particularly desirable for problems in which the boundary conditions involve both the value of the function and its normal gradient. For then the mechanical analogy becomes somewhat complicated, especially in the case of solids of revolution, for which the analogy of variably tensioned nets is not attractive.
2. The finite difference approach to the relaxation method. For square meshes (Fig. 1), we have the approximations

$$
\begin{gather*}
2 a \frac{\partial w}{\partial x} \div w_{1}-w_{3}, \quad a^{2} \frac{\partial^{2} w}{\partial x^{2}} \doteqdot w_{1}+w_{3}-2 w_{0} \\
a^{2} \nabla^{2} w \doteqdot w_{1}+w_{2}+w_{3}+w_{4}-4 w_{0} \tag{2}
\end{gather*}
$$

Similar formulac are easily obtainable for triangular meshes. The order of the error in these approximations is known in each case, and decreases with the mesh-length $a$.

There is a finite-difference equation of type (2) for every mesh point, and the solution of these equations gives a numerical value of $w$ at each mesh point.

At points close to a curved boundary, such as 0 in Fig. 2, we obtain the finitedifference equation as follows, in the case when $w$ is given on the boundary. The


Fig. 1.


Fig. 2.


Fig 3.
arm 01 passes outside the boundary, and a linear interpolation along 01 yields $w_{B}=w_{D}+\left(w_{1}-w_{0}\right) h$, from which we obtain for the finite-difference equation at 0

$$
\begin{equation*}
w_{2}+w_{3}+w_{4}+\frac{w_{B}}{h}-\left(3+\frac{1}{h}\right) w_{0}=0 . \tag{3}
\end{equation*}
$$

All these formulae are reproduced identically by the net analogy, but a more accurate formula than (3) given by Christopherson (4), and obtained by a parabolic interpolation, has not been deduced by analogy.

When $\partial w / \partial \nu$, rather than $w$, is given on the boundary, the same general equations hold as before, but there is a different procedure for points adjacent to the boundary. To write down the finite-difference equation at the point 0 (Fig. 3), we require $w_{A}$ and $w_{B}$. For $w_{A}$, we draw the normal APE and use the approximation

$$
\begin{equation*}
\mathrm{AE} \cdot\left(\frac{\partial w}{\partial \nu}\right)_{P}=w_{A}-w_{E} \tag{4}
\end{equation*}
$$

A linear interplation along $O G$ gives

$$
\mathrm{OG} \cdot w_{E}=\mathrm{OE} \cdot w_{G}+\mathrm{EG} \cdot w_{O},
$$

and elimination of $w_{E}$ between these two equations yields

$$
w_{A}=\frac{O E}{O G} w_{G}+\frac{O E}{O G} w_{O}+\mathrm{AE}\left(\frac{\partial w}{\partial \nu}\right)_{P} .
$$

Similarly, we obtain $w_{B}$ by drawing the normal $B Q F$.
The normal can be terminated on any convenient line. Thus in Fig. 3 we could produce AE to K on OD, to obtain $w_{A}$ in terms of $w_{o}, w_{D}$, instead of $w_{o}, w_{G}$. The shorter the normal, however, the more accurate is the approximation (4), so in this case $E$ is the best place to stop the normal. On the other hand, the normal from $B$ is continued to $F$, because a termination on the diagonal HO would involve the value $w_{H}$, itself fictitious, and the calculation would become more cumbersome.

The approximations employed to date have assumed $w$ to be linear along any line near the boundary. Improvements in accuracy can be made at the cost of additional labour. Thus, if the normal is stopped so that it is bisected by the boundary (Fig. 4), the formula


Fig. 4.

$$
\begin{equation*}
A C \cdot\left(\frac{\partial w}{\partial \nu}\right)_{B}=w_{A}-w_{C} \tag{5}
\end{equation*}
$$

assumes a parabolic variation of $w$. The point C in general no longer lies on a diagonal or mesh line, but its value can easily be found, by double interpolation, to the approximation of Eq. (5). This procedure yields a higher accuracy, but it is better, except when a very high accuracy is required, to use a linear variation together with a finer mesh.
3. Problem I. Let us find the function $w$, harmonic in the circle

$$
x^{2}+y^{2}-2 x-2 y+1=0,
$$

and satisfying the boundary condition

$$
\begin{equation*}
\frac{\partial w}{\partial \nu}=\frac{y-x}{x^{2}+y^{2}} \tag{6}
\end{equation*}
$$

This is one of the problems attacked by Vaisey and Southwell [2:VIII]. It has the exact solution

$$
w=\tan ^{-1} y / x
$$

whence we have a guide to the accuracy of our method. The solution is unique, except for an arbitrary constant, and we choose the constant so that $w=\pi / 4$ at the centre of the circle.

First, we take the mesh-length equal to the radius of the circle. The mesh contains only five points. Fig. 5 shows this mesh, the finite-difference equations, and the values of $w$ multiplied by 1000. External mesh points are denoted by open circles. For comparison, the exact values are entered under the approximate values.

When the mesh-length is halved, there are twelve fictitious mesh points and thirteen simultaneous equations. The solution is given in Fig. 6.


$$
\begin{align*}
& 2(5)+.5(2)+.5(4)-3(1)-733=0  \tag{1}\\
& 2(5)+.5(1)+.5(3)-3(2)+733=0  \tag{2}\\
& 2(5)+.5(2)+.5(4)-3(3)+2333=0  \tag{3}\\
& 2(5)+.5(1)+.5(4)-3(4)-2333=0  \tag{4}\\
& (1)+(2)+(3)+(4)-4(5)=0 \tag{5}
\end{align*}
$$

Fig. 5.
As the form of $\partial w / \partial v$ indicates, $w$ is skew-symmetrical about the line $x=y$. This feature was not utilized in construction of Figs. 5 and 6, but it is found useful as a labour-saving device in the case of a mesh-length of one-quarter of the radius. The


Fig. 6.
results in this case are shown in Fig. 7, and are very close to those for the exact solution, the error being greatest at points nearest the origin. This was to be expected, since $\partial w / \partial \nu$ changes rapidly in value across the line $x=y$ when $x^{2}+y^{2}$ is small. Comparable errors were found in the treatment by Vaisey and Southwell.


Fig. 7.
An interesting point is the oscillatory nature of the convergence of values of $w$. In the boundary-value-specified problem, values usually converge from one side only.
4. Problem II. Let us find the function $w$, harmonic in the same circle as before, but satisfying the boundary condition

$$
\begin{equation*}
\left(\frac{\partial}{\partial \nu}-\frac{1}{r}\right) w=\frac{1}{r^{2}}\left(y-x-r \tan ^{-1} \frac{y}{x}\right) \tag{7}
\end{equation*}
$$

where $r=\sqrt{x^{2}+y^{2}}$. The exact solution is again $w=\tan ^{-1} y / x$.
Here the boundary condition involves the value as well as the normal slope of
the function, but only a slight extension of the method of the previous problem is needed. As before, fictitious points are eliminated from the governing equation by means of the boundary condition.


Fig. 8.

In order to write down the finite-difference equation at 0 (Fig. 8), we need $w_{A}$ and $w_{B}$. As before, the normal AF yields

$$
\mathrm{AF} \cdot\left(\frac{\partial w}{\partial \nu}\right)_{Q}=w_{A}-w_{F}
$$

and linear interpolation on OE and AF gives

$$
\begin{aligned}
& \mathrm{OF} \cdot w_{F}=\mathrm{OF} \cdot w_{E}+\mathrm{FE} \cdot w_{0} \\
& \mathrm{AF} \cdot w_{Q}=\mathrm{AP} \cdot w_{F}+\mathrm{PF} \cdot w_{A} .
\end{aligned}
$$

From these three equations and the boundary condition (7), $w_{A}$ can be found in


Fig. 9.
terms of $w_{0}$ and $w_{E}$. A similar operation yields $w_{B}$, and hence the finite-difference equation for the point $O$ can be written down, and the problem solved.

As in the previous problem it was found that decrease of the mesh-length results in oscillatory convergence. The oscillation is rather more violent, but the final result shown in Fig. 9 is no less accurate, notwithstanding the additional interpolation.
5. Summary. In this paper solutions are obtained of Laplace's equation with boundary condition involving either the normal gradient only, or both the boundary value and the normal gradient. The need for a technique for problems of this kind has arisen in recent developments of the relaxation method. Two problems are solved, both have a known analytical solution, and good results are obtained in each case. The method used is independent of the analogy of tensioned nets, and can be applied without modification to problems for which analogies may be difficult to use.

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## -NOTES-

## THE METHOD OF STEEPEST DESCENT FOR NON-LINEAR MINIMIZATION PROBLEMS*

By HASKELL B. CURRY (Frankford Arsenal)

1. Introduction. The problem considered here is that of minimizing a function of $n$ real variables, $G\left(x_{1}, \cdots, x_{n}\right)$. The object is to find a practical method for evaluating, approximately at least, a stationary point for $G$.

This problem includes as a special case that of solving a set of simultaneous equations

$$
\begin{equation*}
f_{i}\left(x_{1}, \cdots, x_{n}\right)=0 \quad(i=1,2, \cdots, m) \tag{1}
\end{equation*}
$$

because the function

$$
\begin{equation*}
G\left(x_{1}, \cdots, x_{n}\right)=\sum_{k=1}^{m} f_{k}^{2} \tag{2}
\end{equation*}
$$

has a minimum at a solution of (1). It also includes that of determining the parameters $x_{1}, \cdots, x_{n}$ of a function $f\left(u ; x_{1}, \cdots, x_{n}\right)$ so as to get the best approximation, in a least square sense, to a function $F(u)$ for certain values of $u$; the $G$ in this case is of the form given by

$$
\begin{equation*}
G\left(x_{1}, \cdots, x_{n}\right)^{\dot{p}}=\sum_{k=1}^{p}\left[F\left(u_{k}\right)-f\left(u_{k} ; x_{1}, \cdots, x_{n}\right)\right]^{2} \tag{3}
\end{equation*}
$$

Certain engineering applications of the latter sort of problem arose in the work of the Engineering Research Section, Fire Control Design Division, at Frankford Arsenal. In these applications, the function $f\left(u ; x_{1}, \cdots, x_{n}\right)$ was sufficiently complicated so that the standard method for dealing with non linear least square problems ${ }^{1}$ failed to converge. Two techniques for dealing with this situation were developed by the section under the direction of J. G. Tappert. One of these was an original suggestion of my associate K. Levenberg. ${ }^{2}$ The second method is the subject of this note.

