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QUARTERLY OF APPLIED MATHEMATICS

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ON COMBINED FLEXURE AND TORSION, AND THE FLEXURAL BUCKLING OF A TWISTED BAR*

BY

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1. Introduction. When a straight uniform slender bar is twisted, the straight form becomes unstable at a certain value of the twisting couple, and the center line of the bar becomes a space curve. Elements of the bar are bent about both principal axes of section, and the buckled form thus possesses strain energy of flexure as well as of torsion. If the bar is twisted to the critical configuration, and its end sections then held against further rotation, the jump to the buckled form means the appearance of flexural energy at the expense of the torsional energy. The occurrence of the *flexure* must therefore produce some relief of the *torsion*, that is, it must modify the amount of twist.

It proves to be impossible to account for the transference of strain energy from that of torsion to that of flexure if the strain energy is represented in the accepted form of the theory of small bending and torsion of thin bars—

$$\frac{1}{2}\int_0^l (EI_1u''^2 + EI_2v''^2 + GC\beta'^2)dz,$$

where EI_1, EI_2, GC are the flexural and torsional rigidities, u, v the components of deflection parallel to the principal axes of the section, and β the torsional rotation, as functions of the axial co-ordinate z. Coincidence of shear center and centroid is assumed, and secondary effects of non-uniform torsion[1] are disregarded, for simplicity. If for instance this form is used in the potential energy, and the differential equations of the bar buckled from a state of simple torsion by couples M_3 are found by means of the theorem of stationary potential energy, the correct equations[2]

$$EI_1u'' + M_3v' = 0, \qquad EI_2v'' - M_3u' = 0, \qquad M_3 = GC\beta'$$

are not obtained. The terms M_3v', M_3u' in the first two fail to appear. These equations are nevertheless easily derived directly as conditions of equilibrium.

The comparison with the corresponding problem of the bar under thrust is useful. The bar is compressed to the critical state, and the ends held against further approach. The bar jumps over to the bent form, and energy of bending appears. But

* Received March 24, 1944.

[1] J. N. Goodier, (i) *The buckling of compressed bars by torsion and flexure*, Cornell University Engineering Experiment Station, Bulletin 27, 1942; (ii) *Flexural torsional buckling of bars of open section*, Bulletin 28, 1942.

[2] S. Timoskenko, *Theory of elastic stability*, McGraw-Hill, 1936, p. 168, or (1) (ii) equations 2, 3, 7.

the transition to the bent form involves a lengthening of the bar, and some of the compressional strain energy is thus released to supply the energy of flexure. The Euler problem has been analysed from this point of view by R. V. Southwell.[3]

This lengthening of the bar is of the second order in the derivative of the bending displacement with respect to the axial coordinate. It can be disregarded in writing down the differential equation of equilibrium, but not in energy methods. It is natural to look for something analogous in the torsional problem by investigating the nature of combined torsion and flexure to a higher order of small quantities than formerly. This is done in what follows and the required new terms in the strain energy are found. At the same time the nature of combined torsion and flexure is clarified, and the energy method is made available for more difficult problems of buckling from a twisted state such as those of non-uniform bars.

2. Finite bending and torsion of a thin bar. Let the axis (of centroids) of the undeformed straight bar lie along the z-axis of fixed cartesian axes u, v, z. The bar is now subjected to small bending and twisting. Its axis becomes a space curve, consisting of points of co-ordinates u, v, z. Even if the deflection (u, v) is small, the geometrical torsion of this curve is not small. The bending may be in one plane (the osculating plane) at one point, and in a perpendicular plane at another.

The geometrical torsion τ_c of the curve is distinct from the torsion τ of the bar. When the deflection (u, v) is prescribed the space curve of centroids is definite, with definite curvature and torsion. The cross sections of the bar must be in the normal planes of this curve, but the torsion of the bar remains indefinite until the orientations of the principal axes in these planes are specified.

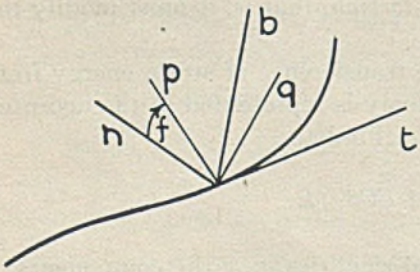

FIG. 1.

In Fig. 1 the tangent, normal and binormal at P are indicated by t, n, b.[4] As the origin of the triad moves along the curve with unit speed, it has a component τ_c of angular velocity about t, and a component κ (the curvature) about b, right handed rotations looking along the positive axes being reckoned positive. Define an angle γ such that $\tau_c = d\gamma/ds$ (s being arc length increasing in the sense of t) and $\gamma = 0$ at some chosen reference section $s = s_0$, as for instance one end of the bar.

Let f be the angle which one principal axis p (Fig. 1) of the cross section at P makes with the principal normal n, positive when this axis is obtained from n by positive rotation about t. Let f_0 be its value at $s = s_0$. Then the rate of rotation of the tpq-triad about t is given by $\tau_c + df/ds$ or $(d\gamma/ds) + (df/ds)$ and this is by definition the torsion of the bar.[5]

Accordingly if the bar is bent but not twisted, $\gamma + f$ is a constant along the bar and in fact $\gamma + f = f_0$, or $f = f_0 - \gamma$. From this state we may derive a twisted form of the

[3] *Introduction to the theory of elasticity*, 2nd ed., Oxford University Press, 1941, p. 443.

[4] The notation and conventions are those of C. E. Weatherburn, *Differential geometry*, vol. 1, Cambridge University Press, Cambridge 1939, p. 15.

[5] The discussion thus far, except for the introduction of the angle γ, corresponds with that of A. E. H. Love, *Mathematical theory of elasticity*, 4th ed., Cambridge University Press, 1934, Ch. XVIII. The further development is different.

bent bar by introducing an angle of twist ϕ with $\phi=0$ at $s=s_0$, so that $f=f_0-\gamma+\phi$. The torsion of the bar is now $d\phi/ds$. The bent and twisted form of the bar is completely specified by the curve of centroids, which defines γ, and the angle $f_0+\phi$ which can be assigned independently.

In the elementary theory of bending, the curvature is related to the bending moments by means of components along the principal axes of cross sections. If κ_1, κ_2 denote these components along p and q (Fig. 1), we have (κ can be regarded as an angular velocity about b)

$$\kappa_1 = \kappa \sin f, \qquad \kappa_2 = \kappa \cos f \tag{1}$$

or

$$\kappa_1 = \kappa \sin (f_0 - \gamma + \phi), \qquad \kappa_2 = \kappa \cos (f_0 - \gamma + \phi). \tag{2}$$

We have also

$$\tau = d\phi/ds. \tag{3}$$

But

$$\kappa = (u''^2 + v''^2 + z''^2)^{1/2}, \tag{4}$$

primes denoting differentiation with respect to s. Also γ is defined through $d\gamma/ds=\tau_c$ and we have

$$\tau_c = \kappa^{-2} \begin{vmatrix} u' & v' & z' \\ u'' & v'' & z'' \\ u''' & v''' & z''' \end{vmatrix}. \tag{5}$$

With these formulas the bending and torsion of the bar are completely specified by the deflection (u, v as given functions of z) and the angles f_0 and ϕ. The orientations of the principal normal and binormal are defined by the deflection curve, and the orientations of the principal axes relative to these are defined by f_0 and ϕ. The formulas (2) and (3) may be used to specify not only the deformed state of the bar, but also an initial "bent and twisted" but unstressed state. The differences between the values of κ_1, κ_2 and τ then represent the changes of curvature and torsion to which the components of bending moment, and the twisting moment, will be respectively proportional.

To illustrate this, and also the significance of f_0, let the bar be circular and in a horizontal plane, with the principal axis p of all cross sections also in the horizontal plane. Then we may take for the initial state $\gamma=f_0=\phi=\tau=\kappa_1=0$, $\kappa_2=\kappa=1/r$ where r is the radius of the circle. Let each cross section now be rotated by the same angle α about t. For the deformed state $f_0=\alpha$ and

$$\kappa_1 = r^{-1} \sin \alpha, \qquad \kappa_2 = r^{-1} \cos \alpha, \qquad \tau = 0.$$

The changes of the components of curvature are

$$r^{-1} \sin \alpha, \qquad r^{-1}(\cos \alpha - 1).$$

When α is small, the second, the change in κ_2, is negligible. The bending moment induced is proportional to $r^{-1}\alpha$, and corresponds to κ_1, that is, its axis is n, in the plane of the ring.[6]

[6] This problem is analysed from first principles in Timoshenko, *Strength of materials*, Part II, 2nd ed., Van Nostrand, 1941, p. 177.

3. Small bending and torsion of a straight bar. The formulas (2) and (3) must yield such expressions as d^2u/dz^2, d^2v/dz^2, $d\phi/dz$ as their principal parts for small deformation. The object of the present investigation is to obtain terms of higher order as well.

Let u', v', ϕ be small compared with 1, and let l be a suitable length such as the length of the bar, or the wavelength of a periodic deflection. The formulas (4) and (5) involve u'', v'', u''', v'''. If the greatest absolute value of u''' and v''' is denoted by η/l^2, u'' and v'' do not exceed η/l and u', v' do not exceed η, which is small. Let ϵ denote the largest absolute value of η and ϕ. Quantities not exceeding ϵ, ϵ/l, etc., or quantities differing from them only by terms involving higher powers of ϵ, will be denoted by $O(\epsilon)$, $O(\epsilon/l)$, etc.

The relation $u'^2+v'^2+z'^2=1$ yields $z'^2=1-O(\epsilon^2)$ and so $z'=1-O(\epsilon^2)$. It yields also

$$z'' = -(u'u''+v'v'')(1-u'^2-v'^2)^{-1/2} = O(\epsilon^2/l) \tag{6}$$

and

$$z''' = O(\epsilon^2/l^2). \tag{7}$$

Then (4) yields $\kappa=[u''^2+v''^2+O(\epsilon^4/l^2)]^{1/2}$. Since u''^2, v''^2 are $O(\epsilon^2/l^2)$ we have as an approximation

$$\kappa = (u''^2+v''^2)^{1/2} \tag{8}$$

in which the error is of order ϵ^2, relative to the part retained.

The determinant of (5) yields $u''v'''-u'''v''$ with an error of order ϵ^2. Then (5) becomes

$$\tau_c = (u''v'''-u'''v'')(u''^2+v''^2)^{-1} \tag{9}$$

with an error or order ϵ^2.

Now the right of (9) may be identified as $(d/ds)\tan^{-1}(v''/u'')$ and, in view of the equations defining γ ($d\gamma/ds=\tau_c$, $\gamma=0$ when $s=s_0$) we have

$$\gamma = \tan^{-1}\frac{v''}{u''} - \tan^{-1}\frac{v_0''}{u_0''}, \tag{10}$$

where u_0'', v_0'' are the values of u'', v'' at $s=s_0$. The inverse tangents are principal values. The values of $\sin\gamma$ and $\cos\gamma$ are required. From (10)

$$\tan\gamma = (u_0''v''-v_0''u'')(u_0''u''+v_0''v'')^{-1}$$

and therefore

$$\left.\begin{aligned}\sin\gamma &= \left(\frac{v''}{u''}-\frac{v_0''}{u_0''}\right)\\ \cos\gamma &= \left(1+\frac{v_0''v''}{u_0''u''}\right)\end{aligned}\right\}\times\left(1+\frac{v_0''^2}{u_0''^2}\right)^{-1/2}\left(1+\frac{v''^2}{u''^2}\right)^{-1/2}. \tag{11}$$

The ambiguity of sign involved in obtaining the sine and cosine from the tangent is disposed of by the consideration that if v''/u'' slightly exceeds v_0''/u_0'', both being positive, γ must be a small positive angle.

4. Expressions for small curvature and torsion. Expanding the first of (2) in the form

$$\kappa_1 = \kappa\left\{(\sin f_0 \cos\gamma - \cos f_0 \sin\gamma)\left(1 - \frac{\phi^2}{2}\cdots\right)\right.$$
$$\left. + (\cos f_0 \cos\gamma + \sin f_0 \sin\gamma)\left(\phi - \frac{\phi^3}{6}\cdots\right)\right\}$$

and substituting for κ, $\cos\gamma$, $\sin\gamma$ from (8), (11) we find

$$\kappa_1 = u''\sin(f_0+\delta) - v''\cos(f_0+\delta) + \phi[u''\cos(f_0+\delta) + v''\sin(f_0+\delta)] + \cdots \tag{12}$$

and similarly

$$\kappa_2 = u''\cos(f_0+\delta) + v''\sin(f_0+\delta) - \phi[u''\sin(f_0+\delta) - v''\cos(f_0+\delta)] + \cdots \tag{13}$$

where $\cos\delta = u_0''(u_0''^2 + v_0''^2)^{-1/2}$, $\sin\delta = v_0''(u_0''^2 + v_0'^2)^{-1/2}$. In these developments the errors are of order ϵ^2 relative to the leading terms. They are therefore accurate as far as explicitly carried.

Since $dz/ds = 1 - O(\epsilon^2)$, replacement of differentiation with respect to s by differentiation with respect to z, to any order, will involve errors of order ϵ^2. Thus the primes in the terms set out in (12) and (13) may be taken to indicate differentiation with respect to z, and the developments remain correct to this order. In the same way the torsion $d\phi/ds$ may be replaced by $d\phi/dz$ with an error of order ϵ^2.

The angle f_0, while significant of course when the axis of the bar is appreciably deflected, tends to become merely a rigid body rotation when the bar is nearly straight. In order to eliminate such a rigid-body rotation, we observe that there is as yet no connection between the u-axis and the principal axis p. If these axes coincide when the bar is undeformed, small torsion and bending, free of large rigid body rotations, will restrict the angle between them to be of the same order as ϕ. Then the direction cosines of p, relative to the u, v, z axes must be $1 - O(\epsilon^2)$, $O(\epsilon)$, $O(\epsilon)$ at most.

The direction cosines of n, the principal normal, are u''/κ, v''/κ, z''/κ so that, if $\boldsymbol{n}$ is the unit vector along n, $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ unit vectors along the axes of u, v and z,

$$\boldsymbol{n} = \kappa^{-1}(u''\boldsymbol{i} + v''\boldsymbol{j} + z''\boldsymbol{k}).$$

The direction cosines of b, the binormal, are used as the coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ in

$$\boldsymbol{b} = \kappa^{-1}[(v'z'' - z'v'')\boldsymbol{i} + (z'u'' - u'z'')\boldsymbol{j} + (u'v'' - v'u'')\boldsymbol{k}]$$

where $\boldsymbol{b}$ is the unit vector along the binormal.

Since the principal axis p (Fig. 1) is in the plane of b and n, and is derived from n by a rotation f towards b, the unit vector along it is given by $\boldsymbol{n}\cos f + \boldsymbol{b}\sin f$ or

$$\begin{aligned}&\kappa^{-1}[u''\cos f + (v'z'' - z'v'')\sin f]\boldsymbol{i}\\ &\quad + \kappa^{-1}[v''\cos f + (z'u'' - u'z'')\sin f]\boldsymbol{j}\\ &\quad + \kappa^{-1}[z''\cos f + (u'v'' - v'u'')\sin f]\boldsymbol{k}\end{aligned} \tag{14}$$

and the coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ give the direction cosines of p.

The first of these is of order 1 without restriction on f. The second may be represented as

$$O(l/\epsilon)\left[O(\epsilon/l)\cos f + O(1)O(\epsilon/l)\sin f - O(\epsilon)O(\epsilon^2/l)\sin f\right],$$

from which it is apparent that these direction cosines will not be small of order ϵ unless

$$\kappa'(v'' \cos f + u'' \sin f)$$

is small of this order. This expression may be developed, by the processes which led to (12) and (13) as

$$\kappa^{-2}v''\{u'' \cos (f_0 + \delta) + v'' \sin (f_0 + \delta) - \phi[u'' \sin (f_0 + \delta) - v'' \cos (f_0 + \delta)] + \cdots\}$$
$$+ \kappa^{-2}u''\{u'' \sin(f_0 + \delta) - v'' \cos (f_0 + \delta) + \phi[u'' \cos (f_0 + \delta) + v'' \sin (f_0 + \delta)] + \cdots\}$$

and will be small of order ϵ only if $f_0 + \delta$ is small of this order.

This result simplifies (12) and (13) to

$$\kappa_1 = -v'' + u''(\phi + f_0 + \delta), \qquad \kappa_2 = u'' + v''(\phi + f_0 + \delta), \tag{15}$$

and with $\tau = \phi'$ these constitute approximations to κ_1, κ_2, and τ with errors of order ϵ^2 relative to the leading terms. It is now implied of course that one principal axis (p) coincides with the u-axis in the undeformed state, and that in the deformation it rotates from it by an angle of the same order as u', v' and ϕ. This is the case if one section of the bar is fixed against rotation, or against rotation of the type ϕ only.

The third direction cosine in (14) is of order ϵ without further conditions.

5. An alternative torsional co-ordinate. The angle ϕ represents a rotation of the cross section about t, from the torsionless configuration associated with the deflection u, v. This torsionless state is far from being geometrically obvious, and the terminal values of ϕ and f corresponding to various types of simple end constraints are not immediately obtainable.

A representation of the torsion and flexure to the second order which does not suffer from these disadvantages is desirable. A straight bar (initially along the z-axis, Fig. 2) may be imagined brought to a bent and twisted state by supposing it cut into thin discs. Let a typical disc be translated without rotation so that its centroid is brought to its final position P on the deflected curve and the principal axes are brought to x_1, y_1 parallel to x, y. It must now be rotated so that the tangent at P to the deflection curve is normal to it, in accordance with the theory of flexure of thin bars. Let this rotation consist of a rotation about y_1 bringing x_1 to x_2 in the normal plane at P, followed by a rotation about x_2 bringing y_1 to y_2 in the normal plane. The configuration so produced is evidently a possible state of bending and torsion. The principal axis x_2 is still parallel to the plane xz. This configuration is to be used as a reference from which to measure the torsional rotation of cross sections. To the first order the torsion is zero, but to the second order it is not.

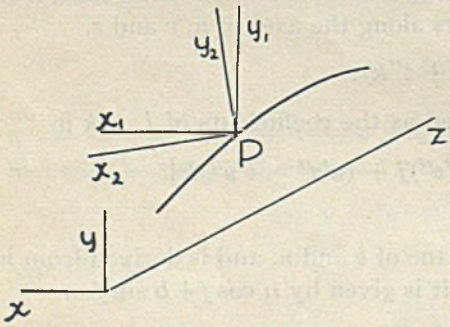

FIG. 2.

To determine its value, let the x, y axes in Fig. 2 correspond with the u, v axes, and let x_2 be the principal axis p. Then, in the proposed configuration, p is everywhere normal to the y, or v, axis. Thus the coefficient of j in (14), which represents the direc-

tion cosine of p with the v axis, must vanish, so that the value of f is determined by the equation

$$f_1 = \tan^{-1} \frac{v''}{u'z'' - z'u''}. \tag{16}$$

The torsion of the bar is $\tau_c + df_1/ds$, τ_c being given by (5), and is thus expressed in terms of the derivatives of u and v. When expanded in powers of these derivatives its leading term is $u''v'$. This is an approximation to the torsion with error of order ϵ. Thus if ϕ_1 is the value of ϕ corresponding to this configuration $\phi_1' = u''v'[1+O(\epsilon)]$. Also, f_0 is obtained from (16) by putting u_0, v_0 for u, v and it is easily found that $\tan f_0 = -(v_0''/u_0')+O(\epsilon^2)$. Since $\tan \delta = v_0'/u_0''$ it follows that $f_0+\delta = O(\epsilon^2)$. This being so $f_0+\delta$ in (15) ceases, for this particular configuration, to be significant, since its products with u'', v'' are of the order of the terms neglected.

Now consider an arbitrary state of (small) flexure and torsion specified by u, v, ϕ. It may be derived from the reference state just defined simply by rotating cross-sections about t in order to convert ϕ_1 to ϕ. Let β be the amount of such rotation. Then $\phi - \phi_1 = \beta$, and $\tau = \phi' = \beta' + \phi_1'$, that is

$$\tau = \beta' + u''v' \tag{17}$$

with error of order ϵ^2.

Let s now be measured from one end of the bar so that $s_0 = 0$. Then ϕ_1 like ϕ is zero at $s=0$ and $\phi_1 = \int_0^s u''v'ds$. Thus $\phi = \beta + \int_0^s u''v'ds$ and the integral is of order ϵ^2. Moreover $f_0+\delta$ is not altered by the rotation β so that it is still of order ϵ^2. The first of (15) becomes in consequence

$$\kappa_1 = -v'' + u''\left(\beta + \int_0^s u''v'ds\right).$$

The first term is or order ϵ/l, $u''\beta$ is of order ϵ^2/l and $u''\int_0^s u''v'ds$ is of order ϵ^3/l. Therefore, with an error of order ϵ^2 the new formulas for the components of curvature are

$$\kappa_1 = -v'' + \beta u'', \qquad \kappa_2 = u'' + \beta v''. \tag{18}$$

These with (17) give an alternative representation of the torsion and flexure, convenient because f_0 and δ have been eliminated, and β is relatively easily envisaged-being the angle by which the cross section must be rotated, about the deflected, tangent, to bring p from the position parallel to the axial plane in which it originally lies, to its final position. At fixed ends β is clearly zero.

6. Energy considerations. The strain energy is given by

$$\frac{1}{2}\int_0^l (EI_1\kappa_2^2 + EI_2\kappa_1^2 + GC\tau^2)dz. \tag{19}$$

The integration with respect to z rather than s will involve an error of order ϵ^2.

Consider now the problem referred to in the introduction—the straight bar twisted until it buckles. Let the state just prior to buckling be

$$\beta = B, \qquad u = 0, \qquad v = 0,$$

and after buckling

$$\beta = B + \beta_1, \qquad u = u_1, \qquad v = v_1.$$

Then B is small in the sense of ϕ in the preceding analysis. But β_1, u_1, v_1 are to be true infinitesimals, since we seek a buckled form which comes to the straight form as a limit. Thus they are to approach zero after a fixed value has been assigned to B.

The expressions (17) and (18) are now used in (19), and terms to the second order in u_1, v_1, β_1 and their derivatives, without regard to B, are retained. The result is

$$\frac{1}{2}\int_0^l [EI_1(u_1''^2 + 2Bu_1''v_1'') + EI_2(v_1''^2 - 2Bu_1''v_1'') + GC(B'^2 + 2B'\beta_1' + \beta_1'^2 + 2Bu_1''v_1')]dz. \tag{20}$$

Let M_3 be the critical torsional couple GCB'. On buckling, some work is done by this couple, but exactly how much, in terms of β_1, u_1, v_1 depends on the end constraints of the bar.

If the ends are in bearings which constrain the axis of the bar to remain fixed in direction at the ends—i.e., the ends are "built-in" with respect to flexure—the rotation of one end may be set as zero, and that of the other about the axis is then the value of β_1 at that end. The potential energy of M_3 in the buckled form is $-M_3\int_0^l \beta_1' dz$ referred to the twisted but unbuckled form as zero. The total potential energy is thus this term together with (20), omitting $\frac{1}{2}\int_0^l GCB'^2 dz$ which is the energy of the unbuckled twisted form.

If the potential energy is now varied by varying u_1 to $u_1+\epsilon_1\eta_1(z)$ the coefficient of ϵ_1 in the variation of the potential energy is

$$\int_0^l [-EI_2Bv_1'' + EI_1(u_1' + Bv_1'') + GCB'v_1']\eta_1' dz$$

and this must vanish if the buckled state is a possible state of equilibrium. Since Bv_1'' is small compared with u_1'', on account of the smallness of B, the conclusion is that the equation

$$EI_1u_1'' + M_3v_1' = 0 \tag{21}$$

must be satisfied. Similarly variation of v_1 yields

$$EI_2v_1'' - M_3u_1' = 0. \tag{22}$$

Variation of β_1 yields $GCB'+GC\beta_1'-M_3=0$, that is $\beta_1'=0$. Equations (21) and (22) are identical with the equations obtainable by direct equilibrium considerations. They are derived in this manner here in order to show that the terms M_3v_1', $-M_3u_1'$ arise from terms in the strain energy of torsion which are of higher order than the term $\frac{1}{2}\int_0^l GC\beta'^2 dz$ hitherto accepted. It is to be expected therefore that in (17) and (18) the terms of the second order will be required in energy calculations in other problems where torsional loads cause, or contribute to, buckling.

When the equilibrium of the straight twisted form is neutral, the work done by M_3 during buckling is equal to the gain of strain energy. Then

$$M_3\int_0^l \beta_1' dz = \frac{1}{2}\int_0^l [EI_1(u_1''^2 + 2Bu_1''v_1'') + EI_2(v_1''^2 - 2Bu_1''v_1'') + GC(2B'\beta_1' + \beta_1'^2 + 2B'u_1''v_1')]dz. \tag{23}$$

The term $Bu_1''v_1''$ in the flexural terms is small compared with $u_1''^2$ or $v_1''^2$ and will be dropped. Introducing $M_3 = GCB'$ the resulting equation yields

$$M_3 = -\frac{1}{2}\frac{\int_0^l (EI_1u_1''^2 + EI_2v_1''^2 + GC\beta_1'^2)dz}{\int_0^l u_1''v_1'\,dz}. \tag{24}$$

Now equations (21), (22) (after one differentiation) together with $\beta_1' = 0$ are the Euler differential equations for the functions u_1, v_1, β_1 making the right of (24) a minimum. Since $\beta_1' = 0$ the term $GC\beta_1'^2$ in the numerator of (24) may be dropped. The critical M_3 is the least value of the right of (24) with or without this term. Without it the equation may be interpreted as showing that the energy of flexure which appears when buckling occurs is accounted for by a decrease of torsional energy of amount $M_3\int_0^l u_1''v_1'\,dz$.

The same equation is suitable for the approximate determination of the critical torque by the Rayleigh method—assuming simple plausible forms for u_1 and v_1 and adjusting the parameters of these forms to obtain a least value of M_3. This method is applicable to non-uniform bars.

Equation (23) would in general require modification if the ends are not "built-in," for instance if they are attached to Hooke's joints. For then the work of M_3 is not done merely on a rotation $\int_0^l \beta_1'\,dz$. Certain terms of higher order must be added to β_1', and these can be of the same order as $u_1''v_1'$. Such terms would be significant in (24). Nevertheless (24) is appropriate in the Rayleigh method whatever the end constraints, for its minimizing conditions are the differential equations of equilibrium which must be satisfied irrespective of end constraints.

There are expressions other than the right of (24) which yield the critical M_3 as a minimum value. If (21) and (22) are multiplied respectively by u_1'', v_1'', integrated along the bar, and added, the result yields another in the form

$$M_3 = \frac{\int_0^l (EI_1u_1''^2 + EI_2v_1''^2)dz}{\int_0^l (u_1'v_1'' - v_1'u_1'')dz}.$$

MEMBRANE STRESSES IN SHELLS OF CONSTANT SLOPE*

BY

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1. A surface S of constant slope may be generated by a straight line L sliding along a plane curve C_0 (say, in the xy plane), maintaining a right angle with the tangent to C_0 and a constant angle θ with its binormal (i.e., with the z axis). When a closed curve C_0 is chosen, the surface is an obvious generalization of a circular cone[2] (see Fig. 1). Since "near-conical" shells occur often in practice,[3] it may be of interest to discuss such effects as fall within the scope of the membrane theory of shells.

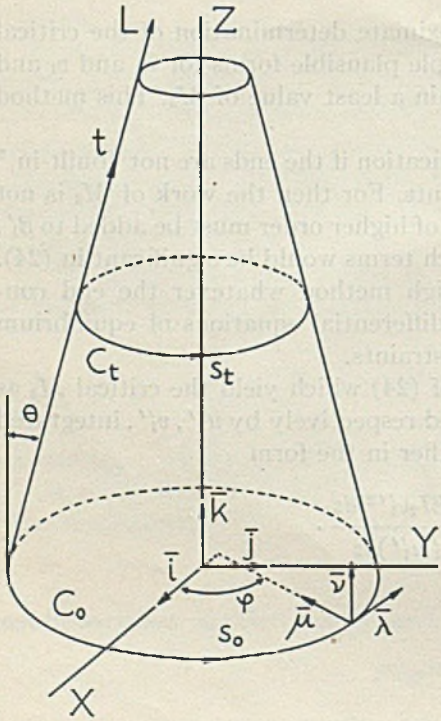

FIG. 1.

We introduce the following notations:

$\bar{i}, \bar{j}, \bar{k}$, unit vectors in fixed rectangular directions x, y, z;

$\bar{\lambda}, \bar{\mu}, \bar{\nu}$, unit tangent, normal, and binormal of curve C_0;

t, length along generators L;

s_t, ρ_t, arc length and radius of curvature of a horizontal section C_t of the surface S; subscripts 0 and 1 will designate corresponding quantities in the end sections C_0 and C_1 of the shell;

$\bar{r} = \bar{r}(s_0)$, vector equation of curve C_0;

φ, angle between the positive x axis and the outward normal of C_0;

E, ν, G, Young's modulus, Poisson's ratio, and shear modulus;

h, thickness of shell having the surface S for middle surface;

N_s, N_t, normal forces per unit length of sections of the shell which are perpendicular to s- and t-directions respectively (Fig. 3);

N_{st}, shearing force in s-direction per unit length of shell section perpendicular to t-direction;

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[1] The author wishes to express his appreciation to Professor W. Prager for proposing the problem and for other valuable suggestions.

[2] Non-circular cones (for which the generators meet in one point while their "slope" varies) have been considered recently by A. Pflüger, Z. angew. Math. Mech. 22, 99–116 (1942).

[3] The fuselages of some aeroplanes, for instance, can be approximated by one or several shells of different slopes connected by stiff bulkheads. The construction of models is relatively simple because each portion forms a developable surface.

e_{ss}, e_{tt}, e_{st}, strains corresponding to N_s, N_t, and N_{st}, respectively.

We note some simple relationships:

$$\frac{d\bar{r}}{ds_0} = \bar{\lambda}; \qquad \bar{\nu} = \bar{k}; \tag{1.1}$$

$$\bar{\lambda} = -\bar{\imath}\sin\varphi + \bar{\jmath}\cos\varphi; \qquad \bar{\mu} = -\bar{\imath}\cos\varphi - \bar{\jmath}\sin\varphi. \tag{1.2}$$

Since $\rho_0 = ds_0/d\varphi$, we obtain from (1.2) the Frenet-Serret formulae for a plane curve:

$$d\bar{\lambda}/ds_0 = \bar{\mu}/\rho_0; \qquad d\bar{\mu}/ds_0 = -\bar{\lambda}/\rho_0. \tag{1.3}$$

The vector equation of the surface of constant slope S has the form:

$$\bar{R}(s_0, t) = \bar{r}(s_0) + t(\bar{\mu}\sin\theta + \bar{\nu}\cos\theta). \tag{1.4}$$

For a constant value of t, (1.4) is the vector equation of the horizontal section C_t. Then, $\partial\bar{R}/\partial s_t$ is the unit vector tangent to C_t. Since $\partial\bar{R}/\partial s_t = \bar{\lambda}(\rho_0 - t\sin\theta)ds_0/\rho_0 ds_t$, C_t is parallel to C_0 at corresponding points (see Fig. 2), and

$$ds_t/ds_0 = (\rho_0 - t\sin\theta)/\rho_0. \tag{1.5}$$

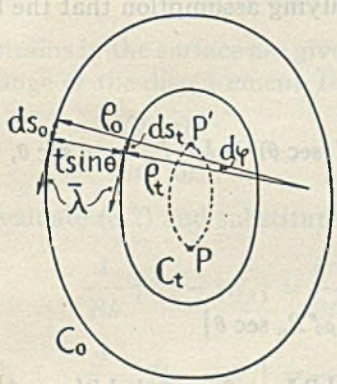

FIG. 2.

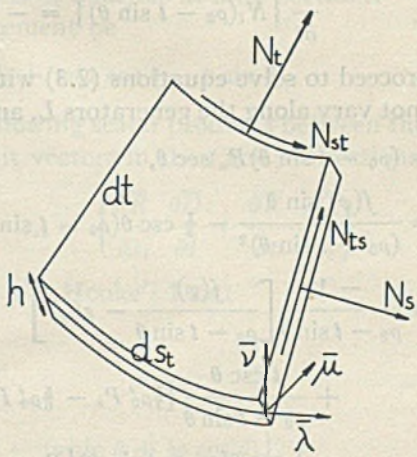

FIG. 3.

Hence, for corresponding points, the centers of curvature of C_0 and C_t coincide, and

$$\rho_t = \rho_0 - t\sin\theta. \tag{1.6}$$

If the shell is long, it may happen that at some point $\rho_t = 0$. At such a point the tangent to C_t ceases to turn continuously (see points P, P' in Fig. 2). We shall discuss only the portion of the shell where $t\sin\theta < \rho_0$, i.e., the open shell without the "tail edge."

