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## QUARTERLY OF APPLIED MATHEMATICS

# LIFTING-LINE THEORY FOR A WING IN NON-UNIFORM FLOW* 

BY<br>THEODORE VON KARMAN and HSUE-SHEN TSIEN<br>California Institute of Technology

1. Introduction. Prandtl's theory of the lifting line gave the answer to most of the questions in the aerodynamic design of airplane wings. Thus the three-dimensional wing theory became a standard tool of airplane designers. One restriction involved in the conventional wing theory is the uniformity of the undisturbed flow in which the wing is placed. Now there are many important cases which do not satisfy this condition. For instance, in the case of a wing spanning an open jet wind tunnel, the velocity of the air stream has a maximum at the center of the jet and drops to zero outside of the jet. Another example is the problem of the influence of the propeller slip-stream on the characteristics of the wing. Here the higher velocity of the propeller slip-stream makes the application of the Prandtlowing theory difficult. Such cases led several authors to investigate the problem of a wing in non-uniform flow. Some investigators found a satisfactory solution of the problem for the case of "stepwise" velocity distribution. In this case the flow in regions of uniform velocity can be determined by using Prandtl's concepts with additional continuity conditions at the boundaries between such regions. On the other hand, the problem of a continuously varying velocity field seems to need an appropriate treatment. K. Bausch ${ }^{1}$ has tried to modify the Prandtl theory for the case of small inhomogeneity in the air stream; however, besides the restriction of slight deviation from uniform flow, his method encounters a further difficulty in estimating the error introduced by the approximations. The seriousness of this difficulty becomes evident when one tries to compare the results of Bausch with that of F. Vandrey. ${ }^{2}$ Vandrey considers the problem with variable velocity as the limiting case of a wing in a stepwise velocity field, and his result seens to differ from that of Bausch. Recently R. P. Isaacs ${ }^{3}$ has investigated the same problem, but the authors have not yet had the opportunity to study his work.

It seems to the authors that a general and more satisfactory solution for the flow of a wing in a non-uniform stream can be obtained by studying the threc-dimensional problem anew in this generalized case, introducing the modifications of Prandtl's fundamental concepts. The first fundamental concept is the following: the span of

[^0]the wing is sufficiently large compared with the chord so that the variation of the velocities in the spanwise direction is small when compared with the variation of the velocities in a plane normal to the span; then the flow at each sectional plane perpendicular to the span can be considered as a two-dimensional flow around an airfoil. The only additional feature for the flow in this sectional plane is the modification of the geometrical angle of attack, as defined by the undisturbed flow, on account of the so-called induced velocity. The second fundamental concept of Prandtl is the replacement of the wing by a lifting line having the same distribution of lifting forces along the span as the wing. This concept,


Fig. 1. Lifting line in a non-uniform flow. with the additional assumption that the disturbance caused by the lifting line is small, i.e., that the wing is lightly loaded, makes the calculation of the induced velocity relatively simple. In this paper the authors will study the flow around a lightly loaded lifting line placed in a parallel stream whose velocity is perpendicular to the span (Fig. 1) and is assumed to vary in both directions normal to the flow. Due to the rather complicated character of the flow, the usual concept of the picturesque system of trailing vortices encountered in Prandtl's wing theory is not very useful here. A method, which is mathematically more convenient, has to be adopted. This method has already been used by the senior author ${ }^{4}$ in explaining the similarity between Prandtl's wing theory and the theory of planning surfaces. After the general theory is formulated, the problem of minimum induced drag will be considered. Finally a general expression for calculating the induced drag of a wing in a stream of varying velocity will be presented.

Of course, the complete solution of the problem of a wing in a non-uniform stream requires a knowledge of the "section characteristic" or the two-dimensional properties of the airfoil sections of the wing. If the velocity of the main stream is varying only in the direction of the span, the required section characteristics are those of an airfoil in a two-dimensional uniform flow, and are common knowledge in applied aerodynamics. However, if the velocity of the main stream is also varying in a direction perpendicular to the span and to the velocity itself, the required section characteristics are those of an airfoil in a two-dimensional non-uniform flow. Such flow problems have not yet been studied extensively. ${ }^{5}$
2. General theory of a lifting line. Let the $x$-axis be parallel to the direction of the main flow, the $y$-axis coincide with the lifting line and the $z$-axis be normal to the

[^1]lifting line (Fig. 1). If $p$ is the pressure, $\rho$ the density, and $v_{1}, v_{2}, v_{3}$ the components of the velocity, the dynamical equations for the steady motion of an inviscid, incompressible fluid without external forces are
\[

$$
\begin{align*}
& v_{1} \frac{\partial v_{1}}{\partial x}+v_{2} \frac{\partial v_{1}}{\partial y}+v_{3} \frac{\partial v_{1}}{\partial z}=-\frac{1}{\rho} \frac{\partial p}{\partial x}  \tag{1}\\
& v_{1} \frac{\partial v_{2}}{\partial x}+v_{2} \frac{\partial v_{2}}{\partial y}+v_{3} \frac{\partial v_{2}}{\partial z}=-\frac{1}{\rho} \frac{\partial p}{\partial y},  \tag{2}\\
& v_{1} \frac{\partial v_{3}}{\partial x}+v_{2} \frac{\partial v_{3}}{\partial y}+v_{3} \frac{\partial v_{3}}{\partial z}=-\frac{1}{\rho} \frac{\partial p}{\partial z} . \tag{3}
\end{align*}
$$
\]

The equation of continuity is

$$
\begin{equation*}
\frac{\partial v_{1}}{\partial x}+\frac{\partial v_{2}}{\partial y}+\frac{\partial v_{3}}{\partial z}=0 \tag{4}
\end{equation*}
$$

Equations (1) to (4) constitute a system of four simultaneous equations for the four unknowns $v_{1}, v_{2}, v_{3}$ and $p$.

For the particular problem of a lightly loaded lifting line, the velocity components can be expressed in the following forms:

$$
\begin{equation*}
v_{1}=U+u, \quad(5) ; \quad v_{2}=v, \quad(6) ; \quad v_{3}=w \tag{5}
\end{equation*}
$$

Here $u, v, w$ are the velocity components due to the presence of the lifting line and $U$ is the main stream velocity assumed to be a function of $y$ and $z$ but independent of $x$. Since the lifting line is assumed to be lightly loaded, $u, v$ and $w$ are small compared with the main velocity $U$. By substituting Eqs. (5) to (7) into the dynamical equations and neglecting higher order terms, a set of linear equations for $u, v$ and $w$ is obtained. Thus

$$
\begin{array}{cc}
U \frac{\partial U}{\partial x}+v \frac{\partial U}{\partial y}+w \frac{\partial U}{\partial z}=-\frac{1}{\rho} \frac{\partial p}{\partial x} \\
U \frac{\partial v}{\partial x}=-\frac{1}{\rho} \frac{\partial p}{\partial y}, & (9) ;  \tag{10}\\
U \frac{\partial w}{\partial x}=-\frac{1}{\rho} \frac{\partial p}{\partial z}
\end{array}
$$

Then the equation of continuity becomes

$$
\begin{equation*}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{11}
\end{equation*}
$$

If Eqs. (8), (9) and (10) are differentiated with respect to $x, y$ and $z$ respectively and the results added, the sum can be simplified by using Eq. (11) and can, finally, be written in the form

$$
\begin{equation*}
\frac{1}{U^{2}} \frac{\partial^{2} p}{\partial x^{2}}+\frac{\partial}{\partial y}\left(\frac{1}{U^{2}} \frac{\partial p}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{U^{2}} \frac{\partial p}{\partial z}\right)=0 \tag{12}
\end{equation*}
$$

This is now an equation for the pressure $p$ only and can be used conveniently as the starting point of the solution. If the pressure of the undisturbed main flow is chosen
as the reference pressure and set equal to zero, one of the boundary conditions to be satisfied by $p$ is

$$
\begin{equation*}
p=0, \text { for }|x| \rightarrow \infty, \quad|y| \rightarrow \infty, \quad \text { or } \quad|z| \rightarrow \infty . \tag{13}
\end{equation*}
$$

The condition at the lifting line, or $y$-axis, is that the lifting force is represented by a suction force on the "upper surface" of the lifting line and a pressure force of equal magnitude on the "lower surface" (Fig. 2). Hence the pressure $p$ must satisfy the following expressions


Fig. 2. Representation of lift as pressure forces acting on the two "surfaces" of the lifting line.

$$
\begin{align*}
& \qquad \int_{-e}^{e} p d x=-\frac{1}{2} l(y), \text { for } z=+0,  \tag{14}\\
& \text { and } \\
& \qquad \int_{-e}^{e} p d x=\frac{1}{2} l(y), \quad \text { for } z=-0, \tag{15}
\end{align*}
$$

where $l(y)$ is the lift per unit length of the lifting line at the point $y$. Furthermore, on account of the symmetry of the flow,

$$
\begin{equation*}
p=0 \quad \text { for } z=0, \quad|x|>\epsilon \tag{16}
\end{equation*}
$$

To solve Eq. (12) together with the boundary conditions given by Eqs. (13) to (16), the Fourier integral theorem can be used to build up the solution of the problem from the elementary solutions of Eq. (12) of the form

$$
P(y, z, \lambda) \cos \lambda x .
$$

The equation to be satisfied by $P$ is

$$
\begin{equation*}
U^{2} \frac{\partial}{\partial y}\left(\frac{1}{U^{2}} \frac{\partial P}{\partial y}\right)+U^{2} \frac{\partial}{\partial z}\left(\frac{1}{U^{2}} \frac{\partial P}{\partial z}\right)-\lambda^{2} P=0 . \tag{17}
\end{equation*}
$$

To determine $P$ uniquely, it is convenient to impose the following conditions

$$
\begin{align*}
& P=0, \text { for }|y| \rightarrow \infty,|z| \rightarrow \infty,  \tag{18}\\
& P=-\frac{1}{2} l(y) \text { for } z=+0,  \tag{19}\\
& P=\frac{1}{2} l(y) \text { for } z=-0 . \tag{20}
\end{align*}
$$

The required solution for $p$ can then be written as

$$
\begin{equation*}
p=\frac{1}{\pi} \int_{0}^{\infty} \cos \lambda x P(y, z, \lambda) d \lambda . \tag{21}
\end{equation*}
$$

By substituting Eq. (21) into Eqs. (9) and (10), the "induced velocities" $v$ and $w$ are obtained;

$$
\begin{equation*}
v(x, y, z)=v(0, y, z)-\frac{1}{\rho U} \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin \lambda x}{\lambda} \frac{\partial}{\partial y} P(y, z, \lambda) d \lambda, \tag{22}
\end{equation*}
$$

$$
\begin{equation*}
w(x, y, z)=w(0, y, z)-\frac{1}{\rho U} \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin \lambda x}{\lambda} \frac{\partial}{\partial z} P(y, z, \lambda) d \lambda . \tag{23}
\end{equation*}
$$

Because the integrals are odd functions of $x$, the following relations hold for velocities far ahead of the lifting line and far behind the lifting line:

$$
\frac{1}{2}[v(-\infty, y, z)+v(\infty, y, z)]=v(0, y, z), \frac{1}{2}[w(-\infty, y, z)+w(\infty, y, z)]=w(0, y, z) .
$$

However, it is evident that the induced velocities far ahead of the lifting lines must be zero. Hence

$$
v(0, y, z)=\frac{1}{2} v(\infty, y, z), \quad(24) ; \quad w(0, y, z)=\frac{1}{2} w(\infty, y, z)
$$

The induced velocities $v$ and $w$ at the lifting line are then one-half of those far downstream. This is in accordance with the usual wing theory based upon the concept of trailing vortices.

One meets an apparent difficulty if the $x$ component of the induced velocity is calculated; integration of Eq. (8) with respect to $x$ furnishes the $x$-component of the induced velocity:

$$
\begin{equation*}
u=-\frac{1}{\rho U} p-\frac{1}{U} \frac{\partial U}{\partial y} \int_{-\infty}^{x} v d x-\frac{1}{U} \frac{\partial U}{\partial z} \int_{-\infty}^{x} w d x . \tag{26}
\end{equation*}
$$

Since $p$ tends to zero, $v$ and $w$ tend to finite quantities as $x$ tends to infinity, and $u$ increases indefinitely as $x$ tends to infinity. This is in contradiction to the assumption of small disturbances introduced at the beginning of the present investigation. However, it is believed that this difficulty does not prevent the application of the theory to practical cases, since the apparent large value of the $u$ component is due to the distortion of the variable main stream by the induced cross flow and the infinite value for $x \rightarrow \infty$ is due to the linearization of the differential equations. Some further remarks on this point are given in Section 4.
3. Conditions far downstream. For the application of the lifting-line theory to the wing problem, the quantity of primary interest is the $z$ component of the induced velocity at the lifting line. The simple relations given by Eqs. (24) and (25) suggest a possible simplification of the calculation by considering conditions far downstream, or the "Trefftz plane" according to the terminology of the conventional wing theory. To abbreviate the notation, we let

$$
\left.\begin{array}{rl}
v_{0}=v(0, y, z), & w_{0}=w(0, y, z),  \tag{27}\\
v_{1}=v(\infty, y, z), & w_{1}=w(\infty, y, z) .
\end{array}\right\}
$$

Then, according to (24) and (25), $v_{0}=\frac{1}{2} v_{1}, w_{0}=\frac{1}{2} w_{1}$. Therefore, Eqs. (22) and (23) give

$$
\begin{aligned}
& v_{1}=-\frac{1}{\rho U} \lim _{x \rightarrow \infty} \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin \lambda x}{\lambda} \frac{\partial}{\partial y} P(y, z, \lambda) d \lambda \\
& w_{1}=-\frac{1}{\rho U} \lim _{x \rightarrow \infty} \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \lambda x}{\lambda} \frac{\partial}{\partial z} P(y, z, \lambda) d \lambda
\end{aligned}
$$

Let us consider $P(y, z, \lambda)$ as a regular function of $\lambda$; then

$$
P(y, z, \lambda)=P(y, z, 0)+\lambda\left[\frac{\partial P}{\partial \lambda}\right]_{\lambda=0}+\cdots .
$$

By using the variable $t=\lambda x$, the expressions for $v_{1}$ and $w_{1}$ can be rewritten,

$$
\begin{aligned}
& v_{1}=-\frac{1}{\rho U} \lim _{z \rightarrow \infty} \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin t}{t} \frac{\partial}{\partial y}\left[P(y, z, 0)+\frac{t}{x}\left(\frac{\partial P}{\partial \lambda}\right)_{\lambda=0}+\cdots\right] d t, \\
& w_{1}=-\frac{1}{\rho U} \lim _{x \rightarrow \infty} \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin t}{t} \frac{\partial}{\partial z}\left[P(y, z, 0)+\frac{t}{x}\left(\frac{\partial P}{\partial \lambda}\right)_{\lambda=0}+\cdots\right] d t .
\end{aligned}
$$

At the limit, only the first terms of the integrands are significant, and furthermore

$$
\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin t}{t} d t=1
$$

Hence

$$
\begin{equation*}
v_{1}=-\frac{1}{\rho U} \frac{\partial}{\partial y} P(y, z, 0), \quad(28) ; \quad w_{1}=-\frac{1}{\rho U} \frac{\partial}{\partial z} P(y, z, 0) . \tag{29}
\end{equation*}
$$

Equations (28) and (29) simplify the problem of calculating the induced velocities at the Trefftz plane considerably. In fact, by introducing a "potential function" $\phi$ defined by the relation

$$
\begin{equation*}
\phi(y, z)=-P(y, z, 0), \tag{30}
\end{equation*}
$$

the problem can be formulated as follows: the differential equation to be satisfied by $\phi$ can be deduced from Eq. (17) by setting $\lambda=0$; thus

$$
\begin{equation*}
\frac{\partial}{\partial y}\left(\frac{1}{U^{2}} \frac{\partial \phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{U^{2}} \frac{\partial \phi}{\partial z}\right)=0 \tag{31}
\end{equation*}
$$

The boundary conditions to be satisfied by $\phi$ are

$$
\begin{align*}
& \phi=0 \text { for }|y| \rightarrow \infty,|z| \rightarrow \infty,  \tag{32}\\
& \phi=l(y) / 2 \text { for } z=+0,  \tag{33}\\
& \phi=-l(y) / 2 \text { for } z=-0 . \tag{34}
\end{align*}
$$

Then

$$
\begin{equation*}
v_{1}=\frac{1}{\rho U} \frac{\partial \phi}{\partial y}, \quad(35) ; \quad \quad w_{1}=\frac{1}{\rho U} \frac{\partial \phi}{\partial z} \tag{35}
\end{equation*}
$$

By substituting Eqs. (35) and (36) into Eq. (31), one has

$$
\begin{equation*}
\frac{\partial}{\partial y}\left(\frac{v_{1}}{U}\right)+\frac{\partial}{\partial z}\left(\frac{w_{1}}{U}\right)=0 \tag{37}
\end{equation*}
$$

This equation has a very simple physical meaning. Since $v_{1}$ and $w_{1}$ are considered to be small quantities, the ratios $v_{1} / U$ and $w_{1} / U$ are the angles of inclination, $\beta$ and $\gamma$, of the stream lines with respect to the $z x$ and $x y$ planes. Consider parallel planes perpendicular to the $x$-axis and $d x$ apart (Fig. 3). If the width of the stream tube at the section $x$ is $\delta_{y}$, then at the section $x+d x$, the width of the stream tube is
$\delta_{y}[1+d x \partial \beta / \partial y]$. If the height of the stream tube at the section $x$ is $\delta_{z}$, then at the section $x+d x$, the height of the stream tube is $\delta_{z}[1+d x \partial \gamma / \partial z]$. The total increase in the cross-sectional area of the stream tube from $x$ to $x+d x$ is then approximately


Fig. 3. Stream tube far downstream from the lifting line.

$$
\delta_{y} \delta_{z}\left(\frac{\partial \beta}{\partial y}+\frac{\partial \gamma}{\partial z}\right) d x \text {. }
$$

Now at the Trefftz plane, the flow field can be considered as settled into a uniform condition; i.e., the pressure is constant in the $x$-direction. Hence, the velocity of the flow along any stream tube is constant. Then the cross-sectional area of the stream tube must be also constant. Therefore,

$$
\frac{\partial \beta}{\partial y}+\frac{\partial \gamma}{\partial z}=0,
$$

which is simply Eq. (37). From this point of view, Eq. (37) is really the equation of continuity, simplified under the conditions prevailing at the Trefftz plane.

On the other hand, $\phi$ can be eliminated from Eqs. (35) and (36). The result is

$$
\begin{equation*}
\frac{\partial}{\partial z}\left(U v_{1}\right)-\frac{\partial}{\partial y}\left(U w_{1}\right)=0 \tag{38}
\end{equation*}
$$

This equation can be considered as the modified vorticity equation. It actually holds for all values of $x$ under the approximation assumed in the present investigation. This can be seen in the following way: since $U$ is a function of $y$ and $z$ but independent of $x$, Eqs. (9) and (10) can be written in the form

$$
\frac{\partial}{\partial x} U v=-\frac{1}{\rho} \frac{\partial p}{\partial y} ; \quad \frac{\partial}{\partial x} U w=-\frac{1}{\rho} \frac{\partial p}{\partial z} .
$$

By differentiating the first equation with respect to $z$ and the second equation with respect to $y$ and then subtracting, the result is

$$
\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}(U v)-\frac{\partial}{\partial y}(U w)\right]=0
$$

Thus

$$
\frac{\partial}{\partial z}(U v)-\frac{\partial}{\partial y}(U w)=\text { a function of } y \text { and } z
$$

But for points far upstream, or for $x=-\infty, v$ and $w$ vanish; therefore the function of $y$ and $z$ on the right of above equation must be identically zero. Hence for all values of $x$,

$$
\begin{equation*}
\frac{\partial}{\partial z}(U v)-\frac{\partial}{\partial y}(U w)=0 \tag{39}
\end{equation*}
$$

It should be noted here that Eqs. (37), (38) and (39) are obtained without any reference to the lifting line and hence they are true for more general cases. However, the complete determination of $v_{1}$ and $w_{1}$ requires a knowledge of the relation between the induced velocities and the lift on the wing. This relation depends upon the type of lift distribution. For the particular case of a lifting line, this relation is supplied by Eqs. (33) and (34).

Equation (37) can be identically satisfied by introducing the "stream function" $\psi$ defined by

$$
\begin{equation*}
v_{1}=U \frac{\partial \psi}{\partial z}, \quad w_{1}=-U \frac{\partial \psi}{\partial y} \tag{40}
\end{equation*}
$$

Then Eq. (38) gives the differential equation for $\psi$ :

$$
\begin{equation*}
\frac{\partial}{\partial y}\left(U^{2} \frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial z}\left(U^{2} \frac{\partial \psi}{\partial z}\right)=0 \tag{41}
\end{equation*}
$$

Both Eq. (31) and Eq. (41) reduce to the Laplace equation for the conventional wing theory when $U$ is a constant.
4. Minimum induced drag. The induced downwash angle at the lifting line is equal to $w_{0} / U$ or $\frac{1}{2} w_{1} / U$, according to Eq. (25). Therefore, Eq. (36) gives the downwash angle at the lifting line as $\left[1 / 2 \rho U^{2}\right](\partial \phi / \partial z)_{z=0}$, and the induced drag $D_{i}$ can then be expressed as

$$
\begin{equation*}
D_{\mathbf{i}}=-\frac{1}{2 \rho} \int[\phi(y,+0)-\phi(y,-0)] \frac{1}{U^{2}}\left(\frac{\partial \phi}{\partial z}\right)_{z=0} d y=\frac{1}{2 \rho} \int_{C} \frac{\phi}{U^{2}} \frac{\partial \phi}{\partial z} d s \tag{42}
\end{equation*}
$$

The first integral is evaluated across the span of the lifting line. The second integral is calculated along a contour following the upper and lower "surface" of the horizontal strip shown in Fig. 4. Since $\phi \rightarrow 0$ for points far from the lifting line, the contour integral can be transformed into an area integral by Green's theorem, and

$$
\begin{equation*}
D_{i}=\frac{1}{2 \rho} \iint\left\{\frac{\partial}{\partial y}\left(\frac{1}{U^{2}} \phi \frac{\partial \phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{U^{2}} \phi \frac{\partial \phi}{\partial z}\right)\right\} d y d z \tag{43}
\end{equation*}
$$

This integral extends throughout the region outside of the lifting line. Since $\phi$ satisfies the differential equation (31), Eq. (43) reduces to

$$
\begin{equation*}
D_{i}=\frac{\rho}{2} \iint\left\{\left(\frac{1}{\rho U} \frac{\partial \phi}{\partial y}\right)^{2}+\left(\frac{1}{\rho U} \frac{\partial \phi}{\partial z}\right)^{2}\right\} d y d z . \tag{44}
\end{equation*}
$$

Therefore, the induced drag is represented by the kinetic energy corresponding to the velocity components $v_{1}$ and $w_{1}$ at the Trefftz plane. It is seen that the $u$ component of the velocity does not appear in the expression for the induced drag. This is due to the fact that the increase of $u$ with increasing $x$ does not represent a real acceleration


Fig. 4. Contour integration in the Trefftz plane.
of a fluid element in the $x$ direction. Rather, it is due to the fact that the cross flow transports fluid elements from regions of lower main velocity to regions of higher main velocity and vice versa. This is in accordance with the modified continuity equation (37) which clearly indicates that the cross section of the individual stream tubes has a definite limiting value for $x \rightarrow \infty$, and therefore the velocity component in the direction of the stream tube tends to a finite value.

The problem of minimum induced drag requires the determination of the minimum of $D_{i}$ as given by Eq. (44) together with the condition that the total lift $L$ remains fixed. Thus

$$
\begin{equation*}
L=\int l d y=\int[\phi(y,+0)-\phi(y,-0)] d y=-\int_{C} \phi d s=\text { constant } \tag{45}
\end{equation*}
$$

By using the method of Lagrange's multiplier, the above problem can be reduced to that of finding the minimum of $D_{i}+K / \rho L$, where $K$ is a constant. Hence,

$$
\begin{equation*}
\delta D_{i}+\frac{K}{\rho} \delta L=0 \tag{46}
\end{equation*}
$$

The variation of the induced drag can be obtained from Eq. (44),

$$
\delta D_{i}=\frac{1}{\rho} \iint\left\{\frac{1}{U} \frac{\partial \phi}{\partial y} \frac{1}{U} \frac{\partial \delta \phi}{\partial y}+\frac{1}{U} \frac{\partial \phi}{\partial z} \frac{1}{U} \frac{\partial \delta \phi}{\partial z}\right\} d y d z .
$$

However, $\phi$ must satisfy the differential equation (31) ; thus

$$
\delta D_{i}=\frac{1}{\rho} \iint\left\{\frac{\partial}{\partial y}\left(\frac{1}{U^{2}} \frac{\partial \phi}{\partial y} \delta \phi\right)+\frac{\partial}{\partial z}\left(\frac{1}{U^{2}} \frac{\partial \phi}{\partial z} \delta \phi\right)\right\} d y d z=\frac{1}{\rho} \int_{c} \frac{1}{U^{2}} \frac{\partial \phi}{\partial z} \delta \phi d s
$$

On the other hand,

$$
\delta L=-\int_{C} \delta \phi d s
$$

By substituting these results into Eq. (46), the condition of minimum induced drag is obtained in the form

$$
\begin{equation*}
\frac{1}{\rho} \int_{C}\left(\frac{1}{U^{2}} \frac{\partial \phi}{\partial z}-K\right) \delta \phi d s=0 \tag{47}
\end{equation*}
$$

The variation of $\delta \phi$ on the lifting line is arbitrary; therefore the minimum induced drag is given by the condition that the induced downwash angle must be constant along the span. If the main stream velocity $U$ is constant, the above condition is reduced to the requirement of constant downwash. This is in agreement with the well-known result of Prandtl's wing theory.
5. Flow with velocity varying in the direction of span only. If the stream velocity varies only in the $y$ direction, i.e., in the direction of the wing span, the calculation of induced velocity and induced drag can be simplified with the aid of characteristic functions connected with the differential equation for the potential function $\phi$. In this case Eq. (31) becomes

$$
\begin{equation*}
\frac{\partial^{2} \phi}{\partial y^{2}}+\frac{\partial^{2} \phi}{\partial z^{2}}-2 \frac{\frac{d U}{d y}}{U} \frac{\partial \phi}{\partial y}=0 \tag{48}
\end{equation*}
$$

To satisfy the boundary condition given by Eq. (32), $\phi$ is expressed by the following integral

$$
\begin{equation*}
\phi(y, z)=\int_{0}^{\infty} f(\lambda) e^{-\lambda z} Y_{\lambda}(y) d \lambda \tag{49}
\end{equation*}
$$

for $z>0 . f(\lambda)$ is an unknown function to be determined. For $z<0$,

$$
\begin{equation*}
\phi(y, z)=-\phi(y,-z) \tag{50}
\end{equation*}
$$

By substituting Eq. (49) into Eq. (48), the differential equation for $Y_{\lambda}(y)$ is obtained,

$$
\begin{equation*}
\frac{d^{2} Y_{\lambda}}{d y^{2}}-2 \frac{\frac{d U}{d y}}{U} \frac{d Y_{\lambda}}{d y}+\lambda^{2} Y_{\lambda}=0 \tag{51}
\end{equation*}
$$

This equation will determine $Y_{\lambda}(y)$ uniquely if proper normalizing and boundary conditions are imposed.

At the span, the condition (33) must be satisfied. Thus

$$
\begin{equation*}
\frac{l(y)}{2}=\int_{0}^{\infty} f(\lambda) Y_{\lambda}(y) d \lambda \tag{52}
\end{equation*}
$$

This relation can be considered as the equation for determining $f(\lambda)$ with the given lift distribution $l(y)$. For example, in the case of constant stream velocity $U$ or Prandtl's case, $Y_{\lambda}(y)$ is a trigonometric function and therefore $f(\lambda)$ can be deter-
mined easily by means of Fourier's inversion theorem. Equation (50) shows that with $f(\lambda)$ so determined, the condition (34) will be automatically satisfied.

The downwash velocity $w_{0}$ at the wing can then be easily calculated by using Eqs. (25), (36) and (49). The result is

$$
\begin{equation*}
w_{0}(y, 0)=-\frac{1}{2 \rho U} \int_{0}^{\infty} \lambda f(\lambda) Y_{\lambda}(y) d \lambda \tag{53}
\end{equation*}
$$

The induced drag $D_{i}$ is given by

$$
D_{i}=-\int_{-\infty}^{\infty} l(y) \frac{w_{0}(y, 0)}{U} d y
$$

Therefore, in terms of $Y_{\lambda}(y)$, the following general expression for the induced drag is obtained:

$$
\begin{equation*}
D_{i}=\int_{-\infty}^{\infty} \frac{1}{\rho U^{2}} d y \int_{0}^{\infty} f(\lambda) Y_{\lambda}(y) d \lambda \int_{0}^{\infty} \eta f(\eta) Y_{\eta}(y) d \eta \tag{54}
\end{equation*}
$$

Thus the problem of calculating the induced drag with a given distribution of lift $l(y)$ is reduced to the problem of solving the integral equation (52) for $f(\lambda)$ and then evaluating the integral given by Eq. (54).

If the chord $c$, the geometrical angle of attack $\alpha$ and the slope $k$ of the lift coefficient are given instead of the lift distribution $l(y)$, then

$$
\begin{equation*}
l(y)=\frac{1}{2} \rho U^{2} c k\left\{\alpha+\frac{w_{0}(y, 0)}{U}\right\} \tag{55}
\end{equation*}
$$

Thus Eq. (52) is replaced by the following equation

$$
\frac{1}{4} \rho U^{2} c k\left\{\alpha-\frac{1}{2 \rho U^{2}} \int_{0}^{\infty} \lambda f(\lambda) Y_{\lambda}(y) d \lambda\right\}=\int_{0}^{\infty} f(\lambda) Y_{\lambda}(y) d \lambda
$$

or

$$
\begin{equation*}
\frac{1}{4} \rho U^{2} c k \alpha=\int_{0}^{\infty}\left(1+\frac{c k}{8} \lambda\right) f(\lambda) Y_{\lambda}(y) d \lambda \tag{56}
\end{equation*}
$$

This is now the integral equation for $f(\lambda)$. When $f(\lambda)$ is determined, the induced drag $D_{i}$ can be again calculated by using Eq. (54).

# ON THEODORSEN'S METHOD OF CONFORMAL MAPPING of NEARLY CIRCULAR REGIONS* 

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1. Introduction. In determining the complex velocity potential of the two-dimensional flow around an airfoil, one is lead to the problem of finding the analytic function which maps the exterior of a circle conformally onto that of a "nearly circular" contour. T. Theodorsen developed a method for the practical computation of this mapping function, a method which was later elaborated on in a joint paper by Theodorsen and I. E. Garrick. ${ }^{1}$ Theodorsen reduces the problem of determining the mapping function to the solution of a certain non-linear integral equation which then is solved by successive approximations. In both papers examples of wing sections of airplanes are calculated demonstrating the use of the process and the rapidity with which it converges. However, the validity of the method from a mathematical point of view, such as the proof of the convergence of the successive approximations, is not discussed. The present paper is an attempt to supply such a discussion. Simple conditions on the nearly circular contour (essentially involving the tangent angle and the curvature) are established which insure the convergence of the process. The absolute value of the difference between the mapping function and the successive approximations is estimated. These estimates serve both to prove the convergence and to appraise the accuracy of the approximation. The analogous problem for the derivative of the mapping function is treated. (The derivative of the mapping function enters in the computation of the velocity and pressure distribution on the surface of the wing.) Finally, conditions are discussed under which the map of the circle by means of the successive approximations is star-shaped.

Although Theodorsen's method is of particular importance in the theory of airfoils, it represents the solution of a general problem in conformal mapping. For this reason all results of the present paper are derived for the "standard" case where the interior of a circle about the origin is mapped onto the interior of the nearly circular contour containing the origin under preservation of the positive line element at the origin. However, all results obtained remain the same for the mapping function of the exteriors and for a different normalization of the mapping function (see §3).

Sections $2-8$ contain the actual results and proofs of the paper. To simplify the presentation some auxiliary results used in the text are listed in $\S 9$.
2. Theodorsen's integral equation and the successive approximations. Let $C$ be a simple closed curve represented in polar co-ordinates by the equation $\rho=\rho(\theta)$ $(0 \leqq \theta \leqq 2 \pi)$, where $\rho(\theta)$ is absolutely continuous ${ }^{2}$ and for some $\epsilon(0<\epsilon<1)$,

[^2]\[

$$
\begin{equation*}
\frac{a}{1+\epsilon} \leqq \rho(\theta) \leqq a(1+\epsilon), \tag{2.1}
\end{equation*}
$$

\]

$a$ being a positive constant, and

$$
\begin{equation*}
\left|\frac{\rho^{\prime}(\theta)}{\rho(\theta)}\right| \leqq \epsilon . \tag{2.2}
\end{equation*}
$$

Any curve $C$ satisfying these conditions will be called a nearly circular contour.
Let us suppose that the function $w=f(z)$ maps the circle $|z|<1$ conformally onto the interior of $C$, and that $f(0)=0, f^{\prime}(0)>0$. The function

$$
\begin{equation*}
F(z)=\log \frac{f(z)}{z}=\log \left|\frac{f(z)}{z}\right|+i \arg \frac{f(z)}{z}, \tag{2.3}
\end{equation*}
$$

which is defined as the real-valued $\log f^{\prime}(0)$ when $z=0$, is single-valued and analytic for $|z|<1$ and continuous for $|z| \leqq 1$. For $z=e^{i \phi}$ we write $\arg \left[f\left(e^{i \phi}\right) e^{-i \phi}\right]=\theta(\phi)-\phi$, and therefore

$$
\begin{equation*}
F\left(e^{i \phi}\right)=\log \rho[\theta(\phi)]+i(\theta(\phi)-\phi) \tag{2.4}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\theta(\phi)-\phi=-\frac{1}{2 \pi} \int_{0}^{\pi}\{\log \rho[\theta(\phi+t)]-\log \rho[\theta(\phi-t)]\} \cot \frac{t}{2} d t \tag{2.5}
\end{equation*}
$$

(The term arg $[f(z) / z]_{z=0}=\arg f^{\prime}(0)$, which should be added to the integral on the right, is zero.) Thus the function $F\left(e^{i \phi}\right)$ and hence $f(z)$ may be found by solving this integral equation for $\theta(\phi)$. The existence of a continuous solution of this integral equation is assured by Riemann's mapping theorem. This solution is also unique as is shown in $\S 9(a)$. In order to compute the solution we follow Theodorsen and form the successive approximations

$$
\left.\begin{array}{rl}
\theta_{0}(\phi) \equiv \phi \\
\theta_{n}(\phi)-\phi=-\frac{1}{2 \pi} \int_{0}^{\pi}\left\{\log \rho\left[\theta_{n-1}(\phi+t)\right]-\log \rho\left[\theta_{n-1}(\phi-t)\right]\right\} \cot \frac{t}{2} d t  \tag{2.0}\\
(n=1,2, \cdots)
\end{array}\right\}
$$

The functions $\theta_{n}(\phi)$ are continuous for $0 \leqq \phi \leqq 2 \pi$; in fact, they are absolutely continuous and the squares of their first derivatives are integrable (for the proof of this, see $\S 9(\mathrm{~b})) ; \theta_{n}(\phi)-\phi$ is a conjugate function of $\log \rho\left[\theta_{n-1}(\phi)\right]$.

We shall show that the sequence $\theta_{n}(\phi)$ converges uniformly to $\theta(\phi)$ as $n \rightarrow \infty$. Hence, also $\log \rho\left[\theta_{n}(\phi)\right]$ converges uniformly to $\log \rho[\theta(\phi)]$ as $n \rightarrow \infty$, so that the functions
$F_{0}\left(e^{i \theta}\right) \equiv \log a, \quad F_{n}\left(e^{i \phi}\right)=\log \rho\left[\theta_{n-1}(\phi)\right]+i\left(\theta_{n}(\phi)-\phi\right) \quad(n \geqq 1)$,
may be used to compute $F\left(e^{i \phi}\right)$ with any desired degree of accuracy.
Let $F_{n}(z)$ denote the function which is analytic for $|z|<1$ and assumes the boundary values $F_{n}\left(e^{i \phi}\right)$ for $|z|=1$. By the principle of the maximum modulus the uniform

[^3]convergence of $F_{n}\left(e^{i \phi}\right)$ to $F\left(e^{i \phi}\right)$ implies that $F_{n}(z)$ converges to $F(z)$ uniformly for $|z| \leqq 1$, and thus the functions $f_{n}(z)=z e^{F_{n}(z)}$ converge uniformly for $|z| \leqq 1$ to the mapping function $f(z)$.

In order to prove the convergence of the functions $\theta_{n}(\phi)$ and $\theta_{n}^{\prime}(\phi)$ we shall derive estimates for the differences $\left|\theta_{n}(\phi)-\theta(\phi)\right|$ and $\left|\theta_{n}^{\prime}(\phi)-\theta^{\prime}(\phi)\right|$ in terms of $\epsilon$ and $n$. These differences approach zero as $n \rightarrow \infty$, and will at the same time permit us to appraise the degree of accuracy of the $n$th approximation.

Remark. Theodorsen considers the case where the exterior of a circle $|\zeta|=R$ is mapped onto the exterior of a "nearly circular" closed curve $\Gamma$ whereby the mapping function $\omega=g(\zeta)$ is so normalized that $\lim _{\zeta \rightarrow \infty} \omega / \zeta=1$. This case is immediately reduced to the one considered above by means of the transformations $w=\omega^{-1}$ and $z=R / \zeta$. Let us suppose that $\Gamma$ is represented by the equation $r=r(\Theta)(0 \leqq \Theta \leqq 2 \pi)$, where, for some positive $b$ and $0<\epsilon<1, b(1+\epsilon)^{-1} \leqq r(\Theta) \leqq b(1+\epsilon)$ and $\left|r^{\prime}(\Theta) / r(\Theta)\right| \leqq \epsilon$. Then the function $w=f(z)=1 / g(\zeta)$, where $\zeta=R / z$, maps the circle $|z|<1$ onto the interior of the nearly circular contour $C$ represented by the equation $\rho=\rho(\theta)=1 / r(\Theta)$, where $\theta=-\Theta$ and $\rho(\theta)$ satisfies the conditions (2.1) and (2.2) with $a=b^{-1}$. For $\zeta=\operatorname{Re}{ }^{i \psi}$, we write arg $[g(\zeta) / \zeta]=\Theta(\psi)-\psi$, where $\arg [g(\zeta) / \zeta]$ is defined as 0 when $\zeta=\infty$. Then, for $\zeta=R e^{i \psi}$ and $z=e^{i \phi}$ where $\phi=-\psi$,

$$
\begin{aligned}
\log \frac{g(\zeta)}{\zeta} & =\log r[\Theta(\psi)]+i(\Theta(\psi)-\psi)-\log R \\
& =-\log \frac{f(z)}{z}-\log R \\
& =-\log \rho[\theta(\phi)]-i(\theta(\phi)-\phi)-\log R
\end{aligned}
$$

Thus one can form the successive approximations $\Theta_{n}(\psi)$ for the function $\Theta(\psi)$ in the same manner as the $\theta_{n}(\phi)$ are formed for $\theta(\phi)$. Furthermore, $\Theta_{n}(\psi)=-\theta_{n}(\phi), \psi=-\phi$ and $\Theta_{n}(\psi)-\Theta(\psi)=-\left(\theta_{n}(\phi)-\theta(\phi)\right)$. Hence any bound obtained for $\left|\theta_{n}(\phi)-\theta(\phi)\right|$ is also a bound for $\left|\Theta_{n}(\psi)-\Theta(\psi)\right|$, and the same remark applies to the derivatives of these differences.
3. Statement of results. We shall prove the following estimates:
I. If $C$ is a nearly circular contour, and if $\theta_{n}(\phi)$ and $\theta(\phi)=\arg f\left(e^{i \phi}\right)$ are defined by (2.6) and (2.4), respectively, then

$$
\begin{equation*}
\left|\theta_{n}(\phi)-\theta(\phi)\right| \leqq 2\left(\frac{\pi^{2}}{1-\epsilon^{2}}\right)^{1 / 4} \epsilon^{(n+2) / 2} \tag{3.1}
\end{equation*}
$$

The bound for $\left|\theta_{n}(\phi)-\theta(\phi)\right|$ obtained here approaches zero as $n \rightarrow \infty$ (since $0<\epsilon<1$ ) and is therefore sufficient to establish the convergence of the functions $\theta_{n}(\phi)$ to $\theta(\phi)$. However, a bound which converges to zero more rapidly can be found if a further assumption regarding $C$ is made.
II. If $C$ is a nearly circular contour and if

$$
\begin{equation*}
\left|\frac{\rho^{\prime}\left(\theta_{2}\right)}{\rho\left(\theta_{2}\right)}-\frac{\rho^{\prime}\left(\theta_{1}\right)}{\rho\left(\theta_{1}\right)}\right| \leqq \epsilon_{\mid}^{\left|\theta_{2}-\theta_{1}\right|, ~} \tag{3.2}
\end{equation*}
$$

$\epsilon$ being the same as in (2.1), then

$$
\begin{equation*}
\left|\theta_{n}(\phi)-\theta(\phi)\right| \leqq(2 \pi A(n+1))^{1 / 2} \epsilon^{n+1} \tag{3.3}
\end{equation*}
$$

where $A=4^{\circ} e^{c^{2}}$.

