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## QUARTERLY OF APPLIED MATHEMATICS

# IMPEDANCE CONCEPT IN WAVE GUIDES* 

BY<br>\section*{S. A. SCHELKUNOFF}<br>Bell Telephone Laboratories

1. Introduction. The impedance concept is the foundation of engineering transmission theory. If wave guides are to be fully utilized as transmission systems or parts thereof, their properties must be expressed in terms of appropriately chosen impedances or else a new transmission theory must be developed. The gradual extension of the concept has necessitated a broader point of view without which an exploitation of its full potentialities would be impossible.

In the course of various private discussions, I have found that there exists some uneasiness with regard to the applicability of the concept at very high frequencies. In part this may be attributed to relative unfamiliarity with the wave guide phenomena and in part to the evolution of the concept itself. Some particular aspects of the concept have to be sacrificed in the process of generalization and although these aspects may be logically unimportant, they frequently become psychological obstacles to understanding in the early stages of the development. For this reason I am going to devote several sections of this paper to a general discussion of the impedance concept before passing to more specific applications; then by way of illustration I shall prove that an infinitely thin perfectly conducting iris between two different wave guides behaves as if between the admittances of its faces there existed an ideal transformer. This theorem is a generalization of another theorem which I proved several years ago to the effect that when the two wave guides are alike, the iris behaves as a shunt reactor. Actual calculation of the admittances and the transformer ratio depends on the solution of an appropriate boundary value problem.

More generally, wave guide discontinuities are representable by $T$-networks. In some special cases these networks lack series branches and in other cases, the shunt branch.
2. Evolution of concepts. Concepts evolve. It is a long way from the primitive to the modern number concept. The primitive number was an integer, a concrete integer at that. In some primitive languages there is no word corresponding to "two." There are words meaning "two men," "two horses," etc.; but the concept of "two" applying either to men, or to horses, is lacking. To a primitive mind the difference between a class comprised of two men and a class comprised of two horses overshadowed the similarity. Seeing similarities requires a degree of abstraction. A resistance to abstract ideas seems to be a characteristic of human minds even in modern times; only the modern mind is quicker to overcome it. An example, pertinent to

[^0]our present discussion, is the following excerpt from a paper Derivation and discussion of the general solution of the current flowing in a circuit containing resistance, selfinductance and capacity with any impressed electromotive force, by Frederick Bedell and Albert C. Crehore, published in the Journal A.I.E.E., 9, 340 (1892):
"From the analogy of this equation to Ohm's law, we see that the expression [ $\left.R^{2}+(1 / C \omega-L \omega)^{2}\right]^{1 / 2}$ is of the nature of a resistance, and is the apparent resistance of a circuit containing resistance, self-inductance and capacity. This expression would quite properly be called 'impedance' but the term impedance has for several years been used as a name for the expression $\left[R^{2}+L^{2} \omega^{2}\right]^{1 / 2}$, which is the apparent resistance of a circuit containing resistance and self-inductance only. We suggest, therefore, that the word 'impediment' be adopted as a name for the expression $\left[R^{2}+(1 / C \omega-L \omega)^{2}\right]^{1 / 2}$ which is the apparent resistance of a circuit containing resistance, self-inductance and capacity, and the term impedance be retained in the more limited meaning it has come to have, that is, $\left[R^{2}+L^{2} \omega^{2}\right]^{1 / 2}$, the apparent resistance of a circuit containing resistance and self-inductance only." The name "impediment" was not adopted. Apparently, it was soon understood that if one really wished to emphasize the difference between the impedances of various circuits, one could simply describe the circuits and, therefore, for most purposes, it was best to emphasize the similarity rather than the difference. And only ten years ago there were some who objected to the use of the word impedance for the ratio $E / H$ in an electromagnetic wave and who wanted a new word for it.

The word "number" now includes fractions, negative numbers, irrational numbers and complex numbers; the impedance is now a complex number, and not its absolute value as originally intended. There are mechanical impedances, acoustic impedances, electromechanical impedances, and finally impedances associated with any wave no matter what its physical nature happens to be. The impedance is now the force/response ratio when the force and response are harmonic functions of time and are represented by complex exponentials. Around this concept has grown the transmission theory of force and response in linear systems. The principal tool of this theory is the theory of functions of a complex variable. This theory is used for engineering purposes as in the design of filters, equalizers, and other transmission systems with prescribed desired properties; and with equal advantage it may be used for general transmission studies. In this paper I am particularly concerned with fundamental ideas applied to wave guides and wave guide elements.
3. General discussion of impedance and admittance. Superficially, it may seem that the impedance concept does not apply to wave guides or if it does it is quite different from the concept as applied to ordinary transmission lines. Actually there is no significant difference; whatever difference there exists is largely psychological rather than logical. In wave guides a characteristic impedance has to be associated with each transmission mode. At first the existence of various transmission modes may strike one as a feature which distinguishes high frequency wave guides from low frequency "ordinary" transmission lines; but soon one will realize that even in ordinary transmission lines it is usual to distinguish between different modes of transmission. Consider, for instance, parallel wires at the same height above ground; there are two obvious transmission modes recognized by communication engincers; in one the currents in the wires are equal and fow in opposite directions and in the other they are equal, flow in the same direction and return through ground. It is the exist-
ence of these two transmission modes that accounts for the important engineering difference between balanced and unbalanced transmission lines. Similarly, there are two obvious transmission modes in a shielded parallel pair. The field patterns are different for different transmission modes and the characteristic impedances are usually different. The existence of transmission modes is not peculiar to hollow tubes and other structures which have become prominent in high frequency transmission; high frequency transmission studies make us merely aware of the fact that any physical wave guide, whether a coaxial pair or a shielded pair or a hollow tube, admits of an infinite number of transmission modes with their characteristic field patterns, characteristic impedances, and propagation constants.

Another cause of worry to some is a degree of indeterminacy connected with an impedance and its associated quantities. The characteristic impedance of a wave guide may be defined in a number of ways giving different values. For each oscillation mode a cavity resonator behaves as an ordinary circuit comprised of inductance and capacitance; but different values are obtained, depending on how $L$ and $C$ are defined. This indeterminacy is really inherent in these conceptions but in elementary theory it is not stressed for the simple reason that no occasion arises for such stressing. In the final analysis, this indeterminacy is of the same kind as that involved in the essential arbitrariness of units and is related to the fact that properties of analytic functions are not affected by a constant factor. Putting it in the language of transmission theory, the essential properties of impedance functions are not affected by ideal transformers. If we have a closed box containing an electric network with two accessible terminals and if we measure a resistance $R$ across these terminals, we cannot be certain that the box contains a resistance $R$; it may contain a resistance $R / 10$ which is then boosted to $R$ by an ideal transformer. It does not really matter which is the case. Similarly, if the measurement seems to indicate that in the box we have a tuned circuit with an inductance $L$ in series with a capacitance $C$, we may actually have a tuned circuit with an inductance $\frac{1}{2} L$ and the capacitance $2 C$ in the secondary of an ideal transformer which then doubles the impedance. More generally, the impedance function is defined by its zeros, infinities, and other singularities except for a constant.

If $V$ is the voltage across an impedance $Z, I$ the current through $Z$, and $W$ the complex power, then

$$
\begin{equation*}
V=Z I, \quad W=\frac{1}{2} V I^{*}=\frac{1}{2} Z I I^{*}, \quad W=\frac{V V^{*}}{2 Z^{*}} \tag{3-1}
\end{equation*}
$$

where the asterisk is used to designate conjugate complex numbers. Now suppose that our voltmeters and ammeters contain concealed ideal transformers; then " $Z$ " will have different values in the above equations and we shall have

$$
\begin{equation*}
V=Z_{V, I} I, \quad W=\frac{1}{2} Z_{W, I} I I^{*}, \quad W=\frac{V V^{*}}{2 Z_{W, V}^{*}} \tag{3-2}
\end{equation*}
$$

These new equations are in effect various definitions of impedance and admittance

$$
\begin{array}{lll}
Z_{V, I}=\frac{V}{I}, & Z_{W, I}=\frac{2 W}{I I^{*}}, & Z_{W, V}=\frac{V V^{*}}{2 W^{*}} \\
Y_{V, I}=\frac{1}{Z_{V, I}}, & Y_{W, I}=\frac{1}{Z_{W, I}}, & Y_{W, V}=\frac{1}{Z_{W, V}} \tag{3-3}
\end{array}
$$

Ordinarily, we make sure that there are no concealed ideal transformers in our measuring instruments. Furthermore, at low frequencies we can measure the voltage across the total capacitance ${ }^{1}$ and the total current through the inductance. There seems to be no question about the meaning of " $V$ " and " $I$ " and it so happens that in this case we are led to equations (3-1). However, in a section of a transmission line or in a cavity resonator the capacitance and inductance are not localized and we are forced to recognize the existence of a certain amount of indeterminacy. There is no harm in this indeterminacy; it does not really matter in which of the following two forms we decide to write the expression for power

$$
\begin{equation*}
W=\frac{1}{2} Z I I^{*}, \quad \text { or } \quad W=\frac{1}{2}\left(n^{2} Z\right) \frac{I}{n} \frac{I^{*}}{n} \tag{3-4}
\end{equation*}
$$

so long as we know how to compute it.
Just as ideal transformers in our "ammeters" and "voltmeters" transform equations (3-1) into equations (3-2) in the case of "ordinary" networks, they may be used to transform equations (3-2) into (3-1) in the case of wave guides and networks with distributed constants.
4. General impedance relations. Eliminating $V, I$, and $W$ from (3-2), we have - the following equation connecting various impedances

$$
\begin{equation*}
Z_{W, I} Z_{W, V}^{*}=Z_{V, I} Z_{V, I}^{*} \tag{4-1}
\end{equation*}
$$

If the impedances are real, then

$$
\begin{equation*}
Z_{W, r} Z_{W, V}=Z_{V, I}^{2} \tag{4-2}
\end{equation*}
$$

In equations (3-2) $V$ and $I$ may be arbitrarily chosen values of the voltage and current associated with a given impedor. If we choose a given definition for $I$, we can define a voltage

$$
\begin{equation*}
V_{W, I}=Z_{W, I} I \tag{4-3}
\end{equation*}
$$

for which equations (3-1) will hold and the impedance $Z_{W, I}$ will become the only impedance associated with the impedor. We can also define

$$
\begin{equation*}
I_{W, V}=\frac{V}{Z_{W, V}} \tag{4-4}
\end{equation*}
$$

so that again we shall have equations (3-1) with $Z_{W, V}$ as the sole impedance.
Since the power is an invariant we have

$$
\begin{equation*}
V_{\Gamma, I} I^{*}=V I_{W, V}^{*} \quad \text { or } \quad \frac{V_{W, I}}{V}=\frac{I_{W, V}^{*}}{I^{*}} \tag{4-5}
\end{equation*}
$$

[^1]From (4-3), (4-4), and (4-5) we have

$$
\begin{equation*}
\left|\frac{V_{W, I}}{V}\right|=\left|\frac{I_{W, V}}{I}\right|=\sqrt{\frac{Z_{W, I}}{Z_{W, V}}}=\sqrt{\frac{Y_{W, V}}{Y_{W, I}}} . \tag{4-6}
\end{equation*}
$$

It is now evident that we can base our calculations on any particular voltage-current pair and then, whenever desirable, we may pass to any other pair simply by inserting in our transmission diagrams an ideal transformer with a proper impedance transformation ratio.
5. Characteristic impedances and admittances of wave guides. The basic impedance associated with the $n$th transmission mode in a wave guide is defined as the ratio of the transverse electric to the transverse magnetic intensity

$$
\begin{equation*}
K_{n}=\frac{E_{t, n}}{H_{t, n}} \tag{5-1}
\end{equation*}
$$

It is called the wave impedance or the specific impedance and enters in the expression for the average power flow per unit area in the direction of the guide

$$
\begin{equation*}
W_{S}=\frac{1}{2} E_{t, n} H_{t, n}^{*}=\frac{1}{2} K_{n} H_{t, n} H_{t, n}^{*} \tag{5-2}
\end{equation*}
$$

The reciprocal of this impedance is the wave admittance

$$
\begin{equation*}
M_{n}=\frac{1}{K_{n}} \tag{5-3}
\end{equation*}
$$

the power flow is then

$$
\begin{equation*}
W_{S}=\frac{1}{2} M_{n}^{*} E_{l, n} E_{t, n}^{*} \tag{5-4}
\end{equation*}
$$

In wave guides with perfectly conducting walls the various transmission modes carry power independently of each other. The field patterns are "orthogonal" to each other and may be "normalized"; that is, the transverse intensities for a typical mode may be expressed as follows

$$
\begin{equation*}
E_{t, n}=V_{n} F_{n}(u, v), \quad H_{t, n}=I_{n} F_{n}(u, v), \quad I_{n}=M_{n} V_{n}, \tag{5-5}
\end{equation*}
$$

where

$$
\begin{gather*}
\iint\left[F_{n}(u, v)\right]^{2} d S=1 \\
\iint F_{m}(u, v) F_{n}(u, v) d S=0, \text { if } m \neq n \tag{5-6}
\end{gather*}
$$

$u$ and $v$ are suitable coordinates in the transverse plane of the wave guide and the integration is extended over the entire cross-section. The coefficients $V_{n}$ and $I_{n}$ may be called respectively the normalized voltage and normalized magnetomotive force or normalized current associated with the $n$th mode.

Calculating the total power carried in the $n$th mode, we obtain

$$
\begin{equation*}
W=\frac{1}{2} K_{n} I_{n} I_{n}^{*}=\frac{1}{2} M_{n}^{*} V_{n} V_{n}^{*} \tag{5-7}
\end{equation*}
$$

Thus, if we express our transmission formulas in terms of normalized voltages and
currents, the same impedance coefficient appears in the alternative expressions (3-1) for power and this impedance is also the ratio of the normalized voltage to the normalized current.

Before going on let us see just what the above formulas mean in one or two special cases. Consider a wave guide consist-


Fig. 1. ing of two parallel metal strips of width $a$, separated by distance $b$, Fig. 1. In the dominant mode the electric intensity is perpendicular to the metal plates and is distributed almost uniformly except near the edges and in the external region where the field is weak and little power is carried by the wave. Neglecting the edge effect, we shall assume that the electric intensity is constant

$$
\begin{equation*}
\cdot E_{\iota}=E_{0} \tag{5-8}
\end{equation*}
$$

The normalized distribution pattern is given by

$$
\begin{equation*}
F_{0}(x, y)=\frac{1}{\sqrt{a b}} \tag{5-9}
\end{equation*}
$$

and, therefore,

$$
\begin{equation*}
E_{t}=V_{0} F_{0}(x, y), \quad V_{0}=E_{0} \sqrt{a b} . \tag{5-10}
\end{equation*}
$$

The wave impedance for transverse electromagnetic waves is $K_{0}=\sqrt{\mu / \epsilon}$ and therefore

$$
\begin{equation*}
H_{t}=I_{0} F_{0}(x, y), \quad I_{0}=H_{0} \sqrt{a b}=\frac{V_{0}}{K_{0}} \tag{5-11}
\end{equation*}
$$

In air $K_{0}=$ approximately 377 ohms. The transverse voltage $V$ between the plates and the longitudinal current $I$ are

$$
\begin{equation*}
V=b E_{0}=V_{0} \sqrt{b / a}, \quad I=a H_{0}=I_{0} \sqrt{a / b} \tag{5-12}
\end{equation*}
$$

consequently the characteristic impedance on the voltage-current basis is

$$
\begin{equation*}
K_{V, I}=\frac{V}{I}=\frac{b V_{0}}{a I_{0}}=\frac{b}{a} K_{0} \tag{5-13}
\end{equation*}
$$

For the total power flow we have

$$
\begin{equation*}
W=\frac{1}{2} K_{0} I_{0} I_{0}^{*}=\frac{1}{2} M_{0} V_{0} V_{0}^{*}=\frac{1}{2} K_{V, I} I I^{*}=\frac{V V^{*}}{2 K_{V, I}} \tag{5-14}
\end{equation*}
$$

so that in the present case

$$
\begin{equation*}
K_{W, I}=K_{W, V}=K_{V, I}=\frac{b}{a} K_{0 .} \tag{5-15}
\end{equation*}
$$

Consider now the $T E_{1,0}$-wave in a rectangular wave guide, Fig. 2; for this wave the field is given by


Fig. 2.

$$
\begin{gather*}
E_{t}=E_{1} \sin \frac{\pi x}{a}, \quad H_{t}=H_{1} \sin \frac{\pi x}{a} \\
E_{1}=K_{1} H_{1}, \quad H_{1}=M_{1} E_{1}, \quad K_{1}=V^{\prime \mu / \epsilon}\left(1-\frac{\lambda^{2}}{4 a^{2}}\right)^{-1 / 2} \tag{5-16}
\end{gather*}
$$

The normalized field distribution function is

$$
\begin{equation*}
F_{1}(x, y)=\sqrt{\frac{2}{a b}} \sin \frac{\pi x}{a} \tag{5-17}
\end{equation*}
$$

so that

$$
\begin{array}{ll}
E_{\ell}=V_{1} F_{1}(x, y), & H_{t}=I_{1} F_{1}(x, y), \\
V_{1}=E_{1} \sqrt{a b / 2}, & I_{1}=H_{1} \sqrt{a / 2 b} \tag{5-18}
\end{array}
$$

In this case the maximum transverse voltage $V$ across the guide and the total longitudinal current $I$ are given by

$$
\begin{align*}
& V=E_{1} b=V_{1} \sqrt{2 b / a}  \tag{5-19}\\
& I=\int_{0}^{a} H_{t} d x=\frac{4}{\pi} I_{1} \sqrt{a b / 2}
\end{align*}
$$

From these equations we have

$$
\begin{equation*}
K_{V, I}=\frac{V}{I}=\frac{\pi b}{2 a} K_{1} \tag{5-20}
\end{equation*}
$$

The power transfer is

$$
\begin{equation*}
W=\frac{1}{2} K_{1} I_{1} I_{1}^{*}=\frac{1}{2} M_{1}^{*} V_{1} V_{1}^{*} ; \tag{5-21}
\end{equation*}
$$

consequently

$$
\begin{equation*}
K_{W, I}=\frac{\pi^{2} b}{8 a} K_{1}, \quad K_{W, V}=\frac{2 b}{a} K_{1} \tag{5-22}
\end{equation*}
$$

Now let us see what happens when we join two wave guides, each consisting of two parallel metal strips. Suppose that the frequency is so low that we do not have to worry about higher transmission modes. At the junction the transverse voltage and the longitudinal current must be continuous. This requirement is responsible for reflection unless the characteristic impedances of the two guides are equal. The coefficients of reflection and transmission depend on the impedance ratio $K_{V, I}^{\prime} / K_{V, I}^{\prime \prime}$ of the two wave guides. As we shall find later the effect of the geometric discontinuity can be calculated equally well by concentrating attention on normalized voltages and currents. With respect to these variables the characteristic impedances of the above
wave guides are equal; but at the wave guide junction there will exist an effective ideal transformer with the impedance transformation ratio equal to $K_{V, I}^{\prime \prime} / K_{V, I}^{\prime}$. In the case of ordinary low frequency transmission lines we prefer to think in terms of total voltages and currents; to think in terms of normalized voltages and currents would be to make simple matters complicated; but it will presently become evident that, in general, it is advantageous to introduce the normalized variables at least in certain stages of the analysis.

Take an iris in a rectangular wave guide. We know that for frequencies between the lowest cut-off frequency and the next higher, the iris can be represented as a shunt susceptance. The value of this susceptance will depend on its definition; but the ratio to the corresponding characteristic admittance of the guide is an invariant. It is this ratio that appears in transmission formulas involving lumped elements inserted in a uniform transmission line. If, however, the iris is between circular and rectangular wave guides, the ratio of the characteristic impedances of the two guides will also be involved and this ratio depends on whether both impedances are defined on the power-voltage basis or the power-current basis. It is evident, therefore, that in this case the iris cannot behave as a simple shunt susceptance. The theory which we are now evolving permits us to prove that in the more general case the equivalent transducer for the iris consists of two shunt susceptances, corresponding to the two faces of the iris, and an ideal transformer between them. The transformer ratio depends on the particular voltage-current set we happen to choose for our analytical work but our final transmission formulas will be independent of this choice. The degree of arbitrariness involved in the choice of " $V$ " and " $I$ " is of the same kind as that involved in the choice of coordinate systems or of units. In elementary analysis, a particular choice was so natural that a mistaken notion spread abroad that this choice was a necessary one.
6. An iris between two wave guides. Let us now obtain an exact equivalent circuit for an infinitely thin perfectly conducting iris between two wave guides of arbitrary


Fig. 3.
cross-section (Fig. 3). The constants of this circuit depend on the particular transmission mode under consideration; that is, there is one equivalent circuit for transition from each transmission mode in wave guide 1 to each mode in wave guide 2 . The most important case is that of transition from the dominant mode in one wave guide to the dominant mode in the other, and in the following analysis we shall keep this case specifically in mind; but the analysis applics to any other case. We shall use Cartesian coordinates in our equations; but this does not mean that our analysis is restricted to rectangular guides.

Suppose that the transverse field of the incident wave at the surface of the iris is

$$
\begin{align*}
& E_{l}^{i}(x, y)=V_{i}^{i} F_{1}(x, y) \\
& H_{l}^{i}(x, y)=M_{1} V_{1}^{i} F_{1}(x, y), \tag{6-1}
\end{align*}
$$

where $V_{1}^{i}$ is the normalized incident voltage. In response to this impressed field, we shall have some field over the aperture of the iris. Let $f(x, y)$ be the tangential electric intensity over the aperture; then in wave guide 2 the "transmitted" tangential electric intensity is

$$
\begin{align*}
\widehat{E}_{l}^{t}(x, y) & =f(x, y) & & \text { over the aperture }  \tag{6-2}\\
& =0 & & \text { over the screen } .
\end{align*}
$$

In wave guide 1 the total tangential electric intensity, that is, the sum of the incident and the reflected intensity, must be

$$
\begin{array}{rlrl}
V_{1}^{i} F_{1}(x, y)+E_{t}^{r}(x, y) & =f(x, y) & & \text { over the aperture, } \\
& =0 \quad \text { over the screen. } \tag{6-3}
\end{array}
$$

The function defined by (6-2) may be expanded into a series of normalized orthogonal functions appropriate to wave guide 2 ; thus

$$
\begin{equation*}
\widehat{E}_{l}^{t}(x, y)=\sum_{n=1}^{\infty} \widehat{V}_{n} \widehat{F}_{n}(x, y) \tag{6-4}
\end{equation*}
$$

The tangential magnetic intensity is then

$$
\begin{equation*}
\widehat{H}_{l}^{l}(x, y)=\sum_{n=1}^{\infty} \widehat{M}_{n} \widehat{Y}_{n} \widehat{F}_{n}(x ; y) \tag{6-5}
\end{equation*}
$$

The function defined by (6-3) can be expanded into a series of normalized orthogonal functions appropriate to the wave guide 1 ; thus

$$
\begin{equation*}
V_{1}^{i} F_{1}(x, y)+E_{t}^{r}(x, y)=\sum_{n=1}^{\infty} V_{n} F_{n}(x, y) \tag{6-6}
\end{equation*}
$$

The reflected tangential intensity is therefore

$$
\begin{equation*}
E_{l}^{r}(x, y)=\left(V_{1}-V_{1}^{i}\right) F_{1}(x, y)+\sum_{n=2}^{\infty} V_{n} F_{n}(x, y) \tag{6-7}
\end{equation*}
$$

The corresponding tangential magnetic intensity is then

$$
\begin{equation*}
H_{l}^{r}(x, y)=-M_{1}\left(V_{1}-V_{1}^{i}\right) F_{1}(x, y)-\sum_{n=2}^{\infty} M_{n} V_{n} F_{n}(x, y) \tag{6-8}
\end{equation*}
$$

The transfer of complex power through the aperture must be continuous; therefore

$$
\begin{equation*}
2 M_{1} V_{1}^{i} V_{1}^{*}-\sum_{n=1}^{\infty} M_{n} V_{n} V_{n}^{*}=\sum_{n=1}^{\infty} \widehat{M}_{n} \widehat{V}_{n} \widehat{V}_{n}^{*} \tag{6-9}
\end{equation*}
$$

The voltage reflection coefficient $q_{v}$ is defined as the ratio of the reflected voltage $V_{1}-V_{1}^{i}$ to the incident voltage $V_{1}^{1}$; it may be obtained from (6-9) if we divide the equation by $2 M_{1} V_{1} V_{1}^{*}$; thus

$$
\begin{equation*}
\frac{1}{1+q_{V}}=\frac{V_{1}^{i}}{V_{1}}=\frac{1}{2}\left[1+\sum_{n=2}^{\infty} \frac{M_{n} V_{n} V_{n}^{*}}{M_{1} V_{1} V_{1}^{*}}\right]+\frac{\widehat{M}_{1} \widehat{V}_{1} \widehat{V}_{1}^{*}}{2 M_{1} V_{1} V_{1}^{*}}\left[1+\sum_{n=2}^{\infty} \frac{\widehat{M}_{n} \widehat{V}_{n} \widehat{V}_{n}^{*}}{\widehat{M}_{1} \widehat{V}_{1} \widehat{V}_{1}^{*}}\right] . \tag{6-10}
\end{equation*}
$$

Consider now the complex power flow into the second wave guide

$$
\begin{equation*}
W^{*}=\frac{1}{2} \widehat{M}_{1} \widehat{V}_{1} \widehat{V}_{1}^{*}+\frac{1}{2} \sum_{n=2}^{\infty} \widehat{M}_{n} \widehat{V}_{n} \widehat{V}_{n}^{*} \tag{6-11}
\end{equation*}
$$

The form of this expression is such that from the input end various transmission modes appear to be in parallel. It is not exactly that the characteristic admittances $\widehat{M}_{1}, \widehat{M}_{2}, \widehat{M}_{3}, \cdots$ are directly in parallel; if we select the first admittance for reference, the others are transformed in the ratio $\widehat{V}_{n} \widehat{V}_{n}^{*} / \widehat{V}_{1} \widehat{V}_{1}^{*}$ before being connected in parallel. In any case the net effect on the input admittance is the same as would be obtained if we had an admittance $\widehat{Y}$ in shunt with a transmission line maintaining only the dominant mode. Thus we can write

$$
\begin{equation*}
W=\frac{1}{2} \widehat{M}_{1} \widehat{V}_{1} \widehat{V}_{1}^{*}+\frac{1}{2} \widehat{Y} \widehat{V}_{1} \widehat{V}_{1}^{*}, \tag{6-12}
\end{equation*}
$$

where

$$
\begin{equation*}
\widehat{Y}=\sum_{n=2}^{\infty} \widehat{M}_{n} \frac{\widehat{V}_{n} \widehat{V}_{n}^{*}}{V_{1} V_{1}^{*}} . \tag{6-13}
\end{equation*}
$$

The ratio of the shunt admittance to the characteristic admittance

$$
\begin{equation*}
\frac{Y}{M}=\frac{\widehat{Y}}{\widehat{M}_{1}}=\sum_{n=2}^{\infty} \frac{\widehat{M}_{n} \widehat{V}_{n} \widehat{V}_{n}^{*}}{\widehat{M}_{1} \widehat{V}_{1} \widehat{V}_{1}^{*}} \tag{6-14}
\end{equation*}
$$

is an invariant. It has the same value regardless of a particular basis for definition of admittances and it depends only on the form of distribution of the tangential electric intensity over the aperture.

Similarly for the admittance ratio looking from the iris into wave guide 1 we have

$$
\begin{equation*}
\frac{V}{M_{1}}=\sum_{n=2}^{\infty} \frac{M_{n} V_{n} V_{n}^{*}}{M_{1} V_{1} V_{1}^{*}} . \tag{6-15}
\end{equation*}
$$

If the frequency is in the interval between the lowest cutoff frequency and the next higher, then $M_{2}, M_{3}, \cdots$ are reactive and the shunt admittances are pure susceptances

$$
\begin{equation*}
Y=i B, \quad \widehat{Y}=i \widehat{B} . \tag{6-16}
\end{equation*}
$$

For frequencies higher than the second cutoff frequency an iris entails some power loss to the dominant wave. The lost power is contributed to one or more higher transmission modes. This is analogous to what happens when a doublet antenna is inserted in shunt with a parallel pair or at the end of it. The plane wave guided by the parallel pair loses power; this power is then carried away by a spherical wave which originates at the junction. One mode of energy transmission is partly transformed into another. Usually there is also an energy exchange between a local field and the plane wave; this results in reactance. At lower frequencies an ordinary coil
(or a capacitor) inserted in a transmission line acts just like an iris; electrically it does not matter just what physical means we happen to provide for a local storage of energy.

We now can rewrite (6-10) as follows

$$
\begin{equation*}
\frac{1}{1+q_{V}}=\frac{1}{2}\left(1+\frac{Y}{M_{1}}\right)+\frac{1}{2} n n^{*} \frac{\widehat{M}_{1}}{M_{1}}\left(1+\frac{\widehat{Y}}{\widehat{M}_{1}}\right) \tag{6-17}
\end{equation*}
$$

where

$$
\begin{equation*}
n n^{*}=\frac{\widehat{V}_{1} \widehat{V}_{1}^{*}}{V_{1} V_{1}^{*}} \tag{6-18}
\end{equation*}
$$

The reciprocal of the voltage transmission coefficient is

$$
\begin{equation*}
\frac{1}{p_{V}}=\frac{V_{1}^{i}}{\widehat{V}_{1}}=\frac{1}{1+q_{V}} \cdot \frac{V_{1}}{\widehat{V}_{1}}=\frac{1}{n}\left[\frac{1}{2}\left(1+\frac{Y}{M_{1}}\right)+\frac{1}{2} m n^{*} \frac{\widehat{M}_{1}}{M_{1}}\left(1+\frac{\widehat{V}}{\widehat{M}_{1}}\right)\right] \tag{6-19}
\end{equation*}
$$

It is a simple matter to prove ${ }^{2}$ that for transmission lines coupled as indicated in Fig. $4, p_{V}$ and $q_{V}$ are given precisely by equations (6-17) and (6-19). In Fig. 4 the


Fig. 4.
transformer ratio $1: n^{2}$ is indicated for the impedances rather than for the admittances in order to conform to the established practice. If $n=1$, which is always the case when the wave guides on both sides of the iris are the same, the admittances $Y$ and $\widehat{Y}$ of the two faces of the iris are just in parallel, and the transformer can be omitted.

The exact numerical values of $n, Y / M_{1}, \widehat{Y} / \widehat{M}_{1}$ are found by solving the appropriate boundary value problems. ${ }^{3}$ The approximate values can be obtained quite easily if we assume a reasonable form of distribution of the tangential electric intensity over the aperture, ${ }^{4}$ and of course, we can always calculate these quantities from measurements of the transmission and reflection coefficients for waves moving from one wave guide into the other. Thus,

$$
\begin{equation*}
n^{*}=\frac{1+a_{V}}{1+q_{V}^{+}} \frac{M_{1}}{\widehat{M}_{1}}=\frac{1+q_{\bar{v}}}{1+q_{v}^{+}} \frac{\widehat{K}_{1}}{K_{1}} \tag{6-20}
\end{equation*}
$$

where $q_{\psi}^{+}$is the voltage reflection coefficient for a wave moving from left to right and $q_{\bar{v}}$ is that for a wave moving in the opposite direction.

[^2]If the iris is not indefinitely thin, there is a section of a wave guide between the two faces of the iris.

While the iris acts effectively as a lumped impedance, the field associated with it is actually distributed. Even if the frequency is such that the iris is reactive, the field extends to some distance on either side of it. Near the cutoff for the second transmission mode this distance may be quite large; but ordinarily the field extends roughly to a distance comparable to the transverse dimensions of the guide. There will exist, therefore, a mutual impedance between those faces of two nearby irises which face each other. For frequencies above the second cutoff, the mutual impedance may, and usually will, exist even between two distant irises. All these considerations do not affect our essential picture of electrical properties of wave guide discontinuities; they affect merely the numerical values of various impedance and admittance functions.

In the above equations we have treated $E_{t}$ and $H_{t}$ as if they were scalars; in general, they are vectors. However, the analysis is similar to the above and the final formulae are the same.

In the case of coaxial pairs or wave guides formed by parallel metal strips the dominant wave is transverse electromagnetic. If the edges of the iris are normal to the lines of force for the dominant wave, the voltage between the edges is equal to the transverse voltage across either guide; the total voltages associated with higher transmission modes are equal to zero; and the transformer ratio is unity provided we base our transmission diagram not on the normalized characteristic impedance but on the conventional impedance $K$ which in this case equals $K_{V, r}, K_{W, I}$ and $K_{W, V}$.
7. Reactances in series with wave guides. An example of a reactance effectively in series with the wave guide is shown in Fig. 5 which represents a circular wave guide


Fig. 5.
and a narrow radial transmission line. ${ }^{5}$ Let us suppose that we are concerned with transmission of a $T M_{0,1}$-wave. For this wave the field is circularly symmetric. Magnetic lines are circles coaxial with the tube, and electric lines are in radial planes. It is practically self-evident that the radial line is in series with the guide, and that in parallel with the radial line there is an impedance associated with the gap. If the frequency is between the lowest cutoff frequency and the next higher, this "gap impedance" or fringing impedance is capacitive and is


Fig. 6. of little importance except when the impedance of the radial transmission line is high. For frequencies above the second cutoff, the gap impedance is in part resistive on account of power transfer from the dominant wave to the higher order waves. As seen from the gap, the impedances of various waves in the guide and the impedance of the radial wave are in parallel; the two halves of the guide are in series; and the impedance diagram looks like that shown
in Fig. 6. The same diagram is shown in Fig. 7 where the characteristic impedances • $K_{1}$ and $K_{1}$ have been "expanded" into semi-infinite transmission lines; the impedance consisting of $2 K_{2}, 2 K_{3}, \cdots$ in parallel is represented simply as the gap impedance $Z_{0}$.


Fig. 7.
Starting with equation (10.17-1) of "Electromagnetic Waves," we can obtain the approximate gap impedance by the method explained there. In this case, however, the following elementary derivation is preferable. To begin with, let us remove the radial line and assume that the electric charge is being transferred across the gap by an impressed voltage $V^{i}$. The total conduction current $I$ in the tube is the sum of currents associated with the various transmission modes. Thus for the input current we have

$$
\begin{equation*}
I=I_{1}+I_{2}+I_{3}+\cdots . \tag{7-1}
\end{equation*}
$$

The input power is then

$$
\begin{align*}
W=\frac{1}{2} V^{i} I^{*} & =\frac{1}{2} V^{i} I_{1}^{*}+\frac{1}{2} V^{i} I_{2}^{*}+\cdots \\
& =\frac{1}{2} Z_{1} I_{1} I_{1}^{*}+\frac{1}{2} Z_{2} I_{2} I_{2}^{*}+\cdots, \tag{7-2}
\end{align*}
$$

where $Z_{1}, Z_{2}, \cdots$ are the input impedances of individual waves; that is,

$$
\begin{equation*}
Z_{1}=\frac{V^{i}}{I_{1}}, \quad Z_{2}=\frac{V^{i}}{I_{2}}, \cdots \tag{7-3}
\end{equation*}
$$

In the above equations we have tacitly assumed that the gap is very small and the current associated with each mode does not vary in the gap. This restriction will presently be removed. The total power contributed to the wave is divided between different modes; one-half of it is carried to the left and the other half to the right. The power carried in one direction in the $n$th mode is $\left.\frac{1}{2} K\right)_{V, I} I_{n} I_{n}{ }^{*}$; thus we have

$$
\begin{equation*}
Z_{n}=2 K_{W, r}^{(n)} . \tag{7-4}
\end{equation*}
$$

Actually the applied voltage is distributed in the interval $(-s / 2, s / 2)$ around the midpoint $z=0$. Assuming that the distribution is uniform, we may write the contribution to the total current associated with the $n$th wave at point $z$ due to an elementary voltage at point $\hat{z}$ as follows

$$
\begin{equation*}
I_{n} e^{-r_{n}|z-\hat{z}|} \frac{d \hat{z}}{s}, \tag{7-5}
\end{equation*}
$$

where $I_{n}$ is the amplitude at the source. The total current at point $z$ is then

$$
\begin{equation*}
I_{n}(z)=\frac{I_{n}}{s} \int_{-\theta / 2}^{0 / 2} e^{-\Gamma_{n}|z-\xi|} \frac{d \hat{z}}{s} \tag{7-6}
\end{equation*}
$$

The power contributed to the $n$th wave is then

$$
\begin{equation*}
W_{n}=\frac{1}{2} V^{i} \int_{-z / 2}^{s / 2} I_{n}^{*}(z) \frac{d z}{s} \tag{7-7}
\end{equation*}
$$

Thus we shall have

$$
\begin{equation*}
W_{n}=\frac{1}{2} \chi_{n} V^{i} I_{n}^{*}, \tag{7-8}
\end{equation*}
$$

where (assuming that $\Gamma_{n}$ is real)

$$
\begin{equation*}
\chi_{n}=\frac{1}{s^{2}} \int_{-s / 2}^{s / 2} d z \int_{-s / 2}^{s / 2} e^{-\Gamma_{n}|z-\hat{z}|} d z=\frac{2}{\Gamma_{n} s}-\frac{2\left(1-e^{-\Gamma_{n} s}\right)}{\Gamma_{n}^{2} s^{2}} . \tag{7-9}
\end{equation*}
$$

On the other hand

$$
\begin{equation*}
W_{n}=K_{W, I}^{(n)} I_{n} I_{n}^{*} ; \tag{7-10}
\end{equation*}
$$

consequently

$$
\begin{equation*}
\frac{V^{i}}{I_{n}}=\frac{2 K_{W, I}^{(n)}}{\chi_{n}} \tag{7-11}
\end{equation*}
$$

Since $\chi_{n}$ decreases with increasing $n$, the successive components of the gap admittance
decrease.

$$
\begin{equation*}
Y_{0}=Y_{2}+Y_{3}+Y_{4}+\cdots=\frac{\chi_{2}}{2 K_{W, I}^{(2)}}+\frac{\chi_{3}}{2 K_{W, I}^{(3)}}+\frac{\chi_{4}}{2 K_{W, I}^{(4)}}+\cdots \tag{7-12}
\end{equation*}
$$

The typical $K_{W, I}^{(n)}$ is given in problem 8.10 on page 509 of "Electromagnetic Waves"

$$
\begin{equation*}
K_{W, I}^{(n)}=\frac{\Gamma_{n}}{4 \pi i \omega \epsilon}, \quad \Gamma_{n}=\left[\frac{k_{n}^{2}}{a^{2}}-\frac{4 \pi^{2}}{\lambda^{2}}\right]^{1 / 2}, \tag{7-13}
\end{equation*}
$$

where $k_{n}$ is the $n$th zero of $J_{0}(x)$. For sufficiently large $n$, therefore, we have

$$
\begin{equation*}
Z_{n}=\Gamma_{n} K_{w, I}^{(n)} s=\frac{\Gamma_{n}^{2} s}{4 \pi i \omega \epsilon}=\frac{k_{n}^{2} s}{4 \pi i \omega \epsilon a^{2}} \tag{7-14}
\end{equation*}
$$

The impedance of the radial line is approximately

$$
\begin{equation*}
Z=60 i \frac{s}{a} \tan \frac{2 \pi l}{\lambda} \tag{7-15}
\end{equation*}
$$

A more accurate expression in terms of Bessel functions may be found on page 269 of. "Electromagnetic Waves."
8. Conclusion. The ideas developed in this paper are adequate for expressing transmission properties of wave guides with discontinuities in terms of impedances and admittances associated with these discontinuities. These impedances are reactive if the frequency is such that the energy in either guide can be transmitted to any distance in only one mode; otherwise, the discontinuities present some resistance for the mode under consideration and a negative resistance to those other modes which
participate in transmission of energy. The finding of exact values of impedances requires solution of corresponding boundary value problems; but frequently good approximations can be found by making reasonable a priori assumptions on physical grounds. In fact, the point of view outlined in this paper makes it easy to make such assumptions.