2. An element of a shell of thickness h having the surface S for middle surface is shown in Fig. 3. According to the usual assumptions of the membrane theory of shells,[4] the bending stresses as well as effects of curvature of S are disregarded and

[4] See for instance S. P. Timoshenko, *Theory of plates and shells*, McGraw-Hill Co., New York, 1940, p. 356; also the first chapter of W. Flügge's *Statik und Dynamik der Schalen*, J. Springer, Berlin, 1934.

one has $N_{st}=N_{ts}$. The total forces acting on the faces hds_t and hdt of the element are respectively:

$$-\left\{N_t(\bar{\mu}\sin\theta+\bar{\nu}\cos\theta)+N_{st}\bar{\lambda}\right\}(\rho_0-t\sin\theta)d\varphi, \tag{2.1a}$$

$$-\left\{N_s\bar{\lambda}+N_{st}(\bar{\mu}\sin\theta+\bar{\nu}\cos\theta)\right\}dt. \tag{2.1b}$$

Let $\bar{P}=P_s\bar{\lambda}+P_t(\bar{\mu}\sin\theta+\bar{\nu}\cos\theta)+P_n(\bar{\mu}\cos\theta-\bar{\nu}\sin\theta)$ represent the load per unit area of the surface. Then the condition of equilibrium of the element of the shell is:

$$\frac{\partial}{\partial t}\left\{\left[N_t(\bar{\mu}\sin\theta+\bar{\nu}\cos\theta)+N_{st}\bar{\lambda}\right](\rho_0-t\sin\theta)\right\}dtd\varphi$$
$$+\frac{\partial}{\partial\varphi}\left\{N_s\bar{\lambda}+N_{st}(\bar{\mu}\sin\theta+\bar{\nu}\cos\theta)\right\}dtd\varphi=(\rho_0-t\sin\theta)\bar{P}dtd\varphi. \tag{2.2}$$

Equating the components of these forces in the n, s, and t directions, we obtain three equations for the determination of the three stress components:

$$\begin{aligned}
&N_s=(\rho_0-t\sin\theta)P_n\sec\theta,\\
&\frac{\partial}{\partial t}\left\{N_{st}(\rho_0-t\sin\theta)\right\}-N_{st}\sin\theta=(\rho_0-t\sin\theta)P_s-\frac{\partial N_s}{\partial\varphi},\\
&\frac{\partial}{\partial t}\left\{N_t(\rho_0-t\sin\theta)\right\}=-\frac{\partial N_{st}}{\partial\varphi}+(\rho_0-t\sin\theta)P_t-N_s\sin\theta.
\end{aligned} \tag{2.3}$$

We proceed to solve equations (2.3) with the simplifying assumption that the load $\bar{P}$ does not vary along the generators L, and obtain:[5]

$$\begin{aligned}
N_s&=(\rho_0-t\sin\theta)P_n\sec\theta,\\
N_{st}&=\frac{f(\varphi)\sin\theta}{(\rho_0-t\sin\theta)^2}-\tfrac{1}{3}\csc\theta(\rho_0-t\sin\theta)(P_s-P_n'\sec\theta)+\tfrac{1}{2}\rho_0'P_n\csc\theta\sec\theta,\\
N_t&=\frac{-1}{\rho_0-t\sin\theta}\left[\frac{f(\varphi)}{\rho_0-t\sin\theta}-g(\varphi)\right]'\\
&\quad+\frac{t\csc\theta}{\rho_0-t\sin\theta}\left[\tfrac{1}{3}\rho_0'P_s-\tfrac{5}{6}\rho_0'P_n'\sec\theta-\tfrac{1}{2}\rho_0''P_n\sec\theta\right]\\
&\quad-\tfrac{1}{2}\csc\theta(\rho_0-t\sin\theta)\left[P_t-P_n\tan\theta-\tfrac{1}{3}P_n''\csc\theta\sec\theta+\tfrac{1}{3}P_s'\csc\theta\right],
\end{aligned} \tag{2.4}$$

where $f(\varphi)$ and $g(\varphi)$ are arbitrary functions of φ and the prime denotes differentiation with respect to φ. If the curve C_0 is closed the continuity of stresses demands that f and g' have a period of 2π.

When the load on the shell is applied only through the end sections C_0 and C_1 the stress system becomes:

$$N_s=0;\quad N_{st}=\frac{f\sin\theta}{(\rho_0-t\sin\theta)^2};\quad N_t=\frac{-1}{(\rho_0-t\sin\theta)}\left[\frac{f}{\rho_0-t\sin\theta}-g\right]'. \tag{2.5}$$

Substituting (2.5) into (2.1a) and integrating between 0 and 2π, we obtain the resultant force $\bar{F}_t$ acting on the section C_t; the expression for $\bar{F}_t$ simplifies readily by virtue of (1.2):

[5] In the case of cylindrical surfaces, $\theta=0$ and integration of (2.3) leads to the special solution: $N_s=\rho P_n$; $N_{st}=f(s)+t(P_s-dN_s/ds)$; $N_t=g(s)-tdf/ds+tP_t+t^2/2(d^2N_s/ds^2-dP_s/ds)$; where $f(s)$ and $g(s)$ are arbitrary functions of the arc length s. In this connection see pp. 66–76 of Flügge's book.

$$\overline{F}_t = \int_0^{2\pi} \left\{ -\left[\frac{f}{\rho_0 - t\sin\theta} (\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) \right]' + g'(\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) \right\} d\varphi$$

$$= -\sin\theta \left\{ \bar{\imath} \int_0^{2\pi} g'\cos\varphi\, d\varphi + \bar{\jmath} \int_0^{2\pi} g'\sin\varphi\, d\varphi \right\} + \bar{k}\cos\theta \{ g(2\pi) - g(0) \}. \tag{2.6}$$

The resultant moment $\overline{M}_t$ about the origin due to the forces on the section C_t is found similarly:

$$\overline{M}_t = \int_0^{2\pi} \overline{R} \times \left\{ -\left[\frac{f}{\rho_0 - t\sin\theta} (\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) \right]' + g'(\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) \right\} d\varphi.$$

It follows by integration by parts that

$$\overline{M}_t = \overline{R}(0) \times \overline{F}_t + \bar{k}\sin\theta \int_0^{2\pi} f d\varphi + \cos\theta \int_0^{2\pi} (\bar{\imath}\cos\varphi + \bar{\jmath}\sin\varphi) f\, d\varphi$$

$$- \int_0^{2\pi} \bar{\lambda} \times \left\{ \int_0^{\varphi} g'(\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) d\varphi \right\} (\rho_0 - t\sin\theta) d\varphi. \tag{2.7}$$

The results (2.6) and (2.7) will form the basis of analysis in later sections.

3. Let the vector of infinitesimal displacement be

$$\overline{D} = u\bar{\lambda} + v(\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) + w(\bar{\mu}\cos\theta - \bar{\nu}\sin\theta). \tag{3.1}$$

The strains in the surface are given by the following scalar products between the rates of change of the displacement $\overline{D}$ and the unit vectors in the t and s directions:

$$e_{tt} = \frac{\partial \overline{R}}{\partial t} \cdot \frac{\partial \overline{D}}{\partial t}, \qquad e_{ss} = \frac{\partial \overline{R}}{\partial s_t} \cdot \frac{\partial \overline{D}}{\partial s_t}, \qquad e_{st} = \left\{ \frac{\partial \overline{R}}{\partial s_t} \cdot \frac{\partial \overline{D}}{\partial t} + \frac{\partial \overline{R}}{\partial t} \cdot \frac{\partial \overline{D}}{\partial s_t} \right\}. \tag{3.2}$$

We evaluate (3.2) and substitute the results into Hooke's Law:

$$\begin{aligned} \frac{1}{Eh}\{N_t - \nu N_s\} &= \frac{\partial v}{\partial t}, \\ \frac{1}{Eh}\{N_s - \nu N_t\} &= \frac{1}{\rho_0 - t\sin\theta}\{u' - (v\sin\theta + w\cos\theta)\}, \\ \frac{2(1+\nu)}{Eh} N_{st} &= \frac{1}{\rho_0 - t\sin\theta}\left\{ (\rho_0 - t\sin\theta)\frac{\partial u}{\partial t} + u\sin\theta + v' \right\}. \end{aligned} \tag{3.3}$$

Equations (3.3) are easily integrated to yield expressions for the displacements:

$$\begin{aligned} v &= \frac{1}{Eh}\int^t (N_t - \nu N_s) dt + A(\varphi), \\ u &= \frac{2(1+\nu)}{Eh}(\rho_0 - t\sin\theta)\int^t \frac{N_{st}}{\rho_0 - t\sin\theta} dt \\ &\quad - \frac{(\rho_0 - t\sin\theta)}{Eh}\int^t \frac{\int^t (N_t' - \nu N_s') dt}{(\rho_0 - t\sin\theta)^2} dt - A'(\varphi)\csc\theta + (\rho_0 - t\sin\theta)B(\varphi), \\ w &= u'\sec\theta + v\tan\theta - \frac{1}{Eh}(\rho_0 - t\sin\theta)(N_s - \nu N_t)\sec\theta, \end{aligned} \tag{3.4}$$

where $A(\varphi)$ and $B(\varphi)$ are arbitrary functions. When the stresses have the form (2.5), the displacements can be expressed directly in terms of the functions f and g:

$$
\begin{aligned}
v &= \frac{\csc\theta}{Eh}\left\{-f'\rho_t^{-1} + \tfrac{1}{2}f\rho'\rho_t^{-2} - g'\ln\rho_t\right\} + A,\\
u &= \frac{\csc^2\theta}{Eh}\left\{(1+\nu)f\,\rho_t^{-1}\sin^2\theta + \tfrac{1}{2}f''\rho_t^{-1} - \tfrac{1}{6}(f\rho'' + 3f'\rho')\rho_t^{-2} + \tfrac{1}{4}f\rho'^2\rho_t^{-3}\right.\\
&\quad \left. + \tfrac{1}{2}g'\rho'\rho_t^{-1} + g''(\ln\rho_t + 1)\right\} - A'\csc\theta + \rho_t B,\\
w &= \frac{\sec\theta\csc^2\theta}{Eh}\left\{\sin^2\theta\left[-\tfrac{3}{2}f\rho'\rho_t^{-2} + 2f'\rho_t^{-1} + g'(\ln\rho_t + \nu)\right] + \tfrac{1}{2}f'''\rho_t^{-1}\right.\\
&\quad - \tfrac{1}{6}(f\rho''' + 4f'\rho'' + 6f''\rho')\rho_t^{-2} + \tfrac{1}{12}(15f'\rho'^2 + 10f\rho'\rho'')\rho_t^{-3}\\
&\quad \left. - \tfrac{3}{4}f\rho'^3\rho_t^{-4} + \tfrac{1}{2}(g'\rho'' + 3g''\rho')\rho_t^{-1} - \tfrac{1}{2}g'\rho'^2\rho_t^{-2} + g'''(\ln\rho_t + 1)\right\}\\
&\quad - \tan\theta(A + A''\csc^2\theta) + B'\rho_t\sec\theta + B\rho'\sec\theta.
\end{aligned}
\tag{3.5}
$$

Expressions for displacements D_x, D_y, D_z in the x, y, z (or any other) directions are best derived by taking a scalar product between a unit vector in the given direction and $\overline{D}$ of (3.1). For instance,

$$D_z = \bar{k}\cdot\overline{D} = v\cos\theta - w\sin\theta. \tag{3.6}$$

4. The current literature on shells contains very little on the boundary conditions in the membrane theory of shells. We recall that local bending of the shell was disregarded according to the simplifying assumptions of the theory. Thus we cannot expect to satisfy all of the usual boundary conditions. For instance, we cannot ask that the heavy end bulkhead be considered rigid; in bending of the shell as a whole this would entail $e_{ss} = N_t = 0$ in the end section which could consequently transmit no bending moment. By allowing deformations in the plane of the end sections we remove the restriction on N_t and the problem of bending has a solution (see section 5). One has to decide in every particular problem which boundary conditions correspond more nearly to the assumption of no local bending.

A casual reader might be tempted to interpret the contribution of A and B to the displacements in (3.4) as that of rigid body motion since it is present when the stresses vanish. However, it is conceivable that a given state of stress induces inextensional displacements other than those of a rigid body as necessitated by the shape of the shell. Thus, in the case of a non-circular cylindrical shell under torsion, A accounts for the warping of the cross-sections.[6]

In general, these inextensional deformations are accompanied by local bending stresses which must be small to be neglected in accordance with our assumptions. One would expect that no energy is expended in the inextensional deformations. The strain energy in shells loaded through the end-sections is

$$V = \frac{1}{2h}\int_0^{t_1}\int_0^{2\pi}\left(\frac{N_t^2}{E} + \frac{N_{st}^2}{G}\right)\rho_t d\varphi\, dt, \tag{4.1}$$

[6] Specifically, $A = (T/2A_0Gh)\left\{\int_0^\varphi \rho d\varphi - (1/2A_0)\int_0^\varphi (x\rho\cos\varphi + y\rho\sin\varphi)d\varphi\right\}$. This expression is found by the method indicated in section 8.

or

$$V = \frac{1}{2}\int_0^{2\pi} (N_{st}u + N_t v)\rho_t d\varphi \Big|_{t=0}^{t=t_1}. \tag{4.2}$$

Substituting (2.5) and the contribution due to A and B into (4.2) and integrating by parts, we find that our expectation is verified. The accompanying local bending, however, absorbs energy and, therefore, places limitations on the inextensional displacements according to the principle of minimum strain energy. The minimum expenditure of energy in bending occurs when the inextensional displacements reduce to rigid body displacements.

One can easily verify that the most general functions A and B corresponding to rigid body displacements have the form

$$\begin{aligned} A &= -a_x \sin\theta\cos\varphi - a_y \sin\theta\sin\varphi + a_z\cos\theta + \alpha_x y_0 \cos\theta \\ &\quad - \alpha_y x_0 \cos\theta + \alpha_z \sin\theta(y_0\cos\varphi - x_0\sin\varphi), \\ B &= \alpha_x \cot\theta\cos\varphi + \alpha_y\cot\theta\sin\varphi + \alpha_z, \end{aligned} \tag{4.3}$$

where a_x, a_y, a_z represent the infinitesimal translations in the x, y, z directions; α_x, α_y, α_z the infinitesimal rotations about the x, y, z axes; and x_0, y_0 the coordinates in the base section C_0.

Instead of imposing conditions on the displacements, one may prescribe a sensible distribution of stresses at the boundary. We note that by (2.5) the state of stress in the whole shell is determined as soon as the stresses N_t and N_{st} are given at *one* end-section. Thus two different stress distributions which are statically equivalent over an end-section will determine distinctly different stress distributions in the rest of the shell.[7]

5. We shall study first the effects of taper[8] as exhibited in a conical shell of circular cross-section; later, we shall discuss the influence of a variable radius of curvature ρ_t of the section C_t.

Let M represent the bending moment (causing tension for $x_t > 0$) applied to the shell through the end-sections C_0 and C_1. We shall try to satisfy the conditions that the end-sections (bulkheads) remain plane, i.e.,

$$D_z = 0 \text{ for } t = 0; \qquad D_z = \beta(x_0 - t_1\sin\theta) \text{ for } t = t_1; \tag{5.1}$$

where β is the (undetermined) angle of bending, and that the displacements due to A and B reduce to rigid body displacements (4.3). By virtue of (3.6), (3.5), and (4.3), we obtain for the first of conditions (5.1)

$$-\frac{\sec\theta\csc\theta}{Eh}\left\{\frac{1}{2r}(2f'(1+\sin^2\theta) + f''') + (g' + g''')\ln r + \nu g'\sin^2\theta + g'''\right\} - \alpha_y r\cos\varphi = 0. \tag{5.2}$$

[7] This is the price that has to be paid for the simplifications due to the assumptions of the membrane theory. A "disturbance" of the state of stress on one end-section (the difference between the equivalent stress distributions) "propagates" itself along the generators without "dying out." The general theory of thin shells would lead to differential equations of higher order; for these one can find solutions representing disturbances that die out with the distance from the end-section.

[8] All the results of sections 5–9 simplify to the corresponding expressions for a cylinder as the taper approaches zero.

By symmetry, the functions g' and f' are odd in x_0; let their Fourier expansions read

$$g' = \sum_0^\infty (2n+1)a_{2n+1} \cos (2n+1)\varphi, \quad f' = \sum_0^\infty (2n+1)b_{2n+1} \cos (2n+1)\varphi. \tag{5.3}$$

Since the resultant force $\overline{F}_0$ on C_0 must vanish, one concludes from (2.6) that $a_1=0$. Equation (2.7) yields $\overline{M}_t = j\pi b_1 \cos\theta$ or $b_1 = -(1/\pi)M \sec\theta$. It follows from the coefficient of $\cos\varphi$ in (5.2) that

$$\alpha_y = (1/2\pi Ehr^2) \sec^2\theta \csc\theta(1 + 2\sin^2\theta)M. \tag{5.4}$$

Substituting (5.4) and (5.3) into the second of conditions (5.1) and equating coefficients of $\cos\varphi$ in the two members, we obtain

$$\beta = \frac{M \sin\theta(2 + \csc^2\theta)}{2\pi Eh \cos^2\theta}\left\{\frac{1}{(r - t_1 \sin\theta)^2} - \frac{1}{r^2}\right\}. \tag{5.5}$$

For the coefficients a_{2n+1} and b_{2n+1}, $n>0$, one obtains a system of two homogeneous equations with a non-vanishing determinant. Therefore $a_{2n+1}=b_{2n+1}=0$, $n>0$, and

$$N_t = \frac{M \sec\theta \cos\varphi}{\pi(r - t\sin\theta)^2}, \qquad N_{st} = -\frac{M \tan\theta \sin\varphi}{\pi(r - t\sin\theta)^2}, \tag{5.6}$$

$$D_z = \frac{M \sin\theta(2 + \csc^2\theta)}{2\pi Eh \cos^2\theta}\left\{\frac{1}{(r - t\sin\theta)^2} - \frac{1}{r^2}\right\} x_t. \tag{5.7}$$

If we designate by I_t the moment of inertia, $\pi(r-t\sin\theta)^3$, of C_t about the neutral axis, we can write $N_t=(1/I_t)Mx_t \sec\theta$. Essentially the stresses in the z direction follow the classical beam formula; the influence of taper is manifested by the presence of the x components of stresses N_t which have to be balanced by N_{st}. From (5.7) we see that all sections C_t remain plane. The rate of change of the angle of bending increases as the shell grows narrower:

$$\frac{d\beta}{dz} = \frac{M(1 + 2\sin^2\theta)}{\pi Eh \cos^3\theta(r - t\sin\theta)^3} = \frac{M(1 + 2\sin^2\theta)}{EhI_t \cos^3\theta}. \tag{5.8}$$

Further effects of taper are apparent in the other displacements:

$$v = \frac{M \tan\theta}{2\pi Eh} \cos\varphi\left\{\frac{2\csc^2\theta}{r - t\sin\theta} - \frac{1}{r}(2\csc^2\theta - \nu)\right\}, \tag{5.9}$$

$$u = \frac{M \sec\theta}{2\pi Eh} \sin\varphi\left\{\frac{\csc^2\theta - 2 - 2\nu}{r - t\sin\theta} - \frac{1}{r}(2\csc^2\theta - \nu) + \frac{1}{r^2}(r - t\sin\theta)(\csc^2\theta + 2)\right\}, \tag{5.10}$$

$$w = \frac{M \sec^2\theta}{2\pi Eh} \cos\varphi\left\{\frac{\csc^2\theta - 4}{r - t\sin\theta} - \frac{1}{r}\cos^2\theta(2\csc^2\theta - \nu) + \frac{1}{r^2}(r - t\sin\theta)(\csc^2\theta + 2)\right\}, \tag{5.11}$$

$$D_x = \frac{M \sec\theta}{2\pi Eh}\left\{\frac{2 - \csc^2\theta + 2\nu\sin^2\varphi}{r - t\sin\theta} + \frac{1}{r}(2\csc^2\theta - \nu) - \frac{1}{r^2}(r - t\sin\theta)(\csc^2\theta + 2)\right\}. \tag{5.12}$$

If we take for X, the fictitious displacement of the axis of the cone, the average of D_x over C_t (by analogy with a cylinder or prism), we obtain for the slope of the deformed axis

$$\frac{dX}{dz} = \frac{M \sin\theta}{2\pi Eh \cos^2\theta}\left\{\frac{2+\nu-\csc^2\theta}{(r-t\sin\theta)^2} + \frac{1}{r^2}(2+\csc^2\theta)\right\}. \tag{5.13}$$

Comparison with equation (5.7) shows that the axis is not perpendicular to the sections C_t as one might expect. Nor is the increment in slope equal to β, the angle between the end sections. In fact, for $\csc^2\theta = 2+\nu$, a large taper, the axis remains altogether straight despite the angle between C_0 and C_1. This is due to a slipping effect caused by an interplay of the shearing forces N_{zt} and the x components of N_t. Finally, let us check (5.5) by the customary[9] application of Castigliano's Principle, $\partial V/\partial M = \beta$. Substituting (5.6), (5.9), and (5.10) into (4.2), we have

$$V\big|_{t=0} = \frac{M^2 \sin\theta}{4\pi Eh\cos^2\theta}\,\frac{2\nu}{r^2}, \tag{5.14}$$

$$V = \frac{M^2\sin\theta(2+\csc^2\theta+2\nu)}{4\pi Eh\cos^2\theta}\left\{\frac{1}{(r-t_1\sin\theta)^2} - \frac{1}{r^2}\right\}, \tag{5.15}$$

$$\beta_V = \frac{M\sin\theta(2+\csc^2\theta+2\nu)}{2\pi Eh\cos^2\theta}\left\{\frac{1}{(r-t_1\sin\theta)^2} - \frac{1}{r^2}\right\}. \tag{5.16}$$

The discrepancy between (5.5) and (5.16) is negligible in practical applications, but is interesting theoretically. It springs from a loose interpretation of Castigliano's Principle above, which is strictly true only for a concentrated couple M. Since M is distributed over the end sections, it does work not only in bending the shell but also in deforming the end-sections within their planes, as seen from (5.14). When the end-sections are alike as in a cylinder or prism, as much energy is spent in the deformation of one end as is gained at the other end; then, Castigliano's Principle holds even for a distributed moment. But to obtain the correct angle of bending in the case of a cone, one must deduct from the total strain energy (5.15) the net energy absorbed in the plane deformation of C_0 and C_1, namely

$$\frac{M^2\nu\sin\theta}{2\pi Eh\cos^2\theta}\left\{\frac{1}{(r-t_1\sin\theta)^2} - \frac{1}{r^2}\right\}.$$

6. We derive easily the expressions for the stresses in a cone twisted by a torque T by making either $D_z = 0$ or $N_t = 0$ at C_0 and C_1 and using (2.6) and (2.7),

$$N_{zt} = \frac{T}{2\pi(r-t\sin\theta)^2}, \qquad N_t = 0. \tag{6.1}$$

From the displacements or the strain energy we obtain the total angle of twist γ and the angle of twist per unit length of the cone

[9] See for instance Timoshenko, *Strength of materials*, vol. 1, D. Van Nostrand, New York, 1940, p. 312.

$$\gamma = \frac{T \csc\theta}{4\pi Gh}\left\{\frac{1}{(r - t_1 \sin\theta)^2} - \frac{1}{r^2}\right\}, \qquad \frac{d\gamma}{dz} = \frac{T \sec\theta}{2\pi Gh(r - t\sin\theta)^3}. \tag{6.2}$$

Here, the effects of taper as manifested in (6.1) and (6.2) are not unexpected.

More interesting is the case of a cone supported at C_0 and bent by a force R (in the x direction) distributed over C_1. We learn from (2.6) that the function $f(\varphi)$ and hence the shear stress N_{st} do not actually contribute to the resultant R acting on any section C_t. Expressions (2.6) and (2.7) show that the term in $\cos\varphi$ of g' alone influences the resultant force as well as moment on C_t. We superpose a state of stress given by (5.6) with $M = R\cot\theta(r - t_1\sin\theta)$ in order to bring the moment across C_1 to zero, and obtain the final result

$$N_t = \frac{-R(t_1 - t)\cos\varphi}{\pi(r - t\sin\theta)^2}, \qquad N_{st} = \frac{-R(r - t_1\sin\theta)\sin\varphi}{\pi(r - t\sin\theta)^2}. \tag{6.3}$$

7. Let us now consider shells with non-circular cross-sections C_t. The coordinates of points on C_t are expressed in terms of ρ_t and φ

$$x_t = x_t(0) - \int_0^\varphi \rho_t \sin\varphi\, d\varphi, \qquad y_t = \int_0^\varphi \rho_t \cos\varphi\, d\varphi. \tag{7.1}$$

It is clear from (7.1) that ρ_t cannot contain any terms in $\cos\varphi$ or $\sin\varphi$ if the shell is closed. If only cosine terms appear in the Fourier expansion

$$\rho_t = r_t - \sum_2^\infty r_n \cos n\varphi, \tag{7.2}$$

the section C_t is symmetric with respect to the x axis. The simple section, for which $r_n = 0$ if $n \neq 3$, approximates the cross-section of many a fuselage:

$$\begin{gathered} x_t = r_t \cos\varphi + \tfrac{1}{4}r_3 \cos 2\varphi - \tfrac{1}{8}r_3 \cos 4\varphi; \\ x_t(0) = r_t + \frac{r_3}{8}, \qquad x_t(\pi) = -r_t + \frac{r_3}{8}; \\ y_t = r_t \sin\varphi - \tfrac{1}{4}r_3 \sin 2\varphi - \tfrac{1}{8}r_3 \sin 4\varphi; \qquad y_t(\pi/2) = r_t. \end{gathered} \tag{7.3}$$

The neutral axis of the sections coincides with the y axis (i.e., is independent of t) if x_t contains no constant term and if

$$\int_0^{2\pi} x_0\rho_0 d\varphi = \frac{\pi}{2}\sum_2^\infty \frac{r_n(r_{n+1} - r_{n-1})}{n} = 0. \tag{7.4}$$

In bending, only sections satisfying (7.4) will be considered.

8. The stresses in a shell of constant slope under torsion are determined from the conditions that the load is applied in such a manner that only shearing stresses are generated at the end-sections. The conditions $N_t = 0$ at $t = 0$ and $t = t_1$ yield $f = k\rho_0(\rho_0 - t_1\sin\theta)$ and $g = k(\rho_0 - t_1\sin\theta)$. Substituting the expression for f into N_{st}, we find the torque T on C_0

$$T = \int_0^{2\pi} \bar{R} \times N_{st}\bar{\lambda}\rho_0 d\varphi = k\sin\theta\left\{\oint \bar{R} \times \frac{d\bar{R}}{ds_0}\, ds_0 - t_1 \sin\theta \int_0^{2\pi} \bar{R} \times \bar{\lambda}\, d\varphi\right\},$$

which reduces to

$$T = k \sin\theta\{2A_0 - t_1L_0 \sin\theta\} = k \sin\theta\{A_0 + A_1 - \pi t_1^2 \sin^2\theta\}. \tag{8.1}$$

Here the A's represent the areas of the sections and L_0 is the length of C_0, all quantities easily measurable. Then,

$$N_{st} = \frac{T}{(A_0 + A_1 - \pi t_1^2 \sin^2\theta)} \cdot \frac{\rho_0\rho_1}{\rho_t^2}, \tag{8.2}$$

$$N_t = \frac{-T}{(A_0 + A_1 - \pi t_1^2 \sin^2\theta)} \cdot \frac{t}{\rho_t}\left(\frac{\rho_1}{\rho_t}\right)' = \frac{-T \sin\theta}{(A_0 + A_1 - \pi t_1^2 \sin^2\theta)} \cdot \frac{t(t_1 - t)\rho'}{\rho_t^3}. \tag{8.3}$$

The effect of the variable radius of curvature of C_t is observed in the expression for N_t; tensile stresses increase directly with ρ' and inversely with ρ_t^3.

The expression (4.2) for strain energy takes the form

$$V = \frac{k^2 \csc\theta}{2Eh}\left\{t_1(1+\nu)\sin^3\theta(L_0 + L_1) + \frac{t_1^2 \sin^2\theta}{12}\int_0^{2\pi}\left(\frac{\rho'}{\rho_t}\right)^2 d\varphi\Bigg|_0^{t_1}\right.$$
$$\left. - \frac{t_1 \sin\theta}{2}\int_0^{2\pi}\rho'^2\left(\frac{1}{\rho_1} + \frac{1}{\rho_0}\right)d\varphi - \int_0^{2\pi}\rho'^2 \ln\left(\frac{\rho_1}{\rho_0}\right)d\varphi\right\}, \tag{8.4}$$

and the angle of twist is

$$\gamma = \frac{T}{Eh(A_0 + A_1 - \pi t_1^2 \sin^2\theta)^2}\left[t_1(1+\nu)(L_0 + L_1) + \csc^3\theta\int_0^{2\pi}\left\{\frac{t_1^2 \sin^2\theta}{12}\rho'^2\left(\frac{1}{\rho_1^2} - \frac{1}{\rho_0^2}\right)\right.\right.$$
$$\left.\left. - \frac{t_1 \sin\theta}{2}\rho'^2\left(\frac{1}{\rho_1} + \frac{1}{\rho_0}\right) - \rho'^2 \ln\left(\frac{\rho_1}{\rho_0}\right)\right\}d\varphi\right]. \tag{8.5}$$

The quantity in the braces is of the order of $\sin^5\theta$. Also, each of its terms contains the factor ρ'^2. In the common case of small taper and nearly circular shell we may use as a good approximation

$$\gamma_{app} = \frac{Tt_1(L_0 + L_1)}{2Gh(A_0 + A_1 - \pi t_1^2 \sin^2\theta)^2}. \tag{8.6}$$

Neglecting the terms in the braces of (8.5) is equivalent to disregarding the effect of the stress N_t; see (4.1).

The inextensional displacements given by A and B in (3.5) can be determined from the twist of the end-sections (centers of twist along z axis)

$$u = 0, \quad t = 0; \qquad u = \gamma(x_1 \cos\varphi + y_1 \sin\varphi), \quad t = t_1. \tag{8.7}$$

These displacements include warping.[10] The actual process of solving (8.7) is quite tedious even when a definite section is given.

9. We conclude with a short discussion of stresses in a general shell of constant slope bent by couples M as in section 5. We assume that the moments at the end-sec-

[10] For a treatment of warping along similar lines see R. V. Southwell, *On the torsion of conical shells*, Proc. Royal Soc. London, (A)163, 337–355 (1937).

tions C_0 and C_1 are applied in such a manner that the stress N_t at these sections is proportional to the distance from the neutral axis:

$$N_t = e_0 x_0 \text{ for } t = 0; \qquad N_t = e_1 x_1 \text{ for } t = t_1. \tag{9.1}$$

Conditions (9.1) and the fact that the moments across C_0 and C_1 are alike lead us to the following expressions:

$$f = \frac{M\rho_0\rho_1}{t_1 \sin\theta \cos\theta}\left\{\frac{Q_0}{I_0} - \frac{Q_1}{I_1}\right\}; \qquad g = \frac{M}{t_1 \sin\theta \cos\theta}\left\{\frac{\rho_0 Q_0}{I_0} - \frac{\rho_1 Q_1}{I_1}\right\}, \tag{9.2}$$

where the I's are the moments of inertia about the neutral axis of the full respective sections and $Q_t = \int_0^\varphi x_t \rho_t d\varphi$ the variable first moment (about the same axis) of the section included between 0 and φ. The expressions for the stresses themselves read:

$$N_{st} = \frac{M\rho_0\rho_1}{t_1\rho_t^2 \cos\theta}\left\{\frac{Q_0}{I_0} - \frac{Q_1}{I_1}\right\}, \tag{9.3}$$

$$N_t = -\frac{M(t_1 - t)t \sin\theta}{t_1 \cos\theta}\cdot\frac{\rho'}{\rho_t^3}\left\{\frac{Q_0}{I_0} - \frac{Q_1}{I_1}\right\}$$
$$+ \frac{M(t_1 - t)x_0}{t_1 I_0 \cos\theta}\cdot\frac{\rho_0^2}{\rho_t^2} + \frac{Mt}{t_1 \cos\theta}\cdot\frac{x_1\rho_1^2}{I_1\rho_t^2}. \tag{9.4}$$

The corresponding expressions for strain energy and displacements are very cumbersome and can hardly be useful in practical applications.

NON-HOMOGENEOUS STRESSES IN VISCO-ELASTIC MEDIA*

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1. Introduction. The purpose of this paper is the extension of the theory of elasticity to include visco-elastic media. The materials considered in this paper are *istropic* and *incompressible*, and are characterized by *linear* relations between the components of stress, strain, and their derivatives with respect to time. As in the classical theory of elasticity, only *small* strains will be considered. Body forces, in particular inertia forces, will be neglected.

In the following, σ_{ik} $(i, k = 1, 2, 3)$ and ϵ_{ik} denote the components of the tensors of stress and strain with respect to a system of rectangular axes x_i. σ_{11}, σ_{22}, σ_{33} are the normal stresses, $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$, $\sigma_{31} = \sigma_{13}$ the shearing stresses. Similarly, ϵ_{11}, ϵ_{22}, ϵ_{33} are the normal strains, $\epsilon_{12} = \epsilon_{21}$, $\epsilon_{23} = \epsilon_{32}$, $\epsilon_{31} = \epsilon_{13}$ the shearing strains. If u_i are the components of the displacement vector,

$$\epsilon_{ik} = \tfrac{1}{2}(u_{i,k} + u_{k,i}), \tag{1}$$

where the index after a comma denotes differentiation with respect to the corresponding coordinate x, i.e., $u_{i,k} = \partial u_i/\partial x_k$; $u_{k,i} = \partial u_k/\partial x_i$.

