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SOME NUMERICAL METHODS FOR LOCATING ROOTS OF POLYNOMIALS*

BY

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1. Introduction. It is the purpose of this paper to discuss the location of the roots of polynomials of high degree, with particular reference to the case of complex roots. This is a problem with which we at the Laboratories have been much concerned in recent years because of the fact that the problem arises rather frequently in the design of electrical networks. I shall not give any attention to strictly theoretical methods, such as the exact solution by elliptic or automorphic functions: nor to the development of roots in series or in continued fractions, though such methods exist and one at least—development of the coefficients of a quadratic factor[1]—is of great value in improving the accuracy of roots once they are known with reasonable approximation.

Instead, we shall deal with just two categories of solutions: one, the solution of the equations by a succession of rational operations, having for their purpose the dispersion of the roots; the other, a method depending on Cauchy's theorem regarding the number of roots within a closed contour.

PART I—MATRIX ITERATION

2. Duncan and Collar. We shall treat the first category by a method recently elaborated by Duncan and Collar in two papers in the Philosophical Magazine.[2] I do not know how thoroughly these writers appreciate the close relationship of their work to that of the other writers whom I shall mention in the course of my presentation. The fact that their interest was primarily concerned with certain broad dynamical problems may perhaps have inhibited them from taking some of the steps which I shall take in their name. But they at least possessed the essential idea, and exhibited quite sufficient ability in the development of it to warrant the assertion that my presentation only differs from theirs in detail—sometimes details of omission, sometimes details of amplification.

* Received Dec. 26, 1944.

[1] The essence of this method is contained in a section of Legendre's *Essai sur la théorie des nombres*. It is also attributed to Bairstow by Frazer and Duncan. It was developed independently, and perhaps somewhat more fully, by the present writer; but the extensions seem so obvious that it has not appeared to warrant separate publication.

[2] W. J. Duncan and A. R. Collar, *A method for the solution of oscillation problems by matrices*, Phil. Mag. (7) 17, 865–909 (1934); *Matrices applied to the motions of damped systems*, Phil. Mag. (7) 19, 197–219 (1935).

3. The fundamental identity. We begin by noting that the λ-determinant

$$D(\lambda) = \begin{vmatrix} a_{11}+\lambda & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22}+\lambda & \cdots & a_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{nn}+\lambda \end{vmatrix} \equiv \prod (\lambda + \lambda_j) \tag{1}$$

is the characteristic function[3] of the matrix

$$M = \|a_{ij}\| \tag{2}$$

and its determinant

$$\Delta = |a_{ij}|. \tag{3}$$

It is obviously a polynomial of degree n, which we may write

$$D(\lambda) = \lambda^n + p_1\lambda^{n-1} - p_2\lambda^{n-2} + \cdots \pm p_n. \tag{4}$$

Any quantity which satisfies the equation

$$D(\lambda) = 0 \tag{5}$$

and obeys the associative and commutative laws of algebra—whether it be a number or not—must also satisfy the relation

$$\lambda^n = - p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots \mp p_n;$$

and if we multiply this by λ throughout and then eliminate λ^n we get

$$\lambda^{n+1} = (p_1^2 + p_2)\lambda^{n-1} - (p_1p_2 + p_3)\lambda^{n-2} + \cdots,$$

which is of the form

$$\lambda^{n+1} = p_1^{(n+1)}\lambda^{n-1} + p_2^{(n+1)}\lambda^{n-2} + \cdots.$$

Similarly, by a continuation of the same process we may get a succession of equations, all of the form

$$\lambda^m = p_1^{(m)}\lambda^{n-1} + p_2^{(m)}\lambda^{n-2} + \cdots. \tag{6}$$

We call the typical polynomial on the right of (6) $f_m(\lambda)$: graphically it represents a curve of degree $n-1$ passing through the n points $-\lambda_j$, $(-\lambda_j)^m$. But we do not wish to emphasize this geometric interpretation but rather the formal algebraic fact that our derivation has required only the elementary rules of algebra and the relation (5), and that when these rules are satisfied

$$\lambda^m = f_m(\lambda). \tag{7}$$

Suppose, now, that we expand the quotient $f_m(\lambda)/D(\lambda)$ in partial fractions. The result is

$$\frac{f_m(\lambda)}{D(\lambda)} = \sum_{j=1}^{n} \frac{f_m(-\lambda_j)}{\lambda + \lambda_j} \frac{1}{\prod_{k \neq j}(-\lambda_j + \lambda_k)}. \tag{8}$$

[3] The unconventional pecularities of sign in (1) and in (4) below happen to be convenient for our purposes later on.

But $f_m(-\lambda_j)=(-\lambda_j)^m$ by (7), and $D(\lambda)=\prod(\lambda+\lambda_k)$ by (1). Hence

$$f_m(\lambda) \equiv \sum_{j=1}^{n} (-\lambda)^m \prod_{k\neq j} \left(\frac{\lambda_k+\lambda}{\lambda_k-\lambda_j}\right). \tag{9}$$

Now (9) is an algebraic identity, and though we have used the process of division in setting it up, it does not require division by λ as a process of verification. Hence it is again true that if λ is any quantity which obeys the distributive and associative laws, such for example as a differential operator, and which satisfies (5),

$$\lambda^m \equiv \sum_{j=1}^{n} (-\lambda_j)^m \pi_j'(\lambda), \tag{10}$$

where

$$\pi_j'(\lambda) = \prod_{k\neq j} \left(\frac{\lambda_k+\lambda}{\lambda_k-\lambda_j}\right). \tag{11}$$

Note that the quantities denoted by $\pi_j(\lambda)$ are polynomials of degree $n-1$ in λ and are *independent of m.*

4. Matrices. We next observe that, though matrix multiplication is not in general commutative, it is so if we restrict ourselves to certain groups. In particular, if we begin with the unit matrix I, any other matrix M, and all scalar quantities (i.e., numbers), then all matrices which can be formed from these by a finite number of additions or multiplications are commutative. For obviously M is commutative with itself and its powers, and with I, and with scalars, which observations together with the associative law are sufficient to warrant the general statement.

Furthermore, we know from the Hamilton-Cayley Theorem[4] that

$$D(M) = 0,$$

where $D(\lambda)$ represents, as in §3, the characteristic function of M. In other words, M satisfies all the requirements imposed upon λ in deriving the identity (10), whence we conclude that

$$M^m \equiv \sum (-\lambda_j)^m \pi'(M), \tag{12}$$

where $\pi_j'(M)$ is a matrix *independent of m.*

As a final step, we multiply this equation throughout by an arbitrary matrix K—which need not be commutative with the rest, since we shall perform no further operations—thus obtaining

$$M^m K \equiv \sum (-\lambda_j) \quad \pi_j(M) \tag{13}$$

where $\pi_j(M)=\pi_j'(M)K$ is again independent of m.

This is the fundamental identity upon which Duncan and Collar rely for their method. It is equivalent to n^2 equations of similar form connecting corresponding elements in the various matrices. For example, if α_m is written for the element in the i-th row and j-th column of M^mK, and e_j for the correspondingly placed element in $\pi_j(M)$, it must be true that

$$\alpha_m = e_1(-\lambda_1)^m + e_2(-\lambda_2)^m + \cdots + e_n(-\lambda_n)^m. \tag{14}$$

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[4] M. Bocher, *Introduction to higher algebra*, MacMillan, New York, 1929, p. 296.

We again recall that $-\lambda_j$ is a root of the characteristic equation of M—that is, of the polynomial $D(\lambda)$—and hence is a number. The set $-\lambda_j$ are, in fact, just the roots which we wish to obtain. Similarly, the e_j's are numbers *independent of m*. But α_m is a numerical function of m.

5. The roots. Suppose now, that one of the roots which we will call $-\lambda_1$, is larger in absolute value than all the rest. Then if we select corresponding elements α_m and α_{m+1} from two consecutive orders of M^mK we will have

$$\frac{\alpha_{m+1}}{\alpha_m} = -\lambda_1 \frac{1 + \frac{e_2}{e_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{m+1} + \frac{e_3}{e_1}\left(\frac{\lambda_3}{\lambda_1}\right)^{m+1} + \cdots}{1 + \frac{e_2}{e_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{m} + \frac{e_3}{e_1}\left(\frac{\lambda_3}{\lambda_1}\right)^{m} + \cdots},$$

and hence obviously

$$\lim_{m\to\infty} \frac{\alpha_{m+1}}{\alpha_m} = -\lambda_1. \tag{15}$$

In other words: *if an arbitrary matrix K is multiplied repeatedly by M, and if its characteristic equation has a largest root, then the ratio of corresponding elements in two consecutive products approaches this largest root as a limit as* $m\to\infty$.

Similarly, we readily find that

$$\begin{vmatrix} \alpha_{m+1} & \alpha_m \\ \alpha_m & \alpha_{m-1} \end{vmatrix} = (-\lambda_1)^m(-\lambda_2)^m \sum_{i\neq j} e_i e_j \left(\frac{\lambda_i\lambda_j}{\lambda_1\lambda_2}\right)^m \left(\sqrt{\frac{\lambda_i}{\lambda_j}} - \sqrt{\frac{\lambda_j}{\lambda_i}}\right)^2; \tag{16}$$

whence if $|\lambda_1|$ and $|\lambda_2|$ are greater than all other $|\lambda\text{'s}|$, (whether they are themselves equal or not), we again have

$$\lim_{m\to\infty} \frac{\begin{vmatrix} \alpha_{m+1} & \alpha_m \\ \alpha_m & \alpha_{m-1} \end{vmatrix}}{\begin{vmatrix} \alpha_m & \alpha_{m-1} \\ \alpha_{m-1} & \alpha_{m-2} \end{vmatrix}} = (-\lambda_1)(-\lambda_2). \tag{17}$$

In the same way it can be shown[5] that provided $|\lambda_1|, \cdots, |\lambda_i|$ are all greater than $|\lambda_{i+1}| \cdots |\lambda_n|$

$$\lim_{m\to\infty} \frac{\begin{vmatrix} \alpha_{m+i} & \alpha_{m+i-1} & \cdots & \alpha_{m+1} \\ \alpha_{m+i-1} & \alpha_{m+i-2} & \cdots & \alpha_m \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{m+1} & \alpha_m & \cdots & \alpha_{m-i+2} \end{vmatrix}}{\begin{vmatrix} \alpha_{m+i-1} & \alpha_{m+i-2} & \cdots & \alpha_m \\ \alpha_{m+i-2} & \alpha_{m+i-3} & \cdots & \alpha_{m-1} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_m & \alpha_{m-1} & \cdots & \alpha_{m-i+1} \end{vmatrix}} = (-\lambda_1)(-\lambda_2)\cdots(-\lambda_i). \tag{18}$$

[5] A. C. Aitken, Proc. Royal Soc. Edinburgh **46**, 289–305 (1926), obtains formulae equivalent to these in a discussion of Bernoulli's method.

These equations are sufficient to determine all the roots in the particular case where

$$|\lambda_1| > |\lambda_2| > |\lambda_3| \cdots > |\lambda_n|.$$

6. Example. As a simple example we may take

$$M = \begin{Vmatrix} 3 & 1 \\ 2 & 2 \end{Vmatrix}, \qquad K = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix},$$

in which case

$$MK = \begin{Vmatrix} 3 \\ 2 \end{Vmatrix}, \qquad M^2K = \begin{Vmatrix} 11 \\ 10 \end{Vmatrix}, \qquad M^3K = \begin{Vmatrix} 43 \\ 42 \end{Vmatrix},$$

$$M^4K = \begin{Vmatrix} 171 \\ 170 \end{Vmatrix}, \qquad M^5K = \begin{Vmatrix} 683 \\ 682 \end{Vmatrix}.$$

Taking the ratios of the first elements of consecutive matrices we get as the successive approximations to $-|\lambda|$,

$$11/3 = 3.667, \qquad 43/11 = 3.909, \qquad 171/43 = 3.977, \qquad 683/171 = 3.992.$$

Similarly we find that

$$\frac{\begin{vmatrix} 171 & 43 \\ 43 & 11 \end{vmatrix}}{\begin{vmatrix} 43 & 11 \\ 11 & 3 \end{vmatrix}} = 4 \quad \text{and} \quad \frac{\begin{vmatrix} 683 & 171 \\ 171 & 43 \end{vmatrix}}{\begin{vmatrix} 171 & 43 \\ 43 & 11 \end{vmatrix}} = 4,$$

which should be the product of λ_1 and λ_2.

The characteristic equation in this case is, however,

$$D(\lambda) = \begin{vmatrix} \lambda + 3 & 1 \\ 2 & \lambda + 2 \end{vmatrix} = \lambda^2 + 5\lambda + 4,$$

and its roots are -1 and -4. The approximation is obvious.

7. Complex roots. So far we have considered only real roots: for obviously, since complex roots occur in conjugate pairs (the coefficients being assumed to be real) there can be no *largest* one. Suppose, then, that $|\lambda_2| = |\lambda_1|$ and that all other roots are smaller in absolute value. Then by (17),

$$\lambda_1\lambda_2 = \lim_{m\to\infty} \frac{\begin{vmatrix} \alpha_{m+1} & \alpha_m \\ \alpha_m & \alpha_{m-1} \end{vmatrix}}{\begin{vmatrix} \alpha_m & \alpha_{m-1} \\ \alpha_{m-1} & \alpha_{m-2} \end{vmatrix}}. \tag{19}$$

This gives us the absolute value of the roots. It does not, however, determine the angles. To get this, we can best return to equation (14) and write (retaining only the leading terms)

$$\alpha_m = e_1(-\lambda_1)^m + e_2(-\lambda_2)^m.$$

Writing the similar equations for $m-1$ and $m+1$, and eliminating e_1 and e_2, we get

$$\lambda_1\lambda_2\alpha_{m-1} + (\lambda_1 + \lambda_2)\alpha_m + \alpha_{m+1} = 0.$$

Substituting the value of $\lambda_1\lambda_2$ as given by (19) we get finally

$$-(\lambda_1 + \lambda_2) = \lim_{m\to\infty} \frac{\begin{vmatrix} \alpha_{m+1} & \alpha_{m-1} \\ \alpha_m & \alpha_{m-2} \end{vmatrix}}{\begin{vmatrix} \alpha_m & \alpha_{m-1} \\ \alpha_{m-1} & \alpha_{m-2} \end{vmatrix}}. \tag{20}$$

This, together with (19) is sufficient to determine the pair of roots.

As written, the formula applies even if the roots are real.[6] When they are complex it is best to write $-\lambda_1 = -\bar{\lambda}_2 = \rho e^{i\phi}$. Then obviously we need only replace the $\lambda_1\lambda_2$ of (19) by ρ^2, and the $-(\lambda_1+\lambda_2)$ of (20) by $2\rho\cos\phi$.

Similar, but more complicated, formulae can be obtained when more than two roots have the same absolute value.

8. The method of Daniel Bernoulli. We now note that any polynomial in λ, which we take in the form

$$D(\lambda) = \lambda^n + p_1\lambda^{n-1} - p_2\lambda^{n-2} + \cdots \mp p_{n-1}\lambda \pm p_n \tag{4}$$

as before, can be written as

$$D(\lambda) = \begin{vmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ p_n & p_{n-1} & p_{n-2} & \cdots & p_2 & p_1 + \lambda \end{vmatrix}. \tag{21}$$

But this is the characteristic function of the matrix

$$M = \begin{Vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ p_n & p_{n-1} & p_{n-2} & \cdots & p_2 & p_1 \end{Vmatrix}. \tag{22}$$

Hence if we choose for K any matrix whatever, we may solve for the largest roots by any of the equations of §§4 and 6.

It is particularly convenient to take K in the form

[6] It is not even necessary that they be equal in absolute value, though unless they are equal (or nearly equal) (15) will obviously be a more convenient formula.

$$K = \left\| \begin{matrix} 0 & 0 & 0 \cdots \alpha_0 \\ 0 & 0 & 0 \cdots \alpha_1 \\ 0 & 0 & 0 \cdots \alpha_2 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots \alpha_{n-1} \end{matrix} \right\| . \tag{23}$$

Then we have

$$MK = \left\| \begin{matrix} 0 & 0 & 0 \cdots \alpha_1 \\ 0 & 0 & 0 \cdots \alpha_2 \\ 0 & 0 & 0 \cdots \alpha_3 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots \sum_{j=0}^{n-1} p_{n-j}\alpha_j \end{matrix} \right\| ,$$

which is again of the same form as K. If we denote $\sum_{j=0}^{n-1} p_{j+1}\alpha_{n-j-1}$ by α_n, we also have

$$M^2K = \left\| \begin{matrix} 0 & 0 & 0 \cdots \alpha_2 \\ 0 & 0 & 0 \cdots \alpha_3 \\ 0 & 0 & 0 \cdots \alpha_4 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots \sum_{j=0}^{n-1} p_{j-1}\alpha_{n-j} \end{matrix} \right\| .$$

And in general

$$M^mK = \left\| \begin{matrix} 0 & 0 & 0 \cdots \alpha_m \\ 0 & 0 & 0 \cdots \alpha_{m+1} \\ 0 & 0 & 0 \cdots \alpha_{m+2} \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots \alpha_{m+n-1} \end{matrix} \right\|$$

where

$$\alpha_{k+1} = \sum_{j=0}^{n-1} p_{j+1}\alpha_{k-j}, \qquad k > n - 2. \tag{24}$$

This entire set of matrices, however, is characterized by a simple sequence of α's, of which the defining equation is (24). Obviously, it is also true that *any set of four consecutive α's in this sequence also constitutes a set of corresponding elements from four consecutive matrices of the set M^mK.* Hence, the use of the symbol α in this connection is consistent with its use in §§3–6. But (24) is the recursion formula used in Bernoulli's method of solution as developed by Euler, Lagrange and Aitken. Hence this particular special case of the results of Duncan and Collar is identical with Bernoulli's method.

Concerning this method Whittaker and Robinson[7] say: "Though hardly now of first-rate importance, it is interesting and worthy of mention." Our tests at the

[7] E. T. Whittaker and G. Robinson, *Calculus of observations*, 2nd ed., Blackie & Son, London, 1929.

Laboratories, however, have shown it as good as any other method in the case of complex roots. Such inferiority as it may have compared to the root-squaring method as regards speed is quite compensated by the fact that it is self-correcting: that is, an error at any stage of the process merely prolongs the calculations, but does not invalidate it.

9. The method of R. L. Dietzold. Another form into which the general results of Duncan and Collar can be thrown is obtained by using the conjugate form of (21) together with the same matrix for K as before. Denoting the conjugate of M by M', we have from (22) and (23)

$$M'K = \begin{Vmatrix} 0 & 0 & 0 & \cdots & p_n \alpha_{n-1} \\ 0 & 0 & 0 & \cdots & \alpha_0 + p_{n-1}\alpha_{n-1} \\ 0 & 0 & 0 & \cdots & \alpha_1 + p_{n-2}\alpha_{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \alpha_{n-2} + p_1\alpha_{n-1} \end{Vmatrix}.$$

If, then, we define

$$\alpha_0' = p_n\alpha_{n-1}, \qquad \alpha_j' = \alpha_{j-1} + p_{n-j}\alpha_{n-1}, \tag{25}$$

$M'K$ becomes identical with (23), except that all the α's are primed. In general, if we set

$$\alpha_j^{(m)} = \alpha_{j-1}^{(m-1)} + p_{n-j}\alpha_{n-1}^{(m-1)}, \tag{26}$$

and understand that $\alpha_{-1}^{(m)}$ is zero for all m, we have

$$M^m K = \begin{Vmatrix} 0 & 0 & 0 \cdots & \alpha_0^{(m)} \\ 0 & 0 & 0 \cdots & \alpha_1^{(m)} \\ 0 & 0 & 0 \cdots & \alpha_2^{(m)} \\ \cdot & \cdot & \cdot \cdots & \cdot \\ 0 & 0 & 0 \cdots & \alpha_{n-1}^{(m)} \end{Vmatrix}. \tag{27}$$

In this case, as in all others, the index m is the one which is to be varied in using formulae such as (16)–(20).

This variant of the general scheme of Duncan and Collar was developed by Mr. R. L. Dietzold of the Bell Telephone Laboratories, but has not been published. As compared with Bernoulli's, it has the merit of using a large number of simple operations instead of a small number of complicated ones. It is approximately as fast, and like all schemes based on Duncan and Collar's results, it is self-correcting.

10. The method of Graeffe. There is also a close connection between Duncan and Collar's processes and the root-squaring method. This method, which is usually attributed to Graeffe, seems actually to have been developed first by Dandelin, and has had the attention of a long list of mathematicians, including Lobachevski, Encke, Brodetsky and Smead, and Hutchinson.

This connection can best be established[8] by recalling that the roots $-\lambda_i$ of the

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[8] M. Bocher, *Introduction to higher algebra*, MacMillan, New York, 1929, p. 283, Theorem 3.

matrix $\lambda I+M$ are invariant under transformations of the type $T^{-1}[\lambda I+M]T$. Furthermore, it is possible to find a transformation of this sort which will throw M into the form

$$M^* = T^{-1}MT = \begin{Vmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{Vmatrix},$$

and hence $\lambda I+M$ into the form $\lambda I+M^*$, since $T^{-1}IT$ is obviously I.

This same transformation, however, carries M^m into M^{*m}, as we readily see from the identity

$$\begin{aligned} T^{-1}M^mT &= T^{-1}(MTT^{-1}MT \cdots T^{-1}M)T \\ &= (T^{-1}MT)(T^{-1}MT) \cdots (T^{-1}MT) \\ &= M^{*m}. \end{aligned}$$

Hence the characteristic equations of M^m and M^{*m} must also have identical roots. But, obviously,

$$M^{*m} = \begin{Vmatrix} \lambda_1^m & & \cdots & 0 \\ 0 & \lambda_2^m & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \lambda_n^m \end{Vmatrix},$$

so that the roots of its characteristic equation, and therefore also those of the characteristic equation of M^m, must be $-\lambda_j^m$.

But if we take $K=I$ in Duncan and Collar's process of matrix iteration, the successive matrices obtained are M^m. Hence the whole process may be regarded as one which sets up a sequence of characteristic equations with roots $-\lambda_j$, $-\lambda_j^2$, $\cdots$ and in general $-\lambda_j^m$.

In the root-squaring process as originally developed only the powers $-\lambda_j^2$, $-\lambda_j^4$, $-\lambda_j^8$, $\cdots$ were obtained, which corresponds in matrix terms to getting first the product of M by M, which is M^2; then the product of M^2 by M^2 which is M^4, and so on. Thus high powers are reached with a smaller number of matrix operations, which is theoretically desirable. Practically, however, the superiority is not so apparent. For the zeros of (22) are rapidly replaced by numbers in forming powers of M, so that a multiplication such as $M^8 \cdot M^8$ involves many more arithmetical operations than a multiplication of the form $M \cdot M^8$. Furthermore, an error at any point of the root-squaring method perpetuates itself, whereas in the other method an error at any stage is merely equivalent to starting over again with a new value of K.

Our experience leads us to believe that the methods of §§8 and 9 are generally to be preferred, at least when computations are to be performed by a clerical staff of computers.

11. The method of Bernoulli as developed by Lagrange. There is also a very close connection between the iterated matrix M^m and a development of Lagrange's which he characterizes as based upon that of Daniel Bernoulli. In it, he notes that

$$\frac{D'(\lambda)}{D(\lambda)} = \sum \frac{1}{\lambda + \lambda_j} = \frac{n}{\lambda} + \frac{s_1}{\lambda^2} + \frac{s_2}{\lambda^3} + \frac{s_3}{\lambda^4} + \cdots, \tag{28}$$

where

$$s_m = (-\lambda_1)^m + (-\lambda_2)^m + \cdots + (-\lambda_n)^m; \tag{29}$$

and it is the quotient s_m/s_{m-1} which Lagrange uses. Obviously, these are just the sums of the elements in the principal diagonals of M^{*m}. But Lagrange's method of obtaining them by dividing $D(\lambda)$ into its derivative is preferable. Besides, in spite of what might at first be assumed, it is self-correcting.

It is of historical interest to note that a very similar development was worked out by Legendre[9] independently of Lagrange, and at about the same time. Both of these writers, however, knew of earlier work by Euler, who had carried out a similar development using instead of $D'(\lambda)$ an arbitrary polynomial of degree $n-1$, which

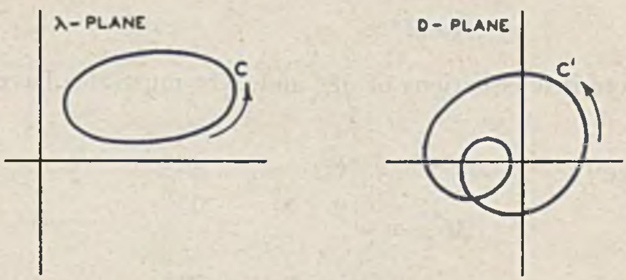

FIG. 1.

merely has the effect of replacing the s_m's in the right-hand member of (28) by the α_m's defined by (14). In other words, the method of Euler was exactly equivalent to that of Duncan and Collar, except that in the former there was no obvious criterion for the choice of a convenient form of numerator, whereas it is easy to choose matrices K which will lead to a simple succession of operations, as we have illustrated in Sections 8 and 9.

PART II—CONFORMAL MAPPING

12. The method of Routh. The second group of methods to which I wish to refer are all founded upon a well-known theorem of Cauchy. If we represent the complex variable λ by one plane, and the complex variable D by another, then the equation

$$D(\lambda) = \lambda^n + p_1\lambda^{n-1} - p_2\lambda^{n-2} + \cdots \pm p_n, \tag{4}$$

may be looked upon as a transformation by means of which the λ-plane is mapped upon the D-plane. The correspondence between λ and D, however, is not 1:1 but in general n:1; and hence a simple closed curve C in the λ-plane (Fig. 1) passes into a much more complicated curve C' in the D-plane. In regard to the curve C' the theorem in question says that the number of times it loops around the origin is exactly equal to the number of roots of $D(\lambda)=0$ which lie inside C.

[9] Legendre's development was in terms of the *reciprocal powers* of the roots, instead of their direct powers. Otherwise the two were identical.

This rule appears first to have been applied by Routh to the problem of determining the number of roots with positive real parts, a problem which interested him because of its relation to the stability of linear dynamical systems. For this purpose he used as the contour in the λ-plane the imaginary axis closed by a semicircle of infinite radius, thus enclosing the entire right half of the plane. For this particular contour he explained in great detail how from the sequence of intersections of C' with the real and imaginary axes the number of roots could be found without more definite information as to the shape of C'. He also developed a sequence of functions, similar to Sturm functions, by means of which the number of roots could be determined from the polynomial directly without even knowing the real and imaginary intercepts of C'. He did not extend either of these studies to the point of locating the roots more exactly, but both are capable of such extension and have actually been used.

13. The method of G. R. Stibitz. The second method—the one using functions similar to the Sturm functions—was developed further by G. R. Stibitz of the Bell Telephone Laboratories. He observes, first, that the method can also be used to find the number of roots with real parts greater than λ_0. To do this, it is merely necessary to replace λ by $\lambda-\lambda_0$ in the polynomial (4), and then proceed as outlined by Routh. By carrying out this process for enough values of λ_0, the roots can be segregated within strips parallel to the imaginary axis. Then by a definite routine (resembling in its essentials the Weierstrass subdivision process in point-set theory) the real values of the roots can be found to any desired degree of approximation. When this has been accomplished, the imaginary parts are determined at once as a ratio of two of the Sturm-like functions.

Stibitz has developed complete schedules for the computations required in solving polynomials by this method, for all values of n up to 10. The method has been tried, and works reasonably well, though perhaps not as rapidly as those explained in Sections 8 and 9. I suspect that the decision in this case, however, must remain a conditional one; for the computational routine of Stibitz' method is complicated (i.e., varied) as compared with the extremely simple (i.e., repetitive) routines of Sections 8 and 9. For this reason, it is not as well adapted to use in an industrial computing laboratory. In the hands of a mathematician who thoroughly understood its theoretical origin it might show up much better.

14. The method of A. J. Kempner. Kempner's methods[10] resemble more nearly the other portion of Routh's work. He chooses as his contour C a circle of radius r about the origin as center. Then $\lambda = re^{i\theta}$, and (4) becomes

$$\begin{aligned} D(\lambda) = {} & [r^n \cos n\theta + p_1 r^{n-1} \cos (n-1)\theta - p_2 r^{n-2} \cos (n-2)\theta + \cdots] \\ & + i[r^n \sin n\theta + p_1 r^{n-1} \sin (n-1)\theta - p_2 r^{n-2} \sin (n-2)\theta + \cdots]. \end{aligned} \tag{30}$$

Thus the real and imaginary parts of D are trigonometric sums, which, as Kempner remarks, could be calculated by means of a harmonic synthesizer, such for example as the Michelson "analyzer." Thus two curves would be obtained, one giving the real part of D, and the other its imaginary part, both as functions of θ. From this point on, Kempner suggests two possible routines. First, to regard these curves as parametric representations of D, and from them construct the curve C' itself. Second, to keep them as separate curves and not bother further about C'. In both cases, he

[10] University of Colorado Studies 16, 75 (1928); Bulletin of the Amer. Math. Society 41, 809 (1935).

develops rules very similar to Routh's for finding the number of roots directly from the sequence of intercepts with the axes.

He uses this routine to segregate the roots in annular rings, and then tracks down their absolute values by a suitably chosen succession of intermediate circles. The angle of any root is, of course, automatically determined as the value of θ at which the real and imaginary parts of (30) vanish when r is given the particular value appropriate to that root.

Kempner also mentions the possibility of applying the method to sectorial instead of annular regions, but does not develop this idea to a significant degree.

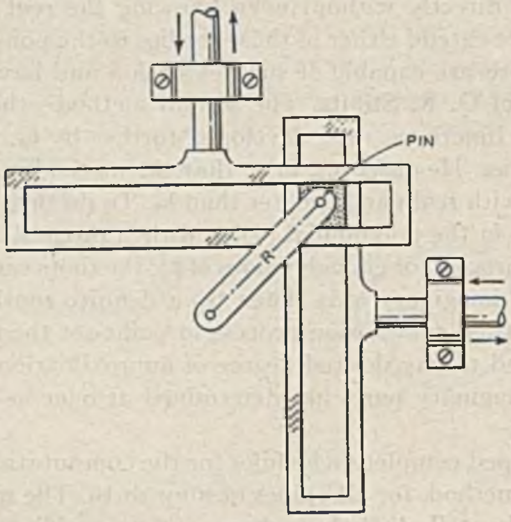

FIG. 2.

15. The isograph. Kempner's method was also developed independently, but somewhat later (1934) at Bell Telephone Laboratories, and led to the construction of a machine, called the isograph, which draws the curve C' corresponding to a circle of any radius r.

Since the independent variable in plotting the curves is an angle, what is required for the isograph is a rotating unit that provides two linear motions—one proportional to the sine and the other to the cosine of the angle. There would have to be ten of these units to provide for the ten variable terms of a tenth degree equation, and while the first unit moves through an angle θ, the second unit must move through an angle 2θ, the third unit through an angle 3θ, and so on. Then by providing a means of summing the sine and cosine motions separately, and allowing these sums to control two perpendicular motions of a pencil and drawing board, a closed curve will be described as θ increased from 0 to 360 degrees.

To secure motions proportional to the sine and cosine of the angle of rotation, the isograph utilizes the "pin and slot" mechanism illustrated in Fig. 2. Here an arm rotating about a fixed point carries a pin arranged to slide, by means of a rectangular block, in rectangular slots cut in two slide-bars, each of which is free to move back and forth in one direction only—the two motions being at right angles to each other. These motions are equal to the length R of the arm times the sine and cosine of the angle of rotation.

The ten units provided are geared to a common driving motor, but the gearing is designed so that when the arm of the first unit moves through an angle θ, that of the second unit will move through an angle 2θ, that of the third through 3θ, and so on.

To provide for summing up all the sine terms and all the cosine terms, the ends of all the slide-bars carry pulleys so that a single wire may be carried around all the sine pulleys and another around all the cosine pulleys as indicated in Fig. 3. Station-

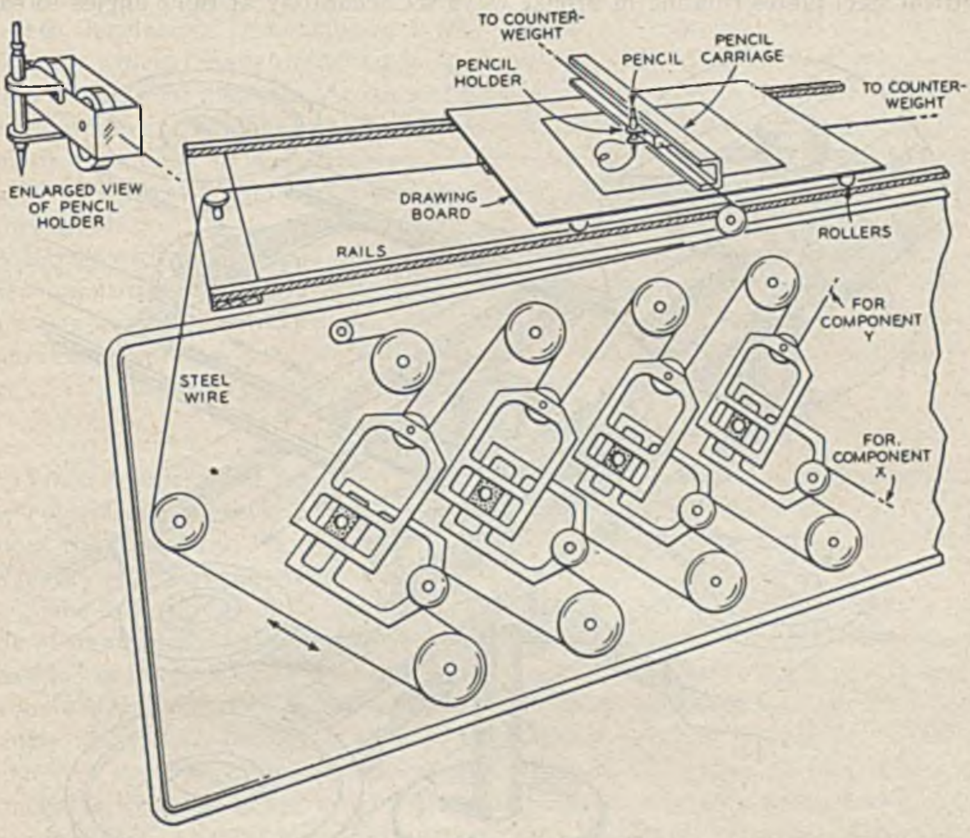

FIG. 3.

ary pulleys are mounted between the movable ones so as to keep the direction of pull on the wires in line with the motion of the slide-bars. These wires control the relative motions of a pencil and drawing board to plot a curve as the angle is varied from zero to three hundred and sixty degrees.

The construction of the rotating elements is shown in Fig. 4. The drive shaft passes through the bed plate and is fastened to the center of a steel bar that acts as the arm of Fig. 2. This bar is grooved to receive the pin of the "pin and slot" mechanism. In order that the pin may be adjusted for different crank lengths, corresponding to the coefficients $p_k r_{n-k}$ of the various terms in the equation, a rack is cut along one edge of the groove so that a pinion attached to the pin may move it along the bar. After adjustment the pin is secured in place by a set-screw.

The top of the bar carries a carefully graduated scale to which the center of the pin must be set accurately. The scale is made visible at the center of the pin by constructing the latter as a hollow cylinder. A vernier scale within the cylinder enables the effective arm length to be adjusted very exactly to the desired value on either side of the center—one side for positive coefficients and the other for negative. The total range of adjustment is three inches.