More complex discontinuities can be analyzed into simpler discontinuities. The discontinuity shown in Fig. 8 is equivalent to an ideal transformer between two wave guides; across the left "winding" of which there is a small shunt capacitance ${ }^{6}$ and across the right winding there is the capacitance ${ }^{7}$ associated with the annular disc


Fig. 9.
looking into the second guide. In parallel with the latter capacitance there is the series combination of the impedance of the radial line and the second guide itself. We may express these ideas by the diagram shown in Fig. 9, where the inductance is used to designate the radial transmission line only because this line, when it is short, is approximately an inductance.

More generally, the discontinuities should be represented by impedances distributed along the guide, as in fact they are. Finally, the section of the guide with the discontinuities may be replaced by an appropriate $T$-network.

Recently J. R. Whinnery and H. W. Jamieson ${ }^{8}$ have obtained explicit expressions for the capacitances of numerous types of "step discontinuities" in transmission lines formed by parallel conducting planes. They show how to apply these results to coaxial conductors. They find theoretical predictions in good agreement with measured values. The equivalent circuits given by Whinnery and Jamieson do not contain ideal transformers; this is because for transmission lines comprised of two conductors, the transformer ratios at discontinuities are equal to unity and the transformers may be omitted.

[^3]
# THE DISTORTION OF THE BOUSSINESQ FIELD DUE TO A CIRCULAR HOLE* 

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1. Introduction. One of the most important problems in the theory of elasticity is the solution of the biharmonic equation $\nabla^{2} \phi=0$, where $\phi$ is Airy's stress function, for a given group of boundary conditions. As is well known, the most common approach to the solution of this problem consists in selecting a system of coordinates particularly suited to the region studied.

Thus, using bipolar coordinates, G. B. Jeffery has given the general solution of the plane problem, that is, of the biharmonic equation in two dimensions, for regions bounded by non-concentric circles (Ref. 1). A clear, but not quite complete, treatment of Jeffery's method can be found in Coker and Filon (Ref. 2). This method has recently been used by R. D. Mindlin for the determination of dead loads on tunnels (Ref. 3).

The present paper is an attempt to apply Jeffery's approach to the problem of the distortion introduced in the so-called plane Boussinesq field by the presence of a circular hole. Starting with the stress function $\phi$ of the undistorted Boussinesq field, an auxiliary stress function $\chi$ will be found such that $\Phi=\phi+\chi$ satisfies the differential equation and all the boundary conditions. The stresses and strains in the discontinuous field can then be directly determined from the derivatives of $\Phi$.
2. The Boussinesq field. Boussinesq and Flamant have given the solution of the biharmonic equation for the case of an isolated force $P$ acting at a point on the boundary of a semi-infinite plane. Their solution, which can be found in all standard texts (see, for instance, Ref. 4, p. 82) is:

$$
\begin{equation*}
\phi_{1}=-\frac{P}{\pi} r \theta \sin \theta \tag{1a}
\end{equation*}
$$

for the case of a normal force, and

$$
\begin{equation*}
\phi_{2}=-\frac{P}{\pi} r \theta \cos \theta \tag{1b}
\end{equation*}
$$

for a force parallel to the boundary. The significance of the symbols is shown in Figs. 1 a and 1 b .

In the simple Boussinesq problem, the only boundary conditions are that the stresses, both normal and shearing, must vanish along the straight boundary (except, of course, at the point of application of the force) and also must tend to zero as one

[^4]moves away from the point of application within the half-plane. With the appearance of the circular discontinuity (Fig. 2), the above conditions remain, and a new one is


Fig. la.


Fig. 1 b.
added, determined by the nature of the discontinuity. Thus, if it is a hole, both normal and shearing stresses must vanish along its periphery.

The region is thus bounded by a circle and a straight line; the latter can be considered as a circle of infinite radius, so that here is a case of a region bounded by two non-concentric circles, to which Jeffery's method is applicable.
3. Bipolar coordinates. Jeffery's method consists essentially in introducing a system of curvilinear coordinates, called bipolar coordinates in works on elasticity. Two poles, $A$ and $B$ (Fig. 3) are taken at abscissas $\pm a$


Fig. 2. along the $X$-axis, and the location of any point is determined with respect to these poles by the quantities

$$
\xi=\log \frac{r_{1}}{r_{2}} \quad \eta=\theta_{1}-\theta_{2}
$$

The lines $\eta=$ constant are circles passing through $A$ and $B$, while $\xi=$ constant are a system of circles with centers on the $X$-axis. Some of these lines are drawn on Fig. 3.

If a circle of diameter $d$ has its center $h$ units from the horizontal axis (Fig. 2), it is easy to show (see Ref. 1) that the proper polar distance $a$ is determined from $a^{2}=h^{2}-d^{2} / 4$ and that the value $\xi_{0}$ of $\xi$ corresponding to the circle is $\xi_{0}=\cosh ^{-1} 2 h / d$. The cartesian coordinates can be expressed as follows in terms of the bipolar:

$$
\begin{equation*}
x=-\frac{a \sinh \xi}{\cosh \xi-\cos \eta}, \quad y=\frac{a \sin \eta}{\cosh \xi-\cos \eta} \tag{2}
\end{equation*}
$$

When the biharmonic equation is expressed in bipolar coordinates, it is found convenient to write it, not in terms of the usual stress function $\chi$, but in terms of $\chi / J$, where $J$ has the value

$$
J=\frac{a}{\cosh \xi-\cos \eta}
$$

the stresses are also expressed as derivatives of $\chi / J$.

The bipolar solution for $\chi / J$ used by Jeffery has the general form:

$$
\begin{align*}
\chi / J= & B \xi \cosh \xi+(-B \xi+G \cosh 2 \xi+H \sinh 2 \xi+F) \cos \eta \\
& +\left(G^{\prime} \cosh 2 \xi+H^{\prime} \sinh 2 \xi+F^{\prime}\right) \sin \eta \\
& +\sum_{k=2}^{\infty}\left\{\left[E_{k} \cosh (k+1) \xi+F_{k} \sinh (k+1) \xi+G_{k} \cosh (k-1) \xi\right.\right. \\
& \left.+H_{k} \sinh (k-1) \xi\right] \cos k_{\eta}+\left[E_{k}^{\prime} \cosh (k+1) \xi+F_{k}^{\prime} \sinh (k+1) \xi\right. \\
& \left.\left.+G_{k}^{\prime} \cosh (k-1) \xi+H_{k}^{\prime} \sinh (k-1) \xi\right] \sin k \eta\right\} \tag{3}
\end{align*}
$$

where all the $B$ 's, $E$ 's, $F$ 's, $G$ 's and $H$ 's are constants. This series will be assumed convergent and differentiable for the time being.

Here the terms independent of $\eta$ and those containing $\cos \eta$ or $\sin \eta$ are used exactly as they appear in Ref. 2 (Eq. 4.066 and paragraph 4.07 ), but those containing functions of multiples of $\eta$ come directly from Ref. 1 (Eq. 21), with some slight changes in nomenclature.


Fig. 3.
4. General procedure. As was said before, the presence of the circular discontinuity causes a modification of the Boussinesq functions $\phi_{1}$ and $\phi_{2}$ into $\Phi_{1}$, and $\Phi_{2}$ the latter having to satisfy the biharmonic equation and all boundary conditions Also $\Phi_{1}=\phi_{1}+\chi_{1}, \Phi_{2}=\phi_{2}+\chi_{2}$ where $\chi_{1}$ and $\chi_{2}$ are auxiliary stress functions of the general form (3). Now since both $\phi$ 's and both $\chi$ 's satisfy the biharmonic equation, which is linear, so do $\Phi_{1}$ and $\Phi_{2}$. As to the boundary conditions, $\phi_{1}$ and $\phi_{2}$ satisfy them along the straight boundary and for remote points. Therefore, $\chi_{1}$ and $\chi_{2}$ must be so selected that:
(1) they give vanishing stresses for remote points ( $\xi \rightarrow 0, \eta \rightarrow 0$ );
(2) they give zero normal and shearing stresses along the straight boundary;
(3) in combination with the known functions $\phi_{1}$ and $\phi_{2}$ they satisfy the boundary conditions at the circular discontinuity.
In the next paragraphs, conditions (1) and (2) will be considered first and their application will determine some of the hitherto arbitrary constants of Eq. (3). Then the function $\chi$ satisfying conditions (1) and (2) will be added to $\phi_{1}$ or $\phi_{2}$ (according to whether a normal or a tangential load is studied), yielding

$$
\Phi_{1}=\phi_{1}+\chi, \quad \Phi_{2}=\phi_{2}+\chi .
$$

Finally the remaining constants of $\chi$ will be determined in each case by the conditions at the inner boundary.
5. First and second boundary conditions. The stresses are expressed as follows in terms of bipolar coordinates:

$$
\begin{align*}
& a \sigma_{\xi}=\left[(\cosh \xi-\cos \eta) \frac{\partial^{2}}{\partial \eta^{2}}-\sinh \xi \frac{\partial}{\partial \xi}-\sin \eta \frac{\partial}{\partial \eta}+\cosh \xi\right]\left(\frac{\chi}{J}\right), \\
& a \sigma_{\eta}=\left[(\cosh \xi-\cos \eta) \frac{\partial^{2}}{\partial \xi^{2}}-\sinh \xi \frac{\partial}{\partial \xi}-\sin \eta \frac{\partial}{\partial \eta}+\cos \eta\right]\left(\frac{\chi}{J}\right),  \tag{4}\\
& a \tau_{\xi \eta}=-(\cosh \xi-\cos \eta) \frac{\partial^{2}}{\partial \xi \partial \eta}\left(\frac{\chi}{J}\right) .
\end{align*}
$$

The first condition necessitates

$$
\sigma_{\xi}=0, \sigma_{\eta}=0, \tau_{\xi \eta}=0 \text { for }(\xi, \eta) \rightarrow 0
$$

and the second

$$
\sigma_{\xi}=0, \tau_{\xi n}=0 \text { for } \xi=0 .
$$

The first condition is seen from Eqs. (4) to be equivalent to

$$
\frac{\chi}{J}=0 \text { for }(\xi, \eta) \rightarrow 0
$$

from which, immediately

$$
G+F=0, G=-F \text { and } E_{k}+G_{k}=0, G_{k}=-E_{k} .
$$

For the second condition, $\tau_{\xi}=0$ for $\xi=0$, which yields

$$
H=\frac{B}{2}, \quad H^{\prime}=0, \quad H_{k}=-\frac{k+1}{k-1} F_{k}, \quad H_{k}^{\prime}=-\frac{k+1}{k-1} F_{k}^{\prime} ;
$$

and from $\sigma_{\xi}=0$ for $\xi=0$,

$$
G_{k}^{\prime}=-E_{k}^{\prime} .
$$

Thus the stress function satisfying boundary conditions (1) and (2) assumes the form: $\frac{\chi}{J}=B \xi \cosh \xi-\left[B(\xi-\sinh \xi \cosh \xi)+2 F \sinh ^{2} \xi\right] \cos \eta+\left(G^{\prime} \cosh 2 \xi+F^{\prime}\right) \sin \eta$
$+\sum_{k=2}^{\infty} \frac{2}{k-1}\left\{\left[E_{k}(k-1) \sinh \xi \sinh k \xi\right.\right.$
$\left.+F_{k}(k \sinh \xi \cosh k \xi-\cosh \xi \sinh k \xi)\right] \cos k \eta$
$\left.+\left[E_{k}^{\prime}(k-1) \sinh \xi \sinh k \xi+F_{k}^{\prime}(k \sinh \xi \cosh k \xi-\cosh \xi \sinh k \xi)\right] \sin k \eta\right\}$.
6. Third boundary condition. The value of $\chi / J$ from (5) is now added to $\phi_{1}$ (for normal load) or $\phi_{2}$ (tangential load), and the remaining arbitrary constants of (5) determined by the conditions at the boundary of the hole, which are that both normal and shear stresses vanish on the periphery, i.e. $\left(\sigma_{\xi}\right)_{\varepsilon_{0}}=0$, and $\left(\tau_{\xi_{\eta}}\right)_{\xi_{0}}=0$.

In order to coordinate the functions $\phi_{1}, \phi_{2}$, on one hand, and $\chi_{1}, \chi_{2}$, on the other, the system of axes shall be selected so that the $y=0$ (or $\eta=0$ ) axis passes through the center of the hole, as shown in Figs. 2 and 3. Then the concentrated force, whether normal or tangential, will act at a point $y=y_{0}$, and the stress functions $\phi_{1}$ and $\phi_{2}$ become

$$
\phi_{1}=-\frac{P}{\pi}\left(y-y_{0}\right) \tan ^{-1} \frac{y-y_{0}}{x}, \quad \phi_{2}=-\frac{P}{\pi} \tan ^{-1} \frac{y-y_{0}}{x} .
$$

Transforming this into bipolar coordinates (Eq. 2) one has

$$
\begin{aligned}
& \phi_{1}=\frac{P}{\pi} \frac{y_{0}(\cosh \xi-\cos \eta)-a \sin \eta}{\cosh \xi-\cos \eta} \tan ^{-1} \frac{y_{0}(\cosh \xi-\cos \eta)-a \sin \eta}{a \sinh \xi} \\
& \phi_{2}=\frac{P}{\pi} \frac{a \sinh \xi}{\cosh \xi-\cos \eta} \tan ^{-1} \frac{y_{0}(\cosh \xi-\cos \eta)-a \sin \eta}{a \sinh \xi}
\end{aligned}
$$

But, as was said before, in treating problems involving bipolar coordinates, it is easier to express stresses not in terms of the stress function itself, but in terms of the stress function divided by the quantity $J$, so that:

$$
\begin{aligned}
& \phi_{1} / J=\frac{P}{\pi a}\left[y_{0}(\cosh \xi-\cos \eta)-a \sin \eta\right] \tan ^{-1} \frac{y_{0}(\cosh \xi-\cos \eta)-a \sin \eta}{a \sinh \xi} \\
& \phi_{2} / J=\frac{P}{\pi} \sinh \xi \tan ^{-1} \frac{y_{0}(\cosh \xi-\cos \eta)-a \sin \eta}{a \sinh \xi}
\end{aligned}
$$

These two expressions must now be written in Fourier series in $\eta$ to be comparable with the auxiliary functions $\chi_{1}$ and $\chi_{2}$ of (5). The coefficients of these series are found by means of the usual integrations which are presented in detail in the Appendix. The results are as follows:

$$
\phi_{1} / J=\frac{T_{0}}{2}+\sum_{k=1}^{\infty}\left(T_{k} \cos k \eta+U_{k} \sin k \eta\right)
$$

with

$$
\frac{T_{0}}{2}=\frac{P}{\pi}\left\{\tan \beta \cosh \xi\left[\tan ^{-1}(\tan \beta \operatorname{coth} \xi)+\left(\frac{\pi}{2}-\beta\right) \sinh \xi\right]-e^{\xi} \frac{1+\cos \beta}{2 \cos \beta}\right\}
$$

$$
\begin{aligned}
& T_{1}=\frac{P}{\pi} \tan \beta\left[e^{\xi} \sin \beta \cosh \xi-\tan ^{-1}(\tan \beta \operatorname{coth} \xi)-\left(\frac{\pi}{2}-\beta\right) \sinh \xi\right] \\
& U_{1}=\frac{P}{\pi}\left\{\frac{\tan \beta}{2}\left[1+\left(1+e^{2 \xi}\right) \cos \beta\right]-\tan ^{-1}(\tan \beta \operatorname{coth} \xi)-\left(\frac{\pi}{2}-\beta\right) \sinh \xi\right\}
\end{aligned}
$$

and for $k \geqq 2$

$$
\begin{gathered}
T_{k}=-\frac{P e^{k \xi}}{\pi}\left\{(-1)^{k} \frac{\tan \beta \sin k \beta}{k} \cosh \xi+\frac{k \sinh \xi-\cosh \xi}{k^{2}-1}\left[1+(-1)^{k} \frac{\cos k \beta}{\cos \beta}\right]\right\} \\
U_{k}=-\frac{P e^{k \xi}}{\pi}\left\{(-1)^{k} \frac{\tan \beta \cos k \beta}{k} \cosh \xi-\frac{k \sinh \xi-\cosh \xi}{k^{2}-1}\left[\frac{\tan \beta}{k}+(-1)^{k} \frac{\sin k \beta}{\cos \beta}\right]\right\} \\
\phi_{2} / J=\frac{R_{0}}{2}+\sum_{k=1}^{k-\infty}\left(R_{k} \cos k \eta+S_{k} \sin k \eta\right)
\end{gathered}
$$

where

$$
\begin{aligned}
& \frac{R_{0}}{2}=\frac{P \sinh \xi}{\pi}\left[\tan ^{-1}(\tan \beta \operatorname{coth} \xi)+\left(\frac{\pi}{2}-\beta\right) \sinh \xi\right] \\
& R_{k}=-\frac{P \sinh \xi}{\pi} \frac{e^{k \xi}}{k}(-1)^{k} \sin k \beta \\
& S_{k}=\frac{P \sinh \xi}{\pi} \frac{e^{k \xi}}{k}\left[1-(-1)^{k} \cos k \beta\right]
\end{aligned}
$$

In these expressions $\beta=\tan ^{-1} y_{0} / a$ (Fig. 2). The above formulas refer to the case $\beta \neq 0$ (see Appendix). For the important case $\beta=0$, i.e., $y_{0}=0$

$$
\frac{T_{0}}{2}=-\frac{P}{\pi} e^{\xi}, \quad T_{1}=0, \quad U_{1}=0
$$

for $k \geqq 2$

$$
T_{k}=-\frac{2 P}{\pi} \frac{e^{k \xi(k \sinh \xi-\cosh \xi)}}{k^{2}-1} \text { for } k \text { even, } \quad T_{k}=0 \text { for } k \text { odd, } U_{k} \doteq 0
$$

and

$$
\frac{R_{0}}{2}=0, \quad R_{k}=0, \quad S_{k}=\frac{2 P}{\pi} \frac{e^{k \xi}}{k} \sinh \xi \text { for } k \text { odd, } \quad S_{k}=0 \text { for } k \text { even. }
$$

The following special cases will be considered in the next section on applications: (A) half-space containing a hole bounded by $\xi=\xi_{0}$ and subjected to normal load; (B) same region subjected to tangential load.

## APPLICATIONS

## A. Hole subject to normal load. Here

$$
\Phi_{1} / J=\phi_{1} / J+\chi_{1} / J,
$$

$\Phi_{1} / J=B \xi \cosh \xi+\frac{P}{\pi}\left\{\tan \beta \cosh \xi\left[\tan ^{-1}(\tan \beta \operatorname{coth} \xi)+\left(\frac{\pi}{2}-\beta\right) \sinh \xi\right]-e^{\left.\frac{1}{2} \frac{1+\cos \beta}{2 \cos \beta}\right\}}\right.$ $-\left\{B(\xi-\sinh \xi \cosh \xi)+2 F \sinh ^{2} \xi\right.$

$$
\left.-\frac{P}{\pi} \tan \beta\left[e^{\xi} \sin \beta \cosh \xi-\tan ^{-1}(\tan \beta \operatorname{coth} \xi)-\left(\frac{\pi}{2}-\beta\right) \sinh \xi\right]\right\} \cos \beta
$$

$+\left\{G^{\prime} \cosh 2 \xi+F^{\prime}+\frac{P}{\pi}\left[\frac{\tan \beta}{2}\left\{1+\left(1+e^{2 \xi}\right) \cos \beta\right\}\right.\right.$
$\left.\left.-\tan ^{-1}(\tan \beta \operatorname{coth} \xi)-\left(\frac{\pi}{2}-\beta\right) \sinh \xi\right]\right\} \sin \beta$
$+\sum_{k=2}^{\infty} \frac{2}{k-1}\left(E_{k}[k-1] \sinh \xi \sinh k \xi+F_{k}[k \sinh \xi \cosh k \xi-\cosh \xi \sinh k \xi]\right.$
$\left.-\frac{k-1}{2 \pi} P e^{k \xi}\left\{(-1)^{k} \frac{\tan \beta \sin k \beta}{k} \cosh \xi+\frac{k \sinh \xi-\cosh \xi}{k^{2}-1}\left[1+(-1)^{k} \frac{\cos k \beta}{\cos \beta}\right]\right\}\right) \cos k \eta$
$+\sum_{k=2}^{\infty} \frac{2}{k-1}\left(E_{k}^{\prime}[k-1] \sinh \xi \sinh k \xi+F_{k}^{\prime}[k \sinh \xi \cosh k \xi-\cosh \xi \sinh \xi]\right.$
$\left.-\frac{k-1}{2 \pi} P e^{k \xi}\left\{(-1)^{k} \frac{\tan \beta \cos k \beta}{k} \cosh \xi-\frac{k \sinh \xi-\cosh \xi}{k^{2}-1}\left[\frac{\tan \beta}{k}+(-1)^{k} \frac{\sin k \beta}{\cos \beta}\right]\right\}\right) \sin k \eta$.
The condition $\left(\tau_{\xi_{\eta}}\right)_{\xi_{0}}=-(\cosh \xi-\cos \eta) \partial^{2} / \partial \xi \partial \eta(\Phi / J)=0$ amounts to equating to zero at $\xi=\xi_{0}$ the derivative with respect to $\xi$ of each term except the one independent of $\eta$. As to $\left(\sigma_{\xi}\right)_{\xi_{0}}=0$, this can be shown to require that for $k \geqq 2$ each term be zero at $\xi=\xi_{0}$. Thus, for each term, two equations are available; and this is sufficient to find all of the remaining constants, with the exception of $F^{\prime}$ in the term in $\sin \beta$. The constant $F^{\prime}$ remains indeterminate, and can therefore be taken as equal to zero. By solving the two equations for each term, the following values are found for the constants:

$$
\begin{aligned}
& B=\frac{P}{2 \pi \sinh ^{2} \xi_{0}}\left\{e^{\xi_{0} \tan \beta \sin \beta\left(\frac{2 \cosh ^{2} \xi_{0}}{\sinh \xi_{0}}-e^{\xi_{0}}\right)-\frac{2 \sin ^{2} \beta\left(\cosh ^{2} \xi_{0}+\frac{1}{2}\right)}{\cosh ^{2} \xi_{0}-\cos ^{2} \beta}} \begin{array}{c}
\left.-(\pi-2 \beta) \tan \beta \cosh \xi_{0}\left(\cosh ^{2} \xi_{0}-\frac{1}{2}\right)-\operatorname{coth} \xi_{0} \frac{1+\cos \beta}{\cos \beta}\right\} \\
F=\frac{P}{2 \pi \sinh ^{2} \xi_{0}}\left\{e^{\xi_{0} \tan \beta \sin \beta \cosh \xi_{0}-\tan \beta \sinh \xi_{0} \cosh \xi_{0}}\right. \\
\left.\quad\left[\frac{\sin \beta \cos \beta}{\cosh ^{2} \xi_{0}-\cos ^{2} \beta}+\left(\frac{\pi}{2}-\beta\right) \cosh \xi_{0}\right]-\frac{1+\cos \beta}{2 \cos \beta}\right\} \\
\begin{array}{rl}
G^{\prime}= & \frac{-P}{2 \pi \sinh 2 \xi_{0}}\left[e^{\left.2 \xi_{0} \sin \beta+\frac{\sin \beta \cos \beta}{\cosh ^{2} \xi_{0}-\cos ^{2} \beta}-\left(\frac{\pi}{2}-\beta\right) \cosh \xi_{0}\right]}\right. \\
F^{\prime}=0,
\end{array} \\
\left.\begin{array}{r}
E_{k}=-\frac{P}{2 \pi}\left\{(-1)^{k} \tan \beta \sin k \beta\left(k \sinh \xi_{0} \cosh \xi_{0}+\sinh ^{2} \xi_{0}-e^{k \xi_{0}} \sinh k \xi_{0}\right\rfloor\right.
\end{array} \quad+\left[1-(-1)^{k} \cos k \beta / \cos \beta\right] k \sinh ^{2} \xi_{0}\right\}\left[\sinh ^{2} k \xi_{0}-k^{2} \sinh ^{2} \xi_{0}\right]^{-1},
\end{array}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \begin{array}{l}
F_{k}=-\frac{P}{2 \pi}\left\{(-1)^{k}(k-1) k^{-1} \tan \beta \sin k \beta\left[k \sinh \xi_{0} \cosh \xi_{0}-e^{k \xi_{0}} \sinh k \xi_{0}\right]\right. \\
+ \\
\left.(k+1)^{-1}\left[1+(-1)^{k} \cos k \beta / \cos \beta\right]\left[k^{2} \sinh ^{2} \xi_{0}-k \sinh \xi_{0} \cosh \xi_{0}-e^{-k \xi_{0}} \sinh k \xi_{0}\right]\right\} \\
\quad \cdot\left[\sinh ^{2} k \xi_{0}-k^{2} \sinh ^{2} \xi_{0}\right]^{-1}, \\
E_{k}^{\prime}= \\
\quad \frac{P}{2 \pi}\left\{(-1)^{k} \tan \beta \cos k \beta\left[k \sinh k \xi_{0} \cosh \xi_{0}+\sinh ^{2} \xi_{0}-e^{k \xi_{0}} \sinh k \xi_{0}\right]\right. \\
\left.\left.\quad-\left[k^{-1} \tan \beta+(-1)^{k} \sin k \beta / \cos \beta\right] k \sinh ^{2} \xi_{0}\right]\right\}\left[\sinh ^{2} k \xi_{0}-k^{2} \sinh ^{2} \xi_{0}\right]^{-1}, \\
F_{k}^{\prime}= \\
=\frac{P}{2 \pi}\left\{( - 1 ) ^ { k } ( k - 1 ) k ^ { - 1 } \operatorname { t a n } \beta \operatorname { c o s } k \beta \left[k \sinh \xi_{0} \cosh \xi_{0}+e^{\left.k \xi_{0} \sinh k \xi_{0}\right]}\right.\right. \\
\quad-(k+1)^{-1}\left[k^{-1} \tan \beta+(-1)^{k} \sin k \beta / \cos \beta\right]\left[k^{2} \sinh ^{2} \xi_{0}-k \sinh \xi_{0} \cosh \xi_{0}-e^{\left.\left.k \xi_{0} \sinh k \xi_{0}\right]\right\}}\right. \\
\quad\left[\sinh ^{2} k \xi_{0}-k^{2} \sinh ^{2} \xi_{0}\right]^{-1} .
\end{array}
\end{aligned}
$$

To test the suitability of this expansion, it is sufficient to examine the terms of the auxiliary functions (Eq. 5) for the values of the constants given above. The coefficient of $\cos k \eta$ in the general term of the latter equation is seen to consist of two parts, one multiplied by $(-1)^{k}$ and the other not. The first part forms an alternating series the general term of which tends to zero, so that the alternating series is convergent by a well-known theorem. The second part is found to converge outside the circle $\xi=\xi_{0}$ by the ratio test. The same is true for the coefficient of $\sin k \eta$. Thus the above expression for $\Phi_{1} / J$ is a uniformly convergent series in $\eta$ in the region considered.

Because of the great complexity of the expression involved, only the case $\beta=0$ will be considered in more detail. For that case
$B^{(0)}=-\frac{P}{\pi} \frac{\cosh \xi_{0}}{\sinh ^{3} \xi_{0}}, \quad F^{(0)}=-\frac{P}{2 \pi \sinh ^{2} \xi_{0}}, \quad G^{\prime(0)}=F^{\prime(0)}=0$,
$E_{k}^{(0)}=-\frac{P}{\pi} \frac{k \sinh ^{2} \xi_{0}}{\sinh ^{2} k \xi_{0}-k^{2} \sinh ^{2} \xi_{0}}$ for $k$ even, $\quad E_{k}^{(0)}=0$ for $k$ odd,
$F_{k}^{(0)}=-\frac{P}{\pi} \frac{k^{2} \sinh ^{2} \xi_{0}-k \sinh \xi_{0} \cosh \xi_{0}-e^{k \xi_{0} \sinh } k \xi_{0}}{(k+1)\left(\sinh ^{2} k \xi_{0}-k^{2} \sinh ^{2} \xi_{0}\right)^{-}}$for $k$ even, $F_{k}^{(0)}=0$ for $k$ odd,
$E_{k}^{(0)}=F_{k}^{\prime(0)}=0$.
The stress function becomes

$$
\begin{aligned}
\Phi_{1}^{(0)} / J= & -\frac{P}{\pi} \frac{\cosh \xi_{0}}{\sinh ^{3} \xi_{0}} \xi \cosh \xi-\frac{P}{\pi} e^{\xi} \\
& +\frac{P}{\pi}\left[\frac{\cosh \xi_{0}}{\sinh ^{3} \xi_{0}}(\xi-\sinh \xi \cosh \xi)+\frac{\sinh ^{2} \xi}{\sinh ^{2} \xi_{0}}\right] \cos \eta \\
& -\sum_{k=2}^{\infty} \frac{2 P}{\pi(k-1)}\left[\frac{k(k-1) \sinh ^{2} \xi_{0}}{\sinh ^{2} k \xi_{0}-k^{2} \sinh ^{2} \xi_{0}} \sinh \xi \sinh k \xi\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{k^{2} \sinh ^{2} \xi_{0}-k \sinh \xi_{0} \cosh \xi_{0}-e^{k \xi_{0}} \sinh k \xi_{0}}{(k+1)\left(\sinh ^{2} k \xi_{0}-k^{2} \sinh ^{2} \xi_{0}\right)}(k \sinh \xi \cosh k \xi-\cosh \xi \sinh k \xi) \\
& \left.+\frac{e^{k \xi}}{k+1}(k \sinh \xi-\cosh \xi)\right] \cos k \eta
\end{aligned}
$$

with the summation extending over even values of $k$ only.
The most significant stress is the hoop stress $\sigma_{\eta}$ at the periphery of the hole, $\xi=\xi_{0}$. Substituting the above value of $\Phi_{1} / J$ into the second of Eqs. (4), the following series is obtained:

$$
\begin{aligned}
a \sigma_{\eta}= & -\frac{P}{\pi}\left(1+\operatorname{coth}^{2} \xi_{0}\right)-\frac{2 P}{\pi} \frac{\cosh \xi_{0}}{\sinh ^{2} \xi_{0}} \cos \eta+\frac{P}{\pi}\left(5 \operatorname{coth}^{2} \xi_{0}-1\right) \cos 2 \eta \\
& -\frac{2 P}{\pi}\left[\frac{2 \sinh \xi_{0} \sinh 2 \xi_{0}}{\sinh ^{2} 2 \xi_{0}-4 \sinh ^{2} \xi_{0}}+\frac{4 \sinh \xi_{0} \sinh 4 \xi_{0}}{\sinh ^{2} 4 \xi_{0}-16 \sinh ^{2} \xi_{0}}\right] \cos 3 \eta \\
& +\frac{4 P}{\pi} \frac{4 \sinh \xi_{0} \sinh 4 \xi_{0}}{\sinh ^{2} 4 \xi_{0}-16 \sinh ^{2} \xi_{0}} \cos 4 \eta \\
& -\frac{2 P}{\pi}\left[\frac{4 \sinh \xi_{0} \sinh 4 \xi_{0}}{\sinh ^{2} 4 \xi_{0}-16 \sinh ^{2} \xi_{0}}+\frac{6 \sinh \xi_{0} \sinh 6 \xi_{0}}{\sinh ^{2} 6 \xi_{0}-36 \sinh ^{2} \xi_{0}}\right] \cos 5 \eta+\cdots
\end{aligned}
$$



Fig. 4.

Fig. 4 is a graphical illustration of the above formula. In that figure, the "stress factor" is plotted for different values of $2 h / d=\cosh \xi_{0}$. By "stress factor" is meant the ratio of the stress $\sigma_{\eta}$ to the stress which would have existed under the same loading at a point corresponding to the center of the hole, if the latter had not been
drilled. If there had been no hole, the point corresponding to its center would have been under a stress $-2 P / \pi h$ (compression) so that the stress factor is the ratio $-\sigma_{\eta} /(2 P / \pi h)$. Therefore, a positive value of the stress factor represents compression, a negative value, tension.

It is seen from the figure that for each curve there exists a tension directly under the load ( $\alpha=0$ ), which becomes a compression as $\alpha$ is increased, reaches a maximum, then decreases, and becomes tension again when $\alpha$ approaches $180^{\circ}$. For low values of $\cosh \xi_{0}$, i.e., of the depth-to-diameter ratio, there exists a secondary maximum of tension in the neighborhood of $\alpha=20^{\circ}$.

As cosh $\xi_{0}$ increases (as the hole gets deeper and deeper), the stress factor curves tend towards the "limit curve," which is simply the graph of $1-2 \cos 2 \alpha$. The latter expression (Ref. 4, p. 77, second of Eqs. (58)) is obtained by assuming the hole to be in a field of uniform compression, equal to the compression $-2 P / \pi h$ at the center of the hole.
B. Hole subjected to tangential load. Now the total stress function has the form

$$
\Phi_{2} / J=\varphi_{2} / J+\chi_{2} / J,
$$

where $\Phi_{2}$ is the total stress function, $\phi_{2}$ is given by (1b) and $\chi_{2}$ is of the general form (5). The heretofore arbitrary constants are determined by the conditions at the inner boundary, which are the same as in the preceding case. The remaining constants are found to be:

$$
\begin{aligned}
B & =\frac{P}{2 \pi}\left[\frac{\sin \beta}{\sinh ^{2} \xi_{0}}+\frac{\sin 2 \beta \operatorname{coth} \xi_{0}}{\cosh ^{2} \xi_{0}-\cos ^{2} \beta}-(\pi-2 \beta) \frac{\cosh ^{2} \xi_{0}}{\sinh \xi_{0}}\right] \\
F & =\frac{P}{2 \pi}\left[\frac{e^{\xi_{0}} \sin \beta}{\sinh \xi_{0}}+\frac{\sin \beta \cos \beta}{\cosh ^{2} \xi_{0}-\cos ^{2} \beta}-\left(\frac{\pi}{2}-\beta\right) \cosh \xi_{0}\right] \\
G^{\prime} & =-\frac{P}{2 \pi} \frac{e^{2 \xi_{0}}}{\sinh 2 \xi_{0}}(1+\cos \beta), \\
F^{\prime} & =0 \\
E_{k} & =-\frac{P}{2 \pi}(-1)^{k} \frac{\sin k \beta}{k} \frac{k^{2} \sinh ^{2} \xi_{0}+k \sinh \xi_{0} \cosh \xi_{0}-e^{k \xi_{0} \sinh k \xi_{0}}}{\sinh ^{2} k \xi_{0}-k^{2} \sinh ^{2} \xi_{0}} \\
F_{k} & =-\frac{P}{2 \pi}(-1)^{k}(k-1) \sin k \beta \frac{\sinh ^{2} \xi_{0}}{\sinh ^{2} k \xi_{0}-k^{2} \sinh ^{2} \xi_{0}}, \\
E_{k}^{\prime} & =\frac{P}{2 \pi} \frac{\left[1-(-1)^{k} \cos k \beta\right]}{k} \frac{k^{2} \sinh ^{2} \xi_{0}+k \sinh \xi_{0} \cosh ^{\prime} \xi_{0}-e^{k \xi_{0} \sinh k \xi_{0}}}{\sinh ^{2} k \xi_{0}-k^{2} \sinh }, \\
F_{k^{\prime}}^{\prime} & =\frac{P}{2 \pi}(k-1)\left[1-(-1)_{0}^{k} \cos k \beta\right] \frac{\sinh }{\sinh ^{2} k \xi_{0}-k_{0}^{2} \sinh ^{2} \xi_{0}}
\end{aligned}
$$

The resulting Fourier series can be shown to converge as in the previous case.
C. Conclusion. In the above paragraphs, a method was presented for computing the distortion of the original Boussinesq field when a hole is introduced. Other interesting results can be derived by simple means; thus, by superposing on the above stress functions $\Phi_{1}$ or $\Phi_{2}$ one of the solutions presented in Refs. 1 and 2 , it is possible
to obtain the stress system for a Boussinesq field containing a hole, the periphery of which is subjected to a uniform pressure. Another extension of the above method, on which the writer is working at present, can be used to solve the case of a Boussinesq field containing a rigid disc.