This method is not new. Levenberg found it set forth in a paper by Cauchy dated 1847. ${ }^{3}$ That it has become a standard procedure in analysis is clear from a recent paper by Courant. ${ }^{4}$ Nevertheless it does not appear to be well known to authorities on nu-

[^26]merical computation. It was used for the case of linear equations by G. Temple; ${ }^{5}$ but he gave no reference to Cauchy's work nor indeed to any previous use of the method. Accordingly there is room for an exposition of the method with emphasis on its practical aspects.

This note also contains an outline of a convergence proof. Cauchy stated that the process converged but gave no proof, at least in the paper cited. Temple's convergence proof applied only to the linear case. Courant (l.c.) gives references to papers dealing with the method; but some of these were not accessible to me under wartime conditions. The convergence proof, as outlined here, is an elementary one and gives a weak result.

The argument is, incidentally, capable of generalization to certain cases where there are infinitely many parameters, i.e., where $G$ is a function of a vector $\mathbf{x}$ belonging to a suitable abstract space. The essential point is that there be a vector function $\mathrm{H}(\mathrm{x})$, the gradient, such that for vectors $\mathrm{x}, \mathrm{y}$ and scalar $t$

$$
\frac{d}{d t}[G(\mathrm{x}+\mathrm{y})]=\mathrm{H}(\mathrm{x}+t \mathrm{y}) \cdot \mathrm{y}
$$

where the dot indicates a scalar product. ${ }^{6}$ Such generalizations will not be considered explicitly.
2. Explanation of the method. The letters $\mathbf{x}, \mathbf{z}$ will be used to stand for the $n$-tuples (vectors) $\left(x_{1}, \cdots, x_{n}\right)$ and ( $z_{1}, \cdots, z_{n}$ ) respectively. It will be convenient also to think of the vector $x$ as a point and $z$ as a set of direction numbers of a direction, viz. the direction $\mathbf{z}$, emanating from $\mathbf{x}$. Superscripts will be used systematically to distinguish different points and their corresponding directions.

Let us suppose, then, that we start at a point $x^{0}$ and determine the direction in which $G$ decreases most rapidly. This direction is given by $z_{i}=-\lambda \partial G / \partial x_{i}$ or, in vector form, $z^{0}=-\lambda \operatorname{grad} G$, where $\lambda$ is an arbitrary positive factor of proportionality. (In practice we should either take $\lambda=1$ or choose $\lambda$ so that the vector $z$ is of unit length.) Then the function $g(t)=G\left(\mathbf{x}^{0}+t z^{0}\right)$ has a negative derivative at $t=0$. It will therefore be possible to find a $t>0$ such that

$$
\begin{equation*}
g(t)<g(0) \tag{4}
\end{equation*}
$$

With such a $t$ we can take $x^{1}=x^{0}+i z^{0}$ as a new starting point and continue. We should then have a sequence of points $\mathbf{x}^{0}, \mathbf{x}^{1}, \mathbf{x}^{2}, \cdots$ such that $G\left(\mathbf{x}^{k+1}\right)<G\left(\mathbf{x}^{k}\right)$. Under suitable restrictions (to be considered later) the sequence will attain or converge to a stationary point of $G$.

The determination of $t$ can be accomplished by trial. If no other indication is availble we can take as first trial value the intercept of the tangent to the curve $y=g(t)$ on the $t$-axis. ${ }^{7}$ If this fails to satisfy (4), it is too large; we can then take half of it, and so on. In this process we can draw a rough graph of $g(t)$, and after a few trials it is

[^27]usually possible to locate a $t$ which is at or near a minimum of $g(t)$. Experience will presumably disclose many ways to shorten the process in individual cases.

If we take for $t$ precisely the smallest positive root of

$$
\begin{equation*}
g^{\prime}(t)=0 \tag{5}
\end{equation*}
$$

the process has the following geometrical interpretation. Starting at $x^{0}$, we determine the direction in which the surface

$$
\begin{equation*}
y=G(\mathbf{x})=G\left(x_{1}, \cdots, x_{n}\right) \tag{6}
\end{equation*}
$$

is descending most rapidly. We continue in that direction until we find ourselves going along a contour (i.e. a horizontal section of the surface). Then we stop, take a new direction of steepest descent, and so continue. Since the direction of steepest descent is always normal to the contour it follows that the directions $\mathbf{z}^{k}$ and $\mathbf{z}^{k+1}$ are at right angles. ${ }^{8}$ This is important in the convergence proof.
3. Proof of Convergence. Let us suppose now that $G\left(x_{1}, \cdots, x_{n}\right)$ is defined and has continuous first partial derivatives at all points within or on the boundary of a region $S$. Let $\mathbf{x}^{0}$ be a point within $S$. Let $C$ be the broken line path starting at $\mathbf{x}^{0}$ and going in the direction of steepest descent at $x^{0}$ until it reaches either the boundary of $S$ or the next approximation $x^{1}$ determined as in $\S 3$ with $t$ the least positive root of (5); in the latter case the broken line goes in the direction of steepest descent at $\mathbf{x}^{1}$ until it reaches the boundary of $S$ or the next approximation $x^{2}$ determined in the same way; and so on. Then $G$ is monotone decreasing along $C$. There are three possibilities: (1) The path $C$ may run into the boundary of $S$. (2) The path $C$ may terminate at a point where the direction of steepest descent does not exist, i.e. at a stationary point of $G$. (3) The process may continue indefinitely. The first possibility will certainly be excluded if the value of $G$ at $x^{0}$ is less than at any point on the boundary of $S$. If the second possibility occurs the case is trivial. I shall make the limitation just stated in regard to $G\left(\mathbf{x}^{0}\right)$ and shall suppose that the process continues indefinitely.

Under these presuppositions let $x^{\infty}$ be a limit point of $x^{0}, x^{1}, \cdots$. Then it is clear that

$$
\begin{equation*}
G\left(\mathrm{x}^{\infty}\right)<G\left(\mathrm{x}^{k}\right) \quad(k=0,1,2, \cdots) . \tag{7}
\end{equation*}
$$

It will now be shown that $x^{\infty}$ is a stationary point of $G$.
Let us suppose the contrary. We write $\mathrm{H}(\mathrm{x})=\operatorname{grad} G, h(\mathrm{x})=|\mathrm{H}(\mathrm{x})|$, and let $\mathbf{z}(\mathrm{x})$ be a unit vector; thus

$$
\begin{equation*}
\mathrm{H}(\mathrm{x})=-h(\mathrm{x}) \mathrm{z}(\mathrm{x}) . \tag{8}
\end{equation*}
$$

According to the supposition, $h\left(\mathbf{x}^{\infty}\right) \neq 0$. Hence it will be possible to find a spherical neighborhood $U$ of $x^{\infty}$ such that for $x$ in $U$,

$$
\left|H(x)-H\left(x^{\infty}\right)\right|<\epsilon h\left(\mathbf{x}^{\infty}\right) .
$$

Then it will follow that $\left|h(x)-h\left(x^{\infty}\right)\right|<\epsilon h\left(x^{\infty}\right),\left|z-z^{\infty}\right|<2 \epsilon$. Hence, from the fact that $\mathbf{z}^{k}$ and $\mathbf{z}^{k+1}$ are at right angles, one can conclude that if $\mathbf{x}^{k}$ is in $U, \mathbf{x}^{k+1}$ is certainly not in $U$ (provided $\epsilon$ is not too large).

Next, let $K$ be the conical sector of $U$ for which

$$
\cos \theta=\frac{x-x^{\infty}}{\left|x-x^{\infty}\right|} \cdot z^{\infty}>\epsilon
$$

[^28]Then it may be shown, by reasonably straight-forward methods, that for x in $K$,

$$
\begin{equation*}
G(\mathrm{x})<G\left(\mathbf{x}^{\infty}\right) . \tag{9}
\end{equation*}
$$

Let $V$ be a subneighborhood of $U$ such that for x in $V$ the ray in the direction $z$ (as given by (8)) from x intersects $K$. Such a $V$ exists if $\epsilon<\frac{1}{3}$. Since $\mathrm{x}^{\infty}$ is a limit point, there exists an $x^{n}$ in $V$. Then the ray in the direction $z^{n}$ from $x^{n}$ will have a point $y$ in $K$. This $y$ cannot be beyond $x^{n+1}$, since $x^{n+1}$ is not in $U$ and $U$ is convex. Hence, by the monotonic character of $G$ on $C$ and by (9), $G\left(\mathrm{x}^{k+1}\right)<G(\mathrm{y})<G\left(\mathrm{x}^{\infty}\right)$, which contradicts (7). This contradiction came from the assumption that $h\left(\mathrm{x}^{\infty}\right) \neq 0$.

The following example shows that we cannot expect a better result without further restrictions on $G$. Let $G(x, y)=0$ on the unit circle and $G(x, y)>0$ elsewhere. Outside the unit circle let the surface have a spiral gully making infinitely many turns about the circle. Then the path $C$ will evidently follow the gully and have all points of the unit circle as limit points.

In a practical problem however we often know in advance that there is a unique minimum of $G$ within $S$; in these cases convergence is assured. If $G$ is given by (2) and the Jacobian of the $f$ 's does not vanish in $S$, then every stationary point of $G$ is a solution of (1); if there is only one such solution the process converges to it.
4. Concluding Remarks. In regard to the practical aspects of the method the following points are to be noted: (1) It does not require any calculation of second derivatives. This is important for the application mentioned, where these second derivatives are numerous and complicated. (2) It involves only direct calculations of $G$ and its first derivatives. (3) The approach to the limit, if any, is along a path $C$ consisting of straight line segments, adjacent segments being approximately at right angles.