The hollow pin turns in a rectangular bronze block which fits the slots of two slide bars, one for the sine motion and one for the cosine motion. The slide bars are identical steel plates running in bronze ways set accurately at right angles to each

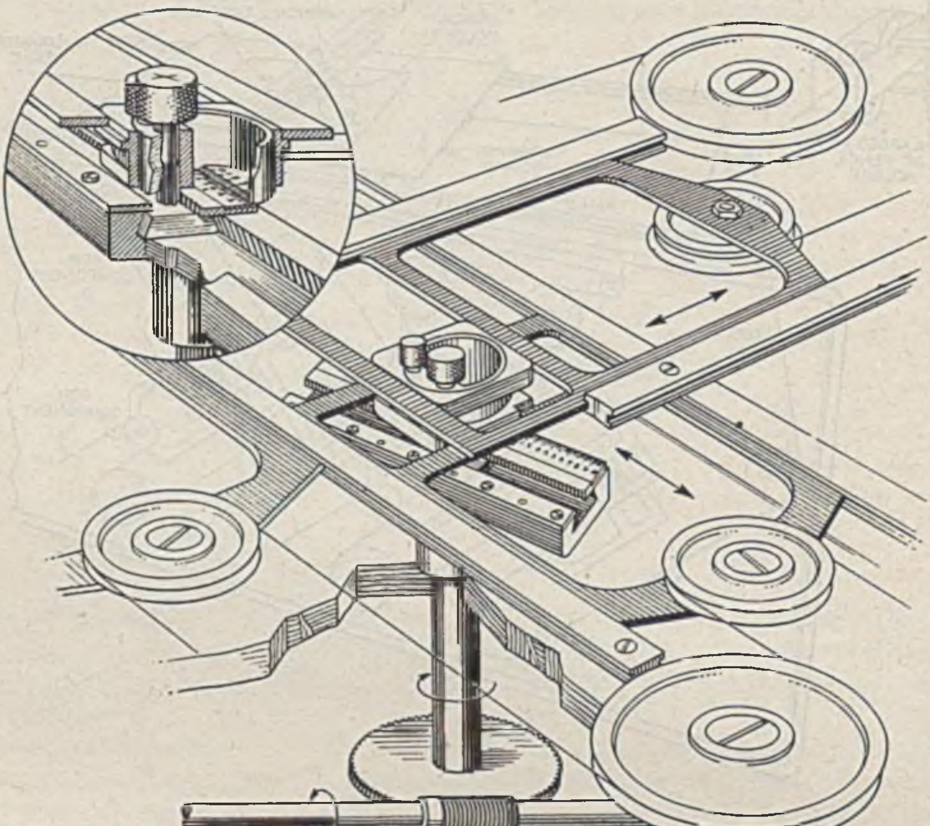

Fig. 4.

other. At the end opposite to the slot each plate carries a pulley around which is passed the wire that sums up the sine or cosine motions of the ten elements. One end of each wire is fixed. The other end of the cosine wire is led by pulleys to the drawing board, which consists of a thin aluminum sheet mounted on ball-bearing rollers so that it is free to move back and forth in only one direction. A counterweight fastened to the other edge of the board keeps the wire under constant tension. The free end of the sine wire is led by pulleys to a counterweighted pencil carriage, which is mounted with ball bearings in a fixed guide crossing the drawing board at right angles to its direction of motion. Thus the board is displaced back and forth in proportion to the sum of the cosine terms, and the pencil is displaced back and forth in a perpendicular direction in proportion to the sum of the sine terms; and this combined motion gives the desired curve.

In operation, the isograph has given accuracies of one per cent or better: and of course gives them quite rapidly. In fact, the most rapid method we have at present is that of using the isograph to obtain this degree of accuracy, and then improving it either by the methods explained in §§8 and 9, or by successive approximation to the quadratic factors.

16. Conclusion. In conclusion I wish merely to point out that in none of the methods which I have described is computation with complex numbers involved. They are all *real* methods. At present this seems to be a fundamental requirement imposed upon us by commercial computing machines, since the multiplication of two complex numbers on such machines requires six, and division eight, separate operations. If this restriction were removed, other methods might conceivably prove to be more rapid.

Partly with this in mind, and partly because we must frequently deal with complex quantities in other connections, we are at present developing a computing machine for complex quantities. When it is completed, as we hope it will be in the course of the present year, we shall undertake a further study of methods which now are clearly ruled out by mechanical limitations.*

POSTSCRIPT BY R. L. DIETZOLD

When the foregoing paper was written, it was intended for immediate publication. By coincidence, however, several other papers of similar character appeared at just about that time, and Dr. Fry concluded that the subject was of too limited interest to justify publishing another.

Since then, the situation has changed in several ways. First, the interest in methods of numerical computation has greatly increased, largely because war activities have led to much work of that kind. Second, the specific problem of root-finding has become a live one because its fundamental importance in linear dynamics is more widely recognized. Finally, a few new methods of iteration have been evolved and some new types of computing machines developed. The paper therefore now has a timeliness which it lacked when written, but a few comments are required to bring it up to date. The most important of these are noted in the following paragraphs.

In the Bell Telephone Laboratories the available computing equipment has been materially improved through the development of the relay computer by Stibitz and this inevitably reacts upon the relative convenience of various methods of solution. Although the relay computer is very flexible in respect to the type of problem it can handle, it is particularly well suited to iterative processes such as Bernoulli's method of root extraction; for once the proper instructions have been set into the control tape which governs the machine, all successive operations are performed without further supervision. The simplicity of Bernoulli's rule, which requires only that the machine accumulate $n-1$ of the α's, each multiplied by the appropriate coefficient from the polynomial, recommends it for mechanization. The instructions are easily

* This machine was placed in service in 1940 and was demonstrated at the summer meeting of the American Mathematical Society in Hanover, New Hampshire in September of that year. The relay computers referred to in Dr. Dietzold's postscript are still more versatile devices which have been developed since that time.

set up, and the machine is not required to recall very many numbers at any stage in the process. Bernoulli's method is likewise well adapted to computing equipment of the punched-card type, provided only that the accumulator is designed to recognize algebraic sign.

One of the routines which may be set up in a relay computer enables the algebraic operations to be performed on complex numbers with the ease that the same operations are performed on real numbers with a mechanical computing machine. The availability of this aid makes Newton's method useful for root improvement in the complex domain, and some on the Laboratories' computing staff prefer it to Bairstow's method, although the margin of choice is not great.

Bairstow's variation of Newton's method avoids computation with complex quantities by improving the coefficients of a trial quadratic factor. The trial factor, say $Q(\lambda)=\lambda^2+a\lambda+b$, is divided twice into the polynomial, and the rates of change of the remainder coefficients found from the second remainder, as in Horner's process. The method has by now been sufficiently publicized;[11] nevertheless, it can be given here, since it is short to state. The polynomial being expressed as

$$D(\lambda) = (r_0\lambda + s_0) + Q(\lambda)(r_1\lambda + s_1) + Q^2(\lambda)(t_0 + t_1\lambda + \cdots),$$

improved coefficients for Q are

$$a' = a - \frac{\begin{vmatrix} r_0 & r_1 \\ s_0 & s_1 \end{vmatrix}}{\begin{vmatrix} ar_1 - s_1 & r_1 \\ br_1 & s_1 \end{vmatrix}}, \qquad b' = b + \frac{\begin{vmatrix} ar_1 - s_1 & r_0 \\ br_1 & s_0 \end{vmatrix}}{\begin{vmatrix} ar_1 - s_1 & r_1 \\ br_1 & s_1 \end{vmatrix}}.$$

Newton's method typifies a class which is deliberately excepted from treatment in Fry's paper; methods in this class are characterized by the property that only sometimes do they lead to a solution. Newton's method, for example, can never lead to a complex root if the iterative process is started from a real trial value. Bairstow's method has a similarly restricted region of convergence and was, quite properly, advanced by him only as a means for improving roots already located approximately.

Methods which sometimes fail to converge may still be very useful if, in application, they converge often enough and fast enough. Newton's method and its variations, however, almost always fail unless they can be started from values closely corresponding to roots. But in 1941, Shih-Nge Lin revealed an algorithm[12] remarkable

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[11] Bairstow gave the method only in Reports and Memoranda No. 154, Advisory Committee for Aeronautics, Oct., 1914 (H. M. Stationer's Office), but it was made generally available by Frazer and Duncan, Proc. Royal Soc. London **125**, 68–82 (1929). Hitchcock offered the method as *An improvement on the G.C.D. method for complex roots*, Jour. Math. Phys. **23**, 69–74 (1944). Hitchcock proposes that the roots be improved by this method after only approximate location by the G.C.D. method, which he gave in Jour. Math. Phys. **17**, 55–58 (1938). The G.C.D. method is nearly identical with the method of G. R. Stibitz, described by Fry. Bairstow's method was also rediscovered by Friedman, whose work is noted in Bull. Amer. Math. Soc. 49, 859–860 (1943). Bairstow's formulae give the leading terms of series developments of the coefficients by Fry, who concluded, after an investigation of the convergence, that the expansion was suitable only for root-improvement.

[12] *A method of successive approximations of evaluating the real and complex roots of cubic and higher order equations*, Jour. Math. Phys. 20, 153 (1941).

both for simplicity and convergence. By Lin's method, the polynomial is divided only once by a trial quadratic factor; if[13]

$$\begin{aligned} D(\lambda) &= p_0 + p_1\lambda + p_2\lambda^2 + \cdots \\ &= (r_0\lambda + s_0) + Q(\lambda)(q_0 + q_1\lambda + q_2\lambda^2 + \cdots), \end{aligned}$$

improved coefficients for Q are

$$a' = \frac{q_0p_1 - q_1p_0}{q_0^2}, \qquad b' = \frac{p_0}{q_0}.$$

If it converges, the process determines the factor corresponding to the roots of least absolute value; thus a suitable initial choice for Q is

$$\lambda^2 + (p_1/p_2)\lambda + (p_0/p_2).$$

In application, the process does very often converge, although sometimes slowly. When the convergence of Lin's method is slow, Bairstow's method offers a valuable supplement. Lin's method is used until the size of the remainder indicates that an approximation to a quadratic factor has been obtained; Bairstow's process, started from a sufficiently close approximation, will converge, and when it converges, it converges rapidly.

The combination of these two methods provides useful, and usually adequate, equipment for the work-a-day solution of polynomial equations. In recalcitrant cases, mechanical aids are particularly helpful. Bernoulli's method is always available, but is quite likely to be slow in cases for which Lin's method has already failed. This makes little difference if the iterative process is performed automatically by a relay computer, but recommends devices to accelerate the convergence if the computation must be performed without aid. An efficient device for accomplishing this is given by A. C. Aitken in a very full discussion[14] of numerical methods for evaluating the latent roots of matrices.

Like most of those who use matrix methods, Aitken is concerned not solely with the solution of polynomial equations, but rather with the more general problem of determining the characteristic roots (and also the characteristic vectors) of matrices. Preliminary reduction of the matrix to the rational canonical form involves so many operations,[15] that one would commonly start the general problem with a matrix M having few vanishing elements. In this event, we lose one of the reasons for preferring Bernoulli's method (i.e., repeated multiplication by M) to matrix powering by the root-squaring method, for the latter method arrives at high powers of M with fewer operations, thus providing another means for hastening the convergence. The advantage is, however, partly illusory except for the limited class of computers who are so unerring that they can afford to sacrifice the self-correcting feature of the former procedure.

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[13] A departure from Fry's notation is convenient here.

[14] Proc. Royal Soc. Edinburgh 57, 172–181 (1937).

[15] Harold Wayland, *Expansion of determinantal equations into polynomial form*, Quarterly Appl. Math. 2, 277–306 (1945).

THE KÁRMÁN-TSIEN PRESSURE-VOLUME RELATION IN THE TWO-DIMENSIONAL SUPERSONIC FLOW OF COMPRESSIBLE FLUIDS*

BY

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1. Introduction. T. v. Kármán and H. S. Tsien[1] have treated the two-dimensional subsonic flow of a perfect, irrotational, compressible fluid by replacing the adiabatic pressure-volume curve by the tangent line drawn at an arbitrary point of this curve.

First, we shall discuss the applicability of the Kármán-Tsien idea in the supersonic range. Secondly, we shall show that when the Kármán-Tsien relation can be used (fairly uniform completely supersonic flow), the characteristics form a Tschebyscheff net (fish net).[2] However, we shall be concerned with those regions of the physical plane which can be mapped into a Tschebyscheff net in a unique one-to-one manner. Hence, we shall not study the onset of shock. Further, we shall show that if the diagonal curves of the net of characteristics are drawn so as to correspond to equidistant values of the arc length parameter along the characteristics, then these diagonal curves will be the families of equipotentials and stream lines. Analytically, this last result means that the determination of the stream lines depends upon two arbitrary functions of one real variable. It is shown that the angle between the characteristics and the angle formed by a tangent to a stream line and the x-axis can be determined in terms of these functions. Further, the magnitude of the velocity and the density depend upon only the former angle and the Mach number of the flow. In particular, if a known stream line coincides with the x-axis, it is shown that only one arbitrary function enters into the problem of determining the stream lines. Even in this last case where the data are of a simple Dirichlet type (symmetric flow about the x-axis and a known external boundary stream line—as in the jet problem), the direct problem cannot be solved easily. Hence, an analytical-geometrical method is outlined for solving the problem indirectly. A particular example is studied. Finally, in an appendix, we furnish another proof (analytical) of the fact that when the Kármán-Tsien relation is applicable, the characteristics form a Tschebyscheff net and conversely.

2. Extension of the Kármán-Tsien method to supersonic flow. In this section, we shall show that the Kármán-Tsien method may be extended to the supersonic flow of a perfect, irrotational, compressible fluid. If we denote the pressure by p, the density by ρ, the ratio of the specific heats by γ, the adiabatic relation is

$$p\rho^{-\gamma} = \text{constant}. \tag{2.1}$$

* Received Oct. 16, 1944.

[1] T. von Kármán, *Compressibility effects in aerodynamics*, Journal of Aeron. Sciences **8**, 337–356 (1941).

H. S. Tsien, *Two-dimensional subsonic flows of compressible fluids*, Journal of Aeron. Sciences **6**, 399–407 (1939).

[2] L. Bianchi, *Lezioni di geometria differentiale*, vol. 1, Enrico Spoerri, Pisa, 1922, pp. 153–162.

Replacing the isentropic curve (2.1) by the tangent line drawn at the point (ρ_1^{-1}, p_1) in the pressure-volume diagram (or by a hyperbola drawn at the point (ρ_1, p_1) in the pressure-density diagram), Tsien obtains the relation

$$p_1 - p = a_1^2\rho_1^2(\rho^{-1} - \rho_1^{-1}), \tag{2.2}$$

where a_1^2 is the velocity of sound corresponding to (ρ_1, p_1). By use of (2.2), it is easily shown that the Bernoulli relation becomes (where w is the velocity)

$$w^2 - w_1^2 = a_1^2\rho_1^2(\rho^{-2} - \rho_1^{-2}). \tag{2.3}$$

Further, by use of (2.2) and the definition of a^2 (that is, $a^2 = dp/d\rho$), it follows that

$$a^2\rho^2 = a_1^2\rho_1^2 = k^2, \tag{2.4}$$

where k is some constant. Hence, (2.3) can be transformed into the following forms:

$$\left(\frac{w}{a_1}\right)^2 - \left(\frac{w_1}{a_1}\right)^2 = \left(\frac{\rho_1}{\rho}\right)^2 - 1, \tag{2.5}$$

$$w^2 - a^2 = w_1^2 - a_1^2 = l^2, \tag{2.6}$$

where l is some constant.

In the following, we shall assume that the point (ρ_1, p_1) corresponds to a supersonic state of the fluid. From (2.5), we see that as w increases, ρ decreases. Further, from (2.4), we see that as ρ decreases, a increases. As noted by Tsien, the first result is in accord with the physical facts; the second result is in contradiction to known physical facts. However, (2.6) furnishes some useful information. Since the density ρ_1 corresponds to a supersonic state of the fluid, the equation (2.1) is valid for this ρ_1 and the corresponding p_1. Hence, by well known results, w_1 is larger than a_1. Thus from (2.6), we see that w is always larger than a. That is, the fluid is always in a supersonic state in this sense of the term. However, by dividing (2.6) by a^2 and noting that as w increases, ρ decreases, and a increases, we see that as w increases, w/a decreases. This ratio approaches the limiting value 1 as w tends to infinity. Hence, the behavior of w/a is contrary to that of a real fluid.

Perhaps the best indication of the permissible values of w which can be used for a given w_1 is obtained by following the procedure of Tsien. If we consider the upper limit of the useful values of w as occurring for $p = 0$, we find that the corresponding ρ is given by

$$\frac{1}{\rho} = \frac{p_1 + a_1^2\rho_1}{a_1^2\rho_1^2}. \tag{2.7}$$

Substituting this value of ρ into (2.3), we obtain the relation

$$\left(\frac{w}{w_1}\right)^2 = 1 + \frac{1}{(w_1/a_1^2)}\left[\left(\frac{p_1}{a_1^2\rho_1} + 1\right)^2 - 1\right]. \tag{2.8}$$

Since the values of ρ_1, p_1 satisfy (2.1), we find

$$a_1^2 = \left(\frac{dp}{d\rho}\right)_1 = \frac{\gamma p_1}{\rho_1}. \tag{2.9}$$

Thus (2.8) becomes

$$\left(\frac{w}{w_1}\right)^2 = 1 + \frac{1}{(w_1/a_1)^2}\left[\left(\frac{1}{\gamma} + 1\right)^2 - 1\right]. \tag{2.10}$$

If γ is taken as 1.4, a simple computation reveals that as w_1/a_1 goes from 1 to ∞, the ratio w/w_1 runs from 1.7 to 1. That is, for large values of the ratio w_1/a_1, the range of applicability of formula (2.2) is severely restricted as regards the upper limit of w. Hence, the Kármán-Tsien relation should be useful in the supersonic range for a fairly uniform fluid flow. Further, as we shall show in the next section, the characteristics in this case form a Tschebyscheff (fish) net. We shall not be concerned with the onset of shocks.

3. The geometry of the characteristics for the relation (2.2). If $u(x, y)$ and $v(x, y)$ denote, respectively, the x and y components of the velocity for the steady flow of a fluid at any point P of the plane region considered and $\rho(x, y)$ denotes the density of the fluid at P, then from the equation of continuity it follows that a stream function $\psi(x, y)$ exists such that

$$\rho u = \frac{\partial\psi}{\partial y}, \qquad \rho v = -\frac{\partial\psi}{\partial x}. \tag{3.1}$$

Further, since the motion is irrotational, a velocity potential exists such that

$$u = \frac{\partial\phi}{\partial x}, \qquad v = \frac{\partial\phi}{\partial y}. \tag{3.2}$$

For a given pressure-density relation, the Bernoulli relation determines ρ as a function of u^2+v^2. Hence (3.1), (3.2) constitute a non-linear system. Eliminating the partial derivatives of ρ from the continuity relation by use of the Euler equations, we find that $\phi(x, y)$ satisfies[3]

$$(a^2 - u^2)\frac{\partial^2\phi}{\partial x^2} - 2uv\frac{\partial^2\phi}{\partial x\partial y} + (a^2 - v^2)\frac{\partial^2\phi}{\partial y^2} = 0. \tag{3.3}$$

In the supersonic range, the equation (3.3) is hyperbolic. Let us denote the equations of the two parameter family of characteristics of (3.3) by

$$x = x(\alpha, \beta), \qquad y = y(\alpha, \beta), \tag{3.4}$$

where $\alpha=$constant and $\beta=$constant are the parametric equations of the characteristics. We denote the arc length element of the net formed by the characteristics by

$$ds^2 = A^2(d\alpha)^2 + B^2(d\beta)^2 + Cd\alpha d\beta, \tag{3.5}$$

where A^2, B^2, C are the metric coefficients of the net. It follows from (3.3) that the projections of the velocity vector on the normals to the characteristics have the magnitude a. This means that the projections of a vector, normal to the velocity vector and of magnitude equal to that of the velocity vector, on the tangents to the characteristics have the magnitude a. Also, the projections of the velocity vector on the tangents to the characteristics have the magnitude $\sqrt{u^2+v^2-a^2}$. Further, from

[3] R. von Mises and K. O. Friedrichs, *Fluid dynamics*, Brown University, Providence, R. I., 1941, p. 230.

(3.1), it follows that ρ^{-1} times the gradient of ψ is a vector, normal to the velocity vector and of magnitude equal to that of the velocity vector; and, from (3.2), it follows that the gradient of ϕ is the velocity vector. From the above properties of the characteristics and those of the gradient, it follows that

$$\frac{\partial\psi}{\partial\alpha} = \rho a A, \qquad \frac{\partial\psi}{\partial\beta} = -\rho a B, \tag{3.6}$$

$$\frac{\partial\phi}{\partial\alpha} = \sqrt{u^2 + v^2 - a^2}\, A, \qquad \frac{\partial\phi}{\partial\beta} = \sqrt{u^2 + v^2 - a^2}\, B. \tag{3.7}$$

We shall prove that *the net of characteristics forms a Tschebyscheff net, when the Kármán-Tsien relation is applicable.*

From (2.4), (2.6) and (3.6), (3.7), we find

$$\frac{\partial\psi}{\partial\alpha} = kA, \quad \frac{\partial\psi}{\partial\beta} = -kB, \quad (3.8) \qquad \frac{\partial\phi}{\partial\alpha} = lA, \quad \frac{\partial\phi}{\partial\beta} = lB. \quad (3.9)$$

The integrability conditions for (3.8), (3.9) furnish the result

$$\frac{\partial A}{\partial\beta} = \frac{\partial B}{\partial\alpha} = 0. \tag{3.10}$$

Hence A and B are functions of α and β, respectively. By proper choice of scale factors, A and B may be assigned the value unity. The new parameters α and β are then arc length parameters and the net is a Tschebyscheff net.

Next, we shall derive a result similar to that obtained by von Mises[4] in plane plasticity: *when the Kármán-Tsien relation is applicable and the diagonal curves of the characteristics are drawn so as to correspond to equi-distant values of the arc length parameter along the characteristics, then these diagonal curves will be the families of equipotentials and stream lines.*

Since (2.2) is valid, we see from our previous result that α and β may be chosen as arc length parameters. Hence, the nets

$$\alpha + \beta = 2\xi = \text{constant}, \qquad \alpha - \beta = 2\eta = \text{constant}, \tag{3.11}$$

represent, respectively, the diagonal curves of the net of characteristics, corresponding to equi-intervaled values of the arc length parameters α and β. Further, from (3.8), (3.9), we obtain

$$\alpha + \beta = \frac{\phi}{l}, \qquad \alpha - \beta = \frac{\psi}{k}. \tag{3.12}$$

Hence, the diagonal curves ξ = constant and η = constant represent, respectively, the equipotentials and stream lines.

With the aid of our previous results and known properties of Tschebyscheff nets, we obtain some additional results. *The general representation of the stream lines in the supersonic range for the Kármán-Tsien relation depends upon two real arbitrary functions. If one stream line coincides with the x-axis, these two functions are equal except for*

[4] R. von Mises, *Bemerkung zur Formulierung des mathematischen Problems der Plastizitätstheorie*, Zeitschr. für angew. Math. u. Mech., 5, 147–149 (1925).

a constant. Further, the velocity and density depend only upon the angle between the characteristics and the Mach number of the flow.

For a Tschebyscheff net, it is well known[2] that (3.5) may be written in the form

$$ds^2 = (d\alpha)^2 + 2\cos\omega d\alpha d\beta + (d\beta)^2, \tag{3.13}$$

where ω is the angle between the two families of characteristics of the net at any point. Further, it is known that ω may be expressed in terms of two arbitrary functions $F(\alpha)$ and $G(\beta)$ by the relation

$$\omega = F(\alpha) + G(\beta). \tag{3.14}$$

Finally, the general representation of the net is given by

$$x = \int^{\alpha} \cos F(t)dt + \int^{\beta} \cos G(t)dt, \tag{3.15}$$

$$y = \int^{\alpha} \sin F(t)dt - \int^{\beta} \sin G(t)dt. \tag{3.16}$$

Introducing the parameters along the equipotentials and stream lines from (3.11), we find that the above equations become

$$ds^2 = 4\cos^2\frac{\omega}{2}(d\xi)^2 + 4\sin^2\frac{\omega}{2}(d\eta)^2, \tag{3.17}$$

$$\omega = F(\xi + \eta) + G(\xi - \eta), \tag{3.18}$$

$$x = \int^{\xi+\eta} \cos F(t)dt + \int^{\xi-\eta} \cos G(t)dt, \tag{3.19}$$

$$y = \int^{\xi+\eta} \sin F(t)dt - \int^{\xi-\eta} \sin G(t)dt. \tag{3.20}$$

Another relation of the form (3.18) can be obtained by introducing the angle $\theta(\xi, \eta)$ which the tangents to the stream lines (η = constant) form with the x-axis. Let the equations of a stream line be

$$x = x(s), \qquad y = y(s), \qquad \eta = \text{constant}, \tag{3.21}$$

where s is the arc length parameter along the stream line. From (3.19), we find by differentiation

$$\frac{dx}{ds} = [\cos F(\xi + \eta) + \cos G(\xi - \eta)]\frac{d\xi}{ds}\cdot \tag{3.22}$$

By use of the well known addition formulas of trigonometry, (3.22) becomes

$$\frac{dx}{ds} = 2\cos\left[\frac{F(\xi + \eta) + G(\xi - \eta)}{2}\right]\cos\left[\frac{F(\xi + \eta) - G(\xi - \eta)}{2}\right]\frac{d\xi}{ds}\cdot \tag{3.23}$$

From (3.17), we find that along η = constant

$$\frac{d\xi}{ds} = \frac{1}{2}\sec\left[\frac{F(\xi + \eta) + G(\xi - \eta)}{2}\right]. \tag{3.24}$$

Substituting (3.24) into the right-hand side of (3.23), we obtain

$$\frac{dx}{ds} = \cos\left[\frac{F(\xi + \eta) - G(\xi - \eta)}{2}\right]. \tag{3.25}$$

From (3.25), we find that except for a constant

$$2\theta = F(\xi + \eta) - G(\xi - \eta). \tag{3.26}$$

By use of (3.14), (3.15), (3.16), and (3.26), the magnitude of the velocity w and the density ρ may be shown to be expressible solely in terms of ω and the Mach number of the flow. Thus, from (3.2), (3.12)

$$u = l\left(\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial x}\right), \qquad v = l\left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial y}\right). \tag{3.27}$$

Hence, by interchanging the independent and dependent variables, it follows that

$$u = \frac{l}{D}\left(\frac{\partial y}{\partial\beta} - \frac{\partial y}{\partial\alpha}\right), \qquad v = \frac{l}{D}\left(\frac{\partial x}{\partial\alpha} - \frac{\partial x}{\partial\beta}\right), \tag{3.28}$$

where

$$D = \frac{\partial x}{\partial\alpha}\frac{\partial y}{\partial\beta} - \frac{\partial x}{\partial\beta}\frac{\partial y}{\partial\alpha}. \tag{3.29}$$

Computing the partial derivatives by use of (3.15), (3.16) and simplifying by use of (3.14), (3.26), we obtain

$$u = 2l\sin\tfrac{1}{2}\omega\cos\theta/\sin\omega, \qquad v = 2l\sin\tfrac{1}{2}\omega\sin\theta/\sin\omega. \tag{3.30}$$

Hence, for the magnitude w of the velocity, we find

$$w = \frac{2l\sin\frac{1}{2}\omega}{\sin\omega}. \tag{3.31}$$

From (2.6), we see that l^2 is equal to $w_1^2 - a_1^2$. Making this substitution in (3.31) and dividing the resulting equation by w_1, we obtain

$$\frac{w}{w_1} = \frac{2\sqrt{1 - (a_1/w_1)^2}\sin\frac{1}{2}\omega}{\sin\omega}. \tag{3.32}$$

In Fig. 1, these curves are plotted for the following values of the Mach number, $w_1/a_1 = 1.5$, 2.0, 2.5, 3.0, 4.0. Note, by the discussion following equation (2.10), as w_1/a_1 varies from 1 to ∞, the permissible values of the upper bound of w/w_1 varies from 1.7 to 1. In each case, the upper bounds are to be determined by use of (2.10). The dotted lines in Fig. 1 denote these upper bounds. Other useful results may be obtained by combining (3.31) with (2.6) and (2.3). Thus, dividing the equation $w^2 - a^2 = l^2$ by w^2 and inserting the value of w as determined from (3.31) into the right-hand member of the resulting equation, we find after a few trigonometric substitutions

$$\frac{w}{a} = \frac{1}{\sin\frac{1}{2}\omega}. \tag{3.33}$$

This equation is of value in determining the lower limit of the ratio w/a, namely, $\sqrt{2}$, for $\omega=\pi/2$. Again, inserting the value of w as determined from (3.31) into the left-hand side of (2.3), dividing the resulting equation by a_1^2, and replacing the term l^2/a_1^2 by w_1^2/a_1^2-1 (see 2.6), we obtain

$$\frac{\rho}{\rho_1}=\frac{\sin\omega}{2\sin^2\frac{1}{2}\omega\sqrt{(w_1/a_1)^2-1}}\cdot \tag{3.34}$$

These curves are plotted in Fig. 2 for the values of the Mach number as indicated above.

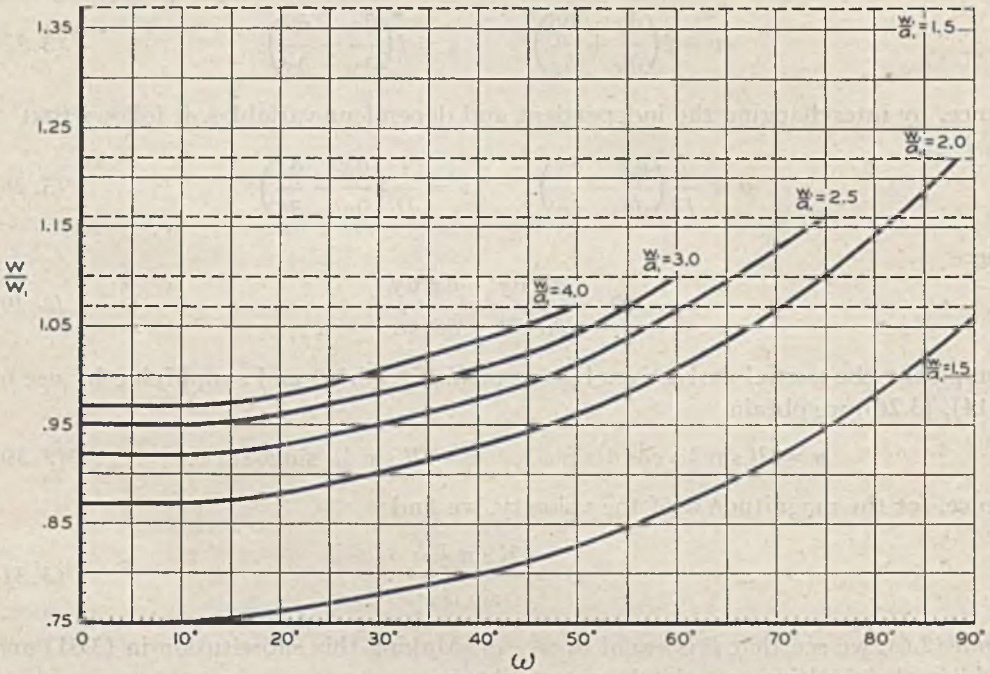

FIG. 1.

An important case in practice[5] (the jet problem) is that for which one stream line is a straight line. In this case, if we assume that the stream line coincides with the x-axis and is $\eta=0$, (3.26) furnishes the result

$$F(\xi) = G(\xi). \tag{3.35}$$

Under properly given Dirichlet data, the function $F(\xi)$ can be determined and the representation of all stream lines can be obtained from (3.19), (3.20). Thus, if $\theta(\xi, c)$ is known along some known stream line $\eta=c$, then the equations (3.26) and (3.35) furnish the result

$$2\theta(\xi, c) = F(\xi+c) - F(\xi-c). \tag{3.36}$$

The equation (3.36) can be solved for $F(\xi)$ by use of the theory of difference equations.

[5] J. Ackeret, *Gasdynamik*, Handbuch der Physik, vol. 7, pp. 318–322.

Unfortunately, $\theta(\xi, c)$ is unknown; $\theta(s)$, where s is the arc length parameter, is known. Hence, one must solve problems by an indirect method. That is, one must introduce a function $F(\xi)$ and then determine the corresponding stream lines.

In calculating the stream lines for particular functions $F(\xi)$, the following analytical-geometrical scheme appears to be the most satisfactory. First, obtain two curves

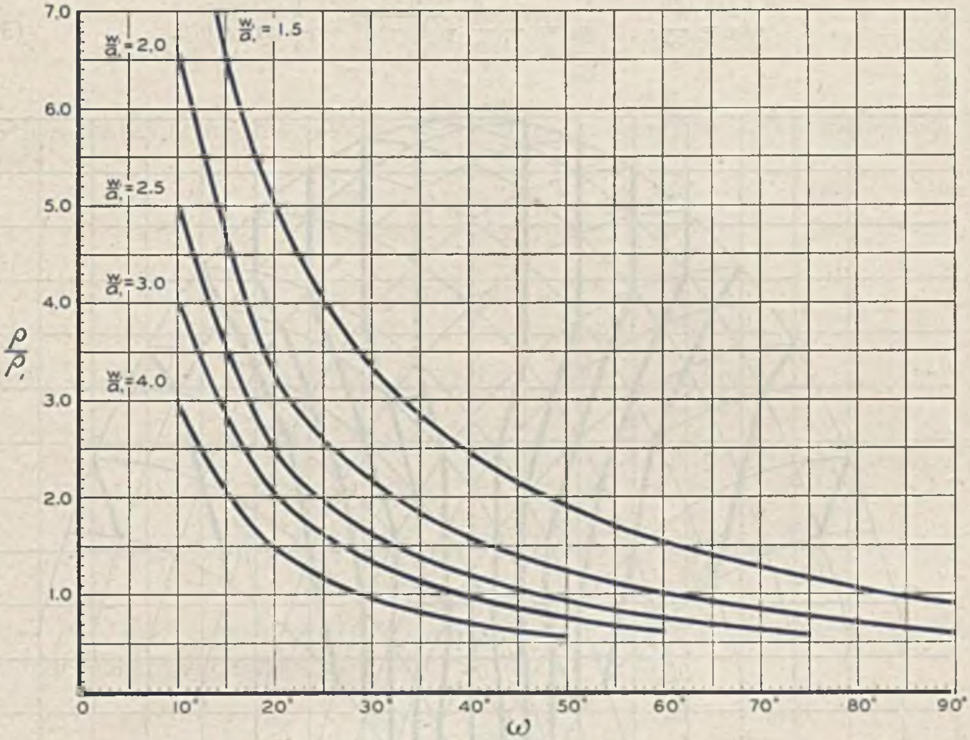

FIG. 2.

of the generating Tschebyscheff net (one of each family) by use of equations (3.15), (3.16). That is, determine the curves

$$x_1 = \int^{\alpha} \cos F(t)dt, \qquad y_1 = \int^{\alpha} \sin F(t)dt, \tag{3.37}$$

$$x_2 = \int^{\beta} \cos F(t)dt, \qquad y_2 = -\int^{\beta} \sin F(t)dt. \tag{3.38}$$

To obtain the initial point of each curve in the x, y plane, we compute one set of values of (x, y) by use of (3.15), (3.16), for some set of values of (α, β) such as $\alpha = 0$, $\beta = 0$. By translating the curves along each other, the complete Tschebyscheff net may be obtained. However, the translation must furnish curves of the families which correspond to equi-distant values of α and β in order that the diagonal curves be stream lines. In view of (3.15), (3.16), this means that the abscissas of the initial points of two corresponding curves must be equal.