## APPENDix

The decomposition of the Boussinesq stress function into a Fourier series in $\eta$. We shall begin by decomposing the shear stress function

$$
\frac{\phi_{2}}{J}=\frac{P}{\pi} \sinh \xi \tan ^{-1} \frac{y_{0}(\cosh \xi-\cos \eta)-a \sin \eta}{a \sinh \xi} .
$$

The different Fourier coefficients are given by

$$
R_{k}+i S_{k}=\frac{1}{\pi} \int_{0}^{2 \pi} \frac{\phi_{2}}{J} e^{i k n d \eta .}
$$

This can be simplified by introducing the angle $\beta=\tan ^{-1} y_{0} / a$.

$$
\begin{aligned}
\frac{\phi_{2}}{J} & =\frac{P}{\pi} \sinh \xi \tan ^{-1} \frac{y_{0}(\cosh \xi-\cos \eta)-a \sin \eta}{a \sinh \xi} \\
& =\frac{P}{\pi} \sinh \xi \tan ^{-1} \frac{\sin \beta \cosh \xi-\sin (\eta+\beta)}{\cos \beta \sinh \xi}
\end{aligned}
$$

Let also $\sin \beta \cosh \xi=p, \cos \beta \sinh \xi=q, \eta+\beta=\psi$. Then

$$
\begin{aligned}
R_{k}+i S_{k} & =\frac{1}{\pi} \int_{0}^{2 \pi} \frac{P}{\pi} \sinh \xi \tan ^{-1} \frac{p-\sin \psi}{q} e^{i k \pi} d \eta \\
& =\frac{P \sinh \xi e^{-i k \beta}}{\pi^{2}} \int_{0}^{2 \pi} \tan ^{-1} \frac{p-\sin \psi}{q} e^{i k \psi} d \psi=\frac{P \sinh \xi e^{-i k \beta}}{\pi^{2}} I_{k}
\end{aligned}
$$

Here the limits of integration need not be changed, since the integrand is a periodic function of period $2 \pi$. To evaluate $I_{k}$, use is made of integration by parts, with

$$
u=\tan ^{-1} \frac{p-\sin \psi}{q}, \quad d v=e^{i k \psi d \psi}
$$

Then

$$
d u=-\frac{q \cos \psi d \psi}{(p-\sin \psi)^{2}+q^{2}}, \quad v=-\frac{i}{k} e^{i k \psi} \quad(k \neq 0)
$$

and

$$
I_{k}=\left.u v\right|_{n} ^{2 \pi}-\int_{0}^{2 \pi} v d u .
$$

But, since both $u$ and $v$ are periodic, their product evaluated over the period $2 \pi$ is zero; then

$$
I_{k}=-\int_{0}^{2 \pi} v d u=-\frac{i q}{k} \int_{0}^{2 \pi} \frac{\cos \psi d \psi}{(p-\sin \psi)^{2}+q^{2}} e^{i k \psi} .
$$

Replacing the sine and cosine by their exponential equivalents and transforming, one has

$$
I_{k}=\frac{2 q}{k} \int_{0}^{2 \pi} e^{i k \psi} \frac{\left(e^{2 i \psi}+1\right) d\left(e^{i \psi}\right)}{\left(e^{2 i \psi}-2 i p e^{i \psi}-1\right)^{2}-4 q^{2} e^{2 i \psi}}
$$

This is easily seen to be a rational function of $e^{i \psi}$, the denominator of which, the difference of two squares, can be decomposed into two quadratic factors with relatively simple roots, so that the transformation by partial fractions can be used to obtain the following result:

$$
I_{k}=\frac{1}{2 k} \int_{0}^{2 x} e^{i k \psi}\left(\frac{1}{e^{i \psi}-e^{\xi} e^{i \beta}}-\frac{1}{e^{i \psi}-e^{-\xi} e^{i \beta}}-\frac{1}{e^{i \psi}+e^{\xi} e^{-i \beta}}+\frac{1}{e^{i \psi}+e^{-\xi} e^{-i \beta}}\right) d\left(e^{i \psi}\right) .
$$

Thus the integral breaks down into four integrals of the form

$$
\int_{0}^{2 \pi} \frac{t^{k}}{t-c} d t
$$

where $t=e^{i \psi}$, and $c$ is a complex constant of the form $\pm e^{ \pm \xi \pm i \beta}$. Now if the indicated division of $t^{k}$ by $t-c$ is performed, a quotient which is a polynomial in $t$ and $a$ remainder $c^{k}$ result. The polynomial is integrated into another polynomial in $t=e^{i \psi}$, and the value of this second polynomial between the limits 0 and $2 \pi$ is zero because of the periodicity of $e^{i \psi}$. Thus the remaining terms are of the type

$$
\int_{0}^{2 \pi} \frac{c^{k} d t}{t-c}=\int_{0}^{2 \pi} \frac{c^{k}}{e^{i \psi}-c} d\left(e^{i \psi}\right)
$$

namely,

$$
\begin{equation*}
I_{k}=\frac{1}{2 k} \int_{0}^{2 \pi}\left[\frac{\left(e^{\xi} e^{i \beta}\right)^{k}}{e^{i \psi}-e^{\xi} e^{i \beta}}-\frac{\left(e^{-\xi} e^{i \beta}\right) k}{e^{i \psi}-e^{-\xi} e^{i \beta}}-\frac{\left(-e^{\xi} e^{-i \beta}\right)^{k}}{e^{i \psi}+e^{\xi} e^{-i \beta}}+\frac{\left(-e^{-\xi} e^{-i \beta}\right)^{k}}{e^{i \psi}+e^{-\xi} e^{-i \beta}}\right] d\left(e^{i \psi}\right) . \tag{A}
\end{equation*}
$$

The value of the resulting terms can be obtained more easily by considering the corresponding complex function of $\zeta=\psi+i \omega$.*

$$
c^{k} \int_{c} \frac{d\left(e^{i \zeta}\right)}{e^{i \zeta}-c}=i c^{k} \int_{C} \frac{e^{i \zeta} d \xi}{e^{i \zeta}-c}
$$

along the contour shown on Fig. 5. It is well known from the theory of the complex variable that the value of the above contour integral is zero if the pole of the integrand falls outside that contour, and is equal to $2 \pi i \times i c^{k} \times$ Res., where Res. is the residue of the integrand, if the pole lies inside the contour.

Performing the integration around the contour, we obtain the following:
(1) along the real axis $-\pi \leqq \psi \leqq \pi$, the complex integral reduces to the real integral to be evaluated (limits $-\pi$ and $\pi$ are equivalent to 0 and $2 \pi$ );
(2) the two integrals along the vertical paths cancel each other;
(3) the integral along $\omega=\Lambda$ has a zero limit for $\Lambda \rightarrow \infty$.

Therefore,

[^5]\[

\int_{0}^{2 \pi} \frac{c^{k}}{e^{i \psi}-c} d\left(e^{i \psi}\right)=\left\{$$
\begin{array}{l}
0 \text { if the pole lies outside the strip } \\
u \geqq 0 \quad-\pi \leqq \psi \leqq \pi \\
2 \pi i \times i c^{k} \times \text { Res. if it lies inside that strip. }
\end{array}
$$\right.
\]

The pole occurs at $e^{i \zeta}-c=0$, or $\zeta=-i \log c ; c$ is of the form

$$
c=e^{ \pm \xi} e^{i \beta} \text { or } c=-e^{ \pm \xi} e^{-i \beta}=e^{ \pm \xi} e^{i(\pi-\beta)}
$$

so that $\log c= \pm \xi+i \beta$ or $\log c= \pm \xi+i(\pi-\beta)$ and $\zeta=\beta \mp i \xi$ or $\zeta=\pi-\beta$ 干 $i \xi$.
But the region $x>0$ corresponds to $\xi<0$, as can be seen from (2), and also $0<\beta<\pi / 2$, so that the integrals whose pole has an imaginary part of the form $+i \xi$, namely the second and fourth of (A), have the value zero. The poles of the first and third, on the contrary, fall inside the region of integration, so that their values are

$$
2 \pi i \times i c^{k} \times \text { Res }
$$

It remains to evaluate the residue. This is found to be $-i$ by methods explained in texts on the complex variable (Ref. 5). Thus the required integrals become

Thus,

$$
2 \pi i \times i c^{k} \times(-i)=2 \pi i c^{k}
$$

$$
I_{k}=\frac{1}{2 k} 2 \pi i\left[\left(e^{\xi} e^{i \beta}\right)^{k}-\left(-e^{\xi} e^{-i \beta}\right)^{k}\right]
$$

and

$$
R_{k}+i S_{k}=\frac{P i}{\pi} \sinh \xi \frac{e^{k \xi}}{k}\left[1-e^{i(\pi-\beta) k}\right] .
$$

Therefore

$$
\begin{aligned}
& R_{k}=-(-1)^{k} \frac{P}{\pi} \sinh \xi \sin k \beta \frac{e^{k \xi}}{k} \\
& S_{k}=\frac{P}{\pi} \sinh \xi \frac{e^{k \xi}}{k}\left[1-(-1)^{k} \cos k \beta\right]
\end{aligned}
$$

When $\beta=0$, the poles shown on Fig. 5 have real parts 0 and $\pi$, respectively. In other words, one of the poles is on the contour itself. Besides, due to the periodicity of the integrand, a third pole appears with a real part equal to $-\pi$. This latter pole has, in general, a real part $-\pi-\beta$, and is identical with the pole at $\pi-\beta$. Thus, there are three poles in all, one wholly within the contour and two others, with equal residues, on the contour itself. Now it is easy to see that each of the latter contributes half its residue to the value of the integral, and since these residues are equal, the situation remains the same as if there were only two poles, both entirely within the contour, so that the case $\beta=0$ is not essentially different from $\beta \neq 0$, and it is sufficient to set $\beta=0$ in the above formulas for $R_{k}$ and $S_{k}$. Thus

$$
R_{k}^{(0)}=0, \quad S_{k}^{(0)}=\frac{2 P}{\pi} \frac{e^{k \xi}}{k} \sinh \xi \quad \text { for } k \text { odd, } \quad S_{k}^{(0)}=0 \text { for } k \text { even }
$$

Case $k=0$. For this case, the procedure is exactly the same up to the integration by parts. There, while $u$ remains as before, $d v=d \psi, v=\psi$, so that $I_{k}$ becomes

$$
I_{0}=\left.\psi \tan ^{-1} \frac{p-\sin \psi}{q}\right|_{0} ^{2 \pi}+\int_{0}^{2 \pi} \frac{q \psi \cos \psi d \psi}{(p-\sin \psi)^{2}+q^{2}}=2 \pi \tan ^{-1} p / q+q J_{0}
$$

To evaluate $J_{0}$, the same method as before is used, exponentials being introduced in place of the trigonometric functions:

$$
J_{0}=2 i \int_{0}^{2 \pi} \frac{\psi\left(e^{2 i \psi}+1\right) d\left(e^{i \psi}\right)}{\left(e^{2 i \psi}-2 i p e^{i \psi}-1\right)^{2}-4 q^{2} e^{2 i \psi}}
$$

This can again be transformed into partial fractions:

$$
J_{0}=\frac{i}{2 \cos \beta} \int_{0}^{2 \pi} \psi\left[\frac{1}{e^{i \psi}-e^{\xi} e^{i \beta}}-\frac{1}{e^{i \psi}-e^{-\xi} e^{i \beta}}-\frac{1}{e^{i \psi}+e^{\xi} e^{-i \beta}}+\frac{1}{e^{i \psi}+e^{-\xi} e^{-i \beta}}\right] d\left(e^{i \psi}\right)
$$

Here we are dealing with integrals of the type

$$
\int_{0}^{2 x} \frac{\psi d\left(e^{i \psi}\right)}{e^{i \psi}-c}=i \int_{0}^{2 \pi} \frac{\psi e^{i \psi} d \psi}{e^{i \psi}-c}
$$

these can be treated, as before, by introducing the complex variable and integrating around the contour of Fig. 5. Since the denominator of the integrand is the same as before, all that was said about the poles of the partial integrals making up $I_{k}$ remains true. Therefore, the second and fourth terms in the expression for $J_{0}$ contribute nothing, and the first and third are each equal to $2 \pi i$ times the residue times constants. The residues,


Fig. 5. however, have here the value $-\log c$, so that

$$
J_{0}=\frac{i}{2 \cos \beta} i 2 \pi i[-\xi-i \beta+\xi+i(\pi-\beta)]=\frac{\pi}{\cos \beta}(\pi-2 \beta)
$$

and the imaginary term vanishes, as could be expected. Then

$$
I_{0}=2 \pi \tan ^{-1}(\tan \beta \operatorname{coth} \xi)+\pi(\pi-2 \beta) \sinh \xi
$$

and

$$
R_{0}=\frac{P \sinh \xi}{\pi}\left[2 \tan ^{-1}(\tan \beta \operatorname{coth} \xi)+(\pi-2 \beta) \sinh \xi\right] .
$$

Half of this expression is the first term of the Fourier series:

$$
\frac{R_{0}}{2}=\frac{P \sinh \xi}{\pi}\left[\tan ^{-1}(\tan \beta \operatorname{coth} \xi)+\left(\frac{\pi}{2}-\beta\right) \sinh \xi\right]
$$

For $\beta=0$, there are again three poles, one on the imaginary axis and the two others with real parts $\pm \pi$. This location is, as in the general case $k \neq 0$, due to the periodicity of the denominator of the integrand. However, here the integrand as a whole is not periodic, so that the residues at the two poles on the contour are not equal, and the situation is not the same as for $\beta \neq 0$. The detailed computations show that $R_{0}=0$.

Series for $\phi_{1} / J$. Since the ratio

$$
\phi_{1} / J: \phi_{2} / J=\phi_{1} / \phi_{2}=\frac{p-\sin (\eta+\beta)}{q}
$$

is a simple trigonometric expression, the series for $\phi_{1} / J$ can be obtained from that for $\phi_{2} / J$ by term-by-term multiplication.

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## THE THERMAL-STRESS AND BODY-FORCE PROBLEMS OF THE INFINITE ORTHOTROPIC SOLID*

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1. Introduction. Elastic problems dealing with orthotropic materials have had considerable investigation in recent years, ${ }^{1}$ but up to the present time, such investigation has been largely limited to a consideration of the problems involving thin plates of this material.

In the present paper, two problems dealing with the stresses and displacements in an infinite elastic orthotropic solid are solved, and in each case the results are obtained in terms of three independent displacement potentials. The two solutions are: 1) the displacement potentials arising from an arbitrary distribution of temperature within a finite region of the solid (the temperature being measured from an arbitrary datum) and 2) the potentials arising from an arbitrary distribution of body force within a finite region. Each of these problems reduces to the solution of three simultaneous partial differential equations, which are transformed, through the use of Fourier integrals, into individual solutions for each potential. The expressions for these potentials are reduced to the form of Newtonian potential integrals for those cases where sufficient symmetry of the material properties exists to allow such a reduction. In the more complicated cases, the results are still expressed in closed form in terms of definite integrals.
2. The thermo-elastic problem. The conditions under which the thermo-elastic problem will be formulated and solved are the following. The material is to be homogeneous, orthotropic, and clastic, throughout the infinite region, and is to be within that class of orthotropic materials which has three coefficients of temperature expansion, $\alpha_{j}$, associated with the three principal directions of the material. The body forces will be taken as vanishing, since any problem involving both thermal and body force effects has a solution which is merely the superposition of the two individual solutions. The temperature distribution is to be an arbitrary function of position with the restrictions that this function must vanish everywhere outside some finite region, be continuous everywhere and be differentiable everywhere except on a finite number of surfaces.

The fundamental relations needed to formulate the problem mathematically are: the equations of equilibrium of an element of the material; the thermo-elastic equations, that is, the relations between strains, stresses and temperature; and the relations between strains and displacements.

The equations of equilibrium are found by a consideration of the equilibrium of a rectangular parallelepiped of the material under general loading. Since these equations are independent of the type of material under consideration, they are given, as

[^6]in the isotropic case for zero body force, ${ }^{2}$ by three equations of the type,
\[

$$
\begin{equation*}
\frac{\partial \sigma_{x}}{\partial x}+\frac{\partial \tau_{x y}}{\partial y}+\frac{\partial \tau_{x z}}{\partial z}=0 \tag{1}
\end{equation*}
$$

\]

where the notation is the conventional one.
The orthotropic material has been defined as one whose Hooke's law has the form indicated by equations (2), when $T$ is identically zero. The effect of temperatures, different from datum, is to produce normal strains in the three principal directions of the material, as specified under the conditions of the problem. Hence, when the coordinate axes are taken parallel to the principal directions, the general formulas for the strains have the form,

$$
\begin{equation*}
e_{x}=a_{11} \sigma_{x}+a_{12} \sigma_{y}+a_{13} \sigma_{z}+\alpha_{1} T, \cdots ; \quad \gamma_{v z}=a_{44} \tau v z, \cdots \tag{2}
\end{equation*}
$$

If we now define three displacement potentials, $\phi_{j}$, such that

$$
u=\frac{\partial \phi_{1}}{\partial x}, \quad v=\frac{\partial \phi_{2}}{\partial y}, \quad w=\frac{\partial \phi_{3}}{\partial z}
$$

and such that $\phi_{j}$ and its derivatives vanish at infinity, the conventional definitions of the strains become,

$$
\begin{equation*}
e_{x}=\frac{\partial u}{\partial x}=\frac{\partial^{2} \phi_{1}}{\partial x^{2}}, \ldots ; \quad \gamma_{y z}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=\frac{\partial^{2}}{\partial z \partial y}\left(\phi_{2}+\phi_{3}\right), \ldots \tag{3}
\end{equation*}
$$

Combining now, equations (1), (2), and (3), we obtain three equations of which the following is the first:

$$
\begin{equation*}
\frac{\partial}{\partial x}\left[\left(b_{11} \frac{\partial^{2}}{\partial x^{2}}+b_{66} \frac{\partial^{2}}{\partial y^{2}}+b_{55} \frac{\partial^{2}}{\partial z^{2}}\right) \phi_{1}+c_{12} \frac{\partial^{2}}{\partial y^{2}} \phi_{2}+c_{13} \frac{\partial^{2}}{\partial z^{2}} \phi_{3}\right]=-\beta_{1} \frac{\partial T}{\partial x} . \tag{4}
\end{equation*}
$$

Each of these may be integrated once to give, ${ }^{3}$

$$
\begin{equation*}
\left(b_{11} \frac{\partial^{2}}{\partial x^{2}}+b_{65} \frac{\partial^{2}}{\partial y^{2}}+b_{B 5} \frac{\partial^{2}}{\partial z^{2}}\right) \phi_{1}+c_{12} \frac{\partial^{2}}{\partial y^{2}} \phi_{2}+c_{13} \frac{\partial^{2}}{\partial z^{2}} \phi_{3}=-\beta_{\mathrm{i}} T, \ldots \tag{4a}
\end{equation*}
$$

The arbitrary functions which appear in each of the foregoing integrations must each vanish, since, for example, in the first equation, all terms vanish when $x$ is infinite and $y, z$ are finite, implying that all functions independent of $x$ must vanish identically.

Due to the convenient form of the boundary conditions, these equations are easily integrated by the following procedure. Multiply each equation through by $e^{-i(x \xi+\nu \pi+z \xi)}$ and integrate over the whole region, integrating by parts those terms containing derivatives of $\phi_{j}$. This operation produces the following three equations, using the abbreviated forms defined below in equations (6).

[^7]\[

$$
\begin{align*}
\left(b_{11} \xi^{2}+b_{66} \eta^{2}+b_{55} \zeta^{2}\right) E_{1}+c_{12} \eta^{2} E_{2}+c_{13} \zeta^{2} E_{3} & =\beta_{1} S \\
c_{12} \xi^{2} E_{1}+\left(b_{66} \xi^{2}+b_{22} \eta^{2}+b_{44} 5^{2}\right) E_{2}+c_{23} E_{3} & =\beta_{2} S  \tag{5}\\
c_{13} \xi^{2} E_{1}+c_{23} \eta^{2} E_{2}+\left(b_{65} \xi^{2}+b_{44} \eta^{2}+b_{33} \zeta^{2}\right) E_{3} & =\beta_{3} S
\end{align*}
$$
\]

where,

$$
\begin{align*}
E_{j} & =\iiint_{-\infty}^{\infty} \phi_{j} e^{-i(x \xi+\nu \eta+z \zeta)} d x d y d z \\
S & =\iiint T e^{-i(x \xi+y \eta+z \zeta)} d x d y d z \tag{6}
\end{align*}
$$

Equations (5) are easily solved for the $E_{j}$, and yield the expressions,

$$
\begin{equation*}
E_{i}=F_{i}(\xi, \eta, \zeta) S \tag{7}
\end{equation*}
$$

where the $F_{j}$ become ratios of homogeneous polynomials in $\xi^{2}, \eta^{2}$ and $\zeta^{2}$.
Noting now, that by their definitions, the $E_{j}$ are the Fourier transforms (in three dimensions) of the $\phi_{j}$, we may write

$$
\begin{align*}
& \phi_{j}(x, y, z)=\frac{1}{8 \pi^{3}} \iiint_{-\infty}^{\infty} F_{i}(\xi, \eta, \zeta) e^{i(x \xi+y \eta+z \zeta)} S(\xi, \eta, \zeta) d \xi d \eta d \zeta \\
& \quad=\frac{1}{8 \pi^{3}} \iiint_{-\infty}^{\infty} F_{j}(\xi, \eta, \zeta) e^{i(x \xi+y \eta+z \zeta)} d \xi d \eta d \zeta \iiint_{-\infty}^{\infty} T(r, s, t) e^{-i(r \xi+s \eta+1 \zeta)} d r d s d t \tag{8}
\end{align*}
$$

and the order of the indicated integrations may be changed to give,
$\phi_{j}=\frac{1}{8 \pi^{3}} \iiint_{-\infty}^{\infty} T(r, s, t) d r d s d t \iiint_{-\infty}^{\infty} F_{j}(\xi, \eta, \zeta) e^{i[(x-r) \xi+(y-s) \eta+(z-t) \zeta]} d \xi d \eta d \zeta$.
Since each $F_{j}$ (as defined by equation 7) is a ratio of second order polynomial in $\xi^{2}, \eta^{2}$, and $\zeta^{2}$, to one of third order, we may write,

$$
F_{j}=B \frac{R_{4}^{2} R_{5}^{2}}{R_{1}^{2} R_{2}^{2} R_{3}^{2}}
$$

where $R_{k}^{2}=\lambda_{k}^{2} \xi^{2}+\mu_{k}^{2} \eta^{2}+\zeta^{2}$, and where the $\lambda_{k}, \mu_{k}$, and $B$, are constants depending on the values of the constants appearing in the determinants defining $F_{j}$, and hence, may be considered as known. Note that the $\lambda_{k}^{2}, \mu_{k}^{2}$, for $k=1,2,3$, must be non-negative, since no singularity may exist except at the origin.

In many cases, the expressions for the $F_{j}$ may be reduced to the form,

$$
\begin{equation*}
F_{j}=\sum_{k=1,2,3} \frac{A_{j k}}{R_{k}^{2}} \tag{10}
\end{equation*}
$$

This will always be true when the problem involves a material which is isotropic in a certain plane (for example, a laminated plastic) unless identical values of $R^{2}$ recur in the denominator. This may be seen by noting that since the denominator of $F_{j}$ must be invariant under a rotation about the $z$ axis due to this isotropy (the plane of isotropy is here taken as the $x, y$ plane), $\xi^{2}$ and $\eta^{2}$ must occur in the combination $\xi^{2}+\eta^{2}$, and hence, $\lambda_{k}=\mu_{k}$, and the $R_{k}^{2}$ become essentially binomials. The reduction of $F_{j}$ to the form of equation (10) is, in this case, merely a matter of evaluating $A_{j k}$.

When equation (10) does hold, the integration proceeds as follows: Using the conventional vector notation and the new coordinates with the subscript $k$ (where $\xi_{k}=\lambda_{k} \xi, x_{k}=x \lambda_{k}^{-1}, r_{k}=r \lambda_{k}^{-1}$, etc. and where $\left.\bar{m}_{k}=\bar{i}\left(x_{k}-r_{k}\right)+j\left(y_{k}-s_{k}\right)+\tilde{k}\left(z_{k}-t_{k}\right)\right)$, the integral over $\xi, \eta$, and $\zeta$, of equation (9) defining Green's function $G$, may be written,

$$
\begin{equation*}
G_{j}(x, y, z, r, s, t)=\iiint_{-\infty}^{\infty} \sum_{k=1,2,3} \frac{A_{j k}}{R_{k}^{2}} e^{i \bar{m}_{k} \cdot \bar{R}_{k}} \frac{d \xi_{k}}{\lambda_{k}} \frac{d \eta_{k}}{\mu_{k}} d \zeta_{k} \tag{11}
\end{equation*}
$$

If we now change to a spherical coordinate system in which $\gamma$ is the angle between $\bar{m}_{k}$ and $\bar{R}_{k}$ and $\delta$ is the polar angle about $\bar{m}_{k}$, this integral becomes,

$$
G_{i}=\iiint \sum \frac{A_{j k}}{\lambda_{k} \mu_{k}} e^{i m_{k} R_{k} \cos \gamma} \sin \gamma d \gamma d \delta d R_{k},
$$

where the integration now takes place over, $0 \leqq \gamma \leqq \pi, 0 \leqq \delta \leqq 2 \pi, 0 \leqq R_{k}<\infty$. The elementary integrations over $\gamma$ and $\delta$ produce

$$
G_{j}=\sum_{k=1,2,3} \frac{4 \pi A_{j k}}{\lambda_{k} \mu_{k} m_{k}} \int_{0}^{\infty} \frac{\sin m_{k} R_{k}}{R_{k}} d R_{k}
$$

which is known to have the value,

$$
G_{i}=2 \pi^{2} \sum_{k=1,2,3} \frac{A_{j k}}{\lambda_{k} \mu_{k}} \frac{1}{m_{k}} .
$$

Now transforming the remaining terms of equation (9) to the coordinates with the subscript $k$, and substituting the above value for Green's function, we obtain,

$$
\begin{equation*}
\phi_{j}=\frac{1}{4 \pi} \iiint \sum_{k=1,2,3} \Lambda_{j k} \frac{T\left(\lambda_{k} r_{k}, \mu_{k} s_{k}, t_{k}\right)}{\sqrt{\left(x_{k}-r_{k}\right)^{2}+\left(y_{k}-s_{k}\right)^{2}+\left(z_{k}-t_{k}\right)^{2}}} d r_{k} d s_{k} d t_{k} \tag{12}
\end{equation*}
$$

Hence, the problem, wherein $T(x, y, z)$ represents the temperature distribution, becomes the problem of evaluating the Newtonian potential function corresponding to a mass distribution of,

$$
\rho=\frac{A_{i k}}{4 \pi} T\left(\lambda_{k} x_{k}, \mu_{k} y_{k,} z_{k}\right)
$$

For an isotropic material, the $\phi_{k}$ become alike, and are given by, ${ }^{4}$

$$
\phi_{i}=\frac{\alpha}{4 \pi} \frac{1+v}{1-v} \iiint \frac{T(r, s, t)}{\sqrt{(x-r)^{2}+(y-s)^{2}+(z-t)^{2}}} d r d s d t .
$$

In the evaluation of Green's function for those cases where the denominator of $F_{j}$ has a multiple root, it is convenient to introduce the notation

$$
G_{i}=\sum_{n} G_{\text {in, }} \quad \Delta_{k}=\frac{\partial^{2}}{\partial x_{k}^{2}}+\frac{\partial^{2}}{\partial y_{k}^{2}}+\frac{\partial^{2}}{\partial z^{2}} .
$$

In this case, integrals of the form,

[^8]\[

$$
\begin{equation*}
G_{j n}=\iiint_{-\infty}^{\infty} \frac{A_{j n} R_{1}^{2}}{R_{2}^{4}} e^{i((x-r) \xi+(\eta-s) n+(z-t))]} d \xi d \eta d \xi \tag{13}
\end{equation*}
$$

\]

must be evaluated, provided $R_{1}^{2} \neq R_{2}^{2}$. Using the above notation, the equivalence of the following to equation (13) may be verified by substitution:

$$
\begin{equation*}
\Delta_{2} G_{i n}=\Delta_{1} \iiint \frac{A_{i n}}{R_{2}^{2}} e^{i((x-r) \xi+(y-s) \eta+(z+t))\}} d \xi d \eta d \xi . \tag{13a}
\end{equation*}
$$

The integral involved in this equation is, however, the same as that appearing in equation (11), so (13a) becomes,

$$
\Delta_{2} G_{\text {in }}=\Delta_{1} \frac{2 C_{\text {in }}}{\sqrt{\left(x_{2}-r_{2}\right)^{2}+\left(y_{2}-s_{2}\right)^{2}+\left(z_{2}-t_{2}\right)^{2}}},
$$

where $C_{j n}$ is an easily evaluated constant. Substitution will again show that,

$$
G_{i n}=C_{i n} \Delta_{1} \sqrt{\left(x_{2}-r_{2}\right)^{2}+\left(y_{2}-s_{2}\right)^{2}+\left(z_{2}-t_{2}\right)^{2}},
$$

is equivalent to the above equation, and hence the $\phi_{i}$ are given by,

$$
\begin{equation*}
\phi_{i}=\frac{1}{8 \pi^{3}} \sum_{n} \iiint T(r, s, t) G_{i n}(r, s, t, x, y, z) d r d s d t \tag{14}
\end{equation*}
$$

In those cases where $F_{j}$ cannot be reduced to one of the foregoing convenient forms, $G_{j}$ is more difficult to evaluate. Since no explicit form has been found for this function, other than complicated definite integrals, it is believed best to leave it in the form defined by equation (9).
3. The body-force problem. As in dealing with isotropic materials, the solution of the body force problem may be shown to reduce to a form analogous to that of the thermo-elastic problem. To show this, we shall consider only the problem where the body force is directed parallel to the $x$ axis, noting that the general solution is obtained by the superposition of three such problems.

Equations (1) and (2) are modified to contain the body-force function, $X$, and to eliminate the temperature terms. Equations (4) are then obtained again, where now the right hand sides are replaced respectively by, $X, 0$, and 0 .

The $\phi_{j}$ will not, in general, vanish at infinity in this problem, hence the procedure needs a slight modification. The second and third of these equations are integrated with respect to $y$ and $z$ respectively and then differentiated with respect to $x$. This yields equations (4a), where again, $X, 0,0$, appear on the right and where the $\phi_{j}$ are replaced by $\partial \phi_{j} / \partial X$. The procedure is now identical with that of the thermal problem, and the $\phi_{j}$ are found by the expressions analogous to equation (14).
4. The two-dimensional problem. If we carry through in two dimensions the procedure used in the previous sections of this paper, we arrive at an equation which is identical to equation (8) except that $z, t$, and $\zeta$, no longer appear. The expressions for $F_{j}$ are now simpler in form, being given by,

$$
F_{i}=\frac{\lambda_{1 \xi^{2}}^{2}+\mu_{1}^{2} \eta^{2}}{\left(\lambda_{2}^{2} \xi^{2}+\mu_{2}^{2} \eta^{2}\right)\left(\lambda_{3}^{2} \xi^{2}+\mu_{3}^{2} \eta^{2}\right)}
$$

which may always be reduced to the form

$$
F_{i}=\sum_{k=1,2} \frac{A_{j k}}{R^{2}}
$$

unless $R_{2}=R_{3}$. $\left(R_{k}^{2}=\lambda_{k}^{2} \xi^{2}+\mu_{k}^{2} \eta^{2}\right)$.
Before changing the order of integration, we differentiate equation (8) with respect to $y$. The integral form of Green's function becomes then

$$
\begin{equation*}
\frac{\partial G_{j k}}{\partial y}=\iint_{-\infty}^{\infty} A_{i k} \frac{i \eta e^{i(x \xi+\nu \eta)}}{\lambda_{k}^{2} \xi^{2}+\mu_{2}^{2} \eta^{2}} d \xi d \eta, \tag{15a}
\end{equation*}
$$

unless $R_{2}=R_{3}$, in which case,

$$
\begin{equation*}
\Delta_{2} \frac{\partial G_{i}}{\partial y}=\Delta_{1} \iint A_{i} \frac{i \eta e^{i\left(x \xi \xi+\eta_{n}\right)}}{\lambda_{2} \xi^{2}+\mu_{2} \eta^{2}} d \xi d \eta . \tag{15b}
\end{equation*}
$$

This latter expression is, of course, derived by the same reasoning used in the three dimensional problem.

Equation (15a), after the introduction of the coordinates with the subscript $k$, can be written in the iterated integral form,

$$
\frac{\partial G_{i k}}{\partial y}=\int_{0}^{\infty} \frac{4 A_{j k}}{\lambda_{k}} \sin \eta_{k} y_{k} d \eta_{k} \int_{0}^{\infty} \frac{\cos \eta_{k} x_{k} \frac{\frac{\xi_{k}}{\eta_{k}}}{1+\left(\frac{\xi_{k}}{\eta_{k}}\right)^{2}} d\left(\frac{\xi_{k}}{\eta_{k}}\right)}{1}
$$

which is known to be equivalent to,

$$
\frac{\partial G_{j k}}{\partial y}=\int_{0}^{\infty} \frac{4 A_{i k}}{\lambda_{k}} \sin y_{k} \eta_{k} \cdot \frac{\pi}{2} e^{-\mid z k \eta_{k} k} d \eta_{k}
$$

and this integral yields,

$$
\frac{\partial G_{j k}}{\partial y}=\frac{2 \pi A_{j k}}{\lambda_{k}} \frac{y_{k}}{x_{k}^{2}+y_{k}^{2}}
$$

or

$$
\begin{equation*}
G_{i k}=\frac{\pi A_{j k}}{\lambda_{k} \mu_{k}} \ln \left(x_{k}^{2}+y_{k}^{2}\right), \tag{16}
\end{equation*}
$$

and we obtain the familiar two-dimensional logarithmic potential.
Equation ( 15 b ), then becomes, in an analogous manner,

$$
\Delta_{2} G_{j}=\frac{\pi A_{i}}{\lambda_{k} \mu_{k}} \Delta_{1} \ln \left(x_{2}^{2}+y_{2}^{2}\right)
$$

or,

$$
\begin{equation*}
G_{i}=\frac{\pi A_{i}}{2 \lambda_{k} \mu_{k}} \Delta_{2}\left[\left(x_{2}^{2}+y_{2}^{2}\right) \ln \left(x_{2}^{2}+y_{2}^{2}\right)\right] . \tag{17}
\end{equation*}
$$

Hence, Green's functions are determined for each two-dimensional problem involving thermal stress or body forces in the infinite plate. The usual methods of superimposing plane stress (or strain) solutions may be utilized, of course, to solve the corresponding problems for the finite body.

## STRESSES IN THE DIAPHRAGMS OF DIAPHRAGM-PUMPS*

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1. Introduction. Complete prevention of leakage from a reciprocating pump is difficult to ensure over a long period of working. When the fluid to be pumped is of such a nature that no leakage whatever is permissible, some modification in the design of the pump is essential, and under these circumstances a diaphragm-pump may conveniently be used. This in its essentials consists of two chambers (Fig. 1) attached to a modified reciprocating pump. The chambers are of conical form rounded off at the apex and at the base, and between them a diaphragm is clamped at its edge. For high-pressure operation the diaphragm is a very thin steel disc. The fluid to be pumped passes through one chamber, connexion to inlet and exhaust valves being made by means of a number of small ports. The other chamber is connected similarly to a single-acting reciprocating pump, which is not fitted with valves. This chamber, the pump cylinder and their connecting ports are filled with a liquid (commonly oil), and thus motion of the piston of the reciprocating pump causes the diaphragm to be pressed alternately against both conical surfaces, thereby producing the desired pumping action. The inevitable leakage of oil past the piston is made good by means of an auxiliary pump.

An approximate method of calculating the stresses in the diaphragm is explained below, hence the size of the chambers may be so designed that the fatigue strength of the diaphragm is not exceeded. In section 2 the deflexion of the diaphragm is taken as sinusoidal, in section 3 as a cubic, and in section 4 as following a Besselfunction relation Attention is confined to the stresses which result from distortion into the same shape as the chamber, no regard being paid to the local stresses round the ports.
2. Stresses when the transverse displacement is sinusoidal. In general the displacement of the diaphragm from its unstrained position has not only a transverse but also a radial component; therefore it does not seem possible (except by relaxation methods) to calculate the stresses for a specified shape of chamber. It is necessary to assume a reasonable expression for the transverse displacement
 $w$, from which the corresponding radial displacement $u$ and the stresses will be obtained; and, when both $u$ and $w$ are known, the shape of the chamber is determined.

With the axes shown in Fig. 1 we shall in this section take $w$ as specified by

$$
\begin{equation*}
w=\frac{w_{0}}{2}\left(1+\cos \frac{\pi r}{a}\right), \tag{1}
\end{equation*}
$$

[^9]where $a$ is the radius of the diaphragm and $w_{0}$ the maximum value of $w$. This expression satisfies the conditions that the slope must vanish at the centre and at the edge. For $u$ we assume as an approximation ${ }^{1}$ that
\[

$$
\begin{equation*}
u=r(a-r)\left(C_{1}+C_{2} r\right), \tag{2}
\end{equation*}
$$

\]

where $C_{1}$ and $C_{2}$ are constants which will be determined by the principle of minimum strain energy. The conditions that $u$ is zero at the centre and at the edge are automatically fulfilled. Now the transverse displacements are many times the thickness of the diaphragm, hence large-deflexion theory must be employed. In Timoshenko's notation (loc. cit.) the radial and tangential strains are thus

$$
\left.\begin{array}{l}
\varepsilon_{r}=\frac{d u}{d r}+\frac{1}{2}\left(\frac{d w}{d r}\right)^{2} \\
e_{t}=\frac{u}{r} \tag{3}
\end{array}\right\}
$$

The diaphragm being very thin in comparison with its radius, the strain energy in it due to bending may be neglected in comparison with that due to the stretching of its middle plane. Hence the strain energy in the diaphragm is

$$
\begin{equation*}
V_{1}=\frac{\pi E h}{1-\nu^{2}} \int_{0}^{a} r\left(e_{r}^{2}+e_{t}^{2}+2 \nu e_{r} e_{t}\right) d r . \tag{4}
\end{equation*}
$$

Here $E$ denotes Young's modulus, $\nu$ Poisson's ratio, and $h$ the uniform thickness of the diaphragm. On putting (1) and (2) into (3) and inserting the results in (4) we obtain

$$
\begin{align*}
V_{1}= & \frac{\pi E h}{1-\nu^{2}}\left[\frac{1}{4} C_{1}^{2} a^{4}+\frac{3}{10} C_{1} C_{2} a^{5}+\frac{7}{60} C_{2}^{2} a^{6}+\frac{3 \pi^{4}}{1024} w_{0}^{2}\right. \\
& \left.+\frac{w_{0}^{2} a}{8}\left\{C_{1}\left(1-\frac{\pi^{2}}{6}+\nu\left(\frac{\pi^{2}}{6}+\frac{1}{2}\right)\right)+C_{2} a\left(\frac{5}{4}-\frac{\pi^{2}}{12}+\nu\left(\frac{\pi^{2}}{12}+\frac{1}{4}\right)\right)\right\}\right] \tag{5}
\end{align*}
$$

Now

$$
\begin{equation*}
\frac{\partial V_{1}}{\partial C_{1}}=\frac{\partial V_{1}}{\partial C_{2}}=0 \tag{6}
\end{equation*}
$$

hence

$$
\left.\begin{array}{l}
C_{1}=\frac{25 w_{0}^{2}}{128 a^{3}}\left\{\frac{\pi^{2}}{3}+\frac{17}{5}-\nu\left(\frac{\pi^{2}}{3}+1\right)\right\}  \tag{7}\\
C_{2}=-\frac{15 w_{0}^{2}}{128 a^{4}}\left\{\frac{\pi^{2}}{3}+13-\nu\left(\frac{\pi^{2}}{3}+1\right)\right\}
\end{array}\right\}
$$

In the remaining calculations we will consider the case $\nu=0.3$ when (7) reduces to

$$
\begin{equation*}
C_{1}=1.06 \frac{w_{0}^{2}}{a^{3}}, \quad C_{2}=-1.76 \frac{w_{0}^{2}}{a^{4}} \tag{8}
\end{equation*}
$$

[^10]The radial and tangential tensile stresses are

$$
\begin{equation*}
\sigma_{r}=\frac{E}{1-\nu^{2}}\left(e_{r}+\nu e_{t}\right), \quad \sigma_{t}=\frac{E}{1-\nu^{2}}\left(e_{t}+\nu e_{r}\right) . \tag{9}
\end{equation*}
$$

Then, with the aid of (3) and (8), (9) becomes

$$
\left.\begin{array}{l}
\sigma_{r}=\frac{E w_{0}^{2}}{a^{2}}\left\{1.51-7.11 \frac{r}{a}+6.38 \frac{r^{2}}{a^{2}}+1.36 \sin ^{2} \frac{\pi r}{a}\right\} \\
\sigma_{t}=\frac{E w_{0}^{2}}{a^{2}}\left\{1.51-4.95 \frac{r}{a}+3.67 \frac{r^{2}}{a^{2}}+0.41 \sin ^{2} \frac{\pi r}{a}\right\} \tag{10}
\end{array}\right\}
$$

These expressions are plotted in Fig. 2, from which it will be seen that the maximum stress occurs at the centre and is given by $1.51 E w_{0}^{2} / a^{2}$. If $w_{0} / a=1 / 35$ and $E=13000$ tons/sq. in., the maximum stress is 17 tons/sq. in., which for a good quality steel is a reasonable working stress. Finally we will examine the shape of the chamber corresponding to (10). It will be noticed from (2) that $u$ is zero not only at the centre and at the edge but also at $r / a=-C_{1} /\left(C_{2} a\right)=0.60$. On differentiating (2) it appears that $u$


Fig. 2. Radial and tangential tensile stresses in the diaphragm.
———— sinusoidal displacement.
--- cubic displacement.
-..-.- Bessel-function displacement.
has a maximum value $0.12 w_{0}^{2} / a$ at $r / a=0.24$ and a minimum value $-0.06 w_{0}^{2} / a$ at $r / a=0.82$. Hence the greatest radial difference between the shape of the chamber
and $w$ as given by (1) is $0.12 w_{0}^{2} / a$, which for $w_{0} / a=1 / 35$ is only $0.98 \times 10^{-4} a$. For a chamber of normal diameter this difference is too small to be appreciable in manufacture.