The following result is obtained for the derivatives $\theta_{n}^{\prime}(\phi)$.
UII. If $C$ is a nearly circular contour, if (3.2) holds, and if $p(\theta)=d\left[\rho^{\prime}(\theta) / \rho(\theta)\right] / d \theta$ salisfies the condition

$$
\begin{equation*}
\left|p\left(\theta_{2}\right)-p\left(\theta_{1}\right)\right| \leqq \epsilon\left|\theta_{2}-\theta_{1}\right| \tag{3.4}
\end{equation*}
$$

$\epsilon$ being the same as in (2.1), then

$$
\begin{equation*}
\left|\theta_{n}^{\prime}(\phi)-\theta^{\prime}(\phi)\right| \leqq \sqrt{2 \pi \sigma_{n}}(A(n+1))^{3 / 2} \epsilon^{n+1} \tag{3.5}
\end{equation*}
$$

where $A=4^{4} e^{\epsilon^{2}}$ and

$$
\begin{equation*}
\sigma_{1}=1+\epsilon, \quad \cdot \sigma_{n}=(1+\epsilon) \prod_{k=2}^{n}\left(1+\epsilon^{k} \sqrt{2 \pi A k}\right) \tag{3.6}
\end{equation*}
$$

For all $n$,

$$
\begin{equation*}
\sigma_{n} \leqq(1+\epsilon) \exp \left[2 \epsilon^{2} \sqrt{\pi A}(1-\epsilon)^{-3 / 2}\right] \tag{3.7}
\end{equation*}
$$

so that $\sigma_{n}$ is bounded if $0<\epsilon<1$.
Estimates for the difference $\left|F_{n}(z)-F(z)\right|,|z| \leqq 1$, may be obtained from those for $\left|\theta_{n}(\phi)-\theta(\phi)\right|$. For by (2.2),

$$
\left|F_{n}\left(e^{i \phi}\right)-F\left(e^{i \phi}\right)\right| \leqq\left\{\epsilon^{2}\left(\theta_{n-1}(\phi)-\theta(\phi)\right)^{2}+\left(\theta_{n}(\phi)-\theta(\phi)\right)^{2}\right\}^{1 / 2}
$$

and for $|z| \leqq 1$

$$
\left|F_{n}(z)-F(z)\right| \leqq \max _{\phi}\left|F_{n}\left(e^{i \phi}\right)-F\left(e^{i \phi}\right)\right|
$$

Thus, for example, in case II we find by use of (3.3) that

$$
\left|F_{n}(z)-F(z)\right| \leqq 2\left(A \pi\left(n+\frac{1}{2}\right)\right)^{1 / 2} \epsilon^{n+1}
$$

Hence, if $0<\epsilon<1$, the successive approximations $F_{n}(z)$ converge uniformly to $F(z)=\log [f(z) / z]$ when $|z| \leqq 1$. An analogous statement applies to the derivatives $\partial\left[F_{n}\left(r e^{i \phi}\right)\right] / \partial \phi$ and $\partial\left[F\left(r e^{i \phi}\right)\right] / \partial \phi$.

To prove the three theorems I, II, and III, we shall first derive bounds for the square means

$$
\left.\begin{array}{c}
M_{n}^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}(\phi)-\theta(\phi)\right)^{2} d \phi, \quad M_{n}^{\prime 2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}^{\prime}(\phi)-\theta^{\prime}(\phi)\right)^{2} d \phi  \tag{3.8}\\
M_{n}^{\prime \prime 2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}^{\prime \prime}(\phi)-\theta^{\prime \prime}(\phi)\right)^{2} d \phi
\end{array}\right\}
$$

The above results will then be obtained by use of the inequalities (see $\S 4(\mathrm{c})$ )

$$
\left|\theta_{n}(\phi)-\theta(\phi)\right| \leqq\left(2 \pi M_{n} M_{n}^{\prime}\right)^{1 / 2}, \quad\left|\theta_{n}^{\prime}(\phi)-\theta^{\prime}(\phi)\right| \leqq\left(2 \pi M_{n}^{\prime} M_{n}^{\prime \prime}\right)^{1 / 2}
$$

The functions $f_{n}(z)=z e^{F_{n}(z)}$ map the circle $|z|=1$ onto closed curves $C_{n}$. Since the functions $f_{n}(z)$ are to be used as approximations to the mapping function $f(z)$, it is essential to know that the $C_{n}$ are simple closed curves. This will certainly be the case if the $C_{n}$ are star-shaped with respect to the origin. (A closed curve is star-shaped with respect to the origin if every ray from the origin intersects the curve in exactly one point.) Knowing that $C_{n}$ is star-shaped has the additional advantage ${ }^{3}$ that $\theta_{n}(\phi)$

[^4]is then an increasing function of $\phi$ and therefore possesses a unique inverse function $\phi=\phi_{n}(\theta)$. This permits us to form immediately the inverse $z=e^{i \phi_{n}(\theta)}$ of the mapping function $w=f(z)$ for $w$ on $C_{n}$. We examine therefore the question when the $C_{n}$ are star-shaped, and obtain the following result:
IV. If $C$ is a nearly circular contour and if the condition (3.2) is satisfed, then the curve $C_{1}$ is star-shaped with respect to the origin if $\epsilon \leqq(2 \log 2)^{-1}, C_{2}$ if $\epsilon \leqq 0.34, C_{3}$ if $\epsilon \leqq 0.31$, and $C_{4}$ if $\epsilon=0.3$. For $n \geqq 4$ all $C_{n}$ are star-shaped if $\epsilon \leqq 0.295$.

This result is derived by examining the values of $\epsilon$ for which $\left|\theta_{n}^{\prime}(\phi)-1\right| \leqq 1$, so that $\theta_{n}^{\prime}(\phi) \geqq 0$ and $\theta_{n}(\phi)$ is therefore monotone increasing. For large values of $n(n \geqq 4)$ a more favorable estimate for $\epsilon$ may be obtained by making use of (3.5) and of a lower bound for $\theta^{\prime}(\phi)$ which is given in $\S 9(\mathrm{~d})$.
4. Proof of I. (a) Estimate of $M_{n}$. Let $F\left(e^{i \phi}\right)$ and $F_{n}\left(e^{i \phi}\right)$ be the functions in (2.4) and (2.7). Because of the representations of $\theta(\phi)-\phi$ and $\theta_{n}(\phi)-\phi$ by means of the integrals (2.5) and (2.6), respectively, we have

$$
\begin{equation*}
\int_{0}^{2 \pi}(\theta(\phi)-\phi) d \phi=0, \quad \int_{0}^{2 \pi}\left(\theta_{n}(\phi)-\phi\right) d \phi=0 \tag{4.1}
\end{equation*}
$$

We now apply the following well known theorem: ${ }^{4}$ If the function $g(\phi)$ is realvalued, periodic (period $2 \pi$ ), and $(g(\phi))^{2}$ is integrable (in the sense of Lebesgue) $0 \leqq \phi \leqq 2 \pi$, and if $\bar{g}(\phi)$ is a conjugate function of $g(\phi)$ (then surely existing), then

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{0}^{2 \pi}[\bar{g}(\phi)]^{2} d \phi+\alpha^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}[g(\phi)]^{2} d \phi+\beta^{2}, \tag{4.2}
\end{equation*}
$$

where

$$
\alpha=\frac{1}{2 \pi} \int_{0}^{2 \pi} g(\phi) d \phi, \quad \beta=\frac{1}{2 \pi} \int_{0}^{2 \pi} \bar{g}(\phi) d \phi .
$$

Applying this with $g(\phi)+i \bar{g}(\phi)=F_{n}\left(e^{i \phi}\right)-F\left(e^{i \phi}\right)$ and observing that $\beta=0$ (because of (4.1)), we obtain

$$
\begin{equation*}
M_{n}^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}(\phi)-\theta(\phi)\right)^{2} d \phi \leqq \frac{1}{2 \pi} \int_{0}^{2 \pi}\left\{\log \rho\left[\theta_{n-1}(\phi)\right]-\log \rho[\theta(\phi)]\right\}^{2} d \phi . \tag{4.3}
\end{equation*}
$$

By hypothesis (2.2),

$$
\left|\log \rho\left[\theta_{n-1}(\phi)\right]-\log \rho[\theta(\phi)]\right| \leqq \epsilon\left|\theta_{n-1}(\phi)-\theta(\phi)\right|,
$$

and therefore

$$
M_{n}^{2} \leqq \epsilon^{2} \frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n-1}(\phi)-\theta(\phi)\right)^{2} d \phi \leqq \epsilon^{2} M_{n-1}^{2},
$$

or

$$
M_{n} \leqq \epsilon M_{n-1}, \quad M_{n} \leqq M_{0} \epsilon^{n} .
$$

For $n=0$ we obtain from (4.2) by use of (2.1),

$$
M_{0}^{2}=\frac{1}{2 \pi} \int_{0}^{i \pi}(\theta(\phi)-\phi)^{2} d \phi \leqq \frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\log \frac{\rho[\theta(\phi)]}{a}\right)^{2} d \phi \leqq \epsilon^{2} .
$$

[^5]Thus we have proved that if $\rho(\theta)$ satisfies hypotheses (2.1) and (2.2),

$$
\begin{equation*}
M_{n} \leqq \epsilon^{n+1} . \tag{4.4}
\end{equation*}
$$

(b). Estimate of $M_{n}{ }^{\prime}$. It follows from $\S 9(b)$ and $\S 9(c)$ that $F_{n}\left(e^{i \phi}\right)$ and $F\left(e^{i \phi}\right)$ are absolutely continuous and that $\left.\left\{d\left[F_{n}\left(e^{i \phi}\right)\right] / d \phi\right]\right\}^{2}$ and $\left\{d\left[F\left(e^{i \phi}\right)\right] / d \phi\right\}^{2}$ are integrable. Furthermore, because of the absolute continuity of $F_{n}\left(e^{i \phi}\right)-F\left(e^{i \phi}\right)$, the imaginary part of the derivative $d\left\{F_{n}\left(e^{i \phi}\right)-F\left(e^{i \phi}\right)\right\} / d \phi$ is a conjugate function of the real part. Finally,

$$
\int_{0}^{2 \pi} \frac{d}{d \phi} F\left(e^{i \phi}\right) d \phi=\left[F\left(e^{i \phi}\right)\right]_{\phi-0}^{\phi-2 \pi}=0, \quad \int_{0}^{2 \pi} \frac{d}{d \phi} F_{n}\left(e^{i \phi}\right) d \phi=0 .
$$

Hence, applying (4.2) with $g(\phi)+i \bar{g}(\phi)=d\left[F_{n}\left(e^{i \phi}\right)-F\left(e^{i \phi}\right)\right] / d \phi$, we obtain ${ }^{5}$

$$
\begin{align*}
M_{n}^{\prime 2} & =\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}^{\prime}(\phi)-\theta^{\prime}(\phi)\right)^{2} d \phi \\
& =\frac{1}{2 \pi} \int_{0}^{2 \pi}\left\{\frac{\rho^{\prime}}{\rho}\left[\theta_{n-1}(\phi)\right] \theta_{n-1}^{\prime}(\phi)-\frac{\rho^{\prime}}{\rho}[\theta(\phi)] \theta^{\prime}(\phi)\right\}^{2} d \phi . \tag{4.5}
\end{align*}
$$

By (2.2) we have (omitting the argument $\phi$ in the integrands)

$$
\begin{equation*}
M_{n}^{\prime 2} \leqq 2 \epsilon^{2} \frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n-1}^{\prime 2}+\theta^{\prime 2}\right) d \phi . \tag{4.6}
\end{equation*}
$$

As is shown in §9(c),

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{0}^{2 \pi} \theta^{\prime 2} d \phi \leqq \frac{1}{1-\epsilon^{2}} \tag{4.7}
\end{equation*}
$$

Furthermore, applying (4.2) with $g(\phi)+i \bar{g}(\phi)=d\left[F_{n}\left(e^{i \phi}\right)\right] / d \phi$, we obtain by (2.2),

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}^{\prime}-1\right)^{2} d \phi=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\frac{\rho^{\prime}}{\rho}\left[\theta_{n-1} \mid \theta_{n-1}^{\prime}\right)^{2} d \phi \leqq \epsilon^{2} \frac{1}{2 \pi} \int_{0}^{2 \pi} \theta_{n-1}^{\prime 2} d \phi,\right.
$$

or

$$
\frac{1}{2 \pi}\left\{\int_{e}^{2 \pi} \theta_{n}^{\prime 2} d \phi-2 \int_{0}^{2 \pi} \theta_{n}^{\prime} d \phi+2 \pi\right\} \leqq \epsilon^{2} \frac{1}{2 \pi} \int_{0}^{2 \pi} \theta_{n-1}^{\prime 2} d \phi .
$$

Since $\int_{0}^{2 \pi} \theta_{n}^{\prime} d \phi=2 \pi$, we find that

If we set

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi} \theta_{n}^{\prime 2} d \phi-1 \leqq \epsilon^{2} \frac{1}{2 \pi} \int_{0}^{2 \pi} \theta_{n-1}^{\prime 2} d \phi
$$

$$
m_{n}^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi} \theta_{n}^{\prime 2} d \phi
$$

we have $m_{n}{ }^{2} \leqq 1+m_{n}{ }^{2}-1 \epsilon^{2}$, and therefore

$$
m_{n}^{2} \leqq 1+\epsilon^{2}+\epsilon^{4}+\cdots+\epsilon^{2 n} m_{0}^{2} .
$$

Since
${ }^{5}$ The notation $\frac{\rho}{\rho}[\theta]$ or $\left(\rho^{\prime} / \rho\right)[\theta]$ means $\rho^{\prime}(\theta) / \rho(\theta)$.

$$
m_{0}^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi} d \phi=1
$$

we obtain

$$
\begin{equation*}
m_{n}^{2} \leqq\left(1-\epsilon^{2}\right)^{-1} \tag{4.8}
\end{equation*}
$$

Thus by (4.6), (4.7), and (4.8),

$$
\begin{equation*}
M_{n}^{\prime 2} \leqq 4 \epsilon^{2} /\left(1-\epsilon^{2}\right) \tag{4.9}
\end{equation*}
$$

(c) Estimate of $\left|\theta_{n}(\phi)-\theta(\phi)\right|$. To complete the proof we now apply the following theorem: If $g(\phi)$ is a real-valued, absolutely continuous and periodic function (period $2 \pi$ ) and if $\left(g^{\prime}(\phi)\right)^{2}$ is integrable, then for any $\phi_{0}$,

$$
\begin{equation*}
[g(\phi)]^{2}-\left[g\left(\phi_{0}\right)\right]^{2} \leqq 2 \pi M M^{\prime} \tag{4.10}
\end{equation*}
$$

where

$$
M^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}[g(\phi)]^{2} d \phi, \quad M^{\prime 2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left[g^{\prime}(\phi)\right]^{2} d \phi
$$

The factor $2 \pi$ is the "best possible" constant; it cannot be replaced by a smaller one. ${ }^{6}$
Let $g(\phi)=\theta_{n}(\phi)-\theta(\phi)$. Since then $\int_{0}^{2 \pi} g(\theta) d \phi=0$, there exists a value $\phi_{0}$ such that $g\left(\phi_{0}\right)=0$. Hence

$$
\left|\theta_{n}(\phi)-\theta(\phi)\right| \leqq\left(2 \pi M_{n} M_{n}^{\prime}\right)^{1 / 2}
$$

Using (4.4) and (4.9), we find (3.1).
5. Proof of II. (a) Estimate of $M_{n}^{\prime}$. Under the present hypotheses an estimate for $M_{n}^{\prime}$ sharper than (4.9) may be obtained. We shall prove that if $\rho(\theta)$ satisfies (2.1), (2.2) and (3.2), then

$$
\begin{equation*}
M_{n}^{\prime} \leqq A(n+1) \epsilon^{n+1}, \quad\left(A=4^{4} e^{\epsilon^{2}}\right) \tag{5.1}
\end{equation*}
$$

Using the relation (4.5) we obtain for $n \geqq 1$,

$$
\begin{aligned}
M_{n}^{\prime}= & \left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left[\left(\frac{\rho^{\prime}}{\rho}\left[\theta_{n-1}\right]-\frac{\rho^{\prime}}{\rho}[\theta]\right) \theta^{\prime}+\frac{\rho^{\prime}}{\rho}\left[\theta_{n-1}\right]\left(\theta_{n-1}^{\prime}-\theta^{\prime}\right)\right]^{2} d \phi\right\}^{1 / 2} \\
\leqq & \left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\frac{\rho^{\prime}}{\rho}\left[\theta_{n-1}\right]-\frac{\rho^{\prime}}{\rho}[\theta]\right)^{2} \theta^{\prime 2} d \phi\right\}^{1 / 2} \\
& +\left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}^{\prime}-1-\theta^{\prime}\right)^{2}\left(\frac{\rho^{\prime}}{\rho}\left[\theta_{n-1}\right]\right)^{2} d \phi\right\}^{1 / 2},
\end{aligned}
$$

${ }^{6}$ To prove (4.10), we note first that for $0 \leqq \phi \leqq 2 \pi, 0 \leqq \phi_{0} \leqq 2 \pi$,

$$
\begin{equation*}
g^{2}(\phi)-g^{2}\left(\phi_{0}\right)=2 \int_{\phi_{0}}^{\phi} g(t) g^{\prime}(t) d t=2 \int_{\phi_{0}}^{\phi-2 \pi} g(l) g^{\prime}(l) d t . \tag{*}
\end{equation*}
$$

Since

$$
\left|\int_{\phi_{0}}^{\phi}\right| g g^{\prime}|d t|+\left|\int_{\phi_{0}}^{\phi-2 \pi}\right| g g^{\prime}|d l|=\int_{\phi-2 \pi}^{\phi}\left|g g^{\prime}\right| d t=\int_{0}^{2 \pi}\left|g g^{\prime}\right| d t
$$

one of the two integrals in ( ${ }^{*}$ ) does not exceed $\frac{1}{2} \int_{0}^{2 \pi}\left|g g^{\prime}\right| d t$. Hence, by the inequality of Schwarz,

$$
[g(\phi)]^{2}-\left[g\left(\phi_{0}\right)\right]^{2} \leqq \int_{0}^{2 \pi}\left|g g^{\prime}\right| d \iota \leqq 2 \pi M M^{\prime}
$$

Applying (4.10) with $g(\phi)=\cos ^{n} \phi\left(\phi_{0}=\frac{1}{2} \pi\right)$ and letting $n \rightarrow \infty$, we see that the constant $2 \pi$ cannot be replaced by a smaller one.
by Minkowski's inequality. Under the present hypotheses we have by $\S 9(\mathrm{~d})$,

$$
0<\theta^{\prime}(\phi) \leqq A=4^{4} e^{e^{2}} .
$$

Hence, by (3.2) and (2.2),

$$
M_{n}^{\prime} \leqq \epsilon A\left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n-1}-\theta\right)^{2} d \phi\right\}^{1 / 2}+\epsilon M_{n-1}^{\prime}=\epsilon\left(A M_{n-1}+M_{n-1}^{\prime}\right),
$$

and, therefore, by (4.4),

$$
\begin{equation*}
M_{n}{ }^{\prime} \leqq \epsilon\left(A \epsilon^{n}+M_{n-1}^{\prime}\right), \quad(n \geqq 1) . \tag{5.2}
\end{equation*}
$$

For $n=0$, we have, using (4.2) and (2.2),

$$
M_{0}^{\prime 2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta^{\prime}-1\right)^{2} \cdot d \phi=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\frac{\rho^{\prime}}{\rho}[\theta] \theta^{\prime}\right)^{2} d \phi \leqq \epsilon^{2} \frac{A}{2 \pi} \int_{0}^{2 \pi} \theta^{\prime} d \phi=\epsilon^{2} A .
$$

This inequality proves (5.1) for $n=0$. For $n \geqq 1$, (5.1) is easily seen to be true by induction. Assuming that it holds for some $n \geqq 0$, we obtain by use of (5.2),

$$
M_{n+1} \leqq \epsilon\left(A \epsilon^{n+1}+A(n+1) \epsilon^{n+1}\right)=A(n+2) \epsilon^{n+2},
$$

i.e., (5.1) is also true for $n+1$.
(b). Estimate of $\left|\theta_{n}(\phi)-\theta(\phi)\right|$. Applying (4.10), (4.4) and (5.1), we find

$$
\left|\theta_{n}(\phi)-\theta(\phi)\right| \leqq\left(2 \pi M_{n} M_{n}^{\prime}\right)^{1 / 2} \leqq(2 \pi A(n+1))^{1 / 2} \epsilon^{n+1} .
$$

6. Proof of III. (a). Some properties of the functions $F\left(e^{i \phi}\right)$ and $F_{n}\left(e^{i \phi}\right)$. Because of the hypothesis (3.4), $F\left(e^{i \phi}\right)$ has a continuous second derivative for ${ }^{7} 0 \leqq \phi \leqq 2 \pi$. The same is true for all $F_{n}\left(e^{i \phi}\right)$ as is shown in $\S 9(e)$. Differentiating $F\left(e^{i \phi}\right)$ and $F_{n}\left(e^{i \phi}\right)$ twice with respect to $\phi$, we obtain

$$
\begin{array}{ll}
\frac{d F}{d \phi}=\frac{\rho^{\prime}}{\rho}[\theta] \theta^{\prime}+i\left(\theta^{\prime}-1\right), & \frac{d^{2} F}{d \phi^{2}}=p[\theta] \theta^{\prime 2}+\frac{\rho^{\prime}}{\rho}[\theta] \theta^{\prime \prime}+i \theta^{\prime \prime}, \\
\frac{d F_{n}}{d \phi}=\frac{\rho^{\prime}}{\rho}\left[\theta_{n-1}\right] \theta_{n-1}^{\prime}+i\left(\theta_{n}^{\prime}-1\right), & \frac{d^{2} F_{n}}{d \phi^{2}}=p\left[\theta_{n-1}\right] \theta_{n-1}^{\prime 2}+\frac{\rho^{\prime}}{\rho}\left[\theta_{n-1}\right] \theta_{n-1}^{\prime \prime}+i \theta_{n .}^{\prime \prime} .
\end{array}
$$

The present proof is similar to that of (II) and we estimate first

$$
M_{n}^{\prime \prime}=\left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}^{\prime \prime}-\theta^{\prime \prime}\right)^{2} d \phi\right\}^{1 / 2}
$$

We prove that, for $n \geqq 1$,

$$
\begin{equation*}
M_{n}^{\prime \prime} \leqq A^{2}(n+1)^{2} \sigma_{n} \epsilon^{n+1}, \tag{6.1}
\end{equation*}
$$

where $A=4^{\prime} e^{e^{2}}$ and $\sigma_{n}$ is defined in (3.6).
(b). Proof of the inequality (6.1). Since

$$
\int_{0}^{2 \pi} \frac{d^{2}}{d \phi^{2}}\left(F_{n}\left(e^{i \phi}\right)-F\left(e^{i \phi}\right)\right) d \phi=0
$$

[^6]we have, applying (4.2),
\[

$$
\begin{align*}
\left(M_{n+1}^{\prime \prime}\right)^{2}= & \frac{1}{2 \pi} \int_{0}^{2 \pi}\left\{\left(p\left[\theta_{n}\right]-p[\theta]\right) \theta^{\prime 2}+p\left[\theta_{n}\right]\left(\theta_{n}^{\prime 2}-\theta^{\prime 2}\right)\right. \\
& \left.+\left(\frac{\rho^{\prime}}{\rho}\left[\theta_{n}\right]-\frac{\rho^{\prime}}{\rho}[\theta]\right) \theta^{\prime \prime}+\frac{\rho^{\prime}}{\rho}\left[\theta_{n}\right]\left(\theta_{n}^{\prime \prime}-\theta^{\prime \prime}\right)\right\}^{2} d \phi \tag{6.2}
\end{align*}
$$
\]

Because of (3.2),

$$
\begin{equation*}
|p(\theta)| \leqq \epsilon . \tag{6.3}
\end{equation*}
$$

Using (9.3), (3.4), (6.3), (3.2), and (2.2), we find that

$$
\left(M_{n+1}^{\prime \prime}\right)^{2} \leqq \frac{\epsilon^{2}}{2 \pi} \int_{0}^{2 \pi}\left\{A^{2}\left|\theta_{n}-\theta\right|+\left|\theta_{n}^{\prime 2}-\theta^{\prime 2}\right|+\left|\theta^{\prime \prime}\right|\left|\theta_{n}-\theta\right|+\left|\theta_{n}^{\prime \prime}-\theta^{\prime \prime}\right|\right\}^{2} d \phi
$$

If $M_{n}$ and $M_{n}^{\prime}$ are defined as in (3.8), we have by Minkowski's inequality:

$$
\begin{align*}
& M_{n+1}^{\prime \prime} \leqq \epsilon\left[A^{2} M_{n}+\left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}^{\prime 2}-\theta^{\prime 2}\right)^{2} d \phi\right\}^{1 / 2}\right. \\
&\left.+\left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}^{\prime \prime}\right)^{2}\left(\theta_{n}-\theta\right)^{2} d \phi\right\}^{1 / 2}+M_{n}^{\prime \prime}\right] \tag{6.4}
\end{align*}
$$

Since by (9.4),

$$
\begin{equation*}
M_{0}^{\prime \prime}=\left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta^{\prime \prime}\right)^{2} d \phi\right\}^{1 / 2} \leqq A^{3 / 2} \sqrt{2} \epsilon, \tag{6.5}
\end{equation*}
$$

and by (3.3),

$$
\left|\theta_{n}(\phi)-\theta(\phi)\right| \leqq(2 \pi A(n+1))^{1 / 2} \epsilon^{n+1}
$$

we have

$$
\begin{equation*}
\left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}-\theta\right)^{2} \theta^{\prime \prime 2} d \phi\right\}^{1 / 2} \leqq(2 \pi A(n+1))^{1 / 2} A^{3 / 2} \sqrt{2} \epsilon^{n+2}=2 A^{2} \epsilon^{n+2} \sqrt{\pi(n+1)} \tag{6.6}
\end{equation*}
$$

Next, applying the theorem of $\S 4(\mathrm{c})$ with $g(\phi)=\theta_{n}^{\prime}(\phi)-\theta^{\prime}(\phi)$, we obtain

$$
\left(\theta_{n}^{\prime}-\theta^{\prime}\right)^{2} \leqq 2 \pi M_{n}^{\prime} M_{n}^{\prime \prime},
$$

and taking the square root and using (9.3), we have

$$
\left|\theta_{n}^{\prime}+\theta^{\prime}\right| \leqq 2 A+\sqrt{2 \pi M_{n}^{\prime} M_{n}^{\prime \prime}} .
$$

Hence

$$
\left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}^{\prime 2}-\theta^{\prime 2}\right)^{2} d \phi\right\}^{1 / 2} \leqq M_{n}^{\prime}\left(2 A+\sqrt{2 \pi M_{n}^{\prime} M_{n}^{\prime \prime}}\right)=2 A M_{n}^{\prime}+M_{n}^{\prime} \sqrt{2 \pi M_{n}^{\prime} M_{n}^{\prime \prime}} .
$$

Applying the inequality ${ }^{8} M_{n}^{\prime} \leqq \sqrt{M_{n} M_{n}^{\prime \prime}}$ to the factor $M_{n}^{\prime}$ of the square root we find that
${ }^{8}$ If we set $g(\phi)=\theta_{n}(\phi)-\theta(\phi)$, we have by integration by parts

$$
\begin{aligned}
\int_{0}^{2 \pi}\left(g^{\prime}(\phi)\right)^{2} d \varphi & =\left|\left[g(\phi) g^{\prime}(\phi)\right]_{0}^{2 \pi}-\int_{0}^{2 \pi} g(\phi) g^{\prime \prime}(\phi) d \phi\right| \\
& \leqq \int_{0}^{2 \pi}\left|g(\phi) g^{\prime \prime}(\phi)\right| d \phi \leqq\left\{\int_{0}^{2 \pi}[g(\phi)]^{2} d \phi \cdot \int_{0}^{2 \pi}\left(g^{\prime \prime}(\phi)\right)^{2} d \phi\right\}^{1 / 2} .
\end{aligned}
$$

This proves the inequality of the text.

$$
\begin{equation*}
\left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{n}^{\prime 2}-\theta^{\prime 2}\right)^{2} d \phi\right\}^{1 / 2} \leqq 2 A M_{n}^{\prime}+M_{n}^{\prime \prime} \sqrt{2 \pi M_{n} M_{n}^{\prime}} \tag{6.7}
\end{equation*}
$$

Thus we obtain from (6.4) using (4.4), (6.7), (6.6) and (5.1),
$M_{n+1}^{\prime \prime} \leqq \epsilon\left\{A^{2} \epsilon^{n+1}+2 A^{2}(n+1) \epsilon^{n+1}+2 A^{2} \epsilon^{n+2} \sqrt{\pi(n+1)}\right.$

$$
\left.+\left(1+\epsilon^{n+1} \sqrt{2 \pi \cdot 4(n+1)}\right) M_{n}^{\prime \prime}\right\}
$$

and therefore
$M_{n+1}^{\prime \prime} \leqq A^{2} \epsilon^{n+2}\left\{1+2(n+1)+2 \epsilon \sqrt{\pi(n+1)}+\left(1+\epsilon^{n+1} \sqrt{2 \pi A(n+1)}\right) \frac{M_{n}^{\prime \prime}}{A^{2} \epsilon^{n+1}}\right\}$.
Assuming now that (6.1) is true for some $n \geqq 2$, we see from this inequality that (6.1) also holds for $n+1$. For, if we substitute in (6.8) for $M_{n}^{\prime \prime}$ the right-hand side of (6.1), we find that

$$
M_{n+1}^{\prime \prime} \leqq A^{2} \epsilon^{n+2}\left\{1+2(n+1)+2 \epsilon \sqrt{\pi(n+1)}+\left(1+\epsilon^{n+1} \sqrt{2 \pi A(n+1)}\right)(n+1)^{2} \sigma_{n}\right\}
$$

For $n \geqq 2$,

$$
1+2(n+1)+2 \epsilon \sqrt{\pi(n+1)}<[1+2(n+1)](1+\epsilon)<[1+2(n+1)] \sigma_{n+1},
$$

and therefore

$$
M_{n+1}^{\prime \prime} \leqq A^{2} \epsilon^{n+2} \sigma_{n+1}\left(1+2(n+1)+(n+1)^{2}\right)=A^{2}(n+2)^{2} \sigma_{n+1} \varepsilon^{n+2} .
$$

To complete the induction we show that (6.1) holds for $n=1$ and $n=2$. From (6.2) with $n=0$, we find that

$$
\begin{aligned}
M_{1}^{\prime} & =\left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left[(p(\phi)-p[\theta(\phi)]) \theta^{\prime 2}+p(\phi)\left(1-\theta^{\prime 2}\right)-\frac{\rho^{\prime}}{\rho}[\theta(\phi)] \theta^{\prime \prime}\right]^{2} d \phi\right\}^{1 / 2} \\
& \leqq \epsilon\left\{A^{2} M_{0}+\left(\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(1-\theta^{\prime 2}\right)^{2} d \phi\right)^{1 / 2}+M_{0}^{\prime \prime}\right\}
\end{aligned}
$$

by Minkowski's inequality and (3.4), (6.3), and (2.2). Applying (4.4), (9.3), (5.1), and observing that by (9.4) $M_{0}^{\prime \prime} \leqq A^{2} \epsilon(1+\epsilon)$, we find that

$$
M_{1}^{\prime \prime} \leqq \epsilon^{2}\left(A^{2}+(1+A) A+A^{2}(1+\epsilon)\right)=A^{2} \epsilon^{2}\left(2+\frac{1+A}{A}+\epsilon\right)
$$

Since $A \geqq 1,(1+A) / A \leqq 2$, and therefore

$$
\begin{equation*}
M_{1}^{\prime \prime} \leqq A^{2} \epsilon^{2}(4+\epsilon)<4 A^{2} \epsilon^{2}(1+\epsilon) \tag{6.9}
\end{equation*}
$$

To prove (6.1) for $n=2$, we apply (6.8) with $n=1$ and $M_{1}^{\prime \prime}$ replaced by $A^{2} \epsilon^{2}(4+\epsilon)$ (see (6.9)), to obtain

$$
M_{2}^{\prime \prime} \leqq A^{2} \epsilon^{3}\left[5+2 \epsilon \sqrt{2 \pi}+\left(1+2 \epsilon^{2} \sqrt{\pi A}\right)(4+\epsilon)\right] .
$$

Since $2 \sqrt{2 \pi}<6$ and $1+2 \epsilon^{2} \sqrt{\pi A}>1$,

$$
\begin{aligned}
5+2 \epsilon \sqrt{2 \pi}+\left(1+2 \epsilon^{2} \sqrt{\pi A}\right)(4+\epsilon) & <(5+6 \epsilon+4+\epsilon)\left(1+2 \epsilon^{2} \sqrt{\pi A}\right) \\
& <9(1+\epsilon)\left(1+2 \epsilon^{2} \sqrt{\pi A}\right)=3^{2} \sigma_{2}
\end{aligned}
$$

and, therefore

$$
M_{2}^{\prime \prime} \leqq 3^{2} \sigma_{2} A^{2} \epsilon^{3}
$$

(c). Estimate of $\left|\theta_{n}^{\prime}(\phi)-\theta^{\prime}(\phi)\right|$. Applying the theorem of $\S 4(\mathrm{c})$ with $g(\phi)$ $=\theta_{n}^{\prime}(\phi)-\theta^{\prime}(\phi)$, we obtain from (5.1) and (6.1)

$$
\left|\theta_{n}^{\prime}(\phi)-\theta^{\prime}(\phi)\right| \leqq \sqrt{2 \pi \sigma_{n}}(A(n+1))^{3 / 2} \epsilon^{n+1} .
$$

(d). Proof of (3.7). To estimate $\sigma_{n}$ we first note that

$$
\prod_{k=2}^{n}\left(1+\epsilon^{k} \sqrt{2 \pi A k}\right) \leqq \exp \left[\sqrt{2 \pi A} \sum_{k=2}^{n} \epsilon^{k} \sqrt{k}\right] .
$$

Now

$$
\sum_{k=2}^{n} \epsilon^{k} \sqrt{k}=\epsilon \sum_{k=2}^{n} \epsilon^{(k-1) / 2}\left(\sqrt{k} \epsilon^{(k-1) / 2}\right) \leqq \epsilon\left\{\sum_{k=2}^{n} \epsilon^{k-1} \sum_{k=2}^{n} k \epsilon^{k-1}\right\}^{1 / 2},
$$

by the inequality of Schwarz. Hence

$$
\sum_{k=2}^{n} \epsilon^{k} \sqrt{k} \leqq \epsilon\left\{\frac{\epsilon}{1-\epsilon} \cdot\left(\frac{1}{(1-\epsilon)^{2}}-1\right)\right\}^{1 / 2}<\frac{\epsilon^{2} \sqrt{2}}{(1-\epsilon)^{3 / 2}}
$$

We find therefore that $\sigma_{n}<(1+\epsilon) \exp \left[2 \sqrt{\pi A} \epsilon^{2}(1-\epsilon)^{-3 / 2}\right]$.
7. An integral representation for $\theta_{n}^{\prime}(\phi)$. We shall discuss now the conditions under which the images $C_{n}$ of the unit circle by means of the functions $w=f_{n}(z)=z e^{F_{n}(z)}$ are star-shaped. For this purpose we shall first establish the following representation for $\theta_{n}^{\prime}(\phi)$. If $C$ is a nearly circular contour and if the function $\rho(\theta)$ which represents $C$ satisfies hypothesis (3.2), then the derivative $\theta_{n}^{\prime}(\phi)$ of $\theta_{n}(\phi)$ is continuous and

$$
\begin{align*}
& \theta_{1}^{\prime}(\phi)-1=-\frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left\{\frac{\rho^{\prime}}{\rho}(t)-\frac{\rho^{\prime}}{\rho}(\phi)\right\} \cot \frac{t-\phi}{2} d t,  \tag{7.1}\\
& \theta_{n}^{\prime}(\phi)-1=-\frac{1}{2 \pi} \int_{-\phi-\pi}^{\phi+\pi}\left\{\frac{\rho^{\prime}}{\rho}\left[\theta_{n-1}(t)\right]-\frac{\rho^{\prime}}{\rho}\left[\theta_{n-1}(\phi)\right]\right\} \theta_{n-1}^{\prime}(t) \cot \frac{t-\phi}{2} d t \\
&-\frac{\rho^{\prime}}{\rho}\left[\theta_{n-1}(\phi)\right] \frac{\rho^{\prime}}{\rho}\left[\theta_{n-2}(\phi)\right] \theta_{n-2}^{\prime}(\phi) \quad(n \geqq 2) . \tag{7.2}
\end{align*}
$$

Proof. The integrand of (7.1) is continuous in both the variables $t, \phi$ except possibly for $t=\phi$, and is bounded because of (3.2). Hence the integral (7.1) is a continuous function of $\phi$. Since this integral represents the conjugate function of ( $\rho^{\prime} / \rho$ ) $[\phi]$ for which the integral over the interval $(0,2 \pi)$ is zero, it is equal to $\theta_{1}^{\prime}(\phi)-1$, at least for almost all $\phi$, and, because of the continuity, for all $\phi$. This proves (7.1).

Let us suppose it were proved that $\theta_{k}^{\prime}(\phi)$ is a continuous function when $k=1,2, \cdots, n(n \geqq 1)$. We then show that the formula (7.2) holds with $n$ replaced by $n+1$, and that $\theta_{n+1}^{\prime}(\phi)$ is continuous. This will then prove the representation (7.2) and the continuity of $\theta_{n}^{\prime}(\phi)$ for all $n$.