A comparison with Levenberg's method in regard to these three respects is now in order. In the first respect the two methods are alike. In the second respect Levenberg's method is more complicated because each stage requires the solution of a set of "normal" equations, as in the traditional method of least squares. In the third respect Levenberg's method is like the present one in that it involves approach along a broken path $C$; but the individual pieces of $C$ are curved. There is evidence that in practical problems these curves follow the natural valleys of the surface, so that each step brings us further toward the goal. As to which of these opposing characteristics is the more important is not yet settled. ${ }^{\text {a }}$

Another point is that the process is not invariant under certain elementary transformations, e.g., a change of scale of one or more of the $x_{i}$. Thus, if the surface (6) (for $n=2$ ) is a hemispherical bowl, the direction of steepest descent is along the meridian, and the minimum is reached in one step. If, however, the scale on the $x_{i}$-axis is changed so that the surface becomes ellipsoidal, the direction of steepest descent is no longer directed toward the minimum. The suggestions inherent in this lack of invariance have not yet been fully worked out.

[^29]
## ON THE BENDING OF A CLAMPED PLATE*

By A. WEINSTEIN (University of Toronto) and D. H. ROCK (Rhode Island State College)

The present paper contains an application of a recently developed variational method ${ }^{1}$ to the boundary value problem of the bending of a clamped plate of arbitrary shape. It will be shown that this problem can be linked to the simpler problem of the equilibrium of a membrane by a chain of intermediate problems, which can be solved explicitly and in finite form in terms of the membrane problem. In the intermediate problems, the deflection converges uniformly in the domain of the plate (including the boundary) to the deflection of the clamped plate, and the derivatives of all orders of the deflection converge uniformly in every domain completely interior to the plate. (In the Ritz method, not even the convergence of the slopes can be guaranteed. ${ }^{2}$ ) The method yields numerical results for plates of all shapes for which the membrane problem (which we shall call the base problem) admits an explicit solution. As an example we shall consider a clamped square plate under a uniform load. This problem has been the object of numerous investigations, ${ }^{3}$ some of which are theoretical, while others are purely numerical, use infinite simple and double series, and operate with an infinite number of linear equations and an infinite number of unknowns. ${ }^{4}$ An inspection of the general formulae derived in the present paper, formulae which become simple in numerical applications, would show how some of the numerical methods might be rendered rigorous. ${ }^{5}$ The convergence of higher derivatives is of great practical interest for the approximate computation of the stresses. (Cf. Handbuch der Physik, Springer, Berlin, Vol. VI, 1928, pp. 220-221.)

Let us denote the domain of a clamped plate by $S$ and its boundary by $C$. The deflection $w(x, y)$ corresponding to a load $q(x, y)$ and to a flexural rigidity $D$ satisfies the differential equation

$$
\begin{equation*}
\Delta \Delta w=q / D \tag{1}
\end{equation*}
$$

with the boundary conditions

$$
\begin{equation*}
w=0, \tag{2}
\end{equation*}
$$

$$
\begin{equation*}
d w / d n=0 \tag{3}
\end{equation*}
$$

on $C$. It is well known that $w$ is the solution of the variational problem $V P_{\text {, }}$

$$
J(w) \equiv \iint_{S}\left[(\Delta w)^{2}-2 \bar{q} w\right] d x d y=\min
$$

[^30]where $\bar{q}=q / D$ and the boundary conditions (2) and (3) are in effect. By withdrawing the condition $d w / d n=0$, we obtain the variational base problem $V P_{0}: J\left(w_{0}\right)=$ min., with the condition $w_{0}=0$ on $C$. The corresponding differential equations problem $D P_{0}$ is: $\Delta \Delta w_{0}=\bar{q}$, with the boundary conditions $w_{0}=0$ and $\Delta w_{0}=0$ on $C$, the latter condition being a natural boundary condition which is automatically satisfied by a solution of $V P_{0}$. It is well known that $D P_{0}$ can be solved in terms of the problem of the equilibrium of a membrane. In fact, putting $\Delta w_{0}=f_{0}$ in $D P_{0}\left(w_{0}=0\right.$ on $\left.C\right)$, we have $\Delta f_{0}=\bar{q}$ in the domain $S$ and $f_{0}=0$ on $C .{ }^{6}$ Let us denote the solution of the equation $\Delta w_{0}=f_{0}$ with the boundary condition $w_{0}=0$ by $w_{0}=G f_{0}$. Thus we have $f_{0}=G \bar{q}$, whence $w_{0}=G G \bar{q}$. (The formula $w_{0}=G f_{0}$ can be written explicitly in the form
$$
w_{0}=-\iint_{s} g(x, y, \xi, \eta) f_{0}(\xi, \eta) d \xi d \eta
$$
where $g(x, y, \xi, \eta)$ is the Green's function for the domain $S$.)
We now link $V P_{0}$ to $V P$ by a chain of intermediate variational problems $V P_{1}, V P_{2}, \cdots$ introduced in the following way. We let $p_{1}(x, y), p_{2}(x, y), \cdots$ denote a complete (but not necessarily orthogonal) sequence of linearly independent harmonic functions in $S$. (It has been shown, l.c., ${ }^{1}$ how a sequence of this kind can be derived from the solutions of the problem of a vibrating membrane for any domain $S$.)
$V P_{m}(m=1,2, \cdots)$ is then defined as the problem of finding the solution $w_{m}$ of $J\left(w_{m}\right)=$ min., with the boundary conditions
$$
w_{m}=0 \text { on } C, \quad \int_{C} p_{k} \frac{d w_{m}}{d n} d s=0, \quad k=1,2, \cdots, m .
$$

By Green's formula these conditions can be replaced by the conditions

$$
\begin{equation*}
w_{m}=0 \text { on } C, \quad\left(p_{k}, \Delta w_{m}\right)=0, \quad k=1,2, \cdots, m, \tag{4}
\end{equation*}
$$

where

$$
\left(p_{k}, \Delta w_{m}\right)=\iint_{s} p_{k} \Delta w_{m} d x d y .
$$

(Similar notations for the "scalar product" of two function, like $p_{k}$ and $\Delta w_{m}$, will be used throughout this paper.) The corresponding differential problem $D P_{m}$ is given by the differential equation

$$
\begin{equation*}
\Delta \Delta w_{m}=\bar{q} \tag{5}
\end{equation*}
$$

with the conditions (4) and the natural boundary condition

$$
\begin{equation*}
\Delta w_{m}=a_{m 1} p_{1}+a_{m 2} p_{2}+\cdots+a_{m m} p_{m} \text { on } C, \tag{6}
\end{equation*}
$$

where $a_{m 1}, a_{m 2}, \cdots, a_{m m}$ are constants to be determined. The solutions of $D P_{m}$ can be easily obtained in terms of solutions of the membrane problem already used to solve $D P_{0}$. In fact, putting $\Delta w_{m}=f_{m}\left(w_{m}=0\right.$ on $\left.C\right)$, we have, in our notation, $w_{m}=G f_{m}$. Also, we obtain from (5),

$$
\begin{equation*}
\Delta f_{m}=\bar{q} \tag{7}
\end{equation*}
$$

[^31]and from the boundary condition (6),
\[

$$
\begin{equation*}
f_{m}=a_{m 1} p_{1}+a_{m 2} p_{2}+\cdots+a_{m m} p_{m} \tag{8}
\end{equation*}
$$

\]

These can be written as follows:

$$
\begin{array}{r}
\Delta\left(f_{m}-\sum_{i=1}^{m} a_{m i} p_{i}\right)=\bar{q} \text { in } S \\
f_{m}-\sum_{i=1}^{m} a_{m i} p_{i}=0 \text { on } C .
\end{array}
$$

Therefore we have in $S$

$$
\begin{equation*}
f_{m}-\sum_{i=1}^{m} a_{m i} p_{i}=G \bar{q} \tag{9}
\end{equation*}
$$

and since $w_{m}=G f_{m}$, it follows that

$$
\begin{equation*}
w_{m}=G G \bar{q}+\sum_{i=1}^{m} a_{m i} G p_{i} \tag{10}
\end{equation*}
$$

where $G G \bar{q}=w_{0}$ is the solution of the base problem.
The conditions ( $p_{k}, \Delta w_{m}$ ) $=0$ yield, in view of (9), the following system of $m$ linear equations for the $m$ constants $a_{m 1}, a_{m 2}, \cdots, a_{m m}$ :

$$
\begin{equation*}
\sum_{i=1}^{m} a_{m i}\left(p_{i}, p_{k}\right)=-\left(\bar{q}, G p_{k}\right), \quad k=1,2, \cdots, m, \tag{11}
\end{equation*}
$$

which can be solved, since their determinant is Gram's determinant of the independent functions $p_{1}, p_{2}, \cdots, p_{m}$, and is different from zero.

In another paper, based on a previously developed method for the computation of frequencies and buckling loads, ${ }^{7}$ it will be proved that the approximate solutions $w_{m}$ and their first derivatives converge to the deflection and slopes of the clamped plate. Here we shall apply our formulae to the case of a uniformly loaded square plate. The domain $S$ will be defined by the inequalities $|x| \leqq \pi / 2,|y| \leqq \pi / 2$. We put $\bar{q}=q / D=1$. Since the deflection of a uniformly loaded square plate is symmetrical with respect to the coordinate axes, we may use a sequence of even harmonic functions $p_{i}(x, y)$ as given by (12) below. ${ }^{8}$ All computations can be performed without the use of Green's function for the square. The deflection $w_{0}$ of the supported plate is given by the well known formulae of Navier.

Calculation of $w_{m}$ for the uniformly loaded square plate.
We use the set of functions

$$
\begin{equation*}
p_{i}(x, y)=\frac{\cosh \alpha_{i} x \cos \alpha_{i} y+\cos \alpha_{i} x \cosh \alpha_{i} y}{\cosh \left(\alpha_{i} \pi / 2\right)}, \quad\left(\alpha_{i}=2 i-1\right) \tag{12}
\end{equation*}
$$

[^32]and denote $G p_{i}$ by $v_{i}$. Then, by the definition of $G$, we have $\Delta v_{i}=\Delta G p_{i}=p_{i}$ in $S$, with $v_{i}=0$ on $C$.