Irrespective of the mechanical properties of the material, the stresses must satisfy the equilibrium conditions

$$\sigma_{ik,k} = 0, \tag{2}$$

where the summation convention of tensor calculus has been used.[1] Similarly, the strain components must satisfy the conditions of compatibility,

$$\epsilon_{ik,lm} + \epsilon_{lm,ik} = \epsilon_{il,km} + \epsilon_{km,il}, \tag{3}$$

where $\epsilon_{ik,lm} = \partial^2\epsilon_{ik}/\partial x_l \partial x_m$, etc. While there are obviously three equations of equilibrium (corresponding to the three values which the subscript i in (2) can assume), it may at first glance appear that there are 3^4 equations of compatibility. On account of the high degree of symmetry in (3), the number of equations of compatibility reduces, however, to six; three equations of the type obtained from (3) when e.g., $i = k = 1$ and $l = m = 2$, and three equations of the type obtained from (3) when e.g., $i = k = 1$, $l = 2$, $m = 3$.

By themselves, Eqs. (2) and (3) are not sufficient to determine the states of stress and strain in a body subject to given surface stresses. A further necessary set of equations are those relating the stress components to the strain components in the general

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[1] According to this convention $\sigma_{ik,k}$ stands for the sum of all the terms obtained by giving k the values 1, 2, 3. In general, whenever a subscript appears twice in the same monomial, this subscript is to be given the values 1, 2, 3 and the resulting terms are to be added. Such a repeated subscript is called a *dummy subscript*.

case of combined stresses. It is through these *stress-strain* relations that the properties of the material enter the problem.

In the case of an incompressible material, $\epsilon_{ii} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = 0$. The stress-strain relations are most easily discussed when the following decomposition of the stress tensor is introduced. Define the *mean normal stress* as

$$\sigma = \tfrac{1}{3}\sigma_{ii} = \tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}), \tag{4}$$

and the *deviatoric part* of the stress tensor as

$$s_{ik} = \sigma_{ik} - \sigma\delta_{ik}, \tag{5}$$

where

$$\delta_{ik} = \begin{cases} 0 & \text{if} \quad i \neq k, \\ 1 & \text{if} \quad i = k. \end{cases}$$

The stress-strain relations of an isotropic, incompressible elastic material can then be written in the form

$$s_{ik} = 2G\epsilon_{ik}, \tag{6}$$

where G denotes the modulus of rigidity.

In view of (5), the equilibrium condition (2) yields

$$s_{ik,k} + \sigma_{,k}\delta_{ik} = s_{ik,k} + \sigma_{,i} = 0. \tag{7}$$

But, according to (6) and (1),

$$s_{ik,k} = 2G\epsilon_{ik,k} = G(u_{i,kk} + u_{k,ik}) = Gu_{i,kk}, \tag{8}$$

since for an incompressible material $u_{k,k} = 0$ and, consequently, $u_{k,ik} = u_{k,ki} = 0$. Comparing (7) and (8), we find

$$\sigma_{,i} = -Gu_{i,kk}. \tag{9}$$

Hence

$$\sigma_{,ii} = -Gu_{i,kki} = 0, \tag{10}$$

on account of the incompressibility of the material.

According to (5),

$$\sigma_{ik,ll} = s_{ik,ll} + \sigma_{,ll}\delta_{ik} = s_{ik,ll},$$

on account of (10). Making use of (6), (1) and (9), we transform this in the following manner:

$$\sigma_{ik,ll} = 2G\epsilon_{ik,ll} = G(u_{i,kll} + u_{k,ill}) = -2\sigma_{,ik}.$$

Thus

$$\sigma_{ik,ll} + 2\sigma_{,ik} = 0. \tag{11}$$

In the case of an incompressible elastic body in equilibrium the boundary conditions may be given in the form of three functions $f_i(x)$ which define the components of the forces (per unit area) applied to the surface of the body. The forces f_i must, of course, be in equilibrium, i.e., the surface integral of $f_i(x)$ must vanish for $i = 1, 2, 3$. If n_k denotes the unit vector directed along the exterior normal of the surface of the body, the stress components at the surface must then satisfy the conditions

$$\sigma_{ik}n_k = f_i. \tag{12}$$

The values of the surface stresses, in conjunction with Eqs. (2) and (11), define the stress distribution in the body and, consequently, also the strain distribution and, to within a rigid body displacement of the entire body, the displacement components.

On the other hand, the displacement components may be given on the surface of the body. These given surface displacements must, of course, be compatible with the assumed incompressibility of the material, i.e., the surface integral of the normal displacement component $u_i n_i$ must vanish. Elimination of σ from (9) furnishes $u_{i,kkl} - u_{l,kki} = 0$, or, after a change of subscripts,

$$u_{i,kll} - u_{k,ill} = 0. \tag{12a}$$

Eqs. (12) in conjunction with the condition of incompressibility, $u_{l,l} = 0$, and the given surface values determine the displacement components.

2. Stress-strain relations of visco-elastic materials. Equations similar to (11) and (12a) may be derived for visco-elastic materials characterized by linear relations between the components of stress, strain and their derivatives with respect to time.

In the case of an incompressible material of the type considered by Voigt[2] we have the stress-strain relations

$$s_{ik} = 2G\epsilon_{ik} + 2\mu\dot{\epsilon}_{ik}, \tag{13}$$

where μ is the *coefficient of viscosity.*

In the case of an incompressible material of the Maxwell type, we have

$$\dot{\epsilon}_{ik} = \frac{1}{2G}\dot{s}_{ik} + \frac{1}{2G\tau}s_{ik}, \tag{14}$$

where dots denote differentiation with respect to time, and τ is the *relaxation time.*[3]

Generalizing, we may consider incompressible materials characterized by stress-strain relations of the form

$$\left(\frac{\partial^m}{\partial t^m} + a_{m-1}\frac{\partial^{m-1}}{\partial t^{m-1}} + \cdots + a_0\right)s_{ik} = \left(b_n\frac{\partial^n}{\partial t^n} + b_{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}} + \cdots + b_0\right)\epsilon_{ik}, \tag{15}$$

where $a_{m-1}, \cdots, a_0, b_n, b_{n-1}, \cdots, b_0$ are constants characteristic of the material.

For such materials two types of boundary value problems may be considered. In the first case the surface forces $f_i(x, t)$ are given as functions of the position x and the time t; for $t=0$ these surface forces and their $m-1$ first derivatives are supposed to vanish as well as all stress components and their derivatives up to the order $m-1$. Moreover, at any given time the forces f_i must be in equilibrium. If, for $t \geqq 0$, the forces are analytic functions of time, this implies that the surface integral of any derivative $\partial^p f_i/\partial t^p$ must vanish for, say, $t=0$. The *first boundary value problem* calls for the determination of the stress distribution $\sigma_{ik}(x, t)$ fulfilling these boundary conditions and initial conditions.

In the second case the surface displacements $u_i(x, t)$ are given as functions of the position x and the time t; for $t=0$ these surface displacements and their $n-1$ first derivatives are supposed to vanish, as well as the displacements in the interior of the

[2] W. Voigt, Abh. Göttingen Ges. Wiss. 36 (1899), 47 pp.

[3] R. Simha has recently used the stress-strain relations which are obtained from Eqs. (14) by substituting the stress tensor σ_{ik} for its deviatoric part s_{ik} [J. Appl. Phys. 13, 201 (1942)]. Such stress-strain relations imply that, at constant strain, the stress decays exponentially with a relaxation time τ which is independent of the geometrical nature of the stress. This treatment ignores the fact that viscous flow, which is the cause of relaxation, is a response to *shearing* stresses only. In an incompressible material a uniform hydrostatic pressure does not produce viscous flow, and, hence, does not tend to relax. Contrary to Simha's stress-strain relations, our Eqs. (14) reflect this behavior.

body and their derivatives up to the order $n-1$. Moreover, on account of the assumed incompressibility of the material, the surface integral of the normal displacement component $u_i n_i$ must vanish for any time. If, for $t \geqq 0$, the displacements are analytic functions of time, this means that the surface integral of all expressions of the form $(\partial^p u_i/\partial t^p) n_i$ must vanish for, say, $t=0$. The *second boundary value problem* calls for the determination of the displacements $u_i(x, t)$ in the interior of the body, fulfilling these boundary conditions and initial conditions.

Let us rewrite the stress-strain relation (15) in the form

$$Ps_{ik} = 2Q\epsilon_{ik}, \tag{16}$$

where P and Q denote the linear differential operators

$$\begin{aligned} P &= \frac{\partial^m}{\partial t^m} + a_{m-1}\frac{\partial^{m-1}}{\partial t^{m-1}} + \cdots + a_1\frac{\partial}{\partial t} + a_0, \\ Q &= b_n\frac{\partial^n}{\partial t^n} + b_{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}} + \cdots + b_1\frac{\partial}{\partial t} + b_0. \end{aligned} \tag{16a}$$

Starting from the stress-strain relations (16) and repeating the various steps which led to the Eqs. (11) and (12a), we obtain

$$P(\sigma_{ik,ll} + 2\sigma_{,ik}) = 0 \tag{17}$$

and

$$Q(u_{i,kll} - u_{k,ill}) = 0 \tag{18}$$

as the equations governing the solution of the first and second boundary value problem, respectively.

For example, consider the first boundary value problem for an incompressible material of the Voigt type. Comparing Eqs. (13), (16) and (16a), we see that for this material

$$P = 1, \qquad Q = \mu\frac{\partial}{\partial t} + G.$$

Eq. (17) consequently takes the same form as for an incompressible elastic material (see Eq. (11)). This means that, *in the case of the first boundary value problem, the stress distribution in an incompressible material of the Voigt type is identical with that in an incompressible elastic material under the same instantaneous surface forces.* This stress distribution does not depend on the past stressing history, although, of course, the displacements do.

This result is readily extended to the case of an incompressible visco-elastic material characterized by a stress-strain relation (16). Consider, for instance, the first boundary value problem for a given set of surface forces $f_i(x, t)$ which, in addition to fulfilling the conditions stipulated above, are supposed to be analytic functions of time for $t \geqq 0$. If $\bar{\sigma}_{ik}(x, t)$ denotes the *static*[4] stress distribution in an incompressible elastic body of the same shape which is subjected to the surface forces $f_i(x, t)$, the required stress distribution in the visco-elastic body is given by

$$\sigma_{ik}(x, t) = \bar{\sigma}_{ik}(x, t).$$

[4] The term "static" is used here to indicate that, though the stresses $\bar{\sigma}_{ik}$ depend on t as do the forces f_i, no inertia effects should be taken into account in computing these stresses. In fact, as far as this elastic body is concerned, t plays the role of a parameter which need by no means be identified with the time.

Indeed, by definition, the stresses $\bar{\sigma}_{ik}$ satisfy the conditions (2), (11) and (12) for any value of t. Since, like the surface forces, these stresses are analytic functions of time, this means that they also satisfy the condition (17). *The result formulated above for the first boundary value problem of an incompressible material of the Voigt type applies, therefore, to any visco-elastic material characterized by stress-strain relations of the form* (16).

A similar result is obtained in the case of the second boundary value problem for an incompressible visco-elastic material obeying stress-strain relations of the form (16), if the prescribed surface discplacements $u_i(x, t)$ fulfill the conditions formulated above and, in addition, are analytic functions of time for $t \geqq 0$. The displacements $u_i(x, t)$ then equal the *static* displacements $\bar{u}_i(x, t)$ of an incompressible elastic body of the same shape, subjected to the given surface displacements $u_i(x, t)$.

3. Determination of the displacements in the first boundary value problem of visco-elasticity. Let us first consider the particularly simple case, where the given surface forces can be factored into the form:

$$f_i = f_i(x)g(t). \tag{19}$$

According to what has been said above, the stress distribution which these surface forces produce in the visco-elastic body has then the form

$$\sigma_{ik}(x, t) = \bar{\sigma}_{ik}(x)g(t), \tag{20}$$

where $\bar{\sigma}_{ik}(x)$ denotes the stresses which the surface forces $f_i(x)$ produce in an incompressible elastic body of the same shape. Introducing the stresses (20) into the stress-strain relation (16), we see that the strains in the visco-elastic body can be written in the form

$$\epsilon_{ik}(x, t) = \bar{\epsilon}_{ik}(x)h(t), \tag{21}$$

where $h(t)$ satisfies the differential equation

$$Qh = Pg, \tag{22}$$

while h and its derivatives up to the order $n-1$ vanish for $t=0$. As regards the quantities $\bar{\epsilon}_{ik}(x)$, they are related to the stresses $\bar{\sigma}_{ik}(x)$ by

$$\bar{s}_{ik} = 2\bar{\epsilon}_{ik}, \tag{23}$$

where $\bar{s}_{ik}$ denotes the deviatoric part of the stress tensor $\bar{\sigma}_{ik}$. In other terms, the quantities $\bar{\epsilon}_{ik}$ are the strains in an incompressible elastic body of the same shape and of unit modulus of rigidity, which is subjected to the surface forces $f_i(x)$. We shall call these strains the *equivalent elastic strains*. In order to obtain the function $h(t)$, all we have to do is to consider the *response* of the visco-elastic material under consideration to a simple shearing stress s varying according to $s=2g(t)$. The shearing strain produced by this stress equals $h(t)$. The strains produced in the visco-elastic body by the surface forces $f_i(x)\ g(t)$ are then obtained by multiplying the equivalent elastic strains by the response function $h(t)$.

Since the differential equations for stresses and strains are linear, solutions of this type may be superimposed on each other. Let us, now, assume that our result holds good even if, contrary to the assumption made above, the surface forces are not analytical functions of time for $t \geqq 0$. In particular consider the case when $f_i = f_i(x)g(\xi, t)$, where $g(\xi, t)$ is Heaviside's unit step function defined by

$$g(\xi, t) = \begin{cases} 0, & \text{if} \quad t < \xi, \\ 1, & \text{if} \quad t \geqq \xi. \end{cases}$$

Let $h(\xi, t)$ denote the response of the visco-elastic material under consideration to a simple shearing stress $s = 2g(\xi, t)$. Since the surface forces $f_i(x, t)$ can be represented in the form

$$f_i(x, t) = \int_0^\infty \dot{f}_i(x, \xi) g(\xi, t) d\xi, \tag{24}$$

the following formal integral representation of the strains produced by these surface forces in the visco-elastic body suggests itself:

$$\epsilon_{ik}(x, t) = \int_0^\infty \bar{\epsilon}_{ik}(x, \xi) h(\xi, t) d\xi, \tag{25}$$

where $\bar{\epsilon}_{ik}(x, \xi) d\xi$ are the equivalent elastic strains corresponding to the surface forces $\dot{f}_i(x, \xi) d\xi$. It can be shown that (25) indeed furnishes the strains of the visco-elastic body whenever the surface forces can be represented in the form (24). Moreover, to within a rigid body displacement the displacements of the visco-elastic body are given by

$$u_i(x, t) = \int_0^\infty \bar{u}_i(x, \xi) h(\xi, t) d\xi, \tag{26}$$

where $\bar{u}_i(x, \xi) d\xi$ are *equivalent elastic displacements* produced in an incompressible elastic body of the same shape and of unit modulus of rigidity, by the surface forces $\dot{f}_i(x, \xi) d\xi$.

Let us consider the following example: *A* thin cantilever beam of length L and cross sectional moment of inertia I is clamped rigidly at the end $x = 0$. The beam consists of an incompressible visco-elastic material of the Voigt type (stress-strain relations (13)), and is subjected to the transverse load

$$f(x, t) = c\left(1 - \frac{x}{L}\right) t^2,$$

per unit of length, c being a constant. At first sight, it may seem that the problem of determining the bending moments and transverse displacements of the beam is outside the scope of our theory, since at the clamped end we have prescribed deformations rather than prescribed forces. However, the system being statically determinate, the transverse reaction and the bending moment at the clamped end are completely determined by the given loads. Consequently, the problem may be considered as a first boundary value problem, if we make the usual assumption that the distribution of stresses over the end section is irrelevant as long as it leads to the resultant and the resultant moment required by the equilibrium of the beam. The displacements of an incompressible elastic cantilever beam of unit modulus of rigidity, loaded by $f(x) = c(1 - x/L)$, are

$$\bar{u}(x) = \frac{cx^2}{360IL}(10L^3 - 10L^2x + 5Lx^2 - x^3),$$

where account has already been taken of the fact that the Young's modulus equals $3G$ for an incompressible elastic material.

Now, in accordance with (13), the response $h(t)$ of Voigt's material to a simple shearing stress $s = 2t^2$ is found from

$$2t^2 = 2Gh + 2\mu\dot{h}, \qquad h(0) = 0.$$

One obtains

$$h(t) = \frac{t^2}{G} - 2\frac{\mu}{G^2}t + 2\frac{\mu^2}{G^3}\left[1 - e^{-Gt/\mu}\right].$$

The deflection of the visco-elastic beam is, therefore, given by

$$u(x, t) = \frac{cx^2}{360GIL}\left[10L^3 - 10L^2x + 5Lx^2 - x^3\right]\cdot\left[t^2 - 2\frac{\mu}{G}t + 2\frac{\mu^2}{G^2}(1 - e^{-\mu t/G})\right].$$

The statically determinate bending moments are completely determined by the given loads.

4. Determination of the stresses in the second boundary value problem of visco-elasticity. A similar procedure leads to the determination of the stresses in the second boundary value problem of visco-elasticity. Consider first the case when the given surface displacements can be factored into the form $u_i = u_i(x)g(t)$, and denote by $\bar{\sigma}_{ik}(x)$ the *equivalent elastic stresses*, i.e., the static stresses set up in an incompressible elastic body by the surface displacements $u_i(x)$. Furthermore, determine the *response function* $h(t)$, i.e., half the shearing stress produced in the visco-elastic material under consideration by a simple shearing strain $g(t)$. The required stress distribution in the visco-elastic body is then given by $\sigma_{ik}(x, t) = \bar{\sigma}_i(x)h(t)$.

In the general case, the stresses in the second boundary value problem may be represented in the form

$$\sigma_{ik}(x, t) = \int_0^\infty \bar{\sigma}_{ik}(x, \xi)h(\xi, t)d\xi, \tag{27}$$

where $\bar{\sigma}_{ki}(x, \xi)d\xi$ are the equivalent elastic stresses corresponding to the surface displacements $\dot{u}_i(x, \xi)d\xi$, and $2h(\xi, t)$ is the response of the visco-elastic material to a simple shearing strain

$$g(\xi, t) = \begin{cases} 0, & \text{if } t < \xi, \\ 1, & \text{if } t \geqq \xi. \end{cases}$$

5. Summary. The solution of the first and second boundary value problems of visco-elasticity is reduced to the solution of equivalent boundary value problems of elasticity, and the determination of the response of the visco-elastic material under consideration to a simple shearing stress or a simple shearing strain. It remains to be seen in how far the technique developed here can be applied to the solution of the third (mixed) boundary value problem where the surface forces are prescribed on part of the surface of the body, and the surface displacement on the rest of this surface.

THE INTRINSIC THEORY OF THIN SHELLS AND PLATES

PART III.—APPLICATION TO THIN SHELLS*

BY

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10. Definitions and method of approximation. The method of approximation used below is essentially the same as in the case of thin plate theory. We define ϵ to be the average reduced thickness of a shell. (We may recall that the reduced thickness of a shell is the ratio of its thickness to a selected lateral dimension of its middle surface). Then for a thin shell, ϵ is a small quantity. This definition of a thin shell is in agreement with that of a thin plate given in Part II.

A thin shell is said to have *finite curvature* when the smallest radius of curvature of its middle surface and the selected lateral dimension are of the same order of magnitude. Furthermore, a thin shell is said to have *small curvature of order b* when the ratio of the selected lateral dimension to the smallest radius of curvature of its middle surface is of the same order of magnitude as ϵ^b, where $b \geqq 1$. Thus a thin plate may be regarded as a thin shell of small curvature of order ∞.

We consider a family of ∞^1 shells of the same material with diminishing reduced thickness, each in a state of stress under (i) external forces applied at the edge, (ii) surface forces and (iii) uniform body forces. We assign to each shell a value of a parameter ϵ $(0 < \epsilon < \epsilon_1)$ denoting the average reduced thickness, so that the thickness is

$$2h = 2\epsilon\bar{h}(x^1, x^2). \tag{10.1}$$

The quantity ϵ_1 is supposed to be small, but the basic idea of the method is that we seek solutions valid for all ϵ in the range $0 < \epsilon < \epsilon_1$. In this theory, ϵ is the only small quantity. All quantities occurring (except Poisson's ratio σ) are functions of ϵ. No quantity is *small* unless it tends to zero with ϵ.

For the greatest generality suppose all quantities to be power series in ϵ. Thus, supposing the middle surface itself to depend on ϵ, we have

$$a_{\alpha\beta} = \sum_{s=0}^{\infty} a_{(s)\alpha\beta}\epsilon^s, \qquad b_{\alpha\beta} = \sum_{s=b}^{\infty} b_{(s)\alpha\beta}\epsilon^s, \tag{10.2a, b}$$

where b is either zero or a positive integer. $a_{(s)\alpha\beta}$ and $b_{(s)\alpha\beta}$ are functions of x^α, independent of ϵ. For $b=0$, we are dealing with thin shells of finite curvature, while for $b \geqq 1$ we are dealing with thin shells of small curvature of order b.

Furthermore, we shall represent Q^i, P^i, $X^i_{[0]}$, $\bar{T}^{\alpha\beta}$, $\bar{T}^{\alpha 0}$, $\bar{L}^{\alpha\beta}$, $p_{\alpha\beta}$, $q_{\alpha\beta}$ by power series as in Part II;

$$Q^0 = \sum_{s=k_0}^{\infty} Q^0_{(s)}\epsilon^s, \qquad Q^\alpha = \sum_{s=k}^{\infty} Q^\alpha_{(s)}\epsilon^s, \tag{10.3a}$$

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$$P^0 = \sum_{s=n_0}^{\infty} P^0_{(s)}\epsilon^s, \qquad P^\alpha = \sum_{s=k}^{\infty} P^\alpha_{(s)}\epsilon^s, \tag{10.3b}$$

$$X^0_{[0]} = \sum_{s=j_0}^{\infty} X^0_{(s)[0]}\epsilon^s, \qquad X^\alpha_{[0]} = \sum_{s=j}^{\infty} X^\alpha_{(s)[0]}\epsilon^s, \tag{10.3c}$$

$$\tilde{T}^{\alpha\beta} = \sum_{s=t}^{\infty} \tilde{T}^{\alpha\beta}_{(s)}\epsilon^s, \qquad \tilde{L}^{\alpha\beta} = \sum_{s=u}^{\infty} \tilde{L}^{\alpha\beta}_{(s)}\epsilon^s, \qquad \tilde{T}^{\alpha 0} = \sum_{s=l}^{\infty} \tilde{T}^{\alpha 0}_{(s)}\epsilon^s, \tag{10.4a, b, c}$$

$$p_{\alpha\beta} = \sum_{s=p}^{\infty} p_{(s)\alpha\beta}\epsilon^s, \qquad q_{\alpha\beta} = \sum_{s=q}^{\infty} q_{(s)\alpha\beta}\epsilon^s. \tag{10.5a, b}$$

Here k, k_0, n, n_0, j, j_0, t, u, l, p are integers greater than zero, and q is zero or a positive integer. The case $q=0$ corresponds to problems of finite deflection. The quantities $Q^0_{(s)}$, $Q^\alpha_{(s)}$, $P^0_{(s)}$ etc. are functions of x^α, independent of ϵ.

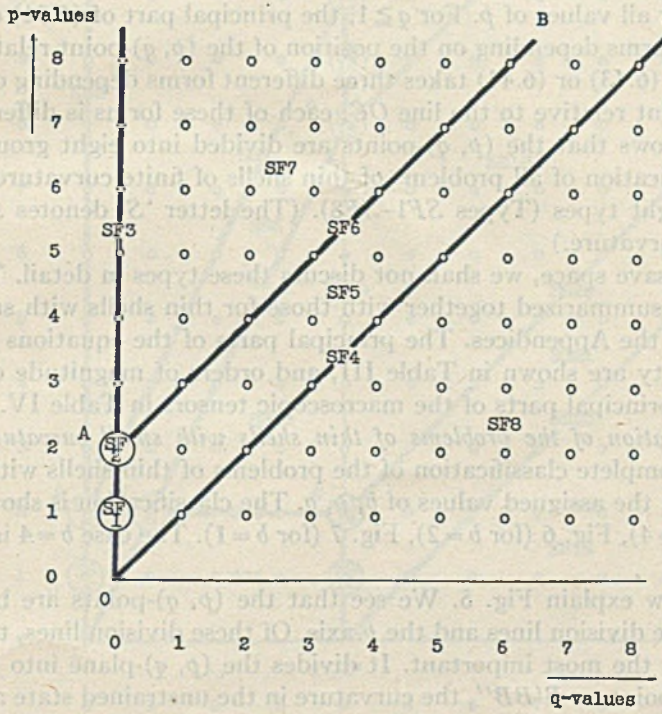

FIG. 4. Classification of problems of thin shells with finite curvature ($b=0$).
p = order of extension of middle surface.
q = order of change of curvature of middle surface.
b = order of initial curvature of middle surface.

Then the problems of thin shells can be classified by assigning integral values to p, q and b. With p, q, b given, the values of k_0, k, n_0, n, j_0, j in (10.3) are fixed by the condition that $X^0_{(j_0)[0]}$, $X^\alpha_{(j)[0]}$, $P^0_{(n_0)}$, $P^\alpha_{(n)}$, $Q^0_{(k_0)}$, $Q^\alpha_{(k)}$ should contribute to the principal parts of (6.34), (6.35), without dominating these equations to the exclusion of $p_{\alpha\beta}$ and $q_{\alpha\beta}$. The values of t, u, l of $T^{\alpha\beta}$, $L^{\alpha\beta}$, $T^{\alpha 0}$ are immediately fixed through the expressions (6.29), (6.30), (6.31). With p, q, b, k, k_0, j, j_0, n, n_0 fixed, the equations of

equilibrium and compatibility in the first approximation are immediately obtained by substituting (10.1)–(10.5) into (6.34), (6.35), (6.43), (6.44), and picking out the principal terms in ϵ from the resulting equations. This gives us six differential equations in six unknowns $p_{(p)\alpha\beta}$ and $q_{(q)\alpha\beta}$. For the various combinations of values of p, q, b, the forms of these differential equations fall into several types. The classification of these types will be given below.

11. Classification of all thin shell problems. *The classification of the problems of thin shells with finite curvature* ($b=0$). The following is a complete classification of the problems of thin shells with finite curvature ($b=0$) based upon assigned values of p, q. The classification is shown graphically in Fig. 4.

It is found that the (p, q)-points in the diagram ($q \geqq 0$, $p \geqq 1$) are broken up into eight groups by the *division lines* AB, OC and the p-axis. For $q=0$, the principal part of (6.34) or (6.35) takes three different forms depending on the position of the point on the p-axis relative to the point A, while the principal parts of (6.43) and (6.44) are the same for all values of p. For $q \geqq 1$, the principal part of (6.34) or (6.35) takes three different forms depending on the position of the (p, q)-point relative to the line AB, and that of (6.43) or (6.44) takes three different forms depending on the position of the (p, q)-point relative to the line OC; each of these forms is different from that for $q=0$. It follows that the (p, q)-points are divided into eight groups and so the complete classification of all problems of thin shells of finite curvature involves consideration of eight types (Types $SF1$–$SF8$). (The letter 'S' denotes *shell*, while 'F' denotes *finite* curvature.)

In order to save space, we shall not discuss these types in detail. The results for these types are summarized together with those for thin shells with small curvature in the tables in the Appendices. The principal parts of the equations of equilibrium and compatibility are shown in Table III, and orders of magnitude of the external forces and the principal parts of the macroscopic tensors in Table IV.

The classification of the problems of thin shells with small curvature ($b \geqq 1$). The following is a complete classification of the problems of thin shells with small curvature based upon the assigned values of b, p, q. The classification is shown graphically in Fig. 5 (for $b=4$), Fig. 6 (for $b=2$), Fig. 7 (for $b=1$). The case $b=4$ is typical of the cases $3 \leqq b < \infty$.

We shall now explain Fig. 5. We see that the (p, q)-points are broken up into 27 groups by the division lines and the p-axis. Of these division lines, the line $B'BB''$ (i.e., $q=b=4$) is the most important. It divides the (p, q)-plane into three main regions. For any point on $B'BB''$, the curvature in the unstrained state and the change of curvature during the strain are of the *same* order of magnitude ($q=b=4$). For any point on the left of $B'BB''$, the magnitude of the curvature in the unstrained state is *smaller* than the magnitude of the change of curvature ($q<b=4$), while for any point on the right of $B'B''$, the magnitude of the curvature in the unstrained state is *greater* than the magnitude of change of curvature ($q>b=4$).

For $q=0$ (i.e., on the p-axis) in Fig. 5, the principal parts of (6.34), (6.35) take three different forms depending on the position of the points on the p-axis relative to the point A, while the principal parts of (6.44), (6.43) are the same for all points on the p-axis. For $1 \leqq q < b = 4$ (i.e., in the region between the p-axis and $B'BB''$), the principal parts of (6.34) or (6.35) or (6.44) take three different forms depending on the position of the (p, q)-point relative to the division line AC or AB or OD respectively,

while the principal part of (6.43) is the same for all the (p, q)-points in this region. It follows that the (p, q)-points in the region on the left-hand side of $B'B''$ are divided into 11 groups (Types $SS1$–$SS11$). (The letters 'SS' denote the *shell* with *small* curvature.)

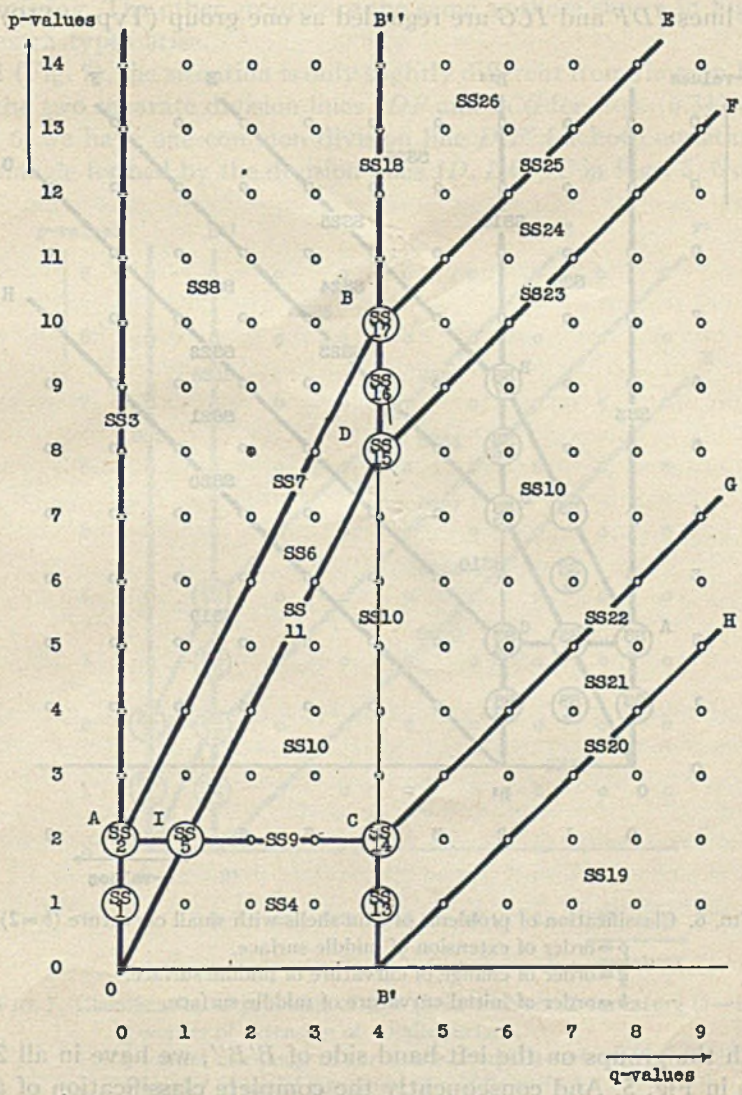

FIG. 5. Classification of problems of thin shells with small curvature ($b=4$).
p = order of extension of middle surface.
q = order of change of curvature of middle surface.
b = order of initial curvature of middle surface.