It is of interest to determine the strain undergone by a radius of the diaphragm, and for this purpose an accurate method of rectification is available. For a chamber of sinusoidal shape a radius is extended to a length $s$ given by

$$
\begin{gather*}
s=2 \int_{0}^{a / 2}\left(1+\frac{\pi^{2} w_{0}^{2}}{4 a^{2}} \cos ^{2} \frac{\pi r}{a}\right)^{1 / 2} d r \\
=\frac{2 a}{\pi}\left(1-p^{2}\right)^{-1 / 2} \int_{0}^{\pi / 2}\left(1-p^{2} \sin ^{2} \frac{\pi r}{a}\right)^{1 / 2} d\left(\frac{\pi r}{a}\right),  \tag{11}\\
p^{2}=\frac{\pi^{2} w_{0}^{2}}{4 a^{2}} /\left(1+\frac{\pi^{2} w_{0}^{2}}{4 a^{2}}\right) .
\end{gather*}
$$

where

Since $p$ is small, the first bracket in (11) may be expanded by the binomial theorem and the complete elliptic integral replaced by

$$
E=\frac{\pi}{2}\left(1-\frac{p^{2}}{4}-\frac{3 p^{4}}{64}-\cdots\right)
$$

The strain of the radius then reduces to

$$
\begin{equation*}
\frac{s-a}{a}=\frac{p^{2}}{4}+\frac{13 p^{4}}{64}+\cdots \tag{12}
\end{equation*}
$$

For $w_{0} / a=1 / 35$ this strain amounts to $0.05 \%$, hence in a steel wire distorted in to this sinusoidal form the tensile stress would be only $0.0005 \times 13000=6.5$ tons/sq.in.
3. Stresses when the transverse displacement follows a cubic relation. To estimate how far the stresses depend on the expression assumed for $w$, we will in this section replace (1) by the cubic

$$
\begin{equation*}
w=w_{0}\left(1-\frac{3 r^{2}}{a^{2}}+\frac{2 r^{3}}{a^{3}}\right) \tag{13}
\end{equation*}
$$

This equation satisfies the same four boundary conditions as (1), and the greatest difference between the two is approximately $0.010 w_{0}$ at $r / a=0.28$ and 0.72 . After employing (2), (3), (4) and (6) we find that

$$
\begin{align*}
V_{1}= & \frac{\pi E h}{1-\nu^{2}}\left[\frac{1}{4} C_{1}^{2} a^{4}+\frac{3}{10} C_{1} C_{2} a^{5}+\frac{7}{60} C_{2}^{2} a^{6}+\frac{9}{35} \frac{w_{0}^{4}}{a^{2}}\right. \\
& \left.+\frac{3}{70} w_{0}^{2} a\left\{2 C_{1}(3 \nu-1)+C_{2} a(3 \nu+1)\right\}\right]  \tag{14}\\
C_{1}= & \frac{-3 w_{0}^{2}}{56 a^{3}}(23-15 \nu), \quad C_{2}=-\frac{9 w_{0}^{2}}{56 a^{4}}(11-3 \nu) . \tag{15}
\end{align*}
$$

For $\nu=0.3$, (15) reduces to

$$
\begin{equation*}
C_{1}=0.99 \frac{w_{0}^{2}}{a^{3}}, \quad C_{2}=-1.62 \frac{w_{0}^{2}}{a^{4}} \tag{16}
\end{equation*}
$$

and the stresses obtained from (3) and (9) are

$$
\begin{align*}
& \sigma_{r}=\frac{E w_{0}^{2}}{a^{2}}\left\{1.42-6.61 \frac{r}{a}+25.67 \frac{r^{2}}{a^{2}}-39.56 \frac{r^{3}}{a^{3}}+19.78 \frac{r^{4}}{a^{4}}\right\} \\
& \sigma_{t}=\frac{E w_{0}^{2}}{a^{2}}\left\{1.42-4.60 \frac{r}{a}+9.32 \frac{r^{2}}{a^{2}}-11.87 \frac{r^{3}}{a^{3}}+5.93 \frac{r^{4}}{a^{4}}\right\} \tag{17}
\end{align*}
$$

From Fig. 2, in which these expressions also are shown, it will be seen that the maximum stress is slightly smaller than that obtained in section 2.
4. Stresses when the transverse displacement follows a Bessel-function relation. Lastly we will take $w$ as given by

$$
\begin{equation*}
w=W_{0}\left\{J_{0}(k r)-m\right\}, \tag{18}
\end{equation*}
$$

where $k=\alpha / a, \alpha=3.83 \cdots$ being the first positive root of $J_{1}(x)=0$,

$$
m=J_{0}(\alpha)=-0.402 \cdots
$$

and

$$
W_{0}=w_{0} /\left\{J_{0}(0)-J_{0}(\alpha)\right\}=w_{0} / 1.402 \cdots
$$

This equation satisfies the four boundary conditions, and it gives a displacement which, unlike those previously considered, is unsymmetrical about the line $w=w_{0} / 2$. Except at $r=0$ and $r=a$ the displacement is everywhere less than that specified by (1), the greatest difference between the two being approximately $0.019 w_{0}$ at $r / a=0.53$. The same procedure as before leads to

$$
\begin{align*}
V_{1}= & \frac{\pi E / 2}{1-\nu^{2}}\left[\frac{1}{4} C_{1}^{2} a^{4}+\frac{3}{10} C_{1} C_{2} a^{5}+\frac{7}{60} C_{2}^{2} a^{6}+\frac{W_{0}^{4} k^{4}}{4} \int_{0}^{a} r J_{1}^{4}(k r) d r\right. \\
& +W_{0}^{2}\left\{\frac{k^{2} \nu a^{3} J_{0}^{2}(k a)}{12}\left(C_{2} a+3 C_{1}\right)\right. \\
& \left.\left.+\frac{3}{8}(2+\nu)\left(C_{2} a-C_{1}\right)\left(\int_{0}^{a} J_{0}^{2}(k r) d r-a J_{0}^{2}(k a)\right)\right\}\right]  \tag{19}\\
C_{1}= & \frac{5 W_{0}^{2}}{4 a^{4}}\left[6(2+\nu)\left\{\int_{0}^{a} J_{0}^{2}(k r) d r-a J_{0}^{2}(k a)\right\}-k^{2} \nu a^{3} J_{0}^{2}(k a)\right], \\
C_{2}= & -\frac{5 W_{0}^{2}}{4 a^{5}}\left[9(2+\nu)\left\{\int_{0}^{a} J_{0}^{2}(k r) d r-a J_{0}^{2}(k a)\right\}-k^{2} \nu a^{3} J_{0}^{2}(k a)\right] . \tag{20}
\end{align*}
$$

If we take $\nu=0.3$ and $\int_{0}^{\alpha} J_{0}^{2}(x) d x=1.2599,{ }^{3}(20)$ reduces to

[^11]\[

$$
\begin{equation*}
C_{1}=1.01 \frac{w_{0}^{2}}{a^{3}}, C_{2}=-1.74 \frac{w_{0}^{2}}{a^{4}} \tag{21}
\end{equation*}
$$

\]

and the stresses are

$$
\left.\begin{array}{l}
\sigma_{r}=\frac{E w_{0}^{2}}{a^{2}}\left\{1.44-6.95 \frac{r}{a}+6.31 \frac{r^{2}}{a^{2}}+4.10 J_{1}^{2}(k r)\right\} \\
\sigma_{t}=\frac{E w_{0}^{2}}{a^{2}}\left\{1.44-4.84 \frac{r}{a}+3.63 \frac{r^{2}}{a^{2}}+1.23 J_{1}^{2}(k r)\right\} \tag{22}
\end{array}\right\}
$$

These stress distributions, which are plotted in Fig. 2, are in close accord with the results obtained in sections 2 and 3.
5. Conclusions. The following conclusions emerge from the above calculations:-
(i) For the three kinds of displacement considered, the maximum stress in the diaphragm is at the centre and is about $1.5 E w_{0}^{2} / a^{2}$.
(ii) The stress distributions due to the three kinds of displacement do not differ widely. Hence, if it is decided to use one kind, and small errors are made in the difficult process of machining the chambers, no great alteration in the stresses will result.

# THE INTRINSIC THEORY OF THIN SHELLS AND PLATES PART II.-APPLICATION TO THIN PLATES* 

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7. The general equations for a thin plate. We shall now investigate the equations of equilibrium and compatibility for a thin plate, not necessarily of constant thickness. First, we shall introduce the condition that the system is a plate, i.e., its middle surface in the unstrained state is plane. We have therefore

$$
\begin{equation*}
b_{\alpha \beta}=0, \quad R_{\alpha \beta \gamma \rho}=0 \tag{7.1}
\end{equation*}
$$

Furthermore, in order to simplify the problem, we assume in the following sections that the body force forms a parallel vector field, and therefore (3.38), (3.39) are satisfied; this is true for most practical problems.

Substituting (7.1) and the conditions on body force into (6.34) and (6.35), we have three equations of equilibrium for a thin plate

$$
\begin{align*}
& -2 A_{(1)}^{\rho \gamma \pi \lambda} q_{\rho \gamma} p_{\pi \lambda} h+\frac{2}{3} A_{(1)}^{\rho \gamma \pi \lambda}\left(q_{\pi \lambda} h^{3}\right)_{a}+A_{(3)}^{\rho \gamma \omega \lambda \delta} q_{\pi \omega} q_{\lambda \delta} q_{\rho \gamma} h^{3}+P^{0} \\
& +2 X_{[0]}^{0} h+\left(Q^{\pi} h\right)_{a \pi}+\frac{1-2 \sigma}{1-\sigma} a^{\pi \lambda} q_{\pi \lambda} Q^{0} h=O_{(42)}^{0}, \tag{7.2a}
\end{align*}
$$

$$
\begin{align*}
& +2 X_{[0]}^{\alpha} h+\frac{\sigma}{1-\sigma} a^{\alpha \rho}\left(Q^{0} h\right)_{\frac{10}{}}+\left(a^{\pi \delta} a^{\alpha \gamma}+2 a^{\alpha \delta} a^{\pi \gamma}\right) Q_{\gamma} q_{\pi \delta} h=O_{(43)}^{\alpha}, \tag{7.2b}
\end{align*}
$$

where the $O$-symbols have the following magnitudes,
$O_{(42)}^{0} ; X^{0} p h, X^{0} \widehat{Q} h, p^{2} q h, \widehat{Q}^{2} q h, \widehat{Q} q p h, \widehat{P} q h^{2}, X^{0} h^{2}, \widehat{Q} h^{3}, q \widehat{P} h^{3}, q p h^{3}, q X h^{3}, q h^{5}$. (7. 3a)
$O_{(43)}^{\alpha} ; p^{2} h, \widehat{Q} p h, \widehat{X} p h, \widehat{X} \widehat{Q} h, \widehat{Q}^{2} h, P q h^{2}, \widehat{Q} h^{3}, \widehat{X} h^{3}, p h^{3}, q h^{5}$.
We recall that

$$
\begin{align*}
A_{(1)}^{\alpha \beta \pi \lambda} & =\frac{1}{1-\sigma^{2}}\left\{\sigma a^{\alpha \beta} a^{\pi \lambda}+(1-\sigma) a^{\alpha \pi} a^{\beta \lambda}\right\}  \tag{7.3c}\\
A_{(3)}^{\alpha \beta \pi \lambda \delta} & =\frac{2(2 \sigma-1)}{3(1-\sigma)} a^{\delta \lambda} A_{(1)}^{\alpha \theta_{\pi \gamma}}+\frac{5}{3} a^{\delta \pi} A_{(1)}^{\alpha \beta \lambda \gamma} . \tag{7.3d}
\end{align*}
$$

Similarly, substituting (7.1) into (6.43) and (6.44), we have three equations of compatibility for a thin plate

$$
\begin{aligned}
2 n_{[0 \mid}^{\beta \gamma} q_{a \beta \mid \gamma}\left(1+2 n_{[0]}^{\pi \lambda} n_{[0]}^{\rho \delta} p_{\delta \lambda \lambda} p_{\pi \rho}\right. & \left.+2 a^{\pi \lambda} p_{\pi \lambda}\right) \\
& -2 n_{[0]}^{\beta \lambda} q_{\beta \pi}\left(a^{\pi \lambda}+2 n_{[0]}^{\pi p} n_{[0]}^{\lambda \delta} p_{\rho \delta}\right)\left(p_{\alpha \lambda \mid \gamma}+p_{\gamma \lambda \mid \alpha}-p_{\alpha \gamma \mid \lambda}\right)=0, \text { (7.4a)}
\end{aligned}
$$

[^12]The macroscopic tensors in (6.29), (6.30), (6.31) can be written as

$$
\begin{align*}
& T^{\alpha \beta}=2 A_{(1)}^{\alpha \beta \pi \lambda} p_{\pi \lambda} h-A_{(3)}^{\alpha \beta \pi \lambda \lambda} q_{\pi \rho} q_{\lambda \delta} h^{3}+\frac{\sigma}{1-\sigma} a^{\alpha \beta} Q^{0} h+O_{(44)}^{\alpha \beta},  \tag{7.5a}\\
& L^{\alpha \beta}=\frac{2}{3} n_{101}^{\rho \beta} a_{\pi \rho} A_{(1)}^{\alpha \pi \lambda \delta} q_{\lambda \delta} h^{3}+O_{(15)}^{\alpha \beta},  \tag{7.5b}\\
& T^{\alpha 0}=\frac{2}{3} A_{(1)}^{\pi \alpha \lambda \lambda}\left(q_{\lambda \delta} h^{3}\right)_{\left.\right|_{a}}+Q^{\alpha} h+\left(a^{\pi \lambda} P^{\alpha}+a^{\alpha \pi} P^{\lambda}\right) q_{\pi \lambda} h^{2}+O_{(46),}^{\alpha 0}, \tag{7.5c}
\end{align*}
$$

where
$O_{(44)}^{\alpha \beta}=O_{(44)}^{\alpha \beta}\left(p^{2} h, \widehat{Q}^{2} h, \widehat{Q} p h, p h^{3}, \widehat{Q} h^{3}, \widehat{X} h^{3}, q^{2} h^{5}\right)$,
$O_{(45)}^{\alpha \beta}=O_{(45)}^{\alpha \beta}\left(q p h^{3}, X^{0} h^{3}, \widehat{Q} h^{3}, q^{2} h^{5}, X q h^{5}\right)$,
$O_{(4 \beta)}^{a 0}=O_{(40)}^{\alpha 0}\left(\widehat{Q} p h, \widehat{Q}^{2} h, \widehat{Q} \widehat{X} h^{2}, \widehat{Q} q h^{2}, \widehat{P} p h^{2}, \widehat{P} \widehat{Q} h^{2}, q p h^{3}, X^{0} h^{3}, \widehat{Q} h^{3}, X q h^{3}, q^{2} h^{5}\right)$.
Equations (7.2a, b) and (7.4a, b) are the six differential equations of a thin plate in the six unknowns $p_{\alpha \beta}$ and $q_{\alpha \beta}$. The next step is to introduce certain systematic approximations based upon the thinness of the plate, so as to obtain a set of differential equations in simpler form.
8. Classification of all thin plate problems. We consider a family of $\infty^{1}$ thin plates of the same material, having an identical middle surface $S_{0}^{\prime}$ in the unstrained state, but different thicknesses; each is subject to the action of (i) external force systems applied at the edges, (ii) surface loadings on its two boundary surfaces, and (iii) uniform body force throughout the plate. (This includes gravity, but excludes a centrifugal field.) We attach to the middle surface of each plate the same system of coordinates $x^{\alpha}$, so that the fundamental tensor $a_{\alpha \beta}$ is the same for all plates in this family. We assign to each plate a value of a parameter $\epsilon$, so that the thickness of all the plates can be represented by

$$
\begin{equation*}
2 h=2 \epsilon \bar{h}\left(\mathrm{x}^{1}, \mathrm{x}^{2}\right) \tag{8.1}
\end{equation*}
$$

where $0<\epsilon<\epsilon_{1}$ and the function $\bar{h}$ is the same for all the plates; for thin plates, $\epsilon_{1}$ is supposed to be small, but the basic idea of the method is that we seek solutions valid for all $\epsilon$ in the range $0<\epsilon<\epsilon_{1}$.

Equation (8.1) implies that the derivatives of the thickness at any point are of the same order of magnitude as the thickness itself. We shall call these plates "regular plates." On the other hand, if the thickness and its derivatives are of different orders of magnitude, we have an "irregular plate." The following theory is limited to regular plates only.

We may suppose $\epsilon$ chosen equal to the ratio of the average thickness to a selected lateral dimension (usually the smallest lateral dimension) of the plate. For a circular plate, $\epsilon$ is the ratio of the average thickness to the diameter of the plate. For a rectangular plate, it may be chosen equal to the ratio of the average thickness to the length of the shorter side.

It is important to observe that $\epsilon$ is the only parameter involved. Except the fundamental tensor $a_{\alpha \beta}$ and Poisson's ratio $\sigma$, all the other quantities occurring are func-
tions of $\epsilon$, and no quantity is "small" unless it tends to zero with $\epsilon$. (Young's modulus does not appear, on account of the use of reduced stresses and body forces.) Thus for any "small quantity" $\Psi$, we must have

$$
\begin{equation*}
\lim _{\mapsto \rightarrow 0} \Psi=0 \tag{8.2}
\end{equation*}
$$

In order that a problem may belong to the theory of small strain, $e_{i j}$ must be a small quantity, and therefore

$$
\begin{equation*}
\lim _{e \rightarrow 0} e_{i j}=0 \tag{8.3}
\end{equation*}
$$

It follows that $p_{a \beta}$ must also be a "small quantity," depending on $\epsilon$ like $\Psi$ in (8.2). But this is not necessarily true for $q_{\alpha \beta}$.

It is understood that all conditions (such as reduced edge forces, reduced surface loadings, and reduced body forces) depend on $\epsilon$ in such a way that (8.3) holds. We shall assume that $Q^{i}, P^{i}, X_{[0]}^{i}$ vanish at least as fast as $\epsilon$, and are in fact power series in $\epsilon$. This assumption implies that the derivatives of any of these quantities with respect to $x^{\alpha}$ are of the same order of magnitude as (or higher order of magnitude than) the quantity itself. Hence we write

$$
\begin{array}{rlrl}
Q^{0} & =\sum_{s=k_{0}}^{\infty} Q_{(s) \epsilon^{s},}^{0} & Q^{\alpha} & =\sum_{s=k}^{\infty} Q_{(s) \epsilon^{s}}, \\
P^{0} & =\sum_{s=n_{0}}^{\infty} P_{(s) \epsilon^{s},}^{0} & P^{\alpha} & =\sum_{s=n}^{\infty} P_{(s) \epsilon^{s},}^{\alpha} \\
X_{[0]}^{0}=\sum_{s=j_{0}}^{\infty} X_{(s)[0] \epsilon^{\epsilon},}^{0} & X_{[0]}^{\alpha} & =\sum_{s=j}^{\infty} X_{(s)[0] \epsilon^{s}}^{\alpha} . \tag{8.4c}
\end{array}
$$

where $k, k_{0}, n, n_{0}, j, j_{0}$ are integers greater than zero, and $P_{(s)}^{t}, Q_{(s)}^{t}, X_{(s)[0]}^{\prime}$ are functions of $x^{\alpha}$, independent of $\epsilon$.

Similarly we assume that the traction, shearing force and bending moment applied on the edge curve can be represented by

$$
\begin{equation*}
\tilde{T}^{\alpha \beta}=\sum_{s=t}^{\infty} \tilde{T}_{(s) \epsilon^{\alpha}}^{\alpha \beta}, \quad(8.5 \mathrm{a}) \quad \tilde{L}^{\alpha \beta}=\sum_{s=1}^{\infty} \tilde{L}_{(s) \epsilon^{s},}^{\alpha \beta} \quad(8.5 \mathrm{~b}) \quad \tilde{T}^{\alpha 0}=\sum_{s=1}^{\infty} \tilde{T}_{(s)}^{\alpha \beta} \epsilon^{s}, \tag{8.5a}
\end{equation*}
$$

where $t, u, l$ are positive integers, and $\bar{T}_{(s)}^{\alpha \beta}, \tilde{L}_{(s)}^{\alpha \beta}, \widetilde{T}_{(s)}^{\alpha 0}$ are functions of position on the edge curve, independent of $\epsilon$.

Now the problem is to find the behaviour of the family of $\infty^{1}$ thin plates under the action of a given family of external force systems (8.4), (8.5). Given an external force system defined by (8.4), (8.5), we seek solutions of the equations of equilibrium (7.2) and the equations of compatibility (7.4) of the form

$$
\begin{equation*}
p_{\alpha \beta}=\sum_{i=p}^{\infty} p_{(z) \alpha \beta \epsilon^{*},} \tag{8.6a}
\end{equation*}
$$

$$
\begin{equation*}
q_{a \beta}=\sum_{s=q}^{\infty} q_{(s) \alpha \beta \varepsilon^{s},} \tag{8.6b}
\end{equation*}
$$

where $p$ and $q$ are zero or positive integers, and $p_{(\rho) \alpha \beta}$ and $q(o) \alpha \beta$ are functions of $x^{\alpha}$, independent of $\epsilon$. Only those problems admitting solutions with $p>0$ belong to the
theory of small strain. On the other hand, $q$ may be zero; then we are dealing with a finite deflection problem.

The usual discussion of plate theory is based on the deflection, i.e. the normal displacement of a particle on the middle surface. The present method is intrinsic, and the general equations contain no explicit reference to the displacement. However, since $q_{\alpha \beta}$ corresponds to change of curvature (i.e. curvature of the middle surface after strain), it is clear that finite values of $q_{\alpha \beta}$ correspond to finite deflection and small values of $q_{\alpha \beta}$ to small deflection. Similar remarks apply in the case of shells. Hence, in classification, we may use the familiar word "deflection" when referring to the order of magnitude of $q_{\alpha \beta}$.

The assumed forms ( $8.6 \mathrm{a}, \mathrm{b}$ ) imply that the derivatives of $p_{\alpha \beta}, \boldsymbol{q}_{\alpha \beta}$ with respect to $x^{\alpha}$ are of the same order of magnitude as the quantities themselves, or of higher order. In fact, $p_{\alpha \beta}$ and $q_{\alpha \beta}$ expressed by ( $8.6 \mathrm{a}, \mathrm{b}$ ) represent the behaviour of the family of $\infty^{1}$ thin plates under the action of the given family of $P^{i}, Q^{i}, X_{[0]}^{i}$, $\tilde{T}^{\alpha \beta}, \tilde{T}^{\alpha 0}, \tilde{L}^{\alpha \beta}$ defined by the equations (8.4), (8.5). It is understood that if $P^{i}, Q^{i}, X_{[0]}^{i}$, $\widetilde{T}^{\alpha \beta}, \bar{T}^{\alpha 0}, \tilde{L}^{\alpha \beta}$ are identically equal to zero (i.e., $k, k_{0}, n, n_{0}, j, j_{0}, t, u, l=\infty$ ), then $p_{\alpha \beta}$ and $q_{\alpha \beta}$ vanish (i.e., $p, q=\infty$ ) everywhere; this corresponds to the unstrained state of the plate. This means that self-strained plates are not discussed.

In a thin plate problem, we are to regard the numbers $k, k_{0}, n, n_{0}, j, j_{0}, t, u, l$ as given; the initial step towards solution would appear to be the determination of $p$ and $q$, for then we could simplify the equations of equilibrium and compatibility in the first approximation by picking out the principal terms in $\in$ from equations (7.2a, b), (7.4a, b). But owing to the partial indeterminacy of $p$ and $q$, this method is not successful.

It is much simpler to solve the problem in the reverse order. First we assign integral values to $p$ and $q$. The values of $k, k_{0}, n, n_{0}, j, j_{0}$ are fixed by the conditions that $X_{\left(U_{0}\right)[0]}^{0}, X_{(j)[(0)}^{\alpha}, P_{\left(n_{0}\right)}^{0}, P_{(n)}^{\alpha}, Q_{\left(k_{0}\right)}^{0}, Q_{(k)}^{\alpha}$ should contribute to the principal parts of ( $7.2 \mathrm{a}, \mathrm{b}$ ), without dominating these equations to the exclusion of $p_{\alpha \beta}$ and $q_{\alpha \beta}$. The equations of equilibrium and compatibility in the first approximation are then obtained by picking out the principal terms in $\epsilon$ from equations (7.2a, b), (7.4a, b). Then the values of $t, u, l$ are automatically fixed through the expressions (7.5).

We shall now discuss the classification of thin plate problems based on assigned values of $p$ and $q$, so that the principal parts of (7.2a, b), (7.4a,b) in the first approximation are different for different "Types." The classification is shown graphically in Fig. 3, where permissible pairs of $(p, q)$-values are represented by circles. As indicated in ( $8.6 \mathrm{a}, \mathrm{b}$ ), we consider only non-negative integral values of $p$ and $q$. Since, however, $p=0$ corresponds to finite extension of the middle surface, we must omit the $(p, q)$ points on the $q$-axis.

It is found that the points in the $(p, q)$-plane break up into twelve groups depending on their positions relative to the division lines $A D, A B, O C$ and the $p$-axis. For any point (except $q=0$ ) on the line $A D$, it is easily seen from inspection of (7.2a) that the first and second terms are of the same order of magnitude and prevail over all the other terms, with possible exception of those involving $X_{[0]}^{i}, P^{i}, Q^{i}$. For any ( $p, q$ ) -point (except $q=0$ ) above $A D$, the second term in (7.2a) dominates, and for any $(p, q)$-point below $A D$, the first term dominates. For the point $A$, the first three terms in (7.2a) are of the same order of magnitude and prevail over the right hand side. For any point on the $p$-axis above $A$, the second and third terms in (7.2a) are
of the same order of magnitude and prevail over the other terms. Thus the principal part of (7.2a) takes five different forms depending on the position of the ( $p, q$ )-point relative to the line $A D$ and the $p$-axis.

Similarly, the form of the dominant part of (7.2b) depends on the position of the ( $p, q$ )-point relative to the line $A B$ and the $p$-axis. Finally, the form of the dominant part of (7.4b) depends on the position of the $(p, q)$-point relative to the line $O C$ and the $p$-axis. The equation (7.4a) has no division line, since the term $n_{[0 \mid}^{\beta \gamma} q_{\alpha \beta \mid \gamma}$ dominates for any position of the $(p, q)$-point.


FIG. 3. Classification of thin plate problems.
$p=$ order of extension of middle surface, $q=$ order of change of curvature of middle surface.
(Type $P 12$ is not indicated in the diagram, since for these problems, $q=\infty$, and consequently the corresponding points lie at infinity to the right hand side.)

It follows that the $(p, q)$-plane is divided into twelve regions, so far as permissible non-negative integral values of $p$ and $q$ are concerned, and so the complete classification of all thin plate problems involves consideration of twelve types (Types P1-P12). Type $P 12$ is not indicated in the diagram, since for these problems, $q=\infty$, and consequently the corresponding points lie at infinity to the right hand side.

Although the classification gives twelve types, four of these (Types P3, P6, P7, $P 8)$ are less important than the others. They represent overdetermined problems, in which the number of equations exceed the number of unknowns. Such cases can occur only when very special relations connect the body forces and surface forces.

These twelve types may be described as follows:
(1) Problems of finite deflection ( $q=0$ ), Types $P 1-P 3$.
(2) Problems of small deflection ( $q \geqq 1, p=1 ; q=1, p=2 ; q \geqq 1, p>2 q$ ), Types P4-P8.
(3) Problems of very small deflection ( $q \geqq 2,2 q \geqq p \geqq 2$ ), Types $P 9-P 11$.
(4) Problems of zero deflection ( $q=\infty$ ), Type $P 12$.

In order to save space, we shall not discuss all the twelve types in detail. The discussion of Types P1, P2, P3 will serve as an example. The results for all types are summarized in the tables in the Appendices at the end of this paper. The principal parts of the equations of equilibrium and compatibility are shown in Table I, and the orders of magnitude of the external forces and the principal parts of the macroscopic tensors in Table II.

It should be noted that the theory of generalized plane stress [1, 2], the LagrangeKirchhoff theory of small deflection $[3,4,5]$, and the von Kármán theory of "large" deflection [6] can be derived respectively from the Types $P 12, P 11, P 5$.

We shall devote the next section to discussing the problems of finite deflection ( $P 1-P 3$ ). All results for these types are new, and may prove particularly interesting.
9. Problems of finite deflection ( $q=0$ ), types $P 1-P 3$.
(a) Type $P 1: q=0, p=1$. Finite deflection with dominant extension in the middle surface
General equations. By the condition that, in the first approximation, (7.2a, b) receive significant contributions from $P_{\left(n_{2}\right)}^{0}, P_{(n)}^{\alpha}, X_{(0) 1)(0),}^{0}, X_{(j)[0]}^{\alpha}, Q_{\left(k_{1}\right)}^{0}, Q_{(k)}^{(\alpha)}$, we must have

$$
\begin{equation*}
n_{0}=n=2, \quad i_{0}=j=1, \quad k_{0}=k=1 \tag{9.1}
\end{equation*}
$$

Therefore, we obtain from (6.23)

$$
\begin{equation*}
h_{(+)}=\bar{h}_{\epsilon}+0\left(\epsilon^{2}\right), \quad h_{(-)}=\bar{h} \epsilon+0\left(\epsilon^{2}\right) ; \tag{9.2}
\end{equation*}
$$

consequently, the common assumption that the middle surface of the unstrained plate is deformed into the middle surface of the strained state is justified in the first approximation.

We now substitute (8.1), (8.4)-(8.6) into (7.2), (7.4). The lowest power of $\epsilon$ occurring is $\epsilon^{2}$ in (7.2), and $\epsilon^{0}$ in (7.4). The corresponding coefficients give rise to equations of equilibrium and compatibility in the first approximation as follows:

$$
\begin{align*}
& \left.-2 A_{(1)}^{0}\right)^{\pi} q_{(0)_{\rho \gamma} P_{(1) \pi \lambda} \bar{h}}+P_{(2)}^{0}+2 X_{(1)(0)}^{0} \bar{h}+\left(Q_{(1)}^{\pi} \bar{h}\right)_{\alpha}+\frac{1-2 \sigma}{1-\sigma} ⿷^{\pi \lambda} q_{(0) \times \lambda} Q_{(1)}^{0} \bar{h}=0, \tag{9.3a}
\end{align*}
$$

$$
\begin{align*}
& +\left(\mathrm{a}^{\pi \lambda} \mathrm{a}^{\alpha \gamma}+2 \mathrm{a}^{\alpha \lambda} \mathrm{a}^{\pi \gamma}\right) q_{(0) \pi \lambda} Q_{(1) \gamma} \bar{h}=0,(9.3 \mathrm{~b}) \\
& \boldsymbol{n}_{(019}^{\beta \beta} q_{(0) a \gamma \gamma_{a}^{1 \beta}}=0,  \tag{9.3c}\\
& \left.\eta_{[00}^{\alpha \beta} n_{[0]}^{\rho \gamma} q_{(0)}\right)_{\beta} q_{(0) \alpha \gamma}=0 . \tag{9.3d}
\end{align*}
$$

We may remark that all quantities in the above equations are finite, i.e. independent of $\epsilon$. The macroscopic tensors in (7.5) can be written as

$$
\begin{align*}
& T^{\alpha \beta}=\left(2 A_{(1)}^{\alpha \beta \pi \lambda} p_{(1) \pi \lambda} \bar{h}+\frac{\sigma}{1-\sigma} a^{\alpha \beta} Q_{(1)}^{0} \bar{h}\right) \epsilon^{2}+O\left(\epsilon^{3}\right),  \tag{9.4a}\\
& L^{\alpha \beta}=\frac{2}{3} n_{[0]}^{\omega \beta} a_{\pi \omega} A_{(1)}^{\alpha \pi \lambda \delta} q_{(0) \lambda \delta} \bar{h}^{3} \epsilon^{3}+O\left(\epsilon^{4}\right),  \tag{9.4b}\\
& T^{\alpha 0}=Q_{(1)}^{\alpha} \bar{h} \epsilon^{2}+O\left(\epsilon^{3}\right) \text { if } Q_{(1)}^{\alpha} \neq 0, \\
& T^{\alpha 0}=\left\{Q_{(2)}^{\alpha} \bar{h}+\frac{2}{3} A_{(1)}^{\pi \alpha \lambda \delta}\left(q_{\left.\left.(0) \lambda \delta \bar{h}^{3}\right)_{\mid \pi}\right\} \epsilon^{3}+O\left(\epsilon^{4}\right) \text { if } Q_{(1)}^{\alpha}=0 .}=0 .\right.\right. \tag{9.4c}
\end{align*}
$$

Equations ( $9.3 \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ ) form a set of six equations for the six unknowns $q_{(0) \alpha \beta}$ and $p_{(1) \alpha \beta}$. From ( 9.3 d ), we see that the total curvature of the middle surface does not change for the first approximation. Consequently the strained middle surface in this type of problem is to be regarded as a developable surface.

Equations (9.3a, b, c, d) can be further simplified. Since for a plate, $R_{\rho \alpha \beta \gamma}=0$, the order of the operations of covariant differentiation is immaterial; consequently, from (9.3c), we have

$$
\begin{equation*}
q_{(0)_{\alpha \beta}}=W_{(0)_{a} \alpha \beta} . \tag{9.5}
\end{equation*}
$$

Here $W_{(0)}$ is an unknown function of $x^{\alpha}$, which satisfies, in consequence of (9.3d),

$$
\begin{equation*}
\left.\left.n_{[0]}^{\alpha \beta} n_{[0]}^{\beta \gamma} W(0)\right|_{a} \mathcal{D}^{W} W(0)\right|_{a} \alpha=0 . \tag{9.6}
\end{equation*}
$$

The existence of $w_{(0)}$, satisfying (9.5), is easily proved by temporary use of special coordinates (rectangular Cartesians). The last equation is, in fact, the famous differential equation [7] of a developable surface in the curvilinear coordinate system, and $w_{(0)}$ may be called the deflection function. If (9.6) is satisfied, $q_{(0) \times \gamma}$ is given by (9.5). There still remain the three equations $(9.3 a, b)$ for the three unknowns $p_{(1) r r}$. We can handle the problem indirectly by means of $T_{(2)}^{\alpha \beta}$. This is the coefficient of the lowest power, $\epsilon^{2}$, in the series for $T^{\alpha \beta}$, and by (9.4a)

$$
\begin{equation*}
T_{(2)}^{\alpha \beta}=2 A_{(1)}^{\alpha \beta \pi \lambda} p_{(1) \pi \lambda} \bar{h}+\frac{\sigma}{1-\sigma} \mathbf{a}^{\alpha \beta} Q_{(1)}^{0} \bar{h} . \tag{9.7}
\end{equation*}
$$

We note that this is a symmetrical tensor, so that it has only three independent components. Substituting (9.5), (9.7) into (9.3a, b), we have

$$
\begin{align*}
- & T_{(2)}^{\pi \lambda} W_{(0) \mid \tau \lambda}+P_{(2)}^{0}+2 X_{(1) \mid 0]}^{0} \bar{h}+\left(Q_{(1)}^{\pi} \bar{h}\right)_{\mid \pi}+a_{a}^{\tau \lambda} W_{(0) \mid \pi \lambda} Q_{(1)}^{0} \bar{h}=0  \tag{9.8a}\\
& T_{(2) \mid \pi}^{\pi \alpha}+P_{(2)}^{\alpha}+2 X_{(1) \mid 0]}^{\alpha} \bar{h}+\left(a^{\pi \lambda} a^{\alpha \gamma}+2 a^{\alpha \lambda} a^{\pi \gamma}\right) W_{(0) \mid \pi \lambda} Q_{(1) \gamma} \bar{h}=0 . \tag{9.8b}
\end{align*}
$$

To sum up, for problems of type $P 1$, we have a set of four equations (9.6), (9.8a, b) in the four unknowns, $W_{(0)}$ and $T_{(2)}^{\alpha \beta}$.