Finally, as an example of this method, let us consider the case $F(t) = \text{arc} \cos t$. From (3.37), (3.38), it follows that

$$x_1 = \frac{\alpha^2}{2}, \qquad y_1 = \frac{\alpha}{2}\sqrt{1-\alpha^2} + \frac{1}{2}\,\text{arc}\sin\alpha, \tag{3.39}$$

$$x_2 = \frac{\beta^2}{2}, \qquad y_2 = -\frac{\beta}{2}\sqrt{1-\beta^2} - \frac{1}{2}\,\text{arc}\sin\beta. \tag{3.40}$$

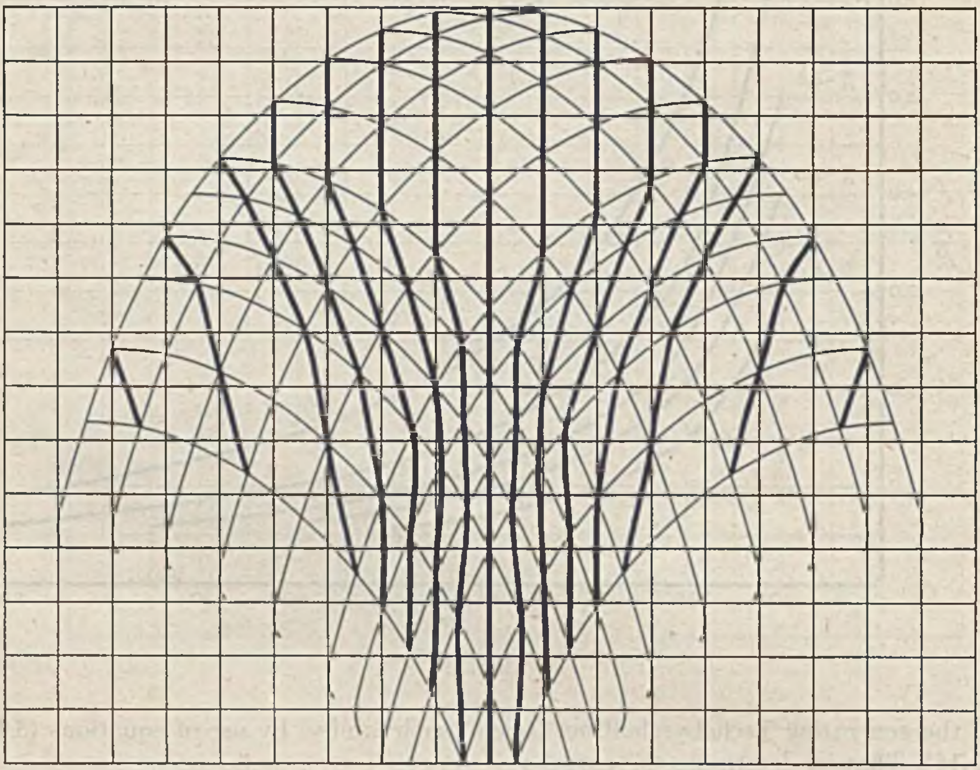

FIG. 3.

The Tschebyscheff net and the resulting stream lines, obtained by the procedure outlined in the preceding paragraph, are illustrated in Fig. 3. By use of a protractor and the graphs of Figs. 1 and 2, the values of w/w_1 and ρ/ρ_1 can be immediately determined at each point of the plane.

In concluding, it should be pointed out that it would be highly desirable to obtain a mechanical method for constructing the Tschebyscheff net when two stream lines are known. This would furnish a direct solution to the problem of the uniform flow of a supersonic jet. It appears that a more thorough understanding of the relation between Tschebyscheff nets and their diagonal curves is needed.

APPENDIX

First, we shall give an analytic derivation of relations (3.6), (3.7). Since the projections of the velocity vector on the normal to the characteristics have the magnitude a, it follows that

$$aA = u\frac{\partial y}{\partial \alpha} - v\frac{\partial x}{\partial \alpha}, \tag{A.1}$$

$$-aB = u\frac{\partial y}{\partial \beta} - v\frac{\partial x}{\partial \beta}. \tag{A.2}$$

Further, by projecting the velocity vector on the tangents to the characteristics, we obtain

$$\sqrt{u^2 - v^2 - a^2}\,A = u\frac{\partial x}{\partial \alpha} + v\frac{\partial y}{\partial \alpha}, \tag{A.3}$$

$$\sqrt{u^2 + v^2 - a^2}\,B = u\frac{\partial x}{\partial \beta} + v\frac{\partial y}{\partial \beta}. \tag{A.4}$$

Solving relations (A.1), (A.2) and (A.3), (A.4) for u, v, we find

$$u = -\frac{a}{D}\left(B\frac{\partial x}{\partial \alpha} + A\frac{\partial x}{\partial \beta}\right) = \frac{\sqrt{u^2 + v^2 - a^2}}{D}\left(A\frac{\partial y}{\partial \beta} - B\frac{\partial y}{\partial \alpha}\right), \tag{A.5}$$

$$v = -\frac{a}{D}\left(B\frac{\partial y}{\partial \alpha} + A\frac{\partial y}{\partial \beta}\right) = \frac{\sqrt{u^2 + v^2 - a^2}}{D}\left(B\frac{\partial x}{\partial \alpha} - A\frac{\partial x}{\partial \beta}\right), \tag{A.6}$$

where D is the Jacobian of the transformation (3.4). By interchanging dependent and independent variables, (A.5), (A.6) become

$$u = a\left(A\frac{\partial \alpha}{\partial y} - B\frac{\partial \beta}{\partial y}\right) = \sqrt{u^2 + v^2 - a^2}\left(A\frac{\partial \alpha}{\partial x} + B\frac{\partial \beta}{\partial x}\right), \tag{A.7}$$

$$v = -a\left(A\frac{\partial \alpha}{\partial x} - B\frac{\partial \beta}{\partial x}\right) = \sqrt{u^2 + v^2 - a^2}\left(A\frac{\partial \alpha}{\partial y} + B\frac{\partial \beta}{\partial y}\right). \tag{A.8}$$

From (A.7), (A.8) and (3.2), we find (3.6), (3.7).

Next, we shall show that *the Kármán-Tsien relation* (2.2) *is valid, if the net of characteristics form a Tschebyscheff net.*

Since the net of characteristics is a Tschebyscheff net, we may consider the metric coefficients A and B as having the value unity. Hence, from (3.6), (3.7) and the chain rule for differentiation, we obtain

$$\frac{\partial \psi}{\partial x} = \rho a\frac{\partial(\alpha - \beta)}{\partial x}, \qquad \frac{\partial \psi}{\partial y} = \rho a\frac{\partial(\alpha - \beta)}{\partial y}, \tag{A.9}$$

$$\frac{\partial \phi}{\partial x} = \sqrt{u^2 + v^2 - a^2}\,\frac{\partial(\alpha + \beta)}{\partial x}, \qquad \frac{\partial \phi}{\partial y} = \sqrt{u^2 + v^2 - a^2}\,\frac{\partial(\alpha + \beta)}{\partial y}. \tag{A.10}$$

With the aid of (3.1), (3.2), we may write the integrability conditions of (A.9), (A.10) in the form

$$\frac{\partial}{\partial s}\rho a = 0, \qquad \text{(A.11)} \qquad\qquad \frac{\partial}{\partial n}\sqrt{u^2 + v^2 - a^2} = 0, \quad \text{(A.12)}$$

where $\partial/\partial s$ represents differentiation along a stream line and $\partial/\partial n$ represents differential along an equipotential. If we assume that a relation exists between p and ρ, then by use of the definition of a^2 (defined as $dp/d\rho$), we find from (11)

$$\rho \frac{d^2 p}{d\rho^2} + 2\frac{dp}{d\rho} = 0. \tag{A.13}$$

Further, from the generalized Bernoulli relation

$$u^2 + v^2 + 2P(\rho) = \text{constant}, \qquad \frac{dP}{d\rho} = \frac{1}{\rho}\frac{dp}{d\rho}, \tag{A.14}$$

we find that (A.12) reduces to (A.13). Integrating (A.13), we obtain (2.2).

The author wishes to thank Professor W. Prager and Dr. L. Bers of Brown University for valuable criticisms and suggestions. Further, he is indebted to the Research Institute of the University of Texas for a grant which permitted the construction of the diagrams.

ON THE STABILITY OF TWO-DIMENSIONAL PARALLEL FLOWS

PART I.—GENERAL THEORY*

BY

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1. Introduction. The study of the stability of laminar motion and its transition to turbulence dates back to the time of Helmholtz and Reynolds [46], and had already attracted great attention at the end of the last century.** Since that time, the subject has not only become a major problem for workers in hydrodynamics, but has also attracted the attention of people like Lord Rayleigh [43–45], Lord Kelvin [20–21], Lorentz [29], Sommerfeld [58], and Heisenberg [14], whose chief interest is not limited to the study of mechanics. Although numerous contributions have since been made, the subject has remained one of considerable dispute, as can be seen from the two general lectures given by Taylor [70] and by Synge [63] as late as 1938. Still more recently, there appeared the work of Görtler [8, 9] and of Thomas [71].

Most of the work on the stability of laminar motions has the following final aims.

1) The first aim is to determine whether a given flow (or a given class of flows) is ultimately unstable for sufficiently large Reynolds numbers. For this purpose, it is desirable to obtain some simple general criterion which will give a rapid classification of velocity profiles according to their stability.

2) The second purpose is to determine the minimum critical Reynolds number at which instability begins. It is often easier to find sufficient conditions for stability than to find the condition for passage from stability to instability.

3) Finally, we want to understand the physical mechanism underlying the phenomena by giving theoretical interpretations and experimental confirmations of the results obtained from mathematical analysis.

Although numerous attempts have been made in these directions, especially for the apparently simplest cases of parallel flows in two dimensions, our knowledge is still very meagre. The classical case of plane Pouiseuilla motion has remained an unsettled problem,† and no satisfactory general results have been reached regarding the stability of a real fluid. The best-known general criterion is that of Rayleigh (1880) and Tollmien [74], classifying profiles according to the occurrence of a flex†† with respect to the stability of a fluid at infinite Reynolds numbers. However, the significance of their results has been too much exaggerated and often misunderstood, and no physical interpretation has ever been given. The present work offers such an interpretation, but also shows that the results can only give some indication regarding

* Received March 3, 1945. An abstract of this paper has already appeared under the same title [27].

** In 1888, the problem was proposed by Rayleigh and Stokes as the subject for the Adams Prize Essay. Cf. p. 321 of Ref. [21], and also the footnote on p. 267 of Ref. [44].

† Cf. Synge's lecture [63].

†† Following Professors Frank Morley and H. Bateman, we shall use the word "flex" for "point of inflection."

the instability of a real (viscous) fluid. This will be discussed in more detail below.

The chief aim of the present work is to try to answer the three questions mentioned above for two-dimensional parallel flows. This work is divided into three parts. Part I (the present paper) deals with the general mathematical theory, with particular emphasis on attempting to clarify the mathematical difficulties involved in the solution of the equation of stability. Part II deals with the stability problem in an inviscid fluid (infinite Reynolds numbers). Part III deals with the problem in a real fluid. The following results have been obtained.

1) It is shown that all velocity distributions of the symmetrical type and of the boundary-layer type are unstable for sufficiently large (but finite) values of the Reynolds number (Part III). The plane Poiseuille motion is included as a special case.

2) A simple approximate method is obtained by which one can calculate the minimum Reynolds number marking the beginning of instability with very little numerical labor (Part III).

3) The tendency toward instability of a profile with a point of inflection is interpreted by considering the distribution of vorticity (Part II). The effect of viscosity is considered as diffusing the disturbance from the "critical layer" inside the fluid and from the solid boundary. A very simple quantity is thereby derived which serves as a measure of the effect of viscosity (Part III). This can also be easily connected with the general mathematical theory.

As numerical examples, we have worked out the curve of neutral stability for the Poiseuille case and the Blasius case. Comparisons with existing results are discussed (Part III). The relation between instability and transition to turbulence is also discussed in Part III of this work.

Since some of the present results differ markedly from customary beliefs, it is necessary to trace the history of the existing lines of thought in order to give proper recognition to earlier ideas and results used in the present work, and to analyze all the results in disagreement with present conclusions. This requires the repetition of some known results when they fall into the present line of treatment. The review of literature is not intended to be exhaustive; only the necessary references are cited. A more complete bibliography up to 1932 has been given by Bateman [2].

2. Historical survey of existing theories. There seem to be two schools of thought in regard to the cause of transition from steady to turbulent conditions. One school contends that transition is due to a definite instability of the flow, i.e., to a condition in which infinitesimal disturbances grow exponentially. The second school regards the motion in most cases as definitely stable for infinitesimal disturbances but liable to be made turbulent by suitable disturbances of finite magnitude or by a large enough pressure gradient. Both schools, however, generally agree that the fluid can be considered as incompressible and that its motion is governed by the Navier-Stokes equations of motion. Since the agreement between theory and experiment has not been very satisfactory, it has also been proposed that the cause of transition must be traced back to the effect of compressibility or to the possible failure of the Navier-Stokes equations. The present work tends to confirm the simplest point of view that the motion in most cases is definitely unstable for infinitesimal disturbances governed by the Navier-Stokes equations for an incompressible fluid.

The theory of finite disturbances dates back to Reynolds [46] and Kelvin [21].

It was developed by Schiller, Taylor and others.* Mathematical investigations of such finite disturbances are mainly based on considerations of energy or of the square of the vorticity of the disturbance, because the solution of the non-linear equations satisfied by the disturbance is extremely difficult. At the end of Part III we shall briefly discuss this line of thought together with the results of the present paper. For more details, the reader is referred to the lecture of Taylor [70] and the papers of Synge [62] and Thomas [71].

For small disturbances, positive definite integrals of the energy and vorticity of the disturbance have been extensively used. These considerations have been discussed by Orr [37], Lorentz [29], von Kármán [18], Synge [63, 64] and others. For excellent accounts of this phase of the theory, the reader is referred to the works of Noether [35], von Kármán [18], Prandtl [42], and Synge [64]. Additional references are cited at the end of this paper. As is now well-known, this method can only give sufficient conditions for stability. Also, since all disturbances are usually allowed, including those which do not satisfy the hydrodynamic equations of motion, a larger viscous decay is required to insure stability than when these disturbances are excluded. Consequently, the limit of stability is always found to be much lower than that indicated by experiment. However, from these considerations, Synge [63] has arrived at a very convenient form of a sufficient condition for the stability of two-dimensional parallel flows with respect to two-dimensional disturbances. This will be found very useful for the discussions in Part III.

To get more concrete results, we have to solve the linearized equations satisfied by the disturbance. The most successful case appeared to be Taylor's treatment of Couette flow [67] between concentric cylinders. His work was verified by the experiments carried out by himself [67, 69] and by others [28]. A rigorous mathematical investigation in this connection was made by Faxon [4]. In fact, it is now known that his analysis is a typical case of the stability of a fluid motion where the centrifugal force plays a dominant part. Such cases were first considered by Lord Rayleigh [45], who gave a condition for the stability of an inviscid fluid. Mathematical proof of a sufficient condition of stability of Couette flow was recently given by Synge [65]. Extension of Taylor's work to the boundary layer over a curved wall was carried out by Görtler [8, 9], who used numerical methods successfully.

While the investigation of curved flows was uneventful, the investigation of axially symmetrical flows was not extensive. The Poiseuille flow in a circular pipe was studied by Sexl [55] with a conclusion of stability. Prandtl [42] gave some discussions of the possible cause of instability in his article in the book "Aerodynamic Theory," edited by Durand.

The most extensive discussion of hydrodynamic stability seems to be the treatment of parallel flows by attempting to solve the eigen-value problem associated with the linearized equations governing the disturbance. This line of development can be easily traced in the work of Helmholtz, Lord Rayleigh [43, 44], Orr [37], Sommerfeld [58], von Mises [31, 32], Hopf [16], Prandtl [41], Tietjens [72], Heisenberg [14], Tollmien [73–75], and Schlichting [52–54]. Other contributions are those of Noether [36], Solberg [57], Southwell [59], Squire [60], Goldstein [6], Pekeris [39, 40], Synge [61–65] and Langer [25].

* See Taylor's lecture [70] for references to the works on finite disturbances.

For convenience, the theory deals with two-dimensional wavy disturbances propagated along the direction of the main flow. Squire [60] has shown that three-dimensional wavy disturbances are more stable than two-dimensional ones. However, Prandtl still mentions the possibility of greater instability of three-dimensional disturbances in his article [42] appearing after Squire's paper.

The first study of two-dimensional hydrodynamic stability seems to have been made by Helmholtz. He proved the instability of wavy disturbances over the surface of discontinuity of two parallel streams of different velocities. Later, Rayleigh [43] realized that Helmholtz's approximation was not good enough to bring out the main features of a flow with continuous velocity distributions. He therefore made an improved approximation consisting of several linear profiles joined up continuously. The vorticity distribution then has constant values in several layers, but has a discontinuity in passing from one layer to another. Investigations with continuous vorticity distributions were also made. Rayleigh's work was mainly concerned with an inviscid fluid. Two main results were obtained. The first is that instability (in an inviscid fluid) can only occur with velocity distributions having a point of inflection. It is usually believed that Lord Rayleigh has also proved that damped disturbances can also only occur with such profiles. The possibility of a disturbance for a profile without a flex then becomes a paradox [5]. Actually, Rayleigh's proof does not lead to such a conclusion. This point will be more fully discussed in §5 and Part II. Rayleigh's second result is obtained from the analysis of broken linear profiles; it substantiates the first result by demonstrating definite instability of broken linear velocity distributions of the type shown in Fig. 1(a), and only stability in the other cases. Rayleigh [43] supported his result by obtaining the condition determining stability in the approximate form

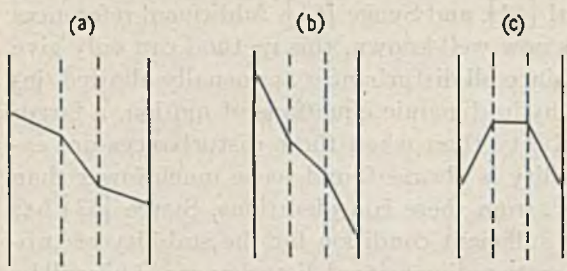

FIG. 1. Broken profiles investigated by Lord Rayleigh. Case (a) may be unstable; the other two cases are stable.

$$\int_{y_1}^{y_2} (w - c)^{-2} dy = 0, \tag{2.1}$$

where $w(y)$ is the velocity distribution, y_1 and y_2 are the coordinates of the solid boundaries, and c is a constant the real part of which represents the wave velocity and the imaginary part of which gives damping or amplification.

Meanwhile, the exact analysis of linear velocity distributions including the effect of viscosity was given by von Mises [31, 32], and Hopf [16] and was also studied by Rayleigh [44]. The results indicate only stability. Prandtl and Tietjens [72] applied Rayleigh's method of approximation to the stability of the boundary layer, taking account of the effect of viscosity. In such an approximation, the inner friction layer mentioned above (§1) for continuous vorticity distributions is left out. The result of Tietjens did not give a minimum critical Reynolds number.

It was Heisenberg [14] who first successfully studied the stability of a variable continuous vorticity distribution. As a particular example, he demonstrated that the plane Poiseuille flow was unstable for sufficiently large Reynolds numbers. Also, using the same equation (2.1) with which Rayleigh supported his approximation with linear profiles, Heisenberg pointed out the fallacy in Rayleigh's method. The essential point is that the corners in the velocity profile introduce extraneous roots of the above equation for c. Consequently, the results of this type of analysis depend upon the manner in which the velocity distribution is approximated.

Heisenberg's numerical computation was, however, incomplete and very rough, and his theory was not generally accepted. Better known are the results of Tollmien and Schlichting. They studied the cases of Blasius [73] and plane Couette flow [48], using Heisenberg's theory essentially. The former case was pursued very much in detail. For the latter case, Schlichting followed the idea of Prandtl, asserting that the instability may be attributed to the initial unsteady distribution prior to the formation of the linear profile. Indeed, the same kind of idea was also suggested by Prandtl to account for the instability of Poiseuille flow by ascribing it to the entrance section where the profile is not yet parabolic [41]. This problem will be discussed in some detail later (§14, Part III).

For an inviscid fluid, Tollmien has also proved the instability of boundary-layer and symmetrical profiles with a point of inflection [74]. For a viscous fluid, the present investigation shows that instability depends upon the general type of these profiles rather than on the appearance of the point of inflection. The inner friction layer plays a dominant role in determining the instability. Attempts to interpret this point physically are given by Prandtl [42] and in the present paper.

3. General formulation of the problem. We shall now formulate the problem of the stability of two-dimensional parallel flows mathematically. In the first place, we note that if the steady motion is strictly two-dimensional and parallel, the velocity distribution must be either linear or parabolic (if body forces are absent). We then have one of the following: 1) the plane Couette flows; 2) the plane Poiseuille flow; 3) a combination of these two flows. The problem would then be very restricted.

However, there are a large number of cases where the flow is *essentially* parallel to one direction. These are the cases where the boundary-layer consideration is permissible. The following are important special cases belonging to this class: 4) inlet flow between parallel walls, flow in a slightly convergent or divergent channel; 5) flow along a flat plate; 6) wake behind a cylindrical body, jet from a narrow slit. Whether these flows can be properly considered as belonging to the same class as the above three is a question of some controversy. Taylor has criticized Tollmien's work with the boundary layer on this ground [70]. In the Appendix to Part III of this work, we shall try to demonstrate that this treatment is generally permissible, but that the interpretation of the results must be taken up with care. A discussion of Tollmien's work will also be found there.

In considering the stability of the main flow, we superpose upon it a hydrodynamically possible small disturbance, and consider its behavior. The disturbance is small in the sense that the inertia forces corresponding to the disturbance alone are negligible and that its behavior is unaltered when its amplitude is (say) doubled or halved. It is then simplest to consider separate harmonic components with respect to time, which may be damped, neutral, or self-excited. By considering disturbances which are also spacially periodic both in the direction of flow and in the direction perpendicular to the plane of symmetry of the main motion, Squire [60] was able to show that two-dimensional disturbances are less stable than three-dimensional disturbances. Hence, important features of the stability problem can be obtained by considering two-dimensional disturbances alone. This is an essential difference between the stability of a parallel flow and of a curved flow. In the latter case, three-dimensional disturbances are of utmost importance.

The consideration of periodic disturbances alone is again a question of some controversy. Justification has been attempted and objection has been raised. We shall see later that at least the difficulties raised are chiefly caused by a misinterpretation of the mathematical results.

Admitting that we can consider two-dimensional disturbances alone, we have a much simplified physical picture at hand. If the effect of viscosity is negligible, we have the well-known fact of conservation of vorticity for two-dimensional motions. Actually, the stability problem is found to depend both on the inertia forces and on the viscous forces. However, the effect of viscosity is also well-known to be one of diffusion of vorticity. Thus, important results can be expected from considerations of vorticity transfer.

Let us now proceed with the mathematical formulation of the problem. We shall give a complete derivation of the stability equations so that we can see how to settle the disputes about the approximations in considering velocity distributions of the boundary-layer type.

Admitting Squire's work as a proper indication that only two-dimensional disturbances need be considered, we may conveniently consider the equation of vorticity

$$\Delta\psi_t + \psi_y\Delta\psi_x - \psi_x\Delta\psi_y = \nu\Delta\Delta\psi, \tag{3.1}$$

with the velocity components

$$u = \psi_y = \frac{\partial\psi}{\partial y}, \qquad v = -\psi_x = -\frac{\partial\psi}{\partial x}, \tag{3.2}$$

and the vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -(\psi_{xx} + \psi_{yy}) = -\Delta\psi. \tag{3.3}$$

As usual, ν is the kinematical viscosity. We may add that Squire's original proof was intended for flow *bounded* between two parallel walls. There is no difficulty in seeing that the proof holds also for a fluid *extending to infinity*,* because the boundary conditions for the disturbance are essentially the same.

Let us put

$$\psi = \Psi(x, y) + \psi'(x, y, t), \tag{3.4}$$

where $\Psi(x, y)$ represents the steady main flow and $\psi'(x, y, t)$ represents the disturbance. Main flows which vary but slowly with time can also be treated this way, but we shall restrict ourselves to steady flows in order to fix our ideas.

If we substitute (3.4) into (3.1) the terms corresponding to the main flow cancel out. If we then drop the terms quadratic in $\psi'(x, y, t)$ and its derivatives, we have the equation

$$\Delta\psi_t' + \Psi_y\Delta\psi_x' - \Psi_x\Delta\psi_y' + \psi_y'\Delta\Psi_x - \psi_x'\Delta\Psi_y = \nu\Delta\Delta\psi'. \tag{3.5}$$

* Cf. Ref. [15].

We shall now assume the flow to be essentially parallel to the x-axis. Using the boundary-layer approximation, we should drop the x-derivative of any quantity connected with the main flow compared with its y-derivative. But for the disturbance we would expect ψ_x' and ψ_y' to be of the same order of magnitude. This will be verified *a posteriori* in the specific examples. Further discussions will be found in the Appendix to Part III. With these considerations, (3.5) reduces to

$$\Delta\psi_t' + \Psi_y \Delta\psi_x' - \psi_x' \frac{\partial^3\Psi}{\partial y^3} = \nu\Delta\Delta\psi'. \tag{3.6}$$

Now we shall make an approximation of the same order by taking for $w = \Psi_y$ and $\partial^2 w/\partial y^2 = \partial^3\Psi/\partial y^3$ their *local* values at a given value x_0 of x. Then we may write

$$\Delta\psi_t' + w(y)\Delta\psi_x' - w''(y)\psi_x' = \nu\Delta\Delta\psi'. \tag{3.7}$$

For the boundary conditions, we shall also consider the *local* boundaries. The problem is then essentially simplified, and can be treated similarly to plane Couette and Poiseuille flows. We consider a main flow between two parallel planes $y = y_1$ and $y = y_2$ with a more or less arbitrary distribution of velocity $w(y)$. Then the disturbance $\psi'(x, y, t)$ must be found as a solution of (3.7) satisfying the conditions $u' = v' = 0$ over the boundaries.

The usual way of dealing with the solution of (3.7) subject to given boundary conditions is to consider periodic disturbances. We shall refer all velocities to a characteristic velocity U and all lengths to a characteristic length l, and define the Reynolds number $R = Ul/\nu$. The two-dimensional periodic disturbance of a field of flow in which the main flow is $w(y)$ may be represented by the stream function $\psi' = \phi(y)e^{i\alpha(x-ct)}$, and the linearized differential equation for $\phi(y)$ is

$$(w - c)(\phi'' - \alpha^2\phi) - w''\phi = -\frac{i}{\alpha R}(\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi), \tag{3.8}$$

as can be easily obtained from (3.7). We shall take α always real and positive, while c may be complex; thus,

$$c = c_r + ic_i. \tag{3.9}$$

To fix our ideas about the boundary conditions, let us consider a flow between the planes $y = y_1$ and $y = y_2$. The equation (3.8) is then to be solved under the boundary conditions

$$\phi(y_1) = 0, \qquad \phi(y_2) = 0, \qquad \phi'(y_1) = 0, \qquad \phi'(y_2) = 0. \tag{3.10}$$

Let us now forget about the physical problem and consider the differential equation (3.8) as a linear differential equation of the fourth order in the *complex* y-plane. To be sure, the function $w(y)$ is defined only for real values of y between y_1 and y_2. We can of course, consider it as defined for other values of y by analytical continuation. We shall assume that the function thus defined is holomorphic in every finite region with which we shall be concerned. The equation (3.8) then has every point in the region under consideration as a regular point, and its coefficients are also *entire* functions of the parameters c, α, and αR (regarded as complex variables). By a well-known theorem in the theory of differential equations, there exists a fundamental system of four solutions of (3.8) which are analytic functions of the variable y and of the parameters c, α, and αR, being in fact *entire* functions of the parameters. The consequences of these simple general analytical considerations appear to have escaped serious attention from earlier investigators. In §§4, 5 of this paper, we shall find this type of consideration very important in settling the controversies about the question of convergence of the series used in the actual solution of equations (3.8) and (3.14).

Let us denote the above-mentioned system of solutions of (3.8) by $\phi_1(y)$, $\phi_2(y)$, $\phi_3(y)$, $\phi_4(y)$, the dependence upon the parameters c, α, αR being understood. The conditions (3.10) then give rise to the secular equation

$$F(c, \alpha, \alpha R) = \begin{vmatrix} \phi_1(y_1) & \phi_2(y_1) & \phi_3(y_1) & \phi_4(y_1) \\ \phi_1(y_2) & \phi_2(y_2) & \phi_3(y_2) & \phi_4(y_2) \\ \phi_1'(y_1) & \phi_2'(y_1) & \phi_3'(y_1) & \phi_4'(y_1) \\ \phi_1'(y_2) & \phi_2'(y_2) & \phi_3'(y_2) & \phi_4'(y_2) \end{vmatrix} = 0. \tag{3.11}$$

Since the function $F(c, \alpha, \alpha R)$ is an entire function of the variables c, α, αR, we may solve for c, obtaining

$$c = c(\alpha, R). \tag{3.12}$$

There may be several branches of the solution, or there may be none as in the case when $F(c, \alpha, \alpha R)$ is (say) $\exp(\alpha Rc)$. In general, we would expect the solution to be unique, or we may consider only one branch of the solution.

Since α and R are later taken to be real and positive, it is convenient to separate (3.12) into its real and imaginary parts. Thus,

$$c_r = c_r(\alpha, R), \qquad c_i = c_i(\alpha, R). \tag{3.13}$$

It is customary to plot curves of constant c_i or αc_i in the αR-plane. The curve $c_i = 0$ gives the limit of stability.

We are particularly interested in the case when the Reynolds number is very large. The study of this case is complicated by the fact that the functions ϕ_1, ϕ_2, ϕ_3, ϕ_4 involved have essential singularities at the infinite point of the αR-plane. From the differential equation (3.8) itself, we see that when $\alpha R \to \infty$, we have the equation

$$(w - c)(\phi'' - \alpha^2\phi) - w''\phi = 0, \tag{3.14}$$

which is only of the second order. Thus, two solutions of (3.8) are lost. From detailed mathematical investigations, we shall find later that two linearly independent solutions of (3.8), say ϕ_1 and ϕ_2, will satisfy (3.14) in the limit of infinite αR, except along certain straight lines through the point $w = c$. The other two linearly independent solutions ϕ_3 and ϕ_4 are highly oscillating for large αR and would therefore disappear in the limit of infinite αR. Furthermore, we shall see that ϕ_3 and ϕ_4 can be so chosen that if $\phi_3(y_1) \gg \phi_4(y_1)$, then $\phi_3(y_2) \ll \phi_4(y_2)$, with corresponding relations for their derivatives. It then appears plausible that the limiting form of (3.11) for infinite αR is

$$\begin{vmatrix} \phi_1(y_1) & \phi_2(y_1) \\ \phi_1(y_2) & \phi_2(y_2) \end{vmatrix} = 0, \tag{3.15}$$

with $\phi_1(y)$, $\phi_2(y)$ satisfying (3.14).

The condition (3.15) states that we are looking for a solution $\phi(y)$ of (3.14) satisfy-

$$\phi(y_1) = 0, \qquad \phi(y_2) = 0, \tag{3.16}$$

with the other two conditions of (3.10) relaxed. Physically, this means that we allow a slipping along the walls $y=y_1$ and $y=y_2$. For very large Reynolds numbers, only a very thin layer of fluid will stick to the solid, and we have naturally an apparent slipping. These points will be taken up again more carefully (§6) after a thorough mathematical investigation of the solutions.

4. Solution of the equation of Orr and Sommerfeld by methods of successive approximation. The stability equation of Orr and Sommerfeld

$$(w - c)(\phi'' - \alpha^2\phi) - w''\phi = -\frac{i}{\alpha R}(\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi) \tag{4.1}$$

has a fundamental system of four solutions, which are analytic functions of y (wherever $w(y)$ is analytic) and which are entire functions of α, c, and αR. In order to obtain useful solutions, it is usual to expand the solutions as power series of a suitable small parameter, say, $(\alpha R)^{-1}$. However, since $(\alpha R)^{-1}$ occurs with the highest derivative in (4.1), the study of such an expansion becomes very complicated. It will be done later.

a) *Solution by convergent series.* An alternative method* is to choose a small parameter ϵ related to $(\alpha R)^{-1}$ and first make a change of variable (y_0 being an arbitrary point so far)

$$y - y_0 = \epsilon\eta, \qquad \phi(y) = \chi(\eta), \tag{4.2}$$

so that (4.1) becomes

$$(w - c)(\chi'' - \alpha^2\epsilon^2\chi) - \epsilon^2 w''\chi = -\frac{i}{\alpha R\epsilon^2}(\chi^{\text{iv}} - 2\alpha^2\epsilon^2\chi'' + \alpha^4\epsilon^4\chi), \tag{4.3}$$

where

$$\left.\begin{aligned} w - c &= (w_0 - c) + w_0'(\epsilon\eta) + \frac{w_0''}{2!}(\epsilon\eta)^2 + \cdots, \\ w'' &= w_0'' + w_0'''(\epsilon\eta) + \frac{w_0^{\text{iv}}}{2!}(\epsilon\eta)^2 + \cdots. \end{aligned}\right\} \tag{4.4}$$

The solution is then obtained in the form

$$\phi(y) = \chi(\eta) = \chi^{(0)}(\eta) + \epsilon\chi^{(1)}(\eta) + \epsilon^2\chi^{(2)}(\eta) + \cdots, \tag{4.5}$$

and the differential equations for the approximations of successive orders can be obtained by substituting (4.4) and (4.5) into (4.3) and equating all the coefficients of the various powers of ϵ to zero.