For $q=b=4$ (i.e., on $B'B''$), the principal parts of (6.34) or (6.35) or (6.44) take three different forms depending on the position of the (p, q)-point relative to C or B or D respectively, while the principal part of (6.43) is the same for all points on this line. Furthermore, for $q>b=4$ (i.e., the region to the right of $B'B''$), the principal parts of (6.34) or (6.35) or (6.43) or (6.44) take three different forms depending on

the position of the (p, q)-point relative to the division line CG or BE or $B'H$ or DF respectively. It follows that the (p, q)-points on the right-hand side of $B'B''$ are divided into 9 groups (Types $SS19$–$SS26$, $SS10$). It should be noted that, as far as the principal parts of (6.34), (6.35), (6.43), (6.44) are concerned, the (p, q)-points lying between the lines IDF and ICG are regarded as one group (Type $SS10$). Therefore,

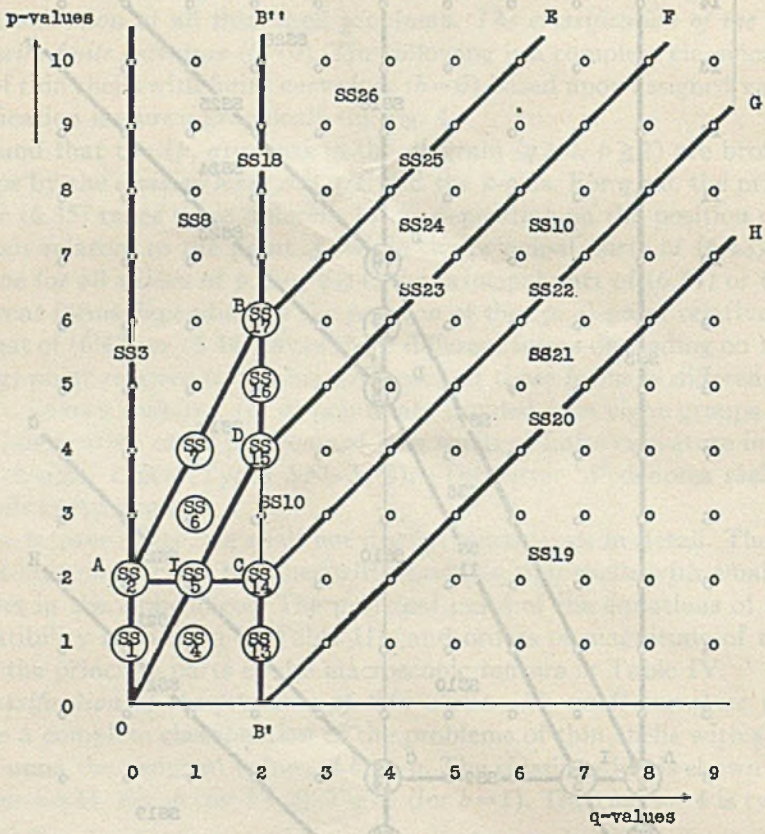

FIG. 6. Classification of problems of thin shells with small curvature ($b=2$).
p = order of extension of middle surface.
q = order of change of curvature of middle surface.
b = order of initial curvature of middle surface.

together with the groups on the left-hand side of $B'B''$, we have in all 25 groups of (p, q)-points in Fig. 5. And consequently the complete classification of all problems of thin shells with small curvature of order $b=4$ involves consideration of 25 types (Types $SS1$–$SS11$, $SS13$–$SS26$).

The general appearance of the classification diagrams for any b satisfying $3 \leqq b < \infty$ is the same as for $b=4$. An increase of b makes the line $B'B''$ shift to the right, while a decrease of b makes it shift to the left. On examining the various groups of (p, q)-points in these diagrams (for any integral value of b in the range of $3 \leqq b < \infty$), it is found that the corresponding groups occupying the same relative positions with respect to the division lines possess the same set of equations of equilibrium and compatibility in the first approximation, and so belong to the same type of problem.

Therefore the complete classification of all problems of thin shells with small curvature of order $3 \leqq b < \infty$ involves consideration of 25 types only.

For $b=2$ (Fig. 6), the situation is almost the same as in Fig. 5, but with the groups $SS9$, $SS11$ missing. The other groups are the same as those shown in Fig. 5 for $b=4$, and so no extra types arise.

For $b=1$ (Fig. 7), the situation is only slightly different from those in Figs. 5 and 6. Instead of the two separate division lines IDF and ICG for Eqs. (6.34) and (6.43) in Figs. 5 and 6, we have one common division line $D'F'$ for both equations. Furthermore, the triangle formed by the division lines ID, DC, IC in Figs. 5, 6 collapses into

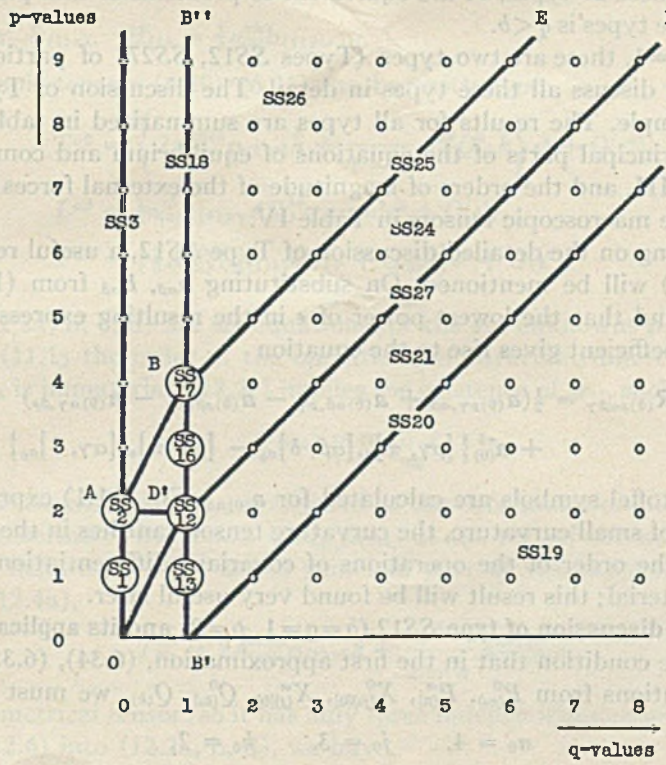

FIG. 7. Classification of problems of thin shells with small curvature ($b=1$).
p = order of extension of middle surface.
q = order of change of curvature of middle surface.
b = order of initial curvature of middle surface.

an isolated point D' in Fig. 7. Thus instead of 25 groups in Fig. 5, or 23 groups in Fig. 6, we have only 15 different groups. Among these groups, 13 belong to the types already mentioned in the case $3 \leqq b < \infty$ (Types $SS1$–$SS3$, $SS13$, $SS16$–$SS21$, $SS24$–$SS26$); the other two are Types $SS12$, $SS27$.

On comparing the classification of (p, q)-points on the left-hand side of $B'B''$ in Figs. 5, 6, 7 with that in the corresponding region of Fig. 3, it is found that they are identical with each other. In fact, for these types, the equations of equilibrium and compatibility in the first approximation are identical with those stated in Table I (Part II) for the corresponding types of thin plate problems. Therefore, we have the

following important conclusion: *A problem of a thin shell with small curvature of order b is effectively equivalent to a problem of a thin plate in the first approximation, if* $q<b$, i.e., if the change of curvature is greater than the curvature of the shell in the unstrained state.

It should be noted that for $b=\infty$, Fig. 5 becomes exactly Fig. 3 for the thin plate problem.

The results are summed up as follows:

(i) The complete classification of the problems of thin shells with small curvature of order $b \geqq 1$ involves the consideration of 27 types (Types *SS*1–*SS*27).

(ii) Among these 27 types, 11 are equivalent to problems of thin plates; the characteristic of these types is $q<b$.

(iii) When $b=1$, these are two types (Types *SS*12, *SS*27) of particular interest.

We shall not discuss all these types in detail. The discussion of Type *SS*12 will serve as an example. The results for all types are summarized in tables in the Appendices. The principal parts of the equations of equilibrium and compatibility are shown in Table III, and the orders of magnitude of the external forces and the principal parts of the macroscopic tensors in Table IV.

Before entering on the detailed discussion of Type *SS*12, a useful result for small curvature ($b \geqq 1$) will be mentioned. On substituting $a_{\alpha\beta}$, $b_{\alpha\beta}$ from (10.2a, b) into (6.39b), it is found that the lowest power of ϵ in the resulting expression is ϵ^0. The corresponding coefficient gives rise to the equation

$$R_{(0)\rho\alpha\beta\gamma} = \tfrac{1}{2}(a_{(0)\rho\gamma,\alpha\beta} + a_{(0)\alpha\beta,\rho\gamma} - a_{(0)\rho\beta,\alpha\gamma} - a_{(0)\alpha\gamma,\beta\rho}) + a^{\pi\delta}_{(0)}\{[\rho\gamma, \pi]_{a_0}[\alpha\beta, \delta]_{a_0} - [\rho\beta, \pi]_{a_0}[\alpha\gamma, \delta]_{a_0}\} = 0, \quad (11.1)$$

where the Christoffel symbols are calculated for $a_{(0)\alpha\beta}$. Eq. (11.1) expresses the fact that in the case of small curvature, the curvature tensor vanishes in the first approximation. Hence the order of the operations of covariant differentiation with respect to $a_{(0)\alpha\beta}$ is immaterial; this result will be found very useful later.

12. Detailed discussion of type *SS*12 ($b=q=1$, $p=2$) and its applications. *General equations.* By the condition that in the first approximation, (6.34), (6.35) receive significant contributions from $P^0_{(n_0)}$, $P^\alpha_{(n)}$, $X^0_{(j_0)[0]}$, $X^\alpha_{(j)[0]}$, $Q^0_{(k_0)}$, $Q_{(k)}$, we must have

$$\begin{aligned} n_0 &= 4, \qquad j_0 = 3, \qquad k_0 = 2, \\ n &= 3, \qquad j = 2, \qquad k = 3. \end{aligned} \quad (12.1)$$

By substituting the ϵ series into (6.34), (6.35), (6.43), (6.44), it is found that the lowest powers in ϵ occurring in the resulting equations are respectively ϵ^4, ϵ^3, ϵ^1, ϵ^2. The corresponding coefficients give rise to the following equations:

$$\begin{aligned} &- A^{\rho\gamma\pi\lambda}_{(01)} b_{(1)\rho\gamma} p_{(2)\pi\lambda}\bar{h} - 2A^{\rho\gamma\pi\lambda}_{(01)} q_{(1)\rho\gamma} p_{(2)\pi\lambda}\bar{h} + \tfrac{2}{3}A^{\rho\gamma\pi\lambda}_{(01)}(q_{(1)\pi\lambda}\bar{h}^3)_{|\rho\gamma}{}_{a_0} \\ &+ P^0_{(4)} + 2X^0_{(3)[0]}\bar{h} + (Q^\lambda_{(3)}\bar{h})_{|\lambda}{}_{a_0} + \frac{2(1-2\sigma)}{1-\sigma} H_{(1)} Q^0_{(2)}\bar{h} \\ &+ \frac{1-2\sigma}{1-\sigma} q_{(1)\pi\lambda} a^{\pi\lambda}_{(0)} Q^0_{(2)}\bar{h} = 0, \end{aligned} \quad (12.2a)$$

$$2A^{\rho\alpha\pi\lambda}_{(01)}(p_{(2)\pi\lambda}\bar{h})_{|\rho}{}_{a_0} + P^\alpha_{(3)} + 2X^\alpha_{(2)[0]}\bar{h} + \frac{\sigma}{1-\sigma} a^{\alpha\rho}_{(0)}(Q^0_{(2)}\bar{h})_{|\rho}{}_{a_0} = 0, \quad (12.2b)$$

$$\mathfrak{n}^{\beta\gamma}_{(0)} q_{(1)\alpha\beta|\gamma} \underset{a_0}{=} 0, \tag{12.2c}$$

$$2\mathfrak{n}^{\rho\alpha}_{(0)}\mathfrak{n}^{\beta\gamma}_{(0)} p_{(2)\rho\gamma|\alpha\beta} \underset{a_0}{+} \mathfrak{n}^{\rho\alpha}_{(0)}\mathfrak{n}^{\beta\gamma}_{(0)} q_{(1)\rho\gamma} q_{(1)\alpha\beta} + (b^{\alpha\beta}_{(1)} - 4H_{(1)} a^{\alpha\beta}_{(0)}) q_{(1)\alpha\beta} = 0, \tag{12.2d}$$

where a_0 under stroke indicates covariant differentiation with respect to the tensor $a_{(0)\alpha\beta}$ and x^α. The other symbols represent

$$A^{\alpha\beta\pi\lambda}_{(01)} = \frac{1}{1-\sigma^2}(\sigma a^{\alpha\beta}_{(0)} a^{\pi\lambda}_{(0)} + (1-\sigma) a^{\alpha\pi}_{(0)} a^{\beta\lambda}_{(0)}), \tag{12.3a}$$

$$\mathfrak{n}^{\alpha\beta}_{(0)} = \epsilon^{\alpha\beta}(a_0)^{-1/2}, \quad a_0 = \det.(a_{(0)\pi\lambda}), \quad \epsilon^{11} = \epsilon^{22} = 0, \quad \epsilon^{12} = -\epsilon^{21} = 1, \tag{12.3b}$$

$$b^{\alpha\beta}_{(1)} = a^{\alpha\pi}_{(0)} a^{\beta\lambda}_{(0)} b_{(1)\pi\lambda}, \quad H_{(1)} = \tfrac{1}{4} a^{\pi\lambda}_{(0)} b_{(1)\pi\lambda}. \tag{12.3c}$$

The macroscopic tensors (6.29)–(6.31) can be written as

$$T^{\alpha\beta} = \left\{ 2A^{\alpha\beta\pi\lambda}_{(01)} p_{(2)\pi\lambda} \bar{h} + \frac{\sigma}{1-\sigma} a^{\alpha\beta}_{(0)} Q^0_{(2)} \bar{h} \right\} \epsilon^3 + O(\epsilon^4), \tag{12.4a}$$

$$L^{\alpha\beta} = \tfrac{2}{3} \mathfrak{n}^{\gamma\beta}_{(0)} a_{(0)\pi\gamma} A^{\alpha\pi\lambda\delta}_{(01)} q_{(1)\lambda\delta} \bar{h}^3 \epsilon^4 + O(\epsilon^5), \tag{12.4b}$$

$$T^{\alpha 0} = \left\{ \tfrac{2}{3} A^{\pi\alpha\lambda\delta}_{(01)} (q_{(1)\lambda\delta} \bar{h}^3)_{|\pi} \underset{a_0}{+} Q^\alpha_{(3)} \bar{h} \right\} \epsilon^4 + O(\epsilon^5). \tag{12.4c}$$

Equations (12.2a, b, c, d) are six equations for the six unknowns $p_{(2)\pi\lambda}$ and $q_{(1)\pi\lambda}$.

Since by (11.1) the order of the operations of covariant differentiation with respect to $a_{(0)\pi\lambda}$ is immaterial, (12.2c) implies the existence of $w_{(1)}$ such that

$$q_{(1)\alpha\beta} = w_{(1)|\alpha\beta}. \tag{12.5}$$

Thus the determination of $q_{(1)\alpha\beta}$ is reduced to the determination of the single function $w_{(1)}$. Furthermore, instead of using $p_{(2)\alpha\beta}$ as the rest of the unknowns, we may use $T^{\alpha\beta}_{(3)}$. By definition, $T^{\alpha\beta}_{(3)}\epsilon^3$ is the principal part of the macroscopic tensor $T^{\alpha\beta}$, namely, by (12.4a),

$$T^{\alpha\beta}_{(3)} = 2A^{\alpha\beta\pi\lambda}_{(01)} p_{(2)\pi\lambda} \bar{h} + \frac{\sigma}{1-\sigma} a^{\alpha\beta}_{(0)} Q^0_{(2)} \bar{h}. \tag{12.6}$$

This is a symmetrical tensor; so it has only three independent components. Substituting (12.5), (12.6) into (12.2a, b, d), we have

$$\begin{aligned} &- T^{\pi\lambda}_{(3)} w_{(1)|\pi\lambda} \underset{a_0}{-} \tfrac{1}{2} b_{(1)\pi\lambda} T^{\pi\lambda}_{(3)} + \tfrac{2}{3} A^{\rho\gamma\pi\lambda}_{(01)} (w_{(1)|\pi\lambda} \bar{h}^3)_{|\rho\gamma} \underset{a_0}{+} P^0_{(4)} \\ &\qquad + 2X^0_{(3)[0]} \bar{h} + (Q^\pi_{(3)} \bar{h})_{|\pi} \underset{a_0}{+} 2H_{(1)} Q^0_{(2)} \bar{h} + w_{(1)|\pi\lambda} a^{\pi\lambda}_{(0)} Q^0_{(2)} \bar{h} = 0, \end{aligned} \tag{12.7a}$$

$$T^{\pi\alpha}_{(3)|\pi} \underset{a_0}{+} P^\alpha_{(3)} + 2X^\alpha_{(2)[0]} \bar{h} = 0, \tag{12.7b}$$

$$\begin{aligned} &\mathfrak{n}^{\rho\alpha}_{(0)}\mathfrak{n}^{\beta\gamma}_{(0)} \left\{ (1+\sigma) a_{(0)\pi\rho} a_{(0)\gamma\delta} - \sigma a_{(0)\rho\gamma} a_{(0)\pi\delta} \right\} \left\{ \frac{1}{\bar{h}} T^{\pi\delta}_{(3)} \right\}_{|\alpha\beta} \underset{a_0}{+} \sigma a^{\pi\lambda}_{(0)} Q^0_{(2)|\pi\lambda} \\ &\qquad + \mathfrak{n}^{\beta\alpha}_{(0)}\mathfrak{n}^{\rho\gamma}_{(0)} w_{(1)|\beta\gamma} w_{(1)|\alpha\rho} + (b^{\alpha\beta}_{(1)} - 4H_{(1)} a^{\alpha\beta}_{(0)}) w_{(1)|\alpha\beta} = 0. \end{aligned} \tag{12.7c}$$

Equations (12.7a, b, c) form a set of four equations for the four unknowns $w_{(1)}$ and $T^{\alpha\beta}_{(3)}$.

Special case. The following special case is interesting. If

$$P^{\alpha}_{(3)} = X^{\alpha}_{(2)[0]} = 0, \tag{12.8}$$

then by (12.7b) there exists a stress function $\chi_{(3)}$ such that

$$T^{\alpha\beta}_{(3)} = \mathbf{n}^{\alpha\pi}_{(0)}\mathbf{n}^{\beta\lambda}_{(0)}\chi_{(3)\underset{a_0}{|}\pi\lambda}. \tag{12.9}$$

Here $\chi_{(3)}$ is a function of x^{α}, having properties similar to those of the Airy function in the thin plate theory. Substituting (12.8), (12.9) into (12.7a, c), we have

$$-\tfrac{1}{2}\mathbf{n}^{\pi\lambda}_{(0)}\mathbf{n}^{\delta\rho}_{(0)}(2w_{(1)\underset{a_0}{|}\pi\delta} + b_{(1)\pi\delta})\chi_{(3)\underset{a_0}{|}\lambda\rho} + \tfrac{2}{3}A^{\rho\gamma\pi\lambda}_{(01)}(w_{(1)\underset{a_0}{|}\pi\lambda}\bar{h}^3)_{\underset{a_0}{|}\rho\gamma}$$
$$+ P^0_{(4)} + 2X^0_{(3)[0]}\bar{h} + 2H_{(1)}Q^0_{(2)}\bar{h} + a^{\pi\lambda}_{(0)}w_{(1)\underset{a_0}{|}\pi\lambda}Q^0_{(2)}\bar{h} = 0, \tag{12.10a}$$

$$\{\sigma a^{\pi\rho}_{(0)}a^{\delta\lambda}_{(0)} - (1+\sigma)a^{\pi\delta}_{(0)}a^{\rho\lambda}_{(0)}\}\left\{\frac{1}{\bar{h}}\chi_{(3)\underset{a_0}{|}\delta\lambda}\right\}_{\underset{a_0}{|}\pi\rho} + \sigma a^{\pi\lambda}_{(0)}Q^0_{(2)\underset{a_0}{|}\pi\lambda}$$
$$+ \mathbf{n}^{\rho\pi}_{(0)}\mathbf{n}^{\beta\gamma}_{(0)}w_{(1)\underset{a_0}{|}\rho\gamma}w_{(1)\underset{a_0}{|}\pi\beta} + (b^{\pi\lambda}_{(1)} - 4H_{(1)}a^{\pi\lambda}_{(0)})w_{(1)\underset{a_0}{|}\pi\lambda} = 0. \tag{12.10b}$$

Equations (13.10a, b) are two equations for the two unknowns $\chi_{(3)}$ and $w_{(1)}$. These equations are valid in general for a shell of non-uniform thickness. For the case of uniform thickness, (12.10a, b) are immediately simplified to the forms

$$-\tfrac{1}{2}\mathbf{n}^{\pi\lambda}_{(0)}\mathbf{n}^{\delta\rho}_{(0)}(2w_{(1)\underset{a_0}{|}\pi\delta} + b_{(1)\pi\delta})\chi_{(3)\underset{a_0}{|}\lambda\rho} + Da^{\pi\gamma}_{(0)}a^{\lambda\delta}_{(0)}w_{(1)\underset{a_0}{|}\pi\gamma\lambda\delta}$$
$$+ P^0_{(4)} + 2X^0_{(3)[0]}\bar{h} + 2H_{(1)}Q^0_{(2)}\bar{h} + a^{\pi\lambda}_{(0)}w_{(1)\underset{a_0}{|}\pi\lambda}Q^0_{(2)}\bar{h} = 0, \tag{12.11a}$$

$$a^{\pi\gamma}_{(0)}a^{\lambda\delta}_{(0)}\chi_{(3)\underset{a_0}{|}\pi\gamma\lambda\delta} - \sigma\bar{h}a^{\pi\lambda}_{(0)}Q_{(2)\underset{a_0}{|}\pi\lambda} + \bar{h}\mathbf{n}^{\pi\rho}_{(0)}\mathbf{n}^{\lambda\gamma}_{(0)}w_{(1)\underset{a_0}{|}\rho\gamma}w_{(1)\underset{a_0}{|}\pi\lambda}$$
$$+ \bar{h}(4H_{(1)}a^{\pi\lambda}_{(0)} - b^{\pi\lambda}_{(1)})w_{(1)\underset{a_0}{|}\pi\lambda} = 0, \tag{12.11b}$$

where D is the reduced flexural rigidity, as given in (9.14). Applications of these two equations will be discussed below.

A circular cylindrical thin shell with small curvature and uniform thickness under end thrust and normal pressure. We shall assume that the external forces and the edge loading are such that the problem is of Type $SS12$. Furthermore let us assume that

$$X^0_{(3)[0]} = Q^0_{(2)} = 0. \tag{12.12}$$

We have in mind the case where body force is negligible and where the shell is loaded normally on one side only. A number of terms disappear from the equations of equilibrium and compatibility (12.11a, b) for Type $SS12$. Thus if we write these equations in terms of the small principal parts instead of in terms of the finite coefficients of the lowest power in ϵ, we have

$$-\tfrac{1}{2}\mathbf{n}^{\pi\lambda}_{[0]}\mathbf{n}^{\delta\rho}_{[0]}(2w_{\underset{a}{|}\pi\delta} + b_{\pi\delta})\chi_{\underset{a}{|}\lambda\rho} + Da^{\pi\gamma}a^{\lambda\delta}w_{\underset{a}{|}\pi\gamma\lambda\delta} + P^0 = 0, \tag{12.13a}$$

$$a^{\pi\gamma}a^{\lambda\delta}\chi_{\underset{a}{|}\pi\gamma\lambda\delta} + h\mathbf{n}^{\pi\rho}_{[0]}\mathbf{n}^{\delta\gamma}_{[0]}w_{\underset{a}{|}\rho\gamma}w_{\underset{a}{|}\pi\delta} + h(4Ha^{\pi\lambda} - b^{\pi\lambda})w_{\underset{a}{|}\pi\lambda} = 0. \tag{12.13b}$$

Here $\underset{a}{|}$ under a stroke indicates covariant differentiation with respect to the tensor $a_{\alpha\beta}$ and x^{α}; also

$$D = \frac{2h^3}{3(1-\sigma^2)}, \qquad 4H = a_{\alpha\beta}b^{\alpha\beta}. \tag{12.14}$$

Let us choose the set of intrinsic rectangular Cartesian coordinates on the middle surface so that $x = x^1$ is the distance measured along the generators of the cylinder and $y = x^2$ is the distance measured perpendicular to the generators. Then we have

$$\begin{aligned} a_{11} &= a^{11} = a_{22} = a^{22} = 1, \qquad a_{12} = a^{12} = 0, \\ b_{11} &= b^{11} = b_{12} = b^{12} = 0, \qquad b_{22} = b^{22} = 2/R, \end{aligned} \tag{12.15}$$

where R is the radius of curvature of the cylindrical middle surface. In these coordinates, Eqs. (12.13a, b) become

$$D\Delta\Delta w + (2w_{,xy}\chi_{,xy} - w_{,xx}\chi_{,yy} - w_{,yy}\chi_{,xx}) - \frac{1}{R}\chi_{,xx} + P^0 = 0, \tag{12.16a}$$

$$\Delta\Delta\chi + 2h(w_{,xx}w_{,yy} - w_{,xy}w_{,xy}) + 2h\frac{1}{R}w_{,xx} = 0, \tag{12.16b}$$

where subscripts preceded by a comma denote partial differentiation. If we let R tend to infinity, we get the von Kármán equations for a flat plate. The equation (12.16b) was recently obtained by von Kármán and Tsien [1] in their treatment of buckling of a thin-walled circular cylindrical shell under compression on the two ends. If we apply the operators $\Delta\Delta$ to (12.16a) and $(1/R)\partial^2/\partial x^2$ to (12.16b) and add the resulting equations, we obtain

$$\begin{aligned} D\Delta\Delta\Delta\Delta w + \frac{2h}{R^2}w_{,xxxx} + \frac{2h}{R}(w_{,xx}w_{,yy} - w_{,xy}w_{,xy})_{,xx} \\ = \Delta\Delta(P^0 + 2w_{,xy}\chi_{,xy} - w_{,xx}\chi_{,yy} - w_{,yy}\chi_{,xx}). \end{aligned} \tag{12.17}$$

This is the equation of equilibrium used by von Kármán and Tsien, except that they omit the term

$$\frac{2h}{R}(w_{,xx}w_{,yy} - w_{,xy}w_{,xy})_{,xx}. \tag{12.18}$$

This term is important when the deflection is comparable with thickness. However, it seems simpler to treat the problem directly by means of (12.16a, b) instead of using the higher-order equation (12.17). Equation (12.16a) appears to be new.

A small segment of a thin spherical shell under external pressure. We shall assume that the solid angle of the segment is small, so that the curvature is small; we shall assume it to be of the same order as the thickness, so that $b = 1$ (cf. section 10). We shall use spherical polar coordinates as in Fig. 8, so that on the middle surface in the unstrained state we have

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta\, d\varphi^2. \tag{12.19}$$

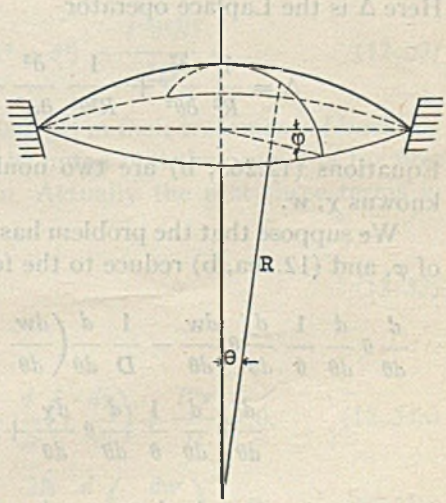

FIG. 8.

Since θ is small, we write

$$ds^2 = R^2 d\theta^2 + R^2\theta^2 d\varphi^2. \tag{12.20}$$

If we put

$$x^1 = \theta, \quad x^2 = \varphi, \tag{12.21}$$

the components of the first and second fundamental tensors are given by

$$a_{11} = R^2, \quad a_{22} = R^2\theta^2, \quad a_{12} = 0, \quad a^{11} = 1/R^2, \quad a^{22} = 1/R^2\theta^2, \quad a^{12} = 0, \tag{12.22}$$

$$b_{11} = 2R, \quad b_{22} = 2R\theta^2, \quad b^{11} = 2/R^3, \quad b^{22} = 2/R^3\theta^2, \quad b_{12} = b^{12} = 0. \tag{12.23}$$

Futhermore, we have from (12.22), (12.23)

$$H = 1/R, \qquad a = R^4\theta^2. \tag{12.24}$$

All the Christoffel symbols are equal to zero, except

$$\left\{ {1 \atop 2\ 2} \right\} = -\theta, \qquad \left\{ {2 \atop 1\ 2} \right\} = 1/\theta. \tag{12.25}$$

We shall assume that the problem is of Type $SS12$. Substituting (12.21)–(12.25) into (12.13a, b), we have

$$-\frac{1}{R^4\theta^2}\left\{ (w_{,\theta\theta} + R)(\chi_{,\varphi\varphi} + \theta\chi_{,\theta}) - 2\left(w_{,\theta\varphi} - \frac{1}{\theta} w_{,\varphi}\right)\left(\chi_{,\theta\varphi} - \frac{1}{\theta}\chi_{,\varphi}\right) \right.$$
$$\left. + (w_{,\varphi\varphi} + R\theta^2 + \theta w_{,\theta})\chi_{,\theta\theta} \right\} + D\Delta\Delta w + P^0 = 0, \tag{12.26a}$$

$$\Delta\Delta\chi + \frac{2h}{R^4\theta^2}\left\{ w_{,\theta\theta}(w_{,\varphi\varphi} + \theta w_{,\theta}) - \left(w_{,\theta\varphi} - \frac{1}{\theta} w_{,\varphi}\right)^2 \right\}$$
$$+ 2h\left\{ \frac{1}{R^3} w_{,\theta\theta} + \frac{1}{R^3\theta^3}(w_{,\varphi\varphi} + \theta w_{,\theta}) \right\} = 0. \tag{12.26b}$$

Here Δ is the Laplace operator

$$\Delta = \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{R^2\theta^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{R^2\theta}\frac{\partial}{\partial\theta} = \frac{1}{R^2\theta}\frac{\partial}{\partial\theta}\theta\frac{\partial}{\partial\theta} + \frac{1}{R^2\theta^2}\frac{\partial^2}{\partial\varphi^2}. \tag{12.27}$$

Equations (12.26a, b) are two nonlinear partial differential equations for two unknowns χ, w.

We suppose that the problem has rotational symmetry. Then w, χ are independent of φ, and (12.26a, b) reduce to the form

$$\frac{d}{d\theta}\theta\frac{d}{d\theta}\frac{1}{\theta}\frac{d}{d\theta}\theta\frac{dw}{d\theta} - \frac{1}{D}\frac{d}{d\theta}\left(\frac{dw}{d\theta}\frac{d\chi}{d\theta}\right) - \frac{R}{D}\frac{d}{d\theta}\left(\theta\frac{d\chi}{d\theta}\right) + \frac{P^0\theta R^4}{D} = 0, \tag{12.28a}$$

$$\frac{d}{d\theta}\theta\frac{d}{d\theta}\frac{1}{\theta}\frac{d}{d\theta}\theta\frac{d\chi}{d\theta} + h\frac{d}{d\theta}\left(\frac{dw}{d\theta}\right)^2 + 2hR\frac{d}{d\theta}\left(\theta\frac{dw}{d\theta}\right) = 0. \tag{12.28b}$$

The equations can be integrated once giving

$$\theta \frac{d}{d\theta} \frac{1}{\theta} \frac{d}{d\theta} \theta \frac{dw}{d\theta} - \frac{1}{D} \frac{dw}{d\theta} \frac{d\chi}{d\theta} - \frac{R}{D} \theta \frac{d\chi}{d\theta} + \frac{P^0 \theta^2 R^4}{2D} = \text{constant}, \tag{12.29a}$$

$$\theta \frac{d}{d\theta} \frac{1}{\theta} \frac{d}{d\theta} \theta \frac{d\chi}{d\theta} + \left(\frac{dw}{d\theta}\right)^2 h + 2hR\theta \frac{dw}{d\theta} = \text{constant}. \tag{12.29b}$$

Since $dw/d\theta$ vanishes for $\theta = 0$, the constants are zero. If we introduce the quantities

$$\alpha = \frac{1}{R} \frac{dw}{d\theta} + \theta, \qquad \beta = \frac{1}{R^2} \frac{d\chi}{d\theta}, \tag{12.30}$$

the equations can be further simplified to the form

$$\theta \frac{d^2\alpha}{d\theta^2} + \frac{d\alpha}{d\theta} - \frac{\alpha}{\theta} - \frac{R^2}{D} \alpha\beta + \frac{P^0 R^3}{2D} \theta^2 = 0, \tag{12.31a}$$

$$\theta \frac{d^2\beta}{d\theta^2} + \frac{d\beta}{d\theta} - \frac{\beta}{\theta} + h(\alpha^2 - \theta^2) = 0. \tag{12.31b}$$

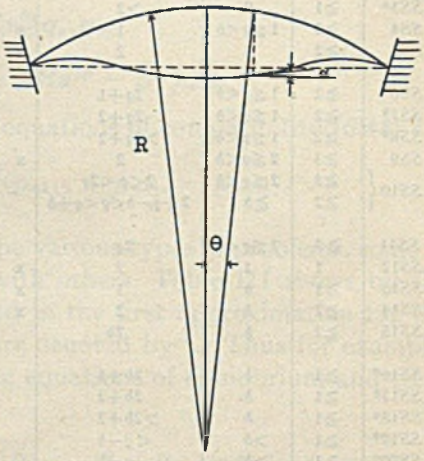

FIG. 9.

The quantity α is the slope of the meridian line in the strained state (Fig. 9). The significance of the quantity β is that β/θ is the radial membrane stress (tension). Equations (12.31a, b) are the fundamental equations for the determination of the buckling pressure of a small segment of spherical shell.