Since $F_{n+1}\left(e^{i \phi}\right)=\log \rho\left[\theta_{n}(\phi)\right]+i\left(\theta_{n+1}(\phi)-\phi\right)$ is absolutely continuous (see $\S 9(\mathrm{~b})$ ) it follows that $\theta_{n+1}^{\prime}(\phi)-1$ is conjugate to $\left(\rho^{\prime} / \rho\right)\left[\theta_{n}(\phi)\right] \theta_{n}^{\prime}(\phi)$, and we have, for almost all $\phi$,

$$
\theta_{n+1}^{\prime}(\phi)-1=-\frac{1}{2 \pi} \int_{0}^{\pi}\left\{\frac{\rho^{\prime}}{\rho}\left[\theta_{n}(\tau)\right] \theta_{n}^{\prime}(\tau)\right\}_{\tau=\phi-t}^{\tau=\phi+t} \cot \frac{t}{2} d t
$$

the integral being convergent in the sense that $\lim _{8=0} \int_{\delta}^{\pi}$ exists. We write

$$
\begin{aligned}
\theta_{n+1}^{\prime}(\phi)-1= & -\frac{1}{2 \pi} \int_{0}^{\pi}\left\{\frac{\rho^{\prime}}{\rho}\left[\theta_{n}(\phi+t)\right]-\frac{\rho^{\prime}}{\rho}\left[\theta_{n}(\phi)\right]\right\} \theta_{n}^{\prime}(\phi+t) \cot \frac{t}{2} d t \\
& +\frac{1}{2 \pi} \int_{0}^{\pi}\left\{\frac{\rho^{\prime}}{\rho}\left[\theta_{n}(\phi-t)\right]-\frac{\rho^{\prime}}{\rho}\left[\theta_{n}(\phi)\right]\right\} \theta_{n}^{\prime}(\phi-t) \cot \frac{t}{2} d t \\
& -\frac{\rho^{\prime}}{\rho}\left[\theta_{n}(\phi)\right] \frac{1}{2 \pi} \int_{0}^{\pi}\left\{\theta_{n}^{\prime}(\phi+t)-\theta_{n}^{\prime}(\phi-t)\right\} \cot \frac{t}{2} d t
\end{aligned}
$$

Because of (3.2) and the continuity of $\theta_{n}^{\prime}(\phi)$, the first two integrals represent continuous functions of $\phi$. The third integral (without the factor $\left.-\left(\rho^{\prime} / \rho\right)\left[\theta_{n}(\phi)\right]\right)$ is equal to ( $\left.\rho^{\prime} / \rho\right)\left[\theta_{n-1}^{\prime}(\phi)\right] \theta_{n-1}^{\prime}(\phi)$, since $\theta_{n}^{\prime}(\phi)-1$ is conjugate to this function. Introducing the variable $\tau=\phi+t$ in the first integral and $\tau=\phi-t$ in the second, we obtain

$$
\begin{aligned}
\theta_{n+1}^{\prime}(\phi)-1= & -\frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left\{\frac{\rho^{\prime}}{\rho}\left[\theta_{n}(\tau)\right]-\frac{\rho^{\prime}}{\rho}\left[\theta_{n}(\phi)\right]\right\} \theta_{n}^{\prime}(\tau) \cot \frac{t-\phi}{2} d \tau \\
& -\frac{\rho^{\prime}}{\rho}\left[\theta_{n}(\phi)\right] \frac{\rho^{\prime}}{\rho}\left[\theta_{n-1}(\phi)\right] \theta_{n-1}^{\prime}(\phi)
\end{aligned}
$$

The right-hand side of this equation represents a continuous function of $\phi$. Hence $\theta_{n+1}^{\prime}(\phi)$ may be defined as a continuous function for all $\phi$, and therefore $\theta_{n+1}(\phi)$ has a continuous derivative for all $\phi$. This completes the proof.
8. Conditions under which $C_{n}$ is star-shaped. Proof of IV. The curve $C_{n}$ is starshaped if $\theta_{n}^{\prime}(\phi) \geqq 0$. By (7.1) and (3.2)

$$
\begin{aligned}
\left|\theta_{1}(\phi)-1\right| & \leqq \frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left|\frac{\rho^{\prime}}{\rho}(t)-\frac{\rho^{\prime}}{\rho}(\phi)\right| \cot \left|\frac{t-\phi}{2}\right| d t \\
& \leqq \frac{\epsilon}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}(t-\phi) \cot \frac{t-\phi}{2} d t=2 \epsilon \log 2
\end{aligned}
$$

Thus, if $2 \epsilon \log 2 \leqq 1$ or $\epsilon \leqq(2 \log 2)^{-1}$, then $\theta_{1}^{\prime}(\phi) \geqq 0$ and $C_{1}$ is star-shaped.
Let us suppose it were proved that $\theta_{n}^{\prime}(\phi) \geqq 0$ for some $n \geqq 1$, provided $\epsilon$ does not exceed some value $\epsilon_{0}<1$. Then we examine $\theta_{n+1}^{\prime}(\phi)$. By (7.2), (3.2), and (2.2),

$$
\begin{equation*}
\left|\theta_{n+1}^{\prime}(\phi)-1\right| \leqq \frac{\epsilon}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left(\theta_{n}(t)-\theta_{n}(\phi)\right) \theta_{n}(t) \cot \frac{t-\phi}{2} d t+\epsilon^{2}\left|\theta_{n-1}^{\prime}(\phi)\right| \tag{8.1}
\end{equation*}
$$

It is to be noted that $\theta_{n}(t)-\theta_{n}(\phi)$ has the same sign as $t-\phi$ since $\theta_{n}^{\prime}(t) \geqq 0$. We find by integration by parts that

$$
\begin{aligned}
m_{n}^{2} & =\frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left(\theta_{n}(t)-\theta_{n}(\phi)\right) \theta_{n}^{\prime}(t) \cot \frac{t-\phi}{2} d t=\frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left(\frac{\theta_{n}(t)-\theta_{n}(\phi)}{2 \sin \frac{1}{2}(t-\phi)}\right)^{2} d t \\
& =\frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left(\frac{\theta_{n}(t)-t-\left[\theta_{n}(\phi)-\phi\right]+t-\phi}{2 \sin \frac{1}{2}(t-\phi)}\right)^{2} d t .
\end{aligned}
$$

Hence by Minkowski's inequality,

$$
\begin{gathered}
m_{n} \leqq\left\{\frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left(\frac{\theta_{n}(t)-t-\left[\theta_{n}(\phi)-\phi\right]}{2 \sin \frac{1}{2}(t-\phi)}\right)^{2} d i\right\}^{1 / 2} \\
+\left\{\frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left(\frac{t-\phi}{2 \sin \frac{1}{2}(t-\phi)}\right)^{2} d t\right\}^{1 / 2}
\end{gathered}
$$

Integrating by parts, we find that

$$
\frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left(\frac{t-\phi}{2 \sin \frac{1}{2}(t-\phi)}\right)^{2} d t=\frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}(t-\phi) \cot \frac{t-\phi}{2} d t=2 \log 2=c^{2}
$$

Furthermore, by the theorem of $\S 9(f)$,

$$
\frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left(\frac{\theta_{n}(t)-t-\left[\theta_{n}(\phi)-\phi\right]}{2 \sin \frac{1}{2}(t-\phi)}\right)^{2} d t=\frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left(\frac{\log \rho\left[\theta_{n-1}(t)\right]-\log \rho\left[\theta_{n-1}(\phi)\right]}{2 \sin \frac{1}{2}(t-\phi)}\right)^{2} d t
$$

By (2 2), the right-hand side of this equation is

$$
\leqq \epsilon^{2} \frac{1}{2 \pi} \int_{\phi-\pi}^{\phi+\pi}\left(\frac{\theta_{n-1}(t)-\theta_{n-1}(\phi)}{2 \sin \frac{1}{2}(t-\phi)}\right)^{2} d t=\epsilon^{2} m_{n-1}^{2}
$$

Hence

$$
m_{n} \leqq \epsilon m_{n-1}+c .
$$

Since $m_{0}=c$, we have

$$
m_{n} \leqq c\left(1+\epsilon+\epsilon^{2}+\cdots+\epsilon^{n}\right)=c \frac{1-\epsilon^{n+1}}{1-\epsilon}
$$

Hence, by (8.1),

$$
\begin{equation*}
\left|\theta_{n+1}^{\prime}(\phi)-1\right| \leqq \epsilon m_{n}^{2}+\epsilon^{2}\left|\theta_{n-1}^{\prime}(\phi)\right| \leqq 2 \epsilon\left[\frac{1-\epsilon^{n+1}}{1-\epsilon}\right]^{2} \log 2+\epsilon^{2}\left|\theta_{n-1}^{\prime}(\phi)\right| \tag{8.2}
\end{equation*}
$$

Applying (8.2) with $n=1$, we find since $\theta_{0}^{\prime}(l)=1$ that,

$$
\begin{equation*}
\left|\theta_{2}^{\prime}(\phi)-1\right| \leqq 2 \epsilon(1+\epsilon)^{2} \log 2+\epsilon^{2} \tag{8.3}
\end{equation*}
$$

and this will be less than 1 if $\epsilon \leqq 0.34$.
For $n=2$ we find, since $\theta_{1}^{\prime}(\phi) \leqq 1+2 \epsilon \log 2$ and $\theta_{1}^{\prime}(\phi)>0$ for $\epsilon<(2 \log 2)^{-1}$, that

$$
\left|\theta_{3}^{\prime}(\phi)-1\right| \leqq 2 \epsilon\left(1+\epsilon+\epsilon^{2}\right)^{2} \log 2+\epsilon(1+2 \epsilon \log 2) .
$$

This expression will be less than 1 , if $\epsilon \leqq 0.31\left(<(2 \log 2)^{-1}\right)$.
By (8.3), $\left|\theta_{2}^{\prime}(\phi)\right| \leqq 1.7927$ if $\epsilon=0.30$. Hence, applying (8.2) for $n=3$ and using this estimate for $\theta_{2}^{\prime}(\phi)$, we find that $\left|\theta_{4}^{\prime}(\phi)-1\right| \leqq 1$, if $\epsilon \leqq 0.3$.

Assuming that, for some $n \geqq 1,0<\theta_{n-1}^{\prime}(\phi) \leqq 2$, we see from (8.2) that

$$
\left|\theta_{n+1}^{\prime}(\phi)-1\right| \leqq \frac{2 \log 2}{(1-\epsilon)^{2}} \epsilon+2 \epsilon^{2}<1
$$

if $\epsilon \leqq 0.295$. Since for $\epsilon \leqq 0.295$ this assumption is certainly satisfied for $n=1$ and $n=2$, it follows that for all $n\left|\theta_{n+1}^{\prime}(\phi)-1\right|<1$ if $\epsilon \leqq 0.295$.

Remark. For large values of $n$ the bound for $\epsilon$ can be improved by use of Theorem IV and the left hand inequality in (9.3).

By Theorem IV, $\left|\theta_{n}^{\prime}:(\phi)-\theta^{\prime}(\phi)\right| \leqq \sqrt{2 \pi \sigma_{n}}(A(n+1))^{3 / 2} \epsilon^{n+1}$, and by (9.3), $\theta^{\prime}(\phi)$ $\geqq A^{-1}\left(1+\epsilon^{2}\right)^{-1 / 2}$. For any given fixed $n, \epsilon_{0}$ can be chosen so that $\sqrt{2 \pi \sigma_{n}}(A(n+1))^{3 / 2} \epsilon_{0}^{n+1}$ $<A^{-1}\left(1+\epsilon_{0}^{2}\right)^{-1 / 2}$. Then, for all $\epsilon \leqq \epsilon_{0}, \theta_{n}^{\prime}(\phi) \geqq \theta^{\prime}(\phi)-A^{-1}\left(1+\epsilon^{2}\right)^{-1 / 2} \geqq 0$.
9. Auxiliary theorems. This section contains the proofs of some of the auxiliary results cited in the text.
(a). Uniqueness of the solution of Theodorsen's Integral Equation. If $C$ is a nearly circular contour, the integral equation (2.5) has at most one continuous solution.

Let us suppose that it had two such solutions, $\theta_{1}(\phi)$ and $\theta_{2}(\phi)$. Since

$$
\int_{0}^{2 \pi}\left(\theta_{1}(\phi)-\phi\right) d \phi=0, \quad \int_{0}^{2 \pi}\left(\theta_{2}(\phi)-\phi\right) d \phi=0,
$$

it follows by use of the theorem cited in §4(a) that

$$
\left.M^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\theta_{1}(\phi)-\theta_{2}(\phi)\right)^{2} d \phi \leqq \frac{1}{2 \pi} \int_{0}^{2 \pi}\left\{\log \rho\left[\theta_{1}(\phi)\right]-\log \rho\left[\theta_{2}(\phi)\right)\right]\right\}^{2} d \phi .
$$

By (2.2),

$$
\left|\log \rho\left[\theta_{1}(\phi)\right]-\log \rho\left[\theta_{2}(\phi)\right]\right| \leqq \epsilon\left|\theta_{1}(\phi)-\theta_{2}(\phi)\right|,
$$

so that we have $M^{2} \leqq \epsilon^{2} M^{2}$. Since $0<\epsilon<1, M=0$ and hence $\theta_{1}(\phi) \equiv \theta_{2}(\phi)$.
(b). A property of the functions $\theta_{n}(\phi)$. If $C$ is a nearly circular contour, then the functions $\theta_{n}(\phi)$ defined by (2.6) are absolutely continuous, and $\left(\theta_{n}^{\prime}(\phi)\right)^{2}$ are integrable (in the sense of Lebesgue) for $0 \leqq \phi \leqq 2 \pi$.

This is clearly true when $n=0$. We suppose that this statement were proved for some $n \geqq 0$. Since $\log \rho(\theta)$ has bounded difference quotients (by (2.2)) and $\theta_{n}(\phi)$ is absolutely continuous, it follows that $\log \rho\left[\theta_{n}(\phi)\right]$ also is absolutely continuous. Furthermore, because of the inequality

$$
\left(\frac{\rho^{\prime}}{\rho}\left[\theta_{n}(\phi)\right] \theta_{n}^{\prime}(\phi)\right)^{2} \leqq \epsilon^{2}\left(\theta_{n}^{\prime}(\phi)\right)^{2},
$$

it follows that the integral

$$
\int_{0}^{2 \pi}\left(\frac{\rho^{\prime}}{\rho}\left[\theta_{n}(\phi)\right] \theta_{n}^{\prime}(\phi)\right)^{2} d \phi
$$

exists. Hence, the conjugate function of $\log \rho\left[\theta_{n}(\phi)\right]$, namely $\theta_{n+1}(\phi)-\phi$, exists and is absolutely continuous and the integral $\int_{0}^{2 \pi}\left(\theta_{n+1}^{\prime}(\phi)-1\right)^{2} d \phi$ exists. ${ }^{9}$
(c). A property of $\theta(\phi)$. If $C$ is a nearly circular contour, then $\theta(\phi)=\arg f\left(e^{i \phi}\right)$ (de-

[^7]fined by (2.4)) is absolutely continuous and $\left(\theta^{\prime}(\phi)\right)^{2}$ is integrable (in the sense of Lebesgue) and
\[

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{0}^{2 \pi} \theta^{\prime 2}(\phi) d \phi \leqq \frac{1}{1-\epsilon^{2}} \tag{9.1}
\end{equation*}
$$

\]

Proof. Since the curve $C$ is rectifiable, the function $F\left(e^{i \phi}\right)$ is absolutely continuous. ${ }^{10}$ Hence

$$
\frac{d}{d \phi} F\left(e^{i \phi}\right)-i=\frac{\rho^{\prime}}{\rho}[\theta(\phi)] \theta^{\prime}(\phi)+i \theta^{\prime}(\phi)
$$

exists almost everywhere for $0 \leqq \phi \leqq 2 \pi$, and is integrable. Furthermore, the function $\partial[F(z)] / \partial \phi-i=u(z)+i v(z), z=r e^{i \phi}$, may be represented by the Poisson Integral in the unit circle,

$$
\begin{equation*}
u(z)+i v(z)=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left\{u\left(e^{i t}\right)+i v\left(e^{i t}\right)\right\} \frac{1-r^{2}}{1+r^{2}-2 r \cos (t-\phi)} d t \tag{9.2}
\end{equation*}
$$

For almost all $\phi(0 \leqq \phi \leqq 2 \pi)$,

$$
\lim _{r \rightarrow 1} u\left(r e^{i \phi}\right)=\frac{\rho^{\prime}}{\rho}[\theta(\phi)] \theta^{\prime}(\phi)=u\left(e^{i \phi}\right), \quad \lim _{r \rightarrow 1} v\left(r e^{i \phi}\right)=\theta^{\prime}(\phi)=v\left(e^{i \phi}\right)
$$

Since $C$ is star-shaped, $\theta^{\prime}(\phi) \geqq 0$, and we have by (2.2),

$$
v\left(e^{i \phi}\right) \pm u\left(e^{i \phi}\right) \geqq \theta^{\prime}(\phi)(1-\epsilon) \geqq 0
$$

Because of the representation (9.2) we conclude that $v(z)+u(z) \geqq 0$ and $v(z)-u(z) \geqq 0$ for $|z|<1$. Hence $v^{2}(z)-u^{2}(z) \geqq 0$ for $|z|<1$. Now

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(v^{2}\left(r e^{i \phi}\right)-u^{2}\left(r e^{i \phi}\right)\right) d \phi=1
$$

Hence, taking the limit as $r \rightarrow 1$, we obtain by Fatou's lemma

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi} \theta^{\prime 2}(\phi)\left[1-\left(\frac{\rho^{\prime}}{\rho}[\theta(\phi)]\right)^{2}\right] d \phi \leqq 1
$$

and by (2.2),

$$
\frac{11}{2 \pi} \int_{0}^{2 \pi}\left(\theta^{\prime}(\phi)\right)^{2} d \phi \leqq \frac{1}{1-\epsilon^{2}}
$$

This proves that $\left(\theta^{\prime}(\phi)\right)^{2}$ is integrable and that (9.1) holds.
(d). AN ESTIMATE FOR $\theta^{\prime}(\phi)$ and $\theta^{\prime \prime}(\phi)$. If $C$ is a nearly circular contour and if in addition (3.2) is satisfied, then

$$
\begin{gather*}
\frac{1}{A \sqrt{1+\epsilon^{2}}} \leqq \theta^{\prime}(\phi) \leqq A=4^{\epsilon} e^{\epsilon^{2}}  \tag{9.3}\\
\left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left\{\theta^{\prime \prime}(\phi)\right\}^{2} d \phi\right\}^{1 / 2} \leqq A^{3 / 2} \epsilon \min (1+\epsilon ; \sqrt{2}) \tag{9.4}
\end{gather*}
$$

[^8]The proofs of these inequalities are contained in a paper to be published elsewhere.
(e). A PROPERTY OF The FUNCTIONS $F_{n}(\phi)$. If $C$ is a nearly circular contour for which (3.2) and (3.4) are satisfied, then the functions $F_{n}\left(e^{i \phi}\right)$ have continuous second derivatives which satisfy a Hölder condition with any fixed exponent $\alpha, 0<\alpha<1$.

The proof may easily be given by induction. Since $\log \rho(\phi)$ and $\theta_{1}(\phi)-\phi$ are conjugate functions and since the second derivative of $\log \rho(\phi)$ satisfies the Lipschitz condition (3.4), it follows from a theorem of I. Privaloff, ${ }^{11}$ that $\theta_{1}^{\prime}(\phi)$ and $\theta_{1}^{\prime \prime}(\phi)$ exist and that $\theta_{1}^{\prime \prime}(\phi)$ satisfies a Hölder condition with any fixed exponent $\alpha, 0<\alpha<1$. Let us suppose now, that it had been shown that $\theta_{n}^{\prime \prime}(\phi)$ exists and satisfies a Hölder condition with any fixed exponent $\alpha, 0<\alpha<1$. Then $\log \rho\left[\theta_{n}(\phi)\right]$ has continuous first and second derivatives, $\left(\rho^{\prime} / \rho\right)\left[\theta_{n}(\phi)\right] \theta_{n}^{\prime}(\phi)$ and $p\left[\theta_{n}(\phi)\right] \theta_{n}^{\prime \prime 2}(\phi)+\left(\rho^{\prime} / \rho\right)\left[\theta_{n}(\phi)\right] \theta_{n}^{\prime \prime}(\phi)$, respectively, and the latter satisfies a Hollder condition with any exponent $\alpha, 0<\alpha<1$, (because of (3.4) and (3.2)). Hence, again by Privaloff's theorem, the conjugate function $\theta_{n+1}(\phi)-\phi$ possesses a second derivative $\theta_{n+1}^{\prime \prime}(\phi)$ which satisfies such a Hölder condition. This completes the proof.
(f). A Theorem on conjugate functions. Let us suppose that $u(t)$ is a periodic function (period $2 \pi$ ) possessing a continuous derivative for $0 \leqq t \leqq 2 \pi$. Let $v(t)$ be conjugate to $u(t)$ and let us suppose that $v(t)$ also possesses a continuous derivative for $0 \leqq t \leqq 2 \pi$. Then for every $\theta$,

$$
\int_{0}^{2 \pi}\left(\frac{u(l)-u(\theta)}{\sin \frac{1}{2}(t-\theta)}\right)^{2} d t=\int_{0}^{2 \pi}\left(\frac{v(t)-v(\theta)}{\sin \frac{1}{2}(t-\theta)}\right)^{2} d t
$$

Proof. Let $G(z)=U(z)+i V(z)$ denote the function which is analytic for $|z|<1$ and assumes the boundary values $g(t) \equiv u(t)+i v(t)-(u(\theta)+i v(\theta))$ for $z=e^{i t}$. Then the real part of $[G(z)]^{2}$ may be represented by the Poisson integral ( $z=r e^{i \phi}$ )

$$
[U(z)]^{2}-[V(z)]^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left\{(u(l)-u(\theta))^{2}-(v(t)-v(\theta))^{2}\right\} \frac{1-r^{2}}{1+r^{2}-2 r \cos (t-\phi)} d t
$$

For $z=r e^{i \theta}$,

$$
\frac{[U(z)]^{2}-[V(z)]^{2}}{1-r^{2}}=\frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{(u(t)-u(\theta))^{2}-(v(t)-v(\theta))^{2}}{(1-r)^{2}+4 r \sin ^{2} \frac{1}{2}(t-\theta)} d t
$$

As is easily seen, the limit of this integral as $r \rightarrow 1$ is

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{(u(t)-u(\theta))^{2}-(v(t)-v(\theta))^{2}}{4 \sin ^{2} \frac{1}{2}(t-\theta)} d t . \tag{9.5}
\end{equation*}
$$

By the mean value theorem (since $U\left(e^{i \theta}\right)=V\left(e^{i \theta}\right)=0$ )

[^9]\[

$$
\begin{align*}
-\frac{[U(z)]^{2}-[V(z)]^{2}}{1-r} & =\frac{\partial}{\partial \rho}\left[U^{2}\left(\rho \epsilon^{i \theta}\right)-V^{2}\left(\rho e^{i \theta}\right)\right]_{\rho=\bar{r}} \\
& =2\left\{U\left(\rho e^{i \theta}\right) \frac{\partial}{\partial \rho} U\left(\rho e^{i \theta}\right)-V\left(\rho e^{i \theta}\right) \frac{\partial}{\partial \rho} V\left(\rho e^{i \theta}\right)\right\}_{\rho=\bar{r}},(r<\bar{r}<1) \tag{9.5}
\end{align*}
$$
\]

Since $g^{\prime}(t)$ exists and is continuous,

$$
\frac{\partial U}{\partial \rho}+i \frac{\partial V}{\partial \rho}=e^{i \theta} G^{\prime}\left(\rho e^{i \theta}\right) \rightarrow-i g^{\prime}(\theta)
$$

as $\rho \rightarrow 1$. Thus $\lim _{\rho \rightarrow 1} \partial\left[U\left(\rho e^{i \theta}\right)\right] / \partial \rho$ and $\lim _{\rho \rightarrow 1} \partial\left[V\left(\rho e^{i \theta}\right)\right] / \partial \rho$ exist. Furthermore, $\lim _{\rho \rightarrow 1} U\left(\rho e^{i \theta}\right)=\lim _{p \rightarrow 1} V\left(\rho e^{i \theta}\right)=0$. Hence, the limit as $r \rightarrow 1$ of (9.6) is zero and therefore the integral (9.5) is zero. This proves the theorem.

# ON ROTATIONAL GAS FLOWS* 

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Introduction. The main body of the science of aerodynamics is based on the classical theory of frictionless, incompressible, irrotational fluids. Recently airplanes have attained such high velocities that this fluid model has proved to be too restricted and interest has centered on the irrotational motion of frictionless, compressible fluids. By the term "compressible fluid" one generally means a fluid for which the density $\rho$ and pressure $p$ are connected by the isentropic relation $p \rho^{-\gamma}=$ const. However, the student of aerodynamics is frequently interested in supersonic phenomena and because of the possible occurrence of shock waves, such flows cannot be described, in general, by isentropic, irrotational flows. Accordingly, it becomes necessary to study the motion of gases under less restricted conditions.

Let us call a fluid barotropic when there is a unique functional relationship between the pressure and the density of the fluid. The most important examples are the incompressible fluid where the same constant density belongs to each pressure and the isentropic fluid where the relation $p \rho^{-\gamma}=$ const. holds. The dynamics of frictionless barotropic fluids is based on a theorem due to Lagrange. If a fluid particle is irrotational al one moment, it will remain so for all subsequent time. One can generally assume in aerodynamics that the air starts from rest. The dynamics of the flow can then be summed up in the single statement that the motion is irrotational. It follows that the velocity distribution admits a potential, and the comparative mathematical simplicity of the dynamics of frictionless barotropic fluids follows from this fact.

Classical fluid dynamics deals almost exclusively with the theory of frictionless barotropic fluids. To find an example of frictionless non-barotropic fluids, we turn to the theory of the propagation of waves. When Newton developed his theory of soundwaves, he assumed that the motion of air was isothermal. Later his theory was superseded by a better one which assumes isentropic motion. Thus, both theories assumed that the transmitting medium was barotropic. The mathematical theory of onedimensional large disturbances, a much more difficult problem, was developed first by Riemann, who again assumed that the flow is isentropic. However, the isentropic theory of shock-waves turns out to be fallacious because it can be shown to violate the law of conservation of energy. When shock-waves are considered the fluid model must be extended to include non-barotropic fluids.

In this connection, let us draw attention to the thermodynamical aspect of the general theory of compressible fluids. In the case of a three-dimensional flow there are six unknowns: three velocity components, pressure, density and temperature. The laws of conservation of matter and momentum together with the equation of state yield only five equations. To get the missing sixth equation the law of conservation of energy, i.e., the first law of thermodynamics, must be used. Flows will be isentropic only when as a consequence of these laws the entropy turns out to be a constant.

[^10]Although the theory of one-dimensional shock-waves requires a non-barotropic fluid model, this fluid model is a very special one. Even if there is an increase of entropy across shock-waves, the flow remains isentropic between shock-waves. Moreover, a one-dimensional fluid motion is always irrotational But when one turns to two or three-dimensional shock-waves, the situation becomes quite different. In this case Hadamard ${ }^{1}$ was the first to point out in 1903, that vortices are generated suddenly by shock-waves and, in general, the flow becomes non-barotropic after shock-waves.

Hadamard determined the sudden change of circulation across a shock-wave but was not interested in the circulation variations occurring in the fluid behind shockwaves. A general circulation theorem for frictionless barotropic fluids was established by Bjerknes ${ }^{2}$ in 1900, for the purposes of his dynamical theory of meteorology. The motion of air masses originating from non-homogeneous conditions is clearly a phenomenon requiring a non-barotropic fluid model.

Crocco, ${ }^{3}$ in 1937, again took up the question of the motion of frictionless fluids behind shock-waves. By restricting himself to the steady state he discovered a very useful theorem. Recently, this theorem was generalized by the author of the present paper. ${ }^{4}$

So far, we have spoken only about frictionless fluids. There are problems with respect to the flow of gases where viscosity cannot be neglected. We mention, for instance, the boundary layer theory and the behavior of a gas within shock-waves. It appears probable that when considering the viscous flow of gases, the conductivity of the gas cannot be neglected in general. Variations in viscosity might have importance also. No general theorems are available for such flows and we shall have to be content with presenting the fundamental differential equations governing these phenomena. Any investigation with respect to the flow of gases must be based on these equations. While in the case of frictionless flows some general consequences of the fundamental equations are available, in the case of viscous flows we must start the investigation of each problem by examining the fundamental equations anew.

Lagrange's theorem plays a fundamental role in our concepts about fluid dynamics. Its validity is restricted, however. The art of acronautics is now at a point where we have to extend our fluid model and thus modify some of our basic concepts. We must accept for instance the fact that vortices can be generated in the midst of a frictionless fluid. Whether this extended fluid model will be able to account for all the phenomena which we may wish to consider, only the future can tell.

## I. THE FUNDAMENTAL EQUATIONS

1. Continuity equation. From the law of conservation of matter it can be proved that

$$
\operatorname{div} \rho q=-\frac{\partial \rho}{\partial \iota}
$$

[^11]where $q$ is the velocity vector. Another useful form of the continuity equation is given by
\[

$$
\begin{equation*}
\operatorname{div} \mathrm{q}=\frac{1}{\rho} \frac{d \rho}{d t} \tag{C}
\end{equation*}
$$

\]

2. The Navier-Stokes equation. From the law of conservation of momentum it can be proved that the equation of motion is given by

$$
\begin{equation*}
\frac{d \mathrm{q}}{d t}=-\frac{1}{\rho} \operatorname{grad} p+\frac{\mu}{\rho} \Delta \mathrm{q}+\frac{\mu}{3 \rho} \operatorname{grad} \operatorname{div} \mathrm{q} \tag{M}
\end{equation*}
$$

where it is assumed that the viscosity $\mu$ is constant.
It will be useful to derive certain other forms of this equation. The specific enthalpy $h$ of a gas is defined by

$$
\begin{equation*}
h=U+p \rho^{-1} \tag{2.1}
\end{equation*}
$$

where $U$ denotes the specific internal energy. The specific entropy $s$ is defined by

$$
\begin{equation*}
T d s=d U+p d\left(\rho^{-1}\right) \tag{2.2}
\end{equation*}
$$

where $T$ denotes the absolute temperature. From Eqs. (2.1) and (2.2) it follows that

$$
\begin{equation*}
T d s=d h-\rho^{-1} d p \tag{2.3}
\end{equation*}
$$

Using vector notation and considering only spacial variations, we may then write

$$
T \operatorname{grad} s=\operatorname{grad} h-\rho^{-1} \operatorname{grad} p
$$

From this last equation and the equation of motion we find that

$$
\frac{d \mathrm{q}}{d t}=T \operatorname{grad} s-\operatorname{grad} h+\frac{\mu}{\rho} \Delta \mathrm{q}+\frac{\mu}{3 \rho} \operatorname{grad} \operatorname{div} \mathrm{q}
$$

Another useful form of the equation of motion can be obtained by using the stagnation enthalpy

$$
\begin{equation*}
h_{0}=h+\frac{1}{2} q^{2} \tag{2.4}
\end{equation*}
$$

and the identity

$$
\begin{equation*}
\frac{d q}{d t}=\frac{\partial q}{\partial t}+\operatorname{grad}\left(\frac{1}{2} q^{2}\right)-q \times \omega \quad(\omega=\operatorname{curl} q) \tag{2.5}
\end{equation*}
$$

together with the equation of motion ( $\mathrm{M}^{\prime}$ ). Thus one obtains

$$
\frac{\partial \mathrm{q}}{\partial t}-\mathrm{q} \times \omega=-\operatorname{grad} h_{0}+T \operatorname{grad} s+\frac{\mu}{\rho} \Delta \mathrm{q}+\frac{\mu}{3 \rho} \operatorname{grad} \operatorname{div} \mathrm{q}
$$

A fourth useful form of the equation of motion can be obtained by introducing the rate of change of the stagnation enthalpy. Differentiating Eq. (2.4) with respect to $t$, we obtain

$$
\begin{equation*}
\frac{d h_{0}}{d t}=\frac{d h}{d t}+\frac{1}{2} \frac{d q^{2}}{d t} \tag{2.6}
\end{equation*}
$$

From Eq. (2.3) it then follows that

$$
\begin{equation*}
\frac{d h_{0}}{d t}=T \frac{d s}{d t}+\frac{1}{\rho} \frac{d p}{d t}+\mathrm{q} \cdot \frac{d \mathrm{q}}{d t}=T \frac{d s}{d t}+\frac{1}{\rho} \frac{\partial p}{\partial t}+\frac{1}{\rho} \mathrm{q} \cdot \operatorname{grad} p+\mathrm{q} \cdot \frac{d \mathrm{q}}{d t} \tag{2.7}
\end{equation*}
$$

Using the continuity equation $\left(\mathrm{C}^{\prime}\right)$, we may write this as follows:

$$
\begin{equation*}
\frac{d h_{0}}{d t}=T \frac{d s}{d t}+\frac{1}{\rho} \frac{\partial p}{\partial t}+q \cdot\left(\frac{1}{\rho} \operatorname{grad} p+\frac{d q}{d t}\right) \tag{2.8}
\end{equation*}
$$

Introducing the last expression into the equation of motion ( $\mathrm{M}^{\prime}$ ), we finally obtain

$$
\frac{d h_{0}}{d t}=T \frac{d s}{d t}+\frac{1}{\rho} \frac{\partial p}{\partial \iota}+\frac{\mu}{\rho} q \cdot\left(\Delta q+\frac{1}{3} \operatorname{grad} \operatorname{div} q\right)
$$

3. The energy equation. From the law of conservation of energy, it can be shown ${ }^{5}$ that

$$
\begin{equation*}
\frac{d U}{d t}+p \frac{d\left(\rho^{-1}\right)}{d t}=\frac{1}{\rho} \phi+Q . \tag{E}
\end{equation*}
$$

The first term on the left-hand side accounts for the rate of change of internal energy; the second term stands for the work required to compress the fluid. The first term on the right-hand side represents the heat generated by the viscous forces. The dissipation function $\phi$ is defined by

$$
\begin{align*}
\phi=\mu\left[2\left(\frac{\partial u}{\partial x}\right)^{2}+2\left(\frac{\partial v}{\partial y}\right)^{2}\right. & +2\left(\frac{\partial w}{\partial z}\right)^{2}+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^{2} \\
& \left.+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^{2}+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^{2}-\frac{2}{3}(\operatorname{div} q)^{2}\right] . \tag{3.1}
\end{align*}
$$

Finally, the last term on the right-hand side accounts for the heat added to the fluid per unit of time per unit of mass. For instance, when all the heat transfer is due to the conductivity of the fluid (no external heat sources, no radiation), $Q$ is given by

$$
\begin{equation*}
Q=\rho^{-1} \operatorname{div}(k \operatorname{grad} T) \tag{3.2}
\end{equation*}
$$

The energy equation can be simplified by introducing the entropy from Eq. (2. 3); thus we have

$$
T \frac{d s}{d t}=Q+\frac{1}{\rho} \phi
$$

The interpretation of this last equation is particularly simple. The right-hand side represents the total heat increase of the fluid, while the left-hand side gives the corresponding product of the temperature by the entropy change. By combining the cquation of motion ( $M^{\prime \prime \prime}$ ) with the energy equation $\left(E^{\prime}\right)$, we obtain the following important relation,

$$
\frac{d h_{0}}{d t}=\frac{1}{\rho} \frac{\partial p}{\partial t}+Q+\frac{1}{\rho} \phi+\frac{\mu}{\rho} q \cdot\left(\Delta q+\frac{1}{3} \operatorname{grad} \operatorname{div} q\right)
$$

In the literature, the energy equation is frequently given in this last form.

[^12]
## II. VORTEX THEOREMS FOR FRICTIONLESS FLUIDS

4. The circulation theorems. The circulation is defined by

$$
\begin{equation*}
\Gamma=\oint \mathrm{q} \cdot d \mathrm{l} \tag{4.1}
\end{equation*}
$$

where the integration is to be taken along a closed curve formed by fluid particles. It is easy to show that the rate of change of the circulation is given by

$$
\begin{equation*}
\frac{d \Gamma}{d t}=\oint \frac{d \mathrm{q}}{d t} \cdot d \mathrm{l} \tag{4.2}
\end{equation*}
$$

Let us combine the last relation with the equation of motion (M). If the viscous terms are omitted, we have

$$
\begin{equation*}
\frac{d \Gamma}{d t}=-\oint \frac{1}{\rho} \operatorname{grad} p \cdot d \mathbf{l}=-\oint \frac{d p}{p} . \tag{4.3}
\end{equation*}
$$

By means of the identity

$$
\begin{equation*}
0=\oint d\left(p^{-1} p\right)=\oint p d\left(p^{-1}\right)+\oint_{p^{-1} d p,} \tag{4.4}
\end{equation*}
$$

Eq. (4.3) can be transformed into

Instead of using Eq. (M), we can use Eq. (M') and thus obtain

$$
\frac{d \Gamma}{d t}=\oint T(\mathrm{grad} s) \cdot d \mathrm{l}=\oint T d s
$$

Sometimes it is preferable to transform the line integrals into surface integrals with the aid of Stokes' theorem. Thus it follows from the last two equations that

$$
\begin{align*}
& \left.\frac{d \Gamma}{d t}=-\oint \oint\left[\operatorname{grad} \rho^{-1}\right) \times \operatorname{grad} p\right] d A  \tag{4.5}\\
& \frac{d \Gamma}{d t}=\oint \oint[(\operatorname{grad} T) \times \operatorname{grad} s] d A
\end{align*}
$$

We now come to the interpretation of the equations for $d \Gamma / d t$. When the fluid is barotropic, the right-hand side is zero in all of these equations, and the theorem of Lord Kelvin is then obtained. The circulation along a closed "fuid line" in a barotropic fluid is constant for all time. In particular, when the circulation is zero at a certain instant, it will remain so for all subsequent time. By applying Kelvin's theorem to an indefinitely small closed line, Lagrange's theorem is obtained.

In the case of non-barotropic fluids, the situation is quite different. The right-hand sides in the equations are not zero in general and Kelvin's theorem does not hold. Bjerknes ${ }^{2}$ gave a simple geometrical interpretation of Eq. (4.5). Let us draw equidistant members of the families of surfaces $p=$ const. and $\rho^{-1}=$ const. and so obtain
a series of tubes bounded by these surfaces. The theorem of Bjerknes states that the rate of change of circulation per unit of time along a fuid line $C$ is proportional to the number of tubes surrounded by $C$. (In the case of a barotropic fluid the surfaces $p=$ const. and $\rho^{-1}=$ const. are identical.)

A very similar interpretation can be given to Eq. (4.5') by considering tubes formed by the families of surfaces $T=$ const. and $s=$ const. In the case of a barotropic fluid, these two families of surfaces are identical, unless the flow is isentropic in which case the surfaces $s=$ const. are no longer defined. Bjerknes' theorem, in this modified form, will be useful in a later part of this paper.
5. Theorems with respect to the rotation. Helmholtz's theorem. With the aid of Eqs. (2.5) and ( $\mathrm{C}^{\prime}$ ) it can be easily proved that

$$
\begin{equation*}
\operatorname{curl} \frac{d \mathrm{q}}{d t}=\frac{d\left(\rho^{-1} \omega\right)}{d t}-(\omega \nabla) \cdot \mathrm{q} \tag{5.1}
\end{equation*}
$$

Applying the operator curl to both sides of the equation of motion (M) or ( $\mathrm{M}^{\prime}$ ), we obtain in the frictionless case

$$
\begin{equation*}
\frac{d\left(\rho^{-1} \omega\right)}{d t}-(\omega \nabla) \cdot q=-\operatorname{grad} \rho^{-1} \times \operatorname{grad} p \tag{5.2}
\end{equation*}
$$

or

$$
\frac{d\left(\rho^{-1} \omega\right)}{d t}-(\omega \nabla) \cdot q=\operatorname{grad} T \times \operatorname{grad} s
$$

In the case of barotropic fluids, the right-hand side equals zero. (For two-dimensional flows the second term on the left-hand side equals zero because $\omega$ is everywhere normal to q .) Thus, for barotropic fluids,

$$
\frac{d\left(\rho^{-1} \omega\right)}{d t}=\left(\rho^{-1} \omega \nabla\right) \cdot \mathbf{q}
$$

A geometrical interpretation of the last equation led Helmholtz to the discovery of his famous vortex theorems. Vortex lines are material lines. The product of the crosssectional area and of the vorticity $\omega$ of a vortex filament is constant both in space and time.* (Holmholtz unnecessarily restricted his investigations to incompressible fluids.) In the case of non-barotropic fluids, ( $5.2^{\prime \prime}$ ) must be replaced by the more general Eq. (5.2) and the Helmholtz vortex theorems do not hold any more. Friedman ${ }^{6}$ derived certain theorems for non-barotropic fluids which are somewhat analogous to the Helmholtz theorems.
6. The theorem of Crocco and its generalization. In the case of steady, frictionless flows the equation of motion ( $\mathrm{M}^{\prime \prime}$ ) simplifies to the important relation

$$
\begin{equation*}
\mathrm{q} \times \omega=\operatorname{grad} h_{0}-T \operatorname{grad} s \tag{6.1}
\end{equation*}
$$

We will see later that for a very important type of flow $h_{0}$ is constant throughout the field. In this case Eq. (6.1) reduces to

[^13]$$
q \times \omega=-T \operatorname{grad} s
$$

This last relation was discovered by Crocco. ${ }^{3}$ When both $h_{0}$ and $s$ are constant the right-hand side of Eq. (6.1) is zero and so the motion must be irrotational. The importance of Eq. (6.1) lies in the fact that it relates the rotation of the fluid to the rates of change of $h_{0}$ and $s$.

## III. ADIABATIC, STEADY, FRICTIONLESS FLOWS

7. General relations. For the flows considered in this chapter the energy equations ( $E^{\prime}$ ) and ( $E^{\prime \prime}$ ) reduce to

$$
\begin{equation*}
\frac{\partial s}{\partial \sigma}=0, \quad(7.1) \quad ; \quad \frac{\partial h_{0}}{\partial \sigma}=0 \tag{7.2}
\end{equation*}
$$

where $\partial / \partial \sigma$ indicates differentiation along a streamline. Accordingly, both the entropy and the stagnation enthalpy are constant along each streamline (but they might vary from one streamline to another). Because of its great importance, we shall write out the integral of Eq. (7.2) in detail for a perfect gas with constant specific heats. One obtains

$$
\begin{equation*}
h_{0}=\frac{1}{2} q^{2}+h=\frac{1}{2} q^{2}+c_{p} T=\frac{1}{2} q^{2}+\frac{c_{p}}{R} \frac{p}{p}=\text { const. along a streamline } \tag{7.2}
\end{equation*}
$$

where the equation of state

$$
\begin{equation*}
p / \rho=R T \tag{7.3}
\end{equation*}
$$

is used.
The modified Bjerknes theorems simplify somewhat for the flows considered in this chapter, because the lines of constant entropy coincide with the streamlines. Similarly the generalized Crocco theorem [Eq. (6.1)] simplifies, because the streamlines coincide with both the lines of constant entropy and the lines of constant stagnation enthalpy.