If we take y_0 to be the point where $w=c$, the proper choice of the parameter ϵ is

$$\epsilon = (\alpha R)^{-1/3}. \tag{4.6}$$

The differential equations for the functions $\chi^{(0)}(\eta)$, $\chi^{(1)}(\eta)$, $\chi^{(2)}(\eta)$, $\cdots$ are as follows:

$$\left.\begin{aligned} &\epsilon^0; \quad w_0'\eta\chi^{(0)''} + i\chi^{(0)\text{iv}} = 0, \\ &\epsilon^n; \quad w_0'\eta\chi^{(n)''} + i\chi^{(n)\text{iv}} = L_{n-1}(\chi), \quad (n \geqq 1), \end{aligned}\right\} \tag{4.7}$$

* This method was first used by Heisenberg, loc. cit. [14], p. 588.

where $L_{n-1}(\chi)$ is a linear combination of $\chi^{(0)}(\eta), \chi^{(1)}(\eta), \cdots, \chi^{(n-1)}(\eta)$ and their derivatives. In particular,

$$L_0(\chi) = w_0''(\chi^{(0)} - \tfrac{1}{2}\eta^2\chi^{(0)''}). \tag{4.8}$$

We note that the homogeneous part is the same for all the differential equations in (4.7). Hence, if we can solve for the first approximation, the rest can all be obtained by *quadratures*. Indeed, the first equation of (4.7) is Stokes' equation* for $\chi^{(0)''}$, and its solution can be readily expressed in terms of Bessel functions of the order 1/3. Thus, for the first equation of (4.7) we have the four particular integrals**

$$\left.\begin{aligned} \chi_1^{(0)} &= \eta, & \chi_3^{(0)} &= \int_{+\infty}^{\eta} d\eta \int_{+\infty}^{\eta} d\eta\, \eta^{1/2} H_{1/3}^{(1)}[\tfrac{2}{3}(i\alpha_0\eta)^{3/2}], \\ \chi_2^{(0)} &= 1, & \chi_4^{(0)} &= \int_{-\infty}^{\eta} d\eta \int_{-\infty}^{\eta} d\eta\, \eta^{1/2} H_{1/3}^{(2)}[\tfrac{2}{3}(i\alpha_0\eta)^{3/2}], \end{aligned}\right| \tag{4.9}$$

where

$$\alpha_0 = (w_0')^{1/3}. \tag{4.10}$$

The higher approximations are given by

$$\chi_i^{(n)} = \frac{\pi}{6}\int^{\eta} d\eta \int^{\eta} d\eta \left\{\chi_4^{(0)''} \int^{\eta} d\eta \chi_3^{(0)''} L_{n-1}(\chi) - \chi_3^{(0)''} \int^{\eta} d\eta \chi_4^{(0)''} L_{n-1}(\chi)\right\}, \tag{4.11}$$

$$(i = 1, 2, 3, 4).$$

These are the explicit formulae for finding the approximations of various orders. In actual calculations, only the initial approximation (4.9) is required. Furthermore, *the series* (4.5) *is convergent provided* ϵ *is restricted so that the series* (4.4) *are convergent.* For then the differential equation (4.3) for $\chi(\eta)$, when normalized, has analytic functions of the parameter ϵ as its coefficients. Hence, a fundamental system of its solutions consists of four analytic functions of ϵ.

It should be mentioned that if y_0 is not taken at the particular point for which $w=c$, the proper parameter to be taken is $(\alpha R)^{-1/2}$ instead of $(\alpha R)^{-1/3}$. In this case, all the approximations can be expressed in terms of elementary transcendental functions. However, it is not found particularly advantageous to do so, because the study of "crossing substitution" (§5) would not be easy. Also, the method is then too much different from those used by earlier investigators to allow an easy comparison of the results.

b) *Solution by asymptotic series.* Although the previous method is theoretically

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* Cf. the exact treatment of (4.1) by Hopf [16] and Rayleigh [44] for the case $w''=0$.

** Note that $\chi_3^{(0)}$ and $\chi_4^{(0)}$ and also $\chi_i^{(n)}$ have *no* branch point at $\eta=0$. The order of the solutions $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ agrees with Tollmien's notation. They are $\{\phi_3, \phi_4, \phi_1, \phi_2\}$ in Heisenberg's notation. Heisenberg gave the solutions ϕ_3 and ϕ_4 in terms of Hankel functions in the form

$$\phi_j = (w - c)\int \frac{1}{\eta} H_{2/3}^{(j)}\left[\frac{2}{3}(i\alpha_0\eta)^{3/2}\right] d\eta, \qquad (j = 1, 2),$$

(p. 289, and Eq. (19a) p. 591). It can be easily verified that these are the same as $\chi_{3,4}^{(0)}$ up to a constant factor $w_0'\epsilon$ and to the proper order of approximation. Note that throughout Heisenberg's paper, i is to be replaced by $-i$ in order to conform to our notation. This can be seen from a comparison of our Eq. (4.1) with his Eq. (7a). The difference arises from a difference of notation in the stream function $\psi'(x, y, t)$.

complete, it is usually more convenient to use asymptotic series for numerical purposes, particularly in dealing with boundary-value problems. Heisenberg has given two asymptotic methods, each of which gives only *two* particular solutions of (4.1). These methods will now be described and investigated mathematically in more detail, because Heisenberg's work has received some criticism in this connection.*

The first of these methods is to develop $\phi(y)$ in powers of $(\alpha R)^{-1}$. We put

$$\phi(y) = \phi^{(0)}(y) + (\alpha R)^{-1}\phi^{(1)}(y) + (\alpha R)^{-2}\phi^{(2)}(y) + \cdots, \tag{4.12}$$

and substitute in (4.1). Comparing corresponding powers of $(\alpha R)^{-1}$, we have the following differential equations

$$\left.\begin{aligned} &(w - c)(\phi^{(0)\prime\prime} - \alpha^2\phi^{(0)}) - w''\phi^{(0)} = 0,\\ &(w - c)(\phi^{(k)\prime\prime} - \alpha^2\phi^{(k)}) - w''\phi^{(k)} = - i[\phi^{(k-1)\mathrm{iv}} - 2\alpha^2\phi^{(k-1)\prime\prime} + \alpha^4\phi^{(k-1)}],\\ &\qquad\qquad (k \geqq 1). \end{aligned}\right\} \tag{4.13}$$

The initial approximation satisfies the inviscid equation and can be solved by developing $\phi^{(0)}$ in powers of α^2. Indeed, two particular integrals of (4.13) are

$$\left.\begin{aligned} \phi_1^{(0)} &= (w - c)[h_0(y) + \alpha^2 h_2(y) + \alpha^4 h_4(y) + \cdots],\\ \phi_2^{(0)} &= (w - c)[k_1(y) + \alpha^2 k_3(y) + \alpha^4 k_5(y) + \cdots], \end{aligned}\right\} \tag{4.14}$$

where

$$\left.\begin{aligned} &h_0(y) = 1, \quad h_{2n+2}(y) = \int_{y_1}^{y} dy(w - c)^{-2}\int_{y_1}^{y} dy(w - c)^2 h_{2n}(y), \quad (n \geqq 0),\\ &k_1(y) = \int_{y_1}^{y} dy(w - c)^{-2}, \quad k_{2n+3}(y) = \int_{y_1}^{y} dy(w - c)^2\int_{y_1}^{y} dy(w - c)^{-2}k_{2n+1}(y), \quad (n \geqq 0). \end{aligned}\right\} \tag{4.15}$$

The point y_1 might have been any fixed point instead of one of the end points; but it is found convenient to take it this way.

Having found two particular integrals for $\phi^{(0)}$, we can obtain the higher approximations by *quadratures*. In actual calculations, this is not necessary

Because of the general nature of the Eq. (4.1), $\phi(y)$ is an entire function of αR. Hence, the infinite point of the αR-plane is a singular point, unless $\phi(y)$ is independent of αR. Consequently, the series (4.12) is asymptotic, unless $\phi(y)$ is a polynomial in $(\alpha R)^{-1}$. We note also that (4.13) is of the second order, so that only *two* solutions are obtained by this method. The solutions of (4.13) are entire functions of α^2 and hence the series (4.14) are uniformly convergent for any finite region of the complex α^2-plane, for a fixed value of y, except when y is the singular point y_0 of the differential equation (4.13).†

* Tollmien, loc. cit., 1929, p. 43.

† This can also be seen from the series itself. So long as it is possible to run a path of finite length from y_1 to y on which $w-c\neq 0$, the general terms $\alpha^{2n}h_{2n}$ and $\alpha^{2n+1}k_{2n+1}$ of the two series are bounded by $A(\alpha M)^{2n}/(2n)!$ and $B(\alpha M)^{2n+1}/(2n+1)!$, respectively, ($A$, B, M being (suitably) fixed constants), and

In fact, the differential equation (4.13) has a logarithmic singularity at the point y_0. This point is, however, an ordinary point in the exact equation (4.1), and the singularity is introduced purely by the method of asymptotic integration. However, the appearance of this singularity gives a serious ambiguity in the determination of the correct path leading from y_1 to y in order that (4.14) may give valid approximations to integrals of (4.1) *all along the path*.* The proper way to settle this question is to compare the solutions (4.14) with the *asymptotic expansions of the regular solutions* obtained by the previous method. This will be done later after we have described the second asymptotic method of Heisenberg for the other *two* particular integrals; for the same kind of problem also arises there.

To obtain *two* other integrals of (4.1) in asymptotic forms, let us make the transformation

$$\phi = \exp\left\{\int g dy\right\}. \tag{4.16}$$

Then, we obtain the non-linear differential equation

$$(w-c)\{(g^2+g')-\alpha^2\} - w'' = -\frac{i}{\alpha R}\{g^4+6g^2g'+3g'^2+4gg''+g'''-2\alpha^2(g^2+g')+\alpha^4\} \tag{4.17}$$

for the function $g(y)$. We try to solve this by putting

$$g(y) = (\alpha R)^{1/2}g_0(y) + g_1(y) + (\alpha R)^{-1/2}g_2(y) + \cdots . \tag{4.18}$$

Then, we obtain the set of equations

$$(w-c)g_0^2 = -ig_0^4, \qquad (w-c)(g_0' + 2g_0g_1) = -i(4g_0^3g_1 + 6g_0^2g_0'),$$
$$(w-c)(g_1' + g_1^2 + 2g_0g_2 - \alpha^2) - w'' = -i(4g_0^3g_2 + 6g_1^2 + 2g_0g_1g_0' + 3g_0'^2 - 2\alpha^2g_0^2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Hence we can obtain the successive approximations without integration. Thus,

$$g_0 = \pm\sqrt{i(w-c)}, \qquad g_1 = -\frac{5}{2}\frac{g_0'}{g_0}, \cdots . \tag{4.19}$$

For definiteness, we define

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hence the series converge like the cosine and the sine series, respectively. Heisenberg did not *prove* the convergence of these series, but stated that their convergence can be hoped to be sufficiently rapid for α^2 of the order of unity (loc. cit., 1924, pp. 584, 587). This was made a point of criticism by Tollmien (loc. cit., 1929, p. 43).

* Considerable dispute has arisen in this connection. Note that it is impossible to dispense with this difficulty by remarking that the two different determinations will differ only by a constant multiple of a particular integral. If we draw two paths from y_1 to y and obtain such a difference in the solution, it is evident that the asymptotic solution cannot be valid on both paths, because the *exact* equation (4.1) has *no* singular point at $y=y_0$ and hence its solution must be single-valued. Although a mistake here would not cause serious difficulties so far as the numerical evaluation of the eigen-value problem is concerned, it does lead to misunderstanding and confusion elsewhere. Even after Heisenberg and Tollmien have analyzed this problem in some detail, they still take the very misleading step of taking the complex conjugate of the inviscid equation (Heisenberg, loc. cit., 1924, p. 596; Tollmien, loc. cit., 1935, p. 88). This point will be discussed more fully later.

$$\arg i = \frac{\pi}{2}, \qquad \arg (w - c) > 0 \quad \text{for} \quad w - c > 0. \tag{4.20}$$

For negative values of $w-c$, we cannot decide, without further investigation, whether $\arg (w-c) = +\pi$ or $\arg (w-c) = -\pi$. The point y_0, where $w=c$, appeared in the previous asymptotic solution as a logarithmic branch point; here it is an algebraic branch point. The determination of the correct path should follow the same criterion as the other two integrals, that (4.18) gives two asymptotic solutions of the exact equation (4.1) *all along the path.* This path might be expected to be the same as that in the previous case. All these will be discussed in the next section.

After such a question is settled, substitution of (4.19) into (4.16) and (4.17) gives the two asymptotic solutions

$$\left.\begin{aligned} \phi_3(y) &= (w - c)^{-5/4} \exp \left\{- \int_{y_0}^{y} \sqrt{i\alpha R(w - c)}\, dy\right\}, \\ \phi_4(y) &= (w - c)^{-5/4} \exp \left\{+ \int_{y_0}^{y} \sqrt{i\alpha R(w - c)}\, dy\right\}, \end{aligned}\right| \tag{4.21}$$

where factors of the order $\exp (\alpha R)^{-1/2} = 1 + O\{(\alpha R)^{-1/2}\}$ are taken as unity.

5. Analytical properties of the solutions. Having thus obtained four asymptotic solutions of the equation (4.1), we must try to correlate them with the four solutions (4.9) and (4.11), and above all to study the correct determination of path around the artificial singularity introduced by the asymptotic methods. For this purpose, we consider the asymptotic expansions of the four regular solutions obtained by the first method and transfer back to the independent variable y.

Let us recall that the asymptotic expansions of the Hankel functions $H_{1/3}^{(1)}(\xi)$, $H_{1/3}^{(2)}(\xi)$ are given by [76],

$$\left.\begin{aligned} H_{1/3}^{(1)}(\xi) &\sim \left(\frac{2}{\pi\xi}\right)^{1/2} \exp \left\{i\left(\xi - \frac{5\pi}{12}\right)\right\} \left\{1 + \sum_{r=1}^{\infty} \frac{(-)^r(1/3, r)}{(2i\xi)^r}\right\}, \\ &\qquad (-\pi < \arg \xi < 2\pi), \\ H_{1/3}^{(2)}(\xi) &\sim \left(\frac{2}{\pi\xi}\right)^{1/2} \exp \left\{-i\left(- \frac{\pi}{12}\right)\right\} \left\{1 + \sum_{r=1}^{\infty} \frac{(1/3, r)}{(2i\xi)^r}\right\}, \\ &\qquad (-2\pi < \arg \xi < \pi). \end{aligned}\right| \tag{5.1}$$

If we put $3\xi = 2(i\alpha_0\eta)^{3/2}$, then (5.1) becomes

$$\left.\begin{aligned} H_{1/3}^{(1)}[\tfrac{2}{3}(i\alpha_0\eta)^{3/2}] &\sim \left(\frac{3}{\pi}\right)^{1/2} (i\alpha_0\eta)^{-3/4} \exp \left\{\tfrac{2}{3}(\alpha_0\eta)^{3/2} e^{5\pi i/4} - \frac{5\pi}{12}\right\} \{1 + O(\eta^{-3/2})\}, \\ &\qquad \left(-\frac{7\pi}{6} < \arg (\alpha_0\eta) < \frac{5\pi}{6}\right), \\ H_{1/3}^{(2)}[\tfrac{2}{3}(i\alpha_0\eta)^{3/2}] &\sim \left(\frac{3}{\pi}\right)^{1/2} (i\alpha_0\eta)^{-3/4} \exp \left\{\tfrac{2}{3}(\alpha_0\eta)^{3/2} e^{\pi i/4} + \frac{5\pi}{12}\right\} \{1 + O(\eta^{-3/2})\}, \\ &\qquad \left(-\frac{11\pi}{6} < \arg (\alpha_0\eta) < \frac{\pi}{6}\right). \end{aligned}\right| \tag{5.2}$$

With the help of these formulae and using the legitimate process of integrating the asymptotic expansions term by term, we obtain

$$
\left.\begin{aligned}
\chi_1^{(0)} + \epsilon\chi_1^{(1)} &\sim \eta + \frac{\epsilon w_0''}{2w_0'}\eta^2 = \frac{1}{w_0'\epsilon}\left(w_0' y + \frac{w_0''}{2} y^2\right), \\
\chi_2^{(0)} + \epsilon\chi_2^{(1)} &\sim 1 + \epsilon\frac{w_0''}{w_0'}\eta \log \eta \sim 1 + \frac{w_0''}{w_0'} y \log y, \\
\chi_3^{(0)} &\sim \text{const. } \eta^{-5/4} \exp\left\{\tfrac{2}{3}(\alpha_0\eta)^{3/2} e^{5\pi i/4}\right\} \\
&= \text{const. } (y - y_0)^{-5/4} \exp\left\{-\int_{y_1}^{y} \sqrt{i\alpha R w_0'(y - y_0)}\, dy\right\}, \\
\chi_4^{(0)} &\sim \text{const. } \eta^{-5/4} \exp\left\{\tfrac{2}{3}(\alpha_0\eta)^{3/2} e^{\pi i/4}\right\} \\
&= \text{const. } (y - y_0)^{-5/4} \exp\left\{\int_{y_0}^{y} \sqrt{i\alpha R w_0'(y - y_0)}\, dy\right\}.
\end{aligned}\right| \tag{5.3}
$$

These formulae can be easily seen to agree with the four asymptotic solutions (4.13) and (4.21) to the proper order of approximation, if we replace y_1 by y_0 in $\phi_1^{(0)}$ (which is permissible).

In evaluating the asymptotic expressions (5.3), the argument $\alpha_0\eta$ must satisfy both requirements specified in (5.2), i.e.,

$$
-7\pi/6 < \arg(\alpha_0\eta) < \pi/6. \tag{5.4}
$$

In this range, the asymptotic solutions (4.13) *and* (4.21) *hold.* Having thus established the range of validity of these solutions, we no longer need to make further comparisons of the two methods of solution.

At least three plans are now possible for further numerical work. First, we may use the four solutions obtained in the approximate form (4.9). Secondly, we may use the four asymptotic solutions (4.14) and (4.21). Thirdly, we may approximate $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ by the four functions $\{\phi_1^{(0)}, \phi_2^{(0)}, \chi_3^{(0)}, \chi_4^{(0)}\}$ given by (4.14) and (4.9). The first method is very similar to the method used by Hopf [16] and Tietjens [72] for linear velocity distributions, where the *exact* solutions are given by functions of the general nature of those in (4.9). For curved velocity distributions, the functions $\chi_1^{(0)}$, $\chi_2^{(0)}$ do not give ϕ_1 and ϕ_2 with sufficient accuracy, and this plan is not good. The second plan was used by Heisenberg in his investigation of the stability of the Poiseuile flow; but he also realized that it served only part of his purpose, and he stated that the third plan should be used.* Tollmien substantially adopted the third plan for his investigation of the stability of the boundary layer, although he did not point out the connection of his method with Heisenberg's work. Instead of the expressions (4.14) for ϕ_1 and ϕ_2, he used solutions in the forms of power series in y. These solutions are easily manageable only for linear and parabolic velocity distributions. Accordingly, he tried to approximate the Blasius profile with such profiles. Since such approximations are not good enough in the neighborhood of the point $y=y_0$, where $w=c$, his discussion becomes very complicated. In the present work, we base our cal-

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* Loc. cit., p. 404.

culations upon the use of (4.14). It will be seen that our method can be applied to any profile with good accuracy. A comparison with Tollmien's method will be discussed in the Appendix to Part III.

It may be added that the adoption of the third plan leaves an error of the order of $(\alpha R)^{-1}$ in ϕ_1 and ϕ_2, and an error of the order of $(\alpha R)^{-1/3}$ in ϕ_3 and ϕ_4. These errors can be reduced by including the higher approximations. In practice, this is hardly necessary. A detailed discussion of numerical accuracy will be found in the Appendix to Part III.

Having thus established the region of validity of the asymptotic solutions, we shall try to settle a few questions of considerable dispute, namely, (a) the "crossing substitution," (b) the inner friction layers, and (c) the complex conjugate of the inviscid solution.

a) *The crossing substitution.* From previous discussions, it is evident that if we pass from $y>\mathrm{Re}(y_0)$ to $y<\mathrm{Re}(y_0)$ along a path *below* the point y_0, we are always in a region of the y-plane where the above asymptotic solutions hold, and no further investigation is necessary. In fact, if $c_i>0$ (and is small) and $\mathrm{Re}(w_0')>0$, the point y_0 is above the real axis, and the asymptotic solutions are *valid along the real axis of y.* In the case of real c, the point y_0 is on the real axis, and there is *one point on the real axis* where the asymptotic solutions fail to be valid. In the case $c_i<0$, and $\mathrm{Re}(w_0')>0$, the point y_0 is below the real axis, and the lines $\arg\{\alpha_0(y-y_0)\}=-7\pi/6,\ \pi/6$ intersect the real axis in two points y_f', y_f'' with* $y_1<y_f'<y_f''<y_2$. Thus, the asymptotic expressions (4.14) and (4.21) represent one solution for $y_1\leqq y<y_f'$ and $y_f''<y\leqq y_2$, but not the same solution for $y_f'<y<y_f''$. It is necessary to obtain a suitable "crossing substitution" in order to obtain the correct solutions for $\pi/6<\arg\{\alpha_0(y-y_0)\}<5\pi/6$ (i.e., in *crossing* the lines $\arg\{\alpha_0(y-y_0)\}=-7\pi/6,\ \pi/6$). For this purpose, we must obtain the asymptotic expansion of the Hankel functions $H_{1/3}^{(j)}[2(i\alpha_0\eta^{3/2})/3]$, $(j=1,2)$, proper to that region. The analytical expression for $H_{1/3}^{(2)}$ would then be quite different from that given in (5.2). Thus, in crossing the *two points y_f' and y_f'' of the real axis*, the asymptotic solutions fail to be analytic. However, it is to be noted that the failure of the asymptotic solutions along the real axis does not exclude their use in the investigation of the boundary value problems to be considered below, so long as we are concerned only with the eigen-value problem. It is only necessary that these solutions be valid in a *connected* region containing the end-points y_1 and y_2. The calculation of the amplitude distribution of the disturbance (the eigen-function) in the neighborhood of the inner friction layers, however, is to be made with the regular solution, or we can calculate the eigen-function for $y_f'<y<y_f''$ by using a proper

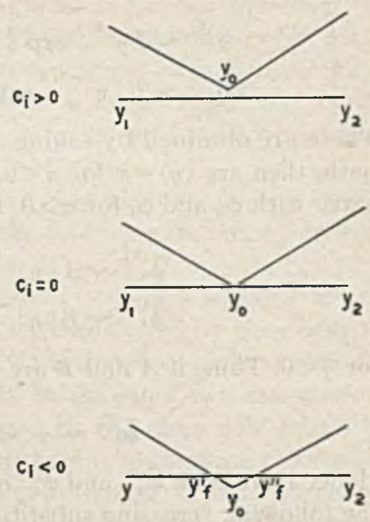

FIG. 2. Diagram showing the relative position of the real axis and the region of validity of the asymptotic solutions in each of the three cases.

* The whole theory must be modified for extremely highly damped solutions for which $y_f'<y_1<y_2<y_f''$.

"crossing substitution." Since we are chiefly concerned with the eigen-value problem, we shall not go into further details.

In order to make the situation still clearer, let us see what would happen if we try to obtain our solutions for $y_1' < y < y_1''$ by going along a path above the point y_0. For simplicity, let us take the case of real c with $w_0' > 0$, and consider the asymptotic expressions $\phi_3^{(0)}$ and $\phi_4^{(0)}$ given by (5.3). We have (A, B being arbitrary constants)

$$\phi_3^{(0)} \sim A\eta^{-5/4} \exp\{\tfrac{2}{3}(\alpha_0\eta)^{3/2} e^{5\pi i/4}\}, \qquad (\eta > 0),$$

$$\phi_3^{(0)} \sim A|\eta|^{-5/4} \exp\{\tfrac{2}{3}(\alpha_0|\eta|)^{3/2} e^{-\pi i/4} + 5\pi i/4\}, \qquad (\eta < 0);$$

$$\phi_4^{(0)} \sim B\eta^{-5/4} \exp\{\tfrac{2}{3}(\alpha_0\eta)^{3/2} e^{\pi i/4}\}, \qquad (\eta > 0),$$

$$\phi_4^{(0)} \sim B|\eta|^{-5/4} \exp\{\tfrac{2}{3}(\alpha_0|\eta|)^{3/2} e^{-5\pi i/4} + 5\pi i/4\}, \qquad (\eta < 0).$$

These are obtained by taking a path below the point y_0. If we had taken the other path, then arg $(\eta) = \pi$ for $\eta < 0$, and we would have the functions $\tilde{\phi}_3^{(0)}$ and $\tilde{\phi}_4^{(0)}$, which agree with ϕ_3 and ϕ_4 for $\eta > 0$, but are defined by

$$\tilde{\phi}_3^{(0)} \sim A|\eta|^{-5/4} \exp\{\tfrac{2}{3}(\alpha_0|\eta|)^{3/2} e^{3\pi i/4} - 5\pi i/4\},$$

$$\tilde{\phi}_4^{(0)} \sim B|\eta|^{-5/4} \exp\{\tfrac{2}{3}(\alpha_0|\eta|)^{3/2} e^{7\pi i/4} - 5\pi i/4\},$$

for $\eta < 0$. Thus, if A and B are taken to be the same, we have

$$\tilde{\phi}_3^{(0)} = -i\phi_4^{(0)}, \quad \tilde{\phi}_4^{(0)} = -i\phi_3^{(0)}, \quad \text{for} \quad \eta < 0.$$

Hence if we took $\tilde{\phi}_3^{(0)}$ and $\tilde{\phi}_4^{(0)}$ as the proper determinations, we would have to make the following "crossing substitution" corresponding to a passage from $\eta > 0$ to $\eta < 0$: $\tilde{\phi}_3^{(0)} \to i\tilde{\phi}_4^{(0)}$; $\tilde{\phi}_4^{(0)} \to i\tilde{\phi}_3^{(0)}$. If we note that $\tilde{\phi}_3^{(0)} \ll \tilde{\phi}_4^{(0)}$ both for $w-c>0$ and for $w-c<0$, we would also have the following equivalent change: $\tilde{\phi}_3^{(0)} \to \tilde{\phi}^{(0)} + i\tilde{\phi}_4^{(0)}$; $\tilde{\phi}_3^{(0)} - i\tilde{\phi}_4^{(0)} \to \tilde{\phi}_3^{(0)}$. These may be compared with Eq. (16), p. 589 of Heisenberg's paper. In making the comparison, we note his definition of the angle of $w-c$ (p. 585), and the difference of notation in the fundamental equation of stability.

The first study of "the crossing substitution" seems to be due to Stokes in connection with the asymptotic expansions of Bessel functions. It may therefore be properly designated as Stokes' phenomenon [76]. We should also compare our results with the work of Jeffreys [17], the *W-K-B* method [23] in quantum mechanics,* and the mathematical investigations of Langer [24] and others.** In those cases, a differential equation of the form $\epsilon^3\phi'' + q(y)\phi = 0$ is considered. If this equation is treated by the method of §4 by writing $\phi = \chi^{(0)}(\eta) + \epsilon\chi^{(1)}(\eta) + \cdots$, $y - y_0 = \epsilon\eta$, and $q(y) = q_0'(\epsilon\eta) + \frac{1}{2}q_0''(\epsilon\eta)^2 + \cdots$, the equation for $\chi^{(0)}(\eta)$ is $\chi^{(0)''} + q_0'\eta\chi^{(0)} = 0$ as compared with (4.7), $\chi^{(0)\text{iv}} - i\eta w_0'\chi^{(0)''} = 0$. It is evident that our η corresponds to $i\eta$ in their case. Kramers has shown that the cuts in their asymptotic expansions are the lines arg $(\eta) = \pm\pi/3$. Thus, in our case, the cuts should be arg $(\eta) = \pi/6$, $5\pi/6$. This agrees with our previous discussions. An important difference is the following. In

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* I am indebted to Professor P. S. Epstein for calling my attention to this comparison.

** For example, S. Goldstein, Proc. Lond. Math. Soc. (2) **28**, 81–90 (1928); C. C. Hurd, Tôhoku Math. Journ. **45**, 58–68 (1939), and the papers of W. J. Trjitzinsky and others quoted there.

their case, the two boundary points on the real axis are separated into two regions of the complex plane by the cuts, so that a *crossing substitution is absolutely necessary.* In our case, the two boundary points on the real axis belong to the same region, and a *crossing substitution is superfluous*, so far as the eigen-value problem is concerned.

b) *The inner friction layers.* There is also a very significant physical interpretation associated with the "crossing substitution" of the asymptotic solutions. The initial approximations $\phi_1^{(0)}$ and $\phi_2^{(0)}$ satisfy the inviscid equation. Hence, if $c_i>0$, these solutions hold throughout the interval (y_1, y_2) of the real axis, and the effect of viscosity is entirely negligible inside the fluid for sufficiently large Reynolds numbers. If $c_i \leqq 0$, the inviscid solution can never hold all along the real axis, and hence the effect of viscosity inside the fluid is not negligible, however large the Reynolds number may be. The singularity of the asymptotic solutions means a very rapid change of velocity within a small distance so that the effect of viscosity is no longer negligible there. Physically, such a point on the real axis corresponds to a layer of fluid where the viscous forces play an important role.

Referring to the foregoing discussions, we see that *there are two inner friction layers for the damped oscillations, one for the neutral oscillations, and none for the self-excited oscillations.*

In the neutral case, the first term of (4.1) disappears at the critical layer $w=c$. The equation then represents a balancing of the vorticity transferred by the disturbance and that diffused away by the effect of viscosity. It is therefore understandable that the effect of viscosity must be predominant there. In the other two cases, $w-c$ never vanishes in the fluid, there is the vorticity carried by the main flow (relative to an observer moving with the phase velocity c_r) and there is always the change of vorticity due to amplification or damping. In the case of amplified oscillations, these two effects can be in equilibrium with the transfer of vorticity due to the disturbance, and the effect of viscosity is completely negligible at very large Reynolds numbers. In the case of damped oscillations, these effects presumably never balance each other, thus resulting in the formation of two critical layers, where the effect of viscosity is not negligible.

c) *The complex conjugate of the solution $\phi(y)$.* It is often argued,* that if $\phi(y)$ is a solution of the inviscid equation with an eigen-value c, then $\bar{\phi}(y)$ is another solution with the eigen-value $\bar{c}$, satisfying the same real boundary conditions on the real axis. Thus, to each damped solution, there is always a corresponding amplified solution, and vice versa. This argument is in direct contradiction to the foregoing discussions, because an amplified solution and a damped solution have entirely different characteristics with respect to inner friction layers. It appears, therefore, that $\bar{\phi}(y)$ should still represent a solution of the same nature as $\phi(y)$.

This is indeed the case, and can be seen more clearly from an examination of the complete equation (4.1). If we take the complex conjugate of that equation, and write y for $\bar{y}$ (which is essentially done in the usual argument), we have

$$\{w(y) - c\}\{\bar{\phi}'' - \alpha^2\bar{\phi}\} - w''(y)\bar{\phi} = \frac{i}{\alpha R}\{\bar{\phi}^{\text{iv}} - 2\alpha^2\bar{\phi}'' + \alpha^4\bar{\phi}\}. \qquad (5.5)$$

* Heisenberg, loc. cit. p. 596; Tollmien, loc. cit., 1935, p. 88. The failure of such an argument would indicate that Heisenberg's classification of velocity profiles on p. 597 of his paper is untenable.

The complete stream function $\bar{\psi}'(x, y, t)$ satisfying this equation is

$$\bar{\psi}'(x, y, t) = \bar{\phi}(y)e^{-i\alpha(x-\bar{c}t)}.$$

Thus, if $\mathrm{Im}(c) < 0$, then $\mathrm{Im}(\bar{c}) > 0$, and we still have a damped solution. This should also hold for the inviscid equation, since it is regarded as a *limiting* case of the viscous equation. From the inviscid equation itself, there is no way of telling whether $\mathrm{Im}(\bar{c}) > 0$ corresponds to damping or to amplification. In fact, the asymptotic solutions of Eq. (5.5) (including the limiting inviscid solutions) hold for

$$-\pi/6 < \arg\left\{\bar{\alpha}_0(y - y_0)\right\} < 7\pi/6, \qquad w(y_0) = \bar{c}. \tag{5.6}$$

Thus, we have a solution $\bar{\phi}(y)$, valid in a region which is quite improper for an asymptotic solution of (4.1). [Compare (5.4) and (5.6).] Hence, it is not legitimate to conclude that a solution of a different nature can be obtained by taking the complex conjugate. The influence of these discussions upon the usual conclusions regarding stability in an inviscid fluid will be discussed fully in the next part of the paper.

6. The boundary value problems. Having fully investigated the solutions of the equation of disturbance, we shall now turn to a study of the boundary-value problems which have been taken up briefly at the end of §3. In general, the boundary conditions are essentially that the velocities of the disturbance should vanish on the solid boundaries, and also at infinity if the field of flow extends to infinity. However, it is often convenient to use equivalent boundary conditions for certain types of velocity distributions.

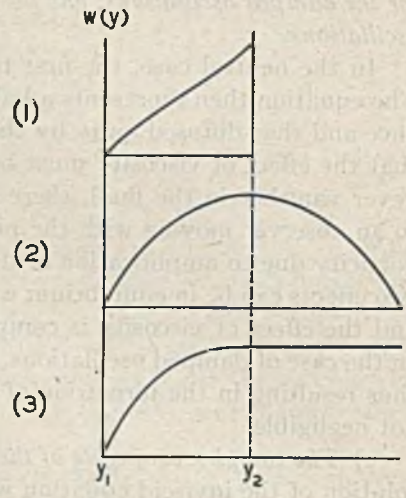

FIG. 3. The three types of velocity distributions.

In order not to be lost in too much generality, we shall limit ourselves to three classes of velocity distributions (as specified below and shown in Fig. 3), and select our fundamental interval (y_1, y_2) so that $w'(y) \geqq 0$ for $y_1 \leqq y \leqq y_2$. We shall define our characteristic length so that $y_2 - y_1 = 1$, and let $\phi_1(y; c, \alpha, \alpha R)$, $\phi_2(y; c, \alpha, \alpha R)$, $\phi_3(y; c, \alpha, \alpha R)$, $\phi_4(y; c, \alpha, \alpha R)$ be a fundamental system of solutions (4.1) arranged in the order discussed above.

CASE (1). *Flow between solid walls in relative motion.* In this case, the boundary conditions are given by

$$\phi(y_1) = \phi'(y_1) = \phi(y_2) = \phi'(y_2) = 0, \tag{6.1}$$

because the velocity of the disturbance should vanish on both the solid boundaries. The determinantal equation corresponding to these conditions is

$$F_1(\alpha, c, \alpha R) = \begin{vmatrix} \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} \\ \phi_{12} & \phi_{22} & \phi_{32} & \phi_{42} \\ \phi_{11}' & \phi_{21}' & \phi_{31}' & \phi_{41}' \\ \phi_{12}' & \phi_{22}' & \phi_{32}' & \phi_{42}' \end{vmatrix} = 0, \tag{6.2}$$

where ϕ_{11}, ϕ_{11}', etc., stand for $\phi_1(y_1)$, $\phi_1'(y_1)$, etc. In this and all later discussions, a subscript 1 or 2 attached to a function of y shall denote the value of that function at $y=y_1$ or $y=y_2$ respectively.