If we assume that the first and second terms in (12.31b) are negligible in comparison with the other terms, then we can solve (12.31b) immediately for β. Substituting the resulting expression for β into (12.31a), we have

$$\theta \frac{d^2\alpha}{d\theta^2} + \frac{d\alpha}{d\theta} - \frac{\alpha}{\theta} = \frac{hR^2}{D} \theta\alpha(\alpha^2 - \theta^2) - \frac{P^0 \theta^2 R^3}{2D}. \tag{12.32}$$

This is the equation used by von Kármán and Tsien [2] in their treatment of buckling of spherical shells by external pressure. It should be noted that the neglect of the first two terms in (12.31b) is a rough approximation. Actually the first three terms in (12.31b) are of the same order of magnitude.

Furthermore, if we introduce

$$r = R\theta, \tag{12.33}$$

Eqs. (12.28a, b) can be written in the form

$$\frac{d}{dr} r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} r \frac{dw}{dr} - \frac{1}{D} \frac{d}{dr}\left(\frac{dw}{dr} \frac{d\chi}{dr}\right) - \frac{1}{RD} \frac{d}{dr}\left(r \frac{d\chi}{dr}\right) + \frac{P^0 r}{D} = 0, \tag{12.34a}$$

$$\frac{d}{dr} r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} r \frac{d\chi}{dr} + h \frac{d}{dr}\left(\frac{dw}{dr}\right)^2 + \frac{2h}{R} \frac{d}{dr}\left(r \frac{dw}{dr}\right) = 0. \tag{12.34b}$$

The quantity r is the radial distance measured along the meridian line from the center of the shell. We see that these two equations are the same as the corresponding von Kármán equations for the circular plate under symmetrical loading [3], with the exception of the terms proportional to $1/R$; this is evident if we make R infinite in (12.34a, b).

A summary of the whole paper was given at the end of the first section (Part I).

APPENDICES

(iii) TABLE III.—Table of the equations of equilibrium and compatibility of thin shell problems.

Types	b	q	p	(6.34)									(6.35)						(6.44)				(6.43)	
				I_1^0	I_2^0	I_3^0	I_4^0	I_5^0	I_6^0	I_7^0	I_8^0	I_9^0	I_1^α	I_2^α	I_3^α	I_4^α	I_5^α	I_6^α	J_1^0	J_2^0	J_3^0	J_4^0	$J_{\alpha 1}$	$J_{\alpha 2}$
$SS1$	$\geqq 1$	0	1	x			x	x					x		x	x				x			x	
$SS2$	$\geqq 1$	0	2	x	x	x	x	x					x	x	x	x				x			x	
$SS3^*$	$\geqq 1$	0	>2		x	x	x	x						x	x	x				x			x	
$SS4$	$\geqq 2$	$1 \leqq q < b$	1	x			x	x					x		x				x				x	
$SS5$	$\geqq 2$	1	2	x	x		x	x					x		x				x	x			x	
$SS6^*$	$\geqq 2$	$1 \leqq q < b$	$2q+1$		x		x						x		x					x			x	
$SS7^*$	$\geqq 2$	$1 \leqq q < b$	$2q+2$		x		x						x	x	x	x				x			x	
$SS8^*$	$\geqq 2$	$1 \leqq q < b$	$>2q+2$		x		x							x	x	x				x			x	
$SS9$	$\geqq 3$	$2 \leqq q < b$	2	x	x		x	x					x		x				x				x	
$SS10$	$\geqq 3$	$2 \leqq q < b$	$2 < p < 2q$																					
	$\geqq 2$	$\geqq b$	$2+q-b < q < q+b$		x		x						x		x				x				x	
$SS11$	$\geqq 3$	$2 \leqq q < b$	$2q$		x		x						x		x				x	x			x	
$SS12$	1	1	2	x	x		x	x	x			x	x		x				x	x		x	x	
$SS13$	$\geqq 1$	b	1	x			x	x	x			x	x		x				x				x	
$SS14$	$\geqq 2$	b	2	x	x		x	x	x			x	x		x				x				x	
$SS15$	$\geqq 2$	b	$2b$		x		x						x		x				x	x		x	x	
$SS16^*$	$\geqq 1$	b	$2b+1$		x		x						x		x					x		x	x	
$SS17^*$	$\geqq 1$	b	$2b+2$		x		x						x	x	x	x	x	x		x		x	x	
$SS18^*$	$\geqq 1$	b	$>2b+2$		x		x							x	x	x	x	x		x		x	x	
$SS19^*$	$\geqq 1$	$>b$	$<q-b$				x		x			x	x		x				x					x
$SS20^*$	$\geqq 1$	$>b$	$q-b$				x		x			x	x		x				x				x	x
$SS21^*$	$\geqq 1$	$>b$	$q-b+1$				x		x			x	x		x				x				x	
$SS22$	$\geqq 2$	$>b$	$q-b+2$		x		x		x			x	x		x				x				x	
$SS23$	$\geqq 2$	$>b$	$q+b$		x		x						x		x				x			x	x	
$SS24^*$	$\geqq 1$	$>b$	$q+b+1$		x		x						x		x							x	x	
$SS25^*$	$\geqq 1$	$>b$	$q+b+2$		x		x						x		x		x	x				x	x	
$SS26^*$	$\geqq 1$	$>b$	$>q+b+2$		x		x								x		x	x				x	x	
$SS27$	1	$>b$	$q+1$		x		x		x			x	x		x				x			x	x	
$SF1$	0	0	1	x			x	x	x			x	x		x	x		x		x		x	x	
$SF2$	0	0	2	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x		x		x	x	
$SF3^*$	0	0	>2		x	x	x	x		x	x	x		x	x	x	x	x		x		x	x	
$SF4$	0	$\geqq 1$	q				x		x			x	x		x			x	x		x	x	x	x
$SF5$	0	$\geqq 1$	$q+1$				x		x			x	x		x			x				x	x	
$SF6$	0	$\geqq 1$	$q+2$		x		x		x	x		x	x		x		x	x				x	x	
$SF7^*$	0	$\geqq 1$	$>q+2$		x		x			x		x			x		x	x				x	x	
$SF8$	0	$\geqq 2$	$<q$				x		x			x	x		x			x	x		x		x	

In this table, the following notation is used:

The terms occurring in the first equation of equilibrium (6.34) are

$$I_1^0 = -2A_{(1)}^{\rho\gamma\pi\lambda} q_{\rho\gamma} p_{\pi\lambda} h, \qquad I_2^0 = \tfrac{2}{3} A_{(1)}^{\rho\gamma\pi\lambda} (q_{\pi\lambda} h^3)_{|\rho\gamma}, \qquad I_3^0 = A_{(3)}^{\rho\gamma\pi\omega\lambda\delta} q_{\pi\omega} q_{\rho\gamma} q_{\lambda\delta} h^3,$$

$$I_4^0 = P^0 + 2X_{[0]}^0 h + (Q^\lambda h)_{|\lambda}, \qquad I_5^0 = \frac{1-2\sigma}{1-\sigma} a_{\pi\lambda} q^{\pi\lambda} Q^0 h, \qquad I_6^0 = -A_{(1)}^{\rho\gamma\pi\lambda} b_{\rho\gamma} p_{\pi\lambda} h,$$

$$I_7^0 = -\tfrac{1}{2} A_{(4)}^{\rho\gamma\pi\omega\lambda\delta} b_{\rho\gamma} b_{\lambda\delta} q_{\pi\omega} h^3, \qquad I_8^0 = A_{(6)}^{\rho\gamma\pi\omega\lambda\delta} q_{\pi\omega} b_{\rho\gamma} q_{\lambda\delta} h^3, \qquad I_9^0 = \frac{1-2\sigma}{1-\sigma} 2HQ^0 h.$$

The terms occurring in the second and third equations of equilibrium (6.35) are

$$I_1^\alpha = 2A_{(1)}^{\rho\alpha\pi\lambda}(p_{\pi\lambda} h)_{|\rho}, \qquad I_2^\alpha = \tfrac{2}{3} a^{\alpha\pi} q_{\pi\gamma} A_{(1)}^{\gamma\rho\lambda\delta}(q_{\lambda\delta} h^3)_{|\rho} - A_{(3)}^{\rho\alpha\pi\omega\delta\lambda}(q_{\pi\omega} q_{\delta\lambda} h^3)_{|\rho},$$

$$I_3^\alpha = P^\alpha + 2X_{[0]}^\alpha h + \frac{\sigma}{1-\sigma} a^{\alpha\rho}(Q^0 h)_{|\rho}, \qquad I_4^\alpha = (a^{\pi\lambda} q_{\pi\lambda} a_\gamma^\alpha + 2a^{\alpha\pi} q_{\pi\gamma}) Q^\gamma h,$$

$$I_5^\alpha = A_{(4)}^{\rho\alpha\pi\omega\lambda\delta}(b_{\lambda\delta} q_{\pi\omega} h^3)_{|\rho} + \tfrac{1}{3} A_{(1)}^{\gamma\rho\lambda\delta} a^{\alpha\pi} b_{\pi\gamma}(q_{\lambda\delta} h^3)_{|\rho}, \qquad I_6^\alpha = (2Ha_\gamma^\alpha + b_\gamma^\alpha) Q^\gamma h.$$

The terms occurring in the first equation of compatibility (6.44) are

$$J_1^0 = 2\mathfrak{n}_{[0]}^{\rho\alpha}\mathfrak{n}_{[0]}^{\beta\gamma} p_{\rho\gamma|\alpha\beta}, \qquad J_2^0 = \mathfrak{n}_{[0]}^{\rho\alpha}\mathfrak{n}_{[0]}^{\beta\gamma} q_{\rho\gamma} q_{\alpha\beta},$$

$$J_3^0 = 2a^{\alpha\beta} p_{\alpha\beta} K, \qquad J_4^0 = -(4Ha^{\alpha\beta} - b^{\alpha\beta}) q_{\alpha\beta}.$$

The terms occurring in the second and third equations of compatibility (6.43) are

$$J_{\alpha 1} = 2\mathfrak{n}_{[0]}^{\beta\gamma} q_{\alpha\beta|\gamma}, \qquad J_{\alpha 2} = \mathfrak{n}_{[0]}^{\beta\gamma} b_{\beta\pi} a^{\pi\lambda}(p_{\alpha\lambda|\gamma} + p_{\gamma\lambda|\alpha} - p_{\alpha\gamma|\lambda}).$$

On account of the conditions which hold in the various types of problems, some of these terms may be negligible in comparison with others. Table III shows by the symbol 'x' those terms which are to be retained in the first approximation for the various types. (The over-determined problems are denoted by *.) Thus for example, for problems of type $SS1$, we have the following equations of equilibrium and compatibility in the first approximation:

$$I_1^0 + I_4^0 + I_5^0 = 0, \qquad I_1^\alpha + I_3^\alpha + I_4^\alpha = 0, \qquad J_2^0 = 0, \qquad J_{\alpha 1} = 0.$$

These equations are written in terms of the small principal parts instead of in terms of the finite coefficients of the lowest power in ϵ.

(iv) In Table IV, the following notation is used:

The terms occurring in the expresion (6.29) for the membrane stress tensor $T^{\alpha\beta}$ are denoted by

$$T_1^{\alpha\beta} = 2A_{(1)}^{\alpha\beta\pi\lambda} p_{\pi\lambda} h, \qquad T_2^{\alpha\beta} = -A_{(3)}^{\alpha\beta\pi\omega\lambda\delta} q_{\pi\omega} q_{\lambda\delta} h^3,$$

$$T_3^{\alpha\beta} = \frac{\sigma}{1-\sigma} a^{\alpha\beta} Q^0 h, \qquad T_4^{\alpha\beta} = A_{(4)}^{\alpha\beta\pi\omega\lambda\delta} b_{\lambda\delta} q_{\pi\omega} h^3.$$

The terms occurring in the expression (6.30) for the bending moment tensor $L^{\alpha\beta}$ are denoted by

$$L_1^{\alpha\beta} = \tfrac{2}{3}\mathfrak{n}_{[0]}^{\rho\beta} a_{\pi\rho} A_{(1)}^{\alpha\pi\lambda\delta} q_{\lambda\delta} h^3,$$

$$L_2^{\alpha\beta} = 2\mathfrak{n}_{[0]}^{\rho\beta} a_{\pi\rho} A_{(5)}^{\alpha\pi\lambda\delta\rho\gamma} b_{\lambda\delta} p_{\rho\gamma} h^3$$

$$+ \frac{\sigma}{6(1-\sigma)} \mathfrak{n}_{[0]}^{\lambda\beta} \left\{ a_\lambda^\alpha \left(\frac{4\sigma}{1-\sigma} HQ^0 - 4X_{[0]}^0 - 2Q_{|\gamma}^\gamma \right) - b_\lambda^\alpha Q^0 \right\} h^3.$$

TABLE IV.—Table of the external force system and the macroscopic tensors for various types of thin shell problems.

Types	n_0	n	j_0	j	k_0	k	$T^{\alpha\beta}$					$L^{\alpha\beta}$			$T^{\alpha 0}$			
							t	$T_1^{\alpha\beta}$	$T_2^{\alpha\beta}$	$T_3^{\alpha\beta}$	$T^{\alpha\beta}$	u	$L_1^{\alpha\beta}$	$L_2^{\alpha\beta}$	l	T_1^{α}	T_2^{α}	T_3^{α}
$SS1$	2	2	1	1	1	1	2	x		x		3	x		2	x		
	2	2	1	1	1	2	2	x		x		3	x		3	x	x	
$SS2$	3	3	2	2	2	2	3	x	x	x		3	x		3	x	x	
$SS3$	3	3	2	2	2	2	3		x	x		3	x		3	x	x	
$SS4$	$q+2$	2	$q+1$	1	1	$q+1$	2	x		x		$q+3$	x		$q+2$	x		
	$q+2$	2	$q+1$	1	1	$q+2$	2	x		x		$q+3$	x		$q+3$	x	x	
$SS5$	4	3	3	2	2	3	3	x		x		4	x		4	x	x	
$SS6$	$q+3$	$2q+2$	$q+2$	$2q+1$	$2q+1$	$q+2$	$2q+2$	x		x		$q+3$	x		$q+3$	x	x	
$SS7$	$q+3$	$2q+3$	$2q+2$	$q+2$	$q+2$	$2q+2$	$2q+3$	x	x	x		$q+3$	x		$q+3$	x	x	
$SS8$	$q+3$	$2q+3$	$2q+2$	$q+2$	$q+2$	$2q+2$	$2q+3$		x	x		$q+3$	x		$q+3$	x	x	
$SS9$	$q+3$	3	$q+2$	2	2	$q+2$	3	x		x		$q+3$	x		$q+3$	x	x	
$SS10$	$q+3$	$p+1$	$q+2$	p	p	$q+2$	$p+1$	x		x		$q+3$	x		$q+3$	x	x	
$SS11$	$q+3$	$2q+1$	$q+2$	$2q$	$2q$	$q+2$	$2q+1$	x		x		$q+3$	x		$q+3$	x	x	
$SS12$	4	3	3	2	2	3	3	x		x		4	x		4	x	x	
$SS13$	$b+2$	2	$b+1$	1	1	$b+1$	2	x		x		$b+3$	x		$b+2$	x		
	$b+2$	2	$b+1$	1	1	$b+2$	2	x		x		$b+3$	x		$b+3$	x	x	
$SS14$	$b+3$	3	$b+2$	2	2	$b+2$	3	x		x		$b+3$	x		$b+3$	x	x	
$SS15$	$b+3$	$2b+1$	$b+2$	$2b$	$2b$	$b+2$	$2b+1$	x		x		$b+3$	x		$b+3$	x	x	
$SS16$	$b+3$	$2b+2$	$b+2$	$2b+1$	$2b+1$	$b+2$	$2b+2$	x		x		$b+3$	x		$b+3$	x	x	
$SS17$	$b+3$	$2b+3$	$b+2$	$2b+2$	$2b+2$	$b+2$	$2b+3$	x	x	x	x	$b+3$	x		$b+3$	x	x	
$SS18$	$b+3$	$2b+3$	$b+2$	$2b+2$	$2b+2$	$b+2$	$2b+3$		x	x	x	$b+3$	x		$b+3$	x	x	
$SS19$	$b+p+1$	$p+1$	$b+p$	p	p	$b+p$	$p+1$	x		x		$b+p+3$		x	$b+p+1$	x		
	$b+p+1$	$p+1$	$b+p$	p	p	$b+p+1$	$p+1$	x		x		$b+p+3$		x	$b+p+2$	x		
	$b+p+1$	$p+1$	$b+p$	p	p	$b+p+2$	$p+1$	x		x		$b+p+3$		x	$b+p+3$	x		x
$SS20$	$b+p+1$	$p+1$	$b+p$	p	p	$b+p$	$p+1$	x		x		$b+p+3$	x	x	$b+p+1$	x		
	$b+p+1$	$p+1$	$b+p$	p	p	$b+p+1$	$p+1$	x		x		$b+p+3$	x	x	$b+p+2$	x		
	$b+p+1$	$p+1$	$b+p$	p	p	$b+p+2$	$p+1$	x		x		$b+p+3$	x	x	$b+p+3$	x	x	x
$SS21$	$b+p+1$	$p+1$	$b+p$	p	p	$b+p$	$p+1$	x		x		$b+p+2$	x		$b+p+1$	x		
	$b+p+1$	$p+1$	$b+p$	p	p	$b+p+1$	$p+1$	x		x		$b+p+2$	x		$b+p+2$	x	x	
$SS22$	$q+2$	$q-b+3$	$q+2$	$q-b+2$	$q-b+2$	$q+2$	$q-b+3$	x		x		$q+3$	x		$q+3$	x	x	
$SS23$	$q+3$	$q+b+1$	$q+2$	$q+b$	$q+b$	$q+2$	$q+b+1$	x		x		$q+3$	x		$q+3$	x	x	
$SS24$	$q+3$	$q+b+2$	$q+2$	$q+b+1$	$q+b+1$	$q+2$	$q+b+2$	x		x		$q+3$	x		$q+3$	x	x	
$SS25$	$q+3$	$q+b+3$	$q+2$	$q+b+2$	$q+b+2$	$q+2$	$q+b+3$	x		x	x	$q+3$	x		$q+3$	x	x	
$SS26$	$q+3$	$q+b+3$	$q+2$	$q+b+2$	$q+b+2$	$q+2$	$q+b+3$			x	x	$q+3$	x		$q+3$	x	x	
$SS27$	$q+3$	$q+2$	$q+2$	$q+1$	$q+1$	$q+2$	$q+2$	x		x		$q+3$	x		$q+3$	x	x	
$SF1$	2	2	1	1	1	1	2	x		x		3	x		2	x		
	2	2	1	1	1	2	2	x		x		3	x		3	x	x	
$SF2$	3	3	2	2	2	2	3	x	x	x	x	3	x		3	x	x	
$SF3$	3	3	2	2	2	2	3		x	x	x	3	x		3	x	x	
$SF4$	$q+1$	$q+1$	q	q	q	q	$q+1$	x		x		$q+3$	x	x	$q+1$	x		
	$q+1$	$q+1$	q	q	q	$q+1$	$q+1$	x		x		$q+3$	x	x	$q+2$	x		
	$q+1$	$q+1$	q	q	q	$q+2$	$q+1$	x		x		$q+3$	x	x	$q+3$	x	x	x
$SF5$	$q+2$	$q+2$	$q+1$	$q+1$	$q+1$	$q+1$	$q+2$	x		x		$q+3$	x		$q+2$	x		
	$q+2$	$q+2$	$q+1$	$q+1$	$q+1$	$q+2$	$q+2$	x		x		$q+3$	x		$q+3$	x	x	
$SF6$	$q+3$	$q+3$	$q+2$	$q+2$	$q+2$	$q+2$	$q+3$	x		x	x	$q+3$	x		$q+3$	x	x	
$SF7$	$q+3$	$q+3$	$q+2$	$q+2$	$q+2$	$q+2$	$q+3$			x	x	$q+3$	x		$q+3$	x	x	
$SF8$	$p+1$	$p+1$	p	p	p	p	$p+1$	x		x		$p+3$		x	$p+1$	x		
	$p+1$	$p+1$	p	p	p	$p+1$	$p+1$	x		x		$p+3$		x	$p+2$	x		
	$p+1$	$p+1$	p	p	p	$p+2$	$p+1$	x		x		$p+3$		x	$p+3$	x		x

The terms occurring in the expression (6.31) for the shearing stress tensor $T^{\alpha 0}$ are denoted by

$$T_1^{\alpha} = Q^{\alpha}h, \qquad T_2^{\alpha} = \tfrac{2}{3}A_{(1)}^{\pi\alpha\lambda\delta}(q_{\lambda\delta}h^2)_{|\pi},$$

$$T_3^\alpha = 2A_{(5)}^{\pi\alpha\lambda\delta\rho\gamma}(b_{\lambda\delta}p_{\rho\gamma}h^3)_{|\pi} + \tfrac{1}{2}(4HP^\alpha + b_\pi^\alpha P^\pi)h^2 + \tfrac{4}{3}HX_{[0]}^\alpha h^3$$
$$+ \frac{\sigma}{6(1-\sigma)}\left\{\left[a^{\alpha\pi}\left(\frac{2\sigma}{1-\sigma}HQ^0 - 4X_{[0]}^0\right) - b^{\alpha\pi}Q^0\right]h^3\right\}_{|\pi}.$$

Furthermore,

n_0 = order of sum of the normal forces acting on the upper and lower boundary surfaces, or order of P^0,

n = order of sum of the tangential forces acting on the upper and lower boundary surfaces, or order or P^α,

j_0 = order of normal component of body force, or order of $X_{[0]}^0$,

j = order of tangential component of body force, or order of $X_{[0]}^\alpha$,

k_0 = order of difference of normal forces acting on the upper and lower surfaces, or order of Q^0,

k = order of difference of tangential components of forces acting on the upper and lower boundary surfaces, or order of Q^α,

t = order of membrane stress tensor $T^{\alpha\beta}$,

u = order of bending moment tensor $L^{\alpha\beta}$,

l = order of shearing stress tensor $T^{\alpha 0}$.

This table gives (a) the values of $n_0, n, j_0, j, k_0, k, t, u, l$, (b) the principal terms in the expressions for $T^{\alpha\beta}$, $L^{\alpha\beta}$, $T^{\alpha 0}$ (denoted by 'x'). The terms not marked with 'x' are negligible in comparison those principal terms. It will be noted that there are two lines in the table for *SS*1, *SS*4, *SS*13, *SS*21, *SF*1, *SF*5, and three lines for *SS*19, *SS*20, *SF*4, *SF*8. This is because, in each case, k may have two or three values.

For example, in the case of Type *SS*1, we have for $T^{\alpha\beta}$, $L^{\alpha\beta}$,

$$T^{\alpha\beta} = T_1^{\alpha\beta} + T_3^{\alpha\beta}, \qquad L^{\alpha\beta} = L_1^{\alpha\beta},$$

while for $T^{\alpha 0}$,

$$T^{\alpha 0} = T_1^\alpha \qquad (\text{if } k = 1),$$

$$T^{\alpha 0} = T_1^\alpha + T_2^\alpha \qquad (\text{if } k = 2).$$

References

1. Th. von Kármán and H. S. Tsien, *The buckling of thin cylindrical shells under axial compression*, J. Aeronautical Sci. 7, 303–312 (1941).

2. Th. von Kármán and H. S. Tsien, *The buckling of spherical shells by external pressure*, J. Aeronautical Sci. 7, 43–50 (1939).

3. K. Friedrichs and J. J. Stoker, *The nonlinear boundary value problem of the circular plate*, Proc. Nat. Acad. Sci. U.S.A. 25, 535–540 (1939); Amer. J. Math. 63, 839–888 (1941).

THE AERODYNAMICS OF A RING AIRFOIL*

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Abstract. The downwash required to produce a given vorticity distribution is computed for a ring airfoil and the results are compared with the corresponding two-dimensional case. From this it appears that if the curvature of the chord plane is small, as is the case with normal amounts of dihedral, the effect of this curvature on the chordwise lift distribution of a wing is extremely small. If the radius of curvature is small compared to the chord, as it is near the vertex of a cranked wing, it is seen that this curvature may cause comparatively large changes in the lift distribution.

1. Introduction. At the present time, the steady state two-dimensional airfoil theory is a highly developed subject; and, subject to the usual limitations of perfect fluid theory, solutions may be obtained with almost any desired degree of accuracy. The steady state three-dimensional airfoil theory is, however, in a much lower state of development. For most engineering problems the "lifting line" theory as developed by Prandtl and others is adequate to provide satisfactory results; however, for certain other problems, such as the flow near a wing tip, the effects of sweepback or of yaw, or the lift of a low aspect ratio wing, the lifting line theory cannot be used. At this time there have been only a fairly small number of solutions of finite wing problems in which lifting surfaces rather than lifting lines have been used and which may thus be used to throw light on these essentially more complicated problems. The best known of these lifting surface theories are those due to Blenk,[1] Kinner,[2] Krienes,[3] and Bollay.[4] As the number of such solutions is so limited almost any special solution involving a lifting surface is of interest.

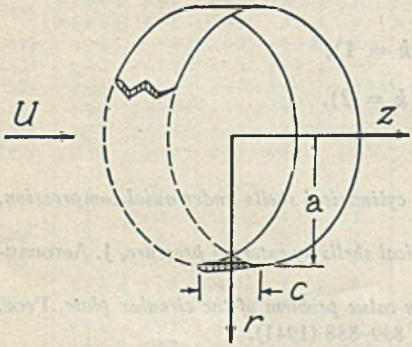

FIG. 1. Ring wing in a uniform flow.

From an analytical viewpoint, probably the simplest lifting surface problem which has not yet been investigated is that of the axially symmetric flow past a ring airfoil as shown in Fig. 1. This flow is especially simple as the vortex lines in the lifting surface are circular rings and there are thus no trailing vortices. The particular purpose of the present paper is to discuss the differences between this problem and the corresponding two dimensional problem. In addition to its intrinsic interest in the theory of the "anti-drag" cowl, the ring airfoil problem possesses a general interest insofar as it demonstrates, at least qualitatively, some of the effects of dihedral on the lift distribution of a wing.

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[1] Blenk, H., Zeit. f. angew. Math. u. Mechanik, 5, 36 (1925).

[2] Kinner, W., Ingenieur Archiv, 8, 47 (1937).

[3] Krienes, K., Zeit. f. angew. Math. u. Mechanik, 20, 65 (1940).

[4] Bollay, W., Zeit. f. angew. Math. u. Mechanik, 19, 21 (1939); also J. Aero. Sci., 4, 294 (1937).

2. The vector potential. The mathematical analysis of this problem may be conveniently carried out by the method of the vector potential. Since the equation of continuity in an incompressible fluid is simply

$$\operatorname{div} \boldsymbol{q} = 0, \tag{1}$$

the velocity vector q may be written as the curl of a vector potential A or

$$\boldsymbol{q} = \operatorname{curl} \boldsymbol{A}. \tag{2}$$

By the Helmholtz decomposition theorem the vector potential may be subjected to the restriction that

$$\operatorname{div} \boldsymbol{A} = 0. \tag{3}$$

The differential equation for the determination of the vector potential is found by curling Eq. (2). This gives

$$\nabla^2 \boldsymbol{A} = -\operatorname{curl} \boldsymbol{q} = -\boldsymbol{\Omega}. \tag{4}$$

If the vorticity Ω is a given function, this is a Poisson equation for the determination of the vector potential. The solution of this equation, which is well-known and may be obtained by the use of Green's theorem, is

$$\boldsymbol{A} = \frac{1}{4\pi} \int \boldsymbol{\Omega} \frac{dv}{r_1} \tag{5}$$

where the volume integral covers the entire region where the vorticity exists and r_1 is the distance from the point at which the vorticity exists to the point P at which the vector potential is being computed. If the vorticity is in the form of a single vortex filament of strength Γ then

$$\boldsymbol{A} = \frac{\Gamma}{4\pi} \int \frac{1}{r_1} d\boldsymbol{s}, \tag{6}$$

where $d\boldsymbol{s}$ is an infinitesimal distance vector along the vortex filament. If there are several vortex filaments the contribution from each one may be found by Eq. (6), and these results must then be summed to obtain the vector potential.

3. The vector potential for a vortex ring. As the vortex filaments are all circles for the axially symmetric flow past a ring wing, the complete vector potential can easily be obtained if the vector potential of a single filament is known. For such a filament of strength Γ and lying in the plane $z=0$ (see Fig. 2), it is obvious that the vector potential is not a function of the meridian angle θ, and it may be calculated at points in the plane $\theta=0$. Since $d\boldsymbol{s}$ is in the plane $z=0$, the vector potential can have no z-component. Furthermore, by considering two vortex elements, one having the negative of the other's θ coordinate, it is evident that the vector potential can have no radial component. The vector potential has thus only the component A_θ which is perpendicular to the meridian planes. By Eq. (6), this is

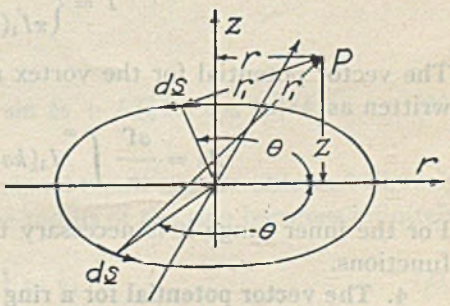

FIG. 2. Vortex ring.

$$A_\theta = \frac{a\Gamma}{2\pi}\int_0^\pi \frac{\cos\theta\, d\theta}{[a^2 + r^2 + z^2 - 2ar\cos\theta]^{1/2}}. \tag{7}$$

With the vector potential expressed in this form as an elliptic integral, it is rather difficult to superimpose the vector potentials for a band of vorticity of radius a and of chord c in order to represent the ring wing. A much more convenient form can be obtained by the use of the Fourier integral. For an even function $f(z)$, the Fourier integral theorem states that

$$f(z) = \frac{2}{\pi}\int_0^\infty \cos kz\left\{\int_0^\infty f(t)\cos kt\, dt\right\} dk. \tag{8}$$

Since A_θ is an even function of z, it follows that

$$A_\theta = \frac{a\Gamma}{\pi^2}\int_0^\infty \cos kz\left[\int_0^\infty \cos kt\left\{\int_0^\pi \frac{\cos\theta\, d\theta}{[a^2 + r^2 - 2ar\cos\theta + t^2]^{1/2}}\right\} dt\right] dk. \tag{9}$$

Since[5]

$$\int_0^\infty \frac{\cos(kt)}{[x^2 + t^2]^{1/2}}\, dt = K_0(kx), \tag{10}$$

the inner two integrals of Eq. (9) become, after inversion of the order of integration,

$$I = \int_0^\pi K_0[k\sqrt{a^2 + r^2 - 2ar\cos\theta}]\cos\theta\, d\theta. \tag{11}$$

The addition theorem for the modified Bessel functions of the second kind (see Ref. 5, p. 74) states that

$$K_0[k\sqrt{a^2 + r^2 - 2ar\cos\theta}] = \begin{cases} I_0(ka)K_0(kr) + 2\sum_{n=1}^{\infty} I_n(ka)K_n(kr)\cos n\theta \text{ if } r > a. \\ I_0(kr)K_0(ka) + 2\sum_{n=1}^{\infty} I_n(kr)K_n(ka)\cos n\theta \text{ if } r < a. \end{cases} \tag{12}$$

Since the trigonometrical functions are orthogonal over the range $0 \leqq \theta \leqq \pi$,

$$I = \begin{cases} \pi I_1(ka)K_1(kr) & \text{if } r > a. \\ \pi I_1(kr)K_1(ka) & \text{if } r < a. \end{cases} \tag{13}$$

The vector potential for the vortex ring in the outer range where $r > a$ can thus be written as

$$A_\theta = \frac{a\Gamma}{\pi}\int_0^\infty I_1(ka)K_1(kr)\cos(kz)dk \qquad (r > a). \tag{14}$$

For the inner range it is necessary to interchange the arguments of the two Bessel functions.