An example illustrating these theorems will be useful.* Consider the discharge of a perfect gas from a container. We assume that the gas is originally in equilibrium, that is, that the pressure $p_{0}$ is constant, but do not assume that the temperature $T_{0}$ is constant. In order to use Bjerknes' theorem (in its modified form) we construct the net formed by the lines of constant entropy and the lines of constant temperature. At the beginning of the experiment the pressure is constant and these lines coincide. Thus it follows from Bjerknes' theorem that $d \Gamma / d t=0$. However, at a subsequent instant, the lines become distinct and so the motion becomes rotational. Let us proceed now to determine the rotation. In order to use the generalized Crocco theorem we consider only steady state flow (infinite container). According to our energy theorem, the entropy and the stagnation enthalpy (and consequently the stagnation temperature) are constant along each streamline. Furthermore, since $p_{0}$ is a constant, it follows from the thermodynamical relation (2.3') that

$$
\begin{equation*}
\operatorname{grad} h_{0}=T_{0} \operatorname{grad} s \tag{7.4}
\end{equation*}
$$

Thus from Eq. (6.1)

[^14]\[

$$
\begin{equation*}
q \times \omega=\left(1-\frac{T}{T_{0}}\right) \operatorname{grad} h_{0} \tag{7.5}
\end{equation*}
$$

\]

or, after simplifications,

$$
\mathrm{q} \times \omega=\frac{1}{2} q^{2} \cdot \operatorname{grad}\left(\ln T_{0}\right)
$$

We observe again that although the flow originates from a resting gas, the motion is rotational in general.
8. Two-dimensional flow. The continuity equation shows that in this case there exists a stream function such that

$$
\begin{equation*}
u=\rho^{-1} \partial \psi / \partial y, \quad v=-\rho^{-1} \partial \psi / \partial x . \tag{8.1}
\end{equation*}
$$

From the definition of the rotation, it follows that the stream function must satisfy the following equation

$$
\begin{equation*}
\partial\left(\rho^{-1} \psi_{x}\right) / \partial x+\partial\left(\rho^{-1} \psi_{y}\right) / \partial y=-\omega \quad(=-\partial v / \partial x+\partial u / \partial y) \tag{8.2}
\end{equation*}
$$

In order to determine $\omega$ we use Eq. (6.1). Because $s$ and $h_{0}$ are constant along a streamline,

$$
\begin{equation*}
q \omega=T \frac{\partial s}{\partial n}-\frac{\partial h_{0}}{\partial n} \tag{8.3}
\end{equation*}
$$

where $\partial / \partial n$ indicates differentiation normal to a streamline. Both the entropy and the stagnation enthalpy are functions of $\psi$ alone. By using the relation

$$
\begin{equation*}
\frac{\partial}{\partial n}=q \rho \frac{\partial}{\partial \dot{\psi}} \tag{8.4}
\end{equation*}
$$

we find from Eq. (8.3),

For a perfect gas this reduces to

$$
\omega=\rho\left(T \frac{\partial s}{\partial \psi}-\frac{\partial h_{0}}{\partial \psi}\right)
$$

$$
\omega=\frac{p}{R} \frac{\partial s}{\partial \psi}-\rho \frac{\partial h_{0}}{\partial \psi}
$$

Noting that $\partial h_{0} / \partial \psi$ and $\partial s / \partial \psi$ are constant along any given streamline, one observes that the rotation on each streamline is a linear combination of the density and the pressure. If $h_{0}$ is constant, throughout the flow the rotation is proportional to the pressure. ${ }^{3}$ If $s$ is constant, throughout the flow the rotation is proportional to the density. ${ }^{7}$ (The constant of proportionality is given by the rate of change of entropy or stagnation enthalpy normal to the streamline.)

It is of some interest to develop Eq. $\left(7.5^{\prime}\right)$ for the two-dimensional case. Here we find that

$$
\begin{equation*}
\omega=\frac{\rho q^{2}}{2} \frac{\partial\left(\ln T_{0}\right)}{\partial \psi} \tag{8.5}
\end{equation*}
$$

and the rotation is thus seen to be proportional to $\rho q^{2}$ along each streamline.
Finally we mention that after rather lengthy computations the differential equation for $\psi$, Eq. (8.2), can be transformed into

[^15]\[

$$
\begin{equation*}
\left(1-\frac{u^{2}}{a^{2}}\right) \psi_{x x}-\frac{2 u v}{a^{2}} \psi_{x y}+\left(1-\frac{v^{2}}{a^{2}}\right) \psi_{v y}=\rho^{2}\left[\frac{\partial h_{0}}{\partial \psi}-\frac{k-1}{k R}\left(h_{0}+\frac{q^{2}}{2}\right) \frac{\partial s}{\partial \psi}\right] . \tag{8.6}
\end{equation*}
$$

\]

When both the entropy and the stagnation enthalpy are constant throughout the field, the right-hand side of Eq. (8.6) becomes zero and one obtains the familiar equation of a steady isentropic irrotational flow.
9. Flow around an obstacle with shock-waves. Shock-waves can be included in our theory by admitting such discontinuities in the flow pattern as are compatible with the laws of conservation of matter, momentum and energy. Thus, the previous theory can be applied for flows between shock-waves For most purposes one can assume that the air comes from a homogeneous condition and in particular that $h_{0}$ and $s$ are constant far ahead of the obstacle. It follows from Eqs. (7.1) and (7.2) that both $h_{0}$ and $s$ are constant at least up to the first shock-wave, and then again along each streamline between consecutive shock-waves. Hence $\omega=0$ on each streamline $u p$ to the first shock-wave. In particular, if a streamline is not intersected by a shock-wave, $\omega$ remains zero all along this streamline. From the law of conservation of energy it can be deduced that $h_{0}$ must be continuous across a shock-wave and thus $h_{0}$ must be a constant throughout the field. Hence from Eq. (8.3")

$$
\begin{equation*}
\omega=\frac{p}{R} \frac{\partial s}{\partial \psi}, \tag{9.1}
\end{equation*}
$$

and the rotation is proportional to the pressure along each streamline between shockwaves. Furthermore it is known that the entropy increases across a shock-wave and the increase depends on the magnitude of the shock. Hence $\partial s / \partial \psi$ is not zero in general after a shock-wave and the motion is rotational. Generally speaking there is always a sudden increase of the rotation across shock-waves (see Hadamard), and then the rotation remains proportional to the pressure (sce Crocco).
10. Flow with axial symmetry. Let the $x$ axis be the axis of symmetry of the flow. Then there is a stream function such that

$$
\begin{equation*}
u=r^{-1} \rho^{-1} \partial \psi / \partial r, \quad v=-r^{-1} \rho^{-1} \partial \psi / \partial x \tag{10.1}
\end{equation*}
$$

where

$$
\begin{equation*}
r=\sqrt{y^{2}+z^{2}} \tag{10.2}
\end{equation*}
$$

and $v$ is the velocity component normal to the $x$ axis.
Quite similarly to the two-dimensional case, it follows from Eq. (6.1) that, in the present case,

$$
\begin{equation*}
\omega=r_{\rho}\left(T \frac{\partial s}{\partial \psi}-\frac{\partial h_{0}}{\partial \psi}\right) \tag{10.3}
\end{equation*}
$$

For a perfect gas, we have

$$
\omega=\frac{r p}{R} \frac{\partial s}{\partial \psi}-r \rho \frac{\partial h_{0}}{\partial \psi}
$$

When $h_{0}$ is a constant throughout the field, one recognizes in Eq. (10.3) a relation discovered by Crocco. ${ }^{3}$

# ON VELOCITY CORRELATIONS AND THE SOLUTIONS OF THE EQUATIONS OF TURBULENT FLUCTUATION* 

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1. Introduction. The theory of turbulence, as developed from Reynolds' point of view, is based upon the equations of turbulent fluctuation [1] and has been applied to the solutions of various special problems $[2,3,4,5,6,7]$. Owing to present circumstances, these papers either have not been submitted to scientific journals for publication or are already printed but have failed to appear before the scientific public. The theory in its original form and its applications has three apparent difficulties: first, the equations of correlation of the second, third or even higher orders constructed out of the equations of turbulent fluctuation contain the unknown terms of correlation between the pressure and velocity fluctuations; secondly, there exist in these equations the terms of decay of turbulence the values of which have to be determined; thirdly, when the differential equations of the velocity correlations of a given order are derived from the equations of turbulent fluctuation, the presence of the inertia terms causes the appearance of the velocity correlations of the next higher order, which are also unknown. This has been pointed out by von Kármán and Howarth [8] in their theory of homogeneous isotropic turbulence.

In the present paper we shall show that the pressure fluctuation can be derived from the equations of turbulent fluctuation, and is expressible as a function of the velocity fluctuation, the mean velocity inside the fluid volume, and the pressure fluctuation on the boundary. We shall also show that the decay terms can be put into simpler and more familiar forms by kinematic considerations. A general equation of vorticity decay will be derived for the determination of Taylor's scale of the microturbulence which appears in the decay term; in the case of homogeneous isotropic turbulence, this equation was given first by von Kármán [8]. To get over the third difficulty we shall compare the orders of magnitudes of the different terms in the equations of triple correlation. We shall find that the term involving the divergence of the quadruple correlation is actually smaller than the correlation between the pressure gradient and the two components of velocity fluctuation, and can therefore be neglected as a first approximation. From this we can also understand why, for the flows in channels and pipes in which the mean velocity profile is comparatively steep, particularly in the neighborhood of the walls, all the equations of mean motion and the equations of double and triple correlation are necessary to describe the phenomena of turbulent motions of fluids. On the other hand, as a consequence of the approximation based on the fact that the divergence of the quadruple correlation is smaller than the correlation between the pressure gradient and the two components of velocity fluctuation, we can stop at the equations of triple correlation instead of building equations of higher orders. As a matter of fact, for the flows in jets [3] and wakes [4]

[^16]where no wall is present, the equations of mean motion and of double correlation are sufficient, after some simple approximations to the triple correlations are made, for the determination of the mean velocity distribution, and the equations of triple correlation can be dispensed with.

From a mathematical point of view the present program indicates that the turbulence problem can be reduced rigorously to a set of non-linear partial integro-differential equations the solutions of which are very difficult to ascertain. In order to facilitate the solution of special problems, approximate forms of the integral parts of the equations have been developed in a general way. These approximations, however, are only valid in regions not too close to the boundary of the moving fluid volume. It may also be worthwhile to point out that the unsatisfactory part of the present theory lies in the uncertain nature of the correlation integrals, as will be seen presently in $\S 8$. A better and more accurate representation of these integrals is possible, provided more accurate experimental information can be obtained as to the distribution of turbulence levels and to the correlation functions between two distinct points in general.

The rigorous way of treating the turbulence problem is probably to solve the Reynolds' equations of mean motion and the equations of turbulent fluctuation simultaneously. This procedure, however, is very difficult owing to the non-linearity of the two sets of equations. Hence we have adopted the method of solving the equations of turbulent fluctuation by setting up the differential equations satisfied by the velocity correlation functions of different orders, a method initiated by von Kármán and Howarth [8] in treating the problem of homogeneous isotropic turbulence. This process of setting up the correlation equations of different orders and seeking their solution can be regarded as a method of successive approximation to the solution of the turbulence problem; it will be explained in the concluding section of the present paper. The correlation functions of higher orders in the various special problems, obtained by this setting-up process, should be verifiable by direct observation with the advance of modern experimental technique; at present experiments have only been performed to measure the mean velocity distribution and the second order stress tensors in a turbulent flow. It should also be noted that although the equations of correlation have a much more complicated mathematical appearance than that of the Navier-Stokes' differential equations from which they are derived, the method of Prandtl's boundary layer approximations can still be used without leading to contradictions for the particular problems $[3,4,5]$ under consideration.

For the sake of convenience we list below the different equations of motion which have been derived heretofore [1]. Reynolds' equations of mean motion and the equation of continuity for an incompressible fluid are given by

$$
\begin{equation*}
\frac{\partial U_{i}}{\partial t}+U^{i} U_{i, j}=-\frac{1}{\rho} \bar{p}_{, i}+\frac{1}{\rho} \tau_{i, j}+\nu \nabla^{2} U_{i}, \quad U_{, j}^{j}=0 \tag{1.1}
\end{equation*}
$$

Here, the tensor notation is employed, and $U_{i}$ are the velocity components of the mean motion, $t$ is the time, $\rho$ is the density, $\bar{p}$ is the mean pressure, $\nu$ is the coefficient of kinematic viscosity, a subscript preceded by a comma denotes the covariant derivative, $\nabla^{2}$ denotes the Laplacian operator, and Reynolds' apparent stress $\tau^{i}{ }_{i}$ is defined by the relation

$$
\begin{equation*}
\tau_{i}^{j}=-\overline{\rho w_{i} w^{i}}, \tag{1.2}
\end{equation*}
$$

$w_{i}$ being the velocity components of the turbulent motion.
The equations of turbulent fluctuation and the equation of continuity for the velocity fluctuation $w^{i}$, which are the differences of the Navier-Stokes' equations and Reynolds' equations (1.1), are
$\frac{\partial w_{i}}{\partial t}+U^{i} w_{i, j}+w^{i} w_{i, j}+w^{i} U_{i, j}=-\frac{1}{\rho} \varpi_{, i}-\frac{1}{\rho} \tau_{i, j}^{j}+\nu \nabla^{2} w_{i}, \quad w^{j}, j=0$,
where $\omega$ is the pressure fluctuation. From the above set of equations we derive the equations of vorticity fluctuation,

$$
\begin{align*}
& \frac{\partial}{\partial t} \omega_{i k}+U^{j} \omega_{i k, j}+U^{j}{ }_{. k} w_{i, j}-U^{j}{ }_{, i} w_{k, j}+w^{j} \omega_{i k, j}+w^{j}{ }_{, k} w_{i, j}-w^{j}{ }_{, i} w_{k, j} \\
&+w^{i} \Omega_{i k, j}+w^{j}{ }_{, k} U_{i, j}-w^{j}{ }_{, i} U_{k, j}=-\frac{1}{\rho}\left(\tau^{j}{ }_{i, j k}-\tau^{j}{ }_{k, j i}\right)+\nu \nabla^{2} \omega_{i k}, \tag{1.4}
\end{align*}
$$

where the mean vorticity $\Omega_{i k}$ and the vorticity fluctuation $\omega_{i k}$ are defined by the equations

$$
\begin{equation*}
\Omega_{i k}=U_{i, k}-U_{k, i}, \quad \omega_{i k}=w_{i, k}-w_{k, i} \tag{1.5}
\end{equation*}
$$

The equations of double velocity correlation derived from (1.3) are

$$
\begin{align*}
-\frac{1}{\rho} \frac{\partial \tau_{i k}}{\partial t}-\frac{1}{\rho}\left(U_{i, j} \tau_{k}^{j}\right. & \left.+U_{k, j} \tau_{i}^{j}\right)-\frac{1}{\rho} U^{j} \tau_{i k, i}+\left(\overline{w^{i} w_{i} w_{k}}\right)_{, j} \\
& =-\frac{1}{\rho}\left(\overline{\left(\omega_{, i} w_{k}\right.}+\overline{\omega, k w_{i}}\right)-\frac{\nu}{\rho} \nabla^{2} \tau_{i k}-2 \nu g^{m n} \overline{w_{i, m} w_{k, n}} \tag{1.6}
\end{align*}
$$

where the superimposed bar denotes the mean. The ten equations of triple correlation are

$$
\begin{align*}
\frac{\partial}{\partial l} \overline{w_{i} w_{k} w_{l}}+U_{i, i} \overline{w^{j} w_{k} w_{l}} & +U_{k, i} \overline{w^{j} w_{l} w_{i}}+U_{l, j} \overline{w^{i} w_{i} w_{k}}+U^{j}\left(\overline{w_{i} w_{k} w_{l}}\right)_{, j}+\left(\overline{w^{i} w_{i} w_{k} w_{l}}\right)_{, i} \\
= & -\frac{1}{\rho}\left(\overline{\omega_{, i} w_{k} w_{l}}+\overline{\omega_{, k} w_{l} w_{i}}+\overline{\omega_{, l} w_{i} w_{k}}\right) \\
& +\frac{1}{\rho^{2}}\left(\tau^{i}{ }_{i, j} \tau_{k l}+\tau^{i} k_{, i} \tau_{l i}+\tau_{l, j}^{j} \tau_{i k}\right)+\nu g^{m n}\left(\overline{w_{i} w_{k} w_{l}}\right)_{, m n} \\
& -2 \nu g^{m n}\left(\overline{w_{i, m} w_{k, n} w_{l}}+\overline{w_{k, m} w_{l, n} w_{i}}+\overline{w_{l, m} w_{i, n} w_{k}}\right) . \tag{1.7}
\end{align*}
$$

2. The pressure fluctuation. Let us take the divergence of the equations of turbulent fluctuation (1.3). Because of the equation of continuity satisfied by $w^{i}$, the pressure fluctuation a satisfies the following Poisson's equation:

$$
\begin{equation*}
\frac{1}{\rho} \nabla^{2} \varpi=-2 U^{m}{ }_{\cdot n} w^{n}{ }_{\cdot m}+\left(\overline{w^{m} w^{n}}-w^{m} w^{n}\right)_{, m n} \tag{2.1}
\end{equation*}
$$

Since any two successive covariant differentiations are commutative in a Euclidean space, the gradient of the pressure fluctuation $\tilde{\omega}_{. k}$ also satisfies a Poisson's equation,

$$
\begin{equation*}
\frac{1}{\rho} \nabla^{2} \varpi, k=-2\left(U^{m}, n w^{n}, m\right)_{, k}+\left(\overline{w^{m} w^{n}}-w^{m} w^{n}\right)_{, m n k} \tag{2.2}
\end{equation*}
$$

The general solution of (2.2) can be written in the form,

$$
\begin{array}{r}
\frac{1}{\rho} \varpi_{, k}=\frac{1}{2 \pi} \iiint\left(U^{\prime m}{ }_{, n} w^{\prime n}{ }_{, m}\right)^{\prime}, k \frac{1}{r} d V^{\prime}-\frac{1}{4 \pi} \iiint\left(\overline{w^{\prime m} w^{\prime n}}-w^{\prime m} w^{\prime n}\right)^{\prime}, m n k \frac{1}{r} d V^{\prime} \\
 \tag{2.3}\\
+\frac{1}{4 \pi \rho} \iint\left\{\frac{1}{r} \frac{\partial \varpi^{\prime}, k}{\partial n^{\prime}}-\varpi_{, k}^{\prime} \frac{\partial}{\partial n^{\prime}}\left(\frac{1}{r}\right)\right\} d S^{\prime}
\end{array}
$$

where the integrations extend over the whole region of the moving fluid, the first two integrals represent the particular integrals, and the third represents the complementary solution which is a harmonic function expressed in terms of the boundary values of itself and its normal derivative; $x^{\prime i}$ are the coordinates of a point $P^{\prime}$ which ranges over the region of the moving fluid, $r$ is the distance from $P^{\prime}$ to the point $P$ with coordinates $x^{i}, d V^{\prime}$ is a volume element, $d S^{\prime}$ is a surface element, $\partial / \partial n^{\prime}$ denotes the normal derivative, and the primes on the various quantities on the right side of (2.3) indicate that these quantities are to be evaluated at $P^{\prime}$. We shall see finally that the surface integral in (2.3) can be neglected for points $P$ where $\omega_{, k}$ is defined and which are not too close to the boundary of the moving fluid.

We now let both $x^{i}$ and $x^{i}$ represent rectangular cartesian coordinates, and let $\xi^{i}$ denote the difference vector of $x^{\prime i}$ and $x^{i}$, i.e.,

$$
\begin{equation*}
\xi^{i}=x^{\prime i}-x^{i} \tag{2.4}
\end{equation*}
$$

Covariant differentiation then reduces to ordinary differentiation, and the difference between covariant and contravariant tensor character disappears. Hence $\xi^{i}$ is equal to $\xi_{i}$ and the distance $r$ between $P$ and $P^{\prime}$ is given by

$$
\begin{equation*}
r^{2}=\xi_{i} \xi^{i} \tag{2.5}
\end{equation*}
$$

The element of volume $d V^{\prime}$ is equal to $d \xi^{1} d \xi^{2} d \xi^{3}$.
The solution (2.3) clearly shows that besides the harmonic function expressed as a surface integral on the boundary, the pressure fluctuation at a point $P$, and its gradient, are determined by the turbulent velocity fluctuation winot only at $P$ but also everywhere within the fluid. However, due to the factor $1 / r$ in the integrands the effect of the velocity fluctuation at distant points $P^{\prime}$ on the pressure fluctuation at $P$ gradually dies away as $P^{\prime}$ recedes farther and farther from $P$.
3. Velocity correlation between two distinct points. The partial differentiations in the integrand functions in (2.3) are taken with respect to the coordinates $x^{\prime k}$ which are independent of $x^{i}$. Hence, if we multiply (2.3) by the velocity fluctuation $w_{i}$ at the point $P$, we obtain the correlation between $w_{i}$ and $\varpi_{, k}$ at the same point $P$ :

$$
\left.\begin{array}{rl}
\frac{1}{\rho} \overline{\omega_{, k} w_{i}}=\frac{1}{2 \pi} \iiint\left[U^{\prime m}{ }_{, n}\left(\overline{w^{\prime n} w_{i}}\right)^{\prime}, m\right.
\end{array}\right]^{\prime}, k \frac{1}{r} d V^{\prime}+\frac{1}{4 \pi} \iiint\left(\overline{w^{\prime m} w^{\prime n} w_{i}}\right)^{\prime}, m_{n k} \frac{1}{r} d V^{\prime} .
$$

We shall neglect, however, the surface integral in the above equation on the
ground that the correlation $\overline{\omega^{\prime}, k \omega_{i}}$ is small provided that the point $P$ where the correlation $\overline{\omega, k w_{i}}$ is under consideration is situated not too close to the boundary. This condition limits the present theory to regions where free turbulence predominates.

Likewise, under the same condition of approximation the correlation function $\rho^{-1} \overline{\omega_{, k} w_{l} w_{i}}$ is given by

$$
\begin{align*}
\frac{1}{\rho} \overline{w_{, k} w_{l} w_{i}}= & \frac{1}{2 \pi} \iiint\left[U^{\prime m}\left(\overline{w^{\prime n} w_{l} w_{i}}\right)^{\prime},\right]^{\prime}, k \frac{1}{r} d V^{\prime} \\
& -\frac{1}{4 \pi} \iiint\left[\overline{w^{\prime m} w^{\prime n}} \overline{w_{l} w_{i}}-\overline{w^{\prime m} w^{\prime n} w_{l} w_{i}}\right]^{\prime}, m n k \frac{1}{r} d V^{\prime} \tag{3.2}
\end{align*}
$$

If we solve for a from (2.1) and form its correlation with $w_{i}$, we find, to the same order of approximation, that
$\frac{1}{\rho} \overline{\varpi w_{i}}=\frac{1}{2 \pi} \iiint U^{\prime m}{ }_{\cdot n}\left(\overline{w^{\prime n} w_{i}}\right)^{\prime} \cdot m \frac{1}{r} d V^{\prime}+\frac{1}{4 \pi} \iiint\left(\overline{w^{\prime m} w^{\prime \prime} w_{i}}\right)^{\prime} \cdot m n \frac{1}{r} d V^{\prime}$.
In the three equations (3.1), (3.2) and (3.3) we recognize three types of functions, namely, $\overline{w^{\prime m} w_{i}}, \overline{w^{\prime m} w w^{\prime m} w_{i}}$ and $\overline{w^{\prime n} w_{i} w_{i}}$, and $\overline{w^{\prime m} w^{\prime n} w_{i} w_{i}}$; they are, according to Taylor [10] and von Kármán [8], the velocity correlations between two distinct points $P$ and $P^{\prime}$ of the second, third and fourth orders respectively. They are usually functions of both the coordinates $x^{i}$ and $x^{\prime k}$ and probably also of the time $t$. The double correlation function $\overline{w^{\prime r} w_{i}}$ between $P$ and $P^{\prime}$ has been measured extensively for isotropic turbulence by several authors [11, 12]; for flow in a channel [13] and in a pipe [14], they have been recorded only in a number of isolated cases and only within limits.

It has been observed that for isotropic turbulence $\overline{w^{\prime n} w_{i}}$ vanishes very rapidly for large values of the quantities $\xi^{i}$ defined in (2.4). This must also hold true for the other two correlation functions $\overline{w^{\prime m} w^{\prime n} w_{i}}$ and $\overline{w^{\prime n} w_{i} w_{i}}$, and also for other types of flow; furthermore their derivatives with respect to $\xi^{k}$ should all approach zero rapidly with increasing $\xi^{i}$.

On the other hand, the quadruple correlation $\overline{w^{\prime m} w^{\prime n} w_{i} w_{i}}$ between the points $P$ and $P^{\prime}$ does not necessarily vanish when $P$ and $P^{\prime}$ are widely separated, for the average values of both $\overline{w^{\prime m} w^{\prime n}}$ and $\overline{w_{i} w_{i}}$ over a period of time $\tau$ are themsclves not separately equal to zero in general. Hence as an analogy to the velocity vector $W^{i}$, we may separate the product $w_{i} w_{i}$ into two parts, the correlation $\overline{w_{i} w_{i}}$ and a symmetric tensor $u_{l i}$ the time average of which vanishes,*

$$
\begin{gather*}
w_{l} w_{i}=\overline{w_{l} w_{i}}+u_{l i},  \tag{3.4}\\
\overline{u_{l i}}=\frac{1}{\tau} \int_{\tau \rightarrow / 2}^{\tau+\tau / 2} u_{l i} d t=0
\end{gather*}
$$

An analogous relation holds good for $\overline{w^{m} w^{\prime n}}$. The quadruple correlation $\overline{w^{\prime m} w^{\prime n} w_{l} w_{i}}$ between the points $P$ and $P^{\prime}$ consequently becomes

$$
\begin{equation*}
\overline{w^{\prime m} w^{\prime n} w_{l} w_{i}}=\overline{w^{\prime m} w^{\prime n}} \overline{w_{l} w_{i}}+\overline{u^{\prime m n} u_{l i}} . \tag{3.5}
\end{equation*}
$$

As $P$ and $P^{\prime}$ recede farther and farther from each other, the correlation function

[^17]$\overline{u^{\prime m n} u_{l i}}$, which behaves like $\overline{w^{\prime n} w_{i}}$, will tend toward zero as a limit. Substitution of (3.5) into (3.2) yields
\[

\left.$$
\begin{array}{rl}
\frac{1}{\rho} \overline{\varpi_{, k} w_{l} w_{i}}= & \frac{1}{2 \pi} \iiint\left[U^{\prime m},\left(\overline{w^{\prime n} w_{l} w_{i}}\right)^{\prime}, m\right.
\end{array}
$$\right]_{, k}^{\prime} \frac{1}{r} d V^{\prime},
\]

If we substitute in the equations of the double and triple correlations (1.6) and (1.7) for $\rho^{-1} \overline{\omega_{, k} w_{i}}$ from (3.1) and $\rho^{-1} \overline{\omega_{, k} w w_{l} w_{i}}$ from (3.6) above, we obtain a set of integrodifferential equations for the mean velocity, the double and triple velocity correlations of a turbulent flow at a point $P$ being the dependent variables with the velocity correlations between two distinct points $\overline{w^{\prime n} w_{i}}$ and $\overline{w^{\prime n} w_{i} w_{i}}$ as kernels. This set of integrodifferential equations is too complicated for solving special problems, so we shall presently develop approximate forms of the integral parts of the equations in a general way.

It should be noted that for homogeneous isotropic turbulence the following relation between the triple correlations holds [8]:

$$
\begin{equation*}
\overline{w_{m}^{\prime} w_{n}^{\prime} w_{i}}=-\overline{w_{i}^{\prime} w_{m} w_{n}} \tag{3.7}
\end{equation*}
$$

4. Conservation relations satisfied by the velocity correlations. The velocity fluctuation $w^{\prime n}$ at the point $P^{\prime}$ satisfies the equation of continuity $w^{\prime n}, n=0$. Let us multiply this equation by $w_{i}$ and average over an interval of time $\tau$. Since $P$ and $P^{\prime}$ are indcpendent, we obtain the conservation equation for the double correlation $\overline{w^{\prime} w_{i}}$ between $P$ and $P^{\prime}$,

$$
\begin{equation*}
\frac{\partial}{\partial x^{\prime n}}\left(\overline{w^{\prime n} w_{i}}\right)_{x}=0 \tag{4.1}
\end{equation*}
$$

where the coordinates are still rectangular cartesian, and the subscript $x$ indicates that the variables $x^{k}$ are to be held constant while the differentiation is carried out.

Instead of $x^{i}$ and $x^{\prime i}$, we can use the coordinates $x^{i}$ and $\xi^{i}$, i.e., we transform from the old variables $x^{i}$ and $x^{\prime i}$ to the new variables $x^{i}$ and $\xi^{i}$ by means of the equations

$$
\begin{equation*}
x^{i}=x^{i}, \quad x^{\prime i}=x^{i}+\xi^{i} . \tag{4.2}
\end{equation*}
$$

In terms of the new coordinates $x^{i}$ and $\xi^{i}$, Eq. (4.1) becomes

$$
\begin{equation*}
\frac{\partial}{\partial x^{\prime n}}\left(\overline{\left(w^{\prime n} w_{i}\right.}\right)_{x}=\frac{\partial}{\partial \xi^{n}}\left(\overline{w^{\prime n} w_{i}}\right)_{x}=0 . \tag{4.3}
\end{equation*}
$$

For the sake of simplicity we shall drop the subscript $x$ in (4.3); it will be understood that the variables $x^{k}$ are regarded as constants during the differentiation. Hence we can write the divergence equation (4.3) in the covariant form,

$$
\begin{equation*}
\left(\overline{w^{\prime n} w_{i}}\right)_{, n}=0 . \tag{4.4}
\end{equation*}
$$

Similarly, from the equation of continuity for $w^{i}$ at the point $P$, we have

$$
\frac{\partial}{\partial x^{i}}\left(\overline{w_{n}^{\prime} w^{w^{\prime}}}\right)_{z^{\prime}}=0 .
$$

In terms of the new coordinates $x^{i}$ and $\xi^{i}$, this relation becomes

$$
\frac{\partial}{\partial x^{i}}\left(\overline{w_{n}^{\prime} w^{i}}\right)_{\xi}-\frac{\partial}{\partial \xi^{i}}\left(\overline{w_{n}^{\prime} w^{i}}\right)_{x}=0 .
$$

After changing the variables from $x^{i}, x^{\prime i}$ to $x^{i}, \xi^{i}$ it can be seen that $\overline{w^{\prime n} w_{i}}$, considered as a function of $x^{i}$ and $\xi^{i}$ rather than of $x^{i}$ and $x^{\prime i}$, varies slowly with $x^{i}$ but rapidly with $\xi^{i}$ for points not too close to the boundary of the fluid volume. Hence, as a first approximation the equation of conservation for the double correlation between $P$ and $P^{\prime}$ in the index $i$ is given by

$$
\begin{equation*}
\left(\overline{w_{n}^{\prime}}{ }_{n}^{w^{i}}\right), i=0 . \tag{4.5}
\end{equation*}
$$

We note that to the first approximation the correlation function $\overline{w_{k}^{\prime} w_{i}}$ satisfies the conservation equation symmetrically with respect to the indices $i$ and $k$.

Likewise, the other two correlation functions $\overline{w^{\prime m} w^{\prime n} w_{i}}$ and $\overline{w^{\prime n} w_{i} w_{k}}$ between $P$ and $P^{\prime}$ can be shown to satisfy the following relations:

$$
\begin{equation*}
\left(\overline{w^{\prime m} w^{\prime n} w^{i}}\right)_{, i}=0, \quad\left(\overline{w^{\prime n} w_{i} w_{k}}\right)_{, n}=0 . \tag{4.6}
\end{equation*}
$$

The first equation in (4.6) is derived by an approximation as was (4.5); the second one is rigorous. We must not forget that all the covariant derivatives in (4.5) and (4.6) are taken with respect to the variables $\xi^{i}$, the coordinates $x^{i}$ being held constant.

It is obvious that since the coordinates $x^{i}$ of the point $P$ are regarded as constants under the integrations in (3.1), (3.2) and (3.3), the covariant derivatives with respect to $x^{\prime k}$ in the integrand functions can all be replaced rigorously by covariant derivatives with respect to the variables $\xi^{k}$, because of the equations of coordinate transformation (4.2). For example, (3.1) then becomes

$$
\begin{align*}
\frac{1}{\rho} \overline{\omega_{, k} w_{i}}= & \frac{1}{2 \pi} \iiint\left[U^{\prime m}{ }_{\cdot n}\left(\overline{w^{\prime n} w_{i}}\right)_{, m}\right]_{, k} \frac{1}{r} d V^{\prime} \\
& +\frac{1}{4 \pi} \iiint\left(\overline{w^{\prime m} w^{\prime n} w_{i}}\right)_{, m n k} \frac{1}{r} d V^{\prime} . \tag{4.7}
\end{align*}
$$

The other two integrals (3.3) and (3.6) can be altered analogously
5. Correlation integrals between the pressure gradient and velocity fluctuations. Let us examine the integral (4.7) more closely. In the integrand function of the first integral on the right hand side, $U^{m_{m}, n}$ is a more slowly varying function of $\xi^{i}$ than its factor $\overline{w^{\prime n} w_{i}}$, both functions being regarded as functions of $x^{k}$ and $\xi^{i}$. Hence, we expand $U^{\prime m}{ }_{, n}$ at the point $P^{\prime}$ in a multiple power series in $\xi^{i}$,

$$
\begin{equation*}
\frac{\partial U^{\prime m}}{\partial x^{\prime n}}=\frac{\partial U^{m}}{\partial x^{n}}+\sum_{s=1}^{\infty} \frac{1}{s!} \frac{\partial^{s+1} U}{\partial x^{l_{1}} \partial x^{l_{2}} \ldots \partial x^{l^{2} \partial x^{n}}} \xi^{l_{1} \xi^{l_{2}}} \cdots \xi^{l_{1}} . \tag{5.1}
\end{equation*}
$$

Substitution of (5.1) into (4.7) would yield a series of integrals which would be too complicated for any practical application. But if we neglect the higher order terms, in (5.1), then we have as a first approximation to (4.7),

$$
\begin{equation*}
\frac{1}{\rho}\left(\overline{\omega_{i}, i w_{k}}+\overline{\omega_{, k} w_{i}}\right)=a^{n} m_{m i k} U^{m}{ }_{, n}+b_{i k}, \tag{5.2}
\end{equation*}
$$

where the functions $a_{m i k}$ and $b_{i k}$ are defined by

$$
\begin{align*}
a_{m i k}^{n} & =\frac{1}{2 \pi} \iiint\left[\left(\overline{w^{\prime n} w_{i}}\right)_{, m k}+\left(\overline{w^{\prime n} w_{k}}\right)_{, m i}\right] \frac{1}{r} d V^{\prime} \\
b_{i k} & =\frac{1}{4 \pi} \iiint\left[\left(\overline{w^{\prime m} w^{\prime n} w_{i}}\right)_{, m n k}+\left(\overline{w^{\prime m} w^{\prime n} w_{k}}\right)_{, m n i}\right] \frac{1}{r} d V^{\prime} . \tag{5.3}
\end{align*}
$$

Owing to the conservation relations (4.5) and (4.6), the above two sets of functions also satisfy the following divergence conditions:

$$
\begin{equation*}
a_{n i k}^{n}=0, \quad g^{i k} a_{m i k}^{n}=0, \quad g^{i k} b_{i k}=0 \tag{5.4}
\end{equation*}
$$

Of these three conservation relations, the first follows from the rigorous continuity equation (4.4) and is hence exact, while the other two follow from (4.5) and the first equation of (4.6) and are hence approximations. The nature of the functions $a_{m i k}^{n}$ and $b_{i k}$ will be discussed in $\S 8$ below.

Because of (5.4), contraction of (5.2) by means of $g^{i k}$ yields,

$$
\begin{equation*}
\frac{1}{\rho} \overline{\omega_{, i} w^{i}}=\frac{1}{\rho}(\overline{\omega w})_{, i}=0 . \tag{5.5}
\end{equation*}
$$

This result is consistent with the correlation (3.3). For we may substitute the series in (5.1) into (3.3) and preserve the largest term; but the latter is smaller than the first term on the right-hand side of (5.2) by a factor of $\lambda$ which is Taylor's scale of micro-turbulence $[10,8]$; the second term on the right-hand side of (3.3) is also smaller than $b_{i k}$ by an analogous factor. Hence the approximate form of (3.3), to the same degree of accuracy as in (5.2), is

$$
\begin{equation*}
\frac{1}{\rho} \overline{\varpi w_{i}}=0 . \tag{5.6}
\end{equation*}
$$

This relation has also been proved to hold true for isotropic turbulence by von Kármán and Howarth [8].

By a similar process, we find from (3.2) that the triple correlation between the pressure gradient and two components of the velocity fluctuations is, to the same degree of approximation,

$$
\begin{equation*}
\frac{1}{\rho}\left(\overline{\varpi_{, i} w_{k} w_{l}}+\overline{\omega_{, k} w_{l} w_{i}}+\overline{\omega_{, l} w_{i} w_{k}}\right)=b_{m i k l}^{n} U^{m_{, n}}+c_{i k l}, \tag{5.7}
\end{equation*}
$$

where the forms of the tensors $b^{n_{m i k l}}$ and $c_{i k l}$ are given by respectively by

$$
\begin{align*}
b_{m i k l}^{n} & =\frac{1}{2 \pi} \iiint\left[\left(\overline{w^{\prime n} w_{i} w_{k}}\right)_{, m l}+\left(\overline{w^{\prime n} w_{k} w_{l}}\right)_{, m i}+\left(\overline{w^{\prime n} w_{l} w_{i}}\right)_{, m k}\right] \frac{1}{r} d V^{\prime} \\
c_{i k l} & =\frac{1}{4 \pi} \iiint\left[\left(\overline{u^{\prime m n} u_{i k}}\right)_{, m n l}+\left(\overline{u^{\prime m n} u_{k l}}\right)_{, m n i}+\left(\overline{u^{\prime m n} u_{l i}}\right)_{, m n k}\right] \frac{1}{r} d V^{\prime} \tag{5.8}
\end{align*}
$$

Because of (4.6), the functions $b^{n}{ }_{m i k l}$ satisfy the rigorous conservation relation,

$$
\begin{equation*}
b_{n i k l}=0 \tag{5.9}
\end{equation*}
$$

We shall discuss the general behaviour of the functions $b_{m i k l}^{n}$ and $c_{i k l}$ in $\S 8$.
6. Terms involving the decay of turbulence in the equations of double and triple correlation. To determine the terms in the decay of turbulence, it is necessary to know explicitly the double and triple velocity correlations between two adjacent points. Physically, the correlation functions between two near-by points must satisfy two conditions: first, they should become the velocity correlations at one point when the two points coincide; secondly, they should degenerate into the isotropic correlations when the flow obeys the condition of isotropy. By two adjacent points we mean that expansions of the double and triple correlation functions in terms of the coordinates $\xi^{i}$ stop after the second and third powers of $\xi^{i}$, respectively. Furthermore, since only approximate expressions of the decay terms are required, conservation equations in the forms (4.4), (4.5) and (4.6) will suffice for the present purpose. In view of the property that the double correlation $\overline{w_{i} w_{n}^{\prime}}$ satisfies the conservation relations (4.4) and (4.5) symmetrically with respect to the two indices $i$ and $n$ as a first approximation, it should also satisfy the supplementary condition that its expansion be symmetrical in the coordinates of $P$ and $P^{\prime}$.

The second order velocity correlation between two adjacent points that satisfies the above two conditions and the supplementary condition of symmetry can be expanded into powers of $\xi^{i}$ in the form,

$$
\begin{align*}
\overline{w_{i} w^{\prime}}=\frac{1}{3} q^{2}\left\{\frac{A}{2 \lambda^{2}} \xi_{i} \xi_{k}\right. & +\frac{\delta_{i k}}{2 \lambda^{2}} B_{m \xi^{m} \xi^{n}}+R_{i k}\left(1+\frac{C_{m n}}{2 \lambda^{2}} \xi^{m} \xi^{n}\right) \\
& \left.-\frac{G}{\lambda^{2}}\left(R_{i l} \xi^{\prime} \xi_{k}+R_{k} \xi^{l} \xi_{i} i\right)+\frac{1}{4!\lambda^{4}} E_{i k j l m n} \xi^{i} \xi^{l} \xi^{m} \xi^{n}+\cdots\right\}, \tag{6.1}
\end{align*}
$$

where $q$ is the mean magnitude of the velocity fluctuation, or the root-mean-square of the velocity fluctuation, defined by

$$
\begin{equation*}
q^{2}=\overline{w_{j} w^{i}}, \tag{6.2}
\end{equation*}
$$

and $R_{i k}$ stands for

$$
\begin{equation*}
R_{i k}=\frac{3}{q^{2}} \overline{w_{i} w w_{k}} . \tag{6.3}
\end{equation*}
$$

The function $\lambda$ is Taylor's scale of micro-turbulence, both $q$ and $\lambda$ being functions of the coordinates $x^{i}$ of $P ; A, B_{m n}, C_{m n}, G$ and $E_{i k j l m n}$ are all independent of $\xi^{\xi}$. The coefficients $B_{m n}$ and $C_{m n}$ are symmetric in $m$ and $n ; E_{i k j l m n}$ is both symmetric in $i$ and $k$ and in the last four indices $j, l, m$ and $n$, but is not symmetric in any one index of the first set of two and any one in the last set of four, e.g., it is not symmetric in $i$ and $j$. Hence this tensor has $6 \times 15=90$ independent components.