Special case. The following special case is interesting. If

$$
\begin{equation*}
P_{(2)}^{\alpha}=X_{(1)[0]}^{\alpha}=Q_{(1)}^{\alpha}=0, \tag{9.9}
\end{equation*}
$$

then by (9.8b) there exists an Airy function $\chi_{(2)}$, so that

$$
\begin{equation*}
T_{(2)}^{\alpha \beta}=n_{(0)}^{\alpha \pi} n_{l 0 \mid}^{\beta \lambda} X_{(2)| | \pi \lambda}^{a} . \tag{9.10}
\end{equation*}
$$

This is easily proved by temporary use of special coordinates (rectangular Cartesians [2]). Consequently, (9.8a) can be reduced to the form

$$
\begin{equation*}
-n_{[0]}^{p \pi} n_{(0]}^{\lambda \delta} \chi_{(2) \mid \pi \delta}^{a} W_{(0) \mid \rho \lambda}^{a}+P_{(2)}^{0}+2 X_{(1)[0]}^{0} \bar{h}+a^{\pi \lambda} W_{(0) \mid r \lambda} Q_{(1)}^{0} \tilde{h}=0 \tag{9.11}
\end{equation*}
$$

The problem is now to find $\chi_{(2)}$ and $w_{(0)}$ as functions of $x^{\alpha}$ satisfying the two nonlinear partial differential equations (9.6) and (9.11). In rectangular Cartesian coordinates, the equations (9.6) and (9.11) may be written as

$$
\begin{gather*}
W(0), 12 W_{(0), 12}-W_{(0), 11}^{W}(0), 22=0  \tag{9.12a}\\
2 \chi_{(2), 12} W_{(0), 12}-\chi_{(2), 11} W_{(0), 22}-\chi_{(2), 22} W_{(0), 11}+Q_{(1)}^{0} \bar{h} \Delta W_{(0)}+P_{(2)}^{0}+2 X_{(1), 01}^{0} \bar{h}=0, \tag{9.12b}
\end{gather*}
$$

where the comma indicates partial differentiation with respect to $x^{\alpha}$, and $\Delta$ is the twodimensional Laplace operator. The macroscopic tensors are given by

$$
\begin{align*}
& \left.\begin{array}{l}
T^{11}=\chi_{(2), 22} \epsilon^{2}+O\left(\epsilon^{3}\right), \quad T^{22}=\chi_{(2), 11 \epsilon^{2}}+O\left(\epsilon^{3}\right), \\
T^{12}=T^{21}=-\chi_{(2), 12} \epsilon^{2}+O\left(\epsilon^{3}\right),
\end{array}\right\}  \tag{9.13a}\\
& \left.\begin{array}{l}
L^{11}=-L^{22}=-D(1-\sigma) \mathbf{w}^{r}(0), 12 \epsilon^{3}+O\left(\epsilon^{4}\right), \\
L^{12}=D\left(W_{(0), 11}+\sigma W(0), 22\right) \epsilon^{3}+O\left(\epsilon^{4}\right), \\
L^{21}=-D\left(W_{(0), 22}+\sigma W_{(0), 11)} \epsilon^{3}+O\left(\epsilon^{4}\right),\right.
\end{array}\right\}  \tag{9.13b}\\
& \left.T^{10}=\left\{(1-\sigma)\left(D W_{(0), 12}\right)_{.2}+\left(D_{(0), 11}+\sigma D_{(0), 22}\right)_{, 1}\right\} \epsilon^{3}+O\left(\epsilon^{4}\right)_{,}\right\} \\
& \left.T^{20}=\left\{(1-\sigma)\left(D w_{(0), 12}\right)_{1}+\left(D w_{(0), 22}+\sigma D W_{(0), 11}\right)_{, 2}\right\} \epsilon^{3}+O\left(\epsilon^{4}\right) .\right\} \tag{9.13c}
\end{align*}
$$

Here the symbol $D$ is defined by

$$
\begin{equation*}
D=\frac{2 \bar{h}^{3}}{3\left(1-\sigma^{2}\right)} \tag{9.14}
\end{equation*}
$$

This is a finite quantity; the ordinary flexual rigidity is $D \epsilon^{3} E$ (where $E$ is Young's modulus). An example of this type of problem is given below.

Example. A long rectangular plate is subjected to a uniform tension $T_{(2)} \epsilon^{2}$ on the two long edges, and a normal load $P_{(2)}^{0} \epsilon^{2}$ on one face; this normal load does not vary along the length of the plate. Find the form of the plate in the strained state.

In this example, we can neglect the edge effect near the end of the plate by considering the plate infinitely long. We assume that the middle surface in the strained state is cylindrical, with the generators of the cylinder parallel to the length of the plate, that is

$$
\begin{equation*}
q_{(0) 11}=W_{(0), 11}=\Omega\left(x^{1}\right), \quad q_{(0) 22}=q_{(0) 12}=0 . \tag{9.15}
\end{equation*}
$$

Here $x^{\alpha}$ are rectangular Cartesian coordinates, such that the $x^{2}$-axis is parallel to the long edges, and the $x^{1}$-axis perpendicular to them; $\Omega$ is an unknown function. Furthermore, in this example,

$$
\begin{equation*}
P_{(2)}^{\alpha}=X_{(1)\{0]}^{\alpha}=Q_{(1)}^{0}=Q_{(1)}^{\alpha}=X_{(1)\{0]}^{0}=0 . \tag{9.16}
\end{equation*}
$$

Then from (9.8b) and the condition that $T_{(2)}^{a \beta}$ are functions of $x^{1}$ only, we have, in consequence of the boundary conditions on the two long edges,

$$
\begin{equation*}
T_{(2)}^{11}=T_{(2)}, \quad(9.17 \mathrm{a}) \quad T_{(2)}^{12}=T_{(2)}^{21}=0, \quad(9.17 \mathrm{~b}) \quad T_{(2)}^{22}=0 \tag{9.17c}
\end{equation*}
$$

Substituting (9.15)-(9.17) into (9.8a), we obtain

$$
\begin{equation*}
\Omega\left(x^{1}\right)=\frac{P_{(2)}^{0}}{T_{(2)}} \tag{9.18}
\end{equation*}
$$

Therefore the curvature at any point of the cylindrical surface is proportional to the normal pressure at the point. For uniformly distributed pressure, the strained middle surface is circular cylindrical. It should be noted that the above conclusion holds in general for plates of non-uniform thickness, with the limitation that $\bar{h}$ is independent of $x^{2}$.
(b) Type $P 2: q=0, p=2$. Finite deflection with small extension in the middle surface

General equations. As in Type P1, we have

$$
\begin{equation*}
n_{0}=n=3, \quad j_{0}=j=2, \quad k_{0}=k=2 . \tag{9.19}
\end{equation*}
$$

By substituting the $\epsilon$ series from (8.1), (8.4)-(8.6) into (7.2a, b), (7.4a, b), it is found that the lowest power of $\epsilon$ occurring in (7.2a,b) is $\epsilon^{3}$, and in (7.4a, b) is $\epsilon^{0}$. The corresponding coefficients give rise to the equations of equilibrium and compatibility in the first approximation as follows:

$$
\begin{align*}
& -2 A_{(1)}^{\rho \gamma \gamma \lambda} q_{(0) \rho \gamma} P_{(2) \pi \lambda} \bar{h}+\frac{2}{3} A_{(1)}^{\rho \gamma \pi \lambda}\left(q_{(0) \pi \lambda} \tilde{h}^{3}\right)_{\mid \rho \gamma}+P_{(3)}^{0}+2 X_{(2)[0]}^{0} \bar{h} \\
& +A_{(3)}^{\rho \gamma \pi \omega \lambda} q_{(0) \pi \omega} q_{(0) \lambda \delta} q_{(0) \rho \gamma} \overline{h^{3}}+\left(Q_{(2)}^{\pi} \bar{h}\right)_{a x}+\frac{1-2 \sigma}{1-\sigma} a^{\pi \lambda} q_{(0) \pi \lambda} Q_{(2)}^{0} \bar{h}=0,  \tag{9.20a}\\
& 2 A_{(1)}^{\rho \alpha \pi \lambda}\left(p_{(2) \pi \lambda} \bar{h}\right)_{a}+A_{(3)}^{\rho \alpha \pi \omega \lambda \delta}\left(q_{(0) \pi \omega} q_{(0) \lambda \delta} \bar{h}^{3}\right)_{\left.\right|_{\rho}}+\frac{2}{3} a^{\alpha \pi} q_{(0) \pi \gamma} A_{(1)}^{\gamma \beta \lambda \delta \delta}\left(q_{(0) \lambda \delta} \overline{h^{3}}\right)_{\mid \beta}+P_{(3)}^{\alpha} \\
& +2 X_{(2)[0]}^{\alpha} \bar{h}+\left(a^{\pi \lambda} a^{\alpha \gamma}+2 a^{\alpha \lambda} a^{\pi \gamma}\right) q_{(0) \pi \lambda} Q_{(2) \gamma} \bar{h}+\frac{\sigma}{1-\sigma} a^{\alpha \beta}\left(Q_{(2)}^{0} \bar{h}\right)_{\mid \beta}=0,  \tag{9.20b}\\
& n_{[01}^{\gamma \beta} q_{(0) \alpha \gamma \mid \beta}=0, \quad(9.20 c) ; \quad n_{[0]}^{\alpha \beta} n_{[0]}^{\rho \gamma} q_{(0)_{\beta \beta} q_{(0) \alpha \gamma}=0 .} \tag{9.20d}
\end{align*}
$$

The macroscopic tensors in ( $7.5 \mathrm{a}, \mathrm{b}, \mathrm{c}$ ) can be written as

$$
\begin{align*}
& T^{\alpha \beta}=\left\{2 A_{(1)}^{\alpha \beta \beta \lambda} p_{(2) \pi \lambda} \bar{h}+\frac{\sigma}{1-\sigma} a^{\alpha \beta} Q_{(2)}^{0} \bar{h}-A_{(3)}^{\alpha \beta \pi \lambda \lambda} q_{(0) \times \omega} q_{(0) \lambda \delta} \bar{h}^{3}\right\} \epsilon^{3}+O\left(\epsilon^{4}\right),  \tag{9.21a}\\
& L^{\alpha \beta}=\frac{2}{3} n_{(0)}^{\mu \beta} a_{x \omega} A_{(1)}^{\alpha \pi \lambda \delta} q_{(0) \lambda \delta \delta \bar{h}^{3} \epsilon^{3}+O\left(\epsilon^{4}\right),}  \tag{9.21b}\\
& T^{a 0}=\left\{Q_{(2) \mid}^{\alpha} \bar{h}+{ }_{3}^{2} A_{(1)}^{\pi \alpha \lambda \delta}\left(q_{\left.\left.(0) \lambda \delta \delta h^{3}\right)_{\mid x}\right\} \epsilon^{3}+O\left(\epsilon^{4}\right) .}\right.\right. \tag{9.21c}
\end{align*}
$$

Here $A_{(1)}^{a \beta \pi \lambda}, A_{(3)}^{\alpha \beta \pi u \lambda)}$ are given as in ( $6.33 \mathrm{a}, \mathrm{b}$ ). Equations ( $9.20 \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ ) form a set of six fundamental equations for the six unknowns $p_{(2) \alpha \beta}$ and $q_{(0) \alpha \beta}$. We see that from ( 9.20 d ) that the middle surface in the strained state is a developable surface.

As in Type P1, the problem can be further simplified by introducing $w_{(0)}$, such that

$$
\begin{equation*}
q_{(0) \alpha \beta}=\left.W(0)\right|_{a} \alpha \beta . \tag{9.22}
\end{equation*}
$$

We have also

$$
\begin{equation*}
T_{(3)}^{\alpha \beta}=2 A_{(1)}^{\alpha \beta \pi \lambda} p_{(2) \pi \lambda} \bar{h}-A_{(3)}^{\alpha \beta \pi \alpha \lambda \delta} q_{(0) \pi \omega} q_{(0) \lambda \delta \delta} \bar{h}^{3}+\frac{\sigma}{1-\sigma} \mathbf{a}^{\alpha \beta} Q_{Q 2}^{0} \bar{h} \bar{h} . \tag{9.23}
\end{equation*}
$$

We note that $T_{(3)}^{\alpha \beta}$ is the coefficient of the lowest power of $\epsilon$ in $T^{\alpha \beta}$. It is a symmetric tensor, and consequently it has only three independent components. Substituting (9.22) and (9.23) into ( $9.20 \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ ), we find that ( 9.20 c ) is identically satisfied, while the other three equations become

$$
\begin{align*}
& +\left(Q_{(2)}^{\pi} \bar{h}\right)_{\mid \pi}+a^{\pi \lambda} w_{(0) \mid{ }_{a} \lambda} Q_{(2)}^{0} \bar{h}=0,  \tag{9.24a}\\
& T_{a}^{\rho \alpha)}+\underset{a}{\alpha}+\frac{2}{3} a^{\alpha \pi} W_{(0) \mid \pi \omega} A_{(1)}^{\omega \lambda \lambda \delta}\left(W_{(0) \mid \lambda \delta}^{a} \bar{h}^{3}\right)_{\mid \rho}+P_{(3)}^{\alpha}+2 X_{(2) \mid[0]^{\alpha} \bar{h}}^{\alpha} \\
& +\left(a^{\pi \lambda} a^{\alpha \gamma}+2 a^{a \lambda} a^{\pi \gamma}\right) w_{(0) \mid \Sigma \lambda} Q_{(2) \gamma} \bar{h}=0, \tag{9.24b}
\end{align*}
$$

To sum up, we have for problems of Type P2 a set of four equations ( $9.24 \mathrm{a}, \mathrm{b}, \mathrm{c}$ ), in the four unknowns, $w_{(0)}$ and $T_{(3)}^{\alpha \beta}$.

The special case of uniform thickness will now be treated. Since $\bar{h}$ is constant, ( $9.24 \mathrm{a}, \mathrm{b}$ ) may be written in the form

$$
\begin{align*}
& -T_{(3)^{W} W_{(0) \mid p \gamma}^{a}}^{\rho T}+D \Delta \Delta W_{(0)}+P_{(3)}^{0}+2 X_{(2) \mid 0]}^{0} \bar{h}+Q_{\substack{(2) \mid \pi \\
a}}^{\pi} h+Q_{(2)}^{0} \bar{h} \Delta W_{(0)}=0 ; \tag{9.25a}
\end{align*}
$$

$$
\begin{align*}
& +\left(a^{\alpha \beta} \Delta w_{(0)}+2 a^{\alpha \lambda} a^{\beta \pi} W_{(0) \mid \pi \lambda}\right) Q_{(2) \beta} \bar{h}=0, \tag{9.25b}
\end{align*}
$$

and (9.24c) remains unchanged. Here $\Delta$ is the two-dimensional Laplace operator, and $D$ is the reduced flexual rigidity as in (9.14).

Furthermore, when

$$
\begin{equation*}
P_{(3)}^{\alpha}=X_{(27(0)}^{\alpha}=Q_{(2)}^{\alpha}=0, \tag{9.26}
\end{equation*}
$$

the equation (9.25b) will be satisfied by putting ( $\phi_{(3)}$ is an arbitrary function of $x^{\alpha}$ )

And consequently, (9.25a) can be reduced to the form

$$
\begin{align*}
&-\left.\left.\boldsymbol{n}_{[0]}^{0 \pi} \boldsymbol{n}_{[(0]}^{\gamma} \phi_{(3)}\right|_{a} \delta W_{(0)}\right|_{a r} \\
&+D \Delta \Delta W_{(0)}+\frac{1}{2} D\left(\Delta W_{(0)}\right)^{3} \tag{9.28}
\end{align*}+P_{(3)}^{0} .
$$

Therefore for a plate of uniform thickness under the condition (9.26), we have in this type of problem a set of two equations, (9.24c) and (9.28), with two unknowns $w_{(0)}$ and $\phi_{(3)}$. In rectangular Cartesians, these two equations may be written as

$$
\begin{gather*}
W_{(0), 12} W_{(0), 12}-W_{(0), 11} W_{(0), 22}=0  \tag{9.29a}\\
2 W_{(0), 12} \phi_{(3), 12}-w_{(0), 11} \phi_{(3), 22}-W_{(0), 22 \phi_{(3), 11}}+D \Delta \Delta W_{(0)} \\
+\frac{1}{2} D\left(\Delta W_{(0)}\right)^{3}+P_{(3)}^{0}+2 X_{(2)[0]}^{0} \bar{h}+Q_{(2)}^{0} \bar{h} \Delta_{(0)}=0 \tag{9.29b}
\end{gather*}
$$

The macroscopic tensors are given by

$$
\begin{align*}
& T^{11}=\left\{\phi_{(3), 22}+\frac{1}{2} D\left(w_{(0), 22}-w_{(0), 11}\right) \Delta w_{(0)}\right\} \epsilon^{3}+O\left(\epsilon^{4}\right), \\
& T^{12}=T^{21}=-\left\{\phi_{(3), 21}+D w_{(0), 12} \Delta w_{(0)}\right\} \epsilon^{3}+O\left(\epsilon^{4}\right),  \tag{9.30a}\\
& T^{22}=\left\{\phi_{(3), 11}+\frac{1}{2} D\left(w_{(0), 11}-W_{(0), 22} \Delta w_{(0)}\right\} \epsilon^{3}+O\left(\epsilon^{4}\right),\right. \\
& L^{11}=-L^{22}=-D(1-\sigma) w_{(0), 12} \epsilon^{3}+O\left(\epsilon^{4}\right), \\
& L^{12}=D\left(w_{(0), 11}+\sigma W_{(0), 22}\right) \epsilon^{3}+O\left(\epsilon^{4}\right),  \tag{9.30b}\\
& L^{21}=-D\left(w_{(0), 22}+\sigma W_{(0), 11)} \epsilon^{3}+O\left(\epsilon^{4}\right),\right. \\
& T^{10}=D\left(\Delta w_{(0)}\right), 1 \epsilon^{3}+O\left(\epsilon^{4}\right), \quad T^{20}=D\left(\Delta w_{(0)}\right), 2 \epsilon^{3}+O\left(\epsilon^{4}\right) . \tag{9.30c}
\end{align*}
$$

An interesting example of this type of problem is given below.
Example: A long rectangular plate of uniform thickness is deformed under the actions of (a) uniform tensions $T_{(3) A} \epsilon^{3}, T_{(3) B} \epsilon^{3}$ and uniform bending moments $L_{(3) A} \epsilon^{3}$, $L_{(3) B} \epsilon^{3}$ on the two long edges, (b) a normal load $P_{(3)}^{0} \epsilon^{3}$ on one face (this load does not vary along the length of the plate). Assuming that $p=2, q=0$, find the form of the middle surface in the strained state.

In this example, we can neglect the edge effect at the two ends by considering the plate infinitely long. Since the given external force system does not vary along the length of the plate, we shall assume that strain and stress are constant along this direction. Hence in the first place, the deformed surface is cylindrical, with the generators of the cylinder parallel to the length of the plate:

$$
\begin{equation*}
q_{(0) 11}=w_{(0), 11}=\Omega\left(x^{1}\right), \quad q_{(0) 12}=w_{(0), 12}=q_{(0) 22}=w_{(0), 22}=0 \tag{9.31}
\end{equation*}
$$

Here $\mathrm{x}^{\alpha}$ are rectangular Cartesians, so that $\mathrm{x}^{2}$-axis is parallel to the long sides and $x^{1}$-axis is perpendicular to them. $\Omega$ is a function of $x^{1}$, to be determined.

In the second place, $T^{\alpha \beta}$ is a function of $x^{1}$ only. Since the ends of the plate are free from tractions, it follows that $T^{12}$ and $T^{22}$ vanish everywhere to the third order:

$$
\begin{equation*}
T_{(3)}^{22}=T_{(3)}^{12}=0 \tag{9.32}
\end{equation*}
$$

The component $T^{11}$ can be written as

$$
\begin{equation*}
T^{11}=T_{(3)}^{11} \epsilon^{3}+O\left(\epsilon^{4}\right), \tag{9.33}
\end{equation*}
$$

where $T_{(3)}^{11}$ is a function of $\mathbf{x}^{1}$, to be determined.
The problem is to determine two unknowns $\Omega$ and $T_{(3)}^{11}$ as functions of $x^{1}$ through Eqs. (9.25a, b) under the conditions

$$
\begin{equation*}
P_{(3)}^{\alpha}=X_{(2)][0]}^{0}=X_{(2)[0]}^{\alpha}=Q_{(2)}^{0}=Q_{(2)}^{\alpha}=0 \tag{9.34}
\end{equation*}
$$

Substituting (9.31)-(9.34) into (9.25a, b), we have

$$
\begin{equation*}
-\Omega T_{(3)}^{11}+D \Omega_{, 11}+P_{(3)}^{0}=0, \quad(9.35) \quad\left(T_{(3)}^{11}+\frac{1}{2} D \Omega^{2}\right)_{.1}=0 \tag{9.35}
\end{equation*}
$$

Integration of (9.36) gives

$$
\begin{equation*}
T_{(3)}^{11}+\frac{1}{2} D \Omega^{2}=C . \tag{9.37}
\end{equation*}
$$

Here $C$ is a constant to be determined by the conditions on the long edges. Substituting $W_{(0), \alpha \beta}$ from (9.31) into ( $9.30 \mathrm{~b}, \mathrm{c}$ ), we get

$$
\begin{array}{rll}
L^{11}=-L^{22}=O\left(\epsilon^{4}\right), \quad L^{12}=D \Omega \epsilon^{3}+O\left(\epsilon^{4}\right), & L^{21}=-\sigma D \Omega \epsilon^{3}+O\left(\epsilon^{4}\right) \\
T^{10}=D \Omega, 1 \epsilon^{3}+O\left(\epsilon^{4}\right), & T^{20}=O\left(\epsilon^{4}\right) . \tag{9.39}
\end{array}
$$

Then (9.37) becomes

$$
\begin{equation*}
T_{(3)}^{11}=C-\frac{\left(L_{(3)}^{12}\right)^{2}}{2 D} \tag{9.40}
\end{equation*}
$$

where, by definition, $L_{(3)}^{12}=D \Omega$. This equation is satisfied everywhere throughout the plate. Therefore it is also satisfied at the two long edges, and consequently $T_{(3) A}$, $T_{(3) B}, L_{(3) A}, L_{(3) B}$ must satisfy the following relation:

$$
\begin{equation*}
T_{(3) A}+\frac{\left(L_{(3) A}\right)^{2}}{2 D}=T_{(3) B}+\frac{\left(L_{(3) B}\right)^{2}}{2 D}=C . \tag{9.41}
\end{equation*}
$$

Therefore we conclude that among $T_{(3) A}, T_{(3) B}, L_{(3) A}, L_{(3) B}$ only three quantitics are independent; when any three are given, the fourth can be calculated through (9.41).

Substituting $T_{(3)}^{11}$ from (9.37) into (9.35), we obtain

$$
\begin{equation*}
\frac{1}{2} D \Omega^{3}-C \Omega+D \Omega_{, 11}=-P_{(3)}^{0} . \tag{9.42}
\end{equation*}
$$

This is a non-linear differential equation of the second order and third degree in $\Omega$. When the boundary values of $\Omega$ are given (or $L_{(3) A}, L_{(3) B}$ are given), the solution is uniquely determined.

If $P_{(3)}^{0}=0$, the problem is identical with the problem of the elastica [8]. For then, if we introduce the new variable $\theta$, so that

$$
\begin{equation*}
\Omega=\theta_{.1,} \tag{9.43}
\end{equation*}
$$

equation (9.42) can be written as

$$
\begin{equation*}
\frac{1}{2} D(0,1)^{3}-C \theta_{.1}+D 0_{.111}=0 . \tag{9.44}
\end{equation*}
$$

The second integral of this equation is

$$
\begin{equation*}
\frac{1}{2} D(0,1)^{2}-C=F \cos 0 . \tag{9.45}
\end{equation*}
$$

Equation (9.45) is in the same form as the well known equation for the elastica. The constant $F$ can be determined by the boundary conditions on the long edges; $\theta$ is a physical quantity which denotes the direction of the tangent to the middle surface in the strained state.

The bending of a rectangular sheet of paper into a cylindrical surface by forces and couples applied to two opposite edges may be considered as a problem of the above type. There is, however, an edge effect in the neighborhood of the free edges.
(c) Type $P 3: q=0, p>2$. Finite deflection with negligible extension in THE MIDDLE SURFACE

General equations. As in type $P 1, P 2$, we have

$$
\begin{equation*}
n_{0}=n=3, \quad j_{0}=j=2, \quad k_{0}=k=2 \tag{9.46}
\end{equation*}
$$

By substituting the $\epsilon$ series from (8.1), (8.4)-(8.6) into (7.2a,b), (7.4a, b), we find that the lowest power in $(7.2 a, b)$ is $\epsilon^{3}$, and in ( $7.4 \mathrm{a}, \mathrm{b}$ ) is $\epsilon^{0}$. The corresponding coefficients give rise to the equations of equilibrium and compatibility in the first approximation as follows:

$$
\begin{align*}
& { }_{3}^{2} A_{(1)}^{\rho \gamma \pi \lambda}\left(q_{(0) \pi \lambda} \bar{h}^{3}\right)_{a}{ }_{a}+A_{(3)}^{\gamma \gamma \pi \omega \lambda} q_{(0) \lambda \delta} q_{(0) \rho \gamma} q_{(0) \pi \omega} \bar{h}^{3}+P_{(0)}^{0} \\
& +\left(Q_{(2)}^{\pi} \bar{h}\right)_{\mid \pi}+2 X_{(2)[0]}^{0 .} \overline{/}+\frac{1-2 \sigma}{1-\sigma} a^{\pi \lambda} q_{(0) \pi \lambda} Q_{(2)}^{0} \hbar=0,  \tag{9.47a}\\
& -A_{(3)}^{\rho \alpha \pi \omega \delta \lambda}\left(q_{(0) \pi \omega} q_{(0) \delta \lambda} \bar{h}^{3}\right)_{\left.\right|_{\rho}}+\frac{2}{3} a^{\alpha \pi} q_{(0) \times \gamma} A_{(1)}^{\gamma \rho \lambda \delta}\left(q_{\left.(0) \lambda_{0} \overline{h^{3}}\right)_{a}}^{\left.\right|_{a}}+P_{(3)}^{\alpha}+2 X_{(2)[0]}^{\alpha} \bar{h}\right. \\
& +\frac{\sigma}{1-\sigma} a^{\alpha \rho}\left(Q_{(2)}^{0} \tilde{h}\right)_{a}+\left(a^{\pi \lambda} a^{\alpha \gamma}+2 a^{\alpha \lambda} a^{x \gamma}\right) q_{(0) \pi \lambda} Q_{(2) \gamma} \bar{h}=0,  \tag{9.47b}\\
& \boldsymbol{n}_{[0]}^{\gamma \beta} q_{(0)_{\alpha \gamma \mid \beta}}^{\alpha}=0, \quad(9.47 \mathrm{c}) \quad \prod_{[0]}^{\alpha \beta} n_{[0]}^{p \gamma} q_{(0)_{\beta} \beta} q_{(0)_{\alpha \gamma}}=0 . \tag{9.47~d}
\end{align*}
$$

The macroscopic tensors in ( $7.5 \mathrm{a}, \mathrm{b}, \mathrm{c}$ ) can be written as

$$
\begin{align*}
& T^{\alpha \beta}=\left\{\frac{\sigma}{1-\sigma} a^{\alpha \beta} Q_{(2)}^{0} \bar{h}-A_{(3)}^{\alpha \beta \pi \lambda \lambda \delta} q_{(0) \pi \gamma} q_{(0) \lambda \delta} \bar{h}^{3}\right\} \epsilon^{3}+O\left(\epsilon^{4}\right),  \tag{9.48a}\\
& L^{\alpha \beta}=\frac{2}{3} n_{(0)}^{\mu \beta} a_{\pi \omega} A_{(1)}^{\alpha \pi \lambda \delta} q_{(0) \lambda \delta} \bar{h}^{3} \epsilon^{3}+O\left(\epsilon^{4}\right),  \tag{9.48b}\\
& T^{\alpha 0}=\left\{\frac{2}{3} A_{(1)}^{\pi \alpha \lambda \delta}\left(q_{(0) \lambda \delta} \bar{h}^{3}\right)_{\mid \pi}+Q_{(2)}^{\alpha} \bar{h}\right\} \epsilon^{3}+O\left(\epsilon^{4}\right) . \tag{9.48c}
\end{align*}
$$

Equations ( $9.46 \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ ) form a set of six equations involving only three unknowns $q_{(0) \alpha \beta}$, so the problem is overdetermined. Let us suppose that $q_{(0) \alpha \beta}$ can be eliminated from these six equations; we get a set of three conditions, which may be written in the form

$$
\begin{equation*}
\Psi_{(i)}=\Psi_{(j)}\left(\bar{h}, Q_{(2)}^{1}, P_{(3)}^{1}, X_{(2)[0]}^{t}\right)=0, \quad(j=1,2,3) \tag{9.49}
\end{equation*}
$$

Equations (9.49) represent the three necessary conditions on the external force system in order that a plate may undergo finite deflection with negligible extension in the middle surface. A special example will be considered as follows.

Example. Under what circumstances can a portion of a plate of uniform thickness be bent by normal pressure into a cylindrical surface of finite curvature with negligible extension in the middle surface? The normal pressure is assumed to be constant along the generators of the cylinder.

In this case,

$$
\begin{equation*}
X_{(2)[0]}^{t}=Q_{(2)}^{\alpha}=P_{(3)}^{\alpha}=0 \tag{9.50}
\end{equation*}
$$

Let us choose the $x^{2}$-axis in the direction of the generators of the assumed cylindrical surface, and the $x^{1}$-axis in the perpendicular direction. Then we have as in the example of Types $P 1, P 2$

$$
\begin{equation*}
q_{(0) 11}=\Omega\left(x^{1}\right), \quad q_{(0) 12}=q_{(0) 22}=0 \tag{9.51}
\end{equation*}
$$

and the equations $(9.47 \mathrm{a}, \mathrm{b})$ become

$$
\begin{gather*}
D \Omega_{, 11}+\frac{3-\sigma}{2(1-\sigma)} D \Omega^{3}+P_{(3)}^{0}+\frac{1-2 \sigma}{1-\sigma} \Omega Q_{(2)}^{0} \bar{h}=0  \tag{9.52a}\\
-D\left(\Omega^{2}\right)_{1}+\sigma \bar{h} Q_{(2), 1}^{0}=0 \tag{9.52b}
\end{gather*}
$$

Integration of ( 9.52 b ) gives

$$
\begin{equation*}
\Omega^{2}=\frac{\sigma \bar{h}}{D}\left(C_{1}+Q_{(2)}\right), \tag{9.53}
\end{equation*}
$$

where $C_{1}$ is an integration constant. Substituting $\Omega^{2}$ from (9.53) into (9.52a), we get

$$
\begin{align*}
(\sigma \bar{h} D)^{1 / 2}\left\{\left(C_{1}+Q_{(2)}^{0}\right)^{1 / 2}\right\}_{11} & +\frac{(3-\sigma)(\sigma \bar{h})^{3 / 2}}{2(1-\sigma) D^{1 / 2}}\left(C_{1}+Q_{(2)}^{0}\right)^{3 / 2} \\
& +P_{(3)}^{0}+\frac{1-2 \sigma}{1-\sigma}\left(\frac{\sigma}{D}\right)^{1 / 2} \bar{h}^{3 / 2} Q_{(2)}^{0}=0 . \tag{9.54}
\end{align*}
$$

This is the required condition to be satisfied by $Q_{(2)}, P_{(3)}^{0}$.
Let us assume that $Q^{0}$ and $P^{0}$ are of the same order of magnitude; then, since $P_{(2)}=0$, we have

$$
\begin{equation*}
Q_{(2)}^{0}=0 . \tag{9.55}
\end{equation*}
$$

Then the condition (9.54) becomes

$$
\begin{equation*}
P_{(3)}^{0}=-\frac{(3-\sigma)(\sigma \bar{h})^{3 / 2}}{2(1-\sigma) D^{1 / 2}} C_{1}^{3 / 2}=\text { constant } . \tag{9.56}
\end{equation*}
$$

Furthermore, since the right hand side of (9.53) is constant, the plate is bent into a circular cylindrical surface; its curvature is given by

$$
\begin{equation*}
\Omega=-\left\{\frac{2(1-\sigma) P_{(3)}}{(3-\sigma) D}\right\}^{1 / 3} . \tag{9.57}
\end{equation*}
$$

When $P_{(3)}^{9}=0$, we get from ( 9.57 ) $\Omega=0$. Therefore we conclude that it is impossible to bend any portion of a plate of uniform thickness into a cylindrical surface of finite curvature with negligible extension in the middle surface, if on that portion of the plate the surface force is of the fourth order, and the body force of the third order, with respect to the thickness of the plate.

## CONCLUSIONS

A systematic method of approximation based upon the thinness of the plate has been developed in this paper. It is found that thin plate problems may be classified into twelve types ( $P 1-P 12$ ) according to the relative orders of magnitude of $p_{\alpha \beta}$, $q_{\alpha \beta}$ and $h$. In each case, the problem reduces to the solution of a set of partial differential equations, different for different types. These differential equations are given in Table I. Furthermore, the principal parts of the macroscopic tensors and the orders of magnitude of the external forces for each case are given in Table II. Among these twelve types, $P 1-P 3$ represent the problems of finite deflection, $P 4-P 8$ the problems of small deflection, $P 9-P 11$ the problems of very small deflection and $P 12$ the problems of zero deflection. The problems of finite deflection are discussed in section 9; these are new problems, and a simple example for each of these types is solved. The problems of small deflections, very small deflection, and zero deflection are familiar; the detailed discussion of these types is therefore not necessary. However, we may note that the theory of generalized plane stress, the Lagrange-Kirchhoff theory of "small" deflection, the von Kármán theory of "large" deflection and the membrane problem can be derived respectively from Types P12, P11, P5, P4.

## APPENDICES

(i) Table I. - Table of the equations of equilibrium and compatibility of thin plate problems.

|  | $q$ | $p$ | (7.2a) |  |  |  |  | (7.2b) |  |  |  | (7.4b) |  | (7.4a) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | $I_{1}^{0}$ | $I_{2}^{0}$ | $I_{3}^{0}$ | $I_{4}^{0}$ | $I_{5}^{0}$ | $I_{1}^{\alpha}$ | $I_{2}^{\alpha}$ | $I_{3}^{\alpha}$ | $I_{4}^{\alpha}$ | $J_{1}^{0}$ |  | $J_{a 1}$ |
| - $P 1$ | 0 | 1 | x |  |  | x | x | X |  | X | x |  | x | x |
| P2 | 0 | 2 | x | x | x | x | X | X | x | X | x |  | x | x |
| $P 3^{*}$ | 0 | $>2$ |  | x | x | x | x |  | X | x | x |  | x | x |
| P4 | $\geq 1$ | 1 | x |  |  | x | x | X |  | x |  | x |  | x |
| P5 | 1 | 2 | x | x |  | x | x | x |  | x |  | x | x | x |
| P6* | $\geqq 1$ | $2 q+1$ |  | $x$ |  | x |  | x |  | X |  |  | X | x |
| P7* | $\geq 1$ | $2 q+2$ |  | x |  | x |  | x | x | x | x |  | x | x |
| $P 8^{*}$ | $\geqq 1$ | $>2 q+2$ |  | X |  | X |  |  | x | x | x |  | x | X |
| $P 9$ | 2 | $>1$ | x | x |  | x | x | x |  | x |  | x |  | x |
| P10 | $\geqq 2$ | $2<p<2 q$ |  | X |  | X |  | X |  | x |  | x |  | X |
| P11 | $\geqq 2$ | $2 q$ |  | x |  | x |  | x |  | X |  | x | x | x |
| P12 | $\infty$ | $\geq 1$ |  |  |  |  |  | x |  | X |  | x |  |  |

In this table, the following notation is used:
The terms occurring in the first equation of equilibrium (7.2a) are

$$
\begin{aligned}
& I_{1}^{0}=-2 A_{(1)}^{\rho \gamma \pi \lambda} q_{\rho \gamma} D_{\pi \lambda} h, \quad I_{2}^{0}=\frac{2}{3} A_{(1)}^{\gamma \gamma \pi \lambda}\left(q_{\pi \lambda} h^{3}\right)_{\mid \rho \gamma,} \quad I_{3}^{0}=A_{(3)}^{\gamma \gamma \pi \alpha \lambda \delta} q_{\rho \gamma} q_{\pi \omega} q_{\lambda \delta} h^{3} \\
& I_{4}^{0}=P^{0}+2 X_{[0]}^{0} h+\left(Q^{\pi} h\right)_{\mid \mathrm{x},} \quad I_{5}^{0}=\frac{1-2 \sigma^{2}}{1-\sigma} a_{\pi \lambda} Q^{\pi \lambda} Q^{0} h .
\end{aligned}
$$

The terms occurring in the second and third equations of equilibrium (7.2b) are

$$
\begin{aligned}
& I_{1}^{\alpha}=2 A_{(1)}^{\alpha \pi \lambda \lambda}\left(p_{\pi \lambda} h\right)_{\mid \rho}, \\
& I_{2}^{\alpha}=\frac{2}{3} a^{\alpha \pi} q_{\pi \gamma} A_{(1)}^{\gamma \alpha)}\left(q_{\lambda \delta} h^{3}\right)_{\mid 0}^{a}-A_{(3)}^{\alpha \pi \pi \alpha \lambda)}\left(q_{\pi \omega} q_{\lambda \delta} h^{3}\right)_{\mid \rho} \\
& I_{3}^{\alpha}=P^{\alpha}+2 X_{[0 \mid}^{\alpha} h+\frac{\sigma}{1-\sigma} a^{\alpha p}\left(Q^{0} h\right)_{\mid \rho}, \quad I_{4}^{\alpha}=\left(a^{\pi \lambda} q_{\pi \lambda} a_{\gamma}^{\alpha}+2 a^{\alpha \pi} q_{\pi \gamma}\right) Q^{\gamma} h .
\end{aligned}
$$

The terms occurring in the first equation of compatibility ( 7.4 b ) are

The term occurring in the second and third equations of compatibility (7.4a) is

$$
J_{\alpha 1}=2 \mathrm{n}_{[0]}^{\beta \gamma} q_{\alpha \beta \mid \gamma}
$$

On account of the conditions which hold in the various types of problem, some of these terms may be negligible in comparison with others. The table shows by the symbol ' $x$ ' those terms which are to be retained in the first approximation for the various types. (The overdetermined problems are denoted by '*'.) Thus for example, for problems of Type $P 1$, we having the following equations of equilibrium and compatibility in the first approximation:

$$
I_{1}^{0}+I_{4}^{0}+I_{5}^{0}=0, \quad I_{1}^{\alpha}+I_{3}^{\alpha}+I_{4}^{\alpha}=0, \quad J_{2}^{0}=0, \quad J_{\alpha 1}=0
$$

These equations are written in terms of the small principal parts instead of in terms of the finite coefficients of the lowest power in $\epsilon$ (sce ( $9.3 a, b, c, d$ )).
(ii) Table II.-Table of the external force system and the macroscopic tensors for various types of thin plate problems.

| Types | $n_{0}$ | $n$ | $j_{0}$ | j | $k_{0}$ | $k$ | $T^{\alpha \beta}$ |  | $L^{\alpha \beta}$ |  | $T^{\alpha 0}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  | $t$ | $T_{1}^{\alpha \beta} T_{2}^{\alpha \beta} T_{3}^{\alpha \epsilon} \mid$ | $u$ | $L_{1}^{\alpha \beta}$ | $l$ | $T_{1}^{\alpha} T_{2}^{\alpha}$ |
| $P 1$ | 2 | 2 | 1 | 1 | 1 | 1 | 2 |  | 3 |  | 2 |  |
| P1 | 2 | 2 | 1 | 1 | 1 | 2 | 2 | $x \quad \mathrm{x}$ | 3 | x | 3 | x x |
| $P 2$ | 3 | 3 | 2 | 2 | 2 | 2 | 3 | $x$ x $x$ | 3 | x | 3 | x x |
| P3 | 3 | 3 | 2 | 2 | 2 | 2 | 3 | $\mathrm{x} \times$ | 3 | x | 3 | x x |
| P4 | $q+2$ | 2 | $q+1$ | 1 | 1 | $q+1$ | 2 | $x \quad \mathrm{x}$ | $q+3$ | x | $q+2$ |  |
| $P_{4}$ | $q+2$ | 2 | $q+1$ | 1 | 1 | $q+2$ | 2 | $\mathrm{x} \quad \mathrm{x}$ | $q+3$ | $x$ | $q+3$ | $x \quad x$ |
| P5 | 4 | 3 | 3 | 2 | 2 | 3 | 3 | $x \quad \mathrm{x}$ | 4 | x | 4 | x x |
| P6 | $q+3$ | $2 q+2$ | $q+2$ | $2 q+1$ | $2 q+1$ | $q+2$ | $2 q+2$ | $x \quad x$ | $q+3$ | x | $q+3$ | x x |
| P7 | $q+3$ | $2 q+3$ | $2 q+2$ | $q+2$ | $q+2$ | $2 q+2$ | $2 q+3$ | $x$ x $x$ | $q+3$ | x | $q+3$ | x x |
| P8 | $q+3$ | $2 q+3$ | $2 q+2$ | $q+2$ | $q+2$ | $2 q+2$ | $2 q+3$ | $\mathrm{x} \times$ | $q+3$ | x | $q+3$ | x x |
| $P 9$ | $q+3$ | 3 | $q+2$ | 2 |  | $q+2$ | 3 |  | $q+3$ | x | $q+3$ |  |
| $P 10$ | $q+3$ | $p+1$ | $q+2$ | $p$ | $p$ | $q+2$ | $p+1$ | $x$ x | $q+3$ | x | $q+3$ | x x |
| P11 | $q+3$ | $2 q+1$ | $q+2$ | $2 q$ | $2 q$ | $q+2$ | $2 q+1$ | $\mathrm{x} \quad \mathrm{x}$ | $q+3$ | x | q+3 | x x. |
| P12 | $\infty$ | $p+1$ | $\infty$ | $p$ | $p$ | $\infty$ | $p+1$ | $\mathrm{x} \quad \mathrm{x}$ | $\infty$ |  | $\infty$ |  |

In this table, the following notation is used:
The terms occurring in the expression (7.5a) for the membrane stress tensor $T^{\alpha \beta}$ are denoted by

$$
T_{1}^{\alpha \beta}=2 A_{(1)}^{\alpha \beta \pi \lambda} p_{\star \lambda} h, \quad T_{2}^{\alpha \beta}=-A_{(3)}^{\alpha \beta \pi \omega \lambda \delta} q_{\lambda \delta} q_{\pi \omega} h^{3}, \quad T_{3}^{\alpha \beta}=\frac{\sigma}{1-\sigma} a^{\alpha \beta} Q^{0} h
$$

The term occurring in the expression (7.5b) for the bending moment tensor $L^{\alpha \beta}$ is denoted by

$$
L_{1}^{\alpha \beta}=\frac{2}{3} n_{[0]}^{\mu \beta} a_{\pi \alpha} A_{(1)}^{\alpha \pi \lambda \delta} q_{\lambda \delta} h^{3} .
$$

The terms occurring in the expression (7.5c) for the shearing stress tensor $T^{\alpha 0}$ are denoted by

$$
T_{1}^{\alpha}=Q^{\alpha} h, \quad T_{2}^{\alpha}=\frac{2}{3} A_{(1)}^{\pi \alpha \lambda \delta}\left(q_{\lambda \delta} h^{3}\right)_{\mid \pi}
$$

Furthermore,
$n_{0}=$ order of sum of the normal forces acting on the upper and lower boundary surfaces, or order of $P^{0}$,
$n=$ order of sum of the tangential forces acting on the upper and lower boundary surfaces, or order of $P^{a}$,
$j_{0}=$ order of normal component of body force, or order of $X_{[0]}^{0}$,
$i=$ order of tangential component of body force, or order of $X_{[0]}^{\alpha}$,
$k_{0}=$ order of difference of normal forces acting on the upper and lower surfaces, or order of $Q^{0}$,
$k=$ order of difference of tangential components of forces acting on the upper and lower boundary surfaces, or order of $Q^{\alpha}$,
$t=$ order of membrane stress tensor $T^{\alpha \beta}$,
$u=$ order of bending moment tensor $L^{\alpha \beta}$,
$l=$ order of shearing stress tensor $T^{\alpha 0}$.
This table gives (a) the values of $n_{0}, n, j_{0}, j, k_{0}, k, i, u, l$, (b) the principal terms in the expressions for $T^{\alpha \beta}, L^{\alpha \beta}, T^{\alpha 0}$ (denoted by ' $x$ '). The terms not marked with ' $x$ ' are negligible in comparison with those principal terms. It will be noted that there are two lines in the table for $P 1$ and also for $P 4$. This is because, in each case, $k$ may have two values.