4. The vector potential for a ring airfoil. A ring airfoil may be considered to be a system of ring vortices of radius a and distributed over the chord c from $z = -c/2$ to $z = c/2$. If the strength of this vortex sheet is $\gamma(z_0)$, then the vector potential for $r > a$ is

[5] Grey, Mathews and MacRobert, *Bessel functions*, Macmillan and Co., London, 1931, p. 52.

$$A_\theta = \frac{a}{\pi}\int_{-c/2}^{c/2}\gamma(z_0)\left\{\int_0^\infty I_1(ka)K_1(kr)\cos k(z-z_0)dk\right\}dz_0. \tag{15}$$

If the vortex strength is known, Eq. (15), after inversion of the order of integration, can conveniently be used to compute the vector potential or the radial or axial velocity components, u_r and u_z respectively. From Eq. (2),

$$u_r = -\frac{\partial A_\theta}{\partial z}, \qquad u_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta). \tag{16}$$

The radial velocity is of the most interest as it corresponds to the downwash velocity in the ordinary two-dimensional airfoil theory. The downwash at the ring airfoil, $r=a$, is

$$u_r = \frac{a}{\pi}\int_0^\infty kI_1(ka)K_1(ka)\left\{\int_{-c/2}^{c/2}\gamma(z_0)\sin k(z-z_0)dz_0\right\}dk. \tag{17}$$

5. Comparison with two-dimensional flat plate airfoil. If the airfoil shape is given, the downwash u_r is known, and Eq. (17) may be considered as an integral equation for the determination of the vortex strength $\gamma(z_0)$. It is, however, an integral equation of a difficult type. The importance of the curvature of the chord plane may be estimated by comparing the downwash for some given vortex distribution with the corresponding two dimensional result. This process will be carried out for the vortex distribution

$$\gamma(z_0) = A\sqrt{\frac{c-2z_0}{c+2z_0}}. \tag{18}$$

In the two-dimensional case, this vorticity distribution corresponds to a flat plate airfoil with its leading edge at $z_0=-c/2$. The downwash is then constant over the airfoil and equal to $A/2$. For this vorticity distribution it can easily be seen by use of the transformation $2z_0=c\sin\theta$ that

$$\int_{-c/2}^{c/2}\gamma(z_0)\sin k(z-z_0)dz_0 = \frac{\pi}{2}Ac[J_0(\tfrac{1}{2}kc)\sin kz + J_1(\tfrac{1}{2}kc)\cos kz]. \tag{19}$$

The downwash velocity is thus

$$u_r = \tfrac{1}{2}Aca\int_0^\infty kI_1(ka)K_1(ka)[J_0(\tfrac{1}{2}kc)\sin kz + J_1(\tfrac{1}{2}kc)\cos kz]dk. \tag{20}$$

It is of interest to note that the two-dimensional result can be obtained directly from this by considering the limiting form as the radius of the ring becomes infinitely large; for

$$\lim_{x\to\infty}\{xI_1(x)K_1(x)\} = \tfrac{1}{2}; \tag{21}$$

so the downwash in the two-dimensional case is given by

$$u_r = \tfrac{1}{4}Ac\int_0^\infty [J_0(\tfrac{1}{2}kc)\sin(kz) + J_1(\tfrac{1}{2}kc)\cos(kz)]dk. \tag{22}$$

TABLE I
Comparison of $F(x)$ and $F_1(x)$

x	$F(x)$	$F_1(x)$	$F(x)-F_1(x)$
0.1	0.0493	0.0132	0.0361
0.5	0.2132	0.2000	0.0132
1	0.3402	0.3637	−0.0235
2	0.4450	0.4571	−0.0121
3	0.4762	0.4800	−0.0038
4	0.4873	0.4885	−0.0012
5	0.4921	0.4926	−0.0005

The first integral vanishes on the airfoil where $z^2 \leqq c^2/4$ and the second is equal to $2/c$ on the airfoil (see Ref. 5, p. 65); so in the two-dimensional case, for this vorticity distribution

$$u_r = \tfrac{1}{2}A \qquad (z^2 \leqq c^2/4). \tag{23}$$

An exact evaluation of the integral of Eq. (20) is rather difficult; however, an approximate evaluation, valid for large values of a/c, may be obtained quite easily. If

$$F(x) = xI_1(x)K_1(x), \tag{24}$$

a very close approximation to $F(x)$ is given by

$$F_1(x) = \frac{x^2/2}{3/8 + x^2}. \tag{25}$$

It may be noted that the asymptotic expansions for $F(x)$ and $F_1(x)$ are the same up through terms of order (x^{-2}). It is shown in Table I and Fig. 3 that $F_1(x)$ is a good approximation to $F(x)$ even for small values of x.

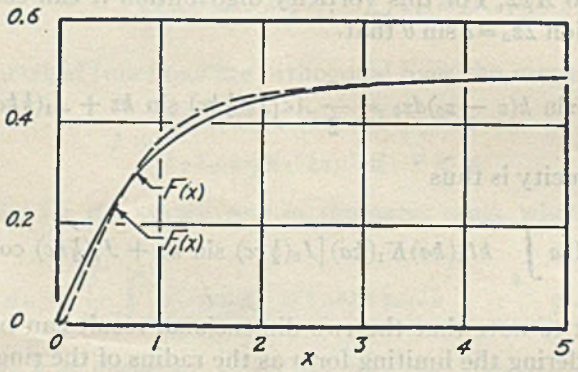

FIG. 3. Comparison of $F(x)$ and $F_1(x)$.

Since

$$F_1(x) - \tfrac{1}{2} = -\frac{3/16}{3/8 + x^2}, \tag{26}$$

an approximate expression for Δu, the difference between the ring airfoil downwash of Eq. (20) and the corresponding two-dimensional case is given by

$$\Delta u = -(3/32)Ac\int_0^\infty [J_0(\tfrac{1}{2}kc)\sin kz + J_1(\tfrac{1}{2}kc)\cos kz]\frac{dk}{3/8 + a^2k^2}. \tag{27}$$

Let $\lambda = kc/2$, $\alpha = \sqrt{3/32}c/a$ and $\beta = 2z/c$. Then

$$\Delta u = -\tfrac{1}{2}A\alpha^2\int_0^\infty [J_0(\lambda)\sin\beta\lambda + J_1(\lambda)\cos\beta\lambda]\frac{d\lambda}{\lambda^2 + \alpha^2}. \tag{28}$$

On the airfoil where $\beta^2 \leqq 1$, this gives (see Ref. 5, p. 78)

$$\Delta u = \tfrac{1}{2}A[\alpha\cosh(\alpha\beta)K_1(\alpha) - \alpha\sinh(\alpha\beta)K_0(\alpha) - 1]. \tag{29}$$

The ratio of the change in downwash to the two dimensional downwash is $2(\Delta u)/A$. For $\alpha = 0.02$ and 0.20 corresponding to $a/c = 15.3$ and 1.53 respectively, this ratio is given in Table 2 for the leading edge ($\beta = -1$), the center of the airfoil ($\beta = 0$), and for the trailing edge ($\beta = 1$).

TABLE 2

Values of $2(\Delta u)/A$ for the ring airfoil

α	0.02	0.20
a/c	15.3	1.53
$\beta = -1$	0.0009	0 .0451
$\beta = 0$	−0.0009	−0.0448
$\beta = 1$	−0.0023	−0.0963

As the downwash velocity is determined by the slope of the camber line, the airfoil camber required to produce the lift distribution of Eq. (18) may be computed by integrating the downwash velocity. The camber lines for $a/c = 1.53$ and for the two dimensional case are shown for comparison in Fig. 4.

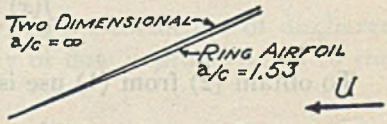

FIG. 4. Airfoil profiles having the same vorticity. See Eq. (18).

6. Conclusions. From Table 2, it is apparent that the effects of the curvature of the chord plane of the ring airfoil are negligibly small if $a/c = 15.3$ while they are fairly large for $a/c = 1.53$. From Fig. 4 it appears that the lift of a ring airfoil having a constant angle of attack across the chord would be somewhat more than that of the corresponding two dimensional airfoil and the lift is shifted away from the leading edge toward the center of the airfoil.

It seems reasonable to suppose that the changes at any given section of the ring wing are caused primarily by the vortex elements near that section. These results may thus be applied in estimating the effects of the dihedral of a wing on the lift distribution over the wing's surface. This indicates that if the curvature of the chord plane is small, as is normally the case, no appreciable changes in the lift distribution need be expected; however, if the radius of curvature of the chord plane is of the same order as the chord, fairly large effects may be expected. This should be particularly noticeable near the vertex of a cranked wing.

THE MATHEMATICS OF WEIR FORMS*

BY

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1. Introduction. This paper aims at making more readily available the results of a study of the mathematics of weir forms, a subject in Hydraulics to which higher mathematics can be applied. Section 2 covers the general application of Abel's integral equation to the forms of weirs by Brenke.[1] Section 3 deals with sectionally analytic weir forms, particularly the Stout-Sutro weir. The writer believes he has made the original application of Abel's integral equation to this corrected weir form. Section 4 deals with cases when the quantity of flow can be expressed as a convergent series.

2. Abel's integral equation. One method of solution of the problem of weir forms involves Abel's integral equation. The natural conditions found in the flow of water through weirs satisfy all the requirements of this integral equation, so it proves a superior mathematical tool in handling the general problem. In 1922 Brenke studied the problem of the weir form when the flow was proportional to some power of the depth. He made the original application of Abel's integral equation. This equation has the form

$$\phi(x) = \int_a^x \frac{f(s)ds}{(x-s)^\lambda}, \qquad 0 < \lambda < 1 \tag{1}$$

and its solution is, under certain conditions,

$$f(x) = \frac{\sin \lambda\pi}{\pi} \int_a^x \frac{\phi'(s)ds}{(x-s)^{1-\lambda}}. \tag{2}$$

To obtain (2) from (1) use is made of two fundamental formulas, namely[2,3]

$$\frac{\pi}{\sin \lambda\pi} = \int_s^z \frac{dx}{(z-x)^{1-\lambda}(x-s)^\lambda}, \qquad 0 < \lambda < 1 \tag{3}$$

$$\int_a^z \int_s^z \frac{\phi'(s)dx}{(z-x)^{1-\lambda}(x-s)^\lambda} ds = \int_a^z \frac{1}{(z-x)^{1-\lambda}} \int_a^x \frac{\phi'(s)ds}{(x-s)^\lambda} dx. \tag{4}$$

$\phi(s)$ is assumed to be continuous and have a continuous derivative in the closed

* Received January 8, 1944. This paper constitutes part of a thesis submitted in partial fulfillment of the requirements for the degree of Master of Arts at the University of Nebraska.

[1] W. C. Brenke, *An application of Abel's integral equation*, Am. Math. Monthly, 29, 58 (1922).

[2] E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th Ed., Cambridge Univ. Press, London, 1927, pp. 211, 229.

[3] Maxime Bôcher, *An introduction to the study of integral equations*, Cambridge Tracts in Math. and Math. Physics, No. 10, Cambridge Univ. Press, London, 1926, p. 8.

interval, a to b. Formula (4) is known as Dirichlet's generalized formula.[4] Multiply (3) by $\phi'(s)ds$ and integrate from a to z, $(a \leqq z \leqq b)$, which gives

$$\frac{\pi}{\sin \lambda\pi}[\phi(z) - \phi(a)] = \int_a^z \int_s^z \frac{\phi'(s)dx}{(z-x)^{1-\lambda}(x-s)^{\lambda}} ds. \tag{5}$$

If (4) is applied to the right hand member of (5), we have

$$\phi(z) - \phi(a) = \frac{\sin \lambda\pi}{\pi} \int_a^z \frac{1}{(z-x)^{1-\lambda}} \int_a^x \frac{\phi'(s)ds}{(x-s)^{\lambda}} dx. \tag{6}$$

Then, if $\phi(a)=0$ and if we replace the inner integral on the right of (6) by its value from (2), we see that (6) becomes (1). Hence (2) is a solution of (1).

The weir is actually symmetrically constructed as in Fig. 3, but for purposes of the present calculation a half section is used (Fig. 1). Letting $y=f(x)$ express the

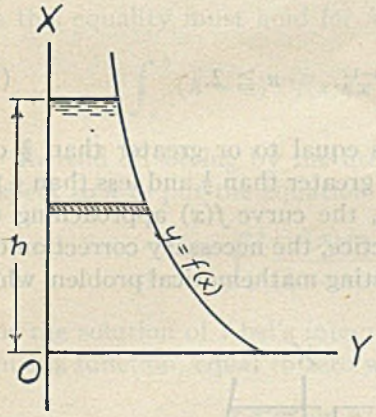

FIG. 1.

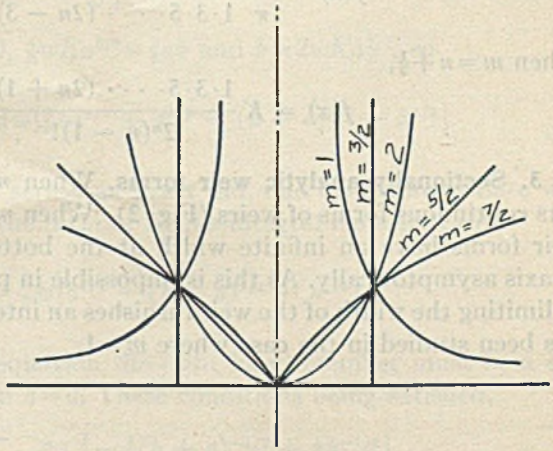

FIG. 2.

distribution of width over depth, h the depth of flow, C_d the coefficient of discharge (approximately 0.6), and assuming that the quantity of flow is proportional to the mth power of the depth of stream, we have

$$C_d \int_0^h [2g(h-x)]^{1/2} f(x)dx = bh^m, \tag{7}$$

or, letting $K=b/C_d(2g)^{1/2}$,

$$\int_0^h (h-x)^{1/2} f(x)dx = Kh^m.$$

Differentiating with respect to h, we have

$$\int_0^h \frac{f(x)dx}{(h-x)^{1/2}} = 2Kmh^{m-1}. \tag{8}$$

This equation has the form of Abel's integral equation.

To find the equation of the weir form when the flow is bh^m, we have

[4] W. A. Hurwitz, *Note on certain iterated and multiple integrals*, Annals of Math., 9, 183 (1907).

$$f(x) = \frac{\sin \pi/2}{\pi} \int_0^x \frac{2Km(m-1)h^{m-2}}{(x-h)^{1/2}}\, dh,$$

or

$$f(x) = \frac{2Km(m-1)}{\pi} \int_0^x \frac{h^{m-2}}{(x-h)^{1/2}}\, dh. \tag{9}$$

By the use of Gamma Functions,[5]

$$f(x) = \frac{2K\Gamma(m+1)}{\pi^{1/2}\Gamma(m-\frac{1}{2})}\, x^{m-3/2}, \qquad m \geqq 2. \tag{10}$$

Let n be a positive integer $\geqq 2$. Then the Gamma Functions become simple products when $m=n$ or $m=n+\frac{1}{2}$. When $m=n$,

$$f(x) = \frac{K}{\pi} \frac{2^n n!}{1\cdot 3\cdot 5\cdot \cdots (2n-3)}\, x^{n-3/2}; \qquad n \geqq 2. \tag{11}$$

When $m=n+\frac{1}{2}$,

$$f(x) = K \frac{1\cdot 3\cdot 5\cdot \cdots (2n+1)}{2^n(n-1)!}\, x^{n-1}; \qquad n \geqq 2. \tag{12}$$

3. Sectionally analytic weir forms. When m is equal to or greater than $\frac{3}{2}$ one gets continuous forms of weirs (Fig. 2). When m is greater than $\frac{1}{2}$ and less than $\frac{3}{2}$ the weir forms have an infinite width at the bottom, the curve $f(x)$ approaching the X-axis asymptotically. As this is impossible in practice, the necessary correction due to limiting the width of the weir furnishes an interesting mathematical problem which has been studied in the case where $m=1$.

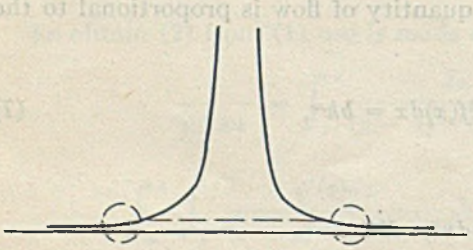

FIG. 3. Copy of Stout's drawing in 1897.

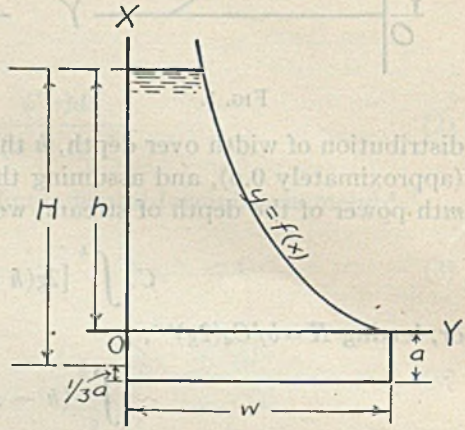

FIG. 4.

The weir in which the flow is proportional to the depth is of engineering value. One of the first records of it is in an article by O. V. P. Stout.[6] Approximate correction was made by circular openings (Fig. 3). A weir of this type was also constructed by Sutro and it is referred to in some texts as the Sutro weir. The modern way to correct

[5] F. S. Woods, *Advanced calculus*, Ginn & Co., 1926, p. 164.

[6] O. V. P. Stout, *A new form of weir notch*, Trans. of the Nebraska Engineering Society, 1, 13 (1897).

the Stout-Sutro weir is to start with a rectangular cross section of depth a and width w (Fig. 4). The upper section is then designed to give a flow proportional to the first power of the depth when the depth of flow equals or exceeds a.

The calculations of E. A. Pratt[7] by series solutions gave a mathematically correct form of weir where $h \geqq a$. In this solution a rectangular section of depth a and width w is first assumed. Soundings are made with the zero point $\frac{1}{3}a$ from the bottom,

$$Q = bH = b(h + \tfrac{2}{3}a).$$

The quantity of water discharged through the rectangular portion of the weir is

$$Q_0 = \tfrac{4}{3}wK[(h + a)^{3/2} - h^{3/2}].$$

Therefore

$$Q = \tfrac{4}{3}wK[(h + a)^{3/2} - h^{3/2}] + 2K\int_0^h (h - x)^{1/2}f(x)dx = b(h + \tfrac{2}{3}a).$$

As this equality must hold for $h=0$, $\frac{4}{3}wKa^{3/2}=\frac{2}{3}ab$ and $b=2wKa^{1/2}$, so

$$\int_0^h (h - x)^{1/2}f(x)dx = \tfrac{2}{3}w[\tfrac{3}{2}ha^{1/2} + a^{3/2} - (h + a)^{3/2} + h^{3/2}].$$

Instead of solving by the use of series, as Pratt did, one may differentiate with respect to h to put the equation in the form of Abel's integral equation; thus

$$\int_0^h \frac{f(x)dx}{(h - x)^{1/2}} = 2w[a^{1/2} - (h + a)^{1/2} + h^{1/2}].$$

For the solution of Abel's integral equation the right hand member must be a continuous function, equal to zero when $h=0$. These conditions being satisfied,

$$\begin{aligned} y = f(x) &= \frac{\sin \pi/2}{\pi} 2w \int_0^x \frac{[-\frac{1}{2}(h + a)^{-1/2} + \frac{1}{2}h^{-1/2}]}{(x - h)^{1/2}} dh \\ &= \frac{w}{\pi}\left[\int_0^x \frac{dh}{[xh - h^2]^{1/2}} - \int_0^x \frac{dh}{[ax + h(x - a) - h^2]^{1/2}}\right] \\ &= \frac{w}{2} + \frac{w}{\pi}\sin^{-1}\frac{a - x}{a + x}, \end{aligned} \tag{13}$$

or

$$y = w - \frac{2w}{\pi}\tan^{-1}\left(\frac{x}{a}\right)^{1/2}. \tag{14}$$

This solution can also be written

$$x = a \tan^2 \frac{\pi(w - y)}{2w}. \tag{15}$$

In the design of the Stout-Sutro weir it is now necessary to choose an a for substitution in the above formulas. One will generally know the average depth of flow

[7] E. A. Pratt, *Another proportional-flow weir, Sutro weir*, Engr. News, 72, 462 (1914).

expected through the weir. It is felt to be better to keep the curve of (13) as close to the curve of the uncorrected weir derived from (10), $y=2Kx^{-1/2}/\pi$, as possible. The scheme is to make the rectangular section of the Stout-Sutro weir have the same dimensions as if the uncorrected weir of (10) were corrected for the average depth of flow by the addition of a rectangular section at the bottom, below the Y-axis, to compensate for limiting its width to $2w$.

One substitutes $y=w$ in the uncorrected formula (10) above and solves for x. This value and that of the h assumed to be average are substituted in

$$\frac{h}{2}\sin^{-1}\frac{h-2x}{h}+\frac{2}{3x^{1/2}}(h+r)^{3/2}-\frac{\pi h}{4}-(hx-x^2)^{1/2}-\frac{2}{3x^{1/2}}(h-x)^{3/2}=0,$$

which is solved for r. One then makes a, the depth of the rectangular section, equal to $x+r$. It must be appreciated that (10) can be corrected for one depth of flow by the addition of a rectangular opening at the bottom, but would not be correct at any other depth of flow. Formula (13) is correct at any depth, $H>\frac{2}{3}a$ (Fig. 4).

4. Series solutions of weir forms. We consider now the forms of weirs when the quantity of flow can be expressed as a convergent series in powers of h. Assume that the quantity of flow, $Q(h)$, can be written

$$Q(h)=\sum_{n=0}^{n=\infty}a_nh^{n+\alpha}, \tag{16}$$

a convergent series not having a constant term, and assume the form of weir to be given by

$$f(x)=\sum_{n=0}^{n=\infty}f_n(x), \tag{17}$$

each term of (17) giving rise to one term of (16).

The general equation is

$$C_d(2g)^{1/2}\int_0^h(h-x)^{1/2}f(x)dx=Q(h).$$

Its solution will involve a series of integral equations of the form

$$C_d(2g)^{1/2}\int_0^h(h-x)^{1/2}f_n(x)dx=a_nh^{n+\alpha}\qquad n=0,1,2,\cdots;\alpha>\tfrac{1}{2}$$

which can be solved by the use of (10), giving

$$f_n(x)=Ca_n\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-\frac{1}{2})}x^{n+\alpha-3/2}, \tag{18}$$

where

$$C=\frac{2}{C_d(2g\pi)^{1/2}}.$$

Substitution of this in (17) gives the formal solution

$$f(x)=C\sum_{n=0}^{n=\infty}a_n\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-\frac{1}{2})}x^{n+\alpha-3/2}. \tag{19}$$

The first term of this series

$$f_0(x) = Ca_0 \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\frac{1}{2})} x^{\alpha-3/2} \tag{20}$$

will be discontinuous at $x=0$ if $\frac{1}{2}<\alpha<\frac{3}{2}$ and continuous if $\alpha \geqq \frac{3}{2}$.

The series formed by all the terms after the first will converge and represent a continuous function of x. This may be proved as follows. By hypothesis the series $\sum_{n=1}^{n=\infty} a_n h^{n+\alpha}$ converges since it is the series for $Q(h)$, (16), with the first term omitted. Let

$$\begin{aligned} c_n &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-\frac{1}{2})}, \qquad \alpha > \tfrac{1}{2} \\ &= \frac{(n+\alpha)(n+\alpha-1)\Gamma(n+\alpha-1)}{\Gamma(n+\alpha-\frac{1}{2})}, \qquad \Gamma(p+1) = p\Gamma(p), \\ &= (n+\alpha)(n+\alpha-1)c_n' \end{aligned}$$

where $c_n' = \Gamma(n+\alpha-1)/\Gamma(n+\alpha-\frac{1}{2})$ and $0<c_n'<1$ since $\Gamma(p)$ increases monotonically for $p>1.46$. Now[8] if the series $\sum_{n=1}^{n=\infty} a_n h^{n+\alpha}$ converges, so also will the series

$$\sum_{n=1}^{n=\infty} c_n' a_n h^{n+\alpha}, \quad \sum_{n=1}^{n=\infty} n c_n' a_n h^{n+\alpha} \quad \text{and} \quad \sum_{n=1}^{n=\infty} n^2 c_n' a_n h^{n+\alpha}.$$

But the series $\sum_{n=1}^{n=\infty} c_n a_n h^{n+\alpha}$ is a simple combination of these three series, hence it also converges. In each case the function represented by the series is continuous.

We have then the form of weir given by

$$f(x) = f_0(x) + g(x), \tag{21}$$

where $f_0(x)$ is given by (20) and

$$g(x) = C \sum_{n=1}^{n=\infty} c_n a_n x^{n+\alpha-3/2},$$

the quantities α, C and c_n being as specified above. The solution $f(x)$ is discontinuous at $x=0$ if $\frac{1}{2}<\alpha<\frac{3}{2}$. It is continuous for $\alpha \geqq \frac{3}{2}$.

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[8] F. S. Woods, *Advanced calculus*, Ginn & Co., 1926, p. 47.

THE NUMERICAL SOLUTION OF LAPLACE'S AND POISSON'S EQUATIONS*

BY

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1. Introduction. A quite common method of solving numerically the Laplace differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1.1}$$

with the boundary values of V prescribed on some contour Γ bounding a region R is to approximate V by the solution u of the Laplace difference equation:

$$4u(x, y) = u(x + h, y) + u(x - h, y) + u(x, y + h) + u(x, y - h). \tag{1.2}$$

Briefly, the method of procedure, commonly called the Liebmann procedure,[1] is to cover the region R with a rectangular network of lines at distances h apart, and to assume values at the interior lattice points of this network. Using these assumed values and the known boundary values, we traverse the region R moving in some definite geometrical pattern from lattice point to lattice point, replacing the assumed values of u at each lattice point by the arithmetic average of the values of u at the four neighboring lattice points. We then repeat the traverse moving in the same pattern to obtain a second improved value of u at each lattice point; and so on until a convergent stage is reached when the values of u are no longer changed materially by continued traversing.

The purpose of this paper is to present a process which yields precisely the convergent values of u obtained by infinitely many traverses of the region. In more precise language, if u_k is the kth approximation of the value of u after k traverses, our process yields the value $u = \lim_{k\to\infty} u_k$.

2. Notation and set up of the problem. Equation (1.2) can be transformed to

$$4u(x, y) = u(x, y + 1) + u(x, y - 1) + u(x + 1, y) + u(x - 1, y) \tag{2.1}$$

by a simple transformation, and we shall concern ourselves with the solution of equation (2.1), with the values of u prescribed on the boundary lines

$$x = 0, \quad x = n, \quad y = 0, \quad \text{and} \quad y = m$$

of the rectangle R.

Unless otherwise stated the numbers m and n are fixed positive integers, and i and j will be used as variable positive integers with the range of values

$$i = 1, 2, \cdots, n - 1; \qquad j = 1, 2, \cdots, m - 1.$$

We shall denote the value of u at the point (i, j) by $u_j(i)$, and we desire to distinguish the known prescribed values of u on the boundary (which values are precisely the

* Received Jan. 7, 1944.

[1] For a fuller description of the process and for references to the literature on the subject, see the paper by Shortley and Weller in J. App. Phys. 9, 334–348 (1938).

same as those of V on the boundary) from the unknown values of u at the interior lattice points. Accordingly, we denote the prescribed values of u on the boundaries as follows:

$$\text{by } \bar{u}_j(0) \text{ at } (0, j); \quad \text{by } \bar{u}_j(n) \text{ at } (n, j);$$
$$\text{by } \bar{u}_0(i) \text{ at } (i, 0); \quad \text{by } \bar{u}_m(i) \text{ at } (i, m);$$

and agree that

$$u_j(0) = u_j(n) = u_0(i) = u_m(i) = 0$$

wherever these terms appear in our equations.

At each interior lattice point we can write

$$4u_j(i) = u_j(i+1) + u_j(i-1) + u_{j+1}(i) + u_{j-1}(i) + \phi_j(i), \tag{2.2}$$

where

$$\phi_j(i) = \delta_{i,1}\bar{u}_j(0) + \delta_{i,n-1}\bar{u}_j(n) + \delta_{j,1}\bar{u}_0(i) + \delta_{j,m-1}\bar{u}_m(i), \tag{2.3}$$

and δ_{ij} (or $\delta_{i,j}$) is the Kronecker delta defined by

$$\delta_{ij} = 1, \quad \text{if} \quad i = j; \qquad \delta_{ij} = 0, \quad \text{if} \quad i \neq j.$$

3. A special system of difference equations. We consider the solution of the system of equations

$$L_c u_j(i) \equiv cu_j(i) - u_j(i+1) - u_j(i-1) = u_{j+1}(i) + u_{j-1}(i) + \phi_j(i), \tag{3.1}$$
$$(i = 1, 2, \cdots, n-1; j = 1, 2, \cdots, m-1),$$

in which c and the $(m-1)(n-1)$ constants $\phi_j(i)$ are prescribed.[2] We assume that

$$u_j(0) = u_j(n) = u_0(i) = u_m(i) = 0,$$

and seek the values of the $(m-1)(n-1)$ unknowns $u_j(i)$.

The system (3.1) represents $(m-1)$ difference equations in the $(m-1)$ functions $u_j(x)$, $(j=1, 2, \cdots, m-1)$, whose values are desired for integral values of the argument x from 1 to $(n-1)$. It can be readily shown that system (3.1) has a unique solution, and of course this solution can be written down by Cramer's Rule, but we shall give the solution in another form.

An immediate property of the operator L_c defined by (3.1) is given by

$$L_{c+a}u(i) = (L_c + a)u(i), \tag{3.2}$$

where a is any constant. We define the inverse operator L_c^{-1} and integral powers L_c^k of the operator L_c in the usual manner.

From (3.2), we obtain

$$[L_c + a]^{-1} = L_{c+a}^{-1} \tag{3.3}$$

and

$$L_c[L_{c+a}^{-1}] = 1 - L_{c+a}^{-1}, \tag{3.4}$$

where 1 is used as the identity operator. The latter is established as follows:

$$L_c[L_{c+a}^{-1}] = [L_{c+a} - a][L_{c+a}^{-1}] = L_{c+a}L_{c+a}^{-1} - aL_{c+a}^{-1} = 1 - aL_{c+a}^{-1}.$$

[2] We assume that at least one of the $\phi_j(i)$ is different from zero; otherwise the system (3.1) has only the trivial solution $u_j(i) = 0$.

The solution of system (3.1) will be given symbolically in terms of the inverse operator L_c^{-1}, and the interpretation of the symbolic solution will depend on the constants $D_c(k)$ and $\lambda_{c,k}(i)$ defined below. To apply the solution obtained to the solution of the Laplace difference system (2.2), we have merely to observe that (2.2) is a special case of (3.1) in which $c=4$ and $\phi_j(i)$ has the value given in (2.3).

Let σ_c and ρ_c be the roots of the characteristic equation

$$c\xi - \xi^2 - 1 = 0. \tag{3.5}$$

We define $D_c(k)$ and $\lambda_{c,k}(i)$ by

$$D_c(k) = \frac{\sigma_c^k - \rho_c^k}{\sigma_c - \rho_c}, \tag{3.6}$$

$$\left.\begin{aligned} \lambda_{c,k}(i) &= \frac{D_c(k)}{D_c(n)} D_c(n-i) \quad \text{when} \quad k \leqq i, \\ \lambda_{c,k}(i) &= \frac{D_c(n-k)}{D_c(n)} D_c(i) \quad \text{when} \quad k \geqq i. \end{aligned}\right\} \tag{3.7}$$

The identities

$$D_c(n) = D_c(i)D_c(n-i+1) - D_c(i-1)D_c(n-i) \tag{3.8}$$

and

$$D_c(i+1) = cD_c(i) - D_c(i-1) \tag{3.9}$$

with $D_c(0)=0$, $D_c(1)=1$ are easily established. In terms of the operator L_c, we may write (3.9) in the form

$$L_c D_c(i) = 0. \tag{3.10}$$

Also if a is any constant, we have

$$L_c D_{c+a}(i) = [L_{c+a} - a]D_{c+a}(i) = L_{c+a}D_{c+a}(i) - aD_{c+a}(i);$$

hence

$$L_c D_{c+a}(i) = -aD_{c+a}(i). \tag{3.11}$$

We shall show that

$$L_c \lambda_{c,k}(i) = \delta_{ik}. \tag{3.12}$$

To establish (3.12) we have three cases to consider:

1) When $k \leqq i-1$, we have

$$L_c \lambda_{c,k}(i) = \frac{D_c(k)}{D_c(n)} L_c D_c(n-i) = 0, \quad \text{by} \quad (3.10).$$

2) When $k \geqq i+1$, we have

$$L_c \lambda_{c,k}(i) = \frac{D_c(n-k)}{D_c(n)} L_c D_c(i) = 0, \quad \text{by} \quad (3.10).$$

3) When $k=i$, we have

$$\begin{aligned} L_c\lambda_{c,i}(i) &= c\lambda_{c,i}(i) - \lambda_{c,i}(i+1) - \lambda_{c,i}(i-1) \\ &= [cD_c(n-i)D_c(i) - D_c(n-i-1)D_c(i) - D_c(i-1)D_c(n-i)]/D_c(n) \\ &= [D_c(i)\{cD_c(n-i) - D_c(n-i-1)\} - D_c(i-1)D_c(n-i)]/D_c(n) \\ &= [D_c(i)D_c(n-i+1) - D_c(i-1)D_c(n-i)]/D_c(n), \quad \text{by} \quad (3.9), \\ &= D_c(n)/D_c(n) = 1, \quad \text{by} \quad (3.8). \end{aligned}$$

Also useful is the relation

$$L_c\lambda_{c+a,k}(i) = \delta_{i,k} - a\lambda_{c+a,k}(i), \tag{3.13}$$

which is a consequence of (3.2) and (3.12).

4. The special cases $m=2$ and $m=3$. As an introduction to the symbolic solution, we consider first the simplest case, $m=2$, in which case there is only one equation in the system (3.1), since by hypothesis $u_0(i)=u_m(i)=0$. This equation is

$$L_c u_1(i) = \phi_1(i). \tag{4.1}$$

Its solution is given by

$$u_1(i) = \sum_{k=1}^{n-1} \lambda_{c,k}(i)\phi_1(k). \tag{4.2}$$

We also write the solution of (4.1) in the symbolic form

$$u_1(i) = L_c^{-1}[\phi_1(i)], \tag{4.3}$$

where the expression on the right side of (4.3) is to be interpreted as being equal to the right side of (4.2). Explicitly,

$$L_c^{-1}[\phi_1(i)] = \sum_{k=1}^{n-1} \lambda_{c,k}(i)\phi_1(k). \tag{4.4}$$

To make actual use of the solution given in (4.2), it is necessary to have a table of values of the constants $\lambda_{c,k}(i)$. In order not to complicate unduly the notation, the dependence of these constants on n has been omitted from our notation. A complete tabulation of these constants would require a great deal of space, since, with m fixed, they still depend on four parameters c, k, i, and n. However, for the application to the solution of (2.2), we have $c=4$, and abridged usable tables requiring only tabulations for $i=1$, 2 and varying k and n are given in Tables 1 and 2 of §9.