The form given in (6.1) for the correlation tensor $\overline{w_{i} w w_{k}^{\prime}}$ between two adjacent points is the most general linear combination of the products of the tensors $\xi_{i} \xi_{k}, \delta_{i k}$ and $w_{i} w_{k}$. The functions $A, B_{i k}, C_{i k}$ and $G$ will be assumed to be constants; it is not necessary to know the exact nature of the separate components of $E_{i k j i m n}$ for our present purpose, but we shall assume for the time being that the invariant $E=g^{i k} g^{i} g^{m n} E_{i k j l m n}$ is constant.

A question naturally arises as to whether the functions $q^{2}$ and $\lambda^{2}$ in (6.1), which vary with the coordinates, should be replaced by expressions which are symmetrical in the coordinates of $P$ and $P^{\prime}$. However, this is not essential, for both $q^{2}$ and $\lambda^{2}$ vary
much more slowly than $\overline{w_{i} w_{k}^{\prime}}$ as a function of $\xi^{i}$; since $P$ and $P^{\prime}$ are close to each other, we may use their values at $P$ as an approximation. Nevertheless, one must be careful with this approximation, whenever differentiation with respect to $x^{\prime \prime}$ is involved.

The function $\overline{w_{i} w_{k}^{\prime}}$ must satisfy the equation of continuity (4.4), or (4.5). By setting the coefficients of $\xi^{k}$ and $\xi^{i \xi^{l} \xi^{m}}$ equal to zero separately, we find that

$$
\begin{gather*}
2 A \delta_{i k}+B_{i k}+R_{i}^{l} C_{i k}-5 G R_{i k}-3 G \delta_{i k}=0  \tag{6.4}\\
g^{k s} E_{i k s i m n}=0 \tag{6.5}
\end{gather*}
$$

Since Eq. (6.4) is symmetric in the indices $i$ and $k$, and since $C_{l k}$, has been assumed to be a constant, we must have

$$
\begin{equation*}
C_{m n}=-C \delta_{m n}, \tag{6.6}
\end{equation*}
$$

where $C$ is a constant. On the other hand if $C_{m n}$ depends upon the correlation tensor $\overline{w_{i} w_{k}}$, then it is possible to have the more general solution $C_{m n}=-C \delta_{m n}+D S_{m n}$, where $S_{m n}$ is the inverse matrix of $R_{m n}$ defined by $S_{i}^{l} R^{k}{ }_{l}=\delta^{k}{ }_{i}$. For the sake of simplicity, we choose $D$ to be zero for the time being. Obviously, the number of independent equations in (6.5) is 30.

In order to give a simpler appearance to the final forms of the decay term in the equations of double correlation and of the equation of vorticity decay, we put

$$
\begin{equation*}
A=1+4 G, \quad C=\frac{1}{3}(k-4 G) \tag{6.7}
\end{equation*}
$$

The first equation amounts to a change of the factor $\lambda$, this factor being arbitrary; the change makes $\lambda$ assume the same numerical value as Taylor's scale of microturbulence, when the correlation tensor obeys the condition of isotropy. The second equation in (6.7) only defines $C$ in terms of a new constant $k$. Utilizing relations (6.4), (6.6) and (6.7), we put (6.1) into the form

$$
\begin{align*}
\overline{w_{i} w_{k}^{\prime}} & =\overline{w_{i} w_{k}}+\frac{q^{2}}{3 \lambda^{2}}\left\{\frac{1}{2}(1+4 G) \xi_{i} \xi_{k}-\frac{1}{2}\left[(2+5 G) r^{2}-\frac{1}{3}(k+11 G) R_{m n} \xi^{m} \xi^{n}\right] \delta_{i k}\right. \\
& \left.-\frac{1}{6}(k-4 G) r^{2} R_{i k}-G\left(R_{i l} \xi^{l} \xi_{k}+R_{k!} \xi^{l} \xi_{i}\right)+\frac{1}{4!\lambda^{2}} E_{i k j l m n} \xi^{i} \xi^{l} \xi^{m} \xi^{n}+\cdots\right\} \tag{6.8}
\end{align*}
$$

where the tensor $E_{i k j l m n}$ satisfies the thirty linear equations (6.5). We shall see presently that, with the form of $\overline{w_{i} w_{k}^{\prime}}$ given in (6.8), only the constant $k$ will appear in the term involving the decay of turbulence (6.14), while only $G$ will be present in the equation for the decay of vorticity (7.11).

For isotropic turbulence we have $\overline{w_{i} w_{k}}=\frac{1}{3} q^{2} \delta_{i k}$, and it is easy to verify that in (6.8) the terms in $\xi_{i} \xi_{k}$ and $r^{2}$ coincide with terms in the isotropic correlation tensor according to von Kármán and Howarth [8]. The validity of formula (6.8) and its properties can be subjected to experimental verification.

For the triple velocity correlation $\overline{w_{i} w_{j} w_{k}^{\prime}}$ between two neighboring points, we have to assume a form which degenerates into $\overline{w_{i} w_{j} w_{k}}$ when the points coincide and becomes the triple correlation for isotropic turbulence when the condition of isotropy is satisfied by the flow. Since the expansion of the triple isotropic correlation function begins with the third powers of $\xi^{i}$, as shown by von Kármán and Howarth [8], the same must hold for the present general case. This expansion must satisfy the equation of continuity (4.6), and the final result obtained is

$$
\begin{equation*}
\overline{w_{i} w_{j} w_{k}^{\prime}}=\overline{w_{i} w_{j} w_{k}}+\frac{F q^{3}}{3!3 \sqrt{3} \lambda^{3}}\left[2 \xi_{i} \xi_{j} \xi_{k}-\frac{5}{2}\left(\delta_{i k} \xi_{j}+\delta_{j k} \xi_{i}\right) r^{2}+\delta_{i j} \xi_{k} r^{2}\right]+\cdots \tag{6.9}
\end{equation*}
$$

This equation tells us that up to this degree of accuracy the correlation function $w_{i} w_{j} w_{k}^{\prime}$ is the sum of $w_{i} w_{j} w_{k}$ and an isotropic correlation tensor; similarly, we have
$\overline{w_{i} w_{m}{ }^{\prime} w_{n}^{\prime}}=\overline{w_{i} w_{m} w_{n}}-\frac{F q^{3}}{3!3 \sqrt{3} \lambda^{3}}\left[2 \xi_{i} \xi_{m} \xi_{n}-\frac{5}{2}\left(\delta_{m i} \xi_{n}+\delta_{n i} \xi_{n}\right) r^{2}+\delta_{m n} \xi_{i} r^{2}\right]+\cdots$
In the above expression the relation (3.7) for isotropic turbulence has been utilized.
In the expansions of (6.9) and (6.10), we have introduced the further assumption that the triple correlation $h$ can be expressed by [8]

$$
\begin{equation*}
h=\frac{F}{3!\lambda^{3}} r^{3} \tag{6.11}
\end{equation*}
$$

where $\lambda$ is Taylor's scale of micro-turbulence and $F$ is a numerical constant which may be different for different flows. This emphasizes the point that this length $\lambda$ plays an important role, not only for double but also for triple correlations as well. The validity of this point should be tested experimentally.

Differentiating the correlation function (6.8) with respect to $x^{\prime s}$, we obtain

$$
\begin{align*}
\frac{\partial}{\partial x^{\prime *}} \overline{w_{i} w_{. k}^{\prime}}= & \overline{w_{i} \frac{\partial w_{k}^{\prime}}{\partial x^{\prime s}}=\frac{\partial}{\partial \xi^{s}}\left(\overline{\left.w_{i} w_{k}^{\prime}\right)_{x}}\right.} \\
= & \frac{q^{2}}{3 \lambda^{2}} \int^{\frac{1}{2}}(1+4 G)\left(\delta_{i s} \xi_{k}+\delta_{k s} \xi_{i}\right)-\left[(2+5 G) \xi_{s}-\frac{1}{3}(k+11 G) R_{s n} \xi^{n}\right] \delta_{i k} \\
& -\frac{1}{3}(k-4 G) \xi_{s} R_{i k}-G\left(R_{i s} \xi_{k}+R_{i k} \delta_{k s} \xi^{l}+R_{k s} \xi_{i}+R_{k l} \delta_{i s} \xi^{l}\right) \\
& \left.+\frac{1}{3!\lambda^{2}} E_{i k j l m s} \xi^{i} \xi^{\prime} \xi^{m}-\cdots\right\} \tag{6.12}
\end{align*}
$$

and furthermore, under the same approximation as in (4.6) where $\partial() \xi / \partial x^{i}$ is neglected, we get

$$
\begin{align*}
\overline{\frac{\partial w_{i}}{\partial x^{p}} \frac{\partial w_{k}}{\partial x^{s}}=} & {\left[\frac{\partial}{\partial x^{p}}\left(\overline{w_{i} \frac{\partial w_{k}^{\prime}}{\partial x^{\prime} \cdot}}\right)\right]_{\xi=0}=-\left(\frac{\partial^{2}}{\partial \xi^{p} \partial \xi^{s}} \overline{w_{i} w_{k}^{\prime}}\right)_{\xi=0} } \\
= & -\frac{q^{2}}{3 \lambda^{2}}\left\{\frac{1}{2}(1+4 G)\left(\delta_{i s} \delta_{k p}+\delta_{k s} \delta_{i p}\right)-\left[(2+5 G) \delta_{s p}-\frac{1}{3}(k+11 G) R_{\varepsilon p}\right] \delta_{i k}\right. \\
& \left.-\frac{1}{3}(k-4 G) \delta_{s p} R_{i k}-G\left(R_{i s} \delta_{k p}+R_{i p} \delta_{k s}+R_{k s} \delta_{i p}+R_{k p} \delta_{i s}\right)\right\} . \tag{6.13}
\end{align*}
$$

Hence the term that represents the decay of turbulence in the equations of double correlation (1.6) is equal to

$$
\begin{equation*}
2 \nu g^{m n} \overline{\frac{\partial w_{i}}{\partial x^{m}} \frac{\partial w_{k}}{\partial x^{n}}}=-\frac{2 \nu}{3 \lambda^{2}}(k-5) q^{2} g_{i k}+\frac{2 \nu k}{\lambda^{2}} \overline{w_{i} w_{k}} . \tag{6.14}
\end{equation*}
$$

If we differentiate the triple correlation (6.9) with respect to the coordinates $x^{\prime m}$, the result is

$$
\frac{\partial}{\partial x^{\prime m}} \overline{w_{i} w_{k} w_{l}^{\prime}}=\overline{w_{i} w_{k} \frac{\partial w_{l}^{\prime} l}{\partial x^{\prime m}}}=\frac{\partial}{\partial \xi^{m}}\left(\overline{\left.w_{i} w_{k} w_{l}^{\prime}\right)_{x}} .\right.
$$

Similarly, to the same order of approximation as in (6.13) the following relation is true:

$$
\frac{\partial}{\partial x^{n}} \overline{w_{i} w_{k} \frac{\partial w_{l}^{\prime}}{\partial x^{\prime m}}}=\overline{w_{i} \frac{\partial w_{k}}{\partial x^{n}} \frac{\partial w_{l}^{\prime} l}{\partial x^{\prime n}}}+\overline{w_{k} \frac{\partial w_{i}}{\partial x^{n}} \frac{\partial w_{l}^{\prime}}{\partial x^{\prime m}}}=-\frac{\partial^{2}}{\partial \xi^{m} \partial \xi^{n}}\left(\overline{\left.w_{i} w_{k} w_{l}^{\prime}\right)_{x}} .\right.
$$

This equation and formula (6.9) then yield

$$
\begin{equation*}
\overline{w_{i} \frac{\partial w_{k}}{\partial x^{n}} \frac{\partial w_{l}}{\partial x^{m}}}+\overline{w_{k} \frac{\partial w_{i}}{\partial x^{n}} \frac{\partial w_{l}}{\partial x^{m}}}=-\left(\frac{\partial^{2}}{\partial \xi^{m} \partial \xi^{n}} \overline{w_{i} w_{k} \cdot w_{l}^{\prime}}\right)_{\xi=0}=0 \tag{6.15}
\end{equation*}
$$

Cyclic permutation of the indices $i, k, l$ in (6.15) gives rise to two similar relations; the sum of the three is identically zero, which shows that the term analogous to the decay of turbulence in the equations of triple correlation vanishes in general:

$$
\begin{equation*}
2 \nu g^{m n}\left[\overline{w_{i, m} w_{k, n} w_{l}}+\overline{w_{k, m} w_{l, n} w_{i}}+\overline{w_{l, m} w_{i, n} w_{k}}\right]=0 . \tag{6.16}
\end{equation*}
$$

7. The equation of vorticity decay. Since Taylor's scale of micro-turbulence $\lambda$ plays a very important role in the decay of turbulence, it is necessary to find the equation which governs the behaviour of this fundamental length. This equation is provided by the decay of vorticity. The root-mean-square of the vorticity fluctuation $\left(\overline{\omega^{2}}\right)^{1 / 2}$ satisfies the equation

$$
\begin{equation*}
\overline{\omega^{2}}=\frac{1}{2} g^{m p} g^{n s} \overline{\omega_{m n} \omega_{p z}} \tag{7.1}
\end{equation*}
$$

where $\omega_{m n}$ is the antisymmetrical tensor defined by (1.5). It is not difficult to derive the equation satisfied by $\overline{\omega^{2}}$ from (1.4) directly. However, this procedure would be too lengthy and we shall pursue an alternative course.

We notice that

$$
\begin{align*}
g^{m p} g^{n s} \overline{\omega_{m n} \omega_{p s}} & =\frac{1}{\tau} \int_{1-\tau / 2}^{t+\tau / 2}\left(w_{m, n}-w_{n, m}\right)\left(w^{m}, ., g^{n s}-w^{n},{ }^{n} g^{m p}\right) d t \\
& =2\left(\overline{w_{m, n} w^{m}, g^{n s}}-\overline{w^{n}, m} w_{, n}^{m}\right. \tag{7.2}
\end{align*} .
$$

On the other hand, to the same order of approximation as in (6.12) and (6.13), the following expressions are true:

$$
g^{n s} \overline{w_{m, n} w^{m i}, s}=-\left(\nabla^{2} \bar{w}_{m} w^{\prime m}\right)_{\xi=0}, \quad \overline{w^{n}, m w^{m}, n}=-\left(\frac{\partial^{2}}{\partial \xi^{m} \partial \xi^{n}} \overline{w^{m} w^{\prime n}}\right)_{\xi=0}=0
$$

where $\nabla^{2} \xi$ stands for the Laplacian operator in the variables $\xi^{i}$. It then follows that

$$
\begin{equation*}
\overline{\omega^{2}}=-\left(\nabla^{2} \overline{w_{m} w^{\prime m}}\right)_{\xi=0} . \tag{7.3}
\end{equation*}
$$

Our next step is to derive the differential equation satisfied by $\left(\nabla^{2} \varepsilon \overline{w_{m} w w^{\prime m}}\right)_{\varepsilon=0}$.
From the equation of turbulent fluctuation at the point $P^{\prime}$, which can be written in the form

$$
\begin{equation*}
\frac{\partial w_{k}^{\prime}}{\partial t}+U^{\prime j} w_{k, j}^{\prime}+w^{\prime j} w_{k, i}^{\prime}+w^{\prime} j U_{k, j}^{\prime}=-\frac{1}{\rho} \varpi^{\prime}, k-\frac{1}{\rho} \tau_{k, j}^{\prime j}+\nu \nabla^{\prime 2} w_{k}^{\prime} \tag{7.4}
\end{equation*}
$$

we derive the equation satisfied by the general double correlation function:

$$
\begin{align*}
& \frac{\partial}{\partial t} \overline{w_{i} w_{k}^{\prime}}+U^{i}\left(\overline{w_{i} w_{k}^{\prime} k}\right)_{, j}+U^{\prime j}\left(\overline{w_{i} w_{k}^{\prime}}\right)^{\prime}, i+\left(\overline{w^{i} w_{i} w_{k}^{\prime} k}\right)_{, j}+\left(\overline{\left.w^{\prime j} w_{i} w_{k}^{\prime}\right)^{\prime}}{ }_{, j}\right. \\
& +\overline{w^{\prime}{ }_{k} w^{i}} U_{i, i}+\overline{w_{i} w^{\prime}{ }^{\prime}} U_{k, i}^{\prime} \\
& =-\frac{1}{\rho}\left(\overline{\omega w_{k}^{\prime}}\right)_{, i}-\frac{1}{\rho}\left(\overline{\omega^{\prime} w_{i}}\right)_{, k}^{\prime}+\nu \nabla^{2}\left(\overline{w_{i} w_{k}^{\prime}}\right)+\nu \nabla^{\prime 2}\left(\overline{w_{i} w_{k}^{\prime} k}\right), \tag{7.5}
\end{align*}
$$

where the covariant derivatives ( ),,$j$ and ()$^{\prime}{ }_{, j}$ are taken with respect to the variables $x^{i}$ and $x^{\prime i}$, respectively.

In Eq. (7.5) we next replace $x^{i}$ and $x^{\prime i}$ by the two new sets of variables $x^{i}$ and $\xi^{i}$ by use of (4.2), and neglect terms involving the partial derivatives with respect to $x^{i}$ when $\xi^{i}$ are held constant, except for the term $U^{i} \partial()_{\xi} / \partial x^{i}$; this exception is made because $U^{i}$ is large when compared with $w^{k}$. Since we are only interested in the correlation functions for two adjacent points, we can write

$$
\left(\overline{w^{\prime} i w_{i} w_{k}^{\prime}}\right)^{\prime}{ }_{i j}=-\frac{\partial}{\partial \xi^{i}}\left(\overline{w^{i} w_{i}^{\prime} w_{k}}\right)_{x},
$$

as in the case of isotropic turbulence (3.7). With all these approximations in view, Eq. (7.5) in rectangular coordinates then becomes

$$
\begin{align*}
& \frac{\partial}{\partial t} \overline{w_{i} w_{k}^{\prime}}+U^{i} \frac{\partial}{\partial x^{j}}\left(\overline{w_{i} w_{k}^{\prime}}\right)_{\xi}-U^{j} \frac{\partial}{\partial \xi^{j}}\left(\overline{w_{i} w_{k}^{\prime}}\right)_{x}+U^{\prime} i \frac{\partial}{\partial \xi^{j}}\left(\overline{w_{i} w_{k}^{\prime}}\right)_{x} \\
&\left.-\frac{\partial}{\partial \xi^{j}} \overline{\left(w^{j} w_{i} w_{k}^{\prime}\right.}+\overline{w^{i} w_{i}^{\prime} w_{k}}\right)_{x}+\overline{w_{k}^{\prime} w^{i}} \frac{\partial U_{i}}{\partial x^{j}}+\overline{w_{i} w^{j i}} \frac{\partial U_{k}^{\prime}}{\partial \xi^{i}} \\
&=\frac{1}{\rho} \frac{\partial}{\partial \xi^{i}}\left(\overline{\left(w_{k}^{\prime}\right.}\right)_{x}-\frac{1}{\rho} \frac{\partial}{\partial \xi^{k}}\left(\overline{\left(\omega^{\prime} w_{i}\right.}\right)_{x}+2 \nu \nabla^{2}\left(\overline{\left(w_{i} w_{k}^{\prime}\right.}\right)_{x} . \tag{7.6}
\end{align*}
$$

For two adjacent points the power series expansion of $\overline{\varpi w_{k}^{\prime}}$ in $\xi^{i}$ is in odd powers of $\xi^{i}$. Hence, by interchanging the two points $P$ and $P^{\prime}$, we should have

$$
\begin{equation*}
\overline{\omega w_{i}^{\prime}}=-\overline{\omega^{\prime} w_{i}} . \tag{7.7}
\end{equation*}
$$

Consequently (7.6) is essentially symmetric in the indices $i$ and $k$.
Next, let us contract the indices $i$ and $k$ in (7.6). As in (4.4), $\overline{\omega w^{\prime}}{ }_{k}$ should satisfy rigorously the equation of continuity,

$$
\begin{equation*}
\frac{\partial}{\partial \xi^{i}}\left(\overline{\omega \psi^{\prime k}}\right)_{x}=0 . \tag{7.8}
\end{equation*}
$$

The result of this contraction then becomes

$$
\begin{align*}
\frac{\partial}{\partial t} \overline{w_{k} w^{\prime k}} & +U^{j} \frac{\partial}{\partial x^{i}}\left(\overline{w_{k} w^{\prime k}}\right)_{\xi}-U^{i} \frac{\partial}{\partial \xi^{i}}\left(\overline{w_{k} w^{\prime k}}\right)_{x}+U^{\prime ;} \frac{\partial}{\partial \xi^{i}}\left(\overline{w_{k} w^{\prime k}}\right)_{x} \\
& -2 \frac{\partial}{\partial \xi^{i}}\left(\overline{w^{i} w_{k} w^{\prime k}}\right)_{x}+\overline{w_{k}^{\prime} w^{\prime}} \frac{\partial U^{k}}{\partial x^{i}}+\overline{w_{k} w^{\prime j}} \frac{\partial U^{\prime k}}{\partial \xi^{i}}=2 \nu \nabla^{2}\left(\overline{w_{k} w^{\prime k}}\right)_{x .} . \tag{7.9}
\end{align*}
$$

Let us operate upon (7.9) with the Laplacian operator $\nabla^{2} \xi$ and then set $\xi^{i}=0$, denoting ( $)_{\xi=0}$ by ( $)_{0}$ for smiplicity. The resulting equation is,

$$
\left.\begin{array}{rl}
\frac{\partial}{\partial t}\left(\nabla^{2} \xi \overline{w_{k} w^{\prime k}}\right)_{0} & +U^{j} \frac{\partial}{\partial x^{j}}\left(\nabla^{2} \xi \overline{w_{k} w^{\prime k}}\right)_{0}+2 U^{j},{ }_{, m} g^{m n}\left(\frac{\partial^{2}}{\partial \xi^{j} \partial \xi^{n}} \overline{w_{k} w^{\prime k}}\right)_{0} \\
& -2\left(\nabla^{2} \xi \frac{\partial}{\partial \xi^{j}} \overline{w^{j} w_{k} w^{\prime k}}\right)_{0}+2 U_{k, j}\left(\nabla^{2} \xi \overline{w^{j}} w^{\prime k}\right. \tag{7.10}
\end{array}\right)_{0}=2 \nu\left(\nabla^{4} \xi \overline{w_{k} w^{\prime k}}\right)_{0} .
$$

It is to be noted that in the above equation we have neglected the term $\overline{w_{k}} w^{j} \nabla^{2}{ }_{x} U^{k}, j$, which is smaller than the term $U_{k, j}\left(\nabla^{2} \overline{w^{i} w^{\prime k}}\right)_{0}$ by a factor which is the square of the ratio of $\lambda$ to a macroscopic length. Equation (7.10) also follows from the equation of vorticity fluctuation (1.4) directly as mentioned before.

By substituting into (7.10) the explicit forms of the correlation functions $\overline{w_{i} w_{k}^{\prime}}$ and $\bar{w}_{i} w_{j} w_{k}$ for two adjacent points given in (6.8) and (6.9), respectively, and then setting $\xi^{\ell}=0$, we obtain the equation of vorticity decay,

$$
\begin{equation*}
5 \frac{\partial}{\partial l}\left(\frac{q^{2}}{\lambda^{2}}\right)+5 U^{j} \frac{\partial}{\partial x^{j}}\left(\frac{q^{2}}{\lambda^{2}}\right)-\frac{14 G}{\lambda^{2}} U_{i, k} \overline{w^{i} w^{k}}-\frac{70 F}{3 \sqrt{3}} \frac{q^{3}}{\lambda^{3}}=-\frac{2 \nu}{3} E \frac{q^{2}}{\lambda^{4}} \tag{7.11}
\end{equation*}
$$

in which $E$ is defined as before,

$$
\begin{equation*}
E=g^{i k} g^{j i} g^{m n} E_{i k j l m n} \tag{7.12}
\end{equation*}
$$

We assume that both $E$ and $F$ are constants which may be different for flows with different Reynolds numbers. In deriving equation (7.11), the equation of continuity $U^{j}{ }_{, j}=0$ for the mean motion has been utilized. It is also readily verifiable that (7.11) agrees with von Kármán's equation of vorticity decay for isotropic turbulence [8].
8. Nature of the correlation integrals and the final forms of the dynamical equations of correlation. Up to the present the only remaining uncertain quantities in the equations of the double and triple correlations (1.6) and (1.7) are the correlation integrals, $a^{n}{ }_{m i k}$, and $b_{i k}$ in (5.3), $b_{m i k l}$ and $c_{i k l}$ of (5.8), and the quadruple velocity correlation $\overline{w_{,} w_{i} w_{k} w_{l}}$. Let us examine the correlation integrals first. The function $a^{n}{ }_{m i k}$ defined in (5.3), for example, would be uniquely determined if the double correlation $\overline{w_{i} w_{k}^{\prime}}$ were known. But unfortunately the equation of continuity (4.4) and the general dynamical equation of double correlation (7.6) are insufficient to yield a definite solution for $\overline{w_{i} w_{k}^{\prime}}$, because of the presence of the triple correlation $\overline{w_{i} w_{j} w_{k}^{\prime}}$ in (7.6).

On the other hand, although the integrand functions of the four kinds of correlation integrals are not known, we are dealing primarily with the integrals themselves and they can only vary slowly with the coordinates involved. This argument can be understood, if we recall that the correlation functions $\overline{w_{i} w_{n}^{\prime}}, \overline{w_{i} w_{k} w_{n}^{\prime}}, \overline{w_{i} w_{m}^{\prime} w^{\prime n}}$ and $\overline{u_{i k} u_{m n}^{\prime}}$ under the integral signs only change slowly when both the point $P$ and the point of integration $P^{\prime}$ undergo a rigid body translation, and that they vary rapidly when the relative displacement of the two points changes. This rapidly varying part of the functions is integrated away, leaving the slowly varying part behind. The neglecting of the term $\partial\left(\overline{w_{n}^{\prime} w^{i}}\right)_{\xi} / \partial x^{i}$ against $\partial\left(\overline{w_{n}^{\prime} w^{i}}\right)_{x} / \partial \xi^{i}$ in (4.5) also follows from this interpretation.

There is another mathematical reason for the fact that the four kinds of integrals
are slowly varying functions of the coordinates. If, for instance, we differentiate with respect to $x^{s}$ the quantities $c_{i k l}$ defined in (5.8), we find that

$$
\begin{align*}
\frac{\partial c_{i k l}}{\partial x^{s}}= & \frac{1}{4 \pi} \iiint\left\{\frac{\partial}{\partial x^{s}}\left(\left[\left(\overline{u^{\prime m n} u_{i k}}\right)_{, m n l}+\left(\overline{u^{\prime m n} u_{k l}}\right)_{, m n i}+\left(\overline{u^{\prime m n} u_{l i}}\right)_{, m n k}\right] \frac{1}{r}\right)_{\xi}\right. \\
& \left.-\frac{\partial}{\partial \xi^{s}}\left(\left[\left(\overline{u^{\prime m n} u_{i k}}\right)_{, m n l}+\left(\overline{u^{\prime m n} u_{k i}}\right)_{, m n i}+\left(\overline{u^{\prime m n} u_{l i}}\right)_{, m n k}\right] \frac{1}{r}\right)_{x}\right\} d V^{\prime} . \tag{8.1}
\end{align*}
$$

The first part of the integrand function is small when compared with the second, and the second can be transformed into a surface integral on the boundary of the fluid by means of the usual divergence theorem of vector analysis. If the point $P$ is not very close to the surface, this surface integral is negligible on the ground that the correlation function $\overline{u^{\prime n \pi} u_{i k}}$ and its derivatives between $P$, the point in the interior of the fluid, and $P^{\prime}$, the point of integration on the boundary, are negligible.

Since the correlation integrals are slowly varying functions of the coordinates, we shall expand them as powers of the coordinates used in the special problems to be solved. From kinematic considerations, the integrands of the integrals may furthermore contain powers of $q$, the root-mean-square of the velocity fluctuation, as factors. Both theory and experiment at present do not assure us of the exact dependence of this factor. Nevertheless, so far as the mean velocity distribution is concerned, this uncertainty is probably not important, as we shall see in the problem of pressure flow between two parallel infinite planes [9].

By substituting into Eqs. (1.6) and (1.7) the approximate forms of the four correlation integrals from (5.2) and (5.7), and the decay terms (6.14) and (6.16), we obtain finally

$$
\begin{align*}
& -\frac{1}{\rho} \frac{\partial \tau_{i k}}{\partial t}-\frac{1}{\rho}\left(U_{i, j} \tau^{i}{ }_{k}+U_{k, \tau^{j} \tau_{i}}\right)-\frac{1}{\rho} U^{i} \tau_{i k, j}+\left(\overline{w^{j} w_{i} w_{k}}\right)_{, j} \\
& =-a_{m i k}^{n} U^{m}, n-b_{i k}-\frac{\nu}{\rho} \nabla^{2} \tau_{i k}+\frac{2 \nu}{3 \lambda^{2}}(k-5) q^{2} g_{i k}-\frac{2 \nu k}{\lambda^{2}} \overline{w_{i} w_{k}},  \tag{8.2}\\
& \frac{\partial}{\partial t} \overline{w_{i} w_{k} w_{l}}+U_{i, j} \overline{w^{j} w_{k} w_{l}}+U_{k, j} \overline{w^{i} w_{l} w_{i}}+U_{l, j} \overline{w^{j} w_{i} w_{k}}+U^{j}\left(\overline{w_{i} w_{k} w_{l}}\right)_{, j}+\left(\overline{w^{i} w_{i} w_{k} w_{l}}\right)_{, j} \\
& =-b_{m i k l}^{n} U^{m}, c_{i k l}+\frac{1}{\rho^{2}}\left(\tau_{i, j}^{j_{k l}}+\tau_{k, j}^{i_{l i}}+\tau_{l, j} \tau_{k i}\right)+\nu g^{m n n}\left(\overline{w_{i} w_{k} w_{l}}\right)_{, m n} . \tag{8.3}
\end{align*}
$$

In the second set of equations we notice that the term involving the quadruple correlation is actually smaller than the terms $b_{m i k l} U^{m}, n$ and $c_{i k l}$ which form the correlation between the pressure gradient and two components of velocity fluctuation. This is due to the fact that the term $\left(\overline{w^{i} w_{i} w_{k} w_{l}}\right)_{, j}$ is equal to a velocity fluctuation raised to the fourth power and divided by a macroscopic length, while on the other hand $c_{i k l}$ is, from its definition (5.8), of the order of a velocity fluctuation raised to the fourth power and divided by a length which has the same order of magnitude as Taylor's scale of micro-turbulence. The permissibility of neglecting the terms ( $w_{i} w_{i} w_{k} w_{l}$ ),j and $\rho^{-2} \tau_{i, j} \tau_{k l}$ as a first approximation, for instance in the problem of pressure flow between two parallel infinite planes [9], can be regarded as a justification of the above approximation and its associated interpretation.

We must not forget that the other dynamical equations necessary for the solution of a turbulence problem are the equations of mean motion (1.1) and the equation of vorticity decay (7.11).
9. Conclusion and summary. It is now not difficult to see that the foregoing development is essentially a method of successive approximation to the solution of the turbulence problem. In the initial approximation we have the well-known Reynolds' equations of mean motion which contain the unknown apparent stress. From the mathematical point of view the momentum and vorticity transport theories connect this stress with the mean velocity by physical arguments, in order to make the mean velocity distribution determinate.

The next approximation in solving the given turbulence problem is to use the equations of mean motion and of double correlation by making certain approximations to the triple velocity correlation in the equations. This procedure has been followed in the determination of the velocity distributions in jets.[3] and wakes [4], where free turbulence predominates; for the triple correlations we use their values at the centers of the flows as an approximation. The mean velocity distributions thus obtained agree with the experimental observations very well over large portions of the flows.

In the third approximation to the solution of the problem we have to solve the equations of mean motion and of both the double and triple correlations simultaneously by assuming approximations for the quadruple correlations. It is obvious that this process of forming the differential equations of the correlations out of the equations of turbulent fluctuation can be generalized to higher orders. Fortunately, as in the problem of pressure flow through a channel [9] where a wall is present, we can stop at the equations of triple correlation and neglect the quadruple correlations as an approximation, so that the solution of the problem is not too unnecessarily complicated from the theoretical point of view. As we shall see, the solution of this particular problem holds true in all parts of the channel, if all the equations of mean motion and of double and triple correlation are used. On the other hand, the solution for the mean velocity based upon the equations of mean motion and of double correlation by using the value of the triple correlation in the center of the channel as in jets and wakes, is only valid in the central part of the channel, and fails when the wall of the channel is approached. This brings up incidentally the important role played by the triple correlation in such problems.

In order to see more clearly how the equations of double and triple correlation in the forms (8.2) and (8.3) and the equation of vorticity decay (7.11) are derived from the equations of turbulent fluctuation, it might be of interest to sum up the conditions and approximations under which they are valid. They are listed below:
(1) The velocity correlation between a point in the interior of the fluid and another on the boundary is negligible. This excludes the immediate neighborhood of the boundary of the fluid as a region of application of the theory.
(2) The variation of the mean velocity is small as compared with the correlation function between two distinct points when the relative displacement between the points changes, so that the higher order terms in the series (5.1) and similar series may be dropped.
(3) The second and third order velocity correlations between two adjacent points
are expansible as power series in $\xi^{i} / \lambda$ with the terms that do not contain $\xi^{i}$ proportional to the Reynolds stress at the points. This brings out the point that Taylor's scale of micro-turbulence $\lambda$ plays an equally important role for both the double and triple velocity correlations.
(4) The slowly varying nature of the functions $a_{m i k}^{n}, b_{i k}, b_{m i k l}^{n}$ and $c_{i k l}$ with the coordinates, and its physical interpretation, have been explained in the preceding section.

With the advance of modern experimental technique the above four conditions and their theoretical consequences, as presented here, can all be tested by direct experimental observation. The less certain part of the theory lies probably in the discussions in $\S 8$ of the slowly varying nature of the correlation integrals with the coordinates; this perhaps could be improved if more accurate experimental evidence were available.

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# QUANTITATIVE INTERPRETATION OF MAPS OF MAGNETIC AND GRAVITATIONAL ANOMALIES BY MATHEMATICAL METHODS* 

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1. Introduction. In geophysical prospecting for oil and other minerals gravitational and magnetic anomalies corresponding to geological phenomena are mapped. The problem of the quantitative interpretation of such empirical maps consists in the determination of numerical values for all the geological parameters (depth, thickness, slope, density, intensity and direction of magnetization, etc.) which characterize a tectonic structure or an ore-body. To illustrate the possibility of such an interpretation it is preferable to avoid the complications involved in the mathematical study of maps of complex anomalies. The complex anomalies are due to the coexistence in the same region of many different geological phenomena. Resulting from the superposition of many simple anomalies, they can be resolved into their simple components each of which corresponds to a single ore-body or tectonic structure. This resolution is the first step which must be performed, since no interpretation of a complex anomaly map as such is possible. The problem of resolution is a very important one since in most cases we have to deal with complex anomalies, the simple ones being exceptions. Special methods devised by the author solve this important problem, but they are not discussed in this paper which deals with the quantitative interpretation of a simple anomaly map. We study here two cases: an axial anomaly created by an anticline and a centered anomaly corresponding to a salt dome. They are sufficiently simple and at the same time have great practical importance.

A new method of interpretation, based as the usual methods on the theory of potential but essentially different from them, is introduced in this paper. In the usual methods ${ }^{1}$ systematic use is made of individual values such as maxima, minima, zeros, inflection points, etc., of the observed and plotted quantity as well as of their distances. The use of such remarkable values and distances is founded on the tacit assumption that they reflect exclusively the physical action of the unknown structure or ore-body whose study is the object of the interpretation. This assumption is permissible for the anomalies of large magnitude but it is doubtful for those of average magnitude and completely wrong for small anomalies.

In the past, geophysical prospecting by gravitational and magnetic methods was directed mostly toward the study of important, clearly pronounced anomalies of large magnitude which correspond to more shallow deposits or to big well defined tectonic structures. But now the geophysicists are obliged to deal with more difficult

[^18]cases and they have to interpret maps of small anomalies. Thus, we must study the obstacles which naturally lead to an erroneous interpretation of an anomaly map by the usual methods of remarkable values and distances if the magnitude of the anomaly is not very large.
2. Punctual anomaly. An anomaly map is plotted on the basis of measurements made at isolated stations. It is supposed to be generally correct, and the usual corrections required by the topography of the region, the fact that the earth is not a sphere, etc., are supposed to have already been made. We must emphasize that a quantitative interpretation presupposes an accurate correction for the so-called regional anomaly, and it cannot be expected to yield good results if the latter correction is made, as is customarily (and very unfortunately) done, simply by smoothing arbitrarily the experimental curves. Special methods exist which ensure a very accurate correction for regional anomaly by deducing it from the map itself, but this important question cannot be discussed here. All usual corrections having been made, each individual value obtained at a station is the combined effect of two anomalies: 1) the anomaly caused by the tectonic structure or the ore-body the study of which is the purpose of the interpretation, and 2) the anomaly generated by local irregularities of mass distribution or of magnetization intensity in the immediate vicinity of and under the point of measurement. This strictly local anomaly-we propose to call it "punctual anomaly"-is in general very small. It affects only a small area around the point and it is precisely this punctual anomaly which is responsible for perceptible variations in the value of the observed quantity which occur for small displacements of the apparatus used around the station. In fact the apparatus used now are extremely sensitive and we cannot neglect any more the existence of punctual anomalies. There is no correction at all for them since the punctual anomalies, affecting every observed individual value, cannot be evaluated. If the magnitude of the studied anomaly is large, the punctual anomalies are negligible and the map can be interpreted with the aid of remarkable values and distances. The positive results achieved by the old interpretation methods must be explained in this way. But, if the magnitude of the anomaly is small, the punctual anomalies not only modify the extremal values but they also displace them, altering all the distances used in the usual interpretation methods. Since these old methods express all geological parameters in terms of remarkable values, their distances and ratios, it is plain that punctual anomalies render these methods completely useless in the interpretation of small anomalies. It is a very important though often disregarded fact that the interpretation based on isolated values can in general be only qualitative and gives exactly nothing in case of small anomalies. This important fact explains the lack of success in dealing with maps of small anomalies and is the reason for the actual ineffectiveness of geophysical prospecting in discovering new oilfields in U.S.A. New methods of interpretation well adapted to small anomalies are now necessary. They must be introduced into practice if the geophysical prospecting by gravitational and magnetic methods is to be applied in the future.

The interpretation errors caused by punctual anomalies can and must be eliminated and there is only one possible way to do $i t$. Considered together, the punctual anomalies in the region covered by the measurements have a random distribution; they oscillate about zero and are independent one from another. Consequently, they must undergo an almost total compensation if we form the average value of some
function of the mapped quantity, the average value with respect to the whole map, using all the observed values at a time. We can eliminate the harmful influence of punctual anomalies and compensate at the same time for the possible residual observation errors only by combining all observed values in an integral. For any map in general the average values express much better the action of the phenomenon under study than do the individual values and their distances. In other words, interpretation rules and formulae based on the systematic use of integrals give much more correct quantitative results in all cases and for all maps. Rules based on special individual values hold only in very rare and exceptional cases of big structures such as, for instance, the shallow salt domes of Texas, the Kursk and Kirunawaara iron ore-bodies or the Great Rhodesian Dyke.

The method described in this paper uses exclusively the average values and, in particular, moment functions and moments of the observed quantity and of its square. This method was applied by the author in France and in Iran with good results. The cases studied here are chosen only for the sake of brevity. The method is elaborated for the most general cases of complex anomaly maps obtained as result of magnetic or gravimetric survey of a completely unexplored region.
3. Center of gravity and first moments. The problem of locating the center of gravity $C$ of disturbing masses is a fundamental one, and its solution is the first step of every interpretation. We solve it in the general cases of an axial anomaly and of a centered anomaly. The coordinates $x^{*}, y^{*}, z^{*}$ of $C$ are expressed with the aid of moments of the observed quantity $Q$, that is in terms of

$$
Q_{m}=\int_{-\infty}^{\infty} x^{m} Q(x) d x
$$

for an axial anomaly and in terms of
$Q_{m n}=\iint_{P} x^{m} y^{n} Q(x, y) d S, \quad Q_{m}=\iint_{P} r^{m} Q(r, \phi) d S, \quad\left(r^{2}=x^{2}+y^{2}, x \tan \phi=y\right)$
for a centered anomaly, the double integration being extended over the infinite plane $x O y$ denoted by $P$.