CASE (2). *Symmetrical flow between solid walls at rest.* In this case, it is easily seen from (4.1) that the disturbance can be separated into two independent parts, one symmetrical with respect to the line $y=y_2$ and the other antisymmetrical. (a) If $\phi(y)$ is a symmetrical function (antisymmetrical disturbance), then the conditions

$$\phi(y_1) = \phi'(y_1) = \phi'(y_2) = \phi'''(y_2) = 0 \tag{6.3}$$

hold, and we have the determinantal equation

$$F_2(\alpha, c, \alpha R) = \begin{vmatrix} \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} \\ \phi'_{11} & \phi'_{21} & \phi'_{31} & \phi'_{41} \\ \phi'_{12} & \phi'_{22} & \phi'_{32} & \phi'_{42} \\ \phi'''_{12} & \phi'''_{22} & \phi'''_{32} & \phi'''_{42} \end{vmatrix} = 0. \tag{6.4}$$

(b) If $\phi(y)$ is an odd function of $y-y_2$ (symmetrical disturbance), then the boundary conditions are

$$\phi(y_1) = \phi'(y_1) = \phi(y_2) = \phi''(y_2) = 0, \tag{6.5}$$

and we have the relation

$$F_3(\alpha, c, \alpha R) = \begin{vmatrix} \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} \\ \phi_{12} & \phi_{22} & \phi_{32} & \phi_{42} \\ \phi'_{11} & \phi'_{21} & \phi'_{31} & \phi'_{41} \\ \phi''_{12} & \phi''_{22} & \phi''_{32} & \phi''_{42} \end{vmatrix} = 0. \tag{6.6}$$

CASE (3). *Flow of the boundary-layer type.* In this case, the point y_2 is taken to correspond to the "edge" of the boundary layer, beyond which the velocity is substantially constant. The boundary conditions to be satisfied at y_1 are the usual ones;

$$\phi(y_1) = \phi'(y_1) = 0. \tag{6.7}$$

The boundary conditions for y becoming infinite are to be replaced as follows.* Since the particular integral ϕ_4 becomes infinite as y becomes infinite, our boundary condition requires that ϕ is a linear combination of ϕ_1, ϕ_2, ϕ_3 alone. Thus,

$$\phi = C_1\phi_1 + C_2\phi_2 + C_3\phi_3, \tag{6.8}$$

where C_1, C_2, C_3 are constants of integration. Also, the integral ϕ_3 makes practically no contribution for $y \geqq y_2$ so that we expect $\phi(y)$ to satisfy the inviscid equation for $y \geqq y_2$. Here $w''=0$, and hence two particular integrals are $e^{\pm\alpha y}$. The condition that $\phi \to 0$ as $y \to \infty$ excludes the integral $e^{\alpha y}$. Hence, ϕ must be proportional to $e^{-\alpha y}$ for $y > y_2$. This may be expressed as follows:

$$\phi' + \alpha\phi = 0 \quad \text{for} \quad y \geqq y_2. \tag{6.9}$$

Hence, we have the determinantal equation

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* Cf. Tietjens [72] and Tollmien [73].

$$F_4(\alpha, c, \alpha R) = \begin{vmatrix} \phi_{11} & \phi_{21} & \phi_{31} \\ \phi'_{12} + \alpha\phi_{12} & \phi'_{22} + \phi\alpha_{22} & 0 \\ \phi'_{11} & \phi'_{21} & \phi'_{31} \end{vmatrix} = 0. \qquad (6.10)$$

We note that the point y_2 can be replaced by any value of $y > y_2$. This is equivalent to the fact that the thickness of the boundary layer cannot be definitely defined. The larger this thickness is taken, the more accurate the results should be .*

The functions F_1, F_2, F_3, and F_4 are entire functions of the parameters α, c, and R.

Reduction of the equations for large values of αR. The equations (6.2), (6.4), and (6.6) can be substantially simplified for large values of αR. Referring to (4.21), we see that $\phi_3(y) = A(y)e^{-Y}$, $\phi_4(y) = B(y)e^{Y}$, where $A(y)$ and $B(y)$ are of the order of unity, and Y is defined by the relation

$$Y = \int_{y_1}^{y} \sqrt{i\alpha R(w - c)}\, dy.$$

Hence, we have the following relations, giving the order of magnitude of certain quantities:

$$\left.\begin{aligned} \frac{\phi'_{31}}{\phi_{31}} &= -\sqrt{i\alpha R(w_1 - c)} + \frac{A'_1}{A_1}, \qquad \frac{\phi_{32}}{\phi_{31}} = \frac{A_2}{A_1} e^{-P}, \\ \frac{\phi'_{32}}{\phi_{31}} &= \left\{ -\sqrt{i\alpha R(w_2 - c)}\, \frac{A_2}{A_1} + \frac{A'_2}{A_1} \right\} e^{-P}, \\ \frac{\phi''_{32}}{\phi_{31}} &= \left\{ i\alpha R(w_2 - c)\, \frac{A_2}{A_1} + O(\sqrt{\alpha R}) \right\} e^{-P}, \\ \frac{\phi'''_{32}}{\phi_{31}} &= \left\{ -\,[i\alpha R(w_2 - c)]^{3/2} \frac{A_2}{A_1} + O(\alpha R) \right\} e^{-P}; \end{aligned}\right| \qquad (6.11)$$

$$\left.\begin{aligned} \frac{\phi'_{41}}{\phi_{41}} &= \sqrt{i\alpha R(w_1 - c)} + \frac{B'_1}{B_1}, \qquad \frac{\phi_{42}}{\phi_{41}} = \frac{B_2}{B_1} e^{P}, \\ \frac{\phi'_{42}}{\phi_{41}} &= \left\{ \sqrt{i\alpha R(w_2 - c)}\, \frac{B_2}{B_1} + \frac{B'_2}{B_1} \right\} e^{P}, \\ \frac{\phi''_{42}}{\phi_{41}} &= \left\{ i\alpha R(w_2 - c)\, \frac{B_2}{B_1} + O(\sqrt{\alpha R}) \right\} e^{P}, \\ \frac{\phi'''_{42}}{\phi_{41}} &= \left\{ [i\alpha R(w_2 - c)]^{3/2} \frac{B_2}{B_1} + O(\alpha R) \right\} e^{P}, \end{aligned}\right| \qquad (6.12)$$

where

$$P = \int_{y_1}^{y_2} \sqrt{i\alpha R(w - c)}\, dy.$$

It then appears that the sign of the real part of P is of consequence. It can be veri-

* In later calculation of the Blasius case, we shall take a boundary layer about 1.19 times as thick as that used by Tollmien.

fied that it is always positive when $c_i>0$. For then the path of integration can be taken along the real axis of y, and we have $-\pi<\arg(w-c)<0$; consequently, $-\pi/4<\arg(P)<\pi/4$. With reference to (4.21), (6.11) and (6.12), we see that the condition $P=n\pi i$ (n = an integer), expresses the fact that $\phi_{31}'\phi_{42}'=\phi_{41}'\phi_{32}'$ when terms of the order $(\alpha R)^{1/2}$ are neglected. This is the corrected form of the first solution of Heisenberg as expressed by the condition (27) on p. 596 of his paper. Heisenberg also stated that such a condition can only be satisfied for damped solutions. In fact, from the condition just obtained for $c_i>0$, we see that $\mathrm{Re}(P)$ can be negative only for highly damped solutions, for which the whole discussion must be modified. (Cf. footnote on p. 29, §5.)

Neglecting quantities of the order e^{-P} and $(\alpha R)^{-1}$ against quantities of the order of unity, we have the following simplifications of Eqs. (6.2), (6.4), and (6.6) for Cases (1) and (2).

CASE (1). *Flow between solid walls in relative motion.* We have

$$\frac{f_1(\alpha, c)}{f_3(\alpha, c)} = \frac{\phi_{31}}{\phi_{31}'} + \frac{\phi_{42}}{\phi_{42}'}\frac{f_2(\alpha, c)}{f_4(\alpha, c)}. \tag{6.13}$$

CASE (2a). *Antisymmetrical disturbance in a symmetrical flow between solid walls.* We have

$$f_2(\alpha, c)/f_4(\alpha, c) = \phi_{31}/\phi_{31}'. \tag{6.14}$$

CASE (2b). *Symmetrical disturbance in a symmetrical flow between solid walls.* We have

$$f_1(\alpha, c)/f_3(\alpha, c) = \phi_{31}/\phi_{31}'. \tag{6.15}$$

In these equations, the functions $f_1(\alpha, c)$, $f_2(\alpha, c)$, $f_3(\alpha, c)$ and $f_4(\alpha, c)$ are defined as follows:

$$f_1(\alpha, c) = \begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix}, \qquad f_2(\alpha, c) = \begin{vmatrix} \phi_{11} & \phi_{12}' \\ \phi_{21} & \phi_{22}' \end{vmatrix},$$
$$f_3(\alpha, c) = \begin{vmatrix} \phi_{11}' & \phi_{12} \\ \phi_{21}' & \phi_{22} \end{vmatrix}, \qquad f_4(\alpha, c) = \begin{vmatrix} \phi_{11}' & \phi_{12}' \\ \phi_{21}' & \phi_{22}' \end{vmatrix}. \tag{6.16}$$

These functions depend only on α and c, because we may take the inviscid solutions for ϕ_1 and ϕ_2, which are accurate up to the order of $(\alpha R)^{-1}$. It may be reiterated that in computing ϕ_{12}, ϕ_{22}, ϕ_{12}', ϕ_{22}' in (6.16), we must take the path leading from y_1 to y_2 in the lower half plane.

CASE (3). *Flow in the boundary layer along a flat plate.* In this case we can reduce (6.10) to

$$\frac{f_2+\alpha f_1}{f_4+\alpha f_3} = \frac{\phi_{31}}{\phi_{31}'}, \tag{6.17}$$

if we also replace ϕ_1 and ϕ_2 by their inviscid approximations. The equations (6.13), (6.14), (6.15), and (6.17) are the final equations based on which the stability investigations are to be made.

The inviscid case. In the limit $\alpha R\to\infty$. The above equations reduce to

$$f_1(\alpha, c) = 0 \quad \text{for Case (1) and Case (2}b\text{)}, \tag{6.18}$$

$$f_2(\alpha, c) = 0 \quad \text{for Case (2a)}, \tag{6.19}$$

$$f_2 + \alpha f_1 = 0 \quad \text{for Case (3)}. \tag{6.20}$$

Mathematically, these are equivalent to the solution of the inviscid equation

$$(w - c)(\phi'' - \alpha^2\phi) - w''\phi = 0 \tag{6.21}$$

subjected to one of the following three sets of boundary conditions

$$\phi(y_1) = \phi(y_2) = 0, \quad \phi(y_1) = \phi'(y_2) = 0, \quad \phi(y_1) = \phi'(y_2) + \alpha\phi(y_2) = 0. \tag{6.22}$$

We have thus arrived at the conclusion that *some asymptotic behavior of the stability conditions can be obtained by neglecting the effect of viscosity* (provided proper care is given to the inner friction layer). This was tacitly assumed in the work of Rayleigh and others, while Heisenberg pointed out [loc. cit., p. 583] that a proof was necessary in accordance with some remarks of Oseen (38); he also virtually gave the proof.

In the next part of the paper, we shall therefore consider the simpler problem of an inviscid fluid. After a thorough investigation of that problem, we shall investigate the effect of viscosity by considering Eqs. (6.13), (6.14), (6.15), and (6.17) in greater detail. These results may be compared with the earlier ones of Heisenberg and Tollmien.

Numerical calculations of the stability limit based upon these equations will also be carried out for certain important special cases. For all these purposes, the evaluation of the six functions $f_1(\alpha, c)$, $f_2(\alpha, c)$, $f_3(\alpha, c)$, $f_4(\alpha, c)$, ϕ_{31}/ϕ_{31}', ϕ_{42}/ϕ_{42}' is necessary. We shall discuss this briefly here.

1) *Evaluation of* $f_1(\alpha, c)$, $f_2(\alpha, c)$, $f_3(\alpha, c)$, $f_4(\alpha, c)$. These quantities are related to the inviscid solutions given by (4.14) with the path of integration subjected to the condition (5.4). Hence, we have

$$\phi_{11} = -c, \qquad \phi_{21} = 0, \qquad \phi_{11}' = w_1', \qquad \phi_{21}' = -\frac{1}{c}, \tag{6.23}$$

$$\left.\begin{aligned}
&\phi_{12} = (1 - c)\sum_{n=0}^{\infty} \alpha^{2n} H_{2n}(c), \qquad \phi_{22} = (1 - c)\sum_{n=0}^{\infty} \alpha^{2n} K_{2n+1}(c),\\
&\phi_{12}' = (1 - c)^{-1}\sum_{n=0}^{\infty} \alpha^{2n} H_{2n-1}(c) + (1 - c)^{-1} w_2' \phi_{12},\\
&\phi_{22}' = (1 - c)^{-1}\sum_{n=0}^{\infty} \alpha^{2n} K_{2n}(c) + (1 - c)^{-1} w_2' \phi_{22},
\end{aligned}\right| \tag{6.24}$$

where

$$\left.\begin{aligned}
H_{2n}(c) &= h_{2n}(y_2), & H_{2n-1}(c) &= (1 - c)^{-2} h_{2n}'(y_2),\\
K_{2n+1}(c) &= k_{2n+1}(y_2), & K_{2n}(c) &= (1 - c)^2 k_{2n+1}'(y_2)
\end{aligned}\right\} \tag{6.25}$$

are functions of c alone. In the above evaluations, we have put $w_1 = 0$, in accordance with the actual conditions in all the cases considered. We have also chosen the characteristic velocity so that $w_2=1$. Referring to (6.23), we have

$$\left.\begin{aligned} f_1(\alpha, c) &= -c\phi_{22}, & f_2(\alpha, c) &= -c\phi_{22}', \\ f_3(\alpha, c) &= w_1'\phi_{22} + \frac{1}{c}\phi_{12}, & f_4(\alpha, c) &= w_1'\phi_{22}' + \frac{1}{c}\phi_{12}'. \end{aligned}\right| \tag{6.26}$$

The actual evaluation therefore depends upon the calculation of the integrals (6.25).

2) *Evaluation of* ϕ_{31}/ϕ_{31}', ϕ_{42}/ϕ_{42}'. These quantities are related to the highly oscillating integrals ϕ_3 and ϕ_4. For values of αR so large that both $(\alpha R)^{1/3}c$, $(\alpha R)^{1/3}(1-c) \gg 1$, the approximate values of ϕ_{31}/ϕ_{31}' and ϕ_{42}/ϕ_{42}' are given by using (4.21). Thus,

$$\frac{\phi_{31}}{\phi_{31}'} = -\frac{e^{\pi i/4}}{\sqrt{\alpha R c}}, \qquad \frac{\phi_{42}}{\phi_{42}'} = \frac{e^{-\pi i/4}}{\sqrt{\alpha R(1-c)}}. \tag{6.27}$$

The condition $(\alpha R)^{1/3}(1-c) \gg 1$ is generally satisfied, because c is usually small and αR is usually very large. This is especially true for Eq. (6.13), for it will be seen from considerations of an inviscid fluid that this case is relatively stable. For the condition $(\alpha R)^{1/3}c \gg 1$, the situation is different, because c is usually small. It is then more convenient to approximate ϕ_3 by $\chi_3^{(0)}(\eta)$ given by (4.9) (or with higher approximations, if so desired). We then have

$$\phi_{31}/\phi_{31}' = (y_1 - y_0)F(z), \tag{6.28}$$

where

$$F(z) = \frac{\int_{+\infty}^{-z} d\zeta \int_{+\infty}^{\zeta} \zeta^{1/2} d\zeta H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right]}{-z\int_{+\infty}^{-z} d\zeta \zeta^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right]}, \qquad z = -\alpha_0(\eta_1 - \eta_0). \tag{6.29}$$

This function has been calculated numerically by Tietjens for real values of z. We have made slightly more extensive and more accurate calculations. The present result differs slightly from that of Tietjens and is here tabulated in Table I and plotted in Fig. 4 together with the related function $\mathcal{F}(z)$ defined by

$$\mathcal{F}(z) = [1 - F(z)]^{-1}. \tag{6.30}$$

The asymptotic form of $F(z)$ is

$$F(z) \sim z^{-3/2}e^{\pi i/4}, \qquad (z \gg 1). \tag{6.31}$$

This agrees with (6.27) if $\alpha_0^3(y_1-y_0) = w_0'(y_1-y_0)$ is replaced by $-c$.

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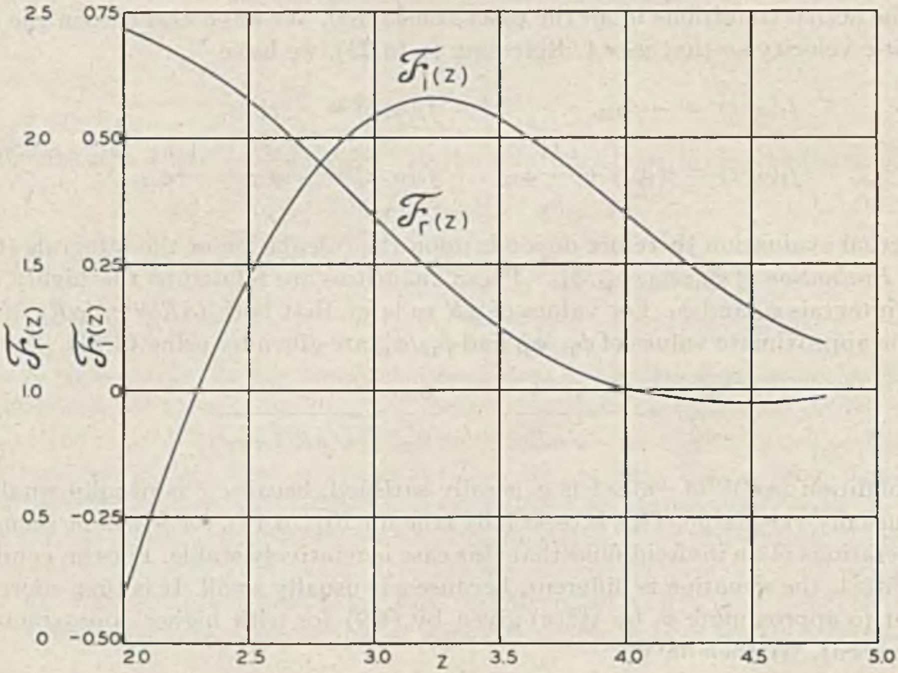

FIG. 4. The function $\mathcal{J}(z)$ shown in its real and imaginary parts (cf. Table I).

TABLE 1.—The functions $F(z)$ and $\mathcal{J}(z)$.

z	F_r	F_i	$\mathcal{J}_r$	$\mathcal{J}_i$
1.0	0.89161	−0.35025	0.80630	−2.60557
1.2	0.78969	−0.27310	1.77012	−2.29854
1.4	0.71970	−0.21213	2.26836	−1.71669
1.6	0.66931	−0.16009	2.44985	−1.18600
1.8	0.63143	−0.11274	2.48104	−0.75892
2.0	0.60144	−0.06741	2.43927	−0.41253
2.2	0.57599	−0.02226	2.35196	−0.12348
2.4	0.55230	−0.02395	2.22724	0.11916
2.6	0.52773	−0.07203	2.06929	0.31558
2.8	0.49952	0.12220	1.88566	0.46043
3.0	0.46456	0.17391	1.68938	0.54872
3.2	0.41947	0.22520	1.49726	0.58082
3.4	0.36110	0.27193	1.32516	0.56401
3.6	0.28802	0.30705	1.18429	0.51074
3.8	0.20352	0.32130	1.07982	0.43560
4.0	0.11800	0.30721	1.01118	0.35220
4.2	0.04698	0.26559	0.97361	0.27133
4.4	0.00240	0.20811	0.96056	0.20038
4.6	0.02160	0.14475	0.95989	0.13601
4.8	0.01477	0.09875	0.97659	0.09503

(*To be continued*)

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METHODS OF REPRESENTING THE PROPERTIES OF VISCOELASTIC MATERIALS*

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Introduction. In a recent paper[1] it has been shown that the solution of the first and second boundary value problem for linear viscoelastic media can be obtained in two steps requiring (a) the solution of an equivalent problem for a perfectly elastic medium, and (b) the determination of the response of the viscoelastic material to an applied shearing stress (or shearing strain) which is a given function of time. The study of the behaviour of viscoelastic materials in pure shear is accordingly seen to be of particular importance. To coordinate various manners of describing this behaviour is the purpose of the present paper.

From the mathematical point of view the behaviour of a viscoelastic material in pure shear is represented by a differential relation between the shear stress s and the shearing strain ϵ. We may write this relation in the form

$$Ps = 2Q\epsilon, \tag{1}$$

where the differential operators P and Q are defined by

$$P = \frac{\partial^m}{\partial t^m} + p_{m-1}\frac{\partial^{m-1}}{\partial t^{m-1}} + \cdots + p_0,$$

$$Q = q_n\frac{\partial^n}{\partial t^n} + q_{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}} + \cdots + q_0.$$

The $m+n+1$ coefficients $p_{m-1}, \cdots, p_0, q_n, \cdots, q_0$ are constants characterizing the mechanical properties of the material. Equation (1) can also be considered as the general stress strain relation of an incompressible viscoelastic medium. In this case, ϵ may be taken as denoting any component of the strain tensor and s as denoting the corresponding component of the deviatoric part of the stress tensor.

While Eq. (1) gives a complete mathematical description of the mechanical behaviour of a viscoelastic material in pure shear, it is often found useful to express this behaviour in terms of a mechanical analogue, or model, consisting of springs and dashpots. Figures 1 and 2 show typical models of this kind.

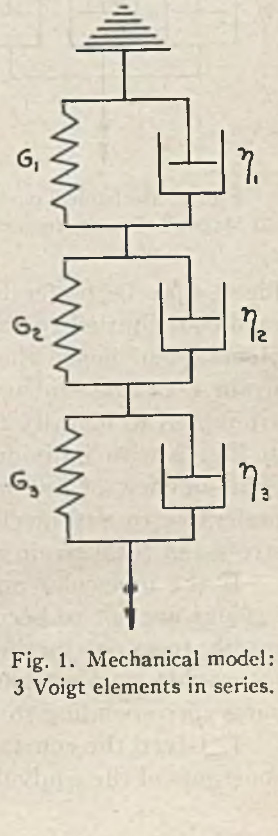

Fig. 1. Mechanical model: 3 Voigt elements in series.

Models of the first type, shown in Fig. 1, consist of retarded elements (Voigt elements) coupled in series. Each

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element is made up of a spring coupled in parallel with a dashpot. In such a model the total extension (corresponding to the strain ϵ) consists of n contributions, one from each of the n Voigt elements. The extension ϵ_i contributed by the ith element is connected with the load s by means of the relation

$$s = 2G_i\epsilon_i + 2\eta_i\dot{\epsilon}_i, \tag{2}$$

where G_i is the spring constant and η_i the dashpot constant of the ith element, and the dot indicates differentiation with respect to time. The load s is the same for all elements coupled in series, and corresponds to the stress in the viscoelastic body. The mechanical behaviour of the model is defined by n equations of the form (2) in conjunction with the relation $\epsilon = \sum \epsilon_i$ which defines the resulting extension ϵ.

Models of the second type, shown in Fig. 2, consist of another kind of composite elements (Maxwell elements) coupled in parallel. Each element is made up of a spring coupled in series with a dashpot. In such a model the total load (corresponding to the stress) is divided among the n elements. The load s_i carried by the ith element is connected with the extension ϵ by means of the relation

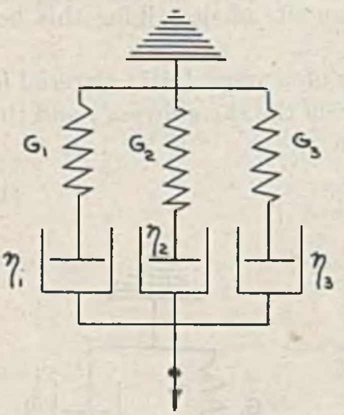

FIG. 2. Mechanical model: 3 Maxwell elements in parallel.

$$\dot{\epsilon} = \frac{1}{2G_i}\dot{s}_i + \frac{1}{2\eta_i}s_i, \tag{3}$$

where G_i and η_i have the same meaning as above. The extension ϵ is the same for all elements coupled in parallel and corresponds to the strain of the viscoelastic material. The mechanical behaviour of the model is defined by n equations of the form (3) together with the relation $s = \sum s_i$ which defines the resulting load s.

In a study of molecular mechanisms of viscoelastic deformation, a model of the type shown in Fig. 1 may be preferable to the general stress-strain relation (1). In such a study, each contribution to the strain may often be identified with some specific molecular process, and hence the strain contributions $\epsilon_1, \epsilon_2, \epsilon_3, \cdots, \epsilon_n$, as well as the total strain ϵ, can be said to possess a physical significance. Likewise some authors have attempted to identify the various stress contributions of a model of the type shown in Fig. 2 with individual "molecular mechanisms of supporting stress." From the point of view of mechanics of continua, on the other hand, the formulation (1) is preferable to any mechanical model, since in any macroscopic study only the total stress and total strain are observable quantities.

If the molecular and the macroscopic methods of approach to viscoelastic behaviour are not to become isolated from one another, it must be possible to change readily from one method of description to the other. It is the purpose of this paper to provide simple techniques for these conversions. The paper is divided into four parts corresponding to the following problems:

1. Given the constants occurring in the stress-strain relation (1), to compute the constants of the equivalent Voigt model.

2. Given the constants occurring in the stress-strain relation (1), to compute the constants of the equivalent Maxwell model.

3. Given the constants of a Voigt model, to compute the constants of the equivalent stress-strain relation.

4. Given the constants of a Maxwell model, to compute the constants of the equivalent stress-strain relation.

1. Determination of the constants of the Voigt model.

A. Nondegenerate case. In the standard or nondegenerate form of the stress-strain relation (1), the operator P is of an order one less than that of Q. The relation (1) thus has the form

$$\frac{\partial^{n-1}s}{\partial t^{n-1}} + p_{n-2}\frac{\partial^{n-2}s}{\partial t^{n-2}} + \cdots + p_0 s = 2q_n \frac{\partial^n \epsilon}{\partial t^n} + \cdots + 2q_0\epsilon. \tag{4}$$

If both the coefficients q_0 and q_n do not vanish, the corresponding mechanical model will consist of n Voigt elements, all nondegenerate. If $q_n=0$, one element of the model consists of a spring only, and if $q_0=0$ one element consists of a dashpot only. These degenerate cases will be considered in the following sections. Cases are also possible where some other coefficient vanishes. This does not affect the form of the resulting model or the nature of the mathematical treatment.

A given Voigt element is defined by its constants G and η. The compliance J is defined as the reciprocal of G; $J=1/G$. The retardation time τ of the element is defined as $\tau=\eta/G=J\eta$. Our problem is to compute, from the $2n$ coefficients of the nondegenerate stress-strain relation the $2n$ parameters of the mechanical model. The method given below depends upon the fact that both the model and the stress-strain relation must give the same prediction as to how the total strain will change with time when a given stress $s(t)$ is applied. It is sufficient to equate the responses to the particular stress $s(t)=t^{n-1}$. The general solution of the equation $P(t^{n-1})=2Q\epsilon$ is the sum of the general solution of the associated homogeneous equation $Q\epsilon=0$ and the particular polynomial solution of the complete equation. In the same way, the response of the model to the stress t^{n-1} is the sum of the general response to a zero stress and a particular polynomial response to the stress t^{n-1}. If the response of the model is to be identical with that predicted by the stress-strain relation, the constants of the model must satisfy certain conditions. First, the retardation times τ_i ($i=1, 2, \cdots, n$) of the Voigt elements are the negative reciprocals of the roots x_i of the characteristic equation $q_n x^n+q_{n-1}x^{n-1}+\cdots+q_0 x+q_0=0$;

$$\tau_i = -\frac{1}{x_i}. \tag{5}$$

Thus, the n retardation times of the model are determined by the general solution of the homogeneous differential equation.

In order to complete the specification of the model the particular polynomial solution must now be used. The particular polynomial solution of the equation $P(t^{n-1}) =2Q\epsilon$ will be of the form

$$2\epsilon(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1}t^{n-1}. \tag{6}$$

The coefficients $a_0, a_1, \cdots, a_{n-1}$ are determined in the usual manner.

In order to obtain the model strain ϵ corresponding to the stress $s=t^{n-1}$, we consider first the behaviour of a single element under this stress. Equation (2) can be written in the form

$$J_i s = 2\epsilon_i + 2\tau_i\dot{\epsilon}_i.$$

Setting $s=t^{n-1}$ and determining the polynomial solution of this differential equation for ϵ_i, we find

$$2\epsilon_i = (n-1)!J_i\left[\frac{t^{n-1}}{(n-1)!} - \tau_i\frac{t^{n-2}}{(n-2)!} + \tau_i^2\frac{t^{n-3}}{(n-3!)} - \cdots - (-1)^{n-1}\tau_i^{n-1}t - (-1)^n\tau_i^n\right]. \qquad (7)$$

Since the total strain is $\epsilon=\sum\epsilon_i$, comparison of (6) and (7) shows that the compliances J_i must satisfy the linear equations

$$\left.\begin{aligned}
a_{n-1} &= \sum_{i=1}^{n} J_i, \\
a_{n-2} &= (n-1)\sum_{i=1}^{n} J_i/x_i, \\
a_{n-3} &= (n-1)(n-2)\sum_{i=1}^{n} J_i/x_i^2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
a_0 &= (n-1)!\sum_{i=1}^{n} J_i/x_i^{n-1},
\end{aligned}\right| \qquad (8)$$

where the retardation times τ_i have been expressed in terms of the roots x_i of the characteristic equation in accordance with (5).

The Voigt model is completely specified when the compliance and retardation time of each Voigt element are determined.

B. Degenerate case; $q_0=0$. If the coefficient q_0 of the differential operator Q is zero, one of the roots, x_1 say, of the characteristic equation vanishes. This indicates that the spring constant G of the first element is zero, the element consisting of a dashpot only. In this case the compliances J_i can no longer be found from the linear equations (8) because of the infinite terms $1/x_1$, $1/x_1^2$, $\cdots$. A parameter may be substituted for the zero coefficient q_0, the $2n$ parameters $\eta_1, G_1, \eta_2, G_2, \cdots, \eta_n, G_n$, may be determined in terms of this parameter, and finally the parameter may be allowed to approach zero and the limiting values of these $2n$ parameters obtained. However, this involves a complicated procedure even in simple cases. The following alternative treatment of this degenerated case seems preferable.

The total extension ϵ of the model consists of the extension ϵ_1 of the degenerate first element and the extension ϵ' of the chain of $n-1$ nondegenerate elements; $\epsilon=\epsilon_1+\epsilon'$. The mechanical behaviour of the degenerate element is given by

$$\frac{1}{\eta_1}s = 2\dot{\epsilon}_1, \qquad (9)$$

and that of the chain of $n-1$ nondegenerate elements, by a relation of the form

$$P's = 2Q'\epsilon', \tag{10}$$

where the differential operators P' and Q' are of the orders $n-2$ and $n-1$, respectively. Applying the operator Q' to both sides of (9), differentiating (10) with respect to time and adding, we obtain

$$P'\dot{s} + \frac{1}{\eta_1}Q's = 2Q'(\dot{\epsilon}' + \dot{\epsilon}_1) = 2Q'\dot{\epsilon}. \tag{11}$$

With

$$\left.\begin{aligned} P' &= \frac{\partial^{n-2}}{\partial t^{n-2}} + p'_{n-3}\frac{\partial^{n-3}}{\partial t^{n-3}} + \cdots + p'_0, \\ Q' &= q'_{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}} + q'_{n-2}\frac{\partial^{n-2}}{\partial t^{n-2}} + \cdots + q'_0, \end{aligned}\right| \tag{12}$$

Eq. (11) becomes

$$\left[\left(1 + \frac{q'_{n-1}}{\eta_1}\right)\frac{\partial^{n-1}}{\partial t^{n-1}} + \left(p'_{n-3} + \frac{q'_{n-2}}{\eta_1}\right)\frac{\partial^{n-2}}{\partial t^{n-2}} + \cdots + \left(p'_0 + \frac{q'_1}{\eta_1}\right)\frac{\partial}{\partial t} + \frac{q'_0}{\eta_1}\right]s$$
$$= 2\left[q'_{n-1}\frac{\partial^n}{\partial t^n} + q'_{n-2}\frac{\partial^{n-1}}{\partial t^{n-1}} + \cdots + q'_0\frac{\partial}{\partial t}\right]\epsilon. \tag{13}$$

When both sides of (13) are divided by the coefficient of the highest order term on the left-hand side, this equation must be identical with the stress-strain relation (1) in which $q_0 = 0$. Comparison of the lowest order terms leads to the relation

$$\eta_1 = q_1/p_0;$$

comparison of the highest order terms leads to

$$1 + \frac{q'_{n-1}}{\eta_1} = \left[1 - \frac{p_0}{q_1}q_n\right]^{-1}.$$

Abbreviating this expression by r, we find by further comparison of coefficients in (13) and (1) that

$$\left.\begin{aligned} q'_{n-1} &= rq_n,\ q'_{n-2} = rq_{n-1},\ \cdots,\ q'_0 = rq_1 \\ p'_{n-3} &= r\left(p_{n-2} - \frac{p_0}{q_1}q_{n-1}\right), \\ p'_{n-4} &= r\left(p_{n-3} - \frac{p_0}{q_1}q_{n-2}\right), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ p'_0 &= r\left(p_1 - \frac{p_0}{q_1}q_2\right). \end{aligned}\right| \tag{14}$$

The differential operators (12) are thus determined, and the procedure outlined under 1A permits the determination of the constants of the $n-1$ nondegenerate elements.

C. Degenerate case; $q_n=0$. If the coefficient q_n of the operator Q is zero, one Voigt element of the corresponding model consists of a spring only. The compliance of this isolated spring can be shown to equal $1/q_{n-1}$. A procedure similar to the one developed above will permit to determine the constants of the nondegenerate elements.

2. Determination of the constants of the Maxwell model.

A. Nondegenerated case; A nondegenerate model of the Maxwell type corresponds to stress-strain relation (1) in which $q_n=0$ and $q_0=0$ (i.e., to a doubly degenerate Voigt model). When the operators are of this standard form, the model will consist of m Maxwell elements in parallel. The $2m$ constants of this model can be computed from the $2m$ coefficients of the stress-strain relation by a method which, except for the interchange of stress and strain, is almost identical with that of Section 1.

For any given imposed strain sequence $\epsilon(t)$, the stress must vary in a definite fashion $s(t)$. The predictions of Eq. (1) and the set of differential equations (3) must be identical for every case—in particular, for the strain sequence $\epsilon(t)=t^m$. The results are as follows:

The m relaxation times of the m Maxwell elements are the negative reciprocals of the m roots of the characteristic equation $x^m+p_{m-1}x^{m-1}+\cdots+p_1x+p_0=0$;

$$\tau_i = -\frac{1}{x_i}. \tag{15}$$

The specification of the model is completed by determination of the m dashpot constants $\eta_1\cdots\eta_m$. These are obtained by solving the following set of m linear equations.

$$\left.\begin{aligned} a_{m-1} &= m\sum_{i=1}^{m}\eta_i,\\ a_{m-2} &= m(m-1)\sum_{i=1}^{m}\eta_i/x_i,\\ &\cdots\cdots\cdots\cdots\\ a_0 &= m!\sum_{i=1}^{m}\eta_i/x_i^{m-1}, \end{aligned}\right| \tag{16}$$

where the a_i are the coefficients of the particular polynomial solution

$$s(t) = a_0 + a_1t + a_2t^2 + \cdots + a_{m-1}t^{m-1}$$

of the differential equation

$$Ps = 2Qt^m.$$

B. Degenerate case; $q_n\neq0$, $q_0=0$. If the order of the operator Q is one greater than that of P (i.e., if $q_n\neq0$), then one Maxwell element of the model consists of a dashpot only. The constant of this isolated dashpot is found to equal q_n. A procedure patterned on that of Section 1B will permit the determination of the constants of the remaining nondegenerate elements.

C. Degenerate case; $q_0\neq0$, $q_n=0$. If the coefficient q_0 does not vanish, then the model contains one element which consists of a spring only. The constant G_1 of this spring, is found to equal q_0/p_0. The constants of the remaining nondegenerate elements can again be determined by a procedure similar to that of Section 1B.

3. Determination of the operators P and Q from the constants of a Voigt model. Consider a material whose behavior in shear is reproduced by a model consisting of n Voigt elements in series. The $2n$ parameters of this model are known. The equivalent relationship between stress and strain can be determined by either of two straightforward methods.

1. The method of part 1A can be used in reverse. This immediately gives an operator which is directly proportional to Q.

$$\lambda Q = \prod_{i=1}^{n}\left(\frac{\partial}{\partial t} + \frac{1}{\tau_i}\right), \tag{17}$$

where λ is an undetermined multiplier.