The entries in Table 1 give the multipliers to be applied to each of the boundary values in the calculation of $u_1(1)$. As an example let us consider a 2 by 10 rectangle. Multiply each boundary value of the first column by the multiplier which appears opposite it in the column $n=10$; add these products, and the sum is the value of $u_1(1)$. Likewise, by interchanging the arguments $(n-i)$ and i, the same multipliers can be used to calculate $u_1(9)$. Next using $u_1(1)$ and $u_1(9)$ as known boundary values and the 2 by 8 rectangle which has the points (1, 1) and (1, 9) on its ends, use the multipliers in column $n=8$ to calculate $u_1(2)$ and $u_1(8)$; and so on.

The number of points at which the values of u are to be calculated by this process can be cut in half by using Table 2. With the entries in this table, using again a 2 by

10 rectangle, the values $u_1(2)$ and $u_1(8)$ can be calculated using the multipliers in column $n=10$; then in the 2 by 6 rectangle with points (2, 1) and (8, 1) as ends, and using multipliers in column $n=6$, calculate $u_1(4)$ and $u_1(6)$. The values $u_1(1), u_1(3), \cdots, u_1(7)$ can then be obtained by the Liebmann formula, each being the average of its four neighbors. For example,

$$u_1(5) = \tfrac{1}{4}[u_1(4) + u_1(6) + \bar{u}_0(5) + \bar{u}_2(5)].$$

In the case $m=3$, system (3.1) reduces to the following:

$$L_c u_1(i) = u_2(i) + \phi_1(i), \qquad L_c u_2(i) = u_1(i) + \phi_2(i), \tag{4.5}$$

with two unknown functions u_1 and u_2.

Treating the operators in (4.5) as algebraic multipliers, and solving for u_1 and u_2, we obtain symbolically

$$\left.\begin{aligned} u_1(i) &= \frac{L_c}{L_c^2 - 1}\phi_1(i) + \frac{1}{L_c^2 - 1}\phi_2(i), \\ u_2(i) &= \frac{1}{L_c^2 - 1}\phi_1(i) + \frac{L_c}{L_c^2 - 1}\phi_2(i). \end{aligned}\right\} \tag{4.6}$$

The following interpretation of the operators in the right members of (4.6) produces actual solutions. Write the operators of the right members in their partial fraction expansions, obtaining

$$\frac{L_c}{L_c^2 - 1} = \tfrac{1}{2}\left[\frac{1}{L_c - 1} + \frac{1}{L_c + 1}\right] = \tfrac{1}{2}[L_{c-1}^{-1} + L_{c+1}^{-1}],$$

$$\frac{1}{L_c^2 - 1} = \tfrac{1}{2}\left[\frac{1}{L_c - 1} - \frac{1}{L_c + 1}\right] = \tfrac{1}{2}[L_{c-1}^{-1} - L_{c+1}^{-1}].$$

The last step in each line follows from (3.3). Consequently, the explicit solution of (4.5) is given by

$$u_1(i) = \sum_{k=1}^{n-1} [\alpha_k(i)\phi_1(k) + \beta_k(i)\phi_2(k)], \qquad u_2(i) = \sum_{k=1}^{n-1} [\beta_k(i)\phi_1(k) + \alpha_k(i)\phi_2(k)]; \tag{4.7}$$

where

$$\alpha_k(i) = \tfrac{1}{2}[\lambda_{c-1,k}(i) + \lambda_{c+1,k}(i)], \qquad \beta_k(i) = \tfrac{1}{2}[\lambda_{c-1,k}(i) - \lambda_{c+1,k}(i)]. \tag{4.8}$$

To verify that (4.7) is the solution of (4.5), we use

$$L_c\alpha_k(i) = \delta_{ik} + \beta_k(i), \qquad L_c\beta_k(i) = \alpha_k(i),$$

which follow from (4.8) and (3.13). Consequently,

$$\begin{aligned} L_c u_1(i) &= \sum_{k=1}^{n-1} \{[L_c\alpha_k(i)]\phi_1(k) + [L_c\beta_k(i)]\phi_2(k)\} \\ &= \sum_{k=1}^{n-1} \delta_{ik}\phi_1(k) + \sum_{k=1}^{n-1} [\beta_k(i)\phi_1(k) + \alpha_k(i)\phi_2(k)] \\ &= \phi_1(i) + u_2(i). \end{aligned}$$

Tables 3 and 4 of §9 are useful in calculating values of $u_1(1)$ and $u_1(2)$ for the case $m=3$, again using $c=4$ which is appropriate for the Laplace equation.

These tables are used in a similar manner as Tables 1 and 2. However, for each value of n, there are two numbers side by side, and also two boundary values in the first column. The interpretation is to multiply the left one of the two boundary values of each line by the left one of the two multipliers of the same line in the appropriate n column.

For a given n, calculate $u_1(1)$ using multipliers of Table 3; then interchanging the subscripts 1 and 2 and the subscripts 0 and 3, calculate $u_2(2)$ using multipliers of Table 4. The value $u_2(1)$ is then obtained by the Liebmann formula

$$u_2(1) = \tfrac{1}{4}[\bar{u}_2(0) + \bar{u}_3(1) + u_1(1) + u_2(2)].$$

As in the case $m=2$, we can also calculate from the end $x=n$ of the rectangle.

5. The general case. We now consider the general system (3.1) for any value of m, and show that its solution can be written symbolically in terms of the polynomial operators defined by

$$P_0 = 0, \qquad P_1 = 1, \qquad P_k = L_c P_{k-1} - P_{k-2} \quad \text{for} \quad k > 1. \tag{5.1}$$

The operator P_k is a polynomial in L_c of the $(k-1)$st degree, which is precisely the same function of L_c as $D_c(k)$ is of c. From this observation, the following analogue of (3.8) can be seen to be valid,

$$P_n = P_i P_{n-i+1} - P_{i-1} P_{n-i}. \tag{5.2}$$

By solving the system (3.1) treating L_c as an algebraic multiplier, we obtain symbolically

$$u_j(i) = \sum_{k=1}^{j} \frac{P_{m-j}P_k}{P_m}\phi_k(i) + \sum_{k=j+1}^{m-1} \frac{P_j P_{m-k}}{P_m}\phi_k(i), \qquad (j = 1, 2, \cdots, m-1), \tag{5.3}$$

in which the operators of the right member are to be interpreted similarly to those of (4.6); that is they are to be expanded into partial fractions. Since each of the operator coefficients in the right member of (5.3) is a proper fraction, their expansions will have the form

$$\frac{P_j P_k}{P_m} = \sum_{l=1}^{m-1} \frac{b_l(j, k)}{L_c - a_l} = \sum_{l=1}^{m-1} b_l(j, k) L_{c-a_l}^{-1}, \tag{5.4}$$

where $a_1, a_2, \cdots, a_{m-1}$ are the roots of $P_m(\xi)=0$ and the numbers $b_l(j, k)$ are uniquely determined. The actual solution of (3.1) is thus given in terms of the operators $L_{c-a_l}^{-1}$ which have the meaning given in (4.4).

That the foregoing actually yields the solution of (3.1) can be established by operating on each side of (5.3) with the operator L_c and using the relations (5.1) and (5.2).

6. Application to solutions of Laplace's equation. We have already observed that (2.2) is a special case of the system (3.1) in which $c=4$ and $\phi_j(i)$ has the value given in (2.3). To apply the preceding results, it becomes necessary to calculate the coefficients of $\phi_k(i)$ in the solution of (3.1) for various values of m. The polynomial operators P_m must be factored, and the operators appearing in (5.3) must be written in the more useful form (5.4).

We have already considered in detail the cases $m=2$, $m=3$ in §4. We now particularize the general solution of §5 to the case $m=4$.

From (5.3) we have symbolically:

$$\left.\begin{aligned} u_1 &= \frac{P_1P_3}{P_4}\phi_1 + \frac{P_1P_2}{P_4}\phi_2 + \frac{P_1P_1}{P_4}\phi_3, \\ u_2 &= \frac{P_1P_2}{P_4}\phi_1 + \frac{P_2P_2}{P_4}\phi_2 + \frac{P_1P_2}{P_4}\phi_3, \\ u_3 &= \frac{P_1P_1}{P_4}\phi_1 + \frac{P_1P_2}{P_4}\phi_2 + \frac{P_1P_3}{P_4}\phi_3, \end{aligned}\right\} \tag{6.1}$$

where

$$\left.\begin{aligned} \frac{P_1P_1}{P_4} &= \frac{1}{L_c^3 - 2L_c} = \tfrac{1}{4}\left[\frac{1}{L_c - \sqrt{2}} + \frac{1}{L_c + \sqrt{2}} - \frac{2}{L_c}\right] = \tfrac{1}{4}\left[L_{c-\sqrt{2}}^{-1} + L_{c+\sqrt{2}}^{-1} - 2L_c^{-1}\right], \\ \frac{P_1P_2}{P_4} &= \frac{L_c}{L_c^3 - 2L_c} = \frac{1}{2\sqrt{2}}\left[\frac{1}{L_c - \sqrt{2}} - \frac{1}{L_c + \sqrt{2}}\right] = \frac{1}{2\sqrt{2}}\left[L_{c-\sqrt{2}}^{-1} - L_{c+\sqrt{2}}^{-1}\right], \\ \frac{P_1P_3}{P_4} &= \frac{L_c^2 - 1}{L_c^3 - 2L_c} = \tfrac{1}{4}\left[\frac{1}{L_c - \sqrt{2}} + \frac{1}{L_c + \sqrt{2}} + \frac{2}{L_c}\right] = \tfrac{1}{4}\left[L_{c-\sqrt{2}}^{-1} + L_{c+\sqrt{2}}^{-1} + 2L_c^{-1}\right], \\ \frac{P_2P_2}{P_4} &= \frac{L_c^2}{L_c^3 - 2L_c} = \tfrac{1}{2}\left[\frac{1}{L_c - \sqrt{2}} + \frac{1}{L_c + \sqrt{2}}\right] = \tfrac{1}{2}\left[L_{c-\sqrt{2}}^{-1} + L_{c+\sqrt{2}}^{-1}\right]. \end{aligned}\right\} \tag{6.2}$$

The explicit form of the solution is given by

$$\left.\begin{aligned} u_1(i) &= \sum_{k=1}^{n-1}\left[Q_k(i)\phi_1(k) + S_k(i)\phi_2(k) + R_k(i)\phi_3(k)\right], \\ u_2(i) &= \sum_{k=1}^{n-1}\left[S_k(i)\{\phi_1(k) + \phi_3(k)\} + T_k(i)\phi_2(k)\right], \\ u_3(i) &= \sum_{k=1}^{n-1}\left[R_k(i)\phi_1(k) + S_k(i)\phi_2(k) + Q_k(i)\phi_3(k)\right], \end{aligned}\right\} \tag{6.3}$$

where

$$\left.\begin{aligned} Q_k(i) &= \tfrac{1}{4}\left[\lambda_{c-\sqrt{2},k}(i) + \lambda_{c+\sqrt{2},k}(i) + 2\lambda_{c,k}(i)\right], \\ R_k(i) &= \tfrac{1}{4}\left[\lambda_{c-\sqrt{2},k}(i) + \lambda_{c+\sqrt{2},k}(i) - 2\lambda_{c,k}(i)\right], \\ S_k(i) &= \frac{1}{2\sqrt{2}}\left[\lambda_{c-\sqrt{2},k}(i) - \lambda_{c+\sqrt{2},k}(i)\right], \\ T_k(i) &= \tfrac{1}{2}\left[\lambda_{c-\sqrt{2},k}(i) + \lambda_{c+\sqrt{2},k}(i)\right], \end{aligned}\right\} \tag{6.4}$$

and

$$\left.\begin{aligned} \phi_1(k) &= \delta_{k,1}\bar{u}_1(0) + \delta_{k,n-1}\bar{u}_1(n) + \bar{u}_0(k), \\ \phi_2(k) &= \delta_{k,1}\bar{u}_2(0) + \delta_{k,n-1}\bar{u}_2(n), \\ \phi_3(k) &= \delta_{k,1}\bar{u}_3(0) + \delta_{k,n-1}\bar{u}_3(n) + \bar{u}_4(k). \end{aligned}\right\} \tag{6.5}$$

Multipliers for calculating $u_2(1)$, $u_2(2)$, and $u_1(2)$ are given in Tables 5, 6, and 7. For a 4 by n rectangle, calculate $u_2(1)$ using Table 5, $u_1(2)$ using Table 7, $u_3(2)$ using

Table 7 with subscripts 0 and 4 and subscripts 1 and 3 interchanged. Then calculate $u_1(1)$, $u_3(1)$ by the Liebmann Formula. Then using known values $u_1(1)$, $u_2(1)$, $u_3(1)$ and treating them as known boundary values for the rectangle one unit shorter, calculate $u_2(3)$ using Table 6; $u_2(2)$ can then be obtained by use of the Liebmann Formula. Then calculate $u_1(4)$, $u_3(4)$ using Table 7; and so on. Of course, as in the case $m=2$ and $m=3$, we can also calculate from the end $x=n$ of the rectangle. By this process the value at only every other point on each line is calculated by the use of the tabular values.

For values of m larger than 4, it is only convenient to use composite values of m. It can be easily shown that when m is composite having q for one factor then P_m contains P_q as a factor. However, for larger values of m, the necessary tables occupy more space and require a tremendous amount of time in their preparation. Theoretically the complete solution for any value of m is given by (5.3), but practically it is sufficient to use tables with $m=4$. A given rectangle can be covered by a lattice 4 units wide and the values of the function u at the interior lattice points calculated by the methods indicated. When the values of u at these lattice points are obtained, each rectangle can then be subdivided again by a finer network and the first calculated values are then used as approximate values for the finer network, and can in turn be improved either by traversing or by the use of the tables given here.

7. The Poisson Equation. The preceding methods and results can be extended with slight modification to apply to the numerical solution of the more general Poisson equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x, y), \tag{7.1}$$

in which $F(x, y)$ is defined in the interior of the region R. The approximating difference equation in this case is

$$4u(x, y) = u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h) - h^2F(x, y). \tag{7.2}$$

Employing our former notation, we can write at each interior lattice point

$$\begin{aligned} 4u_j(i) = {} & u_j(i+1) + u_j(i-1) + u_{j+1}(i) + u_{j-1}(i) - h^2F_j(i) + \delta_{i,1}\bar{u}_j(0) + \delta_{i,n-1}\bar{u}_j(n) \\ & + \delta_{j,1}\bar{u}_0(i) + \delta_{j,m-1}\bar{u}_m(i), \quad (i = 1, 2, \cdots, n-1; j = 1, 2, \cdots, m-1), \end{aligned} \tag{7.3}$$

and again consider

$$u_j(0) = u_j(n) = u_0(i) = u_m(i) = 0.$$

The system (7.3) is again a special case of system (3.1) in which the known function $\phi_j(i)$ is now given by

$$\phi_j(i) = \delta_{i,1}\bar{u}_j(0) + \delta_{i,n-1}\bar{u}_j(n) + \delta_{j,1}\bar{u}_0(i) + \delta_{j,m-1}\bar{u}_m(i) - h^2F_j(i). \tag{7.4}$$

Consequently, the general solution given in §5 applies at once.

In the case $m=2$, for example, we have

$$u_1(i) = \sum_{k=1}^{n-1} \lambda_{c,k}(i)\left[\delta_{k,1}\bar{u}_1(0) + \delta_{k,n-1}\bar{u}_1(n) + \bar{u}_0(k) + \bar{u}_2(k) - h^2F_1(k)\right]. \tag{7.5}$$

To apply Tables 1 and 2 to obtain the values of $u_1(1)$ and $u_1(2)$, first calculate $h^2F_1(k)$, $k=1, 2, \cdots, n-1$, at each interior lattice point. Then apply to these values

the same multipliers as are applied to $\bar{u}_0(k)$ and $\bar{u}_2(k)$. If Table 2 is used to calculate $u_1(2)$, the value $u_1(1)$ can be obtained from the associated Liebmann equation

$$u_1(1) = \tfrac{1}{4}[u_1(2) + \bar{u}_1(0) + \bar{u}_0(1) + \bar{u}_2(1) - h^2F_1(1)]. \tag{7.6}$$

For the case $m=3$, multiply $h^2F_1(k)$ and $h^2F_2(k)$, respectively, by the same multipliers as are used for $\bar{u}_0(k)$ and $\bar{u}_3(k)$.

For the case $m=4$, additional tables are required. The explicit solution in this case is formally the same as that given in (6.3), but $\phi_1(k)$, $\phi_2(k)$, and $\phi_3(k)$ have the following values:

$$\left.\begin{aligned} \phi_1(k) &= \delta_{k,1}\bar{u}_1(0) + \delta_{k,n-1}\bar{u}_1(n) + \bar{u}_0(k) - h^2F_1(k), \\ \phi_2(k) &= \delta_{k,1}\bar{u}_2(0) + \delta_{k,n-1}\bar{u}_2(n) - h^2F_2(k), \\ \phi_3(k) &= \delta_{k,1}\bar{u}_3(0) + \delta_{k,n-1}\bar{u}_3(n) + \bar{u}_4(k) - h^2F_3(k). \end{aligned}\right\} \tag{7.7}$$

The multipliers $Q_k(2)$ and $R_k(2)$ appear in Table 7 and are respectively those multipliers applied to $\bar{u}_0(k)$ and $\bar{u}_4(k)$ in the calculation of $u_1(2)$. The multipliers $S_k(1)$ appear in Table 5 and are those multipliers applied to both $\bar{u}_0(k)$ and $\bar{u}_4(k)$ in the calculation of $u_2(1)$. The multipliers $S_k(2)$ are those multipliers in Table 6 which are applied to both $\bar{u}_0(k)$ and $\bar{u}_4(k)$ in the calculation of $u_2(2)$.

The multipliers $T_k(1)$ and $T_k(2)$ which must be applied to $h^2F_2(k)$ in the calculation of $u_2(1)$ and $u_2(2)$ appear in Tables 8 and 9 respectively.

8. Irregular cases and non-rectangular boundaries. The preceding solutions both in the case of the Laplace equation and the Poisson equation apply only to rectangular boundaries whose dimensions are integral multiples of the lattice unit h. To apply Tables 5, 6, and 7 in the solution of the Laplace equation for a rectangular boundary, divide the smaller dimension of the rectangle by 4 to obtain the lattice unit h. If the longer dimension of the rectangle is an integral multiple of h, the process here outlined for the solution applies directly. We call this the regular case. If the longer dimension is not an integral multiple of h, we call this the irregular case, and a modification of the process here outlined is required. We need an analogue of the Liebmann Formula to express the value of a harmonic function approximately in terms of the values of the function at four non-equidistant neighbors.

Let $H(x, y)$ be an arbitrary harmonic function whose value H_0 at (x_0, y_0) is to be expressed approximately in terms of H_1, H_2, H_3, H_4 which are the values of $H(x, y)$ at the points (x_0+r_1h, y_0), (x_0-r_2h, y_0), (x_0, y_0+r_3h), (x_0, y_0-r_4h) where r_1, r_2, r_3, r_4, and h are positive. When $r_1=r_2=r_3=r_4=1$, and $H(x, y)$ is approximated by its T. S. (Taylor Series) expansion about (x_0, y_0) up to terms including those of the third degree in h, H_0 is found to satisfy the Liebmann equation

$$H_0 = \tfrac{1}{4}(H_1 + H_2 + H_3 + H_4). \tag{8.1}$$

When r_1, r_2, r_3, and r_4 are not equal, and $H(x, y)$ is approximated by its T.S. up to terms including those of the second degree in h, we find[3]

$$H_0 = a_1H_1 + a_2H_2 + a_3H_3 + a_4H_4 \tag{8.2}$$

where

[3] This relation is also given by Shortley and Weller, ibid.; but their method of derivation is slightly different from ours.

$$a_1 = r_2r_3r_4/(r_1 + r_2)(r_1r_2 + r_3r_4), \qquad a_2 = r_1r_3r_4/(r_1 + r_2)(r_1r_2 + r_3r_4),$$
$$a_3 = r_1r_2r_4/(r_3 + r_4)(r_1r_2 + r_3r_4), \qquad a_4 = r_1r_2r_3/(r_3 + r_4)(r_1r_2 + r_3r_4). \tag{8.3}$$

If $H(x, y)$ is approximated by its T.S. up to terms of the first degree in h, we find

$$H_0 = \sum_{k=1}^{4} b_kH_k \quad \text{with} \quad b_k = r_k^{-1}/(r_1^{-1} + r_2^{-1} + r_3^{-1} + r_4^{-1}). \tag{8.4}$$

Either (8.2) or (8.4) expresses H_0 as a weighted average of its four neighbors, and although (8.4) is easier to use, presumably (8.2) gives a better approximation to the value of H_0. However in the application to the irregular case of the rectangle, we require only the simpler forms to which (8.3) and (8.4) reduce when three of the r_k ($k=1, 2, 3, 4$) are equal to unity.

To determine the values of a harmonic function at the interior lattice points of a rectangle whose dimensions are $4h$ by $(n+r)h$ where n is an integer and $0<r<1$, let x, y, z, etc. be the values of the harmonic function at the points indicated in Fig. 1. By means of Tables 5, 6, 7 we can express u, v, and w as linear functions of x, y, z

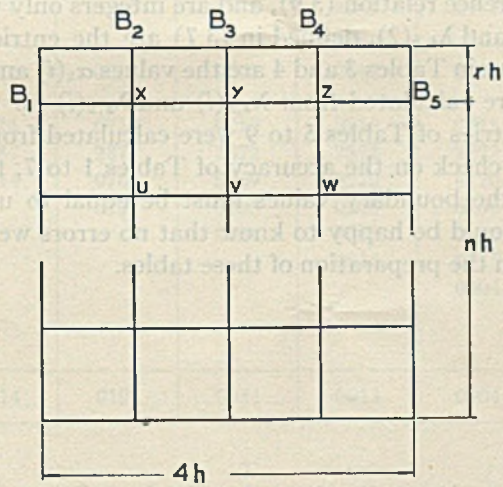

FIG. 1.

and the boundary values. Then by means of (8.2) or (8.4) or any other approximation method determine x, y, and z in terms of u, v, w, and the boundary values B_k ($k=1, 2, 3, 4, 5$) at the indicated points of the figure. This process leads to three linear algebraic equations for the determination of x, y, and z. When these values are determined, the values of the harmonic function at the other interior lattice points can be obtained as the problem is now reduced to the regular case.

A similar method can of course be used for the Poisson equation. In this case the analogue of (8.2) is

$$H_0 = a_1H_1 + a_2H_2 + a_3H_3 + a_4H_4 - a_0F_0 \tag{8.5}$$

where

$$a_0 = \tfrac{1}{2}h^2\left(\frac{r_1r_2r_3r_4}{r_1r_2 + r_3r_4}\right), \tag{8.6}$$

F_0 denotes $F(x_0, y_0)$, and a_1, a_2, a_3, a_4 are given by (8.3).

The foregoing method applies equally well if the top and bottom boundaries of the figure are not straight. We postpone for a later paper the procedure which can be applied for non-rectangular boundaries in general. In this subsequent paper we shall also show the application of our methods to an extension of the Liebmann process in which the values at certain of the interior lattice points are calculated from the arithmetic average of the values at their four normal neighbors while the values at the other lattice points are calculated from the values at their four diagonal neighbors.

9. Tables. The entries in the following tables were rounded off to four decimal places from calculations carried out to a higher number of decimal places. In the tables, the decimal points are not printed but are to be understood to be present just before the first digit. In the compilation of these tables, the values of $D_c(k)$, defined in (3.6), were required for the values $c=4$, 3, 5, $4-\sqrt{2}$, and $4+\sqrt{2}$. These were calculated from the recurrence relation (3.9), and are integers only when c is an integer.

The values $\lambda_{4,k}(1)$ and $\lambda_{4,k}(2)$, defined in (3.7) are the entries in Tables 1 and 2, respectively. The entries in Tables 3 and 4 are the values $\alpha_k(i)$ and $\beta_k(i)$ for $i=1$ and 2, respectively; these were calculated from $\lambda_{3,k}(i)$ and $\lambda_{5,k}(i)$ for $i=1$ and 2 using the relations (4.8). The entries of Tables 5 to 9 were calculated from the relations (6.4).

As one convenient check on the accuracy of Tables 1 to 7, the sum of the multipliers used on all of the boundary values must be equal to unity; this check was applied. The author would be happy to know that no errors were made in the many calculations required in the preparation of these tables.

TABLE 1.—To calculate $u_1(1)$; $m=2$

Boundary values	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n\geqq 9$
$\bar{u}_1(0)$	2667	2679	2679	2679	2679	2679	2679
$\bar{u}_0(1)+\bar{u}_2(1)$	2667	2679	2679	2679	2679	2679	2679
$\bar{u}_0(2)+\bar{u}_2(2)$	0667	0714	0718	0718	0718	0718	0718
$\bar{u}_0(3)+\bar{u}_2(3)$		0179	0191	0192	0192	0192	0192
$\bar{u}_0(4)+\bar{u}_2(4)$			0048	0051	0052	0052	0052
$\bar{u}_0(5)+\bar{u}_2(5)$				0013	0014	0014	0014
$\bar{u}_0(6)+\bar{u}_2(6)$					0003	0004	0004
$\bar{u}_0(7)+\bar{u}_2(7)$						0001	0001
$\bar{u}_0(8)+\bar{u}_2(8)$							0000
$\bar{u}_1(n)$	0667	0179	0048	0013	0003	0001	0000

TABLE 2.—To calculate $u_1(2)$; $m=2$

Boundary values	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n\geqq 10$
$\bar{u}_1(0)$	0714	0718	0718	0718	0718	0718	0718
$\bar{u}_0(1)+\bar{u}_2(1)$	0714	0718	0718	0718	0718	0718	0718
$\bar{u}_0(2)+\bar{u}_2(2)$	2857	2871	2872	2872	2872	2872	2872
$\bar{u}_0(3)+\bar{u}_2(3)$	0714	0766	0769	0770	0770	0770	0770
$\bar{u}_0(4)+\bar{u}_2(4)$		0191	0205	0206	0206	0206	0206
$\bar{u}_0(5)+\bar{u}_2(5)$			0051	0055	0055	0055	0055
$\bar{u}_0(6)+\bar{u}_2(6)$				0014	0015	0015	0015
$\bar{u}_0(7)+\bar{u}_2(7)$					0004	0004	0004
$\bar{u}_0(8)+\bar{u}_2(8)$						0001	0001
$\bar{u}_0(9)+\bar{u}_2(9)$							0000
$\bar{u}_1(n)$	0714	0191	0051	0014	0004	0001	0000

TABLE 3.—To calculated $u_1(1)$; $m=3$

Boundary values		$n=2$		$n=3$		$n=4$		$n=5$		$n=6$	
$\bar{u}_1(0)$	$\bar{u}_2(0)$	2667	0667	2917	0833	2948	0861	2953	0866	2953	0866
$\bar{u}_0(1)$	$\bar{u}_3(1)$	2667	0667	2917	0833	2948	0861	2953	0866	2953	0866
$\bar{u}_0(2)$	$\bar{u}_3(2)$			0833	0417	0932	0497	0945	0509	0947	0511
$\bar{u}_0(3)$	$\bar{u}_3(3)$					0282	0195	0318	0227	0323	0232
$\bar{u}_0(4)$	$\bar{u}_3(4)$							0100	0082	0114	0095
$\bar{u}_0(5)$	$\bar{u}_3(5)$									0037	0033
$\bar{u}_1(n)$	$\bar{u}_2(n)$	2667	0667	0833	0417	0282	0195	0100	0082	0037	0033

TABLE 3.—(continued)

Boundary values		$n=7$		$n=8$		$n=9$		$n=10$		$n\geqq 11$	
$\bar{u}_1(0)$	$\bar{u}_2(0)$	2953	0866	2953	0866	2953	0866	2953	0866	2953	0866
$\bar{u}_0(1)$	$\bar{u}_3(1)$	2953	0866	2953	0866	2953	0866	2953	0866	2953	0866
$\bar{u}_0(2)$	$\bar{u}_3(2)$	0947	0512	0947	0512	0947	0512	0947	0512	0947	0512
$\bar{u}_0(3)$	$\bar{u}_3(3)$	0324	0233	0324	0233	0324	0233	0324	0233	0324	0233
$\bar{u}_0(4)$	$\bar{u}_3(4)$	0116	0097	0116	0097	0116	0097	0116	0097	0116	0097
$\bar{u}_0(5)$	$\bar{u}_3(5)$	0042	0038	0043	0039	0043	0039	0043	0039	0043	0039
$\bar{u}_0(6)$	$\bar{u}_3(6)$	0014	0013	0016	0015	0016	0015	0016	0015	0016	0015
$\bar{u}_0(7)$	$\bar{u}_3(7)$			0005	0005	0006	0006	0006	0006	0006	0006
$\bar{u}_0(8)$	$\bar{u}_3(8)$					0002	0002	0002	0002	0002	0002
$\bar{u}_0(9)$	$\bar{u}_3(9)$							0001	0001	0001	0001
$\bar{u}_0(10)$	$\bar{u}_3(10)$									0000	0000
$\bar{u}_1(n)$	$\bar{u}_2(n)$	0014	0013	0005	0005	0002	0002	0001	0001	0000	0000

TABLE 4.—To calculate $u_1(2)$; $m=3$

Boundary values		$n=3$		$n=4$		$n=5$		$n=6$		$n=7$	
$\bar{u}_1(0)$	$\bar{u}_2(0)$	0833	0417	0932	0497	0945	0509	0947	0511	0947	0512
$\bar{u}_0(1)$	$\bar{u}_3(1)$	0833	0417	0932	0497	0945	0509	0947	0511	0947	0512
$\bar{u}_0(2)$	$\bar{u}_3(2)$	2916	0834	3230	1056	3271	1093	3277	1098	3277	1099
$\bar{u}_0(3)$	$\bar{u}_3(3)$			0932	0497	1045	0591	1061	0606	1063	0608
$\bar{u}_0(4)$	$\bar{u}_3(4)$					0318	0227	0360	0265	0366	0271
$\bar{u}_0(5)$	$\bar{u}_3(5)$							0114	0095	0129	0109
$\bar{u}_0(6)$	$\bar{u}_3(6)$									0042	0038
$\bar{u}_1(n)$	$\bar{u}_2(n)$	2916	0834	0932	0497	0318	0227	0114	0095	0042	0038

TABLE 4.—(continued)

Boundary values		$n=8$		$n=9$		$n=10$		$n=11$		$n\geqq 12$	
$\bar{u}_1(0)$	$\bar{u}_2(0)$	0947	0512	0947	0512	0947	0512	0947	0512	0947	0512
$\bar{u}_0(1)$	$\bar{u}_3(1)$	0947	0512	0947	0512	0947	0512	0947	0512	0947	0512
$\bar{u}_0(2)$	$\bar{u}_3(2)$	3277	1099	3277	1099	3277	1099	3277	1099	3277	1099
$\bar{u}_0(3)$	$\bar{u}_3(3)$	1063	0609	1063	0609	1063	0609	1063	0609	1063	0609
$\bar{u}_0(4)$	$\bar{u}_3(4)$	0367	0272	0367	0272	0367	0272	0367	0272	0367	0272
$\bar{u}_0(5)$	$\bar{u}_3(5)$	0131	0112	0132	0112	0132	0112	0132	0112	0132	0112
$\bar{u}_0(6)$	$\bar{u}_3(6)$	0048	0044	0049	0044	0049	0045	0049	0045	0049	0045
$\bar{u}_0(7)$	$\bar{u}_3(7)$	0016	0015	0018	0017	0018	0017	0018	0017	0018	0017
$\bar{u}_0(8)$	$\bar{u}_3(8)$			0006	0006	0007	0007	0007	0007	0007	0007
$\bar{u}_0(9)$	$\bar{u}_3(9)$					0002	0002	0003	0003	0003	0003
$\bar{u}_0(10)$	$\bar{u}_3(10)$							0001	0001	0001	0001
$\bar{u}_0(11)$	$\bar{u}_3(11)$									0000	0000
$\bar{u}_1(n)$	$\bar{u}_2(n)$	0016	0015	0006	0006	0002	0002	0001	0001	0000	0000