If we set $v_{i}=u_{1}+u_{2}$, where

$$
u_{1}=X(x) \frac{\cos \alpha_{i} y}{\cosh \left(\alpha_{i} \pi / 2\right)}, \quad u_{2}=Y(y) \frac{\cos \alpha_{i} x}{\cosh \left(\alpha_{i} \pi / 2\right)}
$$

then $X(x)$ and $Y(y)$ satisfy the differential equations

$$
X^{\prime \prime}-\alpha_{i}^{2} X=\cosh \alpha_{i} x, \quad Y^{\prime \prime}-\alpha_{i}^{2} Y=\cosh \alpha_{i} y
$$

with the boundary conditions

$$
X( \pm \pi / 2)=0, \quad Y( \pm \pi / 2)=0
$$

The general solution for $X(x)$ is

$$
X(x)=\frac{1}{2 \alpha_{i}} x \sinh \alpha_{i} x+A \cosh \alpha_{i} x+B \sinh \alpha_{i} x
$$

where $A$ and $B$ are determined by the boundary conditions. Hence

$$
X=\frac{1}{2 \alpha_{i}}\left[x \sinh \alpha_{i} x-\frac{\pi}{2} \tanh \frac{1}{2} \alpha_{i} \pi \cosh \alpha_{i} x\right]
$$

The solution for $Y(y)$ is given by a similar expression, so finally

$$
\begin{align*}
& y_{i}=\frac{1}{2 \alpha_{i} \cosh \left(\alpha_{i} \pi / 2\right)}\left\{\left[x \sinh \alpha_{i} x-\frac{\pi}{2} \tanh \frac{1}{2} \alpha_{i} \pi \cosh \alpha_{i} x\right] \cos \alpha_{i} y\right. \\
&\left.+\left[y \sinh \alpha_{i} y-\frac{\pi}{2} \tanh \frac{1}{2} \alpha_{i} \pi \cosh \alpha_{i} y\right] \cos \alpha_{i} x\right\} \tag{13}
\end{align*}
$$

Using (13), we obtain the general formula for $\left(\bar{q}, G p_{i}\right)=\left(1, v_{i}\right)$,

$$
\begin{equation*}
\left(1, v_{i}\right)=\frac{4 \sin \left(\alpha_{i} \pi / 2\right)}{\alpha^{3}}\left[\frac{\pi}{2} \operatorname{sech}^{2} \frac{1}{2} \alpha_{i} \pi-\frac{1}{\alpha_{i}} \tanh \frac{1}{2} \alpha_{i} \pi\right] . \tag{14}
\end{equation*}
$$

For $\left(p_{i}, p_{k}\right)$ we have

$$
\begin{align*}
\left(p_{i}, p_{k}\right) & =\frac{8 \alpha_{i} \alpha_{k}(-)^{i+k}}{\left(\alpha_{i}^{2}+\alpha_{k}^{2}\right)^{2}}, \quad i \neq k \\
& =\pi\left[\frac{\pi}{2} \operatorname{sech}^{2} \frac{1}{2} \alpha_{i} \pi+\frac{1}{\alpha_{i}} \tanh \frac{1}{2} \alpha_{i} \pi\right]+\frac{2}{\alpha_{i}^{2}}, \quad i=k \tag{15}
\end{align*}
$$

From (14) we find that

$$
\begin{gathered}
\left(1, v_{1}\right)=-2.670644, \quad\left(1, v_{2}\right)=0.049299, \quad\left(1, v_{3}\right)=-0.006400 \\
\left(1, v_{4}\right)=0.001666, \text { etc. }
\end{gathered}
$$

and from (15) we get the following table for $\left(p_{i}, p_{k}\right),(i, k=1,2,3,4)$ :

| $k$ | 1 | 2 | 3 | 4 |
| :---: | :---: | ---: | ---: | ---: |
| 1 | 5.665118 | -0.240000 | 0.059172 | -0.022400 |
| 2 |  | 1.270836 | -0.103806 | 0.049941 |
| 3 |  |  | 0.708321 | -0.051132 |
| 4 |  |  |  | 0.489615 |

The equations for the determination of $a_{m i}$ are then

$$
\left.\begin{array}{rr}
5.665118 a_{11}=2.670644, \text { for } a_{11} \\
\begin{array}{r}
5.665118 a_{21}-0.240000 a_{22}
\end{array}=2.670644 \\
-0.240000 a_{21}+1.270836 a_{22}= & -0.049299
\end{array}\right\} \text { for } a_{21} \text { and } a_{22},
$$

These yield the successive values

$$
\begin{aligned}
& a_{11}=0.471419, \\
& a_{21}=0.473564, \quad a_{22}=0.050641 \text {, } \\
& a_{31}=0.473728, \quad a_{32}=0.048762, \quad a_{33}=-0.023392 \text {, } \\
& a_{41}=0.473749, \quad a_{42}=0.048394, \quad a_{43}=-0.022656, \quad a_{44}=0.010970,
\end{aligned}
$$

Then, since the solution for the simply supported plate is given by

$$
w_{0}(x, y)=\frac{16}{\pi^{2}} \sum_{m} \sum_{n} \frac{\sin m(x+\pi / 2) \sin n(y+\pi / 2)}{m n\left(m^{2}+n^{2}\right)^{2}}, \quad(m, n=1,3,5, \cdots)
$$

the successive approximations to the deflection are:

$$
\begin{aligned}
& w_{1}=w_{0}+0.471419 v_{1} \\
& w_{2}=w_{0}+0.473564 v_{1}+0.050641 v_{2}
\end{aligned}
$$

The maximum deflection, which occurs at the center of the plate, is found to be 0.123342 when $w_{2}$ is used. The next approximation affects only the fourth significant figure.

A calculation of the normal derivative of $w_{m}(m=0,1,2,3,4)$ along $x=\pi / 2$ for values of $y$ from 0 to $\pi / 2$ at intervals of $\pi / 16$ yields:

| $m$ | $y=0$ | $\pi / 16$ | $2 \pi / 16$ | $3 \pi / 16$ | $4 \pi / 16$ | $5 \pi / 16$ | $6 \pi / 16$ | $7 \pi / 16$ | $\pi / 2$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | -.41795 | -.41087 | -.38954 | -.35409 | -.30519 | -.24382 | -.17095 | -.08839 | 0 |
| 1 | -.00763 | -.00686 | -.00445 | -.00062 | +.00443 | +.00912 | +.01221 | +.01100 | 0 |
| 2 | +.00197 | +.00122 | -.00047 | -.00189 | -.00208 | -.00060 | +.00216 | +.00355 | 0 |
| 3 | -.00053 | -.00019 | +.00044 | +.00055 | -.00020 | -.00093 | -.00018 | +.00151 | 0 |
| 4 | +.00029 | -.00003 | -.00031 | +.00008 | +.00036 | -.00021 | -.00033 | +.00066 | 0 |

The maximum value of the slope in the interior of the plate is found to be about -0.122 ; this occurs at $x=5 \pi / 16, y=0$. A comparison with the maximum deviation of the normal derivative along the edge for $m=4$ shows that the latter is less than $1 \%$ of the maximum slope in the interior of the plate.

# THE DYNAMICS OF A DIFFUSING GAS* 

By HENRI PUTMAN (Universite Laval, QuEbec)

By use of a hydrodynamical approach, Stefan ${ }^{1}$ derived the following equation for the diffusion of two gases:

$$
\begin{equation*}
\rho_{1} \xi_{1}=-\frac{\partial p_{1}}{\partial x}-A_{12} \rho_{1} \rho_{2}\left(u_{1}-u_{2}\right) \tag{1}
\end{equation*}
$$

Here $p_{1}$ is the partial pressure of the first gas, $\rho_{1}$ is its density, $u_{1}, \xi_{1}$ are respectively the $x$-components of the velocity and acceleration of one of its particles, $u_{2}, \rho_{2}$ refer to the second gas, and $A_{12}$ is a constant. There are two other equations similar to (1) corresponding to the $y$ and $z$-components, and a further set of three equations for the second gas.

Equation (1) is a simplified form of Maxwell's equation of diffusion. ${ }^{2}$ It states that there acts on a particle of the first gas a force due to the pressure gradient of the first gas, and a force proportional to the difference of the velocities of the two gases.

The ordinary equation of diffusion, or Fick's law, ${ }^{3}$ was deduced by Stefan ${ }^{1}$ from (1) and the equation of continuity by assuming that $\xi_{1}$ was negligible.

We shall now assume that $u_{2}$ is negligible. This is the case in which the second gas is immobile and the first gas diffuses through it. Some problems involving two gases can be reduced to just such a problem. ${ }^{4}$

If now in (1) we set $u_{1}=v, \xi_{1}=d v / d t, A_{12} \rho_{2}=a, p_{1}=p, \rho_{1}=\rho$, we obtain

$$
\frac{d v}{d t}=-\frac{1}{\rho} \frac{\partial p}{\partial x}-a v
$$

The corresponding three dimensional form when there is present in addition a body force per unit mass represented by the vector $\mathbf{F}$ is

$$
\begin{equation*}
\frac{d \mathbf{v}}{d t}=\mathbf{F}-\frac{\nabla p}{\rho}-a \mathbf{v}, \tag{2}
\end{equation*}
$$

where $v$ is the velocity vector. This equation is the equation of motion of a viscous fluid with the viscosity terms replaced by a force proportional to the velocity.