For example, in the case of Type $P 1$, we have for $T^{\alpha \beta}, L^{\alpha \beta}$,

$$
T^{\alpha \beta}=T_{1}^{\alpha \beta}+T_{3}^{\alpha \beta}, \quad L^{\alpha \beta}=L_{1}^{\alpha \beta}
$$

while for $T^{a 0}$,

$$
\begin{array}{ll}
T^{a 0}=T_{1}^{\alpha} & (\text { if } k=1) \\
T^{a 0}=T_{1}^{\alpha}+T_{2}^{\alpha} & (\text { if } k=2)
\end{array}
$$

(To be continued)

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# EFFECT OF A SMALL HOLE ON THE STRESSES IN A UNIFORMLY LOADED PLATE* 

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1. Introduction. The National Bureau of Standards has recently made tests on steel columns having perforated cover plates. ${ }^{1}$ Most of the perforations were of socalled ovaloid shape, i.e., that of a square with a semi-circle erected on each of two opposite sides. The tests on the columns included experimental determinations of the distribution of stress in the neighborhood of a perforation, and the results obtained aroused interest in the development of a theory for the distribution of stress in a large, uniformly loaded plate having a single ovaloid hole.

In this paper an exact solution to this problem is obtained for a hole having any boundary of which the equation can be expressed in the parametric form

$$
\begin{equation*}
x=p \cos \beta+r \cos 3 \beta, \quad y=q \sin \beta-r \sin 3 \beta \tag{1}
\end{equation*}
$$

The plate is supposed in a state of generalized plane stress, the stress ${ }^{2}$ at points re-


Fig. 1. Actual and approximate ovaloids.
The dashed line represents the actual ovaloid and the full line the approximate ovaloid of Eqs. (1) and (2). mote from the hole having the constant normal components $\sigma_{x}=S_{x}, \sigma_{y}=S_{y}$, and the constant shearing component $\tau_{x y}=T_{x y}$.

Eq. (1) represents a closed curve having symmetry about the $x$-axis and about the $y$-axis. For certain values of $p, q$, and $r$ the curve is simple, i.e., it does not cross itself. By adjustment of the values of $p, q$, and $r$ a variety of simple closed curves is obtained, including a good approximation to an ovaloid and a good approximation to a square with rounded corners, as well as exact ellipses ( $r=0$ ) of any eccentricity. The approximate ovaloid obtained by taking

$$
\begin{equation*}
\hat{p}=2.063, \quad q=1.108, \quad r=-0.079 \tag{2}
\end{equation*}
$$

is shown compared to the actual ovaloid in Fig. 1. The approximate square obtained by taking

$$
\begin{equation*}
p=q=1, \quad r=-0.14 \tag{3}
\end{equation*}
$$

[^13]is shown in Fig. 2. The sides of the square are parallel to the axes of coordinates. By taking
\[

$$
\begin{equation*}
p=q=1, \quad r=0.14, \tag{4}
\end{equation*}
$$

\]

the same square, but with the diagonals parallel to the axes of coordinates, is obtained. The radius of curvature at the mid-point of the fillet is about 0.086 times the length of the side of the square.
2. Curvilinear coordinates. If two sets of curves are defined by

$$
\begin{equation*}
f_{1}(x, y)=\alpha, \quad f_{2}(x, y)=\beta \tag{5}
\end{equation*}
$$

then a pair of values $(\alpha, \beta)$ defines the points at which the corresponding curves (5) intersect,


Fig. 2. The approximate square of Eqs. (1) and (3). and $(\alpha, \beta)$ are curvilinear coordinates in the $x, y$-plane. As a special case, the functions of Eq. (5) may be obtained by equating real and imaginary parts of both sides of

$$
\begin{equation*}
w=f(z), \tag{6}
\end{equation*}
$$

where $w=\alpha+i \beta$ and $z=x+i y$. In this case the transformation from the $w$-plane to the $z$-plane is conformal and the two families of Eq. (5) are orthogonal. The expression,

$$
\begin{equation*}
\frac{d z}{d w}=\frac{1}{h} e^{i \psi}, \tag{7}
\end{equation*}
$$

defines the stretch ratio, $1 / h$, of the transformation, and gives $\psi$, the inclination of the curve, $\beta=$ constant, to the $x$-axis.

In the absence of body forces, the condition that the stresses satisfy the conditions of equilibrium is that the normal components, $\sigma_{\alpha}$ and $\sigma_{\beta}$, and the shearing component, $\tau_{\alpha \beta}$, can be derived from a stress function, $\phi$, by means of the relations ${ }^{3}$

$$
\begin{align*}
\sigma_{\alpha} & =h^{2} \frac{\partial^{2} \phi}{\partial \beta^{2}}+\frac{1}{2}\left(\frac{\partial \phi}{\partial \beta} \frac{\partial h^{2}}{\partial \beta}-\frac{\partial \phi}{\partial \alpha} \frac{\partial h^{2}}{\partial \alpha}\right) \\
\sigma_{\beta} & =h^{2} \frac{\partial^{2} \phi}{\partial \alpha^{2}}-\frac{1}{2}\left(\frac{\partial \phi}{\partial \beta} \frac{\partial h^{2}}{\partial \beta}-\frac{\partial \phi}{\partial \alpha} \frac{\partial h^{2}}{\partial \alpha}\right),  \tag{8}\\
\tau_{\alpha \beta} & =-h^{2} \frac{\partial^{2} \phi}{\partial \alpha \partial \beta}-\frac{1}{2}\left(\frac{\partial \phi}{\partial \beta} \frac{\partial h^{2}}{\partial \alpha}+\frac{\partial \phi}{\partial \alpha} \frac{\partial h^{2}}{\partial \beta}\right)
\end{align*}
$$

and the condition that the expressions ( 8 ) satisfy the compatibility conditions is

$$
\begin{equation*}
\nabla^{4} \phi=\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right)^{2} \phi=h^{2}\left(\frac{\partial^{2}}{\partial \alpha^{2}}+\frac{\partial^{2}}{\partial \beta^{2}}\right) h^{2}\left(\frac{\partial^{2}}{\partial \alpha^{2}}+\frac{\partial^{2}}{\partial \beta^{2}}\right) \phi=0 . \tag{9}
\end{equation*}
$$

If a function, $F$, satisfies Laplace's equation,

$$
\begin{equation*}
\nabla^{2} F=0, \tag{10}
\end{equation*}
$$

[^14]then $F, x F, y F$, and $\rho^{2} F=\left(x^{2}+y^{2}\right) F$ satisfy Eq. (9). Functions which satisfy Eq. (10) are called harmonic functions, those which satisfy Eq. (9), biharmonic functions.
3. The coordinate system. The solution of the problem is simplified by the use of a coordinate system ( $\alpha, \beta$ ) such that Eq. (1) of the boundary of the hole reduces to the form $\alpha=\alpha_{0}$. Such a system is obtained by writing for Eq. (6)
\[

$$
\begin{equation*}
z=e^{w}+a b e^{-w}+a c^{3} e^{-3 w}, \tag{11}
\end{equation*}
$$

\]

or, separating the real and imaginary parts,

$$
\begin{align*}
& x=\left(e^{\alpha}+a b e^{-\alpha}\right) \cos \beta+a c^{3} e^{-3 \alpha} \cos 3 \beta \\
& y=\left(e^{\alpha}-a b e^{-\alpha}\right) \sin \beta-a c^{3} e^{-3 \alpha} \sin 3 \beta \tag{12}
\end{align*}
$$

For constant $\alpha$, say $\alpha_{0}$, Eq. (12) reduces to Eq. (1) for the boundary of the hole, where

$$
\begin{equation*}
p=e^{\alpha_{0}}+a b e^{-\alpha_{0}}, \quad q=e^{\alpha_{0}}-a b e^{-\alpha_{0}}, \quad r=a c^{3} e^{-3 \alpha_{0}} . \tag{13}
\end{equation*}
$$

From Eqs. (2) and (13) it is easily calculated that for the approximate ovaloid of Eqs. (1) and (2),

$$
e^{\alpha_{0}}=1.585, \quad a b=0.758, \quad a c^{3}=-0.314
$$

By keeping $a b$ and $a c^{3}$ fixed and varying $\alpha$ and $\beta$ the appropriate coordinate system


Fig. 3. Coordinate system for problem of ovaloid hole.
for the ovaloid is obtained. This system is shown in Fig. 3. The appropriate systems for the approximate square with rounded corners are similarly obtained. Figure 4 shows the system corresponding to the case of Eq. (3) and Fig. 2, and Fig. 5 shows the system corresponding to the case of Eq. (4).


Fig. 4. Coordinate system for problem of square hole with rounded corners, sides of square paraliel to Cartesian axes.

The coordinates $(\alpha, \beta)$ approach polar coordinates $(\rho, \theta)$ for large $\alpha$ as follows:

$$
\begin{equation*}
\lim _{\alpha=\infty} \alpha=\log \rho, \quad \lim _{\alpha=\infty} \beta=0 \tag{14}
\end{equation*}
$$

The values of $h^{2}$ and its derivatives may be computed as follows. From Eq. (11)

$$
\frac{d z}{d w}=e^{w}-a b c^{-w}-3 a c^{3} e^{-3 w}
$$

Hence from Eq. (7),

$$
\begin{aligned}
h^{-2}= & e^{2 \alpha}+a^{2} b^{2} e^{-2 \alpha}+9 a^{2} c^{6} e^{-6 \alpha}-2 a b \cos 2 \beta \\
& +6 a^{2} b c^{3} e^{-4 \alpha} \cos 2 \beta-6 a c^{3} e^{-2 \alpha} \cos 4 \beta
\end{aligned}
$$

and

$$
\begin{align*}
\frac{1}{2} \frac{\partial h^{2}}{\partial \alpha}= & -h^{4}\left(e^{2 \alpha}-a^{2} b^{2} e^{-2 \alpha}-27 a^{2} c^{6} e^{-6 \alpha}\right.  \tag{15}\\
& \left.-12 a^{2} b c^{3} e^{-4 \alpha} \cos 2 \beta+6 a c^{3} e^{-2 \alpha} \cos 4 \beta\right) \\
\frac{1}{2} \frac{\partial h^{2}}{\partial \beta}= & -h^{4}\left(2 a b \sin 2 \beta-6 a^{2} b c^{3} e^{-4 \alpha} \sin 2 \beta+12 a c^{3} e^{-2 \alpha} \sin 4 \beta\right)
\end{align*}
$$

4. The boundary conditions. The statement of the problem may be recapitulated as follows. There is given a large plate containing a small hole of the shape given by Eq. (1). The edge of the hole is free from stress. The plate is in a state of (generalized) plane stress and the components of (mean) stress at points remote from the hole are $\sigma_{x}=S_{x}, \sigma_{y}=S_{y}, \tau_{x y}=T_{x y}$; or in polar coordinates,

$$
\begin{align*}
& \sigma_{p}=\frac{S_{x}+S_{y}}{2}+\frac{S_{x}-S_{y}}{2} \cos 2 \theta+T_{x y} \sin 2 \theta \\
& \sigma_{\theta}=\frac{S_{x}+S_{y}}{2}-\frac{S_{x}-S_{y}}{2} \cos 2 \theta-T_{x y} \sin 2 \theta  \tag{16}\\
& \tau_{\rho \theta}=-\frac{S_{x}-S_{y}}{2} \sin 2 \theta+T_{x y} \cos 2 \theta
\end{align*}
$$

The boundary conditions may finally be stated as

$$
\begin{align*}
\sigma_{\alpha} & =\tau_{\alpha \beta}=0,\left(\alpha=\alpha_{0}\right) \\
\sigma_{\alpha} & =\frac{S_{x}+S_{y}}{2}+\frac{S_{x}-S_{y}}{2} \cos 2 \beta+T_{x y} \sin 2 \beta \\
\sigma_{\beta} & =\frac{S_{x}+S_{y}}{2}-\frac{S_{x}-S_{y}}{2} \cos 2 \beta-T_{x y} \sin 2 \beta  \tag{17}\\
\tau_{\alpha \beta} & =-\frac{S_{x}-S_{y}}{2} \sin 2 \beta+T_{x y} \cos 2 \beta,(\alpha=\infty)
\end{align*}
$$

The last three of Eq. 17 are obtained by substitution of Eq. (14) into Eq. (16).
5. The stress function. From the harmonic functions $e^{\alpha} \sin \beta$ and $e^{-\alpha} \sin \beta$ may be constructed the biharmonic functions $y e^{\alpha} \sin \beta$ and $y e^{-\alpha} \sin \beta$. From Eq. (12)

$$
y=e^{\alpha} \sin \beta-a b e^{-\alpha} \sin \beta-a c^{3} e^{-3 \alpha} \sin 3 \beta
$$

Hence
$y e^{\alpha} \sin \beta=\frac{1}{2} e^{2 \alpha}-\frac{1}{2} e^{2 \alpha} \cos 2 \beta-\frac{1}{2} a b+\frac{1}{2} a b \cos 2 \beta+\frac{1}{2} a c^{3} e^{-2 \alpha} \cos 4 \beta-\frac{1}{2} a c^{3} e^{-2 \alpha} \cos 2 \beta$, $y^{\prime} e^{-\alpha} \sin \beta=\frac{1}{2}-\frac{1}{2} \cos 2 \beta-\frac{1}{2} a b e^{-2 \alpha}+\frac{1}{2} a b e^{-2 \alpha} \cos 2 \beta+\frac{1}{2} a c^{3} e^{-4 \alpha} \cos 4 \beta-\frac{1}{2} a c^{3} e^{-4 \alpha} \cos 2 \beta$.

By dropping the harmonic terms from each of these functions and multiplying by 2 the two biharmonic functions,

$$
\begin{aligned}
& \phi_{a}=e^{2 \alpha}+a b \cos 2 \beta+a c^{3} e^{-2 \alpha} \cos 4 \beta \\
& \phi_{b}=-\cos 2 \beta-a b e^{-2 \alpha}-a c^{3} e^{-4 \alpha} \cos 2 \beta
\end{aligned}
$$

are obtained. The biharmonic function.

$$
\phi_{c}=y e^{-\alpha} \cos \beta+x e^{-\alpha} \sin \beta=\sin 2 \beta-a c^{3} e^{-4 \alpha} \sin 2 \beta
$$

is obtained in similar fashion.
The biharmonic function $\rho^{2}$ may be obtained from Eq. (12):

$$
\begin{aligned}
\rho^{2}= & x^{2}+y^{2}=e^{2 \alpha}+a^{2} b^{2} e^{-2 \alpha}+a^{2} c^{6} e^{-6 \alpha}+2 a b \cos 2 \beta \\
& +2 a^{2} b c^{3} e^{-4 \alpha} \cos 2 \beta+2 a c^{3} e^{-2 \alpha} \cos 4 \beta
\end{aligned}
$$



Fig. 5. Coordinate system for problem of square hole with rounded corners, diagonals parallel to Cartesian axes.

The non-harmonic stress functions required by this problem are
and

$$
\begin{aligned}
& \phi_{1}=2 \phi_{a}-2 a b \phi_{b}-o^{2}, \\
& \phi_{2}=-\phi_{b}
\end{aligned}
$$

$$
\phi_{6}=\phi_{c},
$$

or

$$
\begin{aligned}
& \phi_{1}=e^{2 \alpha}+a^{2} b^{2} e^{-2 \alpha}-a^{2} c^{6} e^{-6 \alpha}+2 a b \cos 2 \beta \\
& \phi_{2}=a b e^{-2 \alpha}+\cos 2 \beta+a c^{3} e^{-4 \alpha} \cos 2 \beta,
\end{aligned}
$$

and

$$
\phi_{\mathrm{E}}=\sin 2 \beta-a c^{3} e^{-4 \alpha} \sin 2 \beta
$$

In addition, the harmonic stress functions,
$\phi_{3}=e^{2 \alpha} \cos 2 \beta, \quad \phi_{4}=e^{-2 \alpha} \cos 2 \beta, \quad \phi_{5}=\alpha, \quad \phi_{7}=e^{2 \alpha} \sin 2 \beta, \quad$ and $\quad \phi_{8}=e^{-2 \alpha} \sin 2 \beta$ will be required.

The complete stress function may be written

$$
\begin{equation*}
\phi=C_{1} \phi_{1}+C_{2} \phi_{2}+C_{3} \phi_{3}+C_{4} \phi_{4}+C_{5} \phi_{5}+C_{6} \phi_{6}+C_{7} \phi_{7}+C_{8} \phi_{8} \tag{18}
\end{equation*}
$$

where the $C$ 's are to be adjusted so that the stresses derived from $\phi$ meet boundary conditions (17). Also

$$
\begin{align*}
\frac{\partial \phi}{\partial \alpha}= & 2 C_{1}\left(e^{2 \alpha}-a^{2} b^{2} e^{-2 \alpha}+3 a^{2} c^{6} e^{-6 \alpha}\right)-2 C_{2}\left(a b e^{-2 \alpha}+2 a c^{3} e^{-4 \alpha} \cos 2 \beta\right) \\
& +2 C_{3} e^{2 \alpha} \cos 2 \beta-2 C_{4} e^{-2 \alpha} \cos 2 \beta+C_{5}+4 C_{6} a c^{3} e^{-4 \alpha} \sin 2 \beta \\
& +2 C_{7} e^{2 \alpha} \sin 2 \beta-2 C_{8} e^{-2 \alpha} \sin 2 \beta,  \tag{19a}\\
\frac{\partial \phi}{\partial \beta}= & -4 C_{1} a b \sin 2 \beta-2 C_{2}\left(\sin 2 \beta+a c^{3} e^{-4 \alpha} \sin 2 \beta\right) \\
& -2 C_{3} e^{2 \alpha} \sin 2 \beta-2 C_{4} e^{-2 \alpha} \sin 2 \beta+2 C_{6}\left(\cos 2 \beta-a c^{3} e^{-4 \alpha} \cos 2 \beta\right) \\
& +2 C_{7} e^{2 \alpha} \cos 2 \beta+2 C_{8} e^{-2 \alpha} \cos 2 \beta,  \tag{19b}\\
\frac{\partial^{2} \phi}{\partial \alpha^{2}}= & 4 C_{1}\left(e^{2 \alpha}+a^{2} b^{2} e^{-2 \alpha}-9 a^{2} c^{6} e^{-6 \alpha}\right)+4 C_{2}\left(a b e^{-2 \alpha}+4 a c^{3} e^{-4 \alpha} \cos 2 \beta\right) \\
& +4 C_{3} e^{2 \alpha} \cos 2 \beta+4 C_{4} e^{-2 \alpha} \cos 2 \beta-16 C_{6} a c^{3} e^{-4 \alpha} \sin 2 \beta \\
& +4 C_{7} e^{2 \alpha} \sin 2 \beta+4 C_{8} e^{-2 \alpha} \sin 2 \beta,  \tag{19c}\\
\frac{\partial^{2} \phi}{\partial \beta^{2}}= & -8 C_{1} a b \cos 2 \beta-4 C_{2}\left(\cos 2 \beta+a c^{3} e^{-4 \alpha} \cos 2 \beta\right) \\
& -4 C_{3} e^{2 \alpha} \cos 2 \beta-4 C_{4} e^{-2 \alpha} \cos 2 \beta-4 C_{6}\left(\sin 2 \beta-a c^{3} e^{-4 \alpha} \sin 2 \beta\right) \\
& -4 C_{7} e^{2 \alpha} \sin 2 \beta-4 C_{8} e^{-2 \alpha} \sin 2 \beta,  \tag{19d}\\
\frac{\partial^{2} \phi}{\partial \alpha \partial \beta}= & 8 C_{2} a c^{3} e^{-4 \alpha} \sin 2 \beta-4 C_{3} e^{2 \alpha} \sin 2 \beta+4 C_{4} e^{-2 \alpha} \sin 2 \beta \\
& +8 C_{8} a c^{3} e^{-4 \alpha} \cos 2 \beta+4 C_{7} e^{2 \alpha} \cos 2 \beta-4 C_{8} e^{-2 \alpha} \cos 2 \beta . \tag{19e}
\end{align*}
$$

6. The stresses. Substitution of Eqs. (15) and (19) into Eq. (8) gives the stresses in the form

$$
\begin{align*}
\frac{\sigma_{\alpha}}{h^{4}}= & 2 C_{1}\left(A_{10}+A_{12} \cos 2 \beta+A_{14} \cos 4 \beta\right) \\
& +2 C_{2}\left(A_{20}+A_{22} \cos 2 \beta+A_{24} \cos 4 \beta+A_{26} \cos 6 \beta\right) \\
& +2 C_{3}\left(A_{30}+A_{32} \cos 2 \beta+A_{34} \cos 4 \beta+A_{36} \cos 6 \beta\right) \\
& +2 C_{4}\left(A_{40}+A_{42} \cos 2 \beta+A_{44} \cos 4 \beta+A_{46} \cos 6 \beta\right) \\
& +C_{5}\left(A_{50}+A_{52} \cos 2 \beta+A_{54} \cos 4 \beta\right) \\
& -2 C_{8}\left(A_{62} \sin 2 \beta+A_{64} \sin 4 \beta+A_{66} \sin 6 \beta\right) \\
& -2 C_{7}\left(A_{72} \sin 2 \beta+A_{74} \sin 4 \beta+A_{76} \sin 6 \beta\right) \\
& -2 C_{8}\left(A_{82} \sin 2 \beta+A_{84} \sin 4 \beta+A_{86} \sin 6 \beta\right),  \tag{20}\\
\frac{\sigma_{\beta}}{h^{4}}= & 2 C_{1}\left(B_{10}+B_{12} \cos 2 \beta+B_{14} \cos 4 \beta+B_{16} \cos 6 \beta\right) \\
& +2 C_{2}\left(B_{20}+B_{22} \cos 2 \beta+B_{24} \cos 4 \beta+B_{26} \cos 6 \beta\right) \\
& -2 C_{3}\left(B_{30}+B_{32} \cos 2 \beta+B_{34} \cos 4 \beta+B_{36} \cos 6 \beta\right) \\
& -2 C_{1}\left(B_{40}+B_{42} \cos 2 \beta+B_{44} \cos 4 \beta+B_{46} \cos 6 \beta\right) \\
& -C_{5}\left(B_{50}+B_{52} \cos 2 \beta+B_{54} \cos 4 \beta\right) \\
& -2 C_{6}\left(B_{62} \sin 2 \beta+B_{64} \sin 4 \beta+B_{66} \sin 6 \beta\right) \\
& +2 C_{7}\left(B_{72} \sin 2 \beta+B_{74} \sin 4 \beta+B_{76} \sin 6 \beta\right) \\
& +2 C_{8}\left(B_{82} \sin 2 \beta+B_{84} \sin 4 \beta+B_{86} \sin 6 \beta\right), \tag{21}
\end{align*}
$$

$$
\begin{align*}
\frac{\tau_{\alpha \beta}}{h^{4}}= & 12 C_{1}\left(D_{12} \sin 2 \beta+D_{14} \sin 4 \beta+D_{16} \sin 6 \beta\right) \\
& -2 C_{2}\left(D_{22} \sin 2 \beta+D_{24} \sin 4 \beta+D_{26} \sin 6 \beta\right) \\
& +2 C_{3}\left(D_{32} \sin 2 \beta+D_{34} \sin 4 \beta+D_{36} \sin 6 \beta\right) \\
& -2 C_{4}\left(D_{42} \sin 2 \beta+D_{44} \sin 4 \beta+D_{46} \sin 6 \beta\right) \\
& +2 C_{5}\left(D_{52} \sin 2 \beta+D_{54} \sin 4 \beta\right) \\
& -2 C_{6}\left(D_{80}+D_{62} \cos 2 \beta+D_{64} \cos 4 \beta+D_{66} \cos 6 \beta\right) \\
& +2 C_{7}\left(D_{70}+D_{72} \cos 2 \beta+D_{74} \cos 4 \beta+D_{76} \cos 6 \beta\right) \\
& -2 C s\left(D_{80}+D_{82} \cos 2 \beta+D_{84} \cos 4 \beta+D_{86} \cos 6 \beta\right) \tag{22}
\end{align*}
$$

in which

$$
\begin{aligned}
& A_{10}=e^{4 \alpha}+4 a^{2} b^{2}-\left(24 a^{2} c^{6}+18 a^{3} b^{2} c^{3}-a^{4} b^{4}\right) e^{-4 \alpha}+24 a^{4} b^{2} c^{8} e^{-8 \alpha}-81 a^{4} c^{12} e^{-12 \alpha}, \\
& A_{12}=-4\left[a b e^{2 \alpha}-\left(3 a^{2} b c^{3}-a^{3} b^{3}\right) e^{-2 \alpha}+\left(9 a^{3} b c^{6}-3 a^{4} b^{3} c^{3}\right) e^{-6 \alpha}+9 a^{4} b c^{9} e^{-10 \alpha}\right] \text {, } \\
& A_{14}=2\left(3 a c^{3}+a^{2} b^{2}-6 a^{3} b^{2} c^{3} e^{-4 \alpha}+9 a^{3} c^{9} e^{-8 \alpha}\right), \\
& A_{20}=2 a b-\left(6 a^{2} b c^{3}-a^{3} b^{3}\right) e^{-4 \alpha}+30 a^{3} b c^{6} e^{-8 \alpha} \text {, } \\
& A_{22}=-2\left[e^{2 \alpha}-\left(4 a c^{3}-a^{2} b^{2}\right) e^{-2 \alpha}+6\left(a^{2} c^{6}-a^{3} b^{2} c^{3}\right) e^{-6 \alpha}-18 a^{3} c^{9} e^{-10 \alpha}\right] \text {, } \\
& A_{24}=a b-8 a^{2} b c^{3} e^{-4 \alpha}+9 a^{3} b c^{6} e^{-8 \alpha}, \quad A_{26}=-6 a^{2} c^{6} e^{-6 \alpha}, \\
& A_{30}=3 a b e^{2 \alpha}-15 a^{2} b c^{3} e^{-2 \alpha}, \quad A_{32}=-\left(e^{4 \alpha}-15 a c^{3}+3 a^{2} b^{2}+45 a^{2} c^{6} e^{-4 \alpha}\right) \text {, } \\
& A_{34}=a b e^{2 \alpha}-9 a^{2} b c^{3} e^{-2 \alpha}, \quad A_{36}=3 a c^{3} \text {, } \\
& A_{40}=3 a b e^{-2 \alpha}-3 a^{2} b c^{3} e^{-6 \alpha}, \quad \quad \Lambda_{42}=-\left[3-\left(9 a c^{3}-a^{2} b^{2}\right) e^{-4 \alpha}-9 a^{2} c^{6} e^{-8 \alpha}\right] \text {, } \\
& A_{44}=a b e^{-2 \alpha}+3 a^{2} b c^{3} e^{-6 \alpha}, \quad A_{46}=-3 a c^{3} e^{-4 \alpha} \text {, } \\
& A_{50}=e^{2 \alpha}-a^{2} b^{2} e^{-2 \alpha}-27 a^{2} c^{6} e^{-6 \alpha}, \quad A_{52}=-2 a^{2} b c^{3} e^{-4 \alpha}, \quad A_{54}=6 a c^{3} e^{-2 a}, \\
& A_{62}=2\left[e^{2 \alpha}+\left(a^{2} b^{2}+4 a c^{3}\right) e^{-2 \alpha}+6 a^{2} c^{6} e^{-6 \alpha}+18 a^{3} c^{9} e^{-10 \alpha}\right] \text {, } \\
& A_{64}=-\left(a b-2 a^{2} b c^{3} e^{-4 \alpha}-9 a^{3} b c^{6} e^{-8 \alpha}\right), \quad A_{66}=-6 a^{2} c^{6} e^{-6 \alpha}, \\
& A_{72}=e^{4 \alpha}+3\left(a^{2} b^{2}+5 a c^{3}\right)+45 a^{2} c^{6} e^{-4 \alpha}, A_{74}=-\left(a b e^{2 \alpha}-9 a^{2} b c^{3} e^{-2 \alpha}\right), A_{78}=-3 a c^{3}, \\
& A_{82}=3+\left(a^{2} b^{2}+9 a c^{3}\right) e^{-4 \alpha}-9 a^{2} c^{6} e^{-8 \alpha}, A_{84}=-\left(a b e^{-2 \alpha}+3 a^{2} b c^{3} e^{-6 \alpha}\right), A_{86}=3 a c^{3} e^{-4 \alpha} \text {, } \\
& B_{10}=e^{4 \alpha}+4 a^{2} b^{2}+\left(24 a^{2} c^{6}+6 a^{3} b^{2} c^{3}+a^{4} b^{4}\right) e^{-4 \alpha}-24 a^{4} b^{2} c^{6} e^{-8 \alpha}-81 a^{4} c^{12} e^{-12 \alpha}, \\
& B_{12}=-4\left[a b e^{2 \alpha}-\left(3 a^{2} b c^{3}-a^{3} b^{3}\right) e^{-2 \alpha}-9 a^{3} b c^{6} e^{-6 \alpha}+18 a^{4} b c^{9} e^{-10 \alpha}\right] \text {, } \\
& B_{14}=-2\left(9 a c^{3}-a^{2} b^{2}+6 a^{3} b^{2} c^{3} e^{-4 \alpha}-45 a^{3} c^{9} e^{-8 \alpha}\right), \quad B_{16}=12 a^{2} b c^{3} e^{-2 \alpha}, \\
& B_{20}=2 a b-\left(6 a^{2} b c^{3}-a^{3} b^{3}\right) e^{-4 \alpha}+6 a^{3} b c^{6} e^{-8 \alpha} \text {, } \\
& B_{22}=2\left[2\left(a c^{3}-a^{2} b^{2}\right) e^{-2 \alpha}-3\left(4 a^{2} c^{6}-a^{3} b^{2} c^{3}\right) e^{-6 \alpha}+9 a^{3} c^{9} e^{-10 \alpha}\right] \text {, } \\
& B_{24}=a b-16 a^{2} b c^{3} e^{-4 \alpha}+9 a^{3} b c^{6} e^{-8 \alpha}, \quad B_{2 \theta}=6\left(a c^{3} e^{-2 \alpha}-2 a^{2} c^{6} e^{-6 \alpha}\right) \text {, } \\
& B_{30}=3 a b e^{2 \alpha}-15 a^{2} b c^{3} e^{-2 \alpha}, \quad B_{32}=-\left(e^{4 \alpha}-15 a c^{3}+3 a^{2} b^{2}+45 a^{2} c^{6} e^{-4 \alpha}\right) \text {, } \\
& B_{34}=a b e^{2 \alpha}-9 a^{2} b c^{3} e^{-2 \alpha}, \quad B_{36}=3 a c^{3}, \\
& B_{40}=3 a b e^{-2 \alpha}-3 a^{2} b c^{3} e^{-6 \alpha}, \quad B_{42}=-\left[3-\left(9 a c^{3}-a^{2} b^{2}\right) e^{-4 \alpha}-9 a^{2} c^{6} e^{-8 \alpha}\right], \\
& B_{44}=a b e^{-2 \alpha}+3 a^{2} b c^{3} e^{-6 \alpha}, \quad B_{46}=-3 a c^{3} e^{-4 \alpha}, \\
& B_{50}=e^{2 \alpha}-a^{2} b^{2} e^{-2 \alpha}-27 a^{2} c^{6} e^{-6 \alpha}, \quad B_{52}=-12 a^{2} b c^{3} e^{-4 \alpha}, \quad B_{54}=6 a c^{3} e^{-2 \alpha},
\end{aligned}
$$