TABLE 5.—To calculate $u_2(1)$; $m=4$

Boundary values	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n\geqq 13$
$\bar{u}_2(0)$	2857	3230	3304	3320	3323	3324	3324	3324	3324	3324	3324	3324
$\bar{u}_1(0)+\bar{u}_3(0)$	0714	0932	0982	0993	0996	0997	0997	0997	0997	0997	0997	0997
$\bar{u}_0(1)+\bar{u}_4(1)$	0714	0932	0982	0993	0996	0997	0997	0997	0997	0997	0997	0997
$\bar{u}_0(2)+\bar{u}_4(2)$		0497	0625	0654	0661	0662	0663	0663	0663	0663	0663	0663
$\bar{u}_0(3)+\bar{u}_4(3)$			0268	0332	0346	0349	0350	0350	0350	0350	0350	0350
$\bar{u}_0(4)+\bar{u}_4(4)$				0133	0164	0171	0172	0173	0173	0173	0173	0173
$\bar{u}_0(5)+\bar{u}_4(5)$					0064	0079	0082	0083	0083	0083	0083	0083
$\bar{u}_0(6)+\bar{u}_4(6)$						0031	0038	0039	0040	0040	0040	0040
$\bar{u}_0(7)+\bar{u}_4(7)$							0015	0018	0019	0019	0019	0019
$\bar{u}_0(8)+\bar{u}_4(8)$								0007	0008	0009	0009	0009
$\bar{u}_0(9)+\bar{u}_4(9)$									0003	0004	0004	0004
$\bar{u}_0(10)+\bar{u}_4(10)$										0002	0002	0002
$\bar{u}_0(11)+\bar{u}_4(11)$											0001	0001
$\bar{u}_0(12)+\bar{u}_4(12)$												0000
$\bar{u}_1(n)+\bar{u}_3(n)$	0714	0497	0268	0133	0064	0031	0015	0007	0003	0002	0001	0000
$\bar{u}_2(n)$	2857	1056	0446	0226	0093	0044	0021	0010	0005	0002	0001	0000

TABLE 6.—To calculate $u_2(2)$; $m=4$

Boundary values	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$	$n=14$	$n\geqq 15$
$\bar{u}_2(0)$	1056	1250	1292	1301	1303	1304	1304	1304	1304	1304	1304	1304	1304
$\bar{u}_1(0)+\bar{u}_3(0)$	0497	0625	0654	0661	0662	0663	0663	0663	0663	0663	0663	0663	0663
$\bar{u}_0(1)+\bar{u}_4(1)$	0497	0625	0654	0661	0662	0663	0663	0663	0663	0663	0663	0663	0663
$\bar{u}_0(2)+\bar{u}_4(2)$	0932	1250	1325	1342	1346	1347	1347	1347	1347	1347	1347	1347	1347
$\bar{u}_0(3)+\bar{u}_4(3)$		0625	0788	0825	0833	0835	0835	0835	0835	0835	0835	0835	0835
$\bar{u}_0(4)+\bar{u}_4(4)$			0332	0410	0428	0432	0433	0433	0433	0433	0433	0433	0433
$\bar{u}_0(5)+\bar{u}_4(5)$				0164	0202	0210	0212	0212	0212	0212	0212	0212	0212
$\bar{u}_0(6)+\bar{u}_4(6)$					0079	0097	0101	0102	0102	0102	0102	0102	0102
$\bar{u}_0(7)+\bar{u}_4(7)$						0038	0046	0048	0048	0049	0049	0049	0049
$\bar{u}_0(8)+\bar{u}_4(8)$							0018	0022	0023	0023	0023	0023	0023
$\bar{u}_0(9)+\bar{u}_4(9)$								0008	0010	0011	0011	0011	0011
$\bar{u}_0(10)+\bar{u}_4(10)$									0004	0005	0005	0005	0005
$\bar{u}_0(11)+\bar{u}_4(11)$										0002	0002	0002	0002
$\bar{u}_0(12)+\bar{u}_4(12)$											0001	0001	0001
$\bar{u}_0(13)+\bar{u}_4(13)$												0000	0001
$\bar{u}_0(14)+\bar{u}_4(14)$													0000
$\bar{u}_1(n)+\bar{u}_3(n)$	0932	0625	0332	0164	0079	0038	0018	0008	0004	0002	0001	0000	0000
$\bar{u}_2(n)$	3230	1250	0539	0245	0114	0054	0025	0012	0006	0003	0001	0001	0000

TABLE 7.—To calculate $u_1(2)$; $m=4$

Boundary values		$n=3$		$n=4$		$n=5$		$n=6$		$n=7$		$n=8$	
$\bar{u}_2(0)$		0497		0625		0654		0661		0662		0663	
$\bar{u}_1(0)$	$\bar{u}_3(0)$	0861	0195	0982	0268	1005	0287	1010	0292	1011	0293	1011	0293
$\bar{u}_0(1)$	$\bar{u}_4(1)$	0861	0195	0982	0268	1005	0287	1010	0292	1011	0293	1011	0293
$\bar{u}_0(2)$	$\bar{u}_4(2)$	2948	0282	3304	0446	3365	0494	3377	0506	3380	0508	3381	0509
$\bar{u}_0(3)$	$\bar{u}_4(3)$			0982	0268	1129	0364	1158	0389	1164	0394	1165	0396
$\bar{u}_0(4)$	$\bar{u}_4(4)$					0365	0174	0429	0224	0442	0236	0445	0239
$\bar{u}_0(5)$	$\bar{u}_4(5)$							0148	0097	0177	0122	0183	0128
$\bar{u}_0(6)$	$\bar{u}_4(6)$									0064	0050	0077	0062
$\bar{u}_0(7)$	$\bar{u}_4(7)$											0029	0025
$\bar{u}_1(n)$	$\bar{u}_3(n)$	2948	0282	0982	0268	0365	0174	0148	0097	0064	0050	0029	0025
$\bar{u}_2(n)$		0932		0625		0332		0164		0079		0038	

TABLE 7.—(continued)

Boundary values		$n=9$		$n=10$		$n=11$		$n=12$		$n=13$		$n\geqq 14$	
$\bar{u}_2(0)$		0663		0663		0663		0663		0663		0663	
$\bar{u}_1(0)$	$\bar{u}_3(0)$	1011	0293	1011	0293	1011	0293	1011	0293	1011	0293	1011	0293
$\bar{u}_0(1)$	$\bar{u}_4(1)$	1011	0293	1011	0293	1011	0293	1011	0293	1011	0293	1011	0293
$\bar{u}_0(2)$	$\bar{u}_4(2)$	3381	0509	3381	0509	3381	0509	3381	0509	3381	0509	3381	0509
$\bar{u}_0(3)$	$\bar{u}_4(3)$	1166	0396	1166	0396	1166	0396	1166	0396	1166	0396	1166	0396
$\bar{u}_0(4)$	$\bar{u}_4(4)$	0446	0240	0446	0240	0446	0240	0446	0240	0446	0240	0446	0240
$\bar{u}_0(5)$	$\bar{u}_4(5)$	0184	0129	0185	0129	0185	0130	0185	0130	0185	0130	0185	0130
$\bar{u}_0(6)$	$\bar{u}_4(6)$	0080	0065	0081	0066	0081	0066	0081	0066	0081	0066	0081	0066
$\bar{u}_0(7)$	$\bar{u}_4(7)$	0035	0031	0036	0032	0036	0032	0037	0033	0037	0033	0037	0033
$\bar{u}_0(8)$	$\bar{u}_4(8)$	0013	0012	0016	0015	0017	0016	0017	0016	0017	0016	0017	0016
$\bar{u}_0(9)$	$\bar{u}_4(9)$			0006	0006	0007	0007	0008	0008	0008	0008	0008	0008
$\bar{u}_0(10)$	$\bar{u}_4(10)$					0003	0003	0003	0003	0004	0004	0004	0004
$\bar{u}_0(11)$	$\bar{u}_4(11)$							0001	0001	0002	0002	0002	0002
$\bar{u}_0(12)$	$\bar{u}_4(12)$									0001	0001	0001	0001
$\bar{u}_0(13)$	$\bar{u}_4(13)$											0000	0000
$\bar{u}_1(n)$	$\bar{u}_3(n)$	0013	0012	0006	0006	0003	0003	0001	0001	0001	0001	0000	0000
$\bar{u}_2(n)$		0018		0008		0004		0002		0001		0000	

TABLE 8.—Values of $T_k(1)$

k	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$	$n\geqq 14$
1	2857	3230	3304	3320	3323	3324	3324	3324	3324	3324	3324	3324	3324
2		1056	1250	1292	1301	1303	1304	1304	1304	1304	1304	1304	1304
3			0446	0539	0560	0564	0565	0565	0566	0566	0566	0566	0566
4				0201	0245	0255	0257	0258	0258	0258	0258	0258	0258
5					0093	0114	0119	0120	0120	0120	0120	0120	0120
6						0044	0054	0056	0056	0056	0057	0057	0057
7							0021	0025	0026	0027	0027	0027	0027
8								0010	0012	0012	0013	0013	0013
9									0005	0006	0006	0006	0006
10										0002	0003	0003	0003
11											0001	0001	0001
12												0000	0001
13													0000

TABLE 9.—Values of $T_k(2)$

k	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$	$n=14$	$n\geqq 15$
1	1056	1250	1292	1301	1303	1304	1304	1304	1304	1304	1304	1304	1304
2	3230	3750	3859	3883	3888	3890	3890	3890	3890	3890	3890	3890	3890
3		1250	1493	1546	1558	1561	1561	1562	1562	1562	1562	1562	1562
4			0539	0653	0678	0684	0685	0686	0686	0686	0686	0686	0686
5				0245	0299	0311	0314	0314	0314	0314	0314	0314	0314
6					0114	0140	0145	0146	0147	0147	0147	0147	0147
7						0054	0066	0068	0069	0069	0069	0069	0069
8							0025	0031	0032	0033	0033	0033	0033
9								0012	0015	0015	0015	0015	0015
10									0006	0007	0007	0007	0007
11										0003	0003	0003	0003
12											0001	0002	0002
13												0001	0001
14													0000

—NOTES—

A METHOD FOR THE SOLUTION OF CERTAIN NON-LINEAR PROBLEMS IN LEAST SQUARES*

By KENNETH LEVENBERG[1] (*Frankford Arsenal*)

The standard method for solving least squares problems which lead to non-linear normal equations depends upon a reduction of the residuals to linear form by first order Taylor approximations taken about an initial or trial solution for the parameters.[2] If the usual least squares procedure, performed with these linear approximations, yields new values for the parameters which are not sufficiently close to the initial values, the neglect of second and higher order terms may invalidate the process, and may actually give rise to a larger value of the sum of the squares of the residuals than that corresponding to the initial solution. This failure of the standard method to improve the initial solution has received some notice in statistical applications of least squares[3] and has been encountered rather frequently in connection with certain engineering applications involving the approximate representation of one function by another. The purpose of this article is to show how the problem may be solved by an extension of the standard method which insures improvement of the initial solution.[4] The process can also be used for solving non-linear simultaneous equations, in which case it may be considered an extension of Newton's method.

Let the function to be approximated be $h(x, y, z, \cdots)$, and let the approximating function be $H(x, y, z, \cdots; \alpha, \beta, \gamma, \cdots)$, where $\alpha, \beta, \gamma, \cdots$ are the unknown parameters. Then the residuals at the points, $(x_i, y_i, z_i, \cdots)$, $i = 1, 2, \cdots, n$, are

$$f_i(\alpha, \beta, \gamma, \cdots) = H(x_i, y_i, z_i, \cdots; \alpha, \beta, \gamma, \cdots) - h(x_i, y_i, z_i, \cdots), \tag{1}$$

and the least squares criterion requires the minimization of

$$s(\alpha, \beta, \gamma, \cdots) = \sum_1^n f_i^2. \tag{2}$$

(It is assumed that the weights of the residuals are unity. If not, consider the func-

* This paper was read before the Annual Meeting of the American Mathematical Society in Chicago, Ill., on Nov. 26, 1943. Manuscript received Feb. 2, 1944.

[1] The writer wishes to thank Dr. J. G. Tappert, under whose direction the method of damped least squares was developed, and Dr. H. B. Curry, for valuable suggestions and guidance.

[2] E. T. Whittaker and G. Robinson, *The calculus of observations*, Blackie and Son, London, 1937, p. 214.

[3] E. B. Wilson and R. R. Puffer, *Least squares and laws of population growth*, Proc. Amer. Acad. Arts and Sci. (Boston), 68, 285–382 (1933).

[4] Another extension of the standard method, which requires the use of second partial derivatives, is given by Wilson and Puffer (l.c.).

A different kind of approach, not based upon the standard method, is given by Cauchy, *Méthode générale pour la résolution des systèmes d'équations simultanées*, C. R. Acad. Sci. Paris, 25, 536–538 (1847). See also a paper by H. B. Curry, not yet published, (abstract in Bull. Amer. Math. Soc., 49, 859 (1943), abstract No. 278).

tion f_i to be the product of the residual and the square root of the corresponding weight.) Choosing an initial solution, $p_0=(\alpha_0, \beta_0, \gamma_0, \cdots)$, at which it is assumed that s does not have a stationary value, the first order Taylor expansions of the residuals are taken about p_0, giving a set of linear approximations to the residuals,

$$f_i(\alpha, \beta, \gamma, \cdots) \cong F_i(\alpha, \beta, \gamma, \cdots) = f_i(p_0) + \frac{\partial f_i}{\partial \alpha}\Delta\alpha + \frac{\partial f_i}{\partial \beta}\Delta\beta + \frac{\partial f_i}{\partial \gamma}\Delta\gamma + \cdots, \quad (3)$$

where $\Delta\alpha=\alpha-\alpha_0$, $\Delta\beta=\beta-\beta_0$, $\cdots$, and the partial derivatives are evaluated at p_0. Now, the standard method consists of minimizing

$$S(\alpha, \beta, \gamma, \cdots) = \sum_1^n F_i^2 \quad (4)$$

by setting the partial derivatives of S with respect to the various parameters equal to zero, yielding the usual linear normal equations,

$$\begin{aligned} \frac{1}{2}\frac{\partial S}{\partial \alpha} &= [\alpha\alpha]\Delta\alpha + [\alpha\beta]\Delta\beta + [\alpha\gamma]\Delta\gamma + \cdots + [\alpha 0] = 0, \\ \frac{1}{2}\frac{\partial S}{\partial \beta} &= [\beta\alpha]\Delta\alpha + [\beta\beta]\Delta\beta + [\beta\gamma]\Delta\gamma + \cdots + [\beta 0] = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned} \quad (5)$$

where the notation [ ] is a symbol of summation, so that, e.g.,

$$[\alpha\alpha] = \sum_1^n \left(\frac{\partial f_i}{\partial \alpha}\right)^2, \qquad [\alpha\beta] = \sum_1^n \left(\frac{\partial f_i}{\partial \alpha}\cdot\frac{\partial f_i}{\partial \beta}\right), \qquad [\alpha 0] = \sum_1^n \left(\frac{\partial f_i}{\partial \alpha}\cdot f_i\right), \text{ etc.}$$

However, as pointed out above, the values of the increments, $\Delta\alpha$, $\Delta\beta$, $\Delta\gamma$, $\cdots$, obtained by solving equations (5), may be so large in absolute value as to invalidate the approximations (3) so that the decrease in S may not correspond to a decrease in s.

In such cases, it would seem advisable to limit or "damp" the absolute values of the increments of the parameters in order to improve the first order Taylor approximations (3) and to minimize simultaneously the sum of the squares of the approximating residuals (4) under these damped conditions. In order to make both the increments and the residuals small in absolute value, the least squares idea can be employed. The sum of the squares of both the residuals and the increments may be minimized. More precisely, the expression to be minimized will be

$$\bar{S}(\alpha, \beta, \gamma, \cdots) = wS(\alpha, \beta, \gamma, \cdots) + a(\Delta\alpha)^2 + b(\Delta\beta)^2 + c(\Delta\gamma)^2 + \cdots, \quad (6)$$

where a, b, c, $\cdots$ are a system of positive constants or weighting factors expressing the relative importance of damping the different increments, and w is a positive quantity expressing the relative importance of the residuals and increments in this minimizing process. If we denote the point at which $\bar{S}$ takes its minimum, for any positive value of w, by $p_w=(\alpha_w, \beta_w, \gamma_w, \cdots)$, and set

$$Q(\alpha, \beta, \gamma, \cdots) = a(\Delta\alpha)^2 + b(\Delta\beta)^2 + c(\Delta\gamma)^2 + \cdots, \quad (7)$$

it is seen, under the assumption that s is not stationary at p_0, that

$$wS(p_w) < wS(p_w) + Q(p_w) = \bar{S}(p_w) < \bar{S}(p_0) = wS(p_0) + Q(p_0) = wS(p_0),$$

whence
$$S(p_w) < S(p_0). \tag{8}$$

Also, denoting the standard least squares solution by p_∞ (the reason for the notation is discussed later), we have

$$wS(p_w) + Q(p_w) = \bar{S}(p_w) < \bar{S}(p_\infty) = wS(p_\infty) + Q(p_\infty) < wS(p_w) + Q(p_\infty),$$

whence
$$Q(p_w) < Q(p_\infty). \tag{9}$$

Inequality (8) shows that the minimization of (6) will diminish the sum of the squares of the approximating residuals, S, and (9) shows that the increments given by the standard least squares solution will be improved in the sense that the weighted sum of their squares, Q, will be reduced. That the sum of the squares of the true residuals, s, can be diminished, will be proved shortly.

To minimize (6) and obtain p_w, the partial derivatives of $\bar{S}$ with respect to the various parameters are put equal to zero, and we get

$$\frac{\partial \bar{S}}{\partial \alpha} = w\frac{\partial S}{\partial \alpha} + 2a\Delta\alpha = 0, \qquad \frac{\partial \bar{S}}{\partial \beta} = w\frac{\partial S}{\partial \beta} + 2b\Delta\beta = 0, \quad \cdots .$$

When we divide through by $2w$, and substitute the expressions for the partial derivatives of S from (5), the "damped normal equations" become

$$\begin{aligned} ([\alpha\alpha] + aw^{-1})\Delta\alpha + \quad [\alpha\beta]\Delta\beta + [\alpha\gamma]\Delta\gamma + \cdots + [\alpha 0] &= 0, \\ [\beta\alpha]\Delta\alpha + ([\beta\beta] + bw^{-1})\Delta\beta + [\beta\gamma]\Delta\gamma + \cdots + [\beta 0] &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{10}$$

These equations are seen to be the same as the ordinary normal equations (5), except for the coefficients of the principal diagonal, which are increased by quantities proportional to the weighting factors a, b, c, $\cdots$, respectively. Since the symmetry of the matrix of the coefficients of equations (5) is preserved, simplified methods of solution of linear simultaneous equations, which take full advantage of such symmetry,[5] may be used to solve equations (10). It is to be noted that the standard method of least squares corresponds to $w \to \infty$, and is thus a special case of the method here given, which may be termed the method of "damped least squares."

If we denote the number of parameters by k, it is seen from the determinantal solution of equations (10) that, in the neighborhood of $w = 0$,

$$\Delta\alpha = \alpha_w - \alpha_0 = \frac{-[\alpha 0]w^{1-k}bcd\cdots + \cdots}{w^{-k}abc\cdots + \cdots} = -[\alpha 0]a^{-1}w + \cdots,$$

whence
$$\left(\frac{d\alpha_w}{dw}\right)_{w=0} = -[\alpha 0]a^{-1}, \tag{11}$$

and similarly for the other parameters. Now

$$\frac{ds(p_w)}{dw} = \frac{\partial s}{\partial \alpha}\cdot\frac{d\alpha}{dw} + \frac{\partial s}{\partial \beta}\cdot\frac{d\beta}{dw} + \cdots, \tag{12}$$

[5] P. S. Dwyer, *The solution of simultaneous equations*, Psychometrika, 6, 101–129 (1941).

and, from the definition of the summation symbols, we find that the partial derivatives of s at p_0 are given by

$$\frac{\partial s}{\partial \alpha} = 2[\alpha 0], \qquad \frac{\partial s}{\partial \beta} = 2[\beta 0], \cdots . \tag{13}$$

Hence the substitution of (11) and (13) in (12) yields

$$\left(\frac{ds}{dw}\right)_{w=0} = -2\{[\alpha 0]^2 a^{-1} + [\beta 0]^2 b^{-1} + \cdots\}. \tag{14}$$

This derivative is negative since the partial derivatives in (13) are not all zero, by the assumption that s does not have a stationary value at p_0. Therefore, $s(p_w)$ is decreasing at $w=0$, thus insuring that values of w can be found for which the sum of the squares of the true residuals (2) will be reduced.

The best value of w to use may theoretically be determined directly by solving

$$\frac{ds(p_w)}{dw} = 0; \tag{15}$$

however, this equation is generally complex in practice. By writing

$$s(p_w) \cong s(p_0) + w\left(\frac{ds}{dw}\right)_{w=0}, \tag{16}$$

and setting the left side of (16) equal to zero on the assumption that p_0 was chosen so that the decreased value $s(p_w)$ will be small, the approximate formula,

$$w \cong -\frac{s(p_0)}{ds/dw_{w=0}} = \frac{\frac{1}{2}s(p_0)}{[\alpha 0]^2 a^{-1} + [\beta 0]^2 b^{-1} + \cdots}, \tag{17}$$

is obtained.[6] If necessary, this value may be improved by calculating $s(p_w)$ for several different trial values of w, so that an approximate minimum may be located graphically. Experience with the method, especially in connection with fitting a particular function $H(x, y, z, \cdots; \alpha, \beta, \gamma, \cdots)$, enables one to get an idea of the general order of magnitude of the best value of w so that very few trial values of w should suffice. If so desired, the improved set of values of the parameters may be further improved (if the true minimum has not already been reached), by a repetition of the process, considering this improved set as a new initial solution.

So far, the weighting system $a, b, c, \cdots$ has been left arbitrary, the only restriction being that the weighting factors be positive. If we set the criterion that these factors be chosen so that the directional derivative of s, taken at $w=0$ along the curve $\alpha=\alpha_w$, $\beta=\beta_w$, $\cdots$, should have its minimum value, namely, the negative gradient, we have

$$\frac{ds}{dw}\left\{\left(\frac{d\alpha}{dw}\right)^2 + \left(\frac{d\beta}{dw}\right)^2 + \cdots\right\}^{-1/2} = -\left\{\left(\frac{\partial s}{\partial \alpha}\right)^2 + \left(\frac{\partial s}{\partial \beta}\right)^2 + \cdots\right\}^{1/2}, \tag{18}$$

where the derivatives are taken at $w=0$. Substitution of (14), (11), (13) in (18) gives us

[6] This type of approximation was used by Cauchy (l.c.).

$$\{[\alpha 0]^2 a^{-1} + [\beta 0]^2 b^{-1} + \cdots\}\{\alpha[0]^2 a^{-2} + [\beta 0]^2 b^{-2} + \cdots\}^{-1/2} = \{[\alpha 0]^2 + [\beta 0]^2 + \cdots\}^{1/2}, \quad (19)$$

and this is satisfied when the factors $a, b, c, \cdots$ are all equal. Without loss of generality, they may be taken equal to unity. For this weighting system, the formation of the damped normal equations (10) may be thought of as being accomplished simply by the addition of a positive constant, $1/w$, to the coefficients of the principal diagonal of the standard normal equations (5). Another weighting system which has been used successfully is, $a = [\alpha\alpha]$, $b = [\beta\beta]$, $\cdots$; in this case the damped normal equations are formed by multiplying the principal diagonal coefficients of the standard normal equations by a constant greater than unity, $1 + 1/w$.

The nature of the damping which we have imposed upon the parameter variables can be given a simple geometric interpretation. For instance, if the unity weighting system is considered, the "overshooting" of the solution is prevented by damping the distance (k dimensional) from the initial solution point, since Q is then the square of this distance. By this restriction of k dimensional distance (which would appear to be a natural way to prevent overshooting), we are not obliged to decide on an arbitrary preassigned procedure restricting the variables individually, as is done, for example, by the method of Cauchy (l.c.). The greater freedom given the individual variables by the method of damped least squares may account for the fact that it has solved, with a comparatively rapid rate of convergence, types of problems which are of much greater complexity than those to which the principle of least squares is ordinarily applied.

ON THE DEFLECTION OF A CANTILEVER BEAM*

By H. J. BARTEN (*Washington Navy Yard*)

In spring theory it is sometimes necessary to compute the deflection of a cantilever beam for which the squares of the first derivatives cannot be neglected as is done in classical beam theory. This problem is thus placed in the same category as the problem of the elastica.

The solution given in this note can be applied to a cantilever of any stiffness. The difference between the deflection as found by the classical beam theory and that found by the present method is, however, noticeable only in the case of beams of low stiffness.

The clamped end of the beam is taken as the origin of coordinates and downward deflections are considered as positive. A point on the beam may be identified by four quantities of which only one is independent. These four quantities are the two rectangular coordinates x and y, the arc length s measured from the origin of coordinates, and the deflection angle θ which is the angle between the tangent to the curve at the point under discussion and the horizontal. We may thus identify this point by the symbol (x, y, s, θ). The subscript L is used to identify the value of these quantities at the free end of the beam. Before deflection a vertical load P is applied at the point $(L, 0, L, 0)$. The beam has a uniform cross section of moment of inertia I and is com-

* Received Feb. 21, 1944.

posed of a material whose modulus of elasticity is E. The problem is to find the deflection of the end-point of the beam due to the vertical load P.

The bending moment induced at the point (x, y, s, θ) by the vertical load P is

$$M = P(x_L - x).$$

Therefore

$$d\theta/ds = a(x_L - x), \tag{1}$$

where $a = P/EI$. Using the relation

$$\frac{d\theta}{ds} = \frac{d\theta}{dx}\frac{dx}{ds} = \cos\theta\frac{d\theta}{dx},$$

we obtain

$$\int \cos\theta\, d\theta = \int a(x_L - x)dx$$

or

$$\sin\theta = a(x_L x - \tfrac{1}{2}x^2) + C. \tag{2}$$

The boundary condition at the clamped end of the beam, namely, $\theta = 0$ when $x = 0$, reduces Eq. (2) to

$$\sin\theta = a(x_L x - \tfrac{1}{2}x^2). \tag{3}$$

Thus

$$\sin\theta_L = \tfrac{1}{2}ax_L^2. \tag{4}$$

Combining the latter expression and Eq. (3) we obtain

$$\sin\theta_L - \sin\theta = \tfrac{1}{2}a(x_L - x)^2. \tag{5}$$

Thus

$$x_L - x = [2a^{-1}(\sin\theta_L - \sin\theta)]^{1/2}.$$

Substituting this expression into Eq. (1), we obtain

$$\frac{d\theta}{ds} = \frac{d\theta}{dy}\frac{dy}{ds} = \sin\theta\frac{d\theta}{dy} = [2a(\sin\theta_L - \sin\theta)]^{1/2},$$

or

$$y = \int_0^\theta \frac{\sin\theta\, d\theta}{[2a(\sin\theta_L - \sin\theta)]^{1/2}}.$$

Therefore

$$y_L = \int_0^{\theta_L} \frac{\sin\theta\, d\theta}{[2a(\sin\theta_L - \sin\theta)]^{1/2}}. \tag{6}$$

With the transformation

$$\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \cos\left(\frac{\pi}{4} - \frac{\theta_L}{2}\right)\sin\phi = k\sin\phi,$$

Eq. (6) becomes

$$y_L = a^{-1/2}\int_\delta^{\pi/2} \frac{(2k^2\sin^2\phi - 1)d\phi}{(1 - k^2\sin^2\phi)^{1/2}}, \tag{7}$$

where

$$\sin\delta = \frac{\cos\pi/4}{k}, \qquad k = \cos\left(\frac{\pi}{4} - \frac{\theta_L}{2}\right).$$

Eq. (7) is a combination of incomplete and complete elliptic integrals[1] and may be written

$$y_L = a^{-1/2}[F(k) - F(k, \delta) - 2E(k) + 2E(k, \delta)], \tag{8}$$

where $F(k)$ and $E(k)$ are the first and second complete elliptic integrals respectively and $F(k, \delta)$ and $E(k, \delta)$ are the first and second incomplete elliptic integrals respectively.

As Eq. (8) stands it is useless unless we find θ_L as a function of a and L. This relationship may be obtained in the following manner. From Eq. (1) we get

$$\theta_L = \int_0^L a(x_L - x)ds.$$

Integrating by parts we obtain

$$\theta_L = \int_0^{x_L} as\,dx = \int_0^L as\frac{dx}{ds}\,ds = \int_0^L as\cos\theta\,ds.$$

Differentiating this latter integral with respect to its upper limit, we have

$$d\theta_L/dL = aL\cos\theta_L.$$

The solution to this differential equation is

$$\sin\theta_L = \tanh\frac{aL^2}{2}\cdot \tag{9}$$

This completes the solution to the problem.

In order to compare our results with those of Gross and Lehr[2] we must express our solution in the same dimensionless factors that they employed. By dividing the actual deflection of the beam by the "small deflection" $aL^3/3$ they obtain a deflection factor which is a function of the dimensionless quantity aL^2. We shall call this deflection factor F_y. Thus, from Eq. (8)

$$F_y = \frac{3y_L}{aL^3} = 3(aL^2)^{-3/2}[F(k) - F(k, \delta) - 2E(k) + 2E(k, \delta)]. \tag{10}$$

In order to find the maximum bending stress at the clamped end of the beam we must know the length of the moment arm x_L. Combining Eqs. (4) and (9) we find that

$$x_L^2 = \frac{2}{a}\tanh\frac{aL^2}{2}\cdot \tag{11}$$

Gross and Lehr use the dimensionless contraction factor x_L/L an an aid in finding x_L. We shall define this factor as F_x. Thus

---

[1] Jahnke and Emde, *Funktionentafeln mit Formeln und Kurven*, Dover Publications, 1943.

[2] Gross and Lehr, *Die Federn*, V. D. I. Verlag, 1938.

$$F_x^2 = \frac{2}{aL^2} \tanh \frac{aL^2}{2} \cdot \tag{12}$$

Computations show that Gross and Lehr's values of F_y have a constantly increasing error which deviates about 4% from our results when $aL^2 = 1$.

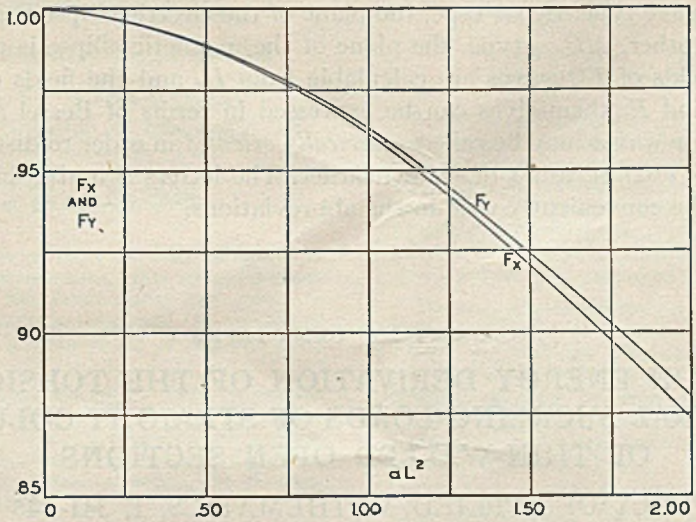

FIG. 1.

The two factors F_x and F_y are very important to the designer. For this reason curves of these two factors with aL^2 as the independent variable are given in Fig. 1. The values of F_y were computed from Jahnke and Emde.

ON WAVES IN BENT PIPES*

BY S. A. SCHELKUNOFF (*Bell Telephone Laboratories*)

In a recent issue of this QUARTERLY,[1] Karlem Riess obtained expressions for the fields of electromagnetic waves in bent pipes of rectangular cross section by the perturbation method. While it is true that in a bent pipe the waves cannot be classified into transverse electric and transverse magnetic types because in general both E and H have components in the direction of wave propagation, a different classification into two types is possible. This permits another method which yields the general solution in terms of Bessel functions.

In the one wave type, the plane of the electric ellipse is normal to the axis of bending (the Y-axis in Figure 1, p. 329 of Riess' paper); these waves have been called *electrically oriented* ($EO_{m,n}$ wave type) and the fields of these waves are obtainable from H_y which may be expressed as the product of Bessel and sine (or cosine) functions.

* Received Feb. 18, 1944.

[1] Vol. 1, No. 4, pp. 328–333.

In the other wave type, the plane of the magnetic ellipse is normal to the axis of bending; these waves are *magnetically oriented* ($MO_{m,n}$ wave type) and their fields are obtainable from E_y. In each case the order of Bessel functions is equal to the angular phase constant.

For a bent pipe formed by the intersection of two concentric spheres and two coaxial cones emerging from the center there is also a solution in terms of known functions. In one wave type, $EO_{m,n}$ type, the plane of the electric ellipse is normal to the radius; in the other, $MO_{m,n}$ type, the plane of the magnetic ellipse is normal to the radius. The fields of EO-waves are calculable from H_r and the fields of MO-waves from E_r; H_r and E_r themselves can be expressed in terms of Bessel and Legendre functions. These waves may be called *spherically oriented* in order to distinguish them from the *plane oriented waves* described earlier. The letters S and P in front of EO and MO may be conveniently used in the abbreviations.

CORRECTIONS TO MY PAPER

A STRAIN ENERGY DERIVATION OF THE TORSIONAL-FLEXURAL BUCKLING LOADS OF STRAIGHT COLUMNS OF THIN-WALLED OPEN SECTIONS

QUARTERLY OF APPLIED MATHEMATICS, 1, 341–345 (1944).

By

N. J. HOFF

In the last term of the right hand side member of Eq. (3) on page 343, n should be raised to the second power and not to the fourth power.

The following equation defining T should be added:

$$T = (1/\rho^2)(n^2R + GC).$$

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By E. L. INCE

Originally pub. at $12.00 3.75

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