We shall now deduce some additional equations which are consequences of (2).
If we multiply (2) scalarly be an arbitrary virtual displacement $\delta e$, we obtain, since $\delta \equiv \nabla \cdot \delta$ e,

$$
\begin{equation*}
\frac{d v}{d t} \cdot \delta e=F \cdot \delta e-\frac{\delta p}{\rho}-a v \cdot \delta e \tag{3}
\end{equation*}
$$

[^33]When the expression $\mathbf{v} \cdot d(\delta \mathrm{e}) / d \iota=\delta\left(\frac{1}{2} v^{2}\right)$ is added to both sides of (3), we obtain

$$
\begin{equation*}
\frac{d}{d t}(\nabla \cdot \delta \mathrm{e})=\mathrm{F} \cdot \delta \mathrm{e}-\frac{\delta p}{\rho}+\delta\left(\frac{1}{2} v^{2}\right)-a \mathrm{~V} \cdot \delta \mathrm{e} . \tag{4}
\end{equation*}
$$

If now we set $\delta e=d e$, where $d e$ follows the natural motion of the system, and assume that $F$ has a potential $U$, then from (3)

$$
\begin{equation*}
d\left(\frac{1}{2} v^{2}\right)=d U-\frac{d p}{\rho}-a \nabla \cdot d e \tag{5}
\end{equation*}
$$

or

$$
\begin{equation*}
d\left(\frac{1}{2} v^{2}\right)+a v^{2} d t=d U-\frac{d p}{\rho} \tag{6}
\end{equation*}
$$

If the temperature of the gas is constant, we have

$$
p=K \rho, \quad \int \frac{d p}{\rho}=K \ln p
$$

$K$ being a constant. Integration of (6) then yields

$$
\begin{equation*}
\frac{1}{2} v^{2}+K \ln p+a \int v \cdot d e=\text { const. } \tag{7}
\end{equation*}
$$

When the only body force is due to gravity, and the $x$-axis is vertically downward, $U=g x$ and (7) becomes

$$
\begin{equation*}
x-\frac{K}{g} \ln p-\frac{v^{2}}{2 g}-\frac{a}{g} \int v \cdot d e=\text { const. } \tag{8}
\end{equation*}
$$

We now return to (3). If there is a straight or curved axis of symmetry such that, if $s$ is the arc length of this axis, $v, p, \rho$ are functions of $s, t$ only, and if we set $\delta \mathrm{e}=\mathrm{u}_{\varepsilon} \delta s$, where $u_{s}$ is a unit vector tangent to the axis of symmetry, then (3) becomes

$$
\frac{d \mathrm{v}}{d t} \cdot \mathrm{u}_{s} \delta s=g \delta x-\frac{\delta p}{\rho}-a \mathrm{v} \cdot \mathrm{u}_{\mathrm{s}} \delta s
$$

or, since $v \cdot u_{s}=v$,

$$
\begin{equation*}
\frac{d v}{d t}=g \frac{\partial x}{\partial s}-\frac{1}{\rho} \frac{\partial p}{\partial s}-a v . \tag{9}
\end{equation*}
$$

Now

$$
d v=\frac{\partial v}{\partial s} d s+\frac{\partial v}{\partial t} d t, \quad \frac{d v}{d t}=\frac{\partial v}{\partial s} v+\frac{\partial v}{\partial t}
$$

whence (9) takes the form

$$
\begin{equation*}
\frac{\partial x}{\partial s}-\frac{1}{\rho g} \frac{\partial p}{\partial s}=\frac{\partial}{\partial s}\left(\frac{v^{2}}{2 g}\right)+\frac{1}{g} \frac{\partial v}{\partial t}+\frac{a}{g} v . \tag{10}
\end{equation*}
$$

The equation of continuity is

$$
\begin{equation*}
\Omega \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial s}(\rho \Omega v)=0 \tag{11}
\end{equation*}
$$

where $\Omega$ is the area of the cross section. When the gas is at constant temperature, $p=K \rho$, and (10) and (11) become

$$
\begin{gather*}
\frac{\partial}{\partial s}\left(x-\frac{K}{g} \ln p\right)=\frac{\partial}{\partial s}\left(\frac{v^{2}}{2 g}\right)+\frac{1}{g} \frac{\partial v}{\partial t}+\frac{a}{g} v,  \tag{12}\\
\Omega \frac{\partial p}{\partial t}+\frac{\partial}{\partial s}(p \Omega v)=0 \tag{13}
\end{gather*}
$$

If the flow is steady, $\partial v / \partial \iota=0$ and (12) becomes

$$
\begin{equation*}
x_{1}-x_{0}+\frac{K}{g} \ln \frac{p_{0}}{p_{1}}=\frac{v_{1}^{2}}{2 g}-\frac{v_{0}^{2}}{2 g}+\frac{a}{g} \int_{s_{0}}^{s_{1}} v d s \tag{14}
\end{equation*}
$$

where the subscripts zero and one refer to two cross sections, both viewed at the same instant. We set $P=\frac{1}{2}\left(p_{0}+p_{1}\right), \pi=p_{0}-p_{1}, \alpha=\frac{1}{2} \pi / P$, whence

$$
\begin{aligned}
p_{0} & =P+\frac{1}{2} \pi=P(1+\alpha), \quad p_{1}=P-\frac{1}{2} \pi=P(1-\alpha) \\
\ln \frac{p_{0}}{p_{1}} & =\ln \frac{1+\alpha}{1-\alpha}=2\left(\alpha+\frac{1}{3} \alpha^{3}+\cdots\right)=2 \alpha\left(1+\frac{1}{3} \alpha^{2}+\cdots\right)
\end{aligned}
$$

When the difference of pressure at these two cross sections is small, $\frac{1}{3} \alpha^{2}$ is much smaller than 1 and we have approximately $\ln \left(p_{0} / p_{1}\right)=2 \alpha=\pi / P$. If $w$ is the specific weight at pressure $P$, then

$$
P=K \rho=\frac{K w}{g}, \quad \frac{K}{g \rho}=\frac{1}{w}, \quad \frac{K}{g} \ln \frac{p_{0}}{p_{1}}=\frac{p_{0}-p_{1}}{w},
$$

and (14) becomes

$$
\begin{equation*}
x_{1}-x_{0}+\frac{p_{0}-p_{1}}{w}=\frac{v_{1}^{2}}{2 g}-\frac{v_{0}^{2}}{2 g}+\frac{a}{g} \int_{s_{0}}^{s_{1}} v d s \tag{15}
\end{equation*}
$$

Let us return once more to (2), and operate on it with $\nabla \times$ and $\nabla \cdot$, to obtain

$$
\begin{align*}
& \frac{d}{d t}\left(\frac{\omega}{\rho}\right)=\left(\frac{\omega}{\rho} \cdot \Delta\right) \nabla-\frac{a}{\rho} \omega  \tag{16}\\
& \frac{d \theta}{d t}=\Delta \psi-a \theta-\nabla_{1} \cdot\left[\left(\mathrm{v}_{1} \cdot \nabla\right) \mathrm{v}\right] \tag{17}
\end{align*}
$$

where $\omega=\nabla \times \nabla, \theta=\nabla \cdot \nabla, \psi=U-p / p, p$ is assumed to be a function of $p$ only, and

$$
\nabla_{1} \cdot\left[\left(\mathrm{v}_{1} \cdot \nabla\right) \mathrm{v}\right]=\theta^{2}-2 \sum \mathrm{k} \cdot\left(\frac{\partial \mathrm{v}}{\partial x} \times \frac{\partial \mathrm{v}}{\partial y}\right)
$$

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ being unit vectors along the $x, y, z$-axes, respectively. The equation of continuity is

$$
\begin{equation*}
\frac{d \rho}{d t}=-\rho \theta \tag{18}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}=-\nabla \cdot(\rho \nabla) \tag{19}
\end{equation*}
$$

If $a$ is not a constant, but a function of $\mathrm{e}(x, y, z)$ for example, we must add the term $-(\nabla \times \nabla a) / \rho$ to the second term in (16).

We shall now examine the propagation of discontinuities in the boundary conditions, in the simple case of one dimension without gravity. Equations (2) and (18) become

$$
\begin{array}{r}
\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x} v+\frac{1}{\rho} \frac{\partial \rho}{\partial x}+a v=0 \\
\frac{\partial \rho}{\partial t}+\frac{\partial \rho}{\partial x} v+\rho \frac{\partial v}{\partial x}=0 . \tag{21}
\end{array}
$$

If the temperature is constant, $p=K \rho$ and these equations become

$$
\begin{align*}
& \frac{\partial v}{\partial t}+v \frac{\partial v}{\partial x}+\frac{K}{\rho} \frac{\partial \rho}{\partial x}=-a v  \tag{22}\\
& \frac{\partial \rho}{\partial l}+v \frac{\partial \rho}{\partial x}+\rho \frac{\partial v}{\partial x}=0 \tag{23}
\end{align*}
$$

which are a system of two simultaneous quasilinear equations of the first order. These equations are usually referred to as Hamburger's equations in two dependent variables. By standard procedures, ${ }^{5}$ the ordinary differential equations for the characteristics are found in the form

$$
\begin{array}{ll}
\frac{d t}{d x}=\left(v+K^{1 / 2}\right)^{-1}, & d v+K^{1 / 2} \rho^{-1} d \rho=-a v\left(v+K^{1 / 2}\right)^{-1} d x \\
\frac{d t}{d x}=\left(v-K^{1 / 2}\right)^{-1}, & d v-K^{1 / 2} \rho^{-1} d \rho=-a v\left(v-K^{1 / 2}\right)^{-1} d x . \tag{25}
\end{array}
$$

If $x, t, v$ are regarded as a rectangular cartesian coordinate system, and $x, t, \rho$ as a second system, the solutions of Eqs. (24) and (25) can be represented graphically as surfaces. We may assign boundary conditions as follows:
(a) In both the $x v$-and the $x \rho$-planes, a curve is given.
(b) In the $t \rho$-plane, a curve is given; the corresponding curve in the $t v$-plane must be determined by means of the characteristics.
(c) In a plane $x=$ const. $=d$ of the $x t \rho$-system, a curve is given; the corresponding curve in the plane $x=d$ of the $x v t$-system must be determined by means of the characteristics.

Let us consider, for example, a gas for which $K=13.7 \times 10^{4} \mathrm{~m} .{ }^{2} \mathrm{sec} .^{-2}, a=2.75 \times 10^{9}$ $\mathrm{sec} .^{-1}$, with the following boundary conditions:
(a) In the $x v$ - and $x \rho$-plane, $v=v_{0}=0, \rho=\rho_{0}=0.073 \mathrm{~kg} \cdot \mathrm{~m} .{ }^{-4} \mathrm{sec} .{ }^{2}$ This corresponds to an initial pressure at rest of $10000 \mathrm{~kg} . \mathrm{m}^{-2}{ }^{-2}$
(b) In the $t \rho$-plane, the curve $\rho=f(t)$ is given.
(c) In the plane $x=2000 \mathrm{~m}$. of the $x t \rho$-system, the curve $\rho=0$ is given.