$B_{62}=2\left[2 a c^{3} e^{-2 a}+3\left(a^{3} b^{2} c^{3}+4 a^{2} c^{6}\right) e^{-6 a}+9 a^{3} c^{9} e^{-10 \alpha}\right]$,
$B_{64}=-\left(a b+4 a^{2} b c^{3} e^{-4 a}-9 a^{3} b c^{6} e^{-8 \alpha}\right), \quad B_{66}=-6\left(a c^{3} e^{-2 \alpha}+2 a^{2} c^{6} e^{-6 \alpha}\right)$,
$B_{72}=e^{4 \alpha}+3\left(5 a c^{3}+a^{2} b^{2}\right)+45 a^{2} c^{6} e^{-4 \alpha}, \quad B_{74}=-\left(a b e^{2 \alpha}-9 a^{2} b c^{3} e^{-2 \alpha}\right), \quad B_{76}=-3 a c^{3}$,
$B_{82}=3+\left(9 a c^{3}+a^{2} b^{2}\right) e^{-4 \alpha}-9 a^{2} c^{6} c^{-8 \alpha}, B_{84}=-\left(a b e^{-2 \alpha}+3 a^{2} b c^{3} e^{-6 \alpha}\right), B_{86}=3 a c^{3} e^{-4 \alpha}$,
$D_{12}=\left(10 a^{3} b c^{6}+a^{4} b^{3} c^{3}\right) e^{-6 \alpha}-3 a^{4} b c^{9} e^{-10 \alpha}, D_{14}=2\left(a c^{3}+3 a^{3} c^{9} e^{-8 \alpha}\right), D_{18}=-a^{2} b c^{3} e^{-2 \alpha}$,
$D_{22}=e^{2 \alpha}+\left(2 a c^{3}+a^{2} b^{2}\right) e^{-2 \alpha}-3\left(2 a^{2} c^{6}+a^{3} b^{2} c^{3}\right) e^{-6 \alpha}+9 a^{3} c^{9} e^{-10 \alpha}$,
$D_{24}=4 a^{2} b c^{3} e^{-4 \alpha}, \quad D_{26}=3\left(a c^{3} e^{-2 \alpha}+a^{2} c^{6} e^{-6 \alpha}\right)$,
$D_{32}=e^{4 \alpha}+15 a c^{3}+3 a^{2} b^{2}+45 a^{2} c^{6} e^{-4 \alpha}, \quad D_{34}=-\left(a b e^{2 \alpha}-9 a^{2} b c^{3} e^{-2 \alpha}\right), \quad D_{36}=-3 a c^{3}$,
$D_{42}=3+\left(9 a c^{3}+a^{2} b^{2}\right) e^{-4 a}-9 a^{2} c^{6} e^{-8 \alpha}, D_{44}=-\left(a b e^{-2 \alpha}+3 a^{2} b c^{3} e^{-6 \alpha}\right), D_{46}=3 a c^{3} e^{-4 \alpha}$,
$D_{52}=a b-3 a^{2} b c^{3} e^{-4 \alpha}, \quad D_{54}=6 a c^{3} e^{-2 \alpha}, \quad D_{60}=12 a^{3} b c^{6} e^{-8 \alpha}$,
$D_{62}=-\left[e^{2 \alpha}-\left(2 a c^{3}+a^{2} b^{2}\right) e^{-2 \alpha}-3\left(2 a^{2} c^{6}+a^{3} b^{2} c^{3}\right) e^{-6 \alpha}-9 a^{3} c^{9} e^{-10 \alpha}\right]$,
$D_{64}=4 a^{2} b c^{3} e^{-4 a}$,

$$
D_{60}=-3\left(a c^{3} e^{-2 \alpha}-a^{2} c^{6} e^{-6 \alpha}\right),
$$

$D_{70}=3\left(a b e^{2 \alpha}-5 a^{2} b c^{3} e^{-2 \alpha}\right), \quad D_{72}=-\left[e^{4 \alpha}-3\left(5 a c^{3}-a^{2} b^{2}\right)+45 a^{2} c^{6} e^{-4 \alpha}\right]$,
$D_{74}=a b e^{2 \alpha}-9 a^{2} b c^{3} e^{-2 \alpha}, \quad D_{76}=3 a c^{3}$,
$D_{80}=3\left(a b e^{-2 \alpha}-a^{2} b c^{3} e^{-6 \alpha}\right), \quad \quad D_{82}=-\left[3-\left(9 a c^{3}-a^{2} b^{2}\right) e^{-4 \alpha}-9 a^{2} c^{6} e^{-8 \alpha}\right]$,
$D_{84}=a b e^{-2 \alpha}+3 a^{2} b c^{3} e^{-6 \alpha}, \quad D_{86}=-3 a c^{3} e^{-4 \alpha}$.
Boundary conditions (17) are satisfied by substitution for the $C$ 's in Eqs. (20), (21), and (22) of

$$
\begin{gather*}
C_{1}=\frac{1}{1}\left(S_{x}+S_{y}\right), \quad C_{3}=-\frac{1}{4}\left(S_{x}-S_{y}\right), \quad C_{7}=-\frac{1}{2} T_{x y} \\
-2\left(1-a c^{3} e^{-4 \alpha_{0}}\right) C_{2}=a b\left(S_{x}+S_{y}\right)-e^{2 \alpha_{0}}\left(S_{x}-S_{y}\right), \\
4\left(1-a c^{3} e^{-4 \alpha_{0}}\right) C_{4}=4 a^{2} b c^{3} e^{-2 \alpha_{0}}\left(S_{x}+S_{y}\right)-\left(e^{4 \alpha_{0}}+3 a c^{3}\right)\left(S_{x}-S_{y}\right) \\
-2\left(1-a c^{3} e^{-4 \alpha_{0}}\right) C_{5}=\left[e^{2 \alpha_{0}}-\left(a c^{3}-a^{2} b^{2}\right) e^{-2 \alpha_{0}}+\left(3 a^{2} c^{6}+a^{3} b^{2} c^{3}\right) e^{-6 \alpha_{0}}\right.  \tag{23}\\
\left.-3 a^{3} c^{9} e^{-10 \alpha_{0}}\right]\left(S_{x}+S_{y}\right)-2 a b\left(S_{x}-S_{y}\right) \\
\left(1+a c^{3} e^{-4 \alpha_{0}}\right) C_{6}=e^{2 \alpha_{0}} T_{x y}, \quad-2\left(1+a c^{3} e^{-4 \alpha_{0}}\right) C_{8}=\left(e^{4 \alpha_{0}}-3 a c^{3}\right) T_{x y}
\end{gather*}
$$

The case $a c^{3} e^{-4 \alpha 0}= \pm 1$, for which some of the $C$ 's in Eq. (23) are infinite, does not correspond to a simple curve for $\alpha=\alpha_{0}$ and hence is excluded.
7. Stresses along the inner boundary. The tangential stress in the boundary $\alpha=\alpha_{0}$ is

$$
\sigma_{\ell}=\left(\sigma_{\beta}\right)_{\alpha=\alpha_{0}} .
$$

However, it is simpler to compute it as follows. From Eq. (8),

$$
\frac{\sigma_{\alpha}+\sigma_{\beta}}{h^{2}}=\frac{\partial^{2} \phi}{\partial \alpha^{2}}+\frac{\partial^{2} \phi}{\partial \beta^{2}}
$$

Hence from Eq. (19),

$$
\begin{align*}
\frac{\sigma_{\alpha}+\sigma_{\beta}}{h^{2}}= & 4 C_{1}\left[e^{2 \alpha}+a^{2} b^{2} e^{-2 \alpha}-9 a^{2} c^{6} e^{-6 \alpha}-2 a b \cos 2 \beta\right] \\
& +4 C_{2}\left[a b e^{-2 \alpha}-\left(1-3 a c^{3} e^{-4 \alpha}\right) \cos 2 \beta\right]-4 C_{6}\left[1+3 a c^{3} e^{-4 \alpha}\right] \sin 2 \beta \tag{24}
\end{align*}
$$



Fig. 6. Ovaloid hole, tension parallel to long axis. Distribution of stress along ovaloid boundary. The dashed curve shows the distribution of stress for the case of an elliptical boundary having the same ratio of major to minor axis and the same rectified length as the ovaloid boundary:


Fig. 7. Ovaloid hole, tension parallel to short axis. Distribution of stress along the ovaloid boundary. The dashed curve shows the distribution of stress for the case of an elliptical boundary having the same ratio of major to minor axis and the same rectified length as the ovaloid boundary.


Fig. 8. "Square" hole, tension parallel to side.
The dashed curve shows the distribution of stress for the case of a circular boundary having the same rectified length as the "square" boundary.

For $\alpha=\alpha_{0}, \sigma_{\alpha}=0$; hence

$$
\sigma_{t}=\left(\sigma_{\beta}\right)_{\alpha-\alpha_{0}}=\left(\sigma_{\alpha}+\sigma_{\beta}\right)_{\alpha=\alpha_{0}}
$$

or from Eq. (24),

$$
\begin{align*}
\frac{\sigma_{t}}{h_{0}^{2}}= & 4 C_{1}\left[e^{2 \alpha_{0}}+a^{2} b^{2} e^{-2 \alpha_{0}}-9 a^{2} c^{6} e^{-6 \alpha_{0}}-2 a b \cos 2 \beta\right] \\
& +4 C_{2}\left[a b e^{-2 \alpha_{0}}-\left(1-3 a c^{3} e^{-4 \alpha_{0}}\right) \cos 2 \beta\right]-4 C_{6}\left[1+3 a c^{3} e^{-4 \alpha_{0}}\right] \sin 2 \beta \tag{25}
\end{align*}
$$

in which $h_{0}$ denotes the value of $h$ for $\alpha=\alpha_{0}$.
Substitution into Eq. (25) of $h_{0}$ from Eq. (15), of $C_{1}, C_{2}$, and $C_{6}$ from Eq. (23), and replacement of the constants $a, b, c$, and $\alpha_{\downarrow}$ by their values obtained from Eq. (13) gives, finally

$$
\begin{align*}
& {\left[\left(p^{2}+6 r q\right) \sin ^{2} \beta+\left(q^{2}+6 r p\right) \cos ^{2} \beta-6 r(p+q) \cos ^{2} 2 \beta+9 r^{2}\right] \sigma_{t} } \\
= & \left(S_{x}+S_{y}\right)\left(p^{2} \sin ^{2} \beta+q^{2} \cos ^{2} \beta-9 r^{2}\right)-T_{x y}(p+q)^{2} \frac{p+q+6 r}{p+q+2 r} \sin 2 \beta \\
- & \frac{\left(p^{2}-q^{2}\right)\left(S_{x}+S_{y}\right)-(p+q)^{2}\left(S_{x}-S_{y}\right)}{p+q-2 r}\left[(p-3 r) \sin ^{2} \beta-(q-3 r) \cos ^{2} \beta\right] . \tag{26}
\end{align*}
$$

8. Some special cases. The components of stress at any point in the plate may be computed from Eqs. (20), (21), (22), and (23). Of especial interest, however, are the values of $\sigma_{l}$, the tangential stress along the inner boundary, $\alpha=\alpha_{0}$, at points of which the numerically greatest normal and shearing stresses may be expected to occur.

In this section the values of $\sigma_{t}$ are computed and shown for several simple cases.
Case 1 (Fig. 6). Ovaloid hole, tension parallel to long axis. In this case $\sigma_{l}$ is obtained from Eq. (26) with $S_{v}=T_{x v}=0$ and $p, q$, and $r$ as given by Eq. (2). Then

$$
\frac{\sigma_{t}}{S_{x}}=\frac{4.915-7.133 \cos 2 \beta}{3.723-2.316 \cos 2 \beta+\cos 4 \beta}
$$

Case 2 (Fig. 7). Ovaloid hole, tension parallel to short axis. Here $S_{x}=T_{x v}=0$. Then

$$
\frac{\sigma_{t}}{S_{y}}=\frac{1.079+7.517 \cos 2 \beta}{3.723-2.316 \cos 2 \beta+\cos 4 \beta}
$$



Fig. 9. "Square" hole, tension parallel to diagonal. The dashed curve shows the distribution of stress for the case of a circular boundary having the same rectified length as the "square" boundary.

Case 3 (Fig. 8). "Square" hole, tension parallel to side. Here $S_{y}=T_{x u}=0$ and $p, q$, and $r$ are given by Eq. (3). Then

$$
\frac{\sigma_{1}}{S_{x}}=\frac{.981-2.967 \cos 2 \beta}{1.401+\cos 4 \beta}
$$

Case 4 (Fig. 9). "Square" hole, tension parallel to diagonal. Here $S_{v}=T_{x y}$ $=0$ and $p, q$, and $r$ are as given by Eq. (4). Then

$$
\frac{\sigma_{i}}{S_{x}}=\frac{.981-1.606 \cos 2 \beta}{1.401-\cos 4 \beta}
$$

In each of Figs. 6, 7, 8, and 9 the values of $\sigma_{t} / S_{x}$ or $\sigma_{t} / S_{y}$ are plotted along the development of one quadrant of the inner boundary of the plate. For comparison, there is shown by means of the dashed curve in each figure the distribution of $\sigma_{t} / S_{x}$ or $\sigma_{t} / S_{y}$ for the case of an elliptical boundary having the same ratio of major axis to minor axis and the same rectified length as the actual boundary.

# SOME PRESENT NONLINEAR PROBLEMS OF THE ELECTRICAL AND AERONAUTICAL INDUSTRIES ${ }^{1}$ 

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1. Introduction. The accelcrated growth of research in the field of nonlinearity is due to different causes. The general advancement of science requires increasingly more precise expressions for the laws of science. Accurate nonlinear equations frequently depart from the linearized or postulated linear equations which have been previously used for approximate results. The quest for perfection and generalization and the love of difficult investigations by professional mathematicians play a large part in this growth. Another incentive is the increasingly exacting requirements of modern manufacturing. These requirements are born of the competitive necessity of producing ever improved machines and equipment in the most economical manner. The greatest incentive is necessity. Manufactured equipment and devices must be designed to work.

A nonlinear problem ${ }^{2}$ has been defined as "one which, when formulated mathematically, reduces to (one or) a system of differential, integral, or integro-differential equations such that at least one of the three quantities, a derivative, an integral, or a dependent variable, is involved transcendentally or algebraically to a power other than the first in at least one equation of the system or in at least one boundary condition of the system." Of course, in dealing with applied problems, a physical definition independent of all mathematical concepts is preferable, but such is difficult to formulate.

Nonlinear problems resolve themselves into two general types, continuous and discrete. The first type deals with the behavior of quantities in a field or in at least one continuous region of space and, more often than not, reduces mathematically to systems of nonlinear partial differential equations. Problems relating primarily to this field have been treated by Dr. Theodore von Kármán in his Josiah Willard Gibbs Memorial lecture. ${ }^{3}$ This paper is both a milestone and a beacon of progress in that it is an admirable exposition and inventory of the nonlinear problems of continuous fields and at the same time an inspiration and invitation to both the engineer and mathematician for further advancement in this difficult field. Among other subjects, the von Kármán lecture treated relaxation oscillations, subharmonic resonance, nonlinear problems in the theory of elasticity in which the hypotheses of (a) small deflections are abandoned, (b) Hooke's law no longer holds, plasticity, hydrodynam-

[^15]ics; and aerodynamics of (a) ideal fluids, (b) viscous fluids, and (c) compressible fluids. The bibliography of the paper contains 178 entries.

The second type of nonlinear problem is called discrete. Discrete nonlinear problems are characterized by the fact that they possess only a finite number of degrees of freedom. They are frequently reducible mathematically to systems of nonlinear ordinary (total) differential equations or to systems of nonlinear integral equations.
2. Nature of industrial discrete nonlinear problems. Solutions of nonlinear problems in industry are usually in the "small"; i.e., a solution of a system is not required for every magnitude whatsoever of the parameters involved. In the solution of such problems the greatest single body of theory contributing to nonlinear analysis of discrete systems is that which grew out of the attempts of the great French and German mathematicians of the last century to solve the problem of three bodies. While their objective, in its complete generality, was never realized, the pure mathematics developed (theory of differential equations, convergence, dominant functions, singularities, removable singularities, etc.) is today directly applicable in the study of nonlinear equations of electrical circuits, rotating electrical machines, and various nonlinear dynamical and aerodynamical devices. The two second largest bodies of theory are those of nonlinear integral equations as developed by T. Lalesco ${ }^{4}$ and others ${ }^{5}$ and the methods of Galerkin ${ }^{6}$ and Ritz along with the modifications of these techniques. ${ }^{7}$

There are at least three salient characteristics of nonlinear engineering problems which distinguish them from purely theoretical problems. First, oscillograms, differential analyzer solutions, ${ }^{8}$ or other records frequently indicate the nature of the solution of the mathematical systems in question. Such mechanical or electrical solutions for the same system often differ so much among themselves that there is the risk of concluding erroneously that the solution is not unique. (For example, the differential equations which yield the two solutions represented in Figs. 3 and 4 also possess sinusoidal solutions. Yet the solutions are unique; i.e., the solution in (4) is identical with the sinusoidal solution.) Of course, there are systems which do not possess a unique solution. In general, even when a solution is unique it may have so many manifestations that it is often necessary to integrate the system to determine the effect of the various parameters involved. A second characteristic of industrial nonlinear problems is that frequently the methods of mathematics are not powerful enough to yield a complete solution of the problem in sufficiently simple form to be usable. Tricks and devices, born of physical concepts, must guide the mathematics if a usable solution is to be attained. The mathematics is surely necessary and it is just as surely not sufficient. The solution is mathematics plus. A third distinction of industrial nonlinear problems is the fact that the derivation of the equations of performance requires, in addition to a knowledge of mathematics, mathematical physics,

[^16]and engineering, inventive ability in thought. A system of equations may be an invention of the highest order. It is not always necessary to integrate a system of nonlinear equations. Often it is necessary only to determine under what conditions the physical system is stable. Of course, no single stability criterion exists for nonlinear systems such as exists for linear systems. When a solution of a nonlinear system cannot be obtained with sufficient rapidity or when it can be obtained but is worthless because the time consumed in applying it is too great, it may be possible to obtain the information desired by integrating a dominant and a "subordinate" system such that the solution of the actual problem is bounded or limited by the solutions of the dominant and subordinate system. The use of dominant and "subordinate" systems will be clear in the following problems.
3. Some representative discrete nonlinear problems of industry. In this paper a number of representative nonlinear systems are treated which illustrate the principles enumerated in the last section. These systems are either original, appearing here for the first time, or else of very recent date. Some of them pertain to electrical manufacturing, others to aircraft development. Although, as stated above, the derivation of the equations of a system is often more important than the solution, none of the equations considered are derived here. Some systems are derived in the literature and to these references are given. The derivations of the remaining ones can not be given for military reasons. These are viewed here merely as hypothetical nonlinear systems.

1. Nonlinear control circuits. As is well known, the volt-ampere characteristic of a nonlinear series circuit (Fig. 1) is represented by the curve in Fig. 2. Such circuits have numerous industrial applications due to their rugged mechanical simplicity and at the same time their electrical sensitivity. The characteristic in Fig. 2 displays the fact that there exists a so-called critical or resonant voltage $E_{0}$ at which the R.M.S. value of the current suddenly increases many fold. For a value of $E<E_{0}$ (see $E \sin$ wt in Fig. 1) the current is sinusoidal. For $E>E_{0}$ the current has the wave form displayed in Fig. 3.


In industrial applications $E_{0}$ is prescribed. It is required to design a circuit which will be sensitive for this prescribed value of $E_{0}$. A simple slide rule formula is desired which will express $E_{0}$ as a function of the circuit parameters and of the nonlinear reactor employed. The equation of performance for the circuit in Fig. 1 is

$$
\begin{equation*}
L(i) \frac{d i}{d t}+R i+\frac{1}{C} \int i d t=-E \cos \omega\left(t-t_{0}\right) \tag{1}
\end{equation*}
$$

For the range of interest the current $i$ is such that the saturation curve of the reactor is single-valued and represented by the equation

$$
\begin{equation*}
H=k i=x-a_{3} x^{3}+a_{5} x^{5} \tag{2}
\end{equation*}
$$

With $\theta=\omega t$, Eqs. (1) and (2) yield

$$
\begin{equation*}
M \frac{d x}{d \theta}+R\left(x-a_{3} x^{3}+a_{5} x^{5}\right)+x_{c} \int\left(x-a_{3} x^{3}+a_{5} x^{5}\right) d \theta=-E k \cos \left(\theta-\theta_{0}\right) \tag{3}
\end{equation*}
$$

where, for a given $\omega, M$ and $x_{c}$ are constants. The integration of the nonlinear Eq. (3) and the development of $E_{0}$ as a function of the parameters of the physical problem are carried out elsewhere and need not be repeated here. ${ }^{9}$
2. Nonlinear transmission line phenomena. If a series capacitor is employed in the primary side of a transmission line to improve the power factor, curious wave forms of voltage and current ensue. The system becomes unstable as far as possessing a periodic solution is concerned. This is to be expected since the maximum flux density attains a value close to that of the knee of the saturation curve if the transformers


Fig. 4.
are operating efficiently. The addition of series capacitance is thus likely to create an unstable system. In this unstable system, the current and voltage taken on an indefinitely large number of wave forms such as shown in Fig. 4. Synchronous motors which require sinusoidal applied voltages cannot operate on currents and voltages of the type shown in Fig. 4.

If the capacitor of the system is shunted by a resistor as indicated in Fig. 5, then the equations of performance are

[^17]\[

$$
\begin{gather*}
\qquad \frac{d \psi}{d \theta}+R_{1} i_{1}+L \frac{d i_{1}}{d t}+\frac{1}{C} \int\left(i_{1}-i_{2}\right) d t=E \sin \left(\theta+\theta_{0}\right), \\
\qquad i_{2} R_{2}=\bar{x}_{c} \int\left(i_{1}-i_{2}\right) d \theta, \quad\left(\bar{x}_{c}=1 / C\right),  \tag{4}\\
\text { or } \quad \alpha i_{1}=x-a_{3} x^{3}+a_{5} x^{5} ; H=\alpha i ; x=B / B_{0} ; B_{0}=d B / d H \text { at } H=0, \\
\begin{array}{c}
\alpha B_{0} \frac{d^{2} x}{d \theta^{2}}+\left[\frac{\bar{x}_{c} B_{0} \alpha}{R_{2}}+\right. \\
\left.R_{1}\left(1-3 a_{3} x^{2}+5 a_{5} x^{4}\right)\right] \frac{d x}{d \theta}+\frac{\bar{x}_{c}\left(R_{1}+R_{2}\right)}{R_{2}}\left(x-a_{3} x^{3}+a_{5} x^{5}\right) \\
=\frac{\alpha}{R_{2}}\left[R_{2}^{2}+\bar{x}_{c}^{2}\right]^{1 / 2} E \cos \left(\theta+\theta_{0}-\tan ^{-1} \frac{\bar{x}_{c}}{R_{2}}\right) .
\end{array}
\end{gather*}
$$
\]

Now $R_{2}$ must have the smallest possible value consistent with stability, since it represents a perpetual loss of power. There are ten parameters and two variables.


Fig. 5.
There are infinitely many values of the parameters for which the system is unstable and equally as many for which it is stable, i.e., for which the solutions are sinusoidal. A convenient slide rule formula is desired giving the above smallest value of $R_{2}$ as functions of the other nine parameters of the system. The equations of the system are derived and solved elsewhere. ${ }^{10}$
3. Nonlinear springs. It is sufficient to say that, in general, the differential equations involving nonlinear springs are integrable by hyperelliptic functions. ${ }^{11}$ If damping is large a combination of variation of parameters and hyperelliptic functions will usually afford sufficient accuracy.
4. Electric locomotive oscillations. Experience classifies the five oscillatory motions of an electric locomotive as pitch, roll, plunge, nose, and rear-end lash. The last two are especially important because their pronounced existence in a locomotive produces a tendency to derail. Considered superficially, characteristic oscillations of an electric locomotive would seem to be similar to those of an ordinary vehicle such as an automobile, but experimental data and observation indicate the existence of dangerous

[^18]nose and rear-end lash which are not common to an automobile. If the tendency to nose exists in an electric locomotive and if the locomotive noses for a speed $V_{0}$, then it will nose for all speeds greater than $V_{0}$. Consequently, nosing is not a resonance phenomenon and cannot be avoided by running at a slightly different speed. It might be supposed that nosing is due to the coning of the wheels or to the staggering of the rails or to a combination of these two possible causes. Such causes, however, would produce resonance frequencies for definite discrete values of $V$ instead of instability for all values of $V$ exceeding $V_{0}$. Rails on European railroads are not staggered and yet electric locomotive nosing still persists. The tendency to nose and the violence of the oscillation increase with the weight and power of the locomotive.

In seeking the source of the phenomenon, consider first an elementary experiment. Let a miniature set of driving wheels and axle be constructed from two rubber paste bottle stoppers and a lead pencil. If the miniature drivers are forced down against two rulers as rails, if a torque is applied tending to rotate the wheels, and if further in the forward motion slight lateral motion is permissible then an oscillating torque will be experienced tending to rotate the axle about a line through the center of axle and perpendicular to the plane of the track. The creepage forces between the rubber wheels and the rails produce an oscillatory torque.

The weight of an electric locomotive is so great that it effectively rolls on elastic wheels on elastic rails. Making use of this fact and whatever additional facts are necessary the equations of motion ${ }^{12}$ can be shown to be

$$
M \tilde{x}_{0}=0
$$

$$
M \ddot{y}_{0}=-F_{2}-f_{2}-2 f\left(\frac{\dot{y}_{2}}{V}-\zeta\right)-2 f\left(\frac{\dot{y}}{V}-\zeta\right)-F_{1}-f_{1}-2 f\left(\frac{\dot{y}_{1}}{V}-\zeta\right),
$$

$$
M \tilde{z}_{0}+\lambda_{1}\left(z_{0}-b_{1} \eta\right)+\lambda_{2}\left(z_{0}-c \xi+b_{2} \eta\right)+\lambda_{2}\left(z_{0}-c \xi+b_{2 \eta}\right)+k_{1}^{2} \dot{z}_{0}=0,
$$

$$
A \ddot{\xi}+\lambda_{2} c\left(z_{0}+c \xi+b_{2} \eta\right)-\lambda_{2} c\left(z_{0}-c \xi+b_{3} \eta\right)+k_{2 \xi}^{2} \xi
$$

$$
=-b_{5}\left(F_{1}+F_{2}+f_{1}+f_{2}\right)-2 b_{5} \frac{f}{V}\left(\dot{y}_{1}+\dot{y}+\dot{y}_{2}\right)+6 b_{5} f \zeta
$$

$$
\begin{equation*}
B \ddot{\eta}-\lambda_{1} b_{1}\left(z_{0}-b_{1} \eta\right)+\lambda_{2} b_{2}\left(z_{0}+c \xi+b_{2} \eta\right)+\lambda_{2} b_{2}\left(z_{0}-c \xi+b_{2} \eta\right)+k_{3}^{2} \dot{\eta}=0, \tag{5}
\end{equation*}
$$

$$
C \zeta=-d_{3}\left(F_{1}-F_{2}\right)-d_{1}\left(f_{1}-f_{2}\right)-\frac{2 f d_{3}}{V}\left(\dot{y}_{1}-\dot{y}_{2}\right)-\frac{6 f b^{2}}{V} \zeta
$$

$$
=\frac{2 f \lambda b}{r}\left(\bar{y}+y_{1}+y_{2}\right)+F_{1}\left(\bar{y}_{1}\right)
$$

To the accuracy required, the flange forces are given by

$$
\begin{align*}
& F_{1}=H\left(\frac{y_{1}}{\delta_{1}}\right)^{3}+I\left(\frac{y_{1}}{\delta_{1}}\right)^{5}+J\left(\frac{y_{1}}{\delta_{1}}\right)^{7}+\cdots \\
& F_{2}=H\left(\frac{y_{2}}{\delta_{1}}\right)^{3}+I\left(\frac{y_{2}}{\delta_{1}}\right)^{5}+J\left(\frac{y_{2}}{\delta_{1}}\right)^{7}+\cdots \tag{6'}
\end{align*}
$$

[^19]\[

$$
\begin{align*}
& f_{1}=h\left(\frac{y_{3}}{\delta_{2}}\right)^{3}+i\left(\frac{y_{3}}{\delta_{2}}\right)^{5}+j\left(\frac{y_{3}}{\delta_{2}}\right)^{7}+\cdots, \\
& f_{2}=h\left(\frac{y_{1}}{\delta_{2}}\right)^{3}+i\left(\frac{y_{4}}{\delta_{2}}\right)^{5}+j\left(\frac{y_{4}}{\delta_{2}}\right)^{7}+\cdots,
\end{align*}
$$
\]

where the constants $H, I_{1} J, j, i, h$ are determined from force curves, and $\delta_{1}$ and $\delta_{2}$ are lengths shown in Fig. 6.

The variables $y_{1}, y_{2}, y_{3}, y_{4}$, and $\bar{y}$ are eliminated from (5) by means of the relations

$$
\begin{gathered}
y_{1}=y_{0}+b_{5} \xi+d_{3} \xi, \quad y_{2}=y_{0}+b_{5} \xi-d_{3} \xi, \\
y_{3}=y_{0}+h_{1} \xi+d_{4} \zeta, \quad y_{4}=y_{0}+h_{1} \xi-d_{4} \zeta, \\
\bar{y}=y_{0}+b_{5} \xi,
\end{gathered}
$$

where $b_{5}, d_{3}, d_{4}, h_{1}$ and $h_{2}$ are lengths defined in reference 12 .
If in (5) $F_{1}=F_{2}=f_{1}=f_{2}=0$, then the equations are linear; and the solution can be written down at once. This solution is either stable or unstable as indicated by the roots of the characteristic equation. The nature of the roots are, of course, a function of $V$, the operating speed of the locomotive. Even if the locomotive is unstable with vanishing flange forces, it is stable with non-vanishing flange forces. In this case the locomotive is operating roughly and damaging the track needlessly.

In practical applications, then, it is not necessary to integrate the nonlinear system (5). As a check on the validity of the theory, however, it is necessary to integrate the nonlinear system and compare the predicted motion with actual motion as determined by runs on a test track. Evidently the solution of (5) for $F_{1}=F_{2}=f_{1}=f_{2}=0$ cannot be used as a generating solution for the case of the non-vanishing of the flange forces because the stability or unstability of this generating solution is carried over in to the complete solution.



Fig. 6.
Since nosing and rear end lash are the two motions of most importance, it is sufficient to resort to elementary means. Consequently, $F_{i}$ and $f_{i}$ [Eqs. (6)] are replaced by segments of straight lines as shown in Fig. 6, and the second and sixth equations of (5) are solved for $y_{0}$ and $\zeta$ by operational methods (general operational methods where both initial charges and initial currents exist must be used.) Since the flange
forces are taken as functions with discontinuous slopes, the system of differential equations and its characteristic equation change as the flange forces, as functions of $y_{1}, y_{2}, y_{3}, y_{4}$, change at points $a, b, c, d, e, f$ shown in Fig. 6. The first set of initial conditions are chosen by trial and error in such a way that the resulting motion is periodic in $y_{0}$ and $\zeta$. The solution for a complete cycle is sufficient. The check of the theory is the approximate agreement of computed and test periods.
5. Nonlinear differential equations of dynamic braking of a synchronous machine. The equations of dynamic braking are

$$
\begin{gather*}
\frac{d I}{d t}=\frac{(E-I R)}{L} \frac{\left[\left(r s_{0} / s\right)^{2}+x_{d} x_{q}\right]}{\left[\left(r s_{0} / s\right)^{2}+x_{d} x_{q}\right]}-\frac{\mu_{0}}{s^{4}} \frac{I^{3}\left[x_{q}^{2}+\left(r s_{0} / s\right)^{2}\right]}{\left[\left(r s_{0} / s\right)^{2}+x_{d} x_{q}\right]\left[\left(r s_{0} / s\right)^{2}+x_{d} x_{q}\right]^{3}}, \\
\frac{d s}{d t}=-\frac{\mu_{1} I^{2}}{s} \frac{x_{q}^{2}+\left(r s_{0} / s\right)^{2}}{\left[x_{d} x_{q}+\left(r s_{0} / s\right)^{2}\right]^{2}}, \tag{7}
\end{gather*}
$$

where

$$
\mu_{0}=\frac{2 K P r^{3} x_{q}\left(x_{d}-x_{d^{\prime}}\right) s_{0}^{2}}{J I_{0}^{2}}, \quad \mu_{1}=\frac{K P r}{J I_{0}^{2}},
$$

$I$ being the field current, $s$ the rotor speed, $t$ the time in seconds, all other symbols being constant parameters. It is desired to obtain an expression for the time of stopping of the rotor as a function of the parameters of the machine.

The last term in the right member of the first of Eqs. (7) has in all cases a magnitude of approximately ten per cent of its predecessor. Thus a solution as a power series in a parameter which vanishes with $\mu_{0}$ is to be expected. Neglecting the term containing $\mu_{0}$ in (7) and dividing the first equation by the second, we obtain a solution of the resulting equation immediately. However, this solution is an implicit function of $I$ and $s$ and such that it is solvable explicitly for either $I$ or $s$ only as a slowly convergent series. An attempted solution by the method of variation of parameters is equally cumbersome.

It is known from oscillograms, however, that both $I$ and $s$ are monotone decreasing functions of the time for the interval with in which (7) is valid. Change of dependent variables by

$$
I=\frac{E}{R}+I_{1} e^{-y} \text { and } s=s_{0} e^{-z}
$$

yields

$$
\begin{align*}
& \frac{d y}{d t}=\frac{R}{L}+\frac{R\left(A^{2}-A_{0}^{2}\right)}{L\left[A_{0}^{2}+\left(r e^{z}\right)^{2}\right]}+\mu_{0} \frac{\left[x_{q}^{2}+\left(r e^{2}\right)^{2}\right]\left[E / R+I_{1} e^{-y}\right]^{3}}{\left[A_{0}^{2}+\left(r e^{2}\right)^{2}\left[A^{2}+\left(r e^{2}\right)^{2}\right]^{3}\right.} \frac{I_{1} s_{0}^{4}}{}, \\
& \frac{d z}{d t}=\mu_{1} \frac{\left[(E / R)+I_{1} e^{-\nu}\right]^{2}\left[x_{Q}^{2}+\left(r e^{2}\right)^{2}\right]}{\left[A^{2}+\left(r e^{2}\right)^{2}\right]^{2} s_{0}^{2}} e^{2 z}, \tag{8}
\end{align*}
$$

where

$$
A_{0}^{2}=x_{d} x_{q}, \quad A^{2}=x_{d} x_{2}, \quad A>A_{0}
$$

The number of revolutions before the rotor of the machine comes to rest is

$$
\begin{equation*}
N=\frac{1}{2 \pi} \int_{0}^{\infty} s d t=\frac{1}{2 \pi} \int_{0}^{\infty} s_{0} e^{-z} d t . \tag{9}
\end{equation*}
$$

Now it is sufficient for practical purposes to set an upper limit to $N$ as given by (9) provided the upper limit is sufficiently small and provided the results display the effect of each parameter of the system. To accomplish this (8) may be replaced by a simpler system of equations. Evidently,

$$
\begin{equation*}
\left[A_{0}^{2}+\left(r e^{z}\right)^{2}\right] \leqq\left(A_{0}^{2}+r^{2}\right) e^{2 z}, \quad\left[A^{2}+\left(r e^{2}\right)^{2}\right] \leqq\left(A^{2}+r^{2}\right) e^{2 z} \tag{10}
\end{equation*}
$$

for $z \geqq 0$. Employing (10) in (7) and integrating, we have

$$
\begin{align*}
& \begin{aligned}
y= & \frac{R}{L} t
\end{aligned}+\frac{R\left(A^{2}-A_{0}^{2}\right)}{L\left(A_{0}^{2}+r^{2}\right)} \int_{0}^{t} e^{-2 z} d t \\
& \quad+\frac{1}{s_{0}^{4} I_{1}} \frac{\left(r^{2}+x_{q}^{2}\right) \mu_{0}}{\left(A_{0}^{2}+r^{2}\right)\left(A^{2}+r^{2}\right)^{3}} \int_{0}^{t}\left[\frac{E}{R}+I_{1} e^{-y}\right]^{3} e^{y-4 z} d t,  \tag{11}\\
& z= \frac{\mu_{1}\left(r^{2}+x_{q}^{2}\right)}{s_{0}^{2}\left(A^{2}+r^{2}\right)^{2}} \int_{0}^{t}\left[\frac{E}{R}+I_{1} e^{-y}\right]^{2} e^{-2 z} d t
\end{align*}
$$

where the instantaneous values of $y$ and $z$ as given by (11) are always less than those given by the solution of (7) for $0<t<\infty$.

The system (11) is of the form

$$
u_{k}(t)=\phi_{k}(t)+\int_{0}^{t} K_{k}\left[t, \xi, u_{1}(\xi), \cdots, u_{n}(\xi)\right] d \xi \quad(k=1,2, \cdots, n)
$$

which is Lalesco's system of nonlinear integral equations. The solution of this is the limit of the sequences

$$
\begin{gathered}
u_{k}^{(0)}=\phi_{k}(t), \\
u_{k}^{(1)}=\phi_{k}(l)+\int_{0}^{t} K_{k}\left[t, \xi, \phi_{1}(\xi), \cdots, \phi_{\pi}(\xi)\right] d \xi \quad(k=1,2, \cdots, n),
\end{gathered}
$$

In the present application $\phi_{1}(t)=R t / L$ and $\phi_{2}(t)=0$. For small synchronous machines the second approximations $u_{1}^{(1)}$ and $u_{2}^{(2)}$ give values of $y$ and $z$ such that $N$ in (9) is in error by five per cent. The integration in (9) is carried out numerically. Because of bearing friction and other decelerating factors not included in (7) the upper limit in (9) is finite.
6. A double-valued nonlinear problem. Consider the integration of the equation

$$
\begin{equation*}
I \ddot{\theta}+\beta \dot{\theta}+\mu\left[k_{1} \theta+k_{2} \tan ^{-1} k_{3}(\theta \pm a)\right]=0 . \tag{12}
\end{equation*}
$$

This equation was derived ingeniously by W. W. Beman to express an important phenomenon in aerodynamics. The quantities $I, \beta, \mu, k_{1}, k_{2}, k_{3}$, and $a$ are all positive numbers and the plus or minus sign in $(\theta \pm a)$ is used according as $\dot{\theta}<0$ or $\dot{\theta}>0$.