Axial anomaly. If the geologic feature being considered is much longer in one dimension (strike, axis of anomaly), the corrected map of the axial anomaly created by such a structure is a family of nearly parallel lines and the interpretation deals with a curve describing the behaviour of the plotted quantity on a typical profile perpendicular to the anomaly axis. Cartesian coordinates $x, y, z$ are introduced with the $z$-axis directed vertically downward, the origin $O$ being at the surface and the $x$-axis being perpendicular (the $y$-axis being parallel) to the anomaly axis, as shown in Fig. 1. The excess of the density of disturbing masses over the density of their environment is called the density-contrast and is denoted by $\sigma(\sigma \lessgtr 0)$. We represent the disturbing structure as a homogencous cylindrical body of normal cross section $S$, denoting the area of $S$ by $A$. At a point $(x, z)$ in the plane $y=0$ the potential $U$ of the body is given by

$$
\begin{equation*}
U(x, z)=-\frac{1}{2} k \iint_{S} \log \left[(x-\xi)^{2}+(z-\zeta)^{2}\right] d S+\text { const. } \tag{1}
\end{equation*}
$$

where $\xi, \zeta$ are running coordinates on $S$ and $k=2 f \sigma, f$ being the gravitational constant
( $66.7 \times 10^{-9} \mathrm{cgs}$.) and $\sigma$ the density-contrast. On the surface (plane $P$ ), $z=0$, the partial derivatives $U_{x}$ and $U_{z}=D g$ of $U$ are given by

$$
\begin{equation*}
U_{x}-i D g=-k \iint_{S}(x-\rho)^{-1} d S=k \oint_{\Gamma} \log (x-\rho) d \zeta \tag{2}
\end{equation*}
$$

where $\rho=\xi+i \zeta$ and $\Gamma$ is the boundary of $S$. The second derivatives $U_{x x}$ and $U_{x z}$ are called the curvature $K$ and the gradient $G$ respectively: $K=U_{x x}, G=U_{x z}$. Thus

$$
\begin{equation*}
K-i G=k \iint_{S}(x-\rho)^{-2} d S=k \oint_{\mathrm{F}}(x-\rho)^{-1} d \zeta \tag{3}
\end{equation*}
$$

Using in (2), (3) the binomial expansion, we deduce for large $|x|$ the approximations which hold for any form of the section $S$. Their first terms for instance are

$$
\begin{equation*}
U_{x} \sim-k A x^{-1}, \quad D g \sim k A z^{*} x^{-2}, \quad G \sim-2 k A z^{*} x^{-3}, \quad K \sim k A x^{-2} \tag{4}
\end{equation*}
$$

where $z^{*}$ is the depth of the center of gravity $C$. Denoting an arbitrarily chosen origin of the coordinate $x^{\prime}$ on the profile by $O$, we regard the function $D g=D g\left(x^{\prime}\right)$ as known from the measurements. To find $x^{*}=O O^{*}, z^{*}=O^{*} C$ we shall use the first three moments of $D g$,

$$
\begin{align*}
M_{0}= & \int_{-\infty}^{\infty} D g\left(x^{\prime}\right) d x^{\prime}=\pi k A, \quad M_{1}=\int_{-\infty}^{\infty} x^{\prime} D g\left(x^{\prime}\right) d x^{\prime}=\pi k A x^{*}  \tag{5}\\
& \int_{-\infty}^{\infty}\left[x^{2} D g(x)-k A z^{*}\right] d x=\pi k \iint_{S}\left(\xi^{2}-\zeta^{2}\right) d S \tag{6}
\end{align*}
$$

where in (6) the origin of the coordinate $x$ is the point $O^{*}$, the projection of $C$ on the profile, $x^{*}$ being considered as already found with the aid of (5). In fact, from (2) we deduce that

$$
\begin{equation*}
x^{\prime} U_{x}+k A-i x^{\prime} D g\left(x^{\prime}\right)=k \iint_{S}\left(\rho-x^{\prime}\right)^{-1} \rho d S \tag{7}
\end{equation*}
$$

Integrating (2) and (7) with respect to $x^{\prime}$ in $(-\infty, \infty)$ and observing that the integral of $\left(x^{\prime}-\rho\right)^{-1} d x^{\prime}$ equals $i \pi$ since $\zeta$ in $\rho=\xi+i \zeta$ is positive, we have (5). To prove (6) we integrate in $(-\infty, \infty)$ for $x^{*}=0$ the relation

$$
x\left(x U_{x}+k A\right)-i\left(x^{2} D g(x)-k A z^{*}\right)=k \iint_{S}(\rho-x)^{-1} \rho^{2} d S
$$

and compare the coefficients of imaginary terms.
From (5) we deduce the rule $x^{*}=M_{1} / M_{0}$. However, in practice the integration can be carried out only over a finite interval $(-R, R)$, where the known length $R$ is at least four or five times the depth $z^{*}$ of $C$. The contributions from the intervals $(-\infty,-R),(R, \infty)$ can be computed by means of the expansion of $D g(x)$ in powers of $x^{-1}$,

$$
\begin{equation*}
D g(x)=k A z^{*} x^{-2}\left\{1+2 c_{11} z^{*} x^{-1}+\left(3 c_{21}-c_{03}\right) z^{* 2} x^{-2}+0\left(x^{-3}\right)\right\} \tag{8}
\end{equation*}
$$

where the constants $c_{m n}$ are given by

$$
\left(z^{*}\right)^{m+n} A c_{m n}=\iint_{S} \xi^{m} \xi^{n} d S
$$

In general, unless $S$ is very irregular, we have $z^{*} c_{11} \doteqdot x^{*}, z^{* 2} c_{21} \div x^{* 2}, c_{03} \doteqdot 1$, whence (8) takes the form

$$
\begin{equation*}
D g(x) \doteqdot k A z^{*} x^{-2}\left\{1+2 x^{*} x^{-1}+\left(3 x^{* 2}-z^{* 2}\right) x^{-2}+0\left(x^{-3}\right)\right\} \tag{9}
\end{equation*}
$$

If $M_{0}^{*}$ and $M_{1}^{*}$ denote moments computed for the interval $(-R, R)$, i.e.,

$$
M_{0}^{*}=\int_{-R}^{R} D g\left(x^{\prime}\right) d x^{\prime}, \quad M_{1}^{*}=\int_{-R}^{R} x^{\prime} D g\left(x^{\prime}\right) d x^{\prime}
$$

and if we neglect terms of the relative order $O\left\{\left(z^{*} / R\right)^{2}\right\}$, we find that

$$
\begin{equation*}
\pi k A=M_{0}=\left(1+\frac{2 z^{*}}{\pi R}\right) M_{0}^{2}, \quad M_{1}=\left(1+\frac{4 z^{*}}{\pi R}\right) M_{1}^{*} \tag{10}
\end{equation*}
$$

whence

$$
\begin{equation*}
x^{*}=\left(1+\frac{2 z^{*}}{\pi R}\right) \frac{M_{1}^{*}}{M_{0}^{*}} \tag{11}
\end{equation*}
$$

$M_{0}^{*}$ and $M_{1}^{*}$ can be obtained from the experimental curve for $D g\left(x^{\prime}\right)$ by means of a planimeter. From (11), we note that $M_{1}^{*} / M_{0}^{*}$ is a first approximation for $x^{*}$.

If we set $x^{*}=0$, by (9) we easily see that the contribution from the intervals $(-\infty,-R),(R, \infty)$ to the integral on the left side of $(6)$ is of order $\left(z^{*} / R\right)^{2}$. If we neglect terms of order $\left(z^{*} / R\right)^{2}$, (6) can be written in the form

$$
\frac{\pi}{2 R} M_{2}^{*}=M_{0 z} z^{*}+\frac{k \pi^{2}}{2 R} \iint_{S}\left(\xi^{2}-\zeta^{2}\right) d S,
$$

where, in the usual notation,

$$
M_{2}^{*}=\int_{-R}^{R} x^{2} D g(x) d x
$$

If we consider only those cases in which the horizontal dimension of $S$ is much smaller than its depth, then $\iint_{S}\left(\xi^{2}-\zeta^{2}\right) d S \doteqdot-z^{* 2} A$. Using this, and substituting for $M_{0}$ from (10), we have
$z^{*}=\frac{\pi M_{2}^{*}}{2 R M_{0}^{*}}\left[1+\frac{\left(\pi^{2}-4\right) z^{*}}{2 \pi R}\right]$.
Equations (11) and (12) permit us to compute $x^{*}$ and $z^{*}$ by successive approximations, the successive values being denoted by $x_{n}{ }^{*}, z_{n}{ }^{*}$ ( $n=1,2,3, \cdots$ ) and the corresponding positions of $O^{*}$ (Fig. 1) by $O_{n}^{*}$. The steps are as follows: (a) We choose a position for $O$ and compute $M_{0}{ }^{*}, M_{1}^{*}$, integrating over the interval $-R>x^{\prime}<R$ with a planimeter.
(b) We obtain $x_{1}^{*}$ by setting $z^{*}=0$ in (11) and plot $O_{1}^{*}$. (c) We compute $M_{2}^{*}$


Fig. 1.
and new values of $M_{0}^{*}$ and $M_{1}^{*}$. (d) Using (12) with $z^{*}=0$ in the right member we compute $z_{1}^{*}$. (e) Using the values of $M_{0}^{*}, M_{1}^{*}, M_{2}^{*}$ obtained in (c), and replacing $z^{*}$ in the right members of (11) and (12) by $z_{1}^{*}$, we obtain $x_{2}^{*}, z_{2}^{*}$ and plot $O_{2}^{*}$. The steps (c) and (e) are repeated until a stabilization of the points $0_{n}^{*}$ and the values $x_{n}^{*}, z_{n}^{*}$ is reached.

The same method can be applied to maps obtained by means of a torsion-balance, giving the curves of the gradient $G(x)$ and the curvature $K(x)$. Denoting the $n$th moment of $G$ by $H_{n}$, we have $H_{0}=0$. Also, integration by parts yields $H_{1}=-M_{0}$, $H_{2}=-2 M_{1}, H_{3}=-3 M_{2}$. For the corresponding reduced moments we have $M_{0}^{*}=$ $-H_{1}^{*}\left(1+2 z^{*} \pi^{-1} R^{-1}\right), 2 M_{1}^{*}=-H_{2}^{*}\left(1+2 z^{*} / \pi R\right), 3 M_{2}^{*}=-H_{3}^{*}+2 k A R z^{*}$. Thus (11), (12) can be transformed into

$$
\begin{equation*}
x^{*}=\frac{H_{2}^{*}}{2 H_{1}^{*}}\left(1+\frac{2 z^{*}}{\pi R}\right), \quad z^{*}=\frac{\lambda H_{3}^{*}}{R H_{1}^{*}}\left(1+\frac{\mu z^{*}}{R}\right), \tag{13}
\end{equation*}
$$

where $\lambda=\frac{1}{2} \pi /(3 \pi-1)=0.187, \mu=\frac{1}{2}\left(3 \pi^{2}-24 \pi+8\right) /\left(3 \pi^{2}-\pi\right)=0.485$. Equations (13) permit us to determine the center of gravity $C$ from the gradient map only.

The first two moments $L_{0}$ and $L_{1}$ of the curvature $K$ vanish. Since for $|x|$ very large $K(x) \sim k A x^{-2}$, we define the moment $L_{2}$ and the reduced moment $L_{2}^{*}$ by the integrals

$$
L_{2}=\int_{-\infty}^{\infty}\left(x^{2} K-\frac{H_{1}}{\pi}\right) d x, \quad L_{2}^{*}=\int_{-R}^{R}\left(x^{2} K-\frac{H_{1}^{*}}{\pi}\right) d x .
$$

If we multiply (3) by $x^{2}$, subtract $k A$ from both sides and integrate over $(-\infty, \infty)$, we find that $L_{2}=2 H H_{1} z^{*}$, the contribution from the intervals $(-\infty,-R),(R, \infty)$ being $-6 k A z^{* 2} / R$. Since $H_{1}=H_{1}^{*}\left(1+4 z^{*} / \pi R\right)$, we then have for $z^{*}$

$$
\begin{equation*}
H_{1}^{*} z^{*}=\alpha \mathrm{L}_{2}^{*}\left(1+\beta z /{ }^{*} R\right), \tag{14}
\end{equation*}
$$

where $\alpha=\frac{1}{2} \pi^{2} /\left(\pi^{2}-4\right)=0.84, \beta=\pi /\left(\pi^{2}-4\right)=0.535$. Equation (14) supplies a control on Eqs. (13).

The magnetic anomaly created by a cylindrical body of section $S$ is related to the gravitational anomaly generated by the same body, and the equations relating to maps of $G$ and $K$ can be transformed into equations relating to maps of the horizontal and vertical components $X$ and $Z$ of the abnormal magnetic field created by the body. If $I$ and $\psi$ denote the magnitude and inclination of the magnetization vector, we have the classical relation (Poisson)

$$
\begin{equation*}
k(X+i Z)=2 I(K+i G) e^{-i \psi}, \tag{15}
\end{equation*}
$$

where $k=2 f \sigma, f$ being the constant of gravitation and $\sigma$ the density-contrast. Multiplying (15) by $x$ and integrating over the interval $(-R, R)$, we obtain for the reduced moments $X_{1}^{*}, Z_{1}^{*}$ the relation $k\left(X_{1}^{*}+i Z_{1}^{*}\right)=2 I\left(L_{1}^{*}+i H_{1}^{*}\right) e^{-i \psi}$. Since $L_{1}=0$, we easily find that $\pi R L_{1}^{*}=4 x^{*} H_{1}^{*}\left(1+4 z^{*} / \pi R\right)$. Thus

$$
k X_{1}^{*}=2 I H_{1}^{*}\left(\sin \psi+\frac{4 x^{*}}{\pi R} \cos \psi\right), \quad k Z_{1}^{*}=2 I H_{1}^{*}\left(\cos \psi-\frac{4 x^{*}}{\pi R} \sin \psi\right),
$$

whence we obtain for the two parameters $I A$ and $\psi$,

$$
I A=-\frac{I H_{1}^{*}}{k \pi}=\frac{1}{2 \pi}\left(X_{1}^{* 2}+Z_{1}^{* 2}\right)^{1 / 2}, \quad Z_{1}^{*} \tan \psi=X_{1}^{*}\left(1-\frac{c x^{*}}{R}\right)
$$

where $c$ is defined by the relation $\pi X_{1}{ }^{*} Z_{1}^{*} c=4\left(X_{1}{ }^{* 2}+Z_{1}^{* 2}\right)$. To find $x^{*}, z^{*}$ we need second moments. Since the principal term of the right side of (15) involves $K$, and $K \sim k A x^{-2}$, for large $|x|$ we have $\pi X \sim Z_{1}^{*} x^{-2}, \pi Z \sim-X_{1}{ }^{*} x^{-2}$. Therefore we define the second reduced moments $X_{2}^{*}, Z_{2}^{*}$ by the integrals

$$
X_{2}=\int_{-R}^{R}\left(x^{2} X-Z_{1}^{*} \pi^{-1}\right) d x, \quad Z_{2}^{*}=\int_{-R}^{R}\left(x^{2} Z+X_{1}^{*} \pi^{-1}\right) d x
$$

Their numerical values can be obtained from the experimental data as areas under curves deduced from the curves $X=X(x), Z=Z(x)$.

Integrating (15) after multiplication by $x^{2}$, and using the defintions of $X_{2}{ }^{*}, Z_{2}{ }^{*}, L_{2}{ }^{*}$, we obtain

$$
k\left(X_{2}^{*}+i Z_{2}^{*}\right)=2 I e^{-i \psi}\left[L_{2}^{*}+i\left(H H_{2}^{*}-2 R L_{1}^{*} \pi^{-1}\right)\right] .
$$

Substituting in this result the values of $H_{2}^{*}, L_{2}^{*}$ obtained by solving (13) and (14), and using the relation $2 \pi R L_{1}^{*}=8 x^{*} H_{1}^{*}\left(1+4 z^{*} / \pi R\right)$, we find that

$$
x^{*}=\alpha\left(1+\gamma z^{*} R^{-1}\right) N_{x}, \quad z^{*}=\alpha\left(1+\beta z^{*} R^{-1}\right) N_{z}+4 x^{* 2} \pi^{-1} R^{-1},
$$

where $\alpha$ and $\beta$ are as in (12), $\gamma=2\left(16-\pi^{2}\right) /\left(\pi^{3}-4 \pi\right)=0.665$, and

$$
N_{x}=\left(X_{1}{ }^{*} X_{2}{ }^{*}+Z_{1}{ }^{*} Z_{2}^{*}\right)\left(X_{1}{ }^{* 2}+Z_{1}^{* 2}\right)^{-1}, \quad N_{2}=\left(X_{2}^{*} Z_{1}{ }^{*}-X_{1}{ }^{*} Z_{2}^{*}\right)\left(X_{1}{ }^{* 2}+Z_{1}^{* 2}\right)^{-1} .
$$

The numbers $N_{x}, N_{z}$ can be deduced from the maps. Thus the moments of $X, Z$ give the four parameters $I A, \psi, x^{*}, z^{*}$.

It is to be noted that the above results can be obtained with the aid of the theory of Fourier transforms. All our results are based on (2), which can be written as a Fourier transform. Now $\rho=\xi+i \zeta$, and since min. $(\zeta)>0$, we have

$$
i(\rho-x)^{-1}=\int_{0}^{\infty} e^{i t(\rho-x)} d t=\int_{-\infty}^{0} e^{-i t(\rho-x)} d t .
$$

This proves that $D g+i U_{x}$ is the Fourier transform of a function $\omega(t)$, vanishing for positive $t$ and defined in the interval $(-\infty, 0)$ by the relation $\omega(t)=k(2 \pi)^{1 / 2} \iint_{S} e^{-i \rho} t d S$. On the other hand, $D g-i U_{x}$ is the transform of $\overline{\omega(-l)}$, which vanishes for negative $t$, and $D g(x)$ appears as the transform of the function $f(t)$ defined for all values of $t$ by

$$
\begin{align*}
f(t) & =k\left(\frac{1}{2} \pi\right)^{1 / 2} \iint_{S} e^{i \xi t-\xi|t|} d S ;  \tag{16}\\
D g(x) & =(2 \pi)^{-1 / 2} \int_{-\infty}^{\infty} e^{i x t f(t) d t .} \tag{17}
\end{align*}
$$

This expression will enable us to find easily the moments of $D \mathrm{~g}$. The moments of the square of $D \mathrm{~g}$, which will be required presently can also be deduced easily from (17) with the aid of the Parseval theorem.

Centered anomalies. In the case of a centered anomaly, we represent the disturbing structure as a homogeneous irregular body $B$. Cartesian coordinates $x, y, z$ are introduced, with the $z$-axis directed vertically downward, the origin $O$ being arbitrarily chosen on the plane $P$ (Fig. 2). $C\left(x^{*}, y^{*}, z^{*}\right)$ is the center of mass of $B$, and $O^{*}$ is its projection on the plane $P$ of the map.

The gravitational anomaly is $\operatorname{Dg}(x, y)$. In the old methods, $O^{*}$ is placed at the
maximum of $D \mathrm{~g}$. If the body is a solid of revolution with a vertical axis, this is correct; but if the body is irregular or inclined, the maximum of $D g$ occurs somewhere above its uppermost part. In the present paper, we shall locate $C$ by means of integrals.


Fig. 2.
If $\delta V$ is an element of volume of $B$ at a point $(\xi, \eta, \zeta)$, and if $\delta[D g(x, y)]$ is the contribution to $D g$ from $\delta V$, then $\delta[D g]=f \sigma \zeta \delta V\left[(x-\xi)^{2}+(y-\eta)^{2}+\zeta^{2}\right]^{-3 / 2}$. Now $\iint_{p} \delta[D g] d S=2 \pi f \sigma \delta V$, which is independent of $\xi, \eta$, $\zeta$. Integration of this over $B$ yields $\iint_{P} D g d S=\pi k V$, where $k=2 f \sigma$ and $V$ is the volume of $B$. This result holds for any homogeneous irregular body or bodies.

Because of symmetry $\iint_{P}[x-\xi+i(y-\eta)] \delta[D g(x, y)] d S=0$. Thus

$$
\iint_{P}(x+i y) \delta[D g] d S=(\xi+i \eta) \iint_{P} \delta\left[D_{g}\right] d S=\pi k(\xi+i \eta) \delta V
$$

and integration over the body $B$ yields

$$
\begin{equation*}
x^{*} \iint_{P} D g d S=\iint_{P} x D g d S, \quad y^{*} \iint_{P} D g d S=\iint y D g d S \tag{18}
\end{equation*}
$$

These are two equations for $x^{*}$ and $y^{*}$. They hold for complex anomalies as well as simple ones, In practice, integration can be carried out only over a finite part of the plane $P$. We choose that part lying inside a circle with center $O$ and radius $R$, where $R$ is a constant at least four or five times the depth $z^{*}$ of $C$. The equations of this circle are $r=R, z=0$, where $r^{2}=x^{2}+y^{2}$. We denote its interior by $L$. The contribution to the above integrals from the infinite region $r \geqq R$ can be easily evaluated, since at such large distances the gravitational action of the body is approximately the same as that of a punctual mass $\sigma V$ located at the point $C\left(x^{*}, y^{*}, z^{*}\right)$. Therefore, for $r \geqq R$ we use the approximate formula

$$
\begin{align*}
D g= & \int \sigma z^{*} V r^{-3}\{1+
\end{align*}
$$

where $r, \theta$ are polar coordinates in the plane $P$, with origin at $O$. Neglecting terms of order $\left(z^{*} / R\right)^{-3}$ and higher, we have with the aid of (19),

$$
\begin{aligned}
\left(1-\frac{z^{*}}{R}\right) \iint_{P} D g d S & =\iint_{L} D g d S \\
\left(1-\frac{3 z^{*}}{2 R}\right) \iint_{P}(x+i y) D g d S & =\iint_{L}(x+i y) D g d S
\end{aligned}
$$

Thus (18) can be written in the form

$$
\begin{equation*}
\left(x^{*}+i y^{*}\right) \iint_{L} D g d S=\left(1+\frac{z^{*}}{2 R}+\frac{3 z^{* 2}}{4 R^{2}}\right) \iint_{L}(x+i y) D g d S \tag{20}
\end{equation*}
$$

Equations (20), applied in the first approximation with $z^{*}=0$, give a first position $O_{1}$ for $O^{*}$. If we choose the origin $O$ at this point, then $d=O_{1} O^{*}$ is small.

To obtain an equation for $z^{*}$, we integrate $r^{2} D g d S$ over $L$. Neglecting terms of relative order $\left(z^{*} / R\right)^{2}$ and higher, we need only the two first principal terms of this integral. Thus we can evaluate the contribution of an elemental volume $\delta V$ at $(\xi, \eta, \zeta)$ by integrating $r^{2} \delta[D g] d S$ over the region $r^{\prime} \leqq R$ instead of $L$, the origin of polar coordinates $\left(r^{\prime}, \theta^{\prime}\right)$ being at the point $(\xi, \eta)$ above the point $(\xi, \eta, \zeta)$. In fact, the difference between two integrals over $L$ and $r^{\prime} \leqq R$ is of relative order $\left(z^{*} / R\right)^{2}$. Now $r^{2}=r^{\prime 2}+\rho^{2}-2 \rho r^{\prime} \cos \left(\theta-\theta^{\prime}\right)$, and integration over $r^{\prime} \leqq R$ gives

$$
\iint_{L} r^{2} \delta[D g] d S=\pi k R \delta V\left\{\zeta+\left(\rho^{2}-2 \zeta^{2}\right) R^{-1}+\zeta 0\left(z^{* 2} / R^{2}\right)\right\}
$$

Integrating this result over $V$ and replacing the integral of the second term by its approximate value $-2 k \pi z^{* 2} V$, we obtain

$$
\iint_{\Sigma} r^{2} D g d S=\pi k V z^{*} R\left\{1-2 z^{*} R^{-1}+0\left(z^{* 2} / R^{2}\right)\right\}
$$

Dividing by $R \iint_{L} D g d S=\pi k V R\left(1-z^{*} R^{-1}\right)$, we find that

$$
\begin{equation*}
z^{*} R \iint_{L} D g d S=\left(1+z^{*} R^{-1}\right) \iint_{L} r^{2} D g d S \tag{21}
\end{equation*}
$$

The term $\frac{1}{2} z^{* 2} R^{-2}$ must be added to the factor $1+z^{*} R^{-1}$ if the terms neglected are of order $\left(z^{*} / R\right)^{3}$ and higher.

When the measurements are performed with the aid of a gradiometer or torsion balance, the resulting maps of $U_{x z}$ and $U_{y z}$ give not only $x^{*}, y^{*}, z^{*}$ but also a control, since each of these two maps can be used to locate the point $C$. Applying the same reasoning as for $D g$, i.e., first integrating $\delta U_{x z}$ and $\delta U_{y z}$ corresponding to an elemental volume $\delta V$, and then integrating the result over $V$, we find that

$$
\begin{array}{ll}
x^{*} \iint_{2} x U_{z z} d S=\frac{1}{2} \iint_{Z} x^{2} U_{x z} d S, & x^{*} \iint_{L} y U_{y z} d S=\iint_{L} x y U_{y z} d S,  \tag{22}\\
y^{*} \iint_{L} x U_{x z} d S=\iint_{L} x y U_{x z} d S, & y^{*} \iint_{L} y U_{y z} d S=\frac{1}{2} \iint_{L} y^{2} u_{y z} d S .
\end{array}
$$

All the integrals in (22) have the same reduction factor $1+3 z^{*} / 2 R$. Hence it does not appear. The third reduced moments give equations for $z^{*}$. They are
$9 R z^{*} \iint_{L} x U_{x z} d S=8\left(1+\frac{7 z^{*}}{6 R}\right) \iint_{L} x^{3} U_{x z} d S=24\left(1+\frac{7 z^{*}}{6 R}\right) \iint_{L} x y^{2} U_{x z} d S$,
$9 R z^{*} \iint_{L} y U_{y z} d S=8\left(1+\frac{7 z^{*}}{6 R}\right) \iint_{L} y^{3} U_{y z} d S=24\left(1+\frac{7 z^{*}}{6 R}\right) \iint_{L} x^{2} y U_{y z} d S$.
If we neglect only terms of order $\left(z^{*} / R\right)^{3}$ and higher, the term $11 z^{* 2} /\left(18 R^{2}\right)$ must be added to the reduction factor $1+7 z^{*}(6 R)^{-1}$. Equations (22) and (23) are general. For a solid of revolution with a vertical axis, they reduce to

$$
\begin{align*}
\left(x^{*}+i y^{*}\right) \iiint_{L} G d S & =\left(1+\frac{3 c z^{*}}{4 R}\right) \iint_{L}(x+i y) G d S  \tag{*}\\
3 R z^{*} \iint_{L} r G d S & =2\left(1+\frac{7 z^{*}}{6 R}+\frac{11 z^{* 2}}{18 R^{2}}\right) \iint_{L} r^{3} G d S \tag{*}
\end{align*}
$$

where $c$ is such that $R c \iint_{L} G d S=\iint_{L} r G d S$.
In the general case of a complex magnetic anomaly we assume that the magnetization vector is the same for all particular isolated bodies which create this anomaly. Its intensity $I$, inclination $\psi$, and azimuth $\phi$ are three unknowns; $I$ cannot be separated from the total volume $V$ and it is the product $V I$ which is deduced from the maps; $\phi$ is defined with respect to arbitrary cartesian axes of $x$ and $y$ on the surface of the earth; $X, Y$ and $Z$ are the components of the anomaly.

We shall now deduce expressions for the six parameters $V I, \psi, \phi, x^{*}, y^{*}, z^{*}$ which characterize the magnetization vector and locate the common center of gravity of all the disturbing magnetic bodies. This can be done by computing the moments of $X, Y, Z$ from maps showing the distribution of $X, Y, Z$ over the surface of the earth, and by use of the classical Eötvös formulae,

$$
\begin{align*}
\sigma(\mathrm{i} X+\mathrm{j} & Y+\mathrm{k} Z) \\
& =I\left(\mathrm{i} \frac{\partial}{\partial x}+\mathrm{j} \frac{\partial}{\partial y}+\mathrm{k} \frac{\partial}{\partial z}\right)\left[\left(U_{x} \cos \phi+U_{\nu} \sin \phi\right) \cos \psi+U_{z} \sin \psi\right] \tag{*}
\end{align*}
$$

where $i, j, k$ are unit vectors on the coordinate axes. We shall use moments of the first four orders, denoting them by subscripts. For example, $X_{m n}=\iint_{L} X x^{m} y^{n} d S$. It is to be noted that $X_{m n}$ are reduced moments. Neglecting terms of relative order $\left(z^{*} / R\right)^{2}$, and using the same method as in the case of (21) in the computation of integrals of the second derivatives of $U$, we obtain
$-X_{00} \sec \psi \sec \phi=-Y_{00} \sec \psi \csc \phi=2 Z_{00} \csc \psi=2 \pi V I / R$.
The first approximate values of $V I, \psi, \phi$ are then given by
$\tan \phi=Y_{00} / X_{00}, \tan \psi=\frac{1}{2} Z_{00}\left(X_{00}^{2}+Y_{00}^{2}\right)^{-1 / 2}, \pi V I=R\left(X_{00}^{2}+Y_{00}^{2}+\frac{1}{2} Z_{00}^{2}\right)^{1 / 2}$.
The zero moments are small quantities, and approach zero as $R$ approaches infinity. Hence (24) yield poor approximations. Nevertheless, the first value of $\phi$ must be used to rotate the axes of coordinates, directing the $x$-axis nearly parallel to the horizontal component of the magnetization vector, so that $\phi$ will be very small.

Among the six moments of the first order, we do not use $X_{01}$ and $Y_{10}$ since they are of order $1 / R$ compared with the other four, which are given by

$$
\begin{array}{ll}
\frac{X_{10}}{2 \pi V I}=-\sin \psi\left(1+\frac{x^{*}}{2 R} \cot \psi \cos \phi\right), & \frac{Y_{01}}{2 \pi V I}=-\sin \psi\left(1+\frac{y^{*}}{2 R} \cot \psi \sin \phi\right), \\
\frac{Z_{10}}{2 \pi V I}=-\cos \psi\left(\cos \phi-\frac{x^{*}}{R} \tan \psi\right), & \frac{Z_{01}}{2 \pi V I}=-\cos \psi\left(\sin \phi-\frac{y^{*}}{R} \tan \psi\right) .
\end{array}
$$

Hence we obtain better values for $V I$ and $\tan \psi$,

$$
\begin{aligned}
& 2 \pi V I=\left(Z_{10}^{2}+Z_{01}^{2}+\frac{1}{2} X_{10}^{2}+\frac{1}{2} Y_{01}^{2}\right)\left(1+\frac{c_{1}}{R}\right) \\
& \tan \psi=2^{-1 / 2}\left(X_{10}^{2}+Y_{01}^{2}\right)^{1 / 2}\left(Z_{10}^{2}+Z_{01}^{2}\right)^{-1 / 2}\left(1-\frac{c_{2}}{R}\right)
\end{aligned}
$$

where

$$
\begin{gathered}
8 c_{1}=3 \sin 2 \psi\left(x^{*} \cos \phi+y^{*} \sin \phi\right)=6 \cos ^{2} \psi \cdot c_{3} \\
4 c_{2}=(\cot \psi+4 \tan \psi)\left(x^{*} \cos \phi+y^{*} \sin \phi\right)=\left(4+\operatorname{cotan}^{2} \psi\right) c_{3} \\
c_{3}=2^{-3 / 2}\left(X_{10}^{2}+Y_{10}^{2}\right)^{1 / 2}\left(Z_{01}^{2}+Z_{10}^{2}\right)^{-2}\left[\left(Z_{20}-Z_{02}\right)\left(Z_{01}^{2}-Z_{10}^{2}\right)-4 Z_{01}^{2} Z_{11} Z_{10}\right] .
\end{gathered}
$$

The coordinates $x^{*}, y^{*}$ of the point $O^{*}$ are obtained with the aid of second moments, as indicated by the equations

$$
\begin{align*}
& x^{*}=\frac{1}{2}\left(Z_{10}^{2}+Z_{01}^{2}\right)^{-1}\left[\left(Z_{20}-Z_{02}\right) Z_{10}+2 Z_{01} Z_{11}\right]\left(1-\frac{c_{3}}{R}\right)+\frac{y^{* 2}}{R} \tan \psi \cos \phi \\
& y^{*}=\frac{1}{2}\left(Z_{10}^{2}+Z_{01}^{2}\right)^{-1}\left[2 Z_{10} Z_{11}-\left(Z_{20}-Z_{02}\right) Z_{01}\right]\left(1-\frac{c_{3}}{R}\right)+\frac{x^{* 2}}{R} \tan \psi \sin \phi \tag{25}
\end{align*}
$$

The expression

$$
Y_{20}=\frac{5}{2} R \cos \psi \sin \phi\left(1-\frac{12 z^{*}}{5 R}\right)
$$

shows that $Y_{20}$ is zero if the $x$-axis is parallel to the horizontal component of the magnetization vector $I$. If $Y_{20}$ is different from zero, the value of $\phi$ is given with good precision by the important equation $X_{02}=Y_{20} \tan \phi$. Thus, by use of this equation and Eqs. (25), we can change our axes so that $x^{*}=y^{*}=\phi=0$. Computing the third moments in this new system of coordinates, we find that $X_{21}=X_{03}=Y_{30}=Y_{12}=Z_{21}=Z_{03}$ $=0,3 Z_{12}=Z_{30}=X_{30} \cot \psi$,

$$
3 X_{12}=3 Y_{21}=X_{30}=Y_{03}=-\frac{9}{8} R z^{*}\left(1-\frac{7 z^{*}}{6 R}\right) \sin \psi
$$

These expressions give for $\tan \psi$ the eight values $X_{30} / Z_{30}, Y_{21} / Z_{12}$, etc., and the mean of these eight values can be considered as the final value of $\tan \psi$. Now, with this choice of axes, the second moments take the values $X_{11}=Y_{20}=Y_{02}=Z_{11}=0$,

$$
\begin{array}{ll}
X_{20}=\frac{5}{2} R\left(1-\frac{16 z^{*}}{5 R}\right) \cos \psi, & X_{02}=\frac{5}{2} R\left(1-\frac{12 z^{*}}{5 R}\right) \cos \psi \\
Y_{11}=\frac{3}{8} R\left(1-\frac{8 z^{*}}{3 R}\right) \cos \psi, & Z_{20}=Z_{02}=-5 R\left(1-\frac{14 z^{*}}{5 R}\right) \sin \psi
\end{array}
$$

Dividing any third moment by one of the five second moments different from zero, we obtain an approximate value for $z^{*}$, and the mean of all such values can be considered as the final value of $z^{*}$. Thus, we have not only solved the problem, but have also found a control of the solution, since the degree of concordance of many different values found for the same parameter, such as $z^{*}$, characterizes the reliability of the solution.

Fom a practical point of view, a solution involving only the moments of the vertical component $Z$ is important, since in many cases magnetic surveys are limited to measurements of $Z$ only. In such cases we can find $\psi$ and $\phi$ and locate the point $O^{*}$ by use of Eqs. (25) together with

$$
\begin{aligned}
Z_{10} \tan \phi & =Z_{01}\left(1+\frac{y^{*}-x^{*}}{R} \tan \psi\right), \\
5 R\left(Z_{10}^{2}+Z_{01}^{2}\right)^{1 / 2} \tan \psi & =\left(1+\frac{c_{4}}{R}\right)\left[\left(Z_{20}-2 x^{*} Z_{10}\right)^{2}+\left(Z_{02}-2 y^{*} Z_{01}\right)^{2}\right]^{1 / 2},
\end{aligned}
$$

where $c_{4}$ is such that $5 R c_{4}=14 z^{*}-\left(x^{*} \cos \phi+y^{*} \sin \phi\right) \tan \psi$. For VI we have

$$
10 \pi R V I=\left(1+\frac{c_{4}}{R}\right)\left(Z_{20}^{2}+Z_{02}^{2}\right)^{1 / 2}
$$

Also, $z^{*}$ is given by

$$
z^{*}=-\frac{8}{9}\left(1+\frac{7 z^{*}}{6 R}\right) \frac{Z_{30}}{R}\left(Z_{10}^{2}+Z_{01}^{2}\right)^{1 / 2},
$$

where the moments are related to the origin $O^{*}$ with $\phi=0$. It is plain that the accuracy of the approximate values given by the above equations increases with $R$.
4. Interpretation of the map of an anticline with the aid of moments. Let us represent the structure of an anticline as a cylindrical body of normal cross section $S$. We have first to choose for $S$ some simple geometrical form which expresses the gravitational action of the anticline adequately. Our choice is the region between two concentric elliptic cylinders, the major axes of which have an angle $\alpha$ of inclination (Fig. 3). This choice is good, except in the vicinity of the deepest part of the region, which part is at a considerable depth; the author has verified that such a choice leads to good results in practise. We shall denote the outer and inner ellipses by $E_{1}$ and $E_{\gamma}$, respectively, and their semi-axes by $a, b$ and $\gamma a, \gamma b$, respectively, where $\gamma$ is a positive constant less than unity.

Equations (10)-(14) give values for $k A, x^{*}, z^{*}$. We shall now show how $\gamma, \alpha$ and the eccentricity $e$ can be deduced from gravitaticnal anomaly maps.

Let us replace $e$ by a parameter $r=2 z^{*} / c$, where $2 c$ is the focal distance of the outer ellipse. We define a function $n(s)$ by the relation

$$
\begin{equation*}
n(s)=M_{0}^{2 n-1} M(-s) D^{-s}, \tag{26}
\end{equation*}
$$

where $s$ is a constant and $0<s<1, M_{0}$ is as defined in $\S 3, M(-s)$ is the moment of $D g$ of order $-s$, and $D$ is the zero moment of $[D g(x)]^{2}$ :

$$
M(-s)=\int_{-\infty}^{\infty}|x|^{-} D g d x, \quad D=\int_{-\infty}^{\infty}[D g(x)]^{2} d x .
$$

The graphs of $|x|^{-s} D g(x)$ and $[D g(x)]^{2}$ can be plotted from the experimental curve of $D g(x)$, and thus the reduced moments can be computed with the aid of a planimeter. On the other hand $n(s)$ can be tabulated for various values of $s$. We shall require $n\left(\frac{1}{4}\right), n\left(\frac{1}{2}\right), n\left(\frac{3}{4}\right)$.