The operator P can subsequently be determined by equating the particular polynomial responses to a stress $s = t^{n-1}$.

2. The mechanical behaviour of the Voigt model is expressed by the following set of equations:

$$\left.\begin{aligned} s &= 2G_1\epsilon_1 + 2\eta_1\dot{\epsilon}_1, \\ s &= 2G_2\epsilon_2 + 2\eta_2\dot{\epsilon}_2, \\ &\;\;\vdots \\ s &= 2G_n\epsilon_n + 2\eta_n\dot{\epsilon}_n, \\ \epsilon &= \sum_{i=1}^{n}\epsilon_i. \end{aligned}\right\} \tag{18}$$

The nth equation can be rewritten, as

$$s = 2G_n\left(\epsilon - \sum_{i=1}^{n-1}\epsilon_i\right) + 2\eta_n\left(\dot{\epsilon} - \sum_{i=1}^{n-1}\dot{\epsilon}_i\right). \tag{19}$$

If each of these equation is differentiated $(n-1)$ times, a total of n^2 equations will result. These equations will contain (n^2-1) derivatives of the form $\partial^r\epsilon_i/\partial t^r$. All of these derivatives can be eliminated, leaving a differential relation between s and ϵ, by multiplying each of the n^2 equations by an appropriate factor and adding. The determination of the factors may, of course, be rather cumbersome.

3. The problem can, however, be simplified by a judicious combination of methods (1) and (2). We determine first the operator λQ in accordance with (17). We then formulate the set of n^2 equations considered above. n of the necessary n^2 factors can immediately be written down. They are obtained from the coefficients of the operator (17). The form of the n^2 equations is such that the remaining factors can be evaluated one at a time if the above set of n factors is known. The result of this procedure is the desired operator equation.

4. Determination of the operators P and Q from the constants of a Maxwell Model. Consider a material whose behaviour in shear can be reproduced by a model consisting of n Maxwell elements in parallel. The $2n$ parameters of this model are known. The equivalent relationship $Ps = 2Q\epsilon$ can be determined by methods almost identical with those of Section 3. Only the simplified third method will be repeated here.

The operator P is given by the equation

$$\lambda P = \prod_{i=1}^{n} \left(\frac{\partial}{\partial t} + \frac{1}{\tau_i} \right), \tag{20}$$

where λ is again an undetermined multiplier. The mechanical behaviour of the model is expressed by the equations

$$\left. \begin{aligned} \dot{\epsilon} &= \frac{1}{2G_1} \dot{s}_1 + \frac{1}{2\eta_1} s_1 \\ &\vdots \\ \dot{\epsilon} &= \frac{1}{2G_n} \left(\dot{s} - \sum_{i=1}^{n-1} \dot{s}_i \right) + \frac{1}{2\eta_n} \left(s - \sum_{i=1}^{n-1} s_i \right) \end{aligned} \right| \tag{21}$$

If each of these equations is differentiated $(n-1)$ times, a total of n^2 equations are obtained, involving n^2-1 derivatives of the form $\partial^r s/\partial t^r$. All of these derivatives can be eliminated, leaving the desired stress-strain relation, by multiplying each equation by an appropriate factor and adding. The n coefficients of the operator (20) provide n of these factors. The remaining (n^2-n) factors can then be obtained one at a time.

RAYLEIGH WAVES AND FREE SURFACE REFLECTIONS*

BY

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1. Introduction. The theory of waves associated with the plane boundary of a semi-infinite, isotropic, homogeneous, perfectly elastic medium was first given by Lord Rayleigh,[1] who discussed the problem for plane waves of fixed frequency. Many papers have been written giving treatments of variations of the problem studied by Rayleigh but the treatment in Rayleigh's original paper contained most of the results of interest for plane waves.

Had Lord Rayleigh realized the great practical importance of his surface waves, he would doubtless have included more numerical results in his original paper, and the material of the present paper would have been more or less completely included therein. Rayleigh waves are important in the seismic method of oil exploration since they generally occur as a troublesome noise on reflection seismograms.

In the present paper, we are interested in the theory of the reflection of a plane compressional incident wave at the free surface. In this theory a cubic expression occurs which also occurs in the theory of the Rayleigh waves. Furthermore our interest is primarily in obtaining numerical results, so that examples may be readily pictured. The results are primarily of theoretical interest since our waves are never plane and the medium is only rarely approximately homogeneous.

2. Reflection at the free surface. This problem originally treated by Knott[2] and Zoeppritz[3] leads to the relation

$$\frac{R}{I} = \frac{v^2 \sin 2r_1 \sin 2i - V^2 \cos^2 2r_1}{v^2 \sin 2r_1 \sin 2i + V^2 \cos^2 2r_1},$$

where R, I, r_1, i, v, and V are respectively the amplitude of the reflected compressional wave, the amplitude of the incident compressional wave, the reflection angle of the shear wave, the angle of incidence, the velocity of the shear wave, and the velocity of the compressional wave.

We may make the substitutions $w = \sin^2 r_1 = p^2v^2$ and $s = \lambda/\mu = 2\sigma/(1-2\sigma)$ and obtain

$$\frac{R}{I} = \frac{4w(1-w)^{1/2}[1-(s+2)w]^{1/2} - (s+2)^{1/2}(1-2w)^2}{4w(1-w)^{1/2}[1-(s+2)w]^{1/2} + (s+2)^{1/2}(1-2w)^2}$$

$$= \frac{N(s, w)}{D(s, w)} = \frac{N^2}{ND} = \frac{ND}{D^2}. \tag{1}$$

Now real values of both $N(s, w)$ and $D(s, w)$ are graphed on Fig. 1, for various values

* Received Dec. 2, 1944.

[1] Lord Rayleigh, *On waves propagated along the plane surface of an elastic solid*, Proc. Lond. Math. Soc. 17, 4–11 (1887).

[2] C. G. Knott, *Reflection and refraction of elastic waves*, Phil. Mag. (5) 48, 64–97 (1899).

[3] K. Zoeppritz, *Über Erdbebenwellen VIIB*, Göttinger Nachrichten 1919, 66–84.

of s or σ. It is seen that the graphs break up into two sets, one for $w \leqq 1/(s+2)$ and the other for $w \geqq 1$. When $w=1/(s+2)$ or $w=1$, the graphs of N and D both have vertical tangents. It will be observed that N has two or no real zeros between 0 and $1/(s+2)$ while D has one real zero $\geqq 1.09574$. The double zero corresponds to

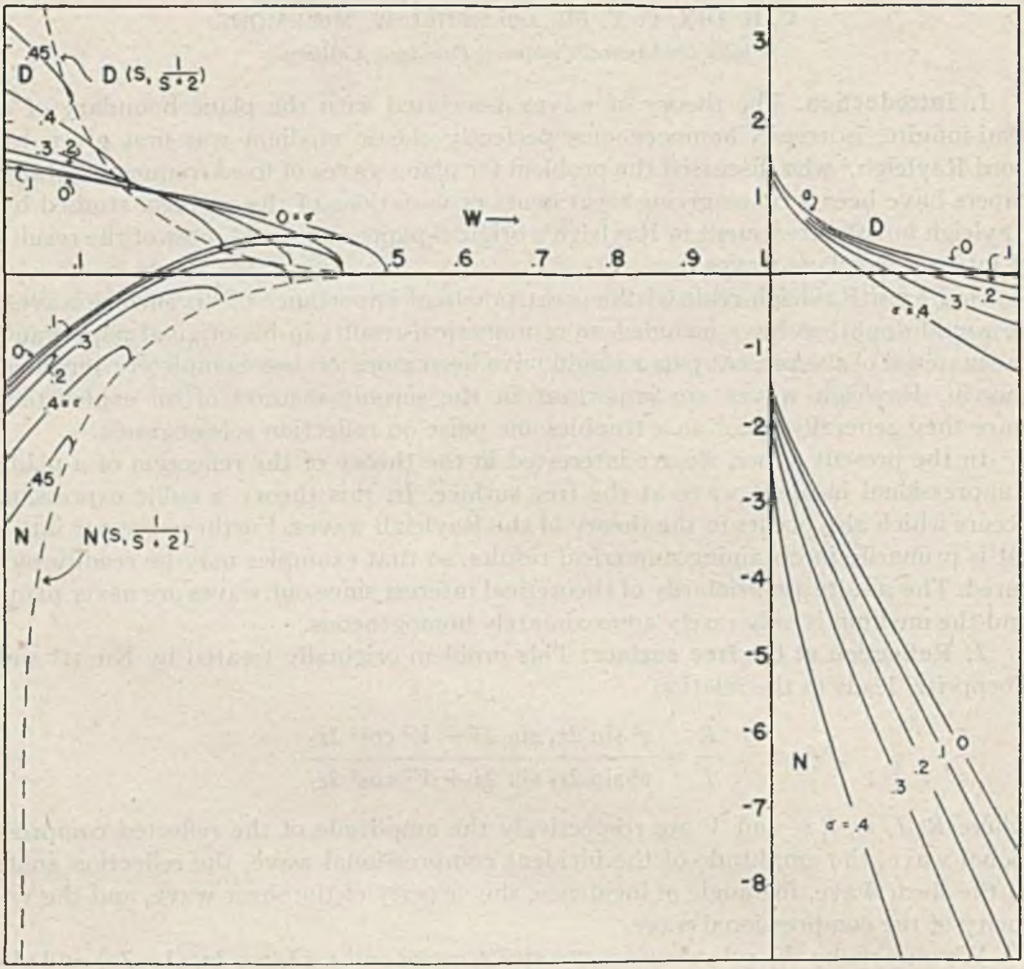

FIG. 1.

$\sigma=0.26308207$, $w=0.27969015$. For large values of w we have $D \sim -2(s+1)(s+2)^{-1/2}w$ and $N \sim -8(s+2)^{1/2}w^2$. Hence the real zeros for $w>0$ are fully accounted for.

There is some interest in the plot of R/I against i for various σ's. This is shown in Fig. 2. Observe that the tangent to the curves at $i=0°$ is horizontal and is vertical at $i=90°$. Observe also that a discontinuity exists for $\sigma=0$ at $i=90°$ since $\lim_{i\to 90°}(R/I)_{\sigma=0}=+1$ and $(R/I)_{i=90°, 0<\sigma<1/2}=-1$. One may note that the zeros of (R/I) correspond to i's, for the given σ, for which there is no reflection of energy in the wave of compressional type. The change of sign of (R/I) corresponds to a phase reversal.

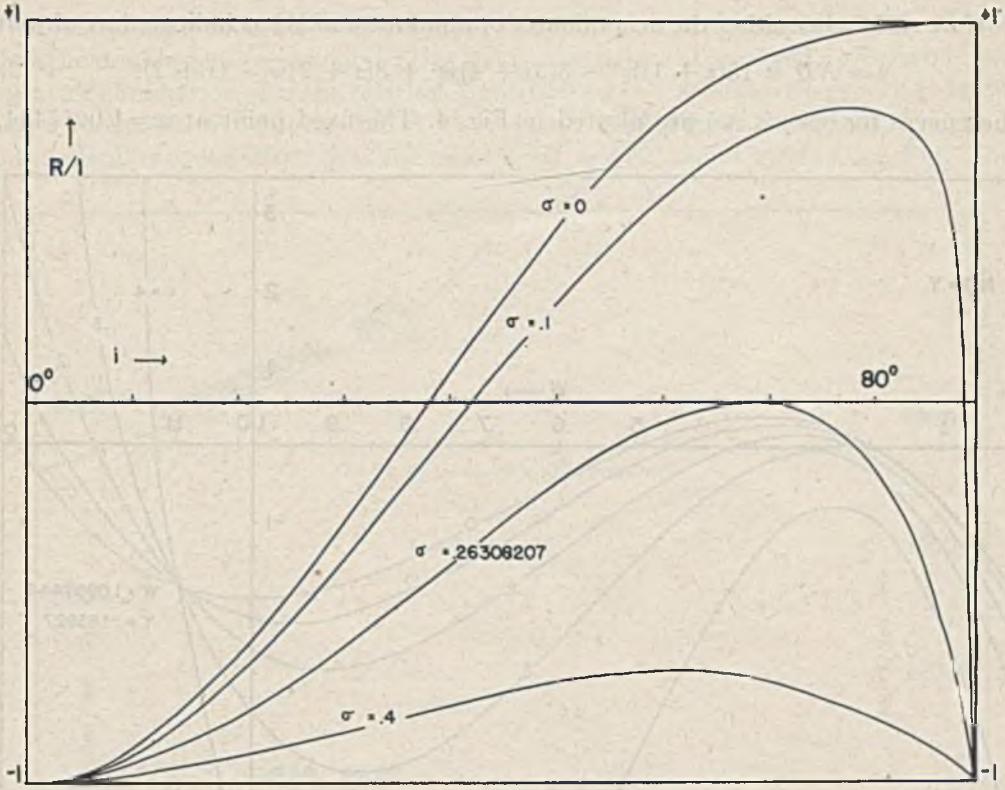

Fig. 2.

Since the zeros of $N(s, w)$ are of considerable interest we have plotted i_0 against σ (where i_0 is incidence angle for which $N=R=0$) in Fig. 3. This graph brings out a point which is indeed curious, namely if $\sigma=0.15$ then for $i_0=87.76°$ all the reflected energy appears in the shear wave, whereas, if we add only 2.24° to i all the reflected energy appears in the compressional wave. Birch[4] records Poisson ratios for granite blocks of 0.093, 0.096, 0.116, 0.086 and 0.109. These are selected low values. It is established in a later paragraph that $(di_0/d\sigma)_{\sigma=0}=-\infty$ for the upper branch and $(di_0/d\sigma)_{\sigma=0}=+0.178$ for the lower branch and $(di_0/d\sigma)_{\sigma=0.263}=\infty$ for the upper and lower branches.

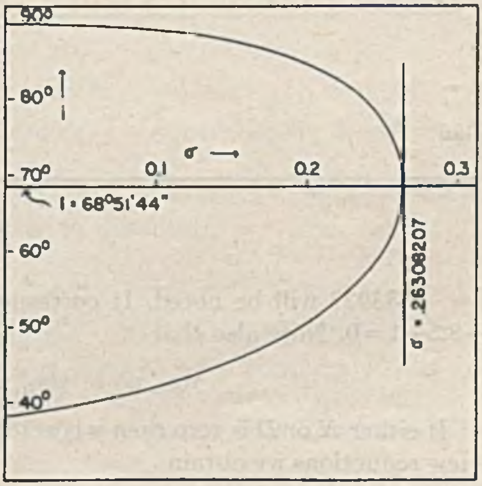

Fig. 3.

[4] Francis Birch, *Handbook of physical constants*, Special Paper no. 36, Geol. Soc. of Amer., pp. 73–74 (1942).

If we rationalize either the denominator or numerator of (1) as indicated we obtain

$$y \equiv ND \equiv 16(s+1)w^3 - 8(3s+4)w^2 + 8(s+2)w - (s+2). \tag{2}$$

The curves for $y = y(s, w)$ are plotted in Fig. 4. The fixed point at $w = 1.0957444$,

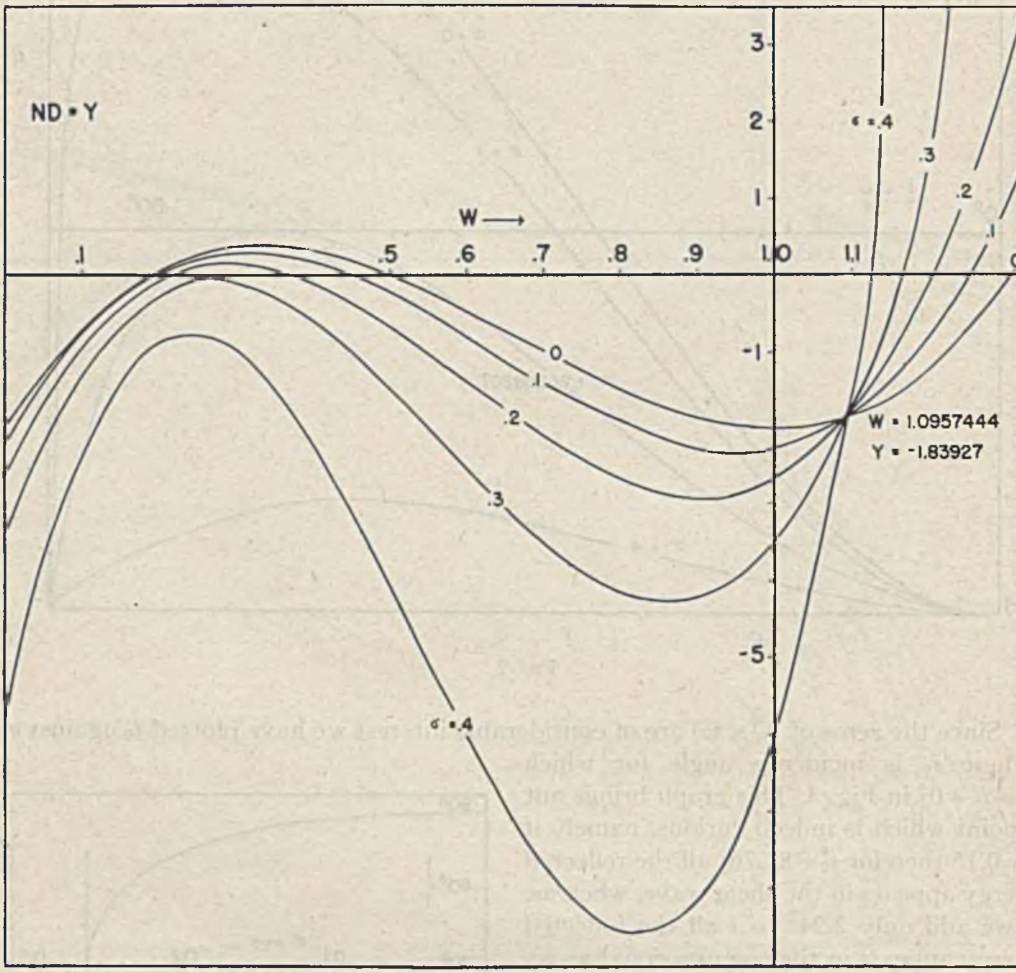

FIG. 4.

$y = -1.83927$ will be noted. It corresponds to the w for which $\partial y/\partial s = 16w^3 - 24w^2 + 8w - 1 = 0$. Note also that

$$y(s_1, w) - y(s_2, w) = (s_1 - s_2)(\partial y/\partial s). \tag{3}$$

If either N or D is zero then y is zero. Assume $y = 0$ while s and w vary. Then after a few reductions we obtain

$$\frac{di_0}{d\sigma} = -2\frac{(s+1)^2(s+2)}{\sin 2i_0}\cdot\frac{(\partial y/\partial s)}{(\partial y/\partial w)_{w=w_0}}. \tag{4}$$

This relationship should be kept in mind in connection with the plot of Fig. 2. Note that the double zero of the cubic (2) is the common zero of $y=0$ and $\partial y/\partial w=0$ (yielding after elimination of w the relation $33s^3+12s^2-27s-30=0$, so that $s_0=1.11043541$ and $\sigma=0.26308207$ while $i_0=68°51'44''$;[5] hence the $\partial y/\partial w$ in (4) is zero, leading to the vertical tangent. Note that the point $\sigma=0$, $i_0=90°$ is not attained on Fig. 3 but

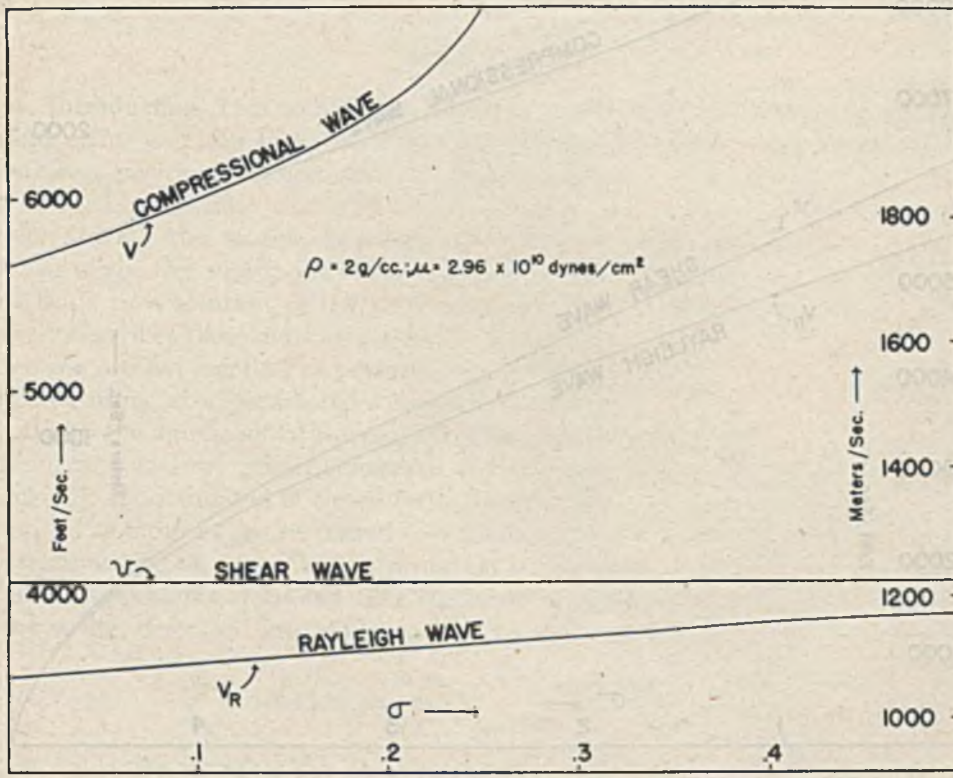

FIG. 5.

that as we approach this point from $\sigma>0$, $di_0/d\sigma\to-\infty$ because $\sin 2i_0\to 0$ whereas the other factors are bounded away from zero.

3. Rayleigh waves. In the usual theory of Rayleigh waves[6] a cubic equation occurs which, using our s-notation, can be written in the form,

$$16(s+1)-8(3s+4)\mathfrak{w}+8(s+2)\mathfrak{w}^2-(s+2)\mathfrak{w}^3=0, \tag{5}$$

where $\mathfrak{w}=V_R^2/v^2$, v and V_R being respectively the velocities of the shear wave and Rayleigh wave. Thus when $y=0$, $w=\mathfrak{w}^{-1}$. But $w=\sin^2 r_1=p^2v^2$ where $p=\sin i/V=\sin r_1/v$. Hence $p^2v^2=v^2/V_R^2$ so $p=1/V_R$. Thus p, which for real values of i or r_1, can be interpreted in terms of the reciprocal of the velocity with which the wave

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[5] B. Gutenberg, *Energy ratio of reflected and refracted seismic waves*, Bull. Seis. Soc. Amer. 34, 85–102 (1944).

[6] J. B. Macelwane, *Theoretical seismology*, Part I, Wiley & Sons, New York, 1936, p. 114, Eq. (5.33).

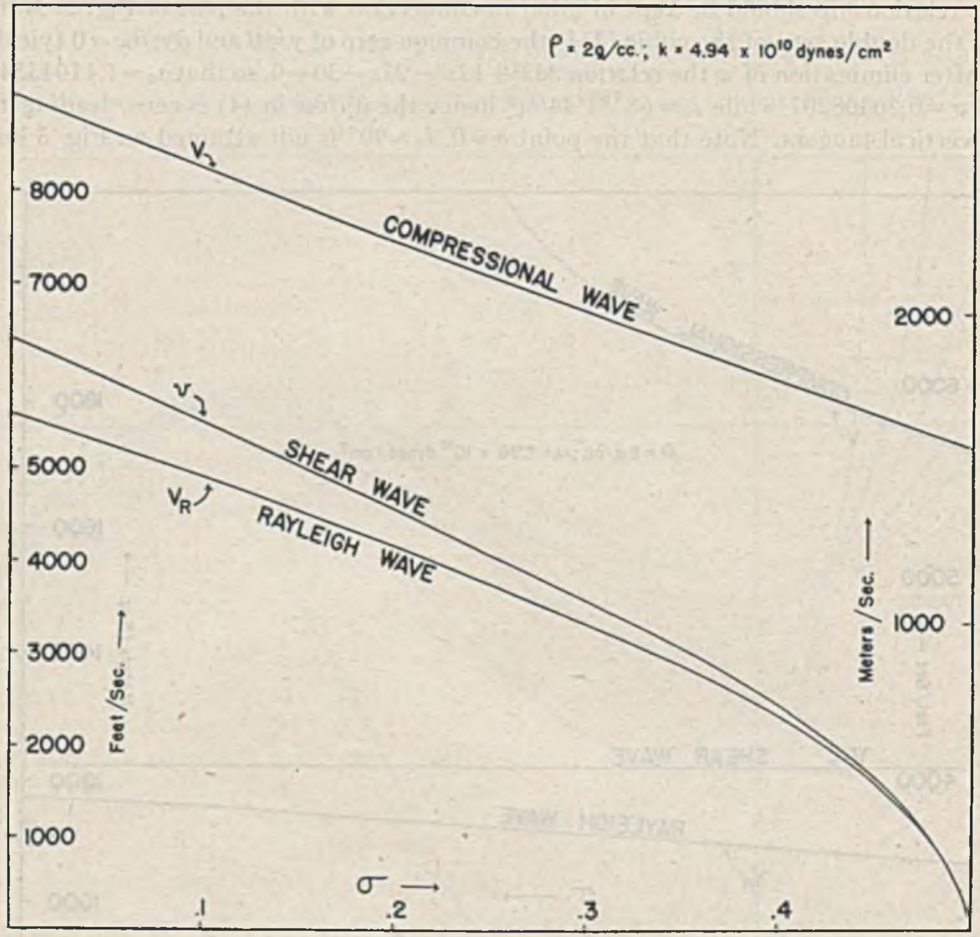

Fig. 6.

sweeps along the surface, $\partial t/\partial x$, has the same interpretation in the case of the Rayleigh wave.

The Rayleigh wave case corresponds to the zero of $D(s, w)$ which in turn yields a poristic problem when $I\equiv 0$.[7]

The Rayleigh wave velocities can be readily computed by solving the cubic equation or by an inspection of Fig. 4. However, Figs. 5 and 6 show V, v, and V_R for the respective cases where $\mu=\text{constant}=2.96\times 10^{10}$ dynes/cm.^{2}, and where $k=(\mu/3)(3s+2)=\text{constant}=4.94\times 10^{10}$ dynes/cm.^{2} = modulus of compression for various values of σ.

[7] T. Sakai, *On the propagation of tremors over plane surface*, Geophysical Magazine, Tokyo, 8, 1–71 (1934).

ON THE NON-LINEAR VIBRATION PROBLEM OF THE ELASTIC STRING*

BY

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1. Introduction. It is well known that the classical linearized analysis of the vibrating string can lead to results which are reasonably accurate only when the minimum (rest position) tension and the displacements are of such magnitude that the relative change in tension during the motion is small. The following analysis of the free vibrations of the string with fixed ends leads to a solution of the problem which adequately describes those motions for which the changes in tension are not small. The perturbation method is adopted, using as a parameter a quantity which is essentially the amplitude of the motion. The periodic motions arising from initial sinusoidal deformations are closely approximated in closed form. The method is applied to motions not restricted to a single plane and finally the exact solution for the transmission of a localized deformation is indicated.

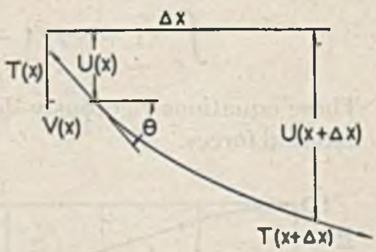

FIG. 1. Displaced element of string.

2. The equations of motion. The equations of dynamic equilibrium of an element of the string, deformed into a plane curve as shown in Fig. 1, are

$$\frac{\partial}{\partial x}\left[T\sin\theta\right] = \rho A\frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial}{\partial x}\left[T\cos\theta\right] = \rho A\frac{\partial^2 v}{\partial t^2}, \tag{1}$$

where ρ denotes the mass per unit volume, A the cross-sectional area of the string in the rest position, and $\theta = \text{arc tan } [u'/(1+v')]$, the primes indicating differentiation with respect to x. The condition of fixed ends implies that,

$$\int_0^l v' dx = 0 \quad \text{for all } t. \tag{2}$$

The stress-strain relation of the string is assumed in the form,

$$T - T_0 = EA\left\{\left[(1+v')^2 + (u')^2\right]^{1/2} - 1\right\}, \tag{3}$$

where T_0 is the tension in the rest position and E is a constant characteristic of the string material. The following dimensionless quantities are introduced to simplify the algebraic work

$$\alpha^2 = \frac{T_0}{EA}, \qquad \tau = \frac{T - T_0}{T_0}, \qquad \xi = \frac{\pi x}{l}, \qquad \eta = \frac{\pi}{l}\left(\frac{T_0}{\rho A}\right)^{1/2} t.$$

After differentiating Eqs. (1) with respect to x, setting

* Received Jan. 3, 1945.

$$\sin\theta = \frac{u'}{[(u')^2 + (1+v')^2]^{1/2}} = \frac{u'}{1+\alpha^2\tau} = \varphi,$$
$$\cos\theta = (1-\varphi^2)^{1/2},$$

and eliminating v' between Eqs. (1) and (2), we obtain

$$\frac{\partial^2}{\partial\xi^2}[(1+\tau)\varphi] = \frac{\partial^2}{\partial\eta^2}[(1+\alpha^2\tau)\varphi], \tag{4a}$$

$$\frac{\partial^2}{\partial\xi^2}[(1+\tau)(1-\varphi^2)^{1/2}] = \frac{\partial^2}{\partial\eta^2}[(1+\alpha^2\tau)(1-\varphi^2)^{1/2}], \tag{4b}$$

$$\int_0^\pi (1+\alpha^2\tau)(1-\varphi^2)^{1/2}d\xi = \pi. \tag{4c}$$

These equations rigorously define the motion of the string which is acted on by on external forces.

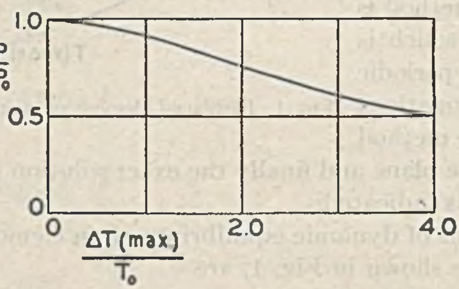

FIG. 2. Comparison of periods obtained by linear and non-linear theories.

$$\frac{P}{P_0} = \frac{\text{non-linear period}}{\text{linear period}};$$

$$\frac{\Delta T_{max}}{T_0} = \frac{\epsilon^2}{4} = \left(\frac{\text{“amplitude”}}{2\alpha}\right)^2.$$

Motion defined by Eqs. (15). $\alpha \neq 0$.

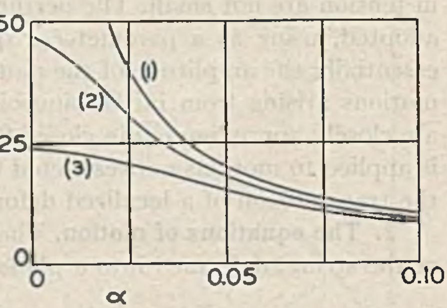

FIG. 3. Period v.s. initial tension.*

$$\frac{P}{P^*} = \frac{\text{non-linear period}}{(E\pi^2/\rho l^2)^{1/2}}; \quad \alpha^2 = T_0/EA$$

(1) vanishing amplitude (linear theory)
(2) $a = \alpha\epsilon =$ “amplitude” $= 0.05$
(3) $a = 0.10$

3. The perturbation procedure. It is convenient to choose, as the perturbation parameter of the problem, a number ϵ which is essentially the amplitude of the motion.** The two functions φ and τ are therefore expanded in powers of this parameter as follows:

$$\varphi = \alpha[\epsilon\varphi_1 + \epsilon^3\varphi_3 + \epsilon^5\varphi + \cdots], \qquad \tau = \epsilon^2\tau_2 + \epsilon^4\tau_4 + \cdots \tag{5}$$

It is easily seen that a reversal of the sign of ϵ should merely reverse the sign of φ. Hence the omission of the even powers of ϵ is justified. In a similar manner the funtions $\tau_1, \tau_3, \cdots$ can be seen to vanish. That τ_0 vanishes is seen by inspection of Eq. (4c). The expressions for φ and τ are now substituted into Eqs. (4), the coeffi-

* In Fig 3. the ordinates should be labeled P/P^*.

** Equation (15) indicates more precisely the meaning of ϵ.

cients of each power of ϵ are equated to zero, and the following system of equations is obtained:

$$L_0(\varphi_1) = 0, \quad \text{(6a)}, \qquad L_1(\tau_2) = -\alpha^2 L_0\left(\frac{\varphi_1^2}{2}\right), \quad \text{(6b)}$$

$$L_0(\varphi_3) = L_1(\tau_2\varphi_1), \quad \text{(6c)}, \qquad L_1(\tau_4) = -\alpha^2 L_0\left(\varphi_1\varphi_3 - \frac{\alpha^2\varphi_1^4}{8}\right), \quad \text{(6d)}$$

$$L_0(\varphi_5) = L_1(\tau_4\varphi_1 + \tau_2\varphi_3), \quad \text{(6e)}, \qquad L_1(\tau_6) = -\alpha^2 L_0\left(\varphi_1\varphi_5 + \frac{\varphi_3^2}{2} + \alpha^2\frac{\varphi_1^3\varphi_3}{2} + \alpha^4\frac{\varphi_1^6}{48}\right), \quad \text{(6f)}$$

. . . . ,

where

$$L_0 = \frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2}, \qquad L_1 = \alpha^2\frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial\xi^2};$$

and

$$\int_0^\pi \left(\tau_2 - \frac{\varphi_1^2}{2}\right) d\xi = 0, \quad \text{(7a)}$$

$$\int_0^\pi \left(\tau_4 - \varphi_1\varphi_3 + \frac{\alpha^2\varphi_1^4}{8}\right) d\xi = 0, \quad \text{(7b)}$$

. . . .

Since each of the operators in the foregoing equations is linear, it is now a simple matter to evaluate successively the φ_i and the τ_i. For the moment, we confine our attention to the motion defined by choosing as a solution to Eq. (6a) the function,

$$\varphi_1 = \cos\xi\cos\eta. \quad \text{(8a)}$$

Note that for $\varphi_1 = \cos n\xi \cos n\eta$ the same solution will exist when l is replaced by l/n in the definitions of ξ and η. Solving successively Eqs. (6), starting with the foregoing definition of φ_1, and using Eqs. (7) to determine the arbitrary terms appearing in the τ_i, we obtain

$$\tau_2 = \frac{1}{4}\cos^2\eta + \frac{\alpha^2}{8}\cos 2\xi, \quad \text{(8b)}$$

$$\varphi_3 = \cos\xi\left[-\frac{3 - 2\alpha^2 - \alpha^4}{32}\eta\sin\eta + \frac{1 - 9\alpha^2}{128}\cos 3\eta - \frac{1}{128}\cos\eta\right] - \frac{\alpha^2(9 - \alpha^2)}{128}\cos 3\xi\cos\eta, \quad \text{(8c)}$$

$$\tau_4 = \left[-\frac{3 - 2\alpha^2 - \alpha^4}{128}\eta\sin 2\eta + \frac{1 - 9\alpha^2}{512}(\cos 2\eta + \cos 4\eta) - \frac{3\alpha^2}{512}\cos^4\eta - \frac{1}{256}\cos^2\eta\right] + \frac{\alpha^4(21 - \alpha^2)}{512}\cos 2\xi + \cdots - \frac{3\alpha^2}{2048}\frac{13 - \alpha^2}{4 - \alpha^2}\cos 4\xi\cos 2\eta, \quad \text{(8d)}$$

$$\varphi_5 = \cos\xi\left[-\frac{9(1+\alpha^2+\cdots)}{2048}\eta^2\cos\eta - \frac{3}{512}\eta\sin\eta - \frac{9}{4096}\eta\sin 3\eta + 2^{-14}\cos 5\eta\right] + \cdots, \tag{8e}$$

. . . .