Evidently, for a particular amplitude of $\theta$, Eq. (12) possesses a periodic solution. The period and amplitude of this solution are desired. Eq. (12) in the normal form is

$$
\begin{equation*}
\dot{\theta}=\theta_{1}, \quad \dot{\theta}_{1}=-(\mu / I)\left[k_{1} \theta+k_{2} \tan ^{-1} k_{3}(\theta \pm a)\right]-\beta \theta_{1} / I . \tag{13}
\end{equation*}
$$

An integral of (13) for $\beta=0$ is

$$
\theta_{1}^{2}=c-\frac{2 \mu}{I}\left\{\frac{k_{1} \theta^{2}}{2}+k_{2}\left[(\theta \pm a) \tan ^{-1} k_{3}(\theta \pm a)-\frac{1}{2 k_{3}} \log \left(1+k_{3}^{2}\left(\theta_{1} \pm a\right)^{2}\right)\right]\right\}
$$

or

$$
\begin{equation*}
\theta_{\mathbf{1}} \equiv \pm \sqrt{c-f(\theta)} \tag{14}
\end{equation*}
$$

where $c=f\left(\theta_{0}\right)$ and where $\theta_{0}$ is the maximum positive displacement for $t=t_{0}$. If (14) is used as an equation of change of variable, the method of variation of parameters yields

$$
\frac{\partial \theta_{1}}{\partial c} \frac{\partial c}{\partial \iota}+\frac{\partial \theta_{1}}{\partial \iota}=-\frac{\mu}{I}\left[k_{1} \theta+k_{2} \tan ^{-1} k_{3}(\theta \pm a)\right]-\frac{\beta}{I} \theta_{1}
$$

whence

$$
\begin{equation*}
\dot{c}=-2 \beta \theta_{1}^{2} / I \tag{15}
\end{equation*}
$$

or

$$
c=-\frac{2 \beta}{I} \int_{t_{0}}^{t} \theta_{1}^{2} d t+d=-\frac{2 \beta}{I} \int_{\theta_{0}}^{\theta} \dot{\theta} d \theta+d
$$

where $d$ is an arbitrary constant. From the last equation

$$
\begin{equation*}
\frac{d c}{d \theta}=-\frac{2 \beta}{I} \theta_{1}=-\frac{2 \beta}{I}( \pm \sqrt{c-f(\theta)}) \tag{16}
\end{equation*}
$$

To determine the signs in (16) it is evident from (15) that $c$ is a decreasing function of the time. Consequently, for $\dot{\theta}<0$

$$
\frac{d c}{d \theta}=\frac{d c}{d t} \frac{d t}{d \theta}=\frac{d c}{d t}(-\sqrt{c-f(\theta)}) .
$$

Thus Eq. (16) is

$$
\begin{equation*}
\frac{d c}{d \theta}= \pm \frac{2 \beta}{I} \sqrt{c-f(\theta)} \tag{17}
\end{equation*}
$$

according as $\hat{\theta}_{1}<0$ or $\dot{\theta}_{1}>0$.
For the integration of (17) it is sufficient to replace $\sqrt{c-f(\theta)}$ by $k[c-f(\theta)]$ where $k$ is determined graphically by

$$
\int \sqrt{c-f(\theta)} d \theta=k \int[c-f(\theta)] d \theta
$$

$c=f\left(\theta_{0}\right)$ or $c=f\left(\theta_{0}^{\prime}\right)$ according as $\theta_{0} \leqq \theta \leqq \theta_{0}^{\prime}$ or $\theta_{0}^{\prime} \leqq \theta \leqq \theta_{0}^{\prime \prime}$, and $\theta_{0}, \theta_{0}^{\prime}$, and $\theta_{0}^{\prime \prime}$ are shown in Fig. 7. The curve in Fig. 7 is the solution (14). With this replacement and simple integration

$$
\begin{array}{ll}
c=e^{2 \beta k\left(\theta-\theta_{0}\right) / I}\left[c_{0}-\frac{2 \beta k}{I} \int_{\theta_{0}}^{\theta} e^{-2 \beta k \theta / I} f(\theta) d \theta\right] & \left(\theta_{0} \leqq \theta \leqq \theta_{0}^{\prime}\right)  \tag{18}\\
c=e^{-2 \beta k\left(\theta-\theta_{0}^{\prime}\right) / I}\left[c_{0}^{\prime}-\frac{2 \beta k}{I} \int_{\theta_{0}^{\prime}}^{\theta} e^{2 \beta k \theta / I} f(\theta) d \theta\right] & \left(\theta_{0}^{\prime} \leqq \theta \leqq \theta_{0}^{\prime \prime}\right),
\end{array}
$$

where $c_{0}=f\left(\theta_{0}\right)$ and $c_{0}^{\prime}=f\left(\theta_{0}^{\prime}\right)$. The ahove values of $c$ are substituted in Eq. (14). The
solution is periodic when $\theta_{0}$ is chosen such that $\theta_{0}^{\prime \prime}$ turns out to be equal to $\theta_{0}$. The period of the motion is then given by

$$
\begin{equation*}
P=2 \int_{\theta_{0}}^{\theta_{0}^{\prime}} \frac{d \theta}{\theta_{1}} . \tag{19}
\end{equation*}
$$

The numerical integration of (19) presents no difficulty at the limits $\theta_{0}$ and $\theta_{0}^{\prime}$ since $\theta_{1}$ in the vicinity of $\theta_{0}$ and $\theta_{0}^{\prime}$ can be replaced by an integrable function $f_{0}$ such that the limit of $\left(f_{0} / \theta_{1}\right)=1$ at $\theta_{1}=\theta_{0}$ and $\theta_{1}=\theta_{0}^{\prime}$.


Fig. 7.


Fig. 8.
7. A nonlinear problem of two oleo-pneumatically coupled masses one of which is subject to impact. It can be shown without difficulty that the nonlinear differential equations of motion of $m_{1}$ and $m_{2}$ shown in Fig. 8 are

$$
\begin{align*}
& m_{1} \tilde{s}_{1}-w_{1}-\frac{p_{0} S}{\left[1-\frac{\left(s_{2}-s_{1}\right)}{D}\right]^{1.2}}-\frac{\rho\left(S-s_{m}\right)^{3}\left(s_{2}-s_{1}\right)^{2}}{2 g c^{2}[A(r)]^{2}}+k_{2} s_{1}+f\left(s_{1}\right)=0, \\
& m_{2} s_{2}-n w_{2}+\frac{p_{0} S}{\left[1-\frac{\left(s_{2}-s_{1}\right)}{D}\right]^{1.2}}+\frac{\rho\left(S-s_{m}\right)^{3}\left(s_{2}-s_{1}\right)^{2}}{2 g c^{2}[A(r)]^{2}}=0, \tag{20}
\end{align*}
$$

where

$$
\begin{equation*}
[A(r)]^{2}=\pi^{2}\left\{R^{2}-\left[r_{i}+b_{i}\left(s_{2}-s_{1}\right)\right]^{2}\right\}, \quad f\left(s_{1}\right)=\left(K_{0}+k_{0} s_{1}\right), \tag{21}
\end{equation*}
$$

and $n=(1-m), 0 \leqq m \leqq 1$. In Eqs. (20) and (21), $s_{1}, s_{2}$, and $t$ are the dependent and independent variables, the remaining symbols being constants.

A solution of (20) is desired for the initial conditions $s_{1}(0)=s_{2}(0)=v_{0}$. The time $t$ is counted from the instant when the lower end of the spring is in contact with a fixed horizontal surface. For suitable values of $R, r_{i}$, and $b_{i}$ a graph of $[A(r)]^{2}$ is either Fig. 9a or 9b.


Fig. 9a.


Fig. 10.

$$
\begin{align*}
s_{1}+K_{0}+k_{0} s_{1}-\alpha\left[A_{0}+m_{0}\left(s_{2}-s_{1}\right)\right]-g & =\beta\left(s_{2}-s_{1}\right)^{2},  \tag{22}\\
\ddot{s}_{2}+\gamma\left[A_{0}+m_{0}\left(s_{2}-s_{1}\right)\right]-n g & =-\delta\left(s_{2}-s_{1}\right)^{2},
\end{align*}
$$

where

$$
\alpha=\frac{p_{0} S}{m_{1}}, \quad \beta=\frac{\rho\left(S-s_{m}\right)^{3}}{2 w_{1} c^{2} \pi^{2} R^{2}}, \quad \gamma=\frac{p_{0} S}{m_{2}}, \quad \delta=\frac{\rho\left(S-s_{m}\right)^{2}}{2 w_{2} c^{2} \pi^{2} R^{2}}
$$

${ }^{13}$ Lord Rayleigh, Theory of sound, (2nd ed.) vol. I, Macmillan, London, 1894, p. 81.
and $A_{0}, m_{0}, K_{0}$, and $k_{0}$ are shown in the figures. The ordinate of $R_{3}$ represents the value of $(1-\xi)^{-1.20}$ when the system is at rest under the force of gravity. The points $R_{2}$ and $R_{1}$ are located such that no ordinate on the secant lines exceeds the corresponding ordinate on the arc by more than ten per cent.

The solution of Eq. (20) is now broken up into two time intervals. We reduce Eq. (21) to the normal form by the substitutions

$$
s_{1}=\xi_{1}+a_{1}, \quad s_{2}=\xi_{2}+a_{2}, \quad \xi_{1}=\xi_{3}, \quad \dot{\xi}_{2}=\xi_{4},
$$

where $a_{1}$ and $a_{2}$ are constants such that no constant term remains in the resulting differential equations. Then the equations are

$$
\begin{align*}
& \xi_{1}=\xi_{3}, \\
& \xi_{2}=\xi_{4},  \tag{23}\\
& \xi_{3}=-\left(\alpha m_{0}+k_{0}\right) \xi_{1}+\alpha m_{0} \xi_{2}+\beta\left(\xi_{4}-\xi_{3}\right)^{2}, \\
& \xi_{4}=\gamma m_{0} \xi_{1}-\gamma m_{0} \xi_{2}-\delta\left(\xi_{4}-\xi_{3}\right)^{2},
\end{align*}
$$

and

$$
\begin{aligned}
& a_{1}=\gamma\left[g+\left(\alpha A_{0}-K_{0}\right)\right]-\alpha\left[\gamma A_{0}-n g\right] / \gamma k_{0} \\
& a_{2}=\left\{m_{0}\left[\gamma\left(g+\alpha A_{0}-K_{0}\right)-\alpha\left(\gamma A_{0}-n g\right)\right]-k_{0}\left(\gamma A_{0}-n g\right)\right\} / m_{0} \gamma k_{0}
\end{aligned}
$$

and $\alpha A_{0}=K_{0}$ in order that $\xi_{1}$ may not be positive in its initial motion. That is, the origin of time is taken to be the instant at which the upward force of the spring $S$ is equal to the downward force due to gas pressure on $m_{1}$.

The general solution (23) (with squared terns suppressed) is

$$
\begin{align*}
& \xi_{1}=A_{1} \sin \omega_{1} t+A_{2} \cos \omega_{1} t+A_{3} \sin \omega_{2} t+A_{4} \cos \omega_{2} t, \\
& \xi_{2}=b_{1} A_{1} \sin \omega_{1} t+b_{1} A_{2} \cos \omega_{1} t+b_{2} A_{3} \sin \omega_{2} t+b_{2} A_{4} \cos \omega_{2} t, \\
& \xi_{3}=\omega_{1} A_{1} \cos \omega_{1} t-\omega_{1} A_{2} \sin \omega_{1} t+\omega_{2} A_{3} \cos \omega_{2} t-\omega_{2} A_{4} \sin \omega_{2} t,  \tag{24}\\
& \xi_{4}=b_{1} \omega_{1} A_{1} \cos \omega_{1} t-b_{1} \omega_{1} A_{2} \sin \omega_{1} t+b_{2} \omega_{2} A_{3} \cos \omega_{2} t-b_{2} \omega_{2} A_{4} \sin \omega_{2} t,
\end{align*}
$$

where $\omega_{1}$ and $\omega_{2}$ are the roots of the characteristic equation and

$$
b_{1}=\left(\alpha m_{0}+k_{0}-\omega_{1}^{2}\right) / \alpha m_{0}, \quad b_{2}=\left(\alpha m_{0}+k_{0}-\omega_{2}^{2}\right) / \alpha m_{0}
$$

The nonlinear terms in (22) are taken into account by the method of variation of parameters. Employing (24) as equations of change of variables and remembering that (24) satisfies

$$
\begin{gathered}
\frac{\partial \xi_{1}}{\partial t}=\xi_{3}, \quad \frac{\partial \xi_{2}}{\partial t}=\xi_{4} \\
\frac{\partial \xi_{3}}{\partial t}=-\left(\alpha m_{0}+k_{0}\right) \xi_{1}+\alpha m_{0} \xi_{2}, \quad \frac{\partial \xi_{4}}{\partial t}=\gamma m_{0} \xi_{1}-\gamma m_{0} \xi_{2}
\end{gathered}
$$

we have

$$
\begin{align*}
& \frac{\partial \xi_{1}}{\partial A_{1}} A_{1}+\frac{\partial \xi_{1}}{\partial A_{2}} A_{2}+\frac{\partial \xi_{1}}{\partial A_{3}} A_{3}+\frac{\partial \xi_{1}}{\partial A_{4}} A_{4}=0 \\
& \frac{\partial \xi_{2}}{\partial A_{1}} A_{1}+\frac{\partial \xi_{2}}{\partial A_{2}} A_{2}+\frac{\partial \xi_{2}}{\partial A_{3}} A_{3}+\frac{\partial \xi_{2}}{\partial A_{4}} A_{4}=0
\end{align*}
$$

$$
\begin{align*}
& \frac{\partial \xi_{3}}{\partial A_{1}} A_{1}+\frac{\partial \xi_{3}}{\partial A_{2}} A_{2}+\frac{\partial \xi_{3}}{\partial A_{3}} A_{3}+\frac{\partial \xi_{3}}{\partial A_{4}} A_{4}=\beta\left(\xi_{4}-\xi_{3}\right)^{2} \\
& \frac{\partial \xi_{4}}{\partial A_{1}} A_{1}+\frac{\partial \xi_{4}}{\partial A_{2}} A_{2}+\frac{\partial \xi_{4}}{\partial A_{3}} A_{3}+\frac{\partial \xi_{4}}{\partial A_{4}} A_{4}=-\delta\left(\xi_{4}-\xi_{3}\right)^{2}
\end{align*}
$$

The solutions of (25), after some rather lengthy trigonometric manipulations, are
$A_{1}=\frac{\delta+b_{2} \beta}{\omega_{1}\left(b_{2}-b_{1}\right)}\left(\xi_{4}-\xi_{3}\right)^{2} \cos \omega_{1} l_{1} \quad \quad A_{2}=-\frac{\delta+b_{2} \beta}{\omega_{1}\left(b_{2}-b_{1}\right)}\left(\xi_{4}-\xi_{3}\right)^{2} \sin \omega_{1} t_{1}$,
$A_{3}=-\frac{\delta+b_{1} \beta}{\omega_{2}\left(b_{2}-b_{1}\right)}\left(\xi_{4}-\xi_{3}\right)^{2} \cos \omega_{2} t, A_{4}=\frac{\delta+b_{1} \beta}{\omega_{2}\left(b_{2}-b_{1}\right)}\left(\xi_{4}-\xi_{3}\right)^{2} \sin \omega_{2} t$.
The solution of (26) is obtained with sufficient approximation by using a device common in celestial mechanics; i.e., for small values of the time, the $A_{i}$ entering (26) through $\xi_{3}$ and $\xi_{4}$ may be considered constants having the values obtained by the solution of (24) for $\xi_{1}=-a_{1}, \xi_{2}=-a_{2}, \xi_{3}=\xi_{4}=v_{0}$ at $t=0$. Thus the solution of (26), to the accuracy required, is reduced to quadratures. Moreover, since the interval for which this solution is valid is small $(0 \leqq t \leqq 0.01)$ the trigonometric functions involved may be expanded as power series in $t$ before the quadratures are performed. The solution of (25) is

$$
\begin{equation*}
A_{i}=C_{i}+f_{i}(t) \quad(i=1, \cdots, 4) \tag{27}
\end{equation*}
$$

where $f_{i}(0)=0$. The substitution of (27) in (24) gives the complete solution for $0 \leqq t \leqq t_{1}$, where $\omega_{2} t_{1}<\frac{1}{2}$ and $\omega_{2}>\omega_{1}$. The values of $C_{i}=A_{i}$ as determined above.

The value of $\left[\xi_{4}\left(t_{1}\right)-\xi_{3}\left(t_{1}\right)\right]^{2}$ locates the point $Q_{1}$ in Fig. 10. The ordinate of $Q_{2}$ is $v_{0}^{2}$. The ordinate of $R_{1}$ is given by $s_{2}\left(t_{1}\right)-s_{1}\left(l_{1}\right)$. The ordinate of $R_{4}$ is the value of $(1-\xi)^{-1.2}$ when the air chamber is decreased to 0.7 of its initial value.


Fig. 11.


Fig. 12.*

[^20]In the solution for the second interval ( $t_{1} \leqq t<t_{2}$ ) of motion it is sufficient to replace the arcs $R_{1} R_{2} R_{3} R_{4}$ and $Q_{1} Q_{2}$ by the secant lines $R_{1} R_{4}$ and $Q_{1} Q_{2}$. The quantity ( $\left.s_{2}-s_{1}\right)^{2}$ on the interval $Q_{1} Q_{2}$ may be written

$$
\begin{equation*}
\left(s_{2}-s_{1}\right)^{2}=-B_{0}+n_{0}\left(s_{2}-s_{1}\right), \tag{28}
\end{equation*}
$$

and (22) becomes

$$
\begin{gather*}
\left(p^{2}+\alpha m_{0}+k_{0}\right) s_{1} \quad-\alpha m_{0} s_{2}=g+\alpha A_{0}-\beta B_{0}+\beta n_{0}\left(s_{2}-s_{1}\right)-K_{0} \\
-\gamma m_{0} s_{1}+\left(p^{2}+\gamma m_{0}\right) s_{2}=n g-\gamma A_{0}+\delta B_{0}-\delta n_{0}\left(s_{2}-s_{1}\right), \tag{29}
\end{gather*}
$$

where any additional constants are shown in the figures. At the new origin of time for (29), $s_{1}(0)=s_{2}(0)=0$ and $s_{1}(0)=s_{1}\left(t_{1}\right)=v_{1}, s_{2}(0)=s_{2}\left(l_{1}\right)=v_{2}$.

The proper determination of the constants $b_{1}$ and $b_{2}$ in the substitution $s_{1}=\xi_{1}+b_{1}$, $s_{2}=\xi_{2}+b_{2}$ in (29) yields

$$
\begin{align*}
\left(p^{2}+\beta n_{0} p+\alpha m_{0}+k_{0}\right) \xi_{1}-\left(\beta n_{0} p+\alpha m_{0}\right) \xi_{2}=0  \tag{30}\\
-\left(\delta n_{0} p+\gamma m_{0}\right) \xi_{1}+\left(p^{2}+\delta n_{0} p+\gamma m_{0}\right) \xi_{2}=0 .
\end{align*}
$$

While the characteristic equation of $(30)$ is of the fourth degree, yet its roots are widely separated in practical cases and quickly found by Graeffe's method.

The values of $s_{1}=\xi_{1}+b_{1}, s_{2}=\xi_{2}+b_{2}$ as given by the solution of (30) do not yield the equilibrium positions of $m_{1}$ and $m_{2}$, because when $\left(s_{2}-s_{1}\right)^{2}$ becomes small the relation (28) and Eqs. (29) are no longer valid. This is no defect of the solution because its purpose is the determination of the maximum accelerations acting on $m_{1}$ and $m_{2}$. These maxima occur in the interval $0 \leqq t \leqq t_{2}$. The equilibrium positions of $m_{1}$ and $m_{2}$ are determined from static considerations.

A point of special interest is the determination of the effects of the factor $n g$ upon the solution. The above solution is constructed with this in mind.

The roots of the characteristic equation of (30) have special physical significance. In practical cases these are usually one or two pairs of complex roots. If there are four complex roots, one pair gives a high frequency oscillation of moderate magnitude for $m_{1}$. This is to be avoided.

If $[A(r)]^{2}$ is given by the graph shown in Fig. $9 b$ the above method is still applicable. The solution is very sensitive with respect to $[A(r)] .{ }^{2}$ Of course, the intervals of solution will exceed two in number, but in each interval the value of $\left(s_{2}-s_{1}\right)^{2}$ will be given by the ordinates of the arc $Q_{0} Q_{1}$ or the secant $Q_{1} Q_{2}$.

The most complicated process involved in solving (20) is the solution of a quartic, equation.
4. Concluding remarks. The seven problems presented above are representative of the nonlinear discrete problems of industry in so far as one nonlinear problem can represent a group the members of which differ greatly. No bibliography is given for the reason stated in footnote 7.

Methods of handling industrial nonlinear problems of continuous systems arising in industry are reserved for a subsequent paper.

## -NOTES-

## A GEOMETRICAL INTERPRETATION OF THE RELAXATION METHOD*

By J. L. SYNGE (The Ohio State University)

Let $a_{i j}, B_{i}(i, j=1,2, \cdots, n)$ be given constants such that $a_{i j}=a_{j i}$ and $\sum_{i, j=1}^{n} a_{i j} x_{i} x_{j}$ is a positive definite form. Consider the equations

$$
\begin{equation*}
\sum_{i=1}^{n} a_{i j} x_{i}-B_{i}=0 \quad(i=1,2, \cdots, n) \tag{1}
\end{equation*}
$$

The solution is easily expressed as a set of quotients of determinants. However, as $n$ increases, the task of calculating the determinants becomes excessively burdensome. The relaxation method ${ }^{1}$ provides a set of easy steps by which the solution of (1) is approached. The method has been compactly described by Temple. ${ }^{2}$

The purpose of the present note is to give a geometrical description of the relaxation method. For the trivial case $n=2$ the geometrical description may be displayed accurately in a diagram. For $n=3$ a model may be visualized. For $n>3$ we pass beyond the region of simple concrete geometrical representation, but in many ways geometry in an $n$-space is closely analogous to geometry in 2 -space or 3 -space, and the geometrical description continues to serve as a general guide to procedure.

Let us regard $x_{i}$ as rectangular Cartesian coordinates in a Euclidean $n$-space. Let us define

$$
\begin{equation*}
H(x)=\frac{1}{2} \sum_{i, j=1}^{n} a_{i j} x_{i} x_{i}-\sum_{i=1}^{n} B_{i} x_{i} . \tag{2}
\end{equation*}
$$

The equation $H(x)=$ const. represents a family of ellipsoids $E$; these ellipsoids have a common center, common directions for their principal axes, and common values for the ratios of their principal axes. They form, in fact, a family of similar and similarly situated ellipsoids.

The equations (1) represent a set of planes (i.e., flats of $n-1$ dimensions). The point of intersection of these planes is the common center $G$ of $E$. Thus the problem of solving (1) is the problem of finding the center of an ellipsoid when its equation is given.

It is important to note that $H(x)$ takes a minimum value at $G . H$ is constant over each ellipsoid, and increases steadily as we pass out from $G$.

It is not possible to define precisely what procedures are to be regarded as permissible. It is a question of ease of computation. Let us follow Southwell and consider an approach to $G$ by steps each of which is parallel to one of the axes of coordinates $x_{i}$.

Fig. 1 shows Southwell's procedure. It is a schematic diagram in which the ellip-

[^21]soids are represented by circles. (The ellipsoids can of course be transformed into concentric spheres by a linear transformation, which however destroys the orthogonality of perpendicular lines.) We start with an arbitrary point $P_{0}$ (the zero approximation). Let $E_{0}$ be the ellipsoid which passes through $P_{0}$. Through $P_{0}$ we draw a straight line $L$ parallel to one of the coordinate axes. Let $Q_{1}$ be the second point in which $L$ cuts $E_{0}$. Let $P_{1}$ be the middle point of the chord $P_{0} Q_{1}$. Then $P_{1}$ is the first approximation.

Since the ellipsoid is a convex surface, $P_{1}$ lies inside $E_{0}$ and so $H\left(P_{1}\right)<H\left(P_{0}\right)$. Moreover it is easy to sce that $P_{0} Q_{1}$ is tangent at $P_{1}$ to the ellipsoid $E_{1}$ which passes through $P_{1}$. Thus, of all points on the chord $P_{0} Q_{1}$, the point $P_{1}$ gives the smallest value of $H$.

The process is repeated, starting from $P_{1}$. The second approximation $P_{2}$ is the middle point of a chord $P_{1} Q_{2}$ of $E_{1}$, drawn parallel to another of the coordinate axes. In this way we get a sequence of points $P_{0}, P_{1}, \cdots$. The success of the method depends on the rapidity of the convergence of this sequence to $G$.

In one important respect the above


Fig. 1. procedure is incompletely defined. When we have reached $P_{m}$, in which of the directions defined by the coordinate axes are we to proceed in order to get $P_{m+1}$ ? There are $n$ coordinate axes. Of these one cannot be used, viz., that which gave the direction of the step $P_{m-1} P_{m}$. But, of the remaining $n-1$ directions, which should we use?

Gaskell ${ }^{3}$ has suggested the following plan. Write

$$
\begin{equation*}
C_{i}(x)=\sum_{j=1}^{n} a_{i j} x_{i}-B_{i} \tag{3}
\end{equation*}
$$

Having reached the point $P_{m}$, we calculate the quantities $C_{i}\left(P_{m}\right)$. Let $C_{k}\left(P_{m}\right)$ be the greatest of these in absolute value. Then we choose for the step $P_{m} P_{m+1}$ the direction of the axis of $x_{k}$.

This procedure is called the liquidation of the greatest error, since we obtain $C_{k}\left(P_{m+1}\right)=0$. It is interesting to see how this result fits into the geometrical discussion. The plane $C_{k}(x)=0$ is the plane through $G$ conjugate to the direction of the axis $x_{k}$. The line $P_{m} P_{m+1}$ is parallel to this axis and tangent at $P_{m+1}$ to one of the ellipsoids, $E_{m+1}$. But the point of contact of a line with an ellipsoid lies on the central plane conjugate to the direction of the line. Hence $P_{m+1}$ lies on $C_{k}(x)=0$, i.e., $C_{k}\left(P_{m+1}\right)=0$.

[^22]But it may well be asked whether the quantities $C_{i}$ themselves possess any deep significance. It is true that $G$ satisfies $C_{i}(x)=0$, but the quantity $C_{i}(x)$ for a general point does not represent the perpendicular distance of that point from the plane $C_{i}(x)=0$. This perpendicular distance is

$$
\begin{equation*}
f^{\prime}(x)=\frac{\left|C_{i}(x)\right|}{\left(\sum_{i=1}^{n} a_{i j}^{2}\right)^{1 / 2}} \tag{4}
\end{equation*}
$$

Should we not liquidate the greatest $p_{i}$ rather than the greatest $C_{i}$ ? Or is there a better plan than either?

The following plan is suggested. Having reached the point $P_{m}$, we have an option on $n-1$ next points. Each of these points lies on an ellipsoid of the family E. Choose that point which lies on the innermost ellipsoid. This is equivalent to saying: Choose that point which gives the smallest value to $H$.

Now ${ }^{4}$ for a step in the direction of the axis $x_{i}$ the decrease in $H$ is $\frac{1}{2} C_{i}^{2} / a_{i i}$. This is to be made as great as possible, and so we should pick the direction of the step $P_{m} P_{m+1}$ according to the following rule: Proceed in the direction of the axis of $x_{k}$ where $C_{k}^{2} / a_{k k}$ is the greatest of the quantities $C_{i}^{2} / a_{i i}(i=1,2, \cdots, n)$.

Thus $C_{i}^{2} / a_{i i}$ is made the criterion rather than Gaskell's $C_{i}$. The calculation of the former quantities involves slightly more computation, but this may be taken care of by making the initial transformation

$$
\begin{equation*}
x_{i}^{\prime}=\left(a_{i i}\right)^{1 / 2} x_{i} . \tag{5}
\end{equation*}
$$

Then

$$
\begin{equation*}
H=\frac{1}{2} \sum_{i, j=1}^{n} a_{i j}^{\prime} x_{i}^{\prime} x_{i}^{\prime}-\sum_{i=1}^{n} B_{i}^{\prime} x_{i}^{\prime} \tag{6}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{i i}^{\prime}=1, \quad a_{i j}^{\prime}=a_{i i} /\left(a_{i i} a_{i j}\right)^{1 / 2}, \quad B_{i}^{\prime}=B_{i} /\left(a_{i i}\right)^{1 / 2} \tag{7}
\end{equation*}
$$

Now, with

$$
\begin{equation*}
C_{i}^{\prime}\left(x^{\prime}\right)=\sum_{j=1}^{n} a_{i j}^{\prime} x_{j}^{\prime}-B_{i,}^{\prime} \tag{8}
\end{equation*}
$$

the criterion for the direction of the next displacement is $C_{i}^{\prime 2}$ or $\left|C_{i}^{\prime}\right|$, the same as Gaskell's. Moreover the transformation from $P_{m}$ to $P_{m+1}$ is now extremely simple. It is ${ }^{5}$

$$
\begin{array}{ll}
P_{m}: & x_{1}^{\prime}, \cdots, x_{k}^{\prime}, \cdots, x_{n}^{\prime} \\
P_{m+1}: & x_{1}^{\prime}, \cdots, x_{k}^{\prime}-C_{k}^{\prime}\left(x^{\prime}\right), \cdots, x_{n}^{\prime}
\end{array}
$$

[^23]
## PROPOSED SYMBOLS FOR THE MODIFIED COSINE AND EXPONENTIAL INTEGRALS

## By S. A. SCHELKUNOFF (Bell Telephone Laboratories)

The standard sine and cosine integrals are defined as follows

$$
\text { Si } x=\int_{0}^{x} \frac{\sin t}{t} d t, \quad \text { Ci } x=\int_{\infty}^{x} \frac{\cos t}{t} d t .
$$

The cosine integral has a logarithmic singularity at $x=0$. Now in problems of electromagnetic radiation $x$ is proportional to the frequency but the impedance functions involving $\mathrm{Ci} x$ are free from logarithmic singularities at $f=0$. Thus one expects and actually encounters logarithmic functions which cancel the singular parts of the cosine integrals.

For this reason the more suitable function is the following modified cosine integral

$$
\operatorname{Cin} x=\int_{0}^{x} \frac{1-\cos t}{t} d t
$$

which is an entire function. This function has already been used quite frequently, and we wish only to suggest that a standard notation be adopted for it.

Inasmuch as one is frequently interested in the analytic properties of impedance functions over the entire oscillation constant plane, the following modified exponential integral is suggested

$$
\operatorname{Ein} z=\int_{0}^{z} \frac{1-e^{-w}}{w} d w
$$

The independent variable $z$ will be proportional to $p=\xi+i \omega$ where $\omega=2 \pi$ times the frequency. Then, on the imaginary axis we have

$$
\operatorname{Ein}(i y)=\operatorname{Cin} y+i \operatorname{Si} y
$$

where $y$ is proportional to the frequency.
The even part of Ein $z$ may be designated as Cinh $z$ and the odd part Sih $z$.

## BOOK REVIEWS

## The mathematics of physics and chemistry. By Henry Margenau and George Moseley Murphy. D. Van Nostrand Company, Inc. New York, 1943. xii +581 pp. $\$ 6.50$.

Contents: 1. The mathematics of thermodynamics. 2. Ordinary differential equations. 3. Special functions. 4. Vector analysis. 5. Vectors and curvilinear coordinates. 6. Calculus of variations. 7. Partial differential equations of classical physics. 8. Eigenvalues and eigenfunctions. 9. Mechanics of molecules. 10. Matrices and matrix algebra. 11. Quantum mechanics. 12. Statistical mechanics. 13. Numerical calculations. 14. Linear integral equations. 15. Group theory.

The need for comprehensive manuals of mathematical tools is widely felt by workers in various applied fields. The readers of this Quarterly may, therefore, envy the theoretical physicists and chemists for whom the present book is primarily intended. However, it appears from the above table of contents that the book covers such a great variety of topics that almost everyone will find some chapter of particular interest. In this connection the chapters on special functions and special coordinate systems deserve particular mention.

The authors have well succeceded in making the book appear as a homogeneous unit although the individual chapters are independent and show a refreshing lack of formal uniformity. In some chapters physical theories are treated at very considerable length, while other chapters are quite mathematical in form. Formal deductions are given in general, but often it seemed more desirable merely to record formulas or facts. "The degree of difficulty of the treatment is such that a Senior majoring in physics or chemistry would be able to read most parts of the book with understanding."

Occasionally, a more daring departure from customary lines would have made the book still more useful. Thus some numerical methods which are often presented and hardly ever used would better have been omitted in favor of a more thorough presentation of the really useful techniques. The modern statistician will regret to find the theory of errors treated along conventional, obsolete lines. The magic spell of purely conventional but impressive terms such as "probable error" has proved very dangerous indeed and inspires an unjustified confidence. The physicist who still believes in the normalcy of observational errors should consult W. A. Shewhart's "Statistical Method From the Viewpoint of Quality Control" (Washington 1939). There, starting on p. 66, he will find a most interesting analysis of some measurements among the very clite (velocity of light, the gravitational constant, Planck's constant). They all show complete lack of statistical control, and even the simplest methods of industrial quality control could be used for an improvement.

In general, the presentation is very clear. Only occasionally an attempt at mathematical sophistication makes itself felt. Thus the authors first introduce vectors in the usual (most satisfactory) manner. Then (pp. 134-135), rather unclear references are made to a more restrictive analytical definition. The passage culminates in the puzzling statement that $[y, x]$ (which, by the way, is the gradient of the function $x y$ ) "does not define a vector." It does. And the authors themselves make free use of gradients and, on the other hand, they (p.135) "do assume that all of the vectors discussed are proper vectors."
W. Feller

## Navigational trigonometry. By P. R. Rider and Ch. A. Hutchinson. The Macmillan Company. New York, 1943. ix +232 pp. $\$ 2.00$.

The revicwer has considered this book more from the standpoint of a person studying the principles underlying the art of navigation, either for the first time or as a refresher, than as a mathematical textbook.

The book, as the authors say, is "a revision and expansion of part of Rider's Plane and Spherical Trigonometry." The general arrangement of the material is very good, both as to the sequence of topics taken up by chapters and the presentation of the material in each chapter itself. Chapter by chapter it leads the student from fundamental definitions through the solutions of right spherical triangles and oblique spherical triangles which are necessary for the student to know if he is to thoroughly understand his navigation. Admitting that one can learn to navigate and use the short cuts common to practical navigation without a very thorough background of spherical trigonometry, nevertheless the more complete his knowledge of this branch of mathematics, the better navigator he will be and the more he will
enjoy working out navigational problems. This phase has, in the reviewer's opinion, been very well handled by the authors, who have shown good judgment in maintaining the proper balance between the amount of detail used in "proofs" and the confidence shown in the intelligence of the student in assuming that he will either accept certain facts or will be able to complete the detailed proofs himself.

The chapters on The Terrestrial Sphere, Charts, The Sailings, Astronomical Triangle and Lines of Position are presented in clear, concise English and in logical order, giving the student the information necessary for him to understand the problems which will confront him later when he takes up navigation as a working tool. The authors very sensibly do not attempt to include in these chapters everything that a man must know in order to actually navigate, but leave that to other books written especially for this purpose.

Throughout the book the method of presentation of material is excellent. Each chapter contains certain proofs and facts followed by problems or exercises based on preceding information, giving the student an opportunity to apply the principles discussed. The fact that the answers to certain problems are given in the back of the book gives the student the chance to know whether or not he has used the proper method of solution, and also the satisfaction of knowing that he has successfully accomplished his task.

The inclusion of an appendix discussing briefly the standards of accuracy is, in the reviewer's opinion, very well worth while. This subject, often neglected, is not well understood by students who have had little experience in mathematics, and is all too often not recognized even by those who have had such experience.

The problems throughout the book are well thought out and the authors have given careful study to the matter, laying special emphasis on the authenticity of materials and assumptions so that the problems are as practical as possible in a text of this size.

The book contains a complete five place table of natural and logarithmic haversines with one minute intervals, which is a notable and welcome innovation in a textbook on trigonometry. It has been the reviewer's experience that the beginner finds it confusing to use the table in Bowditch with its variable interval. This table, along with the table of Common Logarithms of the Trigonometric Functions makes it necessary for the student to make less frequent use of Bowditch, which, because of its size, is rather awkward to manipulate.

All in all, the authors have accomplished what they set out to do. The book fulfils their claims even better than might be expected and should prove to be very popular in the teaching and studying of the basic mathematical problems underlying the principles of navigation.

Leighton T. Bohl
Table of circular and hyperbolic tangents and colangents for radian arguments. Prepared by the Mathematical Tables Project, Work Projects Administration of the Federal Works Agency; conducted under the sponsorship of the National Bureau of Standards. Official Sponsor: Lyman J. Briggs. Technical Director: Arnold N. Lowan. Columbia University Press. New York, 1943. xxxviii +410 pp. $\$ 5.00$.

The main table gives the values of $\tan x, \tanh x, \cot x$ and $\operatorname{coth} x$ over the range $x=0$ to $x=2$ at intervals of 0.0001 . Circular and hyperbolic tangents are given to 8 significant figures for $0<x \leqq 0.01$ and for $0.1 \leqq x \leqq 2$, and to 9 decimal places for $0.01 \leqq x \leqq 0.1$. Circular cotangents are given to 8 significant figures for $0.1 \leqq x \leqq 1.57$ and $1.575 \leqq x \leqq 2$, to 8 decimal places for $0<x \leqq 0.1$ and to 13 decimal places for $1.57 \leqq x \leqq 1.575$. Hyperbolic cotangents are given to 8 decimal places for $0 \leqq x \leqq 0.1$ and to 8 significant figures for $0.1 \leqq x \leqq 2$. The second central differences for all these functions are given wherever linear interpolation is not sufficient. Auxiliary tables contain the values of the circular and hyperbolic tangents and cotangents to 10 decimal places over the range $x=0$ to $x=10$ at intervals of 0.1 ; the values of the interpolation coefficients for the formulas of Gregory-Newton and of Everett; the values of $n \pi / 2$ for integer values of $n$ from 1 to 100 ; and values facilitating the conversion from radians to degrees and from degrees to radians.

## IV. Prager



## SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The Editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections,"

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Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.
Bibliography: References should be given as footnotes. Only in longer expository articles may references be grouped together in a bibliography at the end of the manuscript. The arrangement should be as follows: (for books)-author, title, volume, publisher, place of publication, year, page referred to; (for periodicals)-awthor, title, name of periodical, volume, page, year. All references should be complete and thoroughly checked.

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Deals with the application of the methods of plane and solid analytic geometry in the design, lofting, tooling, and engineering of airplanes.

Send for copies on approval


[^0]:    * Received Sept. 16, 1943.

[^1]:    ${ }^{1}$ Or almost across the total capacitance.

[^2]:    ${ }^{2}$ See for instance S. A. Schelkunoff, Electromagnetic waves, D. Van Nostrand Company, Inc., New York, 1943, p. 212.
    ${ }^{3}$ For example, see S . A. Schelkunoff, The impedance of a transverse wire in a rectangular wave guide, Quarterly of Applied Mathematics, 1, 78-85 (1943).
    ${ }^{4}$ S. A. Schelkunoff, Electromagnetic waves, p. 491.

[^3]:    - In the first approximation this capacitance may be neglected.
    ${ }^{3}$ We assume that we are operating below the second frequency cutoff; otherwise there will also be a conductance.
    ${ }^{8}$ J. R. Whinnery and H. W. Jamicson, Equivalent circtuits for discontinuities in transmission lines, I.R.E. Proc., February 1944, pp. 98-114.

[^4]:    * Received Sept. 23, 1943.

    1 The writer wishes to express his thanks to the following persons who have assisted him in the preparation of this paper: Professor M.S. Ketchum of Case School of Applied Science, who suggested the problem; Dr. W. N. Dudley, also of Case, for several helpful hints; and especially Dr. H. G. Baerwald whose assistance in many mathematical details has been invaluable.

[^5]:    *This treatment was indicated to the writer by his friend and colleague, Dr. H. G. Baerwald.

[^6]:    * Received Sept. 1, 1943.
    ${ }^{1}$ See, for example, A. E. Green, and G. I. Taylor, Stress distributions in aeolotropic plates, Proc. Roy. Soc. A 173, 163 (1939).

[^7]:    : A. E. H. Love, A treatise on the mathematical theory of elasticity, Cambridge, 1934, p. 125.
    ${ }^{2}$ The $b_{i j}, c_{i j}$ and $\beta_{i}$ are combinations of elastic and thermal constants arising from the above operation. The manner in which these constants appear in the second and third of these is easily deduced from equations (5).

[^8]:    ${ }^{4}$ J. N. Goodier, On the integration of the thermo-elastic equations, Phil. Mag. (7), 23, 1017 (1937).

[^9]:    * Received Sept. 29, 1943.

[^10]:    ${ }^{1}$ Cf. S. Timoshenko, Theory of Plates and Shells, McGraw-Hill, New York and London, 1940, chap. IX.

[^11]:    ${ }^{2}$ After some fruitless attempts to evaluate this integral, I asked Professor G. N. Watson whether it was expressible in any simple form; his reply was that he thought not, and he computed its value to 15 places of decimals, his result being 1.259909735905768 . The value 1.2599 is sufficiently accurate for our present purpose.

[^12]:    * Received June 12, 1943. Part I of this paper appeared in this Quarterly, 1, 297-327 (1944).

[^13]:    * Received Nov. 13, 1943.
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