Applying the theory of Fourier transforms to (16), (17), we have

$$
\begin{align*}
\int_{-\infty}^{\infty} D g(x) e^{i x t} d x=\pi k \iint_{S} e^{-\zeta i+1+i \xi t} d S  \tag{27}\\
2 \int_{-\infty}^{\infty}[D g(x)]^{2} e^{-i x t} d x=\pi k^{2} \iint_{S} \iint_{S} d S d S^{\prime} \int_{-\infty}^{\infty} e^{i \phi(u)} d u, \tag{28}
\end{align*}
$$



Fig. 3.
where $\phi(u)=\xi t+\left(\xi^{\prime}-\xi\right) u-\zeta^{\prime}|u|-\zeta|t-u|$, and $(\xi, \zeta),\left(\xi^{\prime}, \zeta^{\prime}\right)$ are two sets of running coordinates on $S$. We note that

$$
\int_{1}^{\infty} e^{i v v}(y-t)^{-\delta} d y=(i v)^{\delta-1} e^{i v t} \Gamma(1-\delta)
$$

for $t \geqq 0,0<\delta<1, \operatorname{Re}[i v] \leqq 0$, where $\operatorname{Re}[i v]$ denotes the real part of $i v$. Consequently, the application to (27) of fractional integration of order $1-\delta$ yields for $t \geqq 0$,

$$
\begin{equation*}
\int_{-\infty}^{\infty} D g(x)|x|^{\delta-1} e^{i x t} d x=\pi k \iint_{S} \rho^{\delta-1} e^{i \rho t} d S \tag{29}
\end{equation*}
$$

The inversion of order of integration is permissible, since all integrals are absolutely convergent. Replacing in (29) the exponent $\delta-1$ by $-s,(0<s<1)$, multiplying both sides of this by $e^{i r s / 2}$ and letting $t$ tend to zero, we obtain

$$
M(-s)=\int_{-\infty}^{\infty}|x|-s D g(x) d x=\pi k \sec \left(\frac{1}{2} \pi s\right) \operatorname{Re}[\Phi(A ;-s)]
$$

where $\Phi(A ;-s)=e^{i \pi s / 2} \iint_{S} \rho^{-} d S$. For our region $S$ we have
$\Phi(A ;-s)=A\left(z^{*}\right)^{-s}\left[F\left(\frac{1}{2} s, \frac{1}{2}+\frac{1}{2} s, 2 ;-4 r^{-2} e^{-2 i \alpha}\right)\right.$

$$
\left.-\gamma^{2} F\left(\frac{1}{2} s, \frac{1}{2}+\frac{1}{2} s, 2 ;-4 r^{-2} \gamma^{2} e^{-2 i \alpha}\right)\right]
$$

where $F(a, b, c ; x)$ is the classical hypergeometric function. The function $M(-s)$, which depends on the four parameters $s, \alpha, \gamma, r$, can be tabulated for $-1<s<1$, $0 \leqq \gamma \leqq 1,0 \leqq \alpha \leqq \frac{1}{2} \pi, r \geqq 2 \sin \alpha$. We shall require tabulations for $s=\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

If in the right side of (28) we carry out the integration with respect to $u$ for $t \geqq 0$, and then let $t$ approach zero, we find that

$$
2 D=2 \int_{-\infty}^{\infty}[D g(x)]^{2} d x=i \pi k^{2} \iint_{S} \iint_{S}\left(\rho-\bar{\rho}^{\prime}\right)^{-1} d S d S^{\prime}
$$

where $\rho=\xi+i \zeta, \bar{\rho}^{\prime}=\xi^{\prime}-i \zeta^{\prime}$. In the present case, we have

$$
D=M_{0}^{2}\left(z^{*}\right)^{-1} \phi(\alpha, \gamma ; r),
$$

where

$$
\begin{aligned}
& \phi(\alpha, \gamma ; r)=f\left(e^{-i \alpha}, e^{i \alpha} ; r\right)-2 \operatorname{Re}\left[f\left(\gamma e^{-i \alpha}, e^{i \alpha} ; r\right)\right]+f\left(\gamma e^{-i \alpha}, \gamma e^{i \alpha} ; r\right) \\
& 3 f(u, v ; r)=16 r u^{3} v^{3}(u+v)^{-2}\left[C_{1} E-C_{2} K+C_{3} K E\left(b, \lambda^{\prime}\right)\right]
\end{aligned}
$$

$K$ and $E$ being complete elliptic integrals of the first and second kind of modulus $\lambda$ given by $4 u v=\lambda^{2}\left[r^{2}+(u+v)^{2}\right], \lambda^{\prime}$ and $b$ being such that $\lambda^{\prime}=\left(1-\lambda^{2}\right)^{1 / 2}, 2(u v)^{1 / 2} d n\left(b, \lambda^{\prime}\right)$ $=\lambda \cdot(u+v)$, and $C_{1}, C_{2}, C_{3}$ such that

$$
\begin{aligned}
& C_{3}=3 \lambda^{\prime}\left[1-\lambda^{\prime} \operatorname{sn}\left(b, \lambda^{\prime}\right) c n\left(b, \lambda^{\prime}\right) d n\left(b, \lambda^{\prime}\right)\right] \\
& C_{1}=3 d n^{2}\left(b, \lambda^{\prime}\right)+3 \lambda^{\prime} s n\left(b, \lambda^{\prime}\right)-1-\lambda^{2}+b C_{3} \\
& C_{2}=3 \lambda^{\prime 2} d n^{2}\left(b, \lambda^{\prime}\right) c n^{2}\left(b, \lambda^{\prime}\right)+3 \lambda^{\prime} s n\left(b, \lambda^{\prime}\right)\left[1+\lambda^{\prime 2} c n^{2}\left(b, \lambda^{\prime}\right)\right]+\lambda^{\prime 2}+b C_{3}
\end{aligned}
$$

Thus $\phi(\alpha, \gamma ; r)$ can be tabulated.
We have now derived expressions for $M(-s)$ and $D$ occurring in the right side of (26). Substitution yields the equation

$$
\begin{equation*}
E(s ; \alpha, \gamma ; r)=n(s) \tag{30}
\end{equation*}
$$

where $E(s ; \alpha, \gamma ; r)=[\phi(\alpha, \gamma ; r)]^{-s} m(-s ; \alpha, \gamma ; r)$, with

$$
m(-s ; \alpha, \gamma ; r)=\sec \left(\frac{1}{2} \pi s\right) A^{-1} z^{* s} \operatorname{Re}[\Phi(A ;-s)]
$$

In (30) we set $s=\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, to obtain three equations which we can solve for the required quantities $\alpha, \gamma, r$.

Once $\alpha, \gamma, r$ have been determined, we can deduce a new value for $z^{*}$ from the relation

$$
z^{*}=M_{0}^{2} D^{-1} \phi(\alpha, \gamma ; r)
$$

This serves as a control on the value obtained from (12).
In actual computations based on experimental data, we can use only a finite region on the surface of the earth. Hence reduced moments must be introduced, as before, and the parameters of the problem must be deduced by successive approximations.

The same procedure can be used in treating maps of the gradient $G$ and the curva-
ture $K$. Each map leads to an independent set of values for $x^{*}, z^{*}, \alpha, \gamma, r, k A$. In the case of a magnetic anomaly, the map of $X$ or of $Z$ could be used. In all cases the moments can be expressed in terms of $\Phi(A ;-s)$.

We note that the above method yields $k A$. Now $A$ is the area of the cross section and $k=2 \sigma$, where $f$ is the gravitational constant and $\sigma$ is the density-contrast. We define the average thickness $T$ of the disturbing layer as the geometric mean of the extreme thicknesses; $T=(1-\gamma)(a b)^{1 / 2}$. Since $A=\pi a b\left(1-\gamma^{2}\right)$, we then have $A(1-\gamma)$ $=\pi(1+\gamma) T^{2}$. Hence, if $\sigma$ is known, $A$ can be found and then $T$. Often $T$ is known. In this case, $A$ can be found, and then $\sigma$.
5. Interpretation of a centered anomaly created by a salt dome. Before applying the new method it is interesting to see what can be obtained by the old method. Some results can be achieved if we disregard the cap-rock, neglect the slope of the flanks and omit the depth of the salt dome base considering this structure as a homogeneous vertical infinite circular cylinder whose top is at the depth $p$ (Fig. 4). The interpretation problem is reduced to finding the three parameters: $k(\sigma=k / 2 f), p$ and the radius $a$ of the dome. Instead of $a$ we consider the angle $\omega$, letting $a=p \tan \omega$. With the origin at $O^{*}$, just above the center of the top circle (maximum of $D g$, zero for the gradient $G$ ) the expressions for $G$ and $D g$ are:

$$
\begin{aligned}
G(r) & =k(a / r)^{1 / 2}\left[2 \lambda^{-1}(K-E)-\lambda K\right] \\
D g(r) & =k\left[C_{1} E+C_{2} K-\pi p f(r)\right]
\end{aligned}
$$



Fig. 4.
where $C_{1}=b p \operatorname{sign}(a-r)+2(a r)^{1 / 2} \lambda^{-1}$, $C_{2}=\lambda\left(a^{2}-r^{2}\right)(4 a r)^{-1 / 2}+p\left[E\left(b, \lambda^{\prime}\right)-b\right] \operatorname{sign}(a-r)$ and $2 f(r)=1+\operatorname{sign}(a-r)$, sign 0 being defined by $\operatorname{sign} 0=0$. The elliptic function $E\left(b, \lambda^{\prime}\right)$ and the complete integrals $K$, $E$ have the moduli $\lambda^{\prime}=\left(1-\lambda^{2}\right)^{1 / 2}$ and $\lambda$ is defined by $\lambda\left[p^{2}+(a+r)^{2}\right]^{1 / 2}=2(a r)^{1 / 2}$, $r$ being the distance from the origin. The argument $b\left(0 \leqq b \leqq K^{\prime}\right)$ is that in $d n\left(b, \lambda^{\prime}\right) \cdot\left[p^{2}+(a+r)^{2}\right]^{1 / 2}=a+r$. Either of two curves $G=G(r), D g=D g(r)$ can be deduced from the other by graphical differentiation or integration. Therefore, we can use both of them for the interpretation.

Here it is the maximum of the product $F(r)=r^{1 / 2}|G(r)|$ which it is important to locate and therefore the curve $G$ must be transformed into the graph of the function $F(r)$. In fact, the equation $F^{\prime}(r)=0$ reduces to $\lambda^{3} E\left(p^{2}+a^{2}-r^{2}\right)=0$ and the maximum of $F(r)$ corresponds to $r=r_{0}=\left(p^{2}+a^{2}\right)^{1 / 2}$. The corresponding value of the modulus $\lambda$ is $\lambda_{0}=\left[1-\tan ^{2}(\pi / 4-\omega / 2)\right]^{1 / 2}$. The value of the maximum itself is $\max F(r)=F_{0}=k(p \tan \omega)^{1 / 2}\left[2 \lambda_{0}^{-1}\left(E_{0}-K_{0}\right)+\lambda_{0} K_{0}\right]$. Thus, we have at our disposal three experimental data $r_{0}, F_{0}$ and the maximum $D g_{0}=\pi k p(\sec \omega-1)$ of $D g(r)$. First we find the angle $\omega$, solving the equation $B(\omega)=n$, where the function $B(\omega)$ is defined by $\pi \sin (\omega / 2)(1+\sin \omega-\cos \omega) B(\omega)=-2^{1 / 2}\left[(1+\sin \omega) E_{0}-K_{0}\right]$, the modulus $\lambda_{0}$ being a function of $\omega$ only. The value of the number $n$ is deduced from the measure-
ments by the rule $n=r_{0}^{1 / 2} F_{0} / D g_{0}$, this value being simply the ratio of two maxima, multiplied by the square root of the observed distance $r_{0}$. The function $B(\omega)$ is easily tabulated. Knowing $\omega$, we have $p=r_{0} \cos \omega$ and $a=p \tan \omega$. The value of $k$ is deduced from $F_{0}$ or $D g_{0}$. That is all that can be obtained, using the old method, and it is evident that it cannot satisfy a geologist. In practice the best value for $r_{0}$ is $(S / \pi)^{1 / 2}$, where $S$ denotes the area of the closed curve, the locus of maxima of $F(r)$ on all radial profiles through the origin $O^{*}$. This rather rough method of interpretation works if it is applied to an isolated salt dome. The advantage of the inaccurate first approximation which it gives consists in the simplicity of computations. The interpretation requires only the table of values of the function $B(\omega)$ and it can be performed as fieldwork and very rapidly.

We shall now apply the new method. It is assumed that the salt dome is a solid of revolution. Cartesian axes are chosen, with the $z$-axis directed downward along the axis of revolution, and the origin $O^{*}$ on the surface of the earth. We denote the cylindrical coordinates of a general point inside the solid of revolution by ( $\rho, \phi, \zeta$ ), and of a general point outside by $(r, \theta, z)\left(z<\zeta_{\min }\right) ; R$ is the distance between these two points. By means of the classical formula

$$
\int_{0}^{2 \alpha} \frac{d \phi}{R}=2 \pi \int_{0}^{\infty} e^{-(\xi-z) u} J_{0}(r u) J_{0}(\rho u) d u,
$$

where $J_{\mathrm{n}}(t)$ is a Bessel function, we deduce for the potential

$$
U(r, z)=\int_{0}^{\infty} e^{u z} W(u) J_{0}(r u) d u
$$

where

$$
\begin{equation*}
W(u)=\pi k \iint_{A} e^{-\zeta u} J_{0}(\rho u) \rho d \rho d \zeta=\frac{\pi k}{u} \int_{C} e^{-\zeta u} J_{1}(\rho u) \rho d \zeta, \tag{31}
\end{equation*}
$$

$A$ being the region the revolution of which generates the solid, and $C$ being its boundary. Denoting the Hankel transform of order $r$ by $H_{r}$, so that by definition

$$
F(r)=H_{r}[f(u)]=\int_{0}^{\infty} f(u) J_{r}(r u) u d u
$$

for $z=0$ we have $U(r, 0)=H_{0}[W(u) / u]$. Differentiation of (31) yields similar expressions for $U_{z}=D g(r), U_{z z}, G, K$ :
$D g(r)=H_{0}[W(u)], \quad U_{z z}=H_{0}[u W(u)], \quad-G=H_{1}[u W(u)], \quad K=H_{2}[u W(u)]$.
The advantage presented by the expressions (32) consists in the possibility of using the theorems of the transform theory in calculating the moments and moment functions used in the interpretation. Because of lack of space we can give as example only the general expression for the moment function of $D g$ which holds for any form of the solid of revolution, the expression applied below to the interpretation of the gravity map of a salt dome. There is another approach to the mathematical problem of computing the moments and moment functions needed in the interpretation. Indeed, direct integration of the explicit formulae (32) is easily performed with the aid
of divergent and summable integrals of Bessel functions of the general type (with $a$ and $b$ restricted only by $a+b>-1$ ),

$$
\begin{equation*}
\int_{0}^{\infty} t^{a} J_{b}(u t) d t \cong 2^{a} u^{-(a+1)} \Gamma[(a+b+1) / 2]\{\Gamma[(b-a+1) / 2]\}^{-1}, \tag{33}
\end{equation*}
$$

but we prefer to use the transform theory. A salt dome creates also a magnetic anomaly and, for a solid of revolution, (15*) gives the expressions for its components on the ground $z=0$ with the aid of gravitational quantities $G, K$ and $U_{z z}$. Using the results of Section 3, we choose the origin and the $x$-axis so that $x^{*}=y^{*}=0, \phi=0$. Under this assumption

$$
\begin{align*}
k(X+i Y) & =I\left[\cos \psi\left(K e^{2 i \theta}-U_{z z}\right)+2 G \sin \psi e^{i \theta}\right], \\
k Z & =2 I\left(\sin \psi U_{z z}+G \cos \psi \cos \theta\right) . \tag{34}
\end{align*}
$$

Instead of $X, Y$ it is convenient to use in the interpretation the radial and transversal components $A_{r}$ and $A_{\iota}$ of the horizontal anomaly $\left(X^{2}+Y^{2}\right)^{1 / 2}$. Transforming the $X-$ and $Y$-maps into the maps of $A_{r}$ and $A_{l}$, we can consider their moments and moment functions as experimental data. The corresponding expressions in terms of parameters of the problem are deduced with the aid of

$$
\begin{equation*}
k\left(A_{r}+i A_{1}\right)=I\left[\cos \psi\left(K e^{i \theta}-U_{z z} e^{-i \theta}\right)+2 G \sin \psi\right] . \tag{35}
\end{equation*}
$$

With the aid of (34), (35) all the rules for the interpretation of the gravitational anomaly can be adapted to the interpretation of magnetic anomalies produced by a solid of revolution. Far from the origin $O^{*}$ the gravitational action of a body can be approximately expressed as that of a material point of the same excess-mass $M=\sigma V$ located at the center of gravity $C\left(0,0, z^{*}\right)$. Thus, for large $r$ we have the approximate formulae

$$
\begin{equation*}
D g \sim f M z^{*} r^{-3}, \quad-G \sim 3 f M z^{*} r^{-4}, \quad K \sim 3 f M r^{-3}, \quad U_{z z} \sim-f M r^{-3}, \tag{36}
\end{equation*}
$$

the neglected terms being of the relative order $\left(z^{*} / r\right)^{2}$. For the magnetic quantities $A_{r}, Z$ we have, denoting the volume of the salt dome by $V_{8}$ (the caprock does not create a magnetic anomaly),
$A_{r} \sim I V_{s} r^{-3}\left(2 \cos \psi \cos \theta-3 z^{*} r^{-1} \sin \psi\right),-Z \sim I V_{s} r_{r}^{-3}\left(\sin \psi+3 z^{*} r^{-1} \cos \psi \cos \theta\right)$.
The moment functions are defined by the integrals extended over the infinite plane $P$ and the formulae (36), (37) are used in computing the reduction factors introduced by the integration over the finite area $r \leqq R$, denoted by $L$.

The formula (32) $D g(r)=H_{0} W(u)$ gives immediately $W(u)=H_{0} D g(r)$, that is,

$$
\begin{equation*}
2 \pi W(u)=\iint_{P} D_{g}(r) J_{0}(r u) d S \tag{38}
\end{equation*}
$$

If the form of the solid is considered as known, only its dimensions being asked, i.e., if the expression of $W(u)$ as function of the parameters is prescribed, the relation (39) becomes the source of equations since for every particular value of the arbitrary parameter $u$ (which is the reciprocal of a length) it is an equation in the unknown parameters of the problem. In practice the numerical value of the second member is
obtained by integrating over $L$, and adding to the result the contribution of the infinite area $r \geqq R$. This contribution, computed with the aid of (36), is equal to $2 \pi f M\left(z^{*} / R\right)\left[J_{1}(R u) / R u\right]$ where, as has been seen in Section $3,2 \pi f M=\pi k V$ is the value of the first moment of $D \mathrm{~g}$. In this method the map of $D \mathrm{~g}$ is transformed, for each value of $u$, into the map of the product $D g(r) J_{0}(r u)$ and the integration over $L$, giving the second member of (38), is performed on this map. Let us consider as an example the special case when the flanks of the dome are vertical. In this case the area $A$ is formed by two rectangles (see Fig. 5*) and there are seven parameters: the radii $a$ (salt) and $b$ (cap-rock) of two cylinders which together form the dome with its caprock, the depth $p$ at which the salt begins (bottom of the cap-rock), the thickness $h$


Fig. 5. of the cap-rock and that $H$ of the salt, the density-contrasts $k_{1}$ and $k_{2}$ (salt). The expression (31) gives the characteristic function $W(u)$,

$$
\begin{aligned}
\pi^{-1} u^{2} W(u) & =e^{-u p}\left[a k_{2} J_{1}(a u t)\right. \\
\left(1-e^{-u H}\right) & \left.+b k_{1} J_{1}(b u)\left(e^{u / h}-1\right)\right] .
\end{aligned}
$$

Choosing any seven numerical values for $u$ and computing for them the corresponding values of the product $\pi^{-1} u^{2} W(u)$ with the aid of the rule (38), we have a system of seven equations with seven unknowns $a, b, k_{1}, k_{2}, p, H, h$. Solving it we have all the necessary informations about the dome. This method can be applied only if the flanks of the dome are vertical. Let us now consider the case of a deeply buried salt dome, substituting for it as an idealized form a truncated cone with the vertex at $O^{*}$. This assumption is not acceptable for shallow domes. We need the moment function and we begin by giving its general expression which holds for all solids of revolution. A. Erdelyi and H. Kober have recently proved ${ }^{2}$ an important theorem on the Hankel transform which they formulate using Tricomi's form of this transform. In our notation (Hankel's form) this theorem states that if $W=H_{s+2 \alpha} w$, where $s>-1, \alpha>0$, and $u^{1 / 2} w(u)$ belongs to $L_{2}(0, \infty)$ then $T_{s \alpha}(W)=H_{s}\left[T_{s a}(w)\right]$, the operator $T_{s \alpha}$ being defined by

$$
\Gamma(\alpha) T_{s a} f(x)=2 x^{\prime} \int_{x}^{\infty}\left(y^{2}-x^{2}\right)^{\alpha-1} y^{1-s-2 \alpha} f(y) d y .
$$

Applying it to our relation $W(u)=H_{0} D g(r)$ with $-1<s<0, s+2 \alpha=0$, we have

$$
\int_{x}^{\infty}\left(y^{2}-x^{2}\right)^{-(1+\theta / 2)} W(y) y d y=x^{-0} \int_{0}^{\infty} J_{t}(x t) t^{t+1} d t \int_{t}^{\infty}\left(r^{2}-t^{2}\right)^{-(1+s / 2)} D g(r) r d r .
$$

[^19]Interchanging the integrations in the second member (absolutely convergent double integral), using $t=r v^{1 / 2}$ and passing to the limit $x=0$, we get the desired general expression for the moment function $M(s)$ of $D g(r)$ for $-2<s<0$,

$$
\begin{align*}
M(s) & =\iint_{P} D g(r) r^{*} d S=2 \pi \int_{0}^{\infty} D g(r) r^{s+1} d r \\
& =2^{2+s} \Gamma^{2}(1+s / 2) \sin (-s \pi / 2) \int_{0}^{\infty} W(u) u^{-(s+1)} d u \tag{40}
\end{align*}
$$

The same result can be obtained integrating the relation $r^{s+1} D g=H_{0} r^{s+1} W(u)$ under the sign of integration in $u$ and applying the integral (33). Now the integral in the second member of (40) can be calculated, using (31) and the formula (3) p. 385, ch. 13.2 of "Bessel Functions" by G. N. Watson, 1922. Let $\omega$ be the angle the radius vector of a point ( $\rho, \zeta$ ) makes with the $z$-axis; thus $\zeta=\left(\rho^{2}+\zeta^{2}\right)^{1 / 2} \cos \omega, \rho=\zeta \tan \omega$. Let $P_{n}$ be the Legendre function of the first kind $P_{n}(\cos \omega)$. Since $F[(1+s) / 2,-s / 2 ; 1$; $\left.\sin ^{2} \omega\right]=P_{s}$ and $(2+s) \sin ^{2} \omega F\left[(3+s) / 2,-s / 2 ; 2 ; \sin ^{2} \omega\right]=2\left(P_{s}-\cos \omega P_{s+1}\right)$, we have

$$
\begin{equation*}
M(s)=(2+s) c_{s} \iint_{A} k \sec ^{3} \omega P_{s} s^{s} \rho d \rho d \zeta=c_{s} \int_{C} k \sec ^{2+s} \omega\left(P_{s}-\cos \omega P_{s+1}\right) \zeta^{2+\imath} d \zeta \tag{41}
\end{equation*}
$$

where $(2+s) c_{s}=2 \pi \Gamma(1 / 2) \Gamma(1+s / 2) \Gamma(1-s / 2)$. The formula (41) solves the problem for a deeply buried dome. The parameters in this case (see Fig, 6) are: the three lengths $\bar{O}^{*} A=p_{j}, j=1,2,3$, the density contrasts $k_{1}=2 f \sigma_{1}, k_{2}=2 f \sigma_{2}$ and the angle $\omega_{1}$


Fig. 6.
the slope of the dome's flanks. Instead of $p_{j}$ we use the ratios $r=p_{1} / p_{2}, q=p_{3} / p_{2}$ and write $p_{2}=p$. Also, $k=k_{2} / k_{1}$ this ratio being generally negative. The parameters $r<1$, $q>1, k$ verify the equation

$$
\begin{equation*}
4 z^{*}\left[1-r^{3}+\left(q^{3}-1\right) k\right]=3\left[1-r^{4}+\left(q^{4}-1\right) k\right] p \tag{42}
\end{equation*}
$$

In our case $\omega$ is constant in the second integral of (41) since only the integral
along $\overline{B A}$ does not vanish. It is equal to $(3+s) k_{1} p^{3+8} \phi(s ; r, q, k)$, where $\phi(s ; r, q, k)$ $=1-r^{3+s}+k\left(q^{3+s}-1\right)$. Denoting $(3+s)^{-1} c_{s} \sec ^{2+s} \omega\left[P_{s}(\cos \omega)-\cos \omega P_{s+1}(\cos \omega)\right]$ by $f(s ; \omega)$, we have the explicit expression of the moment function $M(s)$

$$
\begin{equation*}
M(s)=k_{1} p^{3+s} f(s ; \omega) \phi(s ; r, q, k) \tag{43}
\end{equation*}
$$

The $f(s ; \omega)$ is easy to tabulate for eleven particular values of $s$, namely for $4 s=m$, where the integer $m$ verifies $-7 \leqq m \leqq 3$ since $-2<s<1$. If $4 s=m$ the function $P_{s}$ reduces to polynomials or to elliptic integrals $K$ and $E$. In fact, letting $\cos \omega=x$, we have $3 P_{-7 / 4}=3 P_{3 / 4}=P_{1 / 4}+2 x P_{-1 / 4}, \quad P_{-5 / 4}=P_{1 / 4}, \quad P_{-3 / 4}=P_{-1 / 4}, \quad 5 P_{5 / 4}=6 x P_{1 / 4}-P_{-1 / 4}$, $21 P_{7 / 4}=10 x P_{1 / 4}+\left(20 x^{2}-9\right) P_{-1 / 4}$, where $\pi \cos (\omega / 4) P_{-1 / 4}=2[\cos (\omega / 2)]^{1 / 4} K$ and $\pi(\cos (\omega / 2)]^{1 / 4} P_{1 / 4}=2 \cos (\omega / 4)\left[4(E-K)+\left(1+\lambda^{2}\right) K\right]$ with the modulus $\lambda=\tan (\omega / 4)$.

Also $P_{-3 / 2}=P_{1 / 2}, 3 P_{3 / 2}=4 x P_{1 / 2}-P_{-1 / 2}$, where $P_{ \pm 1 / 2}$ are elliptic integrals too, but with modulus $\mu=\sin (\omega / 2)$, namely $\pi P_{-1 / 2}=2 K$ and $\pi P_{1 / 2}=4 E-2 K$. Transforming the experimental gravity map into the maps of the quantities $r^{s} D g(r)$ and integrating on these maps over $r \leqq R$, we obtain the reduced moment function $M^{*}(s)$ which differs from $M(s)$ only in the contribution of the infinite area $r \geqq R$. This contribution is computed with the aid of (36), and the reduction factor $\nu(s)$ in

$$
\begin{equation*}
M(s)=M^{*}(s)\left[1+\nu(s)(p / R)^{1-s}\right] \tag{44}
\end{equation*}
$$

is defined by $4(1-s) f(s ; \omega) \phi(s ; r, q, k) \nu(s)=3 f(0 ; \omega) \phi(1 ; r, q, k)$.
We consider equations of the general type $Q(s, t ; \omega, r, q, k)=N(s, t)$, where the function $Q$ of four unknowns $\omega, r, q, k$ is defined by $Q \equiv[f(s ; \omega) \phi(s)]^{-t}[f(t ; \omega) \phi(t)]^{s}$ $[f(0 ; \omega) \phi(0)]^{t-s}$, the $\phi(s)$ denoting $\phi(s ; r, q, k)$. Thus the number $N(s, t)$ is to be calculated by the rule $N(s, t)=[M(s)]^{-t}[M(t)]^{*}[M(0)]^{t-s}$. Its first value is obtained by using the reduced moment functions $M^{*}(s), M^{*}(t), M^{*}(0)$. With the aid of these first values $N^{*}$ we have to find first approximations for our parameters $\omega, r, q, k$. It is sufficient to form four equations of the type $Q=N$, giving to the orders $s$ and $t$ numerical values, for instance $s=-7 / 4$ and $t=-5 / 4,-3 / 2$ and $-1,-5 / 4$ and $-3 / 4,-1$ and $-1 / 2$. Solving such a system with the experimental data $N^{*}(-7 / 4$, $-5 / 4), N^{*}(-3 / 2,-1), N^{*}(-5 / 4,-3 / 4)$ and $N^{*}(-1,-1 / 2)$, we get the first approximate values for $\omega, r, q, k$. The depth $p$ is then found using the ratio $M^{*}(s) / M^{*}(0)$ for any value of $s$ (or $t$ ) and in practice the average value from many such determinations will be taken as $p$. Using the set of first approximations in (44), we improve the first values of the second members $N(s, t)$ in our system of four equations $Q=N$ and solving the improved system, we have second approximations for $\omega, r, q, k, p$. This procedure of successive approximations is continued until the stabilization of seguences. The depths $p_{1}=r p$ and $p_{3}=q p$ are found, as well as $k_{2}=k k_{1}$, since the value of $k_{1}$ can be obtained from the value of any $M(s)$. We observe that the interpretation yields both density contrasts $\sigma_{1}$ (cap-rock) and $\sigma_{2}$ (salt). The depth $z^{*}$ of the center of gravity obtained with the aid of (42), compared with $z^{*}$ computed by ( $23^{*}$ ), gives a control. The control also is obtained with the aid of four other values of the order $s$ which were not used for the interpretation. We choose the negative values of $s$ in order to diminish the reduction factor in (44). It is interesting to add that the same method applies to the interpretation of $G$ - and $K$-maps, their moment functions $G(s), K(s)$ being expressed in terms of $M(s)$. In fact $G(s)=-(2+s) M(s-1)$ and
$K(s)=c_{s}^{\prime} M(s-1)$ with $c_{s}^{\prime} \Gamma(1+s / 2) \Gamma(1 / 2-s / 2)=2 \Gamma(-s / 2) \Gamma(5 / 2+s / 2)$. For the interpretation of the magnetic maps we can use the same function $M(s)$ since the moment functions $A_{r}(s)$ and $Z(r)$ are given by $k_{2}(2+s) Z(s)=-2 I \sin \psi \cdot s K(s)$ and $k_{2} A_{r}(s)=-2 I \sin \psi \cdot(2+s) M(s)$.

In the general case, when the vertex of the cone is on the vertical of the point $O^{*}$ at the distance (unknown) $l$ above ( $l>0$ ) or below ( $l<0$ ) the ground, the moment function has a very complicated expression and is difficult to tabulate. In its place we use the integrals $D_{n}$ of the type

$$
D_{n}=l^{2 n-2} \iint_{P}\left(l^{2}+r^{2}\right)^{-(n+1 / 2)} D g(r) d S,
$$

and the interpretation yields all the seven parameters $k_{1}, k_{2}, p_{1}, p_{2}, p_{3}, \omega, l$, the angle $\omega$ being the slope of the dome's flanks. Lack of space does not permit the development of this general case.

Conclusion. The possibilities offered by the new method for quantitative interpretation of magnetic and gravitational anomalies we attempted to develop in this work seem to be very large. The mathematical tools, used in the proofs of the final interpretation rules, are not needed at all in practical applications. If the tables of functions used and the charts of auxiliary curves for the graphical solution of fundamental equations are calculated once for all and plotted, all that remains to be done in a particular case is the plotting of some auxiliary maps derived from the experimental data and the evaluation of some areas with the aid of a planimeter.

No use of average values in the interpretation is known to the author, with the exception of the work of K. Jung, ${ }^{3}$ where only one integral, namely our M(0), is defined and its value $2 f M$ is used together with remarkable values and distances.

[^20]
# CANTILEVER BEAMS OF UNIFORM STRENGTH* 

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1. The object of the paper, its methods and results. The problem of shaping a beam from a given amount of material in such a manner as to obtain maximum strength requires that the maximum stress of each cross section be constant. In the case of bending, the classical treatment of this problem ${ }^{1,2,3,4}$ is based on the theory of beams of constant cross section, the influence of shearing stresses and of the weight of the beam being neglected. A collection of solutions of this elementary problem, for rectangular and circular cross sections, is given in the Hütte handbook for engineers. ${ }^{5}$ If the strength of the material is relatively low, the weight $W$ of the beam cannot be neglected. This occurs in certain concrete structures, such as reinforced concrete bridges, and was demonstrated by Gaede ${ }^{6}$ in his treatment of a cantilever of rectangular cross section and constant width, the external load being a force $F$ at the free end.

In the present paper, we shall consider cantilevers of more general cross section but with the same type of loading, except in $\S 6$ where more general loading will be considered. Let us denote by $x$ the distance from the free end, by $A(x)$ the area of the cross section and by $S(x)$ the section modulus ( $S=M / \sigma$, where $M$ is the bending moment and $\sigma$ is the maximum stress). The bending moment $M(x)=\sigma S(x)$ at the distance $x$ from the free end is then given by

$$
\begin{equation*}
F x+\gamma \int_{0}^{x}(x-\xi) A(\xi) d \xi=\sigma S(x) \tag{1.1}
\end{equation*}
$$

where $\gamma$ denotes the density of the beam material. The total weight of the beam equals

$$
\begin{equation*}
\gamma \int_{0}^{L} A(\xi) d \xi=W \tag{1.2}
\end{equation*}
$$

where $L$ is the length of the beam. Since $\sigma$ is constant along the beam, differentiation of (1.1) with respect to $x$ yields

[^21]\[

$$
\begin{align*}
F+\gamma \int_{0}^{x} A(\xi) d \xi=\sigma S^{\prime}(x), & S(0)=0 \\
\gamma A(x)=\sigma S^{\prime \prime}(x), & S(0)=0, \quad \sigma S^{\prime}(0)=F,
\end{align*}
$$
\]

where the primes denote derivatives with respect to $x$. By use of (1.1') we can write (1.2) in the form

$$
\sigma S^{\prime}(L)=F+W
$$

We note that ( $1.1^{\prime}$ ) and (1.1 ${ }^{\prime \prime}$ ) are forms of the well-known equations of equilibrium of a beam, $Q=M^{\prime}, q=M^{\prime \prime}$, where $Q, q$ are respectively the shearing force and load per unit length.

If the section modulus is assigned, $A(x)$ is given by ( $1.1^{\prime \prime}$ ) and the problem is solved. In general however there are no criteria for the choice of the function $S(x)$; instead, some geometric characteristics of the cross section are assigned. Problems of this type are treated in the present paper in a general manner. They involve an integral equation (cf. Blasius ${ }^{7}$ ). Its solution may involve almost any of the classical special functions. Some simple cases leading to hyperbolic, Bessel and elliptic functions are discussed. The possibility of using Legendre, hypergeometric, Lamé and some other functions is indicated.
2. The type of beam. Throughout this paper we shall limit ourselves to cantilevers satisfying the following conditions: the line of centroids is a horizontal straight line ( $x$-axis) ; each cross section has a vertical axis of symmetry ( $V$-axis). In the plane of the cross section we choose a system of orthogonal Cartesian coordinates ( $U, V$ ) with origin at the centroid $C$. In the vertical plane through the $x$-axis, we choose a system of Cartesian coordinates $(x, y)$ with origin at the free end and $y$-axis directed downward. We assume that the curves bounding the cross sections are representable by the equations

$$
\begin{equation*}
U=u(x) u_{1}(t), \quad V=v(x) v_{1}(t) \tag{2.1}
\end{equation*}
$$

$t$ being a parameter. The functions $u_{1}(t), v_{1}(t)$ determine the shape of the cross section, whereas the functions $u(x)$ and $v(x)$ represent the change of the cross section along the axis of the beam. Any two cross sections are obtainable from each other by a transformation of dilatation ${ }^{8}$ which depends on the position of the cross sections. We will choose $u_{1}(t)$ and $v_{1}(t)$ in such a manner that $u(x)$ and $v(x)$ be $\geqq 0$.
3. General equations. It is easily seen that $S=I / V_{m}$, where $I$ is the moment of inertia of the cross section about the $U$-axis, and $V_{m}$ is the maximum value of $V$. Thus, if $\alpha$ is the area enclosed by the curve $U=u_{1}(t), V=v_{1}(t)$ and $\beta$ the corresponding section modulus, we have

$$
\begin{equation*}
S=\beta u(x)[v(x)]^{2}, \quad A=\alpha u(x) v(x) . \tag{3.1}
\end{equation*}
$$

If we set $a=\alpha \gamma /(\sigma \beta)$, the substitution of (3.1) into (1.1), (1.1'), (1.2), (1.2') gives

$$
\begin{equation*}
F x+\alpha \gamma \int_{0}^{x}(x-\xi) u(\xi) v(\xi) d \xi=\sigma \beta u(x)[v(x)]^{2} \tag{3.2}
\end{equation*}
$$

[^22]\[

$$
\begin{gather*}
\left(u v^{2}\right)^{\prime \prime}=a u v, \quad\left(u v^{2}\right)_{x=0}=0, \quad \sigma \beta\left(u v^{2}\right)_{x=0}^{\prime}=F, \\
\alpha \int_{0}^{L} u(x) v(x) d x=W / \gamma, \quad(3.3) ; \quad \sigma \beta\left(u v^{2}\right)_{x=L}^{\prime}=F+W .
\end{gather*}
$$
\]

If $v(x)$ is known, (3.2) is a Volterra integral equation in $u(x)$ with the kernel $(x-\xi) /[v(x)]^{2}$. This kernel is a continuous function, within the interval of integration, if $v(0) \neq 0$, because we assume $v(x)$ continuous and by its physical meaning it must be $\neq \theta$ for $x>0$. Therefore, according to the general theory of integral equations, ${ }^{9}$ if $v(0) \neq 0$, Eq. (3.2) has one and only one solution $u(x)$ if $F \neq 0$ and only a meaningless solution $u=0$ if $F=0$. In other words, a cantilever of uniform strength under the action of its own weight alone must be such that $v(0)=0$.
4. Particular types of cantilevers of uniform strength. These are obtained by assuming particular forms for $u(x)$ or $v(x)$.
I). $v$ is constant. The cross sections have constant height. If $F \neq 0$ the integral of (3.2') is

$$
\begin{equation*}
u=F r(\alpha \gamma v)^{-1} \sinh (r x) \tag{4.1}
\end{equation*}
$$

where $r=(a / v)^{1 / 2}$. Substitution from this equation in (3.3') gives the following condition for $W$ :

$$
\begin{equation*}
\cosh (r L)=1+(W / F) \tag{4.2}
\end{equation*}
$$

If $F=0$ no solution exists, which is in accordance with the general statement of $\S 3$, because here $v(0) \neq 0$.
$I I) . v(x)$ is a linear function of $x$. The cross sections have linearly varying height. We may restrict ourselves to the case

$$
\begin{equation*}
v(x)=c \pm x \tag{4.3}
\end{equation*}
$$

since if $x$ had a coefficient different from $\pm 1$, the coefficient could be factored out and included in the function $v_{1}(t)$. Also, since we agreed to take $v(x) \geqq 0$ (cf. §2) and $x=0$ represents a point of the beam, $c$ must be $\geqq 0$. Since $d v= \pm d x$, the solution of (3.2'), (3.3') is $^{10}$

$$
\begin{equation*}
u=v^{-3 / 2} Z_{1}\left(2 i a^{1 / 2} v^{1 / 2}\right), \tag{4.4}
\end{equation*}
$$

where $Z_{1}$ is a cylindrical function of order 1 which must satisfy the conditions

$$
\begin{gather*}
Z_{1}\left(2 i a^{1 / 2} c^{1 / 2}\right)=0, \quad \pm \sigma \beta u^{1 / 2} i Z_{0}\left(2 i a^{1 / 2} c^{1 / 2}\right)=F  \tag{4.5}\\
\pm \sigma \beta a^{1 / 2} i Z_{0}\left[2 i a^{1 / 2}(c \pm L)^{1 / 2}\right]=F+W
\end{gather*}
$$

The second equation in (4.5) and Eq. (4.5') are obtained by use of the formula ${ }^{10}$ $Z_{1}^{\prime}(x)=Z_{0}(x)-x^{-1} Z_{1}(x)$. We put

$$
\begin{equation*}
Z_{1}\left(2 i a^{1 / 2} v^{1 / 2}\right)=A i J_{1}\left(2 i a^{1 / 2} v^{1 / 2}\right)+B H_{1}^{(1)}\left(2 i a^{1 / 2} v^{1 / 2}\right), \tag{4.6}
\end{equation*}
$$

where $J_{1}$ and $H_{1}^{(1)}$ are the Bessel and the Hankel functions of the first kind and first order. ${ }^{10}$ Equations (4.5), (4.5') then give the following conditions for $A, B$ and $c$ :

[^23]\[

$$
\begin{align*}
A i J_{1}\left(2 i a^{1 / 2} c^{1 / 2}\right)+B H_{1}^{(1)}\left(2 i a^{1 / 2} c^{1 / 2}\right) & =0 \\
-A J_{0}\left(2 i a^{1 / 2} c^{1 / 2}\right)+B i H_{0}^{(1)}\left(2 i a^{1 / 2} c^{1 / 2}\right) & = \pm F /\left(\sigma \beta a^{1 / 2}\right) \\
-A J_{0}\left[2 i(a c \pm a L)^{1 / 2}\right]+B i H_{0}^{(1)}\left[2 i(a c \pm a L)^{1 / 2}\right] & = \pm(F+W) /\left(\sigma \beta a^{1 / 2}\right) \tag{4.8}
\end{align*}
$$
\]

We discuss first the case $c=0$, i.e., $v(x)=x$ (the lower sign in (4.3) has no meaning here, since $v$ must be $\geqq 0$ ). The free end of the cantilever is represented by $v=x=0$. Therefore, since $H_{1}^{(1)}(0)=\infty$, the constant $B$ must be zero. Since $J_{1}(0)=0$ and $J_{0}(0)$ $=1$, Eqs. (4.7'), (4.7") require that $A=-F /\left(\sigma \beta a^{1 / 2}\right)$, and Eq. (4.8) gives

$$
J_{0}\left(2 i a^{1 / 2} L^{1 / 2}\right)=1+(W / F)
$$

If $L$ and $W$ are not related by this equation, the constant $c$ must be distinct from zero. The determinant of the coefficients of (4.7 ), (4.7"), considered as equations in $A$ and $B$ is, by a known relation of Bessel functions, ${ }^{10}$

$$
\begin{equation*}
J_{0}(z) H_{1}^{(1)}(z)-H_{0}^{(1)}(z) J_{1}(z)=-\pi^{-1} a^{-1 / 2} c^{-1 / 2}, \tag{4.9}
\end{equation*}
$$

where $z=2 i a^{1 / 2} c^{1 / 2}$. Since this cannot be zero it is seen that there are no solutions if $F=0$, which agrees with the general result of $\S 3$. If $F \neq 0$, the solutions of (4.7 ), (4.7") are

$$
\begin{equation*}
A= \pm \pi(\sigma \beta)^{-1} c^{1 / 2} F H_{1}^{(1)}\left(2 i a^{1 / 2} c^{1 / 2}\right), \quad B=\mp \pi(\sigma \beta)^{-1} c^{1 / 2} F i J_{1}\left(2 i a^{1 / 2} c^{1 / 2}\right) \tag{4.10}
\end{equation*}
$$

Substitution from these into (4.8) gives a relation for $W$, ${ }^{11}$

$$
\begin{equation*}
\pi a^{1 / 2} c^{1 / 2}\left[J_{1}(z) H_{0}^{(1)}(\zeta)-H_{1}^{(1)}(z) J_{0}(\zeta)\right]=1+(W / F) \tag{4.11}
\end{equation*}
$$

where $z=2 i(a c)^{1 / 2}, \zeta=2 i(a c \pm a L)^{1 / 2}$.
III). $u(x)$ is proportional to $[v(x)]^{n}$. This includes a circular cross section $(n=1)$, a rectangular cross section of constant width ( $n=0$ ), a rectangular cross section with the height proportional to the width ( $n=1$ ), an elliptic cross section with axes proportional to each other $(n=1)$. By a suitable choice of $u_{1}(t)$ and $v_{1}(t)$ we may reduce the problem to the case $u=v^{n}$. The first two equations in (3.2') then give

$$
\begin{equation*}
x=\int_{0}^{0} v^{n+1}\left[C^{2}+2 a(n+2)^{-1}(2 n+3)^{-1} v^{2 n+3}\right]^{-1 / 2} d v \tag{4.12}
\end{equation*}
$$

where $C$ is a constant. The last equation of (3.2') and Eq. (3.3') give

$$
\begin{equation*}
\sigma \beta C=F /(n+2), \quad W=\left[F^{2}+(n+2)(2 n+3)^{-1} 2 \gamma \sigma \alpha \beta v_{x=L}^{2 n+3}\right]^{1 / 2}-F \tag{4.13}
\end{equation*}
$$

If $n=-1$ the cross sections have a constant area; this case gives elementary expressions for $u(x)$ and $v(x)$ but the width at the free end is infinite. When $n=1$, the in-

[^24]tegral in (4.12) is hyperelliptic; when $n=0$, it is elliptic. If $F=0$, Eqs. (4.12) and (4.13) give
\[

$$
\begin{equation*}
v=a[2(2 n+3)(n+2)]^{-1} x^{2} \tag{4.14}
\end{equation*}
$$

\]

IV. More general cases. Instead of $u$ and $v$ we introduce new variables $\omega=u v^{2}$, $\tau=1 / v ; \omega$ is directly proportional to the section modulus, and $\tau$ inversely proportional to the radius of gyration of the cross section. From (3.2) we obtain

$$
\begin{equation*}
F x+\alpha \gamma \int_{0}^{x}(x-\xi) \tau(\xi) \omega(\xi) d \xi=\sigma \beta \omega(x) \tag{4.15}
\end{equation*}
$$

This is a Volterra integral equation in $\omega(x)$. From it, or directly from ( $3.2^{\prime}$ ), ( $3.3^{\prime}$ ) we have

$$
\begin{equation*}
\omega^{\prime \prime}(x)=a \tau(x) \omega(x), \quad \omega(0)=0, \quad \sigma \beta \omega^{\prime}(0)=F, \quad \sigma \beta \omega^{\prime}(L)=F+W \tag{4.16}
\end{equation*}
$$

Since most of the so called special functions satisfy linear differential equations of second order, the first equation of (4.16) suggests the possibility of using such functions. The following are some results which may be easily checked. The constants $p, q, s, a$ must satisfy the last three equations in (4.16).
$I V a) . \tau(x)=p-q e^{2 x}, \omega(x)=Z_{m}\left(n e^{x}\right)$, where $Z_{m}=$ a cylindrical function (Bessel, Hankel, etc.), $m^{2}=a p, n^{2}=a q$.
$I V b) . \tau(x)=p-q(\cosh x)^{-2}, \omega(x)=K_{n}^{(m)}(\tanh x)$, where $K_{n}^{(m)}=$ an associate Legendre function $\left(P_{n}^{(m)}, Q_{n}^{(m)}\right), m^{2}=a p, n(n+1)=a q$.
$I V c) . \tau(x)=p-q \cos x, \omega(x)=$ a function of an elliptic cylinder. ${ }^{12}$
$I V d) . \tau(x)=\left(p-q x+x^{2}\right) /\left(4 a x^{2}\right), \omega(x)=$ a confluent hypergeometric function. ${ }^{12}$
$I V e) . \tau(x)=\left(p-q x^{s}\right) / x^{2}, \omega(x)=x^{1 / 2} Z_{m}\left(n x^{s / 2}\right)$, where $Z_{m}=$ a cylindrical function, ${ }^{10}$ $m^{2} s^{2}=1+4 a p, n^{2} s^{2}=4 a q$. If $p=0$, in order that $v$ be finite $s$ must be $<2$.