The arbitrary solutions of Eq. (6a) which may be added to each of the φ_i as they are evaluated have been chosen in such a manner that lim $\alpha\epsilon^j\varphi_j$ exists when α tends to zero and $\alpha\epsilon=\text{constant}=a$.* This limiting process, of course, defines the motion wherein the initial tension T_0 is zero and the amplitude a is non-vanishing. An investigation of this problem will simplify the question of the convergence of the functions φ and τ as defined by Eqs. (5) and (8). When α tends to zero as specified above, the symbols τ and η become meaningless. Hence, we replace them by

$$\sigma = \frac{T}{EA} = \alpha^2\tau, \quad \text{and} \quad \eta = \alpha s.$$

The limiting process then yields the following expressions for the φ_j and the σ_j

$$\begin{aligned}
\alpha\epsilon\varphi_1 &= a\cos\xi,\\
\alpha\epsilon^3\varphi_3 &= a^3\left[-\frac{1}{2!}\left(\frac{s}{2}\right)^2\cos\xi + \frac{9}{128}\cos\xi - \frac{9}{128}\cos 3\xi\right],\\
\alpha\epsilon^5\varphi_5 &= a^5\left[\frac{3}{4!}\left(\frac{s}{2}\right)^4\cos\xi + \frac{5}{2^8}\left(\frac{s}{2}\right)^2\cos\xi - \frac{45}{128}\left(\frac{s}{2}\right)^2\cos 3\xi + f(\xi)\right],\\
\alpha\epsilon^7\varphi_7 &= a^7\left[-\frac{27}{6!}\left(\frac{s}{2}\right)^6\cos\xi + \cdots\right],\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,
\end{aligned} \tag{9}$$

$$\begin{aligned}
\epsilon^2\sigma_2 &= \frac{1}{4}a^2,\\
\epsilon^4\sigma_4 &= a^4\left[-\frac{1}{16}s^2 + \frac{\cos 2\xi}{32} + \frac{37}{256}\right],\\
\epsilon^6\sigma_6 &= a^6\left[\frac{s^4}{128} - \frac{13s^2}{512} + \frac{3s^2}{128}\cos 2\xi\right]\\
&\quad + \left[\sigma_4\frac{\varphi_1^2}{2} + \sigma_2\varphi_1\varphi_3 + \sigma_2\frac{\varphi_1^4}{8}\right]\alpha^2\epsilon^6 + g(\xi),\\
\epsilon^8\sigma_8 &= a^8\left[-\frac{s^6}{1280} + \cdots\right],\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned} \tag{10}$$

These solutions may also be obtained, of course, by assuming α equal to zero at

* Such complementary solutions are usually chosen to be consistent with a given set of initial conditions. However, it is convenient here to choose them so that the solution does not become meaningless when $\alpha\to 0$, $\alpha\epsilon=a$. Equations (15) indicate that this choice leads to a solution corresponding to a nearly sinusoidal initial deformation.

the outset, expanding φ and σ in powers of a parameter a, and proceeding in the foregoing manner.

Note that the leading terms of the φ_j define an absolutely converging series for all a. Note also that the remaining terms of each φ_j are dominated by this leading term. In fact, for sufficiently large s, the sum of the remaining terms in each φ_j is as small as we please compared to this leading term. Although this dominance has not been shown to occur uniformly, it is to be expected that the series defined by Eqs. (9) and (10) will converge over some range of a. The requirement, "sufficiently large s" introduces no difficulty since the initial value of s may be chosen arbitrarily large.

The functions φ and σ are now most conveniently written in the forms

$$\begin{aligned} \varphi(\xi, s, a) &= af_1(as, \xi) + a^3f_3(as, \xi) + \cdots, \\ \sigma(\xi, s, a) &= a^2g_2(as, \xi) + a^4g_4(as, \xi) + \cdots, \end{aligned} \tag{11}$$

where the terms of the series defining the f_j and the g_j are easily chosen from Eqs. (9) and (10). f_1 and g_2 are composed of the previously mentioned leading terms, and it is easily established that they converge to the values

$$f_1 = \operatorname{cn}\left(\frac{as}{2}, \frac{1}{\sqrt{2}}\right)\cos\xi, \qquad g_2 = \frac{1}{4}\operatorname{cn}^2\left(\frac{as}{2}, \frac{1}{\sqrt{2}}\right), \tag{12}$$

where *cn* denotes the elliptic cosine. Energy considerations may be used to show that the remaining f_j and g_j are bounded, and it is to be expected that the motion is closely described by $\varphi = af_1$ and $\sigma = a^2g_2$ when a is sufficiently small. For most materials, a value of a^2 greatly in excess of 10^{-3} will lead to plastic deformations; hence, the motion of such strings is well defined.

The motions arising when T_0 is arbitrary, as defined by Eqs. (5) and (8), can also be written in the form,

$$\begin{aligned} \varphi &= \alpha\epsilon F_1(\xi, \eta, \epsilon) + \alpha^3\epsilon^3F_3(\xi, \eta, \epsilon) + \cdots + P(\xi, \eta, \alpha, \epsilon), \\ \tau &= \epsilon^2[G_2(\xi, \eta, \epsilon) + \alpha^2\epsilon^2G_4(\xi, \eta, \epsilon + \cdots] + Q(\xi, \eta, \alpha, \epsilon), \end{aligned} \tag{13}$$

where P and Q are those parts of φ and τ which vanish when α tends to zero and $\alpha\epsilon = a$. For this case,

$$F_1 = \operatorname{cn}\left(\sqrt{1 + \frac{\epsilon^2}{4}}\,\eta, k\right)\cos\xi, \qquad G_2 = \frac{1}{4}\operatorname{cn}\left(\sqrt{1 + \frac{\epsilon^2}{4}}\,\eta, k\right). \tag{14}$$

where $k = \epsilon[2(4+\epsilon^2)]^{-1/2}$. It is evident, in view of the foregoing results, that

$$\lim_{\epsilon \to \infty} F_j(\xi, \eta, \epsilon) = f_j(\xi, as)$$

and it is to be concluded that since the series defining the F_i converge as ϵ tends to infinity, they will also converge for the smaller values of ϵ. Both α and $\alpha\epsilon$ must be small because of elastic considerations, which indicates that P and Q will also exist. We conclude therefore that the motion of the string, whose "amplitude" $\alpha\epsilon$ is of the order of magnitude required by elastic considerations, is adequately defined by the leading terms of Eq. (13). That is, in the first approximation,

$$\varphi = \alpha\epsilon \,\mathrm{cn}\left[\sqrt{1 + \frac{\epsilon}{4}}\,\eta, k\right]\cos\xi$$
$$\tau = \frac{\epsilon^2}{4}\,\mathrm{cn}^2\left[\sqrt{1 + \frac{\epsilon^2}{4}}\,\eta, k\right]. \tag{15}$$

Figs. 2 and 3 compare the results of this analysis with those of the linear theory.

4. The motion following an arbitrary initial deformation. The motions derived in the preceding section are obviously those corresponding to initial sinusoidal deformations. If the perturbation procedure is again carried out, and if for φ_1 the function $\varphi_1 = \sum_j b_j \cos j\xi \cos j\eta$ is selected, a solution will be obtained, the leading terms of which contain no powers of α greater than unity. The solution so obtained will correspond to an initial deformation, $\varphi_1(\xi, 0) = \sum_j b_j \cos j\xi$. This predominating part of the solution may, however, be obtained by a simpler, less rigorous, procedure which nevertheless leads to identical results. We merely expand $(1-\varphi^2)^{1/2}$ in the conventional power series and omit in Eqs. (4), φ^{n+2} as compared to φ^n, and α^2 as compared to 1. We thus obtain as replacement for Eqs. (4)

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial\xi^2}\left[(1 + \tau)\varphi\right] = \frac{\partial^2\varphi}{\partial\eta^2}\\
&\frac{\partial^2\tau}{\partial\xi^2} = 0, \quad \text{hence} \quad \tau = \tau(\eta),\\
&\int_0^\pi \left(\alpha^2\tau - \frac{\varphi^2}{2}\right)d\xi = 0.
\end{aligned}\right| \tag{16}$$

Finally the first of these becomes

$$\left[1 + \frac{\alpha^{-2}}{2\pi}\int_0^\pi \varphi^2(\xi, \eta)d\xi\right]\frac{\partial^2\varphi}{\partial\xi^2} = \frac{\partial^2\varphi}{\partial\eta^2}. \tag{17}$$

The solution corresponding to the initial conditions specified at the beginning of this section is found by considering that solution of the form $\varphi = \alpha\sum_j b_j \cos j\xi\psi_j(\eta)$, where $\psi_j(0) = 1$ for each j.

Upon substitution of this function, Eq. (17) yields the following set of ordinary differential equations

$$\psi_n'' + n^2\psi_n\left[1 + \frac{1}{4}\sum_j b_j^2\psi_j^2\right] = 0. \tag{18}$$

These may be written in the conventional operational form

$$(D^2 + n^2)\psi_n = -\frac{n^2}{4}\psi_n\sum_j b_j^2\psi_j^2, \tag{19}$$

and standard integration procedure leads immediately to the integral equation

$$\psi_n(\eta) = \cos n\eta - \frac{\eta}{4}\int_0^\eta \sin n(z - \eta)\psi_n(z)\sum_j b_j^2\psi_j^2(z)dz. \tag{20}$$

The method of successive approximations when applied to this equation will produce a converging sequence of solutions. This method is obviously preferable to the direct application of the perturbation method, once the equivalence of the results has been established, since no minor terms are carried along in the algebraic work, no complimentary solutions need be added as the integration proceeds,* and the τ_i do not appear when the function φ is evaluated.

It is of interest to note that when $b_j \equiv 0$ for $j \neq 1$, Eq. (20) assumes the form

$$\psi_1 = \cos\eta - \frac{b_1^2}{4}\int_0^\eta \sin(z-\eta)\psi_1^3(z)dz, \tag{21}$$

and that this equation must generate the elliptic function previously encountered. When the method of successive approximations is applied to this equation, the series obtained is that one found in the first solution obtained in this paper. This function may be obtained more directly by solving Eq. (18) for this particular set of initial conditions.

Perhaps the quickest way to obtain an approximation to the motion for non-sinusoidal initial deformation is to be found in the application of a numerical procedure using finite differences. Equation (17) lends itself readily to such a treatment and the results are considerably easier to interpret than those found by the more rigorous integral equation treatment.

5. The three dimensional problem. If we now allow deflections w normal to the plane of u, the procedure of the foregoing sections of this paper leads to the equation

$$\left\{1 + \frac{\alpha^{-2}}{2\pi}\int_0^\pi [\varphi^2(\xi,\eta) + \chi^2(\xi,\eta)]d\xi\right\}\frac{\partial^2\varphi}{\partial\xi^2} = \frac{\partial^2\varphi}{\partial\eta^2} \tag{22}$$

and to the equation obtained by interchanging φ and χ in (22). τ is given by the integral on the left side of this equation and $\chi = w/(1+\alpha^2\tau)$. It follows immediately from the similarity of Eq. (17) and that given above that the integral equation method previously described will provide the solutions to problems of this nature. In particular, however, the motions wherein the string at any instant lies in a single plane and wherein each particle describes a quasi-elliptical path is easily determined in closed form by considering the deformation expressed in the complex form

$$\varphi = \epsilon\alpha\,\psi(\eta)e^{i\mu(\eta)}\cos\xi,$$

where ψ and μ are each real. Equation (22) assumes the form,

$$\left[1 + \frac{\epsilon^2}{2}\int_0^\pi |\varphi(\xi,\eta)|^2 d\xi\right]\frac{\partial^2\varphi}{\partial\xi^2} = \frac{\partial^2\varphi}{\partial\eta^2} \tag{23}$$

which, when separated into its real and imaginary parts, implies,

$$\mu'(\eta) = c/\psi^2(\eta)$$

* When dealing with the differential equations leading to Eq. (8), it was necessary to choose complementary solutions to conform to given initial (or other auxiliary) conditions of the problem. In the integral equation approach, such conditions are always included in the equations.

and

$$\psi'' + \psi + \frac{\epsilon^2}{4}\psi^3 - c^2\psi^{-3} = 0. \tag{24}$$

Here, c is a constant defined by the initial conditions as follows;

$$\psi(0) = 1, \quad \psi'(0) = 0, \quad \mu(0) = 0, \quad \mu'(0) = c.$$

When $c<1+\epsilon^2/4$, these initial conditions lead to a solution of Eq. (24) given by

$$\psi = \left[1 - (1-\gamma)\,\mathrm{sn}^2\left\{\sqrt{\frac{1+\beta}{8}}\,\epsilon\eta,\ \sqrt{\frac{1-\gamma}{1+\beta}}\right\}\right]^{1/2} \tag{25}$$

$$\mu = c\int_0^\eta \psi^{-2}(s)ds, \qquad \tau = \frac{\epsilon^2}{4}\psi^2,$$

where

$$\beta = \frac{1}{2\epsilon^2}\left[\sqrt{(8+\epsilon^2)^2 + 32\epsilon^2c^2} + (8+\epsilon^2)\right],$$

$$\gamma = \frac{1}{2\epsilon^2}\left[\sqrt{(8+\epsilon^2)^2 + 32\epsilon^2c^2} - (8+\epsilon^2)\right].$$

Note that as c tends to $1+\epsilon^2/4$, ψ becomes identically unity and the motion of each particle is circular. That is,

$$\varphi = \alpha\epsilon\cos\xi e^{i\eta\sqrt{1+\epsilon^2/4}}. \tag{26}$$

When $c>1+\epsilon^2/4$, integration of Eq. (24) yields,

$$\psi^2 - 1 = \frac{(\beta+1)(\gamma-1)Z^2}{\gamma+\beta-(\gamma-1)Z^2}, \tag{27}$$

where

$$Z = \mathrm{sn}\left[\sqrt{\frac{\gamma+\beta}{8}}\,\epsilon\eta,\ \sqrt{\frac{\gamma-1}{\gamma+\beta}}\right].$$

It is interesting to observe that the string never passes through its rest position for values of c different from zero. This follows from the fact that ψ never vanishes.

The function which rigorously defines the transmission of a localized disturbance along the string is easily found by considering those solutions of Eqs. (4) which allow the function τ to assume a constant value. Equations (4) become, under this assumption,

$$p^2\frac{\partial^2 u'}{\partial\xi^2} = \frac{\partial^2 u'}{\partial\eta^2}, \qquad p^2\frac{\partial^2 v'}{\partial\xi^2} = \frac{\partial^2 v'}{\partial\eta^2}, \qquad \int_0^\pi\left[(1+\alpha^2\tau)^2 - (u')^2\right]^{1/2}d\xi = \pi, \tag{28}$$

where

$$p^2 = \frac{1+\tau}{1+\alpha^2\tau}.$$

If we now choose

$$u' = f(\xi - p\eta), \qquad v' = \{(1 + \alpha^2\tau)^2 - (u')^2\}^{1/2} - 1, \tag{29}$$

where τ is determined by

$$\frac{1}{\pi}\int_0^{\pi} \{[1 + \alpha^2\tau]^2 - |u'(\xi, 0)|^2\}^{1/2} d\xi = 1,$$

and where $f(\xi)$ is non-vanishing in a small region in ξ, all equations are satisfied. This solution is valid until the deformation reaches a fixed point in the string. When this occurs, the reflection phenomenon requires a change in τ. This solution is in agreement with that found by the linear theory except that p would assume the value unity in that theory.

A NEW METHOD OF INTEGRATION BY MEANS OF ORTHOGONALITY FOCI*

BY

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1. **Introduction.** This paper contains a new method of integration which is partly graphical, partly analytical.[1] It permits a simple determination of integrals of the form $\int\phi_i(x)\phi_k(x)dx$, where $\phi_i(x)$ is given graphically and $\phi_k(x)$ is given either graphically or analytically. The method requires the construction of certain diagrams, called scales, showing the abscissae of the centroids of certain areas associated with $\phi_k(x)$, and is based on some properties of the so-called orthogonality foci. Finally, the method is applied to interpolation, Fourier analysis, and the evaluation of Mohr integrals in the theory of structures.

2. **Definite integrals.** Let us consider the integral

$$T = \int_0^l \phi_i(x)\phi_k(x)dx. \qquad (2.1)$$

If rectangular cartesian coordinates x, y are introduced, the functions $\phi_i(x)$, $\phi_k(x)$ can be represented by curves, such as in Fig. 1. We now consider a distribution of mass along the curve $y=\phi_i(x)$, $0\leqq x\leqq l$, the mass per unit length in the x-direction being $\phi_k(x)$. The centroid of this mass distribution we shall call the *orthogonality focus.*[2] We shall denote it by F_{ik}, and its coordinates by ρ_k, f_{ik} (neither ρ_k nor the total mass Ω_k of the system depend on $\phi_i(x)$). We have

$$\Omega_k = \int_0^l \phi_k(x)dx. \qquad (2.2)$$

Ω_k is also the area under the curve $y=\phi_k(x)$. Since T represents the mass moment of the mass distribution about the x-axis,

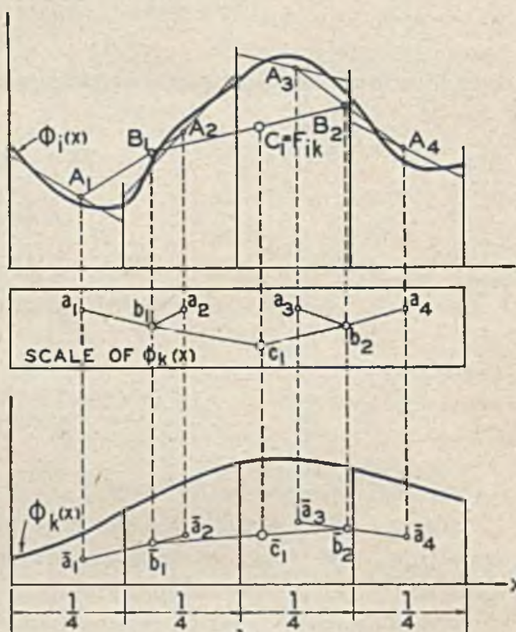

FIG. 1.

* Russian manuscript received Feb. 24, 1944. The present condensed version was prepared by Dr. L. Bers, Brown University, and Professor G. E. Hay, University of Michigan.

[1] This method was announced in the author's note entitled *A new method of graphical integration*, C. R. (Doklady) Acad. Sci. URSS (N.S.) 38, (1943).

[2] This name is justified by the properties discussed in Section 4.

$$\rho_k = \frac{1}{\Omega_k}\int_0^l x\phi_k(x)dx, \qquad (2.3) \qquad\qquad f_{ik} = T/\Omega_k. \qquad (2.4)$$

ρ_k is also the abscissa of the centroid of the area under the curve $y=\phi_k(x)$.

The following lemmas can be verified easily:

(a) *If $\phi_i(x)$ is a linear function, its graph is a straight line and F_{ik} lies on this line; F_{ik} can thus be found immediately if ρ_k is known.*

(b) *If the interval $(0, l)$ is divided into two parts, the orthogonal foci of the two parts and of the whole are collinear.*

These two lemmas permit a graphical determination to any desired degree of accuracy of the point F_{ik} and hence of the integral T. The procedure is as follows:

(*a*) The interval $(0, l)$ is divided into 2^m equal intervals[3] $(0, l_1), (l_1, l_2), \cdots, (l_{2^m-1}, l)$.

(*b*) Operation (*a*) divides the region under the curve $y=\phi_k(x)$ into 2^m regions. We find the centroids $\bar{a}_r$ $(r=1, 2, \cdots, 2^m)$ of these 2^m regions,[4] then combine adjacent pairs of regions and find the centroids $\bar{b}_r$ $(r=1, 2, \cdots, 2^{m-1})$ of the 2^{m-1} regions so formed, then combine adjacent pairs of these 2^{m-1} regions and find the centroids $\bar{c}_r$ $(r=1, 2, \cdots, 2^{m-2})$ of the 2^{m-2} regions so formed, and so on. In the final stage, we find the centroid of the entire region under the curve $y=\phi_k(x)$. In the lower part of Fig. 1, we see these centroids in a case when $m=2$.

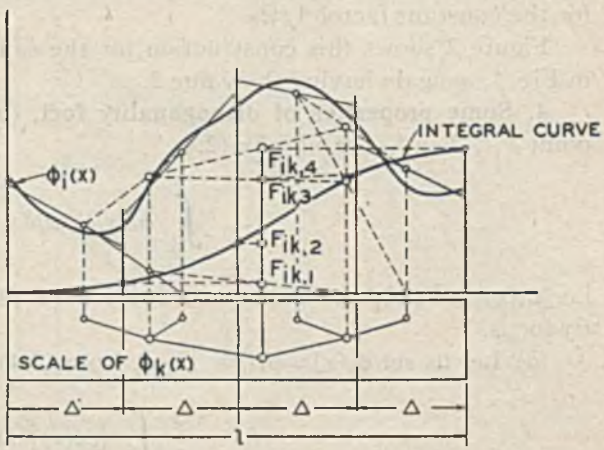

FIG. 2.

(*c*) A diagram, called the *scale* of $\phi_k(x)$, is constructed. The middle part of Fig. 1 shows such a scale. It consists of points $a_r (r=1, 2, \cdots, 2^m)$ vertically above $\bar{a}_r$ and all at the same level, points $b_r (r=1, 2, \cdots, 2^{m-1})$ vertically above $\bar{b}_r$ and all at the same arbitrary level slightly below the points a_r, and so on.

(*d*) Operation (*a*) divides the curve $y=\phi_i(x)$ into 2^m parts. We replace each part by a segment of a straight line, as shown in the upper part of Fig. 1, and assume that the mass is distributed along these segments rather than along the curve.

(*e*) We determine the points $A_r (r=1, 2, \cdots, 2^m)$ of intersection of these line segments with the verticals through the points a_r. We then determine the points $B_r (r=1, 2, \cdots, 2^{m-1})$ of intersection of the straight lines joining adjacent pairs of points A_r with verticals through the points b_r. This process is repeated, until finally we arrive at the final point F_{ik}.

(*f*) f_{ik}, which is the ordinate of F_{ik}, is determined by measurement; the value T of the required integral then follows from (2.4).

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[3] Unequal intervals could also be used.

[4] It is to be noted that areas corresponding to negative values of $\phi_k(x)$ must be considered as corresponding to negative mass.

3. Indefinite integrals. The graphical construction of Section 2 can be applied to the indefinite integral $\int\phi_i(x)\phi_k(x)dx$, in the following manner (see Fig. 2). An interval $(0, l)$ on the x-axis is taken, and the construction of Section 2 is applied to the integral $\int_0^l\phi_{i,1}(x)\phi_k(x)dx$, where $\phi_{i,1}(x)=\phi_i(x)$ in $(0, l_1)$ and vanishes elsewhere. This yields a point $F_{ik,1}$ with ordinate $f_{ik,1}$. The point $(l_1, f_{ik,1})$ is then plotted. The construction of Section 2 is then applied to the integral $\int_0^l\phi_{i,2}(x)\phi_k(x)dx$, where $\phi_{i,2}(x)=\phi_i(x)$ in $(0, l_2)$ and vanishes elsewhere. This yields a point $F_{ik,2}$, and the point $(l_2, f_{ik,2})$ is plotted. In this way we obtain the sequence of points $(l_r, f_{ik,r})$, $(r=1, 2, \cdots, 2^m)$. Since

$$f_{ik,r} = \frac{1}{\Omega_k}\int_0^l \phi_{i,r}(x)\phi_k(x)dx, \tag{3.1}$$

the curve passing through these points is approximately the integral curve, except for the constant factor $1/\Omega_k$.

Figure 2 shows this construction for the same functions $\phi_i(x)$, $\phi_k(x)$ considered in Fig. 1, m again having the value 2.

4. Some properties of orthogonality foci. (a) If the x-axis passes through the point F_{ik}, then $f_{ik}=0$, and by (2.4)

$$\int_0^l \phi_i(x)\phi_k(x)dx = 0, \tag{4.1}$$

i.e., $\phi_i(x)$ and $\phi_k(x)$ are orthogonal. It is for this reason that F_{ik} is called the *orthogonality* focus.

(b) Let us set $\phi_i(x)\equiv\phi_k(x)$. Then, from (2.4),

$$f_{kk} = \frac{1}{\Omega_k}\int_0^l [\phi_k(x)]^2dx = 2f_k \tag{4.2}$$

where f_k is the ordinate of the centroid of the region under the curve $y=\phi_k(x)$, $(0\leqq x\leqq l)$. We shall now prove the following theorem. *The curve $y=h\phi_k(x)$, where h is a constant, has the least mean square deviation from the curve $y=\phi_i(x)$ when*

$$h = f_{ik}/f_{kk}. \tag{4.3}$$

To prove this, we note that the mean square deviation is a minimum when

$$\frac{d}{dh}\int_0^l [\phi_i(x) - h\phi_k(x)]^2dx = 0,$$

i.e., when

$$\int_0^l [\phi_i(x) - h\phi_k(x)]\phi_k(x)dx = 0,$$

or

$$h = \int_0^l \phi_i(x)\phi_k(x)dx \Big/ \int_0^l [\phi_k(x)]^2dx = f_{ik}/f_{kk}.$$

(*c*) Let us draw the horizontal line β through the point F_{ik} (Fig. 3), and then rotate β about F_{ik} through an angle α to a new position β'. A new curve $y=\phi_i'(x)$ is constructed such that the vertical distance from β' to points on this curve is equal to the vertical distance from β to points on the curve $y=\phi_i(x)$. We shall now prove that

$$\int_0^l \phi_i'(x)\phi_k(x)dx = \int_0^l \phi_i(x)\phi_k(x)dx. \tag{4.4}$$

We have $\phi_i'(x)=\phi_i(x)+(\rho_k-x)\tan\alpha$, where, it is recalled, ρ_k is the abscissa of F_{ik}. Thus

$$\int_0^l \phi_i'(x)\phi_k(x)dx = \int_0^l \phi_i(x)\phi_k(x)dx + \tan\alpha \int_0^l (\rho_k - x)\phi_k(x)dx.$$

The last integral vanishes, by the definition of ρ_k, and the desired result is obtained.

It is to be noted that the two curves have a common point I, about which the curve is "rotated."

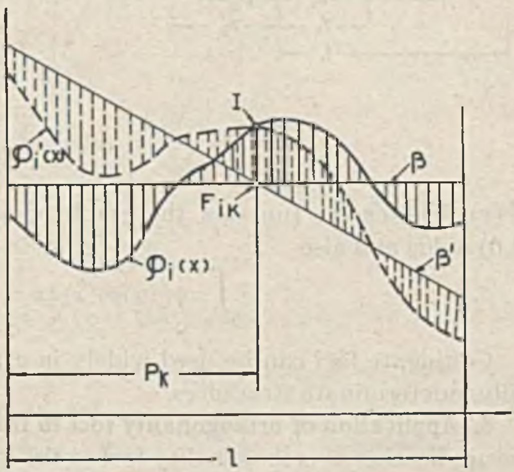

Fig. 3.

(*d*) Let us consider the case when $\phi_i(x)$ is linear in each of the intervals $(0, l_1)$ (l_1, l), so that its graph is a broken line *abcd* (Fig. 4). We shall now show that, *if ab is rotated about a point M on ab to a new position a'b', then* F_{ik} *is unchanged if cd is rotated about a certain point N on cd in such a way that* $bc=b'c'$. The points M, N are called conjugate foci. To prove this theorem, we use Fig. 4, in which A_1, A_2, A_1', A_2' are points leading to the determination of F_{ik}, following the procedure laid down in operation (*e*) of Section 2. From the three pairs of similar triangles Mbb' and MA_1A_1', Ncc' and NA_2A_2', $F_{ik}A_1A_1'$ and $F_{ik}A_2A_2'$, we have

$$\frac{bb'}{A_1A_1'} = \frac{\beta_1}{\beta_1-\gamma_1}, \qquad \frac{cc'}{A_2A_2'} = \frac{\beta_2}{\gamma_2-\beta_2}, \qquad \frac{A_1A_1'}{A_2A_2'} = \frac{\alpha_1}{\alpha_2}.$$

Since $bb'=cc'$, a value for A_1A_1'/A_2A_2' can be determined from the first two equations. Substitution of this value in the third equation yields

$$\alpha_1\gamma_2(1/\beta_2) + \alpha_2\gamma_1(1/\beta_1) = \alpha_1 + \alpha_2. \tag{4.5}$$

Thus β_2 is uniquely determined by β_1; hence N is uniquely determined by M.

It is easily seen that, if rotations of the above type are carried out about conjugate foci, and if $\phi_i'(x)$ denotes the function the graph of which is $a'b'c'd'$, then

$$\int_0^l \phi_i'(x)\phi_k(x)dx = \int_0^l \phi_i(x)\phi_k(x)dx. \tag{4.6}$$

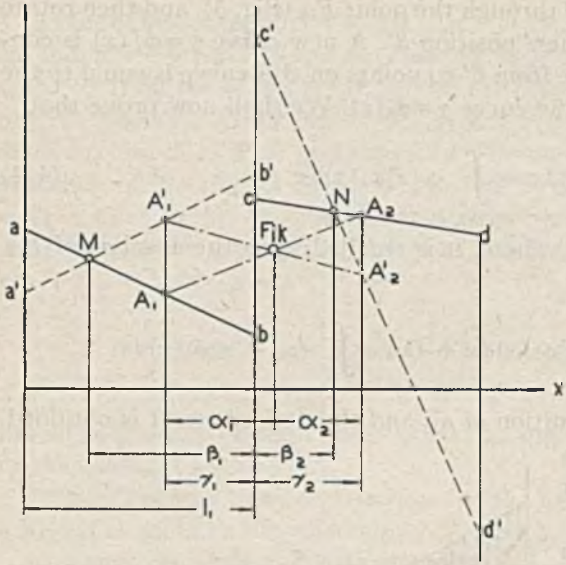

FIG. 4.

(e) Let us denote by F_{ik} and F_{il} the orthogonal foci of the function $\phi_i(x)$ represented by the broken line *abcd* with respect to two functions $\phi_k(x)$ and $\phi_l(x)$. We shall now show that, *there is a unique pair of conjugate foci M (on ab) and N (on cd) such that rotations of ab about M and cd about N (with bc = b'c') leave both F_{ik} and F_{il} unchanged.* This follows from the fact that F_{ik} is unchanged if β_1 and β_2 (Fig. 4) satisfy (4.5), and F_{il} is unchanged if they satisfy a second relation of the same form as (4.5). Since both these relations are linear in $1/\beta_1$, $1/\beta_2$, they can be solved for unique values of β_1 and β_2.

It is easily seen that, if rotations of the above type are carried out about such conjugate foci, and if $\phi_i'(x)$ denotes the function the graph of which is displaced position of *abcd*, then (4.6) holds and also

$$\int_0^l \phi_i'(x)\phi_l(x)dx = \int_0^l \phi_i(x)\phi_l(x)dx.$$

Conjugate foci can be used widely in graphical computations dealing with statically indeterminate structures.

5. Application of orthogonality foci to the interpolation of curves. Let us consider the application of orthogonality foci to the following problem. We are given two functions or curves $\phi_i(x)$ and $\phi_k(x)$. It is required to find a straight line $y = A + Bx$ such that the integral

$$U = \int_0^l [\phi_i - (A + Bx)]^2\phi_k dx, \tag{5.1}$$

will have the least possible value. We shall now show that *U has the least possible value when the straight line $y = A + Bx$ passes through the orthogonality foci F_{ik}, F_{il}, where $\phi_l = x\phi_k$.*

We set $\partial U/\partial A = \partial U/\partial B = 0$, to obtain the equations

$$\left.\begin{aligned} A\int_0^l \phi_k dx + B\int_0^l x\phi_k dx &= \int_0^l \phi_i\phi_k dx, \\ A\int_0^l x\phi_k dx + B\int_0^l x^2\phi_k dx &= \int_0^l x\phi_i\phi_k dx. \end{aligned}\right| \tag{5.2}$$

Let us consider the functions $\phi_0(x) = 1$, $\phi_1(x) = x$, $\phi_2(x) = x^2$. We have $\Omega_0 = l$, $\Omega_1 = \frac{1}{2}l^2$, $\Omega_2 = \frac{1}{3}l^3$. If the orthogonality foci of ϕ_k with ϕ_0, ϕ_1, ϕ_2 are denoted by $F_{k0}(\rho_0, f_{k0})$, $F_{k1}(\rho_1, f_{k1})$, $F_{k2}(\rho_2, f_{k2})$, respectively, and the orthogonality foci of ϕ_i with ϕ_k and ϕ_l are denoted by $F_{ik}(\rho_k, f_{ik})$, $F_{il}(\rho_l, F_{il})$, respectively, then

$$\left.\begin{aligned}
&\int_0^l \phi_k dx = f_{k0}l, \qquad \int_0^l \phi_k x dx = \tfrac{1}{2} f_{k1} l^2, \qquad \int_0^l \phi_k x^2 dx = \tfrac{1}{3} f_{k2} l^3,\\
&\int_0^l \phi_i \phi_k dx = f_{ik} \int_0^l \phi_k dx = f_{ik} f_{k0} l,\\
&\int_0^l \phi_i \phi_l dx = f_{il} \int_0^l \phi_k x dx = \tfrac{1}{2} f_{il} f_{k1} l^2.
\end{aligned}\right| \tag{5.3}$$

Thus (5.2) can be written in the form

$$A + B\frac{f_{k1}l}{2f_{k0}} = f_{ik}, \qquad A + B\frac{2f_{k2}l}{3f_{k1}} = f_{il}. \tag{5.4}$$

Since ρ_k, ρ_l are abscissas of orthogonality foci, by (2.3) we have

$$\left.\begin{aligned}
\rho_k &= \int_0^l x\phi_k dx \Big/ \int_0^l \phi_k dx = \frac{f_{k1}l}{f_{k0}},\\
\rho_l &= \int_0^l x^2\phi_k dx \Big/ \int_0^l x\phi_k dx = \frac{2f_{k2}l}{3f_{k1}}.
\end{aligned}\right| \tag{5.5}$$

Thus (5.5) take the form

$$A + B\rho_k = f_{ik}, \qquad A + B\rho_l = f_{il},$$

whence it follows that A and B must be such that the straight line $y = A + Bx$ passes through the orthogonality foci F_{ik}, F_{il}.

In order to construct the straight line $y = A + Bx$ which is such that U has the least possible value, we can proceed as follows:

(*a*) Scales are constructed for $\phi_0 = 1$, $\phi_1 = x$, $\phi_2 = x^2$. These are as shown in Fig. 5 when the interval $(0, l)$ is divided into 8 equal parts $(m = 3)$.

(*b*) A scale is constructed for ϕ_k. If the function ϕ_k is given, this can be done analytically. In any event, it can be done graphically using the scales in Fig. 5, since it involves integrals of the forms $\int \phi_k dx$, $\int \phi_k x dx$.