When $v=0$, we find from the first of Eqs. (24) that $d t / d x=K^{-1 / 2}=370 \mathrm{~m} . \mathrm{sec} .^{-1}$. Thus the time that the initial discontinuity requires to cover the distance $x=2000 \mathrm{~m}$. is 5.4 sec .

[^34]
# ON THE TREATMENT OF DISCONTINUITIES IN BEAM DEFLECTION PROBLEMS* 

BY ERYK KOSKO (Ecole Polytechnique, Montréal)

In a note on the treatment of discontinuities in beam deflection problems (Quarterly of Applied Mathematics, 1, 349-351) Mr. C. L. Brown suggests the use of Heaviside's unit step function. He thus avoids what he calls the "sectionalizing" treatment in the integration of the differential equation for the deflection of a beam with discontinuous transversal loading.

Mr. Brown's method appears to be equivalent to the procedure which seems to have been first developed by R. Macaulay ${ }^{1}$ and has since been included in several British textbooks. ${ }^{2,3}$ In order to establish expressions for moments with discontinuous variations, Macaulay introduces terms in twisted brackets, such as $\{x-a\}$, with the convention that these terms be neglected when the quantity within the brackets becomes negative. When integrating the term in question, the quantity in brackets is to be regarded as the independent variable instead of $x$; the indefinite integral of $\{x-a\}$ would be $\frac{1}{2}\{x-a\}^{2}$.

Taking Mr. Brown's example, the expression for the bending moment of the beam (1. c., Fig. 1, p. 349) would be with Macaulay's notation:

$$
E I \frac{d^{2} y}{d x^{2}}=M=-M_{1}+R_{1} x-P\{x-a\}
$$

The first integration would give

$$
E I \frac{d y}{d x}=-M_{1} x+\frac{1}{2} R_{1} x^{2}-\frac{1}{2} P\{x-a\}^{2}+C_{1}
$$

and the second integration

$$
E I y=-\frac{1}{2} M_{1} x^{2}+\frac{1}{6} R_{1} x^{3}-\frac{1}{6} P\{x-a\}^{3}+C_{1} x+C_{2}
$$

All the above equations hold at all parts of the span so that there are only two constants of integration to be determined from the conditions at both ends of the span, instead of having two for each section of the span as in the classical treatment.

It is therefore apparent that Macaulay's twisted bracket is but another symbol for the multiplication by the unit step function.

An important remark is to be made about the use of this procedure with regard to distributed loads. These must always be made to extend to the right-hand extremity of the beam, introducing negative loads if necessary. An extension of the method

[^35]due to H.A. Webb ${ }^{3}$ covers the effect of a concentrated bending couple applied at an intermediate point of the span. As an application, R. V. Southwell suggests (Example 14, l.c.) the derivation of the theorem of three moments for a continuous beam by the same method. The use of the method for a beam with a stepwise variation of bending rigidity seems however to be Mr. Brown's original contribution.

It is hoped that the discussed method, whatever the symbols used, will get more attention from engineers on this side of the Atlantic, as it carries with it a very substantial shortening of the computations.

## FORMULAS FOR COMPLEX INTERPOLATION*

By A. N. LOWAN and H. E. SALZER, (Math. Tables Project, Nat. Bureau of Standards)
An analytic function of $z=x+i y$ may be approximated by a complex polynomial of degree $n$ passing through $n+1$ points in accordance with the Lagrange-Hermite formula of interpolation. For the important special case when the given $n+1$ points are equidistantly spaced along any straight line in the $z$-plane, the following tables give the real and imaginary parts of the coefficients $A_{k}(P)$ of the interpolation polynomial $f(z)=A_{k}(P) f\left(z_{k}\right)$, where $P=\left(z-z_{0}\right) / h=p+i q$ and $h$ is the complex tabular interval. The formulas cover the cases ranging from complex quadratic ( 3 points) to complex quintic interpolation ( 6 points).

## Quadratic interpolation (3 points)

$$
\begin{array}{ll}
\operatorname{Re} A_{-1}(P)=\frac{1}{2}\left[p(p-1)-q^{2}\right], & \operatorname{JmA_{-1}(P)=q(p-.5)} \\
\operatorname{Re} A_{0}(P)=1-p^{2}+q^{2}, & J m A_{0}(P)=-2 p q \\
\operatorname{Re} A_{1}(P)=\frac{1}{2}\left[p(p+1)-q^{2}\right], & \operatorname{Jm} A_{1}(P)=q(p+.5)
\end{array}
$$

Cubic interpolation (4 points)

$$
\begin{array}{ll}
\operatorname{Re} A_{-1}(P)=\frac{(1-p)}{6}\left[p(p-2)-3 q^{2}\right], & \operatorname{Jm-A}(P)=\frac{q}{6}\left[q^{2}-2+3 p(2-p)\right] \\
\operatorname{Re} A_{0}(P)=1+\frac{1}{2}\left[p\left(p^{2}-2 p-1\right)+q^{2}(2-3 p)\right], & \operatorname{Jm} A_{0}(P)=\frac{q}{2}\left[p(3 p-4)-q^{2}-1\right] \\
\operatorname{Re} A_{1}(p)=-\frac{1}{2}\left[p(p-2)(p+1)+q^{2}(1-3 p)\right], & \operatorname{JmA_{1}}(P)=\frac{q}{2}\left[p(2-3 p)+q^{2}+2\right], \\
\operatorname{Re} A_{2}(P)=\frac{p}{6}\left(p^{2}-3 q^{2}-1\right), & \operatorname{JmA_{2}(P)=\frac {q}{6}[3p^{2}-q^{2}-1]}
\end{array}
$$

[^36]
## Quartic interpolation (5 points)

$$
\begin{aligned}
& \operatorname{ReA_{-2}}(P)=\frac{1}{24}\left[p\left(p^{2}-1\right)(p-2)+q^{2}\left(q^{2}+1\right)+6 p q^{2}(1-p)\right] \\
& \operatorname{Im}_{m-2}(P)=\frac{q}{12}\left[q^{2}(1-2 p)+2 p^{3}-3 p^{2}-p+1\right]
\end{aligned}
$$

$$
\operatorname{Re} A_{-1}(P)=-\frac{1}{6}\left[p(p-1)\left(p^{2}-4\right)+q^{2}\left(q^{2}+4\right)+3 p q^{2}(1-2 p)\right]
$$

$$
5 m A_{-1}(P)=-\frac{q}{6}\left[4 p^{3}-3 p^{2}-8 p+4-q^{2}(4 p-1)\right]
$$

$$
\operatorname{Re} A_{0}(P)=\frac{1}{4}\left[\left(p^{2}-1\right)\left(p^{2}-4\right)+q^{2}\left(q^{2}-6 p^{2}+5\right)\right]
$$

$$
J m A_{0}(P)=\frac{p q}{2}\left(2 p^{2}-2 q^{2}-5\right),
$$

$$
\operatorname{Re} A_{1}(P)=-\frac{1}{6}\left[p(p+1)\left(p^{2}-4\right)+q^{2}\left(q^{2}+4\right)-3 p q^{2}(1+2 p)\right],
$$

$$
J m A_{1}(p)=-\frac{q}{6}\left[4 p^{3}+3 p^{2}-8 p-4-q^{2}(1+4 p)\right]
$$

$$
\operatorname{Re}_{e} A_{2}(p)=\frac{1}{24}\left[p(p+2)\left(p^{2}-1\right)+q^{2}\left(q^{2}+1\right)-6 p q^{2}(1+p)\right],
$$

$$
J m A_{2}(P)=\frac{q}{12}\left[2 p^{3}+3 p^{2}-p-1-q^{2}(2 p+1)\right]
$$

## Quintic interpolation (6 points)

$\operatorname{ReA}_{-2}(P)=\frac{1}{120}\left[-p\left(p^{2}-1\right)(p-3)(p-2)+5 q^{2}(p-1)\left(2 p^{2}-4 p-1\right)+5 q^{4}(1-p)\right]$,
$J m A_{-2}(P)=\frac{q}{120}\left[-5 p(p-2)\left(p^{2}-2 p-1\right)-\left(q^{2}-6\right)\left(q^{2}+1\right)+10 p q^{2}(p-2)\right]$,
$\operatorname{Re} A_{-1}(P)=\frac{p}{24}\left[(p-1)\left(p^{2}-4\right)(p-3)-q^{2}\left(10 p^{2}-24 p-5 q^{2}-3\right)\right]-\frac{q^{2}}{6}\left(q^{2}+4\right)$,
$5 m A_{-1}(P)=\frac{q}{24}\left[\left(q^{2}+4\right)\left(q^{2}-3\right)+p\left(5 p^{3}-16 p^{2}-3 p+32\right)+p q^{2}(16-10 p)\right]$,
$\operatorname{Re} A_{0}(P)=\frac{1}{12}\left[\left(p^{2}-1\right)\left(p^{2}-4\right)(3-p)+q^{2}\left(10 p^{3}-18 p^{2}-15 p+15\right)+q^{4}(3-5 p)\right]$,
$\Im m A_{0}(P)=-\frac{q}{12}\left[\left(q^{2}+1\right)\left(q^{2}+4\right)+p\left(5 p^{3}-12 p^{2}-15 p+30\right)+p q^{2}(12-10 p)\right]$,

$$
\begin{aligned}
& \operatorname{Re} A_{1}(P)=\frac{1}{12}\left[p(p+1)\left(p^{2}-4\right)(p-3)+q^{2}\left(-10 p^{3}+12 p^{2}+21 p-8\right)+q^{4}(5 p-2)\right] \\
& \operatorname{Jm} A_{1}(P)=\frac{q}{12}\left[\left(q^{2}+4\right)\left(q^{2}+3\right)+p\left(5 p^{2}-8 p^{2}-21 p+16\right)+p q^{2}(8-10 p)\right] \\
& \operatorname{Re} A_{2}(P)=\frac{1}{24}\left[-p\left(p^{2}-1\right)(p-3)(p+2)-q^{2}\left(-10 p^{3}+6 p^{2}+21 p-1\right)+q^{4}(1-5 p)\right] \\
& \operatorname{Jm} A_{2}(P)=-\frac{q}{24}\left[5 p^{4}-4 p^{3}-21 p^{2}+2 p+6+q^{2}\left(q^{2}+7\right)+2 p q^{2}(2-5 p)\right] \\
& \operatorname{Re} A_{3}(P)=\frac{p}{120}\left[\left(p^{2}-4\right)\left(p^{2}-1\right)+5 q^{2}\left(q^{2}-2 p^{2}+3\right)\right] \\
& \operatorname{Jm} A_{3}(p)=\frac{q}{120}\left[\left(q^{2}+4\right)\left(q^{2}+1\right)+5 p^{2}\left(p^{2}-2 q^{2}-3\right)\right]
\end{aligned}
$$

## BOOK REVIEWS

## Ten lectures on theoretical rheology. By Markus Reiner. Nordemann Publishing Co., Inc. New York, 1943. iv +164 pp. $\$ 4.50$.

It is customary to define Rheology as the science of flow and deformation of matter. If this definition is taken literally, Rheology becomes practically identical with Mechanics of Continua, and it becomes hard to understand why a new term has been coined. It seems that a more adequate definition would state Rheology to be a Mechanics of Continua in which the ideal elastic body and the perfect fluid are almost as systematically disregarded, as they are over-emphasized in classical Mechanics of Continua. This undue prominence which the classical scheme gives to the ideal clastic body and the perfect fluid tends to be reflected in textbooks of Mechanics as well as in engineering curricula. The necessity of developing and studying the mechanics of other solids and fluids must therefore be stressed, even to the extent of creating a new term for this part of Mechanics of Continua which, up to a fairly recent past, has been so badly neglected.

There is a definite need for a treatise covering the entire field of Rheology rather than parts, such as plasticity or dynamics of (Newtonian) viscous fluids. The present book is in the nature of an introduction to such a treatise. Four chapters $(1-3,9)$ deal with the analysis of stress and strain and the important decomposition of the tensors of stress and strain into isotropic and deviatoric parts. The author bows to convention in defining the strains as $e_{x x}=\partial u / \partial x, \cdots, e_{x y}=\partial v / \partial x+\partial u / \partial y, \cdots$. This procedure, a remnant from the time when the tensor character of strain and its geometrical implications was not yet fully realized, makes it necessary to denote the components of the strain tensor by $e_{x x}, \cdots, \frac{1}{2} e_{x y}, \cdots$, and deprives many relations of their natural symmetry. Four further chapters $(4,6,8,10)$ are devoted to the discussion of certain rheological idealizations, viz. the ideal elastic solid, the (Newtonian) viscous fluid, and the materials customarily named after Maxwell (viscous fluid with relaxation of stresses), Voigt (visco-elastic solid), Saint-Venant (perfectly plastic solid), and Bingham (viscous fluid with yield limit). The remaining two chapters $(5,7)$ are concerned with the solution of special problems (tension and simple flexure of a prismatical bar), Einstein's law of the viscosity of sols, and rheological models.

Theoretical Rheology is a subject which cannot be treated satisfactorily without using the tool of tensor analysis. The author did obviously not want to suppose the reader to be familiar with this tool. On the other hand, limitations originally imposed on the time available for his lectures seem to have prevented him from presenting even the basic conception of tensors in a precise form. One wonders whether, under these circumstances, it would not have been preferable to restrict the discussion to the mechanical behavior of solids and liquids in pure shear. As it is, the uninitiated reader cannot fail to get the impression that any nine quantities neatly arranged between double vertical bars constitute a tensor. The laws of transformation are touched upon in chapter 9 only, and are nowhere stated to form an integral part of the definition of tensor. On the other hand, the reader familiar with tensor analysis might wish to have a tensorial expression of the yield condition of plastic materials which is more adequate than the cryptic relation $p_{0}=\vartheta$ (p. 111, Eq. (4)), where $p_{0}$ denotes the stress deviator and $\vartheta$ is defined as "the yield stress." Many other instances could be cited where the clarity of exposition has obviously suffered from the tendency to cram too much material into a text which, according to the preface, is intended as a brief introduction. In spite of such occasional shortcomings the book, which fills a patent need, will prove very useful.
W. Prager

Table of reciprocals of the integers from 100,000 through 200,000. Prepared by the Mathematical Tables Project, Work Projects Administration of the Federal Works Agency; conducted under the sponsorship of the National Bureau of Standards. Official Sponsor: Lyman J. Briggs; Technical Director: Arnold N. Lowan. Columbia University Press. New York, 1943, viii +201 pp. $\$ 4.00$.
These tables are a useful supplement to the existing tables of reciprocals. The tabular interval is small enough to permit linear interpolation throughout; the differences decrease slowly from 100 to 25 units of the last place. The arrangement of the tables is very practical. Seven significant figures are given (if a number has $k$ figures before the decimal point, its reciprocal has $k-1$ zeros after the decimal
point, before the first significant figure). Moreover, the tables indicate the direction in which the last digit is rounded; this practical device reduces the relative tabular error to $2.5 \times 10^{-8}$.

## W. Feller

Table of the Bessel functions $J_{0}(z)$ and $J_{1}(z)$ for complex arguments. Prepared by the Mathematical Tables Project, Work Projects Administration of the Federal Works Agency; conducted under the sponsorship of the National Bureau of Standards. Official Sponsor: Lyman J. Briggs; Technical Director: Arnold N. Lowan. Columbia University Press. New York. 1943, xiv +403 pp. $\$ 5.00$.

Many problems of mathematical physics and mechanics lead to Bessel functions of various types. The most important of these problems are sketched in the foreword to the present tables, written by Professor H. Bateman of California Institute of Technology. The number of existing tables of Bessel functions is also legion. The present tables contain a valuable bibliography listing some 65 tables of Bessel functions of orders zero and one. However, the new tables are unique both in range and extent.

The Bessel functions $J_{\nu}(z)$ are defined by

$$
J_{\nu}(z)=\sum_{k=0}^{\infty} \frac{(-1)^{k_{z} z^{\nu+2 k}}}{k!\Gamma(\nu+k+1) 2^{\nu+2 k}}
$$

and satisfy the differential equation

$$
z^{2} u^{\prime \prime}(z)+z u^{\prime}(z)+\left(z^{2}-\nu^{2}\right) u(z)=0 .
$$

One has the recurrence formula

$$
J_{\nu+1}(z)=-J_{\nu-1}(z)+2 \nu J_{\nu}(z) / z
$$

which enables one to compute $J_{\nu}(z)$ for all integers $\nu$ given the values of $J_{0}(z)$ and $J_{1}(z)$. These are now tabulated to ten decimal places. The entries are written in the polar form $z=\rho(\cos \phi+i \sin \phi)$. The functions are tabulated along the rays $\phi=0^{\circ}, 5^{\circ}, 10^{\circ}, \cdots, 90^{\circ}$ for values of $\rho$ from 0 to 10 in the steps of .01 . The values of the functions in all other quadrants follow easily by means of simple symmetry relations.

To facilitate interpolation, the book contains also tables of the cocfficients in Lagrange's interpolation formula which uses five equally spaced points. The coefficients are tabulated to 10 decimal places, the argument varying in steps of .001 . These tables will, of course, be useful for many computations and are quite independent of the main tables.

## W. Ficleer

The methodology of Pierre Duhem. Armand Lowinger. Columbia Univ. Press, 184 pp., 1941. \$2.25.

The French theoretical physicist Duhem (1861-1916) devoted his life-work to thermodynamics, although he is mostly remembered today for his studies in medieval science. He moreover published his views on the methods, aims and significance of physics in a few scattered papers and in one connected account, a book entitled La Theorie plysique, son objet et sa structure (1906). Mr. Lowinger has given an excellent presentation of Duhem's ideas. These are challenging, for on the one hand Duhem was an eminently "classical" physicist, strongly opposed to atomism, opposed also to Maxwell's electromagnetism, so that from our present point of vantage we can prove him wrong on both these counts; but on the other hand, he believed with Kirchhoff that physical theory is a description, not an explanation; with Mach, that its purpose is intellectual economy, and so was on the way, with these and other correlated ideas, towards present-day scientific pragmatism. Duhem's opinions on physical method were rooted in his metaphysical beliefs, a fact obvious to his readers and critics, but which he was very anxious to deny. Mr. Lowinger's presentation of Duhem is faithful and unbiased; he gives us his own views in a last stimulating chapter. There is a good bibliography of Duhem and his critics, to which should be added the rather important books by A. Rey and P. Humbert mentioned on pp. 15 and 8 respectively.

P. Le Corbeiller

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[^0]:    Entered as second class matter March 14, 1944, at the post office at Providence, Rhode Island, under the act of March 3, 1879. Additional entry at Menasha, Wisconsin.

[^1]:    * An invited address presented before the American Mathematical Society at New York, Feb. 25, 1944. Manuscript received March 15, 1944.

[^2]:    * The numbers in square brackets refer to the bibliography at the end of the paper.

[^3]:    * J. Appl. Physics, 15, 598-606 (1944).

[^4]:    * Received Jan. 24, 1944.
    ${ }^{1}$ M. Herzberger, Direct methods in geometrical optics, Trans. Am. Math. Soc., 53, 218-229 (1943).
    ${ }^{2}$ M. Herzberger, A direct image error theory, Quarterly of Applied Mathematics, 1, 69-77 (1943).
    ${ }^{3}$ For the connection of the Lagrangian invariant with different branches of mathematics and physics, see M. Herzberger, Theory of transversal curves and the connections between the calculus of variations and the theory of partial differential equations, Proc. Nat. Acad. Sci. U.S.A., 24, 466-473 (1938).

[^5]:    - Invented by L. Euler in 1776. See M. Herzberger, Gaussian optics and Gaussian brackets, J. Opt. Soc. Amer. 33, 651-655 (1943).