If the function $v(x)=1 / \tau(x)$ is assigned by means of any one of previous expressions for $\tau$, the function $u=\omega \tau^{2}$ is determined by the corresponding expression for $\omega(x)$. In the case of a rectangular cross section, $v(x)$ represents the height and $u(x)$ the width.
5. The deflection curve. The curvature of the geometric axis of a beam of constant strength in bending is ${ }^{1,2,4} 1 / r=h / v(x)$ where $h=\sigma /\left(E v_{m}\right), E$ being the modulus of elasticity and $\nu_{m}$ the value of $\nu_{1}(t)$ at the point of maximum stress (cf. §2). We note that this equation is a form of the well-known relation $\sigma=E y / r$. For small deflections the usual approximation is $1 / r=d^{2} y / d x^{2}$. Thus

$$
\begin{equation*}
y(x)=-\int_{0}^{x} \varphi(x) d x, \quad \varphi(x)=h \int_{x}^{L}[v(x)]^{-1} d x \tag{5.1}
\end{equation*}
$$

since

$$
\begin{equation*}
y(0)=0, \quad(d y / d x)_{x-L}=0 \tag{5.2}
\end{equation*}
$$

A simple formula for the deflection at the free end is obtained through integration of (5.1) by parts. Setting $-y(L)=Y$, we have

$$
\begin{equation*}
Y=h\left\{x \int_{x}^{L}[v(x)]^{-1} d x\right\}_{x=0}^{x=L}+h \int_{0}^{L}[v(x)]^{-1} x d x=h \int_{0}^{L}[v(x)]^{-1} x d x \tag{5.3}
\end{equation*}
$$

[^25]It is seen from (5.3) that, if $v(x)$ tends to zero as $k x^{n}$ with $n \geqq 2$ and $k$ is constant, the deflection $Y$ is infinite. This would occur, for instance, in the case corresponding to Eq. (4.14). Such a physically impossible conclusion may be explained by the fact that a large value of $n$ implies a rapid variation of $v(x)$, i.e., a rapid change of the cross section, whereas the theory which was used is based on bending of beams of constant cross section. ${ }^{13}$ More important still, the theory used in this paper neglects the shearing stresses in comparison with the bending stresses. Such a procedure is not permissible in the vicinity of the free end, and consequently it is understandable that the theoretical results for this part of the beam differ widely from reality.
6. More general loads. If $M(x)$ is the moment of the external load acting on the cantilever, we have instead of Eqs. (1.1'), (1.2')

$$
\begin{gather*}
M^{\prime \prime}(x)+\gamma A(x)=\sigma S^{\prime \prime}(x), \quad \sigma S(0)=M(0), \quad \sigma S^{\prime}(0)=M^{\prime}(0)  \tag{6.1}\\
\sigma S^{\prime}(L)=M^{\prime}(L)+W \tag{6.2}
\end{gather*}
$$

For example, if the beam is acted upon by $F$ and also by a load distributed uniformly along the axis of the beam of intensity $T$, we have $M(x)=F x+\frac{1}{2} T x^{2}$. If $v=$ const., we obtain by (6.1), (3.1), and (6.2),

$$
\begin{gather*}
u=[F r \sinh (r x)+T \cosh (r x)-T] /(\alpha \gamma v)  \tag{6.3}\\
\cosh (r L)+T(F r)^{-1} \sinh (r L)=1+(W+T L) / F, \tag{6.4}
\end{gather*}
$$

where $r=(a / v)^{1 / 2}$. If $T=0$, Eqs. (6.3), (6.4) reduce to (4.1), (4.2). Eqs. (6.3), (6.4) may be easily generalized to the case $M(x)=\sum a_{n} x^{n}$.
7. Numerical examples. We consider a rectangular cross section of width 1 and height $H$. Then (cf. §3) $\alpha=I I, v_{m}=H / 2, \beta=H^{2} / 6, a=6 \gamma(\sigma H)^{-1}$. Let $L=10 \mathrm{ft}$., $F=9000 \mathrm{lbs} ., \sigma=75000 \mathrm{lbs} . / \mathrm{sq}$. ft., $\gamma=150 \mathrm{lbs} . / \mathrm{cu} . \mathrm{ft} ., E=45 \times 10^{7} \mathrm{lbs} . / \mathrm{sq} . \mathrm{ft}$. These values correspond to a certain type of concrete.
I). Cantilever of constant height ( $\$ 4$, Case I). We put $H=1$ and assume the height $v H=v=1.9 \mathrm{ft}$. From (4.2) we obtain the weight $W=3000 \mathrm{lbs}$. We put $R=[\sigma \gamma /(\sigma v)]^{1 / 2}$. Then $R=0.0795$. From (4.1) we obtain the width

$$
u(x)=F R(v \gamma)^{-1} \sinh (R x)=2.51 \sinh (0.0795 x)
$$

At the fixed end we then have $u=u(10) \approx 2.21 \mathrm{ft}$. Equation (5.3) gives for the deflection $Y=\sigma L^{2} / E v \approx 0.1 \mathrm{in}$.
II). Cantilever with a linearly varying height ( $\$ 4$, Case II). Let the height at the fixed end be 2 ft . and at the free end $1 / 4 \mathrm{ft}$. In (4.3) we take $v(x)=c+x$. Since $H c=1 / 4 \mathrm{ft} ., H(c+10)=2 \mathrm{ft}$., we get $H=7 / 40, c=10 / 7 \mathrm{ft}$. Eqs. (4.10) give $A=-69.1$, $B=29.0$, and from Eqs. (4.4), (4.6) we obtain

$$
\begin{equation*}
u=v^{-3 / 2}\left[-69.1 i J_{1}(i \xi)+29.0 H_{1}^{(1)}(i \xi)\right], \text { where } \xi=0.8(3 v / 7)^{1 / 2} \tag{7.1}
\end{equation*}
$$

At the fixed end $x=10$, and we thus obtain $u \approx 2.2 \mathrm{ft}$. At the free end, $v=c$ and by (4.4), (4.5) we have the general result $u=0$. From (4.5') or (4.11) we get the weight $W \approx 4800 \mathrm{lbs}$. From (5.3) the deflection is

$$
Y=2 \sigma(E H)^{-1}\left[L-c \log _{e}\left(1+L \sigma^{-1}\right)\right] \approx 0.2 \mathrm{in} .
$$

[^26]
## -NOTES-

# THE INVERSE OF A STIFFNESS MATRIX* 

By K. E. BISSHOPP (Armour Research Foundalion)

It is well known that torsional vibration problems of ten require the computation of latent roots of matrices. Now the usual methods ${ }^{1}$ give these roots in descending order of magnitude while in torsional vibration problems we require the smallest root of the stiffness matrix and then, perhaps, some of the remaining roots in ascending order of magnitude. It is therefore necessary to find the inverse $u^{-1}$ of the given stiffness matrix; $u^{-1}$ is called the flexibility matrix. If its roots are in descending order $p_{1}, p_{2}, \ldots, p_{k}$ then the required roots of the original stiffness matrix in ascending order are $1 / p_{1}, 1 / p_{2}, \cdots, 1 / p_{k}$.

In gencral it is very difficult to invert a given matrix. The purpose of this note is to show that a special type of stiffness matrix which occurs frequently can be inverted with a small amount of work. Let us consider, for purposes of illustration, $n+1$ discs with moments of inertia $I_{0}, \cdots, I_{n}$ connected by massless elastic shafts of circular** cross section. Let $c_{i}$ be the coefficient of stiffness of the shaft between the $i$ th and $(i+1)$ th discs. Then

$$
\begin{equation*}
\dot{c}_{i}=G J_{z} / l_{i}, \tag{1}
\end{equation*}
$$

where $l_{i}$ is the length of the shaft in question and $G J_{s}$ is a numerical factor depending on the material and the polar sectional moment of inertia. The application of Lagrange's equations ${ }^{2}$ to the functions representing the kinetic and potential energies of the system respectively yields a system of linear differential equations. Therefore the second order time derivatives can be replaced by $-p_{1}^{2}$, so that in the absence of damping we obtain the following system of algebraic equations in matrix form,

$$
\begin{equation*}
\left(p^{2} I-\bar{u}\right) \theta=0 \tag{2}
\end{equation*}
$$

where $\theta$ is the column matrix of the normal mode appropriate to any value of $p^{2}$ for which Eq. (2) is satisfied. Since the system is capable of a rigid body rotation, $p^{2}=0$ is a solution and the degree of system (2) can be reduced by unity with the aid of the substitution

$$
I_{0} \theta_{0}+I_{1} \theta_{1}+\cdots+I_{n} \theta_{n}=0
$$

[^27]which may be looked upon as the orthogonality condition ${ }^{3}$ for the degenerate frequency $p=0$. The stiffness matrix $u$ for the reduced system in which the zero root is absent becomes
\[

\left[$$
\begin{array}{ccccc}
\frac{c_{0}}{I_{0}}+\frac{c_{0}}{I_{1}}+\frac{c_{1}}{I_{1}} & -\frac{c_{1}}{I_{1}}+\frac{c_{0} I_{2}}{I_{0} I_{1}} & \frac{c_{0} I_{3}}{I_{0} I_{1}} & \cdots & \frac{c_{0} I_{n}}{I_{0} I_{1}}  \tag{3}\\
-\frac{c_{1}}{I_{2}} & \frac{c_{1}}{I_{2}}+\frac{c_{2}}{I_{2}} & -\frac{c_{2}}{I_{2}} & \cdots & 0 \\
0 & -\frac{c_{2}}{I_{3}} & \frac{c_{2}}{I_{3}}+\frac{c_{3}}{I_{3}} & \cdots & 0 \\
\cdots \cdots \cdots \cdots & \cdots \cdots \\
0 & \cdots \cdots \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & 0 & \cdots & \frac{c_{n-1}}{I_{n}}
\end{array}
$$\right] .
\]

The inverse matrix $u^{-1}=\left(a_{i j}\right)$ may be stated in a convenient form for numerical computation as follows.

If $A_{r}=\sum_{s-r}^{n} I_{s}$, then the $j$ th element in the first row becomes

$$
\begin{equation*}
a_{1 i}=\left[I_{i} /\left(G J_{s} \sum I\right)\right]\left(I_{0} l_{0}-\sum_{r=1}^{i-1} l_{r} A_{r+1}\right) . \tag{4}
\end{equation*}
$$

When $i \leqq j$ the $i$ th element in the $j$ th column is

$$
\begin{equation*}
a_{i j}=a_{1 i}+\left(I_{i} / G J_{s}\right) \sum_{s=1}^{i-1} l_{s} \tag{5}
\end{equation*}
$$

and when $i>j$,

$$
\begin{equation*}
a_{i j}=a_{i j} . \tag{6}
\end{equation*}
$$

As an application, let us consider the case when $I_{0}=181.306 \mathrm{lb} . \mathrm{in} . \mathrm{sec}^{2}, I_{1}=I_{2}=I_{3}$ $=1,328.61 \mathrm{lb} . \mathrm{in} . \mathrm{sec}^{2}{ }^{2}, I_{4}=21,557.3 \mathrm{lb} . \mathrm{in} . \mathrm{sec} .{ }^{2}, l_{0}=30 \mathrm{in}$., $l_{1}=l_{3}=34 \mathrm{in}$., $l_{3}=62.2 \mathrm{in}$.,

Table I

| $n$ | $(1)$ | $(2)$ | $(3)$ <br> $I_{n}$ | $I_{n}$ | $I_{n} \times 10^{6} / G J_{s}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 181.306 | 30. |  | $(4)$ <br> $A_{n}$ | $I_{0} l_{0}-\sum_{1}^{(5)-1} l_{n} A_{n+1}$ |
| 1 | $1,328.61$ | 34. | 0.076051 | $25,724.4$ |  |
| 2 | $1,328.61$ | 34. | 0.076051 | $24,543.1$ |  |
| 3 | $1,328.61$ | 62.2 | 0.076051 | $22,214.5$ | $-817,850.2$ |
| 4 | $21,557.3$ |  | 1.23396 | $21,557.3$ | $-1,595,970$. |

$G J_{s}=17,470 \times 10^{6} \mathrm{lb}$.in. ${ }^{2}$ All necessary computations are contained in Table I. From the second line of this table ( $n=1$ ), we obtain immediately

$$
10^{6} a_{11}=I_{0} l_{0} I_{1} / G J_{0} A_{0}=[\mathrm{col} \text {. (5) } \times \mathrm{col} \text {. (3) }] / A_{0} .
$$

[^28]Equation (4) shows that the remainder of the elements of the first row of $u^{-1}$ can be calculated in order from each succeeding line of the table by performing the operations indicated on columns (3) and (5) respectively. The other elements in the upper righthand corner of $u^{-1}$ then are computed from Eq. (5) which gives for instance when $i=3$ and $j=4$,

$$
\begin{aligned}
10^{6} a_{34} & =10^{6} a_{14}+I_{4} \times 10^{6} / G J_{s} \sum_{n=1}^{n-2} l_{n}=-140.875+1.23396 \times(34+34) \\
& =-56.966
\end{aligned}
$$

The inverse matrix now can be completed quickly by filling in the lower left-hand corner according to Eq. (6), so that

$$
u^{-1}=10^{-6} \times\left[\begin{array}{rrrr}
0.0160803 & -2.4179 & -4.7183 & -140.875 \\
0.0160803 & 0.1678 & -2.1326 & -98.920 \\
0.0160803 & 0.1678 & 0.4532 & -56.966 \\
0.0160803 & 0.1678 & 0.4532 & 19.786
\end{array}\right]
$$

## ON THE PROBLEM OF HEAT CONDUCTION IN A SEMI-INFINITE RADIATING WIRE*

By ARNOLD N. LOWAN (Math. Tables Project, Nat. Bureau of Standards)

R. V. Churchill ${ }^{1}$ derives the solution of the problem of heat conduction in a semiinfinite radiating wire when the initial temperature is zero, and the boundary temperature is a constant. It is the object of this paper to derive the general solution corresponding to an arbitrary initial temperature distribution when the boundary temperature is a prescribed function of time.

Let $k, c, \rho, s, A, h$ and $\alpha=k / \rho c$ denote the thermal conductivity, specific heat, density, perimeter, cross-sectional area, coefficient of heat transfer, and thermal diffusivity of the wire, respectively. Further, let $a=h s / c \rho A$ and $b=a T_{2}$, where $T_{2}$ is the temperature of the medium. If the wire is sufficiently thin so that the temperature may be assumed to be constant over the entire cross section, the problem becomes one-dimensional and the temperature $T(x, t)$ must satisfy the following differential equation, initial and boundary conditions:

$$
\begin{array}{rlr}
\left(\frac{\partial}{\partial t}-\alpha \frac{\partial^{2}}{\partial x^{2}}+a\right) T(x, t)=b & (x>0, t>0) \\
\lim _{t \rightarrow 0} T(x, t)=f(x), & \text { (2); } & T(0, t)=\varphi(t)
\end{array}
$$

It is easily verified that the expression

$$
\begin{equation*}
T(x, t)=e^{-a t} u(x, t)+v(x, t) \tag{4}
\end{equation*}
$$

[^29]satisfies Eqs. (1), (2) and (3), provided the functions $u(x, t)$ and $v(x, t)$ satisfy the following differential equations, initial and boundary conditions:
\[

$$
\begin{gather*}
\frac{\partial u}{\partial t}-\alpha \frac{\partial^{2} u}{\partial x^{2}}=0 \quad(x>0, t>0)  \tag{5}\\
\lim _{t \rightarrow 0} u(x, t)=f(x), \quad(6) ; \quad u(0, t)=e^{a t} \varphi(t)  \tag{6}\\
\frac{\partial v}{\partial t}-\alpha \frac{\partial^{2} v}{\partial x^{2}}=-a v+b, \quad(8) ; \quad \lim _{t \rightarrow 0} v(x, t)=0, \quad(9) ; \quad v(0, t)=0 \tag{7}
\end{gather*}
$$
\]

From Eqs. (5), (6) and (7), it is clear that $u(x, t)$ is the temperature in a semiinfinite solid initially at the temperature $f(x)$ and with its bounding plane $x=0$ kept at the temperature $e^{a t} \varphi(t)$. Using the expression of $u(x, t)$ given by H. S. Carslaw, ${ }^{2}$ we obtain

$$
\begin{align*}
e^{-a t} u(x, t)= & \frac{e^{-a t}}{2 \sqrt{\pi \alpha t}} \int_{0}^{\infty} f(\xi)\left\{e^{-(x-\xi)^{2} /(4 \alpha t)^{\prime}}-e^{-(x+\xi)^{2} /(4 \alpha t)}\right\} d \xi \\
& +\frac{x}{2 \sqrt{\pi \alpha}} \int_{0}^{t} e^{-a \eta} \varphi(t-\eta) e^{-x^{2} / 4 a \eta \eta^{-3 / 2} d \eta} \tag{11}
\end{align*}
$$

With the aid of the identity

$$
\int_{0}^{\infty} e^{-\alpha \beta^{2} t} \cos \beta(x-\xi) d \beta=\frac{\sqrt{\pi}}{2 \sqrt{\alpha t}} e^{-(x-\xi)^{2} / 4 \alpha t},
$$

the first term of (11) may be written in the alternative form

$$
\frac{2}{\pi} e^{-a t} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\alpha \beta^{2} l f(\xi) \sin \beta x \sin \beta \xi d \beta d \xi . . ~}
$$

Accordingly, an alternative form of (11) is

$$
\begin{align*}
e^{-\alpha t} u(x, t)= & \frac{2 e^{-\alpha t}}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\alpha \beta^{2} t} f(\xi) \sin \beta x \sin \beta \xi d \beta d \xi \\
& +\frac{x}{2 \sqrt{\pi \alpha}} \int_{0}^{t} e^{-\alpha \eta} \varphi(t-\eta) e^{-x^{2} / 4 \alpha \eta} \eta^{-3 / 2} d \eta
\end{align*}
$$

We proceed to the solution of the system (8), (9), (10). The Laplace transform $v^{*}(x, p)$ of the function $v(x, t)$ must satisfy the equations

$$
\begin{equation*}
\frac{\partial^{2} v^{*}}{\partial x^{2}}-\frac{p+a}{\alpha} v^{*}=-\frac{b}{p \alpha}, \quad(12) ; \quad v^{*}(0, p)=0 . \tag{12}
\end{equation*}
$$

The solution of the system (12) and (13) is

$$
\begin{equation*}
v^{*}(x, p)=\frac{b}{p(p+a)}\left(1-e^{-\alpha \sqrt{(p+a) / \alpha})}=\sigma^{*}(p)-\sigma^{*}(p) w^{*}(p)\right. \tag{14}
\end{equation*}
$$

[^30]whence, by a well-known theorem, ${ }^{3}$
\[

$$
\begin{equation*}
v(x, t)=\sigma(t)-\int_{0}^{t} \sigma(t-\eta) w(\eta) d \eta \tag{15}
\end{equation*}
$$

\]

From

$$
\int_{0}^{\infty} e^{-p t} \sigma(t) d t=\sigma^{*}(p)=\frac{b}{p(p+a)}=\frac{b}{a}\left(\frac{1}{p}-\frac{1}{p+a}\right)
$$

it follows that

$$
\begin{equation*}
\sigma(t)=\frac{b}{a}\left(1-e^{-a t}\right) \tag{16}
\end{equation*}
$$

If in the known identity ${ }^{4}$

$$
\int_{0}^{\infty} e^{-p t} \sqrt{\frac{\bar{\lambda}}{\pi}} e^{-\lambda / t t^{-3 / 2} d t=e^{-2 \sqrt{\lambda p}}}
$$

we put $\lambda=x^{2} / 4 \alpha$ and replace $p$ by $p+a$, we obtain

$$
\frac{x}{2 \sqrt{\pi \alpha}} \int_{0}^{\infty} e^{-p t} e^{-a t} e^{-x^{2} / 4 \alpha t} t^{-3 / 2} d t=e^{-x V(p+a) / \alpha}
$$

whence

$$
\begin{equation*}
w(x, t)=\frac{x}{2 \sqrt{\pi \alpha}} e^{-a t} e^{-x^{2} / 4 \alpha t} t^{-3 / 2} \tag{17}
\end{equation*}
$$

In view of (16) and (17), (15) becomes

$$
\begin{equation*}
v(x, t)=-\frac{b}{2 a \sqrt{\pi \alpha}} \int_{0}^{t}\left\{1-e^{-a(t-\eta)}\right\} e^{-a \eta} e^{-x^{2} / 4 a \eta} \eta^{-3 / 2} d \eta+\frac{b}{a}\left(1-e^{-a t}\right) \tag{18}
\end{equation*}
$$

Making use of the identity ${ }^{5}$

$$
\begin{aligned}
\int_{c}^{\infty} e^{-\left(a^{2 \lambda \alpha+}+b^{2}\right) / \lambda^{2}} d \lambda= & \frac{\sqrt{\pi}}{2 a} \cosh 2 a b \\
& +\frac{\sqrt{\pi}}{4 a} e^{-2 a b} \operatorname{Erf}\left(\frac{b}{c}-a c\right)-\frac{\sqrt{\pi}}{4 a} e^{2 a b} \operatorname{Erf}\left(\frac{b}{c}+a c\right)
\end{aligned}
$$

and some elementary transformations, we may write (18) in the alternative form

$$
\begin{align*}
v(x, t) & =\frac{b}{a}\left\{1-e^{-a t} \operatorname{Erf}\left(\frac{x}{2 \sqrt{\alpha t}}\right)\right\} \\
& -\frac{b}{2 a}\left\{2 \cosh x \sqrt{\frac{a}{\alpha}}+e^{-x \sqrt{a / \alpha}} \operatorname{Erf}\left(\sqrt{a t}-\frac{x}{2 \sqrt{\alpha t}}\right)\right. \\
& \left.-e^{\pi \sqrt{a / a}} \operatorname{Erf}\left(\sqrt{a t}+\frac{x}{2 \sqrt{\alpha t}}\right)\right\}
\end{align*}
$$

[^31]where
$$
\operatorname{Erf}(x)=\frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-\beta^{2}} d \beta
$$

It should be noted that in (18') the function $v(x, t)$ is expressed in terms of tabulated functions.

The final solution of our problem is given by (4) in conjunction with (11) and (18) or (11') and (18 ).

## THE SPHERICAL GYROCOMPASS*

## By WALTER KOHN (University of Toronto)

In the existing literature on gyroscopes ${ }^{1}$ the theory of the gyrocompass is developed for the case of a rotor whose ellipsoid of inertia is an ellipsoid of revolution. The mathematics of this treatment is somewhat involved and, in deducing the differential equations of motion, approximations based on the smallness of the earth's angular velocity are made. In the present communication we shall treat a gyrocompass the rotor of which has a spherical ellipsoid of inertia. The motion of such a gyrocompass is, of course, covered by the more general theory usually given, but owing to the symmetry of the sphere this case allows a considerably simpler, separate treatment in which, moreover, no approximations are necessary. At the same time the essential features of gryoscopic motion are preserved.

The following system will serve as a simple model of a spherical gyrocompass. The rotor is a rigid homogeneous sphere rotating freely about a light axle which passes through its centre. The ends of this axle can slide in a smooth horizontal ring which is concentric with the rotor and rigidly attached to the earth. When the rotor is set in rapid revolution about its axle the latter executes oscillations about the meridian which will now be examined.


Fig. 1.

In the figure the right-handed unit triad, $\mathbf{i}, \mathrm{j}, \mathrm{k}$, which is fixed relative to the earth is defined as follows: $O$ is the center of the rotor; $k$ lies in the direction of the upward vertical; i lies along the meridian and points north; $\mathbf{j}$, pointing west, completes the triad. The unit vector, a, lies along the axle of the gyrocompass and the unit vector, e (in the $\mathrm{i}, \mathrm{k}$ plane), is parallel to the earths' axis; thus the angle $\lambda$, between $i$ and $e$, is the latitude of the observer.

[^32]It is clear that the couple exerted by the ring on the rotor must be of the form

$$
\mathbf{G}=G(\mathrm{a} \times \mathrm{k}) .
$$

Further, if $A$ is the moment of inertia of the rotor, $\omega$ its angular velocity and $h$ its angular momentum, we have the relation

$$
\mathrm{G}=\dot{\mathrm{h}}=A \dot{\omega}
$$

Consequently

$$
\begin{equation*}
\dot{\omega} \cdot \mathbf{a}=0, \quad(1) ; \quad \dot{\omega} \cdot \mathbf{k}=0 \tag{2}
\end{equation*}
$$

Since the gyrocompass has only two degrees of freedom, equations (1) and (2), together with initial conditions, completely determine its motion.

We observe that $\omega$ is made up of three parts: the spin of the sphere about its axle; the rotation of the axle relative to the frame $\mathbf{i}, \mathbf{j}, \mathbf{k}$; and finally the absolute rotation of $\mathbf{i}, \mathbf{j}, \mathrm{k}$ or of the earth to which it is attached. Therefore we may write $: \omega=s \mathbf{a}+\dot{\theta} \mathrm{k}+\Omega \mathrm{e}$, where $s$ is the spin of the rotor, $\theta$ the angle between the meridian $i$ and the axle $a$, and $\Omega$ the angular velocity of the earth. Differentiation of this relation gives $\dot{\omega}=\dot{s} \mathbf{a}+s \dot{a}+\ddot{\theta} \mathbf{k}+\dot{\theta} \mathbf{k}$, and since the angular velocity of $\mathbf{a}$ is $\hat{\theta} \mathbf{k}+\Omega \mathrm{e}$ and that of $\mathbf{k}$ is $\Omega \mathrm{e}$, this equation becomes

$$
\begin{equation*}
\dot{\omega}=s \mathbf{a}+s(\hat{\theta} \mathbf{k}+\Omega \mathbf{e}) \times \mathbf{a}+\ddot{\theta} \mathbf{k}+\dot{\theta} \Omega(\mathbf{e} \times \mathbf{k}) . \tag{3}
\end{equation*}
$$

Substituting (3) into (1) and (2), we immediately arrive at

$$
\begin{equation*}
s-\dot{\theta} \Omega \cos \lambda \sin \theta=0, \quad(4) ; \quad s \Omega \cos \lambda \sin \theta+\ddot{\theta}=0 \tag{4}
\end{equation*}
$$

as the required equations of motion.
The solution of these equations may be obtained in the usual way. From (4) it follows that $s=s_{0}+\Omega \cos \lambda\left(\cos \theta_{0}-\cos \theta\right)$, where $s_{0}$ and $\theta_{0}$ are the initial values of $s$ and $\theta$. Inserting this value of $s$ into (5), we obtain a differential equation for $\theta$ alone, which is of the classical type $\ddot{\theta}=f(\theta)$; this can be solved in terms of hyperelliptic functions. If the initial spin $s_{0}$ is great we may replace $s$ in (5) by $s_{0}$ to obtain the well known result: the motion of the axle a is identical with the motion of a simple pendulum, the position of equilibrium being in the direction of the meridian.

An interesting property of the spherical gyrocompass can be deduced directly from Eqs. (4) and (5) which are exact. For, if we multiply by $s$ and $\dot{\theta}$ respectively and add, we obtain $s s=\dot{\theta} \ddot{\theta}=0$, which on integration becomes $s^{2}+\dot{\theta}^{2}=$ constant. ${ }^{2}$ This shows that a spherical gyrocompass has an angular velocity of strictly constant magnitude relative to the earth.

The author is indebted to Prof. A. Weinstein for his advice and criticism.

$$
\begin{aligned}
& { }^{2} \text { Levi Civita and Amaldi, l.c., derive a corresponding result, namely } \\
& \qquad \frac{1}{2} A \dot{\theta}^{2}+\frac{1}{2} C s^{2}=\text { const. }
\end{aligned}
$$

for the general gyrocompass. ( $A$ and $C$ are the transverse and axial moments of inertia respectively.) They then explain it by energy considerations. In fact, however, this equation and therefore also their reasoning is not strictly accurate. The exact form is

$$
\frac{1}{2} A \dot{\theta}^{2}+\frac{1}{2} C s^{2}+\frac{1}{2}(C-A) \Omega^{2} \cos ^{2} \lambda \sin ^{2} \theta=\text { const. }
$$

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[^0]:    * Reccived September 27, 1944.
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    ${ }^{3}$ R. P. Isaacs, Airfoil theory for flows of variable velocity, abstract in Bulletin of the American Mathematical Society, 50, 186 (1944).

[^1]:    - Th. von Kármán, Neue Darṡtellung der Tragflügeliheorie, Zeitschrift f. angew. Math. u. Mech. 15, 56-61 (1935).
    ${ }^{5}$ H. S. Tsien, Symmetrical Joukowsky airfoils in shear flow, Quart. Appl. Math. 1, 130-148 (1943).

[^2]:    * Received May 9, 1944; presented to the American Mathematical Society, November 26, 1943.
    ${ }^{1}$ T. Theodorsen, Theory of wing sections of arbitrary shape, NACA Tech. Rep. No. 411 (1931); T. Theodorsen and I. E. Garrick, General potential theory of arbitrary wing sections, NACA Tech. Rep. No. 452 (1934).
    ${ }^{2}$ A function $g(\theta)$ is absolutely continuous in an interval if its derivative $g^{\prime}(\theta)$ exists for all $\theta$ of this

[^3]:    interval except possibly for a set of Lebesgue measure zero and if $\int_{a}^{b} g^{\prime}(\theta) d \theta=g(b)-g(a)$ for every $a$ and $b$ of this interval. In order to establish the convergence of Theodorsen's method under reasonably general conditions, we employ the integral of Lebesgue.

[^4]:    ${ }^{3}$ Cf. Theodorsen and Garrick, 1.c., pp. 184-185.

[^5]:    4See, for example, A. Zygmund, Trigonomelric series, Warsaw, 1935, p. 76 (Eq. (4)).

[^6]:    ${ }^{7}$ See, for example, S. E. Warschawski, On the higher derivatives at the boundary in conformal mapping, Trans. Amer. Math. Soc. 38, 326 (Theorem III), (1935).

[^7]:    ${ }^{9}$ We are using here the following theorem: if $g(\phi)$ is an absolutely continuous and periodic function (period $2 \pi$ ) for $0 \leqq \phi \leqq 2 \pi$ and if $\left[g^{\prime}(\phi)\right]^{2}$ is integrable, then the conjugate function $\bar{g}(\phi)$ is absolutely continuous and $\left(\bar{g}^{\prime}(\phi)\right)^{2}$ is integrable. (See, for example, W. Seidel, Über die Ränderzuordnung bei konformen Abbildungen, Math. Annalen 104, 223 (1931).

[^8]:    ${ }^{10}$ This follows from a theorem of F. and M. Riesz, Über die Randwerte einer analytischen Funktion, Comptes Rendus du Quatrième Congrès des Mathématiciens Scandinaves à Stockholm (1916) pp. 27-44. See also, F. Riesz, Math. Zeitschrift, 18, 95 (1923).

[^9]:    ${ }^{11}$ I. Privaloff, Sur les fonctions conjuguces, Bull. Soc. Math. France 44, 100-103 (1916); or Zygmund, 1.c. p. 156. Privaloff's Theorem states: if $g(\phi)$ is periodic (period $2 \pi$ ) and satisfies a Hölder condition with the exponent $\alpha, 0<\alpha<1$, for all $\phi$, then any conjugate function of $g(\phi)$ satisfies such a condition.

[^10]:    * Received Oct. 2, 1944.

[^11]:    ${ }^{1}$ J. Hadamard, Sur les tourbillons produit par les ondes de choc, Note III, in Leçons sur la propagation des ondcs, A. Hermann, Paris, 1903, p. 362.
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    - A. Vazsonyi, On two-dimensional rotational gas flows, Bull. of the American Mathematical Society, 50, 188 (1944).

[^12]:    ${ }^{5}$ See J. Ackeret, Handbuch der Physik, vol. 7, Berlin 1927, chap. 5, p. 293, or H. Lamb, l.c. pp. 575 and 637. In the energy equation it is not assumed that $\mu$ is constant.

[^13]:    * A line which at each point is tangent to the vorticity vector $\omega$, at this point, is called a vortex line. An infinitely thin tube formed by vortex lines is called a vortex filament.
    - A. A. Friedmann, Über Wirbelbewegung in einer kompressiblem Fliissigkeit, Zeitschrift f. Ang. Math. und Mech., 4, 102-107 (1924).

[^14]:    * The author is indebted to Professor H. W. Emmons of Harvard University for this example.

[^15]:    ${ }^{7}$ K. O. Friedrichs (and R. von Mises), Fluid dynamics, Brown University, Providence, R. I., 1941, p. 229.

[^16]:    * Received Aug. 21, 1944.
    ** Now at California Institute of Technology.

[^17]:    * The author wishes to express his gratitude to Mr. S. L. Chang for pointing out relation (3.4).

[^18]:    * Received April 11, 1944.
    ** On leave of absence. Now at The New School, New York.
    ${ }^{1}$ For examples of these usual methods see L. L. Nettleton, Geophysical prospecting for oil, McGrawHill Book Co., New York, 1940, Chapters 6 and 12, and also H. Shaw, Interpretation of gravitational anomalies, Trans. Amer. Inst. Min. Met. Eng. 97, 271-366 (1932).

[^19]:    * In Fig. 5 the letters $A_{1}$ and $A_{2}$ denote the centers of the two end surfaces of the cap rock cylinder.
    ${ }^{2}$ Quarterly Journal of Math., 2, 217 (1940), Theorem 5, case b.

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    ${ }^{3}$ S. Timoshenko, Strength of Materials, part I, 2nd Edition, D. Van Nostrand Company, New York, 1940, pp. 209-210.
    ${ }^{4}$ C. Guidi, Teoria dell' elasticità e resistensa dei materiali, 11 th Edition, Torino, 1925, pp. 135-142.
    ${ }^{5}$ Hiutle-Des Ingenieurs Taschenbuch, vol. I, 25th Edition, Wilhelm Ernst, Berlin, 1925, pp. 626-629.
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    ${ }^{8}$ The writer is indebted to Prof. H. Busemann for this geometric terminology.

[^23]:    ${ }^{9}$ See for instance R. Courant-D. Hilbert, Methoden der mathematischen Physik, vol. I, 2nd Edition, Julius Springer, Berlin, 1931, pp. 119, 120, 133; or Riemann-Weber, Die Differential- und Integralgleichungen der Mechanik und Physik, vol. I, 7th Edition, F. Vieweg, Braunschweig, 1925, pp. 426-428 or E. Hellinger-O. Toeplitz, Enc. d. math. Wiss., vol. II. 3, pp. 1459, 1460.
    ${ }^{10}$ E. Jahnke-F. Emde, Funktionentafeln mil Formeln und Kurven, 3rd Edition, B. G. Teubner, Leipzig, 1938, pp. 128, 144-147, 224, 237, 242.

[^24]:    ${ }^{11}$ By (4.3) the possible range of $L$ is $(0,+\infty)$ or $(0, c)$. The left hand side of (4.11) is within this range a function of $L$ which increases continuously from 1 to $+\infty$, as may be shown by the theory of Bessel functions and by $\left(4.7^{\prime}\right),(4.9),(4.10)$. Therefore for any assigned values of $F, W, a, c$ there exists, in each case (4.3), one and only one value of $L$ satisfying (4.11). Problems of this type in which the length $L$ is not assigned but a given amount of material is to be distributed into a cantilever of uniform strength under the action of $F$ and $W$ or $W$ alone have occurred in some biological fields (cf. C. Holtermann, Schwendener's Vorlesungen über mechanische Probleme der Botanik, Leipzig, 1909, pp. 18, 19 and O. Fischer, Enc. d. malh. Wiss., IV.8, p. 119).

[^25]:    ${ }^{12}$ E. T. Whittaker-G. N. Watson, Modern analysis, 4th Edition, Cambridge University Press, 1935, pp. $337,405$.

[^26]:    ${ }^{13}$ A method which takes into account the variability of the cross section was worked out by J. Resal, Résistance des matériaux, Paris, 1898, pp. 393-405 for rectangular and double $T$ cross sections of constant width.

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    ** For other than circular cross sections, suitable factors can be obtained from the appropriate torsion functions.
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    4 J. R. Carson, loc. cit., p. 39.
    ${ }^{6}$ This is a slight generalization of Churchill's formula (4), page 120.

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    ${ }^{1}$ Cf. T. Levi Civita and U. Amaldi, Lezioni di meccanica razionale, vol. 2, Zanichelli, Bologna, 1927, pp. 191-195; or J. L. Synge and B. A. Griffith, Principles of mechanics, McGraw-Hill Book Co., New York, 1942, pp. 430-433.