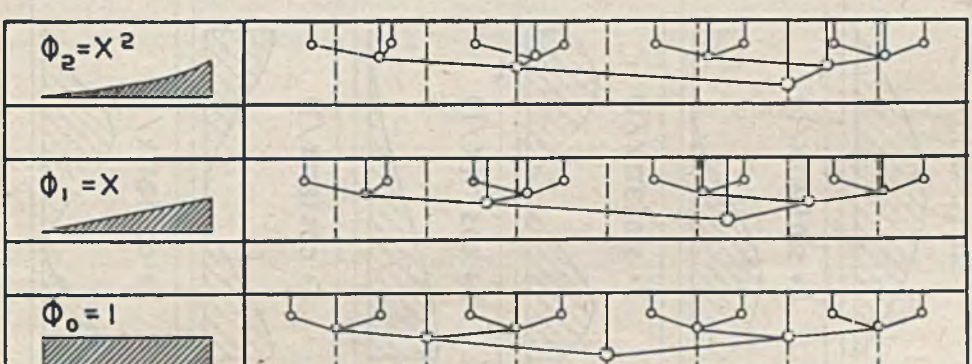

FIG. 5.

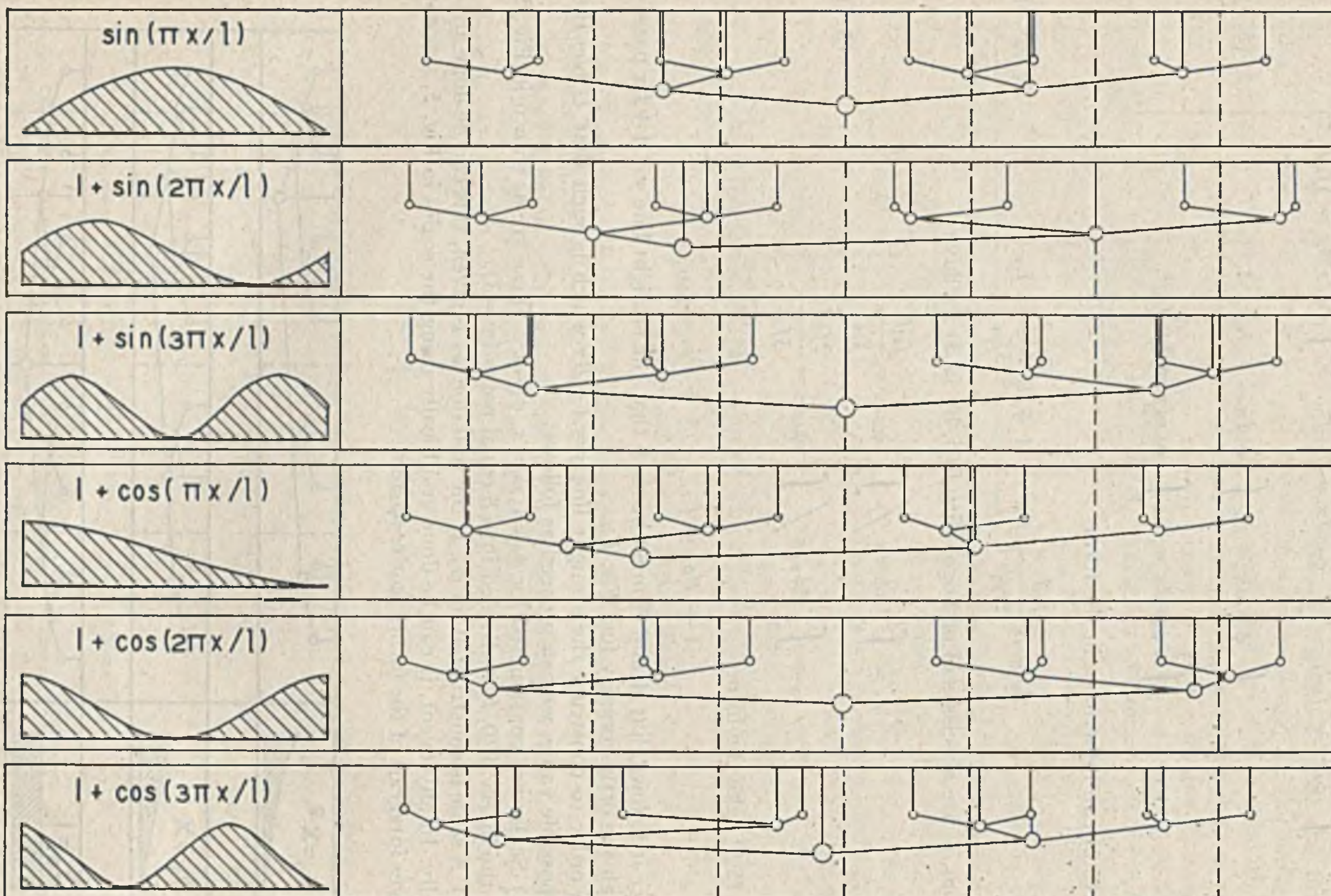

FIG. 6.

(*c*) A scale is constructed for $\phi_l = x\phi_k$. This also can be done graphically by means of the scales in Fig. 5.

(*d*) The foci F_{ik}, F_{il} are found following the procedure outlined in Section 2. The straight line through F_{ik}, F_{il} is the required line.

6. Graphical harmonic analysis. Orthogonality foci can be used to obtain the Fourier series expansion of a function which is given either analytically or graphically. If $\phi(x)$ denotes the function, its expansion into a Fourier series of sines or cosines will involve the integrals

$$\left.\begin{aligned} a_n &= \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx \qquad (n = 1, 2, \cdots), \\ b_n &= \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi x}{l}\,dx \qquad (n = 0, 1, \cdots), \end{aligned}\right\} \tag{6.1}$$

where a_n, b_n are Fourier coefficients.

Difficulties are encountered if the method of orthogonal foci is applied directly to the integrals in (6.1). These difficulties are avoided if we write (6.1) in the form

$$\left.\begin{aligned} a_1 &= \frac{2}{l}\int_0^l \phi(x)\phi_{s1}(x)dx, \\ a_n &= \frac{2}{l}\int_0^l \phi(x)\phi_{sn}(x)dx - \frac{2}{l}\int_0^l \phi(x)dx \qquad (n = 2, 3, \cdots), \\ b_n &= \frac{2}{l}\int_0^l \phi(x)\phi_{cn}(x)dx - \frac{2}{l}\int_0^l \phi(x)dx \qquad (n = 0, 1, \cdots), \end{aligned}\right\} \tag{6.2}$$

where

$$\left.\begin{aligned} &\phi_{s1}(x) = \sin\frac{\pi x}{l}, \qquad \phi_{sn}(x) = 1 + \sin\frac{n\pi x}{l} \qquad (n = 2, 3, \cdots), \\ &\phi_{cn}(x) = \cos\frac{n\pi x}{l} \qquad (n = 0, 1, \cdots). \end{aligned}\right\} \tag{6.3}$$

By the use of the scale of $\phi_0(z) = 1$ (Fig. 5) and the scales of the functions in (6.3) (Fig. 6), the ordinates of the orthogonality foci of $\phi(x)$ with these functions can be found graphically by the procedure of Section 2. If we denote these ordinates by f_0, f_{sn}, f_{cn}, respectively, then (6.2) takes the form

$$\left.\begin{aligned} &a_1 = \frac{2}{l} f_{s1}\Omega_{s1}, \qquad a_n = \frac{2}{l}(f_{sn}\Omega_{sn} - f_0\Omega_0) \qquad (n = 2, 3, \cdots), \\ &b_n = \frac{2}{l}(f_{cn}\Omega_{cn} - f_0\Omega_0) \qquad (n = 0, 1, \cdots), \end{aligned}\right\} \tag{6.4}$$

where Ω_0, Ω_{sn}, Ω_{cn} are respectively the areas under the curve $\phi_0(x) = 1$ and the curves in (6.3) for the interval $(0, l)$. Now

$$\Omega_{s1} = \frac{2l}{\pi}, \qquad \Omega_{sn} = l\left(1 + \frac{1 - \cos n\pi}{n\pi}\right) \qquad (n = 2, 3, \cdots),$$

$$\Omega_0 = l, \qquad \Omega_{cn} = l,$$

whence (6.4) becomes

$$a_1 = \frac{4}{\pi} f_{s1}, \qquad a_n = 2\left[f_{sn}\left(1 + \frac{1 - \cos n\pi}{n\pi}\right) - f_0\right] \qquad (n = 2, 3, \cdots),$$

$$b_n = 2(f_{cn} - f_0) \qquad (n = 0, 1, \cdots).$$

7. Graphical evaluation of Mohr integrals. In the theory of structures, the determination of deflections in bending often requires the evaluation of so-called Mohr

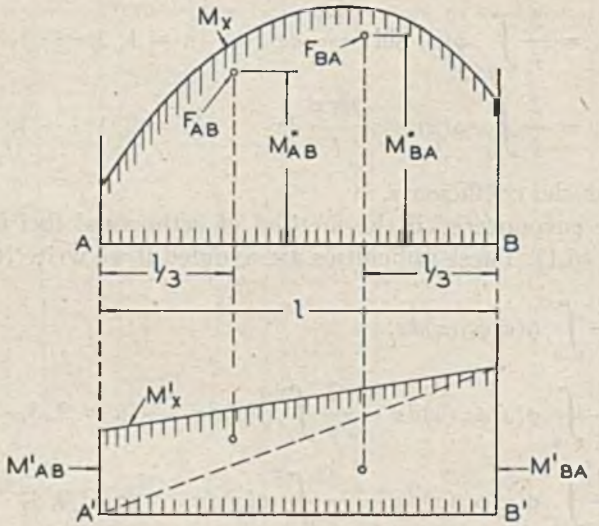

FIG. 7.

integrals, which have the form $T = \int_0^l M_x M'_x\, dx$, where M_x is a function of x given graphically and M'_x is a linear function of x (Fig. 7).

From Fig. 7, we see that

$$M'_x = M'_{AB} \frac{l - x}{l} + M'_{BA} \frac{x}{l} \cdot \tag{7.1}$$

Thus

$$T = \int_0^l M_x \frac{M'_{AB}}{l} (l - x) dx + \int_0^l M_x \frac{M'_{BA}}{l} x dx. \tag{7.2}$$

By use of the scale of x given in Fig. 5, for both of these integrals the orthogonality foci F_{AB} and F_{BA} can be determined graphically. If M^*_{AB} and M^*_{BA} denote the ordinates of these foci, then

$$T = M^*_{AB}(\tfrac{1}{2}M'_{AB}l) + M^*_{BA}(\tfrac{1}{2}M'_{BA}l),$$

or

$$T = \tfrac{1}{2}l(M^*_{AB}M'_{AB} + M^*_{BA}M'_{BA}). \tag{7.3}$$

M^*_{AB} and M^*_{BA} are called the focal moments.

—NOTES—

THE PRESSURE DISTRIBUTION ON A BODY IN SHEAR FLOW*

By M. RICHARDSON (*Brooklyn College*[1])

Problems involving shear flow have been studied recently by Tsien[2] and Kuo.[3] The purpose of the present note is to point out that the pressure distribution on an infinite cylindrical body immersed in a two-dimensional shear flow can be obtained by means of integral equations, at least for a sufficiently smooth contour. A direct attack on the boundary value problem for the stream function is avoided. The method used is essentially that employed by Prager[4] in the case of potential flow.

If the undisturbed shear flow is given by the velocity field

$$v_x = U(1 + ky), \qquad v_y = 0, \tag{1}$$

where U and k are constants, then it has constant vorticity equal to $-kU$. The continuity equation implies the existence of a stream function $\psi(x, y)$ which satisfies the Poisson equation

$$\nabla^2\psi = kU$$

with the boundary conditions

$$\frac{\partial\psi}{\partial x} = -v_y = 0, \qquad \frac{\partial\psi}{\partial y} = v_x = U(1 + ky)$$

at ∞, and $\psi = c$, a constant, on the contour C of the cross section of the cylindrical body immersed in the usual position in the flow. Let us set $\psi = \psi_0 + \psi_1$, where

$$\psi_0 = U\left(y + \frac{k}{2}y^2\right)$$

is the undisturbed stream function and ψ_1 is the disturbance stream function. Then $\nabla^2\psi_0 = kU$, and ψ_1 is harmonic in the region E exterior to C, with the boundary condition

$$\psi_{1s} = c - \psi_{0s},$$

on C, where the parameter s may be the arc length on C measured from any convenient starting point.

---

* Received July 6, 1944.

[1] This note was prepared while the author was a fellow in the Program of Advanced Instruction and Research in Mechanics at Brown University (Summer 1943). The author is indebted to Prof. W. Prager for suggesting the topic and for valuable advice.

[2] H. S. Tsien, *Symmetrical Joukowsky airfoils in shear flow*, Quarterly of Applied Mathematics, 1, 130–148 (1943).

[3] Y. H. Kuo, *On the force and moment acting on a body in shear flow*, Quarterly of Applied Mathematics, 1, 273–275 (1943).

[4] W. Prager, *Die Druckverteilung an Körpern in ebener Potentialströmung*, Physikalische Zeitschrift, 29, 865–869 (1928).

By a well-known theorem of potential theory, we have

$$\psi_1(P) - \psi_1(\infty) = \frac{1}{2\pi}\int_C \left(\psi_1 \frac{\partial}{\partial n}\log\frac{1}{r} - \frac{\partial\psi_1}{\partial n}\log\frac{1}{r}\right) ds \tag{2}$$

where n is the exterior normal, P is a point in E, and r is the distance between P and a variable point whose range will be clear from the context. We now apply Green's theorem

$$\iint_I (u\nabla^2 v - v\nabla^2 u)dA = -\int_C \left(u\frac{\partial v}{\partial n'} - v\frac{\partial u}{\partial n'}\right) ds,$$

where n' is the interior normal and I is the region interior to C, to the functions $u = -\psi_0$ and $v = \log(1/r)$, obtaining

$$kU\iint_I \log\frac{1}{r}\, dA = \int_C \psi_0 \frac{\partial}{\partial n'}\log\frac{1}{r}\, ds - \int_C \frac{\partial\psi_0}{\partial n'}\log\frac{1}{r}\, ds.$$

Using the fact that $\partial/\partial n' = -\partial/\partial n$ and combining this with (2), we obtain

$$\psi_1(P) - \psi_1(\infty) = \frac{1}{2\pi}\int_C \psi\frac{\partial}{\partial n}\log\frac{1}{r}\, ds - \frac{1}{2\pi}\int_C \frac{\partial\psi}{\partial n}\log\frac{1}{r}\, ds + \frac{kU}{2\pi}\iint_I \log\frac{1}{r}\, dA. \tag{3}$$

The first integral in (3) vanishes because $\psi = c$ on C, and because

$$\int_C \frac{\partial}{\partial n}\log\frac{1}{r}\, ds$$

is the angle subtended by C at P, which is zero since P is outside C. In the second integral of (3) we may write $-\partial\psi/\partial n = v(s)$ where $v(s)$ is the (tangential) velocity along C. Hence we have

$$\psi_1(P) - \psi_1(\infty) = \frac{1}{2\pi}\int_C v(s)\log\frac{1}{r}\, ds + \frac{kU}{2\pi}\iint_I \log\frac{1}{r}\, dA. \tag{4}$$

Let us introduce

$$V = \int_C v(s)\log\frac{1}{r}\, ds.$$

Then there exist interior and exterior limits $\partial V_i/\partial n$ and $\partial V_e/\partial n$ such that

$$\frac{1}{2}\left(\frac{\partial V_i}{\partial n} - \frac{\partial V_e}{\partial n}\right) = \pi v(s), \qquad \frac{1}{2}\left(\frac{\partial V_i}{\partial n} + \frac{\partial V_e}{\partial n}\right) = \int_C v(t)\frac{\partial}{\partial n}\log\frac{1}{r}\, dt,$$

so that

$$\frac{\partial V_e}{\partial n} = -\pi v(s) + \int_C v(t)\frac{\partial}{\partial n}\log\frac{1}{r}\, dt. \tag{5}$$

From (4) and (5), we find that the normal derivative of ψ_1 at the exterior edge of C is given by

$$\frac{\partial\psi_{1e}}{\partial n_s} = -\frac{1}{2}v(s) + \frac{1}{2\pi}\int_C v(t)\frac{\partial}{\partial n_s}\log\frac{1}{r}\, dt + \frac{kU}{2\pi}\frac{\partial}{\partial n_s}\iint_I \log\frac{1}{r}\, dA, \tag{6}$$

where the subscript s indicates the point of C at which the quantity is to be evaluated. But

$$v(s) = -\frac{\partial\psi}{\partial n_s} = -\frac{\partial\psi_0}{\partial n_s} - \frac{\partial\psi_{1e}}{\partial n_s}$$

$$= -\frac{\partial\psi_0}{\partial n_s} + \frac{1}{2}v(s) - \frac{1}{2\pi}\int_C v(t)\frac{\partial}{\partial n_s}\log\frac{1}{r}dt - \frac{kU}{2\pi}\frac{\partial}{\partial n_s}\int\int_I \log\frac{1}{r}dA.$$

Therefore, the velocity distribution along C, $v(s)$, satisfies the integral equation

$$v(s) + \frac{1}{\pi}\int_C v(t)\frac{\partial}{\partial n_s}\log\frac{1}{r}dt = -2\frac{\partial\psi_0}{\partial n_s} - \frac{kU}{\pi}\frac{\partial}{\partial n_s}\int\int_I \log\frac{1}{r}dA, \tag{7}$$

or, since the last integral may be differentiated under the integral sign,

$$v(s) + \frac{1}{\pi}\int_C v(t)\frac{\cos(r_{st}, n_s)}{r_{st}}dt = -2\frac{\partial\psi_0}{\partial n_s} - \frac{kU}{\pi}\int\int_I \frac{\cos(r, n_s)}{r}dA, \tag{8}$$

where s and t are points of C, (r_{st}, n_s) is the angle between the direction st and the exterior normal at s, r is the distance from s to a variable point p of dA, and (r, n_s) is the angle between the direction sp and the exterior normal at s. This result reduces to Prager's equation (6a), loc. cit.,[5] for the special case of uniform flow, that is, when $k=0$.

The integral equation (7) or (8) for the velocity distribution on the contour C may be solved in general by approximative methods. Knowledge of the velocity distribution on C is equivalent to knowledge of the pressure distribution on C.

Example. Suppose C is a circle of radius a with center at 0. In this case, the integral equation can be solved explicitly. We have, $\cos(r_{st}, n_s)/r_{st} = -1/2a$. It is not difficult to show that

$$\frac{\partial}{\partial n}\int\int_I \log\frac{1}{r}dA = -\pi a.$$

Finally, $\partial\psi_0/\partial n_s = U\sin\theta + Uka\sin^2\theta$ at the point with polar coordinates (a, θ). Hence (7) or (8) becomes

$$v(s) = \frac{\Gamma}{2a\pi} - 2U\sin\theta - 2Uka\sin^2\theta + Uka, \tag{9}$$

where $\Gamma = \int_C v(t)dt$ is the circulation.

For the same example, Tsien (loc. cit., equation 18) finds the stream function

$$\psi = U\left[\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{k}{2}\left(r^2\sin^2\theta + \frac{a^4}{2r^2}\cos 2\theta\right)\right].$$

Hence,

$$v(s) = -\frac{\partial\psi}{\partial r} = -2U\sin\theta - 2Uka\sin^2\theta + \tfrac{1}{2}Uka. \tag{10}$$

To reconcile this result with (9), we must observe that we can write $\Gamma = \Gamma_0 + \Gamma_1$, where Γ_0 and Γ_1 are the circulations arising from the undisturbed flow and the disturbance

[5] The difference in sign is due to the fact that our (r_{st}, n_s) is the angle supplementary to that so denoted by Prager.

flow, respectively. Hence $\Gamma_0=\int_C v_{0t}dt$ where v_0 is the undisturbed velocity field given by (1) and the subscript t indicates the tangential component. By Stokes' theorem, $\Gamma_0=\int\int_I(\text{curl } v_0)_z dA=-UkA$ where A is the area of I. Hence in our example, $\Gamma_0=-Uk\pi a^2$. If we substitute this for Γ in (9), assuming, as Tsien does,[6] that $\Gamma_1=0$, then our result (9) reduces to (10).

[6] The author is indebted to Dr. Tsien for pointing this out. He had at first mistakenly supposed that Tsien's result was based on the assumption $\Gamma=0$.

ON PLASTIC BODIES WITH ROTATIONAL SYMMETRY*

By C. H. W. SEDGEWICK (*University of Connecticut*)

Introduction. The rotational symmetry problem in plasticity was discussed by H. Hencky[1] in 1923. In the present paper some new results are obtained. Furthermore, the presentation is different from that used by Hencky.

In the following discussion, r and z in the cylindrical coordinate system (r, θ, z) will be replaced by $\alpha(r, z)$ and $\beta(r, z)$ in such a way that α, β, θ form a curvilinear, orthogonal system. The line element ds will be written in the form

$$ds^2 = A^2d\alpha^2 + B^2d\beta^2 + r^2d\theta^2,$$

where A and B are functions of α and β. Furthermore, if the angle between the curve $\beta=$const. and the direction of increasing r is denoted by γ, we will have

$$\frac{\partial r}{\partial \alpha} = A\cos\gamma, \qquad \frac{\partial r}{\partial \beta} = -B\sin\gamma, \tag{1}$$

$$\frac{\partial z}{\partial \alpha} = A\sin\gamma, \qquad \frac{\partial z}{\partial \beta} = B\cos\gamma. \tag{2}$$

From these, we get

$$\frac{\partial A}{\partial \beta} = -B\frac{\partial \gamma}{\partial \alpha}, \tag{3}$$

$$\frac{\partial B}{\partial \alpha} = A\frac{\partial \gamma}{\partial \beta}\cdot \tag{4}$$

The stress components will be designated by $\sigma_{\alpha\alpha}, \sigma_{\beta\beta}, \sigma_{\theta\theta}, \sigma_{\alpha\beta}, \sigma_{\alpha\theta}, \sigma_{\beta\theta}$. In the problem under discussion, $\sigma_{\alpha\theta}=\sigma_{\beta\theta}=0$.

1. **Lines of principal stress.** Along the lines of principal stress, $\sigma_{\alpha\beta}=0$. In this case the equations of equilibrium[2] reduce to

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[1] H. Hencky, *Über einige statisch bestimmte Fälle des Gleichgewichts in plastischen Körpern*, Zeitschr. für angew. Math. u. Mech. 3, 241 (1923).

[2] A. E. H. Love, *The mathematical theory of elasticity*, 4th edition, Cambridge University Press, 1934, p. 90.

$$\frac{1}{ABr}\left[\frac{\partial}{\partial\alpha}(Br\sigma_{\alpha\alpha})\right] - \frac{\sigma_{\beta\beta}}{A}\frac{\partial}{\partial\alpha}(\ln B) - \frac{\sigma_{\theta\theta}}{A}\frac{\partial}{\partial\alpha}(\ln r) = 0,$$

$$\frac{1}{ABr}\left[\frac{\partial}{\partial\beta}(Ar\sigma_{\beta\beta})\right] - \frac{\sigma_{\theta\theta}}{B}\frac{\partial}{\partial\beta}(\ln r) - \frac{\sigma_{\alpha\alpha}}{B}\frac{\partial}{\partial\beta}(\ln A) = 0.$$

On simplifying, these become

$$\frac{\partial\sigma_{\alpha\alpha}}{\partial\alpha} + (\sigma_{\alpha\alpha} - \sigma_{\beta\beta})\frac{\partial}{\partial\alpha}(\ln B) + (\sigma_{\alpha\alpha} - \sigma_{\theta\theta})\frac{\partial}{\partial\alpha}(\ln r) = 0, \tag{5}$$

$$\frac{\partial\sigma_{\beta\beta}}{\partial\beta} - (\sigma_{\alpha\alpha} - \sigma_{\beta\beta})\frac{\partial}{\partial\beta}(\ln A) + (\sigma_{\beta\beta} - \sigma_{\theta\theta})\frac{\partial}{\partial\beta}(\ln r) = 0. \tag{6}$$

Assuming the Tresca yield condition, we have $\sigma_{\alpha\alpha}-\sigma_{\beta\beta}=2k$, where k is constant. Furthermore, in the so called "fully plastic state," $\sigma_{\theta\theta}$ must be equal to either $\sigma_{\alpha\alpha}$ or $\sigma_{\beta\beta}$. Let us assume first that $\sigma_{\theta\theta}=\sigma_{\alpha\alpha}$. Writing $\sigma_{\alpha\alpha}+\sigma_{\beta\beta}+\sigma_{\theta\theta}=3\sigma$, we have $\sigma_{\alpha\alpha}=\sigma_{\theta\theta}=\sigma+2k/3$, $\sigma_{\beta\beta}=\sigma-4k/3$. From (5) and (6), we then get

$$\frac{\partial\sigma}{\partial\alpha} + 2k\frac{\partial}{\partial\alpha}(\ln B) = 0, \tag{7}$$
$$\frac{\partial\sigma}{\partial\beta} - 2k\frac{\partial}{\partial\beta}[\ln(Ar)] = 0. \tag{8}$$

Elimination of σ furnishes

$$\frac{\partial^2}{\partial\alpha\partial\beta}[\ln(ABr)] = 0. \tag{9}$$

If above we had assumed that $\sigma_{\theta\theta}=\sigma_{\beta\beta}$, we would have obtained

$$\frac{\partial\sigma}{\partial\alpha} + 2k\frac{\partial}{\partial\alpha}[\ln(Br)] = 0, \tag{7*}$$
$$\frac{\partial\sigma}{\partial\beta} - 2k\frac{\partial}{\partial\beta}(\ln A) = 0. \tag{8*}$$

These also lead to Eq. (9), the solution of which is

$$ABr = e^{f(\alpha)}e^{g(\beta)}. \tag{10}$$

Let us define α' and β' by[3]

$$d\alpha' = e^{f(\alpha)}d\alpha, \qquad d\beta' = e^{g(\beta)}d\beta. \tag{11}$$

This transformation merely relabels the families of surfaces $\alpha=$const. and $\beta=$const.

Now, the volume bounded by the surfaces α, $\alpha+d\alpha$, β, $\beta+d\beta$, θ, $\theta+d\theta$ is equal to $ABrd\alpha d\beta d\theta=A'B'rd\alpha'd\beta'd\theta$.

Substituting for $d\alpha'$ and $d\beta'$ from (11) and making use of (10), we get

$$A'B'r = 1. \tag{12}$$

Thus, the volume contained between the co-ordinate surfaces α_1', α_2'; β_1', β_2'; θ_1, θ_2 is given by

$$\int_{\alpha_1'}^{\alpha_2'}\int_{\beta_1'}^{\beta_2'}\int_{\theta_1}^{\theta_2} d\alpha'd\beta'd\theta = (\alpha_2' - \alpha_1')(\beta_2' - \beta_1')(\theta_2 - \theta_1).$$

[3] W. Prager, *Theory of plasticity*, mimeographed lecture notes, Brown University, R. I., 1942.

It follows that if the differences $\alpha_2' - \alpha_1'$, $\beta_2' - \beta_1'$, $\theta_2 - \theta_1$ are kept constant for successive co-ordinate surfaces, the resulting volumes will be equal. This result is analogous to that obtained by Boussinesq[4] in the plane problem.

Dropping the primes for the sake of simplicity we may construct a solution by setting $\gamma = g(\beta)$.

From (3), A is then seen to be a function of α alone. We set $A = \phi'(\alpha)$, and obtain from (4) $B = \phi g' + h(\beta)$. The first equation (1) leads to

$$r = \phi \cos g + l(\beta), \qquad \frac{\partial r}{\partial \beta} = - g'\phi(\sin g) + l'.$$

But, according to the second equation (1),

$$\frac{\partial r}{\partial \beta} = - (\phi g' + h) \sin g.$$

Hence $h = -l'/\sin g$. The condition (12) now takes the form

$$\phi'\left(\phi g' - \frac{l'}{\sin g}\right)(\phi \cos g + l) = 1.$$

This can be satisfied by setting $l = 0$, $g' \cos g = c$, $\phi^2\phi' = 1/c$, where c is a constant. Discarding constants of integration we thus obtain

$$\sin g = c\beta, \qquad \phi^3 = \frac{3\alpha}{c},$$

$$\gamma = \sin^{-1}(c\beta), \qquad A = 3^{-2/3}c^{-1/3}\alpha^{-2/3},$$

$$B = 3^{1/3}c^{2/3}\alpha^{1/3}[1 - c^2\beta^2]^{-1/2}, \qquad r = 3^{1/3}c^{-1/3}\alpha^{1/3}[1 - c^2\beta^2]^{1/2}.$$

Equations (2) now give $z = 3^{1/3}c^{2/3}\alpha^{1/3}\beta$. Hence

$$z/r = c\beta[1 - c^2\beta^2]^{-1/2}, \qquad r^2 + z^2 = 3^{2/3}c^{-2/3}\alpha^{2/3}.$$

The curves $\alpha = \text{const.}$ and $\beta = \text{const.}$ are thus seen to be concentric circles around $r = z = 0$, and radial straight lines, respectively.

In the above example, it may easily be verified that, corresponding to a set of equidistant values of α, β and θ, the resulting volumes will be equal.

By substituting the value for B above in (7) and integrating, an expression for σ is obtained.

2. **Lines of maximum shearing stress.** Along the lines of maximum shearing stress, $\sigma_{\alpha\beta} = k$ and $\sigma_{\alpha\alpha} = \sigma_{\beta\beta} = \sigma$. $\sigma_{\theta\theta}$ will be equal to either $\sigma + k$ or $\sigma - k$. Let us assume first that $\sigma_{\theta\theta} = \sigma + k$. In this case, the equations of equilibrium (2) are

$$\frac{1}{ABr}\left[\frac{\partial}{\partial \alpha}(Br\sigma) + \frac{k}{A}\frac{\partial}{\partial \beta}(A^2 r)\right] - \frac{\sigma}{A}\frac{\partial}{\partial \alpha}(\ln B) - \frac{(\sigma + k)}{A}\frac{\partial}{\partial \alpha}(\ln r) = 0,$$

$$\frac{1}{ABr}\left[\frac{\partial}{\partial \beta}(Ar\sigma) + \frac{k}{B}\frac{\partial}{\partial \alpha}(B^2 r)\right] - \frac{1}{B}(\sigma + k)\frac{\partial}{\partial \beta}(\ln r) - \frac{\sigma}{B}\frac{\partial}{\partial \beta}(\ln A) = 0.$$

These reduce to

[4] J. Boussinesq, *Lois géométrique de la distribution des pressions, dans un solide homogène et ductile soumis à des déformations planes.* Comptes Rendus Ac. Sci. Paris **74**, 242 (1872).

$$\frac{\partial\sigma}{\partial\alpha}+\frac{k}{ABr}\left[A^2\frac{\partial r}{\partial\beta}+2Ar\frac{\partial A}{\partial\beta}\right]-k\frac{\partial}{\partial\alpha}(\ln r)=0,$$

$$\frac{\partial\sigma}{\partial\beta}+\frac{k}{ABr}\left[B^2\frac{\partial r}{\partial\alpha}+2Br\frac{\partial B}{\partial\alpha}\right]-k\frac{\partial}{\partial\beta}(\ln r)=0.$$

Making use of Eqs. (1), we obtain

$$\frac{\partial\sigma}{\partial\alpha}+k\left[-\frac{A}{r}\sin\gamma-2\frac{\partial\gamma}{\partial\alpha}\right]-k\frac{\partial}{\partial\alpha}(\ln r)=0, \tag{13}$$

$$\frac{\partial\sigma}{\partial\beta}+k\left[\frac{B}{r}\cos\gamma+2\frac{\partial\gamma}{\partial\beta}\right]-k\frac{\partial}{\partial\beta}(\ln r)=0. \tag{14}$$

Eliminating σ, we find

$$\frac{\partial}{\partial\beta}\left[\frac{-A\sin\gamma}{r}-2\frac{\partial\gamma}{\partial\alpha}\right]=\frac{\partial}{\partial\alpha}\left[\frac{B\cos\gamma}{r}+2\frac{\partial\gamma}{\partial\beta}\right].$$

Carrying out the differentiations and substituting for $\partial A/\partial\beta$, $\partial B/\partial\alpha$, $\partial r/\partial\alpha$, $\partial r/\partial\beta$ from Eqs. (1), (3) and (4), we obtain

$$\frac{\partial^2\gamma}{\partial\alpha\partial\beta}+\frac{1}{2r}\left[A\cos\gamma\frac{\partial\gamma}{\partial\beta}-B\sin\gamma\frac{\partial\gamma}{\partial\alpha}\right]-\frac{AB}{4r^2}\cos 2\gamma=0. \tag{15}$$

We may remark that as $r\to\infty$, Eq. (15) reduces to that governing the case for plane strain, i.e., $\partial^2\gamma/\partial\alpha\partial\beta=0$.

It is easily seen that the only solution of equation (15) having two orthogonal families of straight lines occurs when $\gamma=45°$, i.e., when the two families of straight lines are inclined at an angle of 45° to the axis of symmetry. This result was obtained by Hencky.[1]

If we had assumed above that $\sigma_{\theta\theta}=\sigma-k$, our equilibrium equations would reduce to

$$\frac{\partial\sigma}{\partial\alpha}+k\left[\frac{-A}{r}\sin\gamma-2\frac{\partial\gamma}{\partial\alpha}\right]+k\frac{\partial}{\partial\alpha}(\ln r)=0, \tag{13*}$$

$$\frac{\partial\sigma}{\partial\beta}+k\left[\frac{B}{r}\cos\gamma+2\frac{\partial\gamma}{\partial\beta}\right]-k\frac{\partial}{\partial\beta}(\ln r)=0, \tag{14*}$$

which also lead to Eq. (15).

Let us assume a solution of Eq. (15) in the form $\gamma=f(\alpha)+g(\beta)$. The equation then becomes

$$2r[\{A\cos(f+g)\}g'-\{B\sin(f+g)\}f']-AB\cos 2(f+g)=0.$$

Substituting for A and B from relations (1), we get

$$\frac{\partial r}{\partial\alpha}\frac{\partial}{\partial\beta}\left[\frac{\cos 2(f+g)}{r}\right]+\frac{\partial r}{\partial\beta}\frac{\partial}{\partial\alpha}\left[\frac{\cos 2(f+g)}{r}\right]=0. \tag{16}$$

A solution of (16) is given by $r=C\cos 2(f+g)$ where C is a constant. Without loss in generality we may set $f(\alpha)+g(\beta)=\alpha-\beta$. We then have

$$r = C \cos 2(\alpha - \beta),$$

$$\frac{\partial r}{\partial \alpha} = -2C \sin 2(\alpha - \beta) = -4C \sin (\alpha - \beta) \cos (\alpha - \beta).$$

Equations (1) now furnish

$$A = -4C \sin (\alpha - \beta), \qquad B = -4C \cos (\alpha - \beta),$$

and Eqs. (2) give

$$z = -2C(\alpha + \beta) + C \sin 2(\alpha - \beta).$$

It follows that the curves $\alpha = \text{const.}$ and $\beta = \text{const.}$ are cycloids tangent to the lines $r = C$ and $r = -C$, respectively.

From Eqs. (14) and (15), we are able to determine σ. Substituting our values for γ, A, B, r and integrating, we find that

$$\sigma = 4k(\alpha + \beta) + k \ln [1 - \sin 2(\alpha - \beta)] + \text{const.}$$

Another solution is obtained by