[^6]:    * Received Jan. 29, 1944.
    ** Now at Armour Research Foundation, Chicago, Ill.
    ${ }^{1}$ Timoshenko, Theory of plates and shells, McGraw-Hill, 1st Edition 1940, Art. 54, p. 282.
    ${ }^{2}$ K. E. Bisshopp, Stress coefficients for rotating discs of conical profile, Journal of Applied Mechanics, Vol. 11, No. 1, March 1944, pp. A1-A9.

[^7]:    ${ }^{3}$ Whittaker and Watson, A course of modern analysis, Cambridge, England, 4th Edition, 1927. p. 198.

    * When $\sigma=1 / 3$ both solutions can be expressed in terms of rational algebraic functions.
    - Titchmarsh, Theory of functions, Oxford, England, 1932, p. 224.
    ${ }^{5}$ Forsyth, Theory of differential equations, Cambridge, England, 1902, vol. 4, part 3, p. 147.
    $\dagger$ The numerical value of $c$ is used since it is independent of Poisson's ratio $\sigma$.

[^8]:    ${ }^{6}$ Ref. 3, p. 246.
    ${ }^{7}$ Ref. 3, p. 297, Example 7.

[^9]:    ${ }^{8}$ J. B. Scarborough, Numerical mathematical analysis, The Johns Hopkins Press, Baltimore, Md., 1930, p. 64.
    ${ }^{\circ}$ Ref. 1, p. 477.

[^10]:    * Variation in the slopes of corresponding generators of middle surfaces belonging to the conical discs of the equivalent system is not considered.

[^11]:    *The tables were compiled with the aid of the staff of the Calculation Department of Fairbanks Morse \& Co., Beloit, Wis., to whom acknowledgement hereby is made.

[^12]:    * Reccived Feb. 9, 1944.
    ** On leave of absence; now at Federal Telephone and Radio Laboratories, New York City.
    $\dagger$ Purely dynamical examples of the method of subspaces have been given by Kron in an unpubished manuscript.

[^13]:    ${ }^{1}$ We use the tensor practice of placing the indices of contravariant quantities above the symbol. $x^{2}, x^{2}, x^{3}$ do not stand for $x, x$-squared, $x$-cubed.

[^14]:    ${ }^{2}$ According to the summation convention of the index notation, a repeated index in a single expression indicates summation over its whole range of values.

[^15]:    ${ }^{3}$ When the transformations are non-singular, this is the basis of the important "group property" of tensor transformations. In the absence of inverses the term "group" is inappropriate here, but provided the succession is always to a subspace of the preceding space or subspace, as it always is here, the usual combination properties of tensor transformations are preserved,

[^16]:    - In general one must include all forces that do work, including dissipative forces.

[^17]:    ${ }^{5}$ Kron has suggested another method having advantages when the system and its interconnections are complicated.

[^18]:    * Received Feb. 28, 1944.
    ${ }^{1}$ L. Prandtl, Göttinger Nachrichten, 1918, 451, and 1919, 107. Also N.A.C.A. Report 116, 1921. See also the presentation in Mises and Friedrichs' Pluid dynamics, Brown University mimeographed lecture notes, 1941, p. 108 and foll.
    ${ }^{2}$ E. Trefftz, Zeitschr. f. Angew. Math. und Mechanik, 1, 206 (1921).

[^19]:    ${ }^{3}$ v. Kármán and Burgers, in vol. II of Durand's Aerodynamic theory, Springer, Berlin 1935, p. 171.
    ${ }^{1}$ The aspect ratio $A$ is defined as $b^{2} / S$, where $S$ is the area of the wing. For the calculation of $S$ as a function of $\kappa$, see (22).

[^20]:    ${ }^{5}$ G. Holzmüller, Einfihrung in die Theorie der isogonalen Verwandischaften, 1882, p. 256 and foll. See also Cayley, Collected papers, vol. 13, papers No. 891, 920, and 921.

[^21]:    - More general cases of twisted airfoils are being computed at present and will be reported on elsewhere.
    ${ }^{7}$ See, for instance. Whittaker and Watson, Modern analysis, 4th ed., Cambridge University Press, p. 510 .

[^22]:    ${ }^{8}$ The mapping for $k=3-\sqrt{8}$ is studied and illustrated by figures in Holzmüller's book (see footnote 5).
    ${ }^{9}$ However, $E$ and $K$ depend only on $k^{2}$.

[^23]:    ${ }^{10}$ Formulae (53) and (54) may be derived in a different way without appealing to the completeness relation. The velocity potential $\phi(y, z)$ is only the real part of a complex stream function $F(u)$ in terms of which the total lift may be written in the form

    $$
    L=2 \rho V \operatorname{Re} \oint F(u) d u
    $$

    When transformed into the $Z$-plane, this becomes a contour integral around the rectangle of Fig. 2. This integral can be carried out simply by applying Cauchy's theorem. The same can be done for the rolling moment. This method recalls the Blasius theorem in the theory of infinite span.

[^24]:    ${ }^{11}$ L. M. Milne-Thomson, Die elliptischen Funktionen von Jacobi, Springer, Berlin, 1931.

[^25]:    * Received May 19, 1944.

[^26]:    * Received Jan. 22, 1944.
    ${ }^{1}$ See, for example, W. E. Deming, Some notes on least squares, U. S. Dept. of Agriculture Graduate School, 1938, p. 31 ff., or E. T. Whittaker and G. Robinson, The calculus of observations, Blackie and Son, London, 1940, p. 214. Deming's treatment is also given in his book, Statistical adjustment of data, John Wiley \& Sons, New York, 1943, p. 52 ff.
    ${ }^{2} \mathrm{~K}$. Levenberg, A method for the solution of certain non-linear problems in least squares, Quarterly of Applied Mathematics, 2, 164 (1944).
    ${ }^{3}$ A. L. Cauchy, Methode génêrale pour la resolution des systimes d'équations simultanées, Comptes rendus, Ac. Sci. Paris, 25, 536-538 (1847).
    ${ }^{4}$ R. Courant, Variational methods for the solution of problems of equilibrium and vibrations, Bull. Amer. Math. Soc. 49, 1-23 (1943). See especially pp. 17-20. Courant calls the method the "method of gradients" and ascribes its origin to a paper published by Hadamard in 1907.

[^27]:    ${ }^{5}$ G. Temple, The general theory of relaxation nethods applied to linear systems, Proc. Roy. Soc. London (A) 169, 476-500 (1939). For this reference I am indebted to H. Hotelling.
    ${ }^{6}$ Even the case where such a gradient does not exist, there being only a total differential, can probably be handled by a method which bears the same relation to the present method that Temple's method for gyrostatic systems does to his method of steepest descent.
    ${ }^{7}$ This was done by Cauchy (1.c.). It represents the approximation by Newton's method to the smallest positive zero of $g(l)$. This is a reasonable guess for a $G$ given by (2), where the numerical value is zero. In the least square cases (where $G \geqq 0$ ) the guess is often many times too large.

[^28]:    ${ }^{8}$ This is also easily proved analytically.

[^29]:    ${ }^{9}$ The engineers at Frankford Arsenal prefer the Levenberg method for the problems which have confronted them; but I do not know to what extent they have exploited the present method. Since the Levenberg method is confined to a $G$ of the form (3) and makes use of that representation, it would not be surprising if it should prove superior for that case. On the other hand it is not difficult to concoct artificial examples for which the method of steepest descent is superior in the third respect as well as the second, at least for certain determinations of the weighting factors.

[^30]:    * Received Feb. 8, 1944.
    ${ }^{1}$ A. Weinstein, Mémorial des Sciences Mathématiques, No. 88, 1937.
    ${ }^{2}$ Cf. R. Courant, Variational methods for the solutions of problems of equilibrium and vibrations, Bull. Amer. Math. Soc., 49, 1-23, 1943, especially p. 11. See also K. Friedrichs, Math. Annalen, 98, 217, 1928. The method of finite differences does not give satisfactory numerical results for clamped plates.
    ${ }^{3}$ The bibliography given in S. Timoshenko, Plates and shells, McGraw-Hill, 1940, p. 222 covers papers from 1902 to 1939 and shows the persistent interest in this problem.
    ${ }^{4}$ H. W. March, Trans. Amer. Math. Soc., 27, 307-317, 1925, proves the weak convergence of the approximations given by his series. Cf. I. S. Sokolnikoff, Mathematical theory of elasticity, Brown University, 1941, p. 387.
    ${ }^{6}$ Cf. for instance a note by C. Miranda, Rend. Semin. Mat. di Roma, 1, 262-266, 1937.

[^31]:    ${ }^{6}$ For rectangular plates $D P_{0}$ is the problem of the supported plate.

[^32]:    ${ }^{7}$ N. Aronszajn and A. Weinstein, On the unified theory of eigen-ralues of plates and membranes, Amer. J. Math., 64, 625-645, 1942.
    ${ }^{8}$ For non-uniform loading, the sequence given l.c. 1,416 must be used.

[^33]:    * Received Feb. 8, 1944.
    ${ }^{1}$ J. Stefan, Ber. der Wiener Akad. 63 (2), 63-124 (1871).
    ${ }^{2}$ J. C. Maxwell, Phil. Mag. (4), 35, 185-217 (1868).
    ${ }^{3}$ R. M. Barrer, Diffusion in and through solids, Cambridge University Press, Cambridge, 1941, p. 1.
    ${ }^{4}$ B. Lewis and G. v. Elbe, Combustion flames and explosion of gases, Cambridge University Press, Cambridge, 1938, p. 224.

[^34]:    ${ }^{5}$ A. R. Forsyth, Theory of differential equations, Cambridge University Press, 1906, vol. 5, p. 435.

[^35]:    * Received March 27, 1944.
    ${ }^{1}$ R. Macauly, Note of the deflection of beams, Messenger of Mathematics, 48, 129 (1919).
    ${ }^{2}$ R. V. Southwell, An introduction to the theory of elasticity, Oxford, 1936, \$\$194-196.
    ${ }^{3}$ J. Case, The strength of materials, 2nd edition, Arnold \& Co., London, 1932, 8169.

[^36]:    * Received April 17, 1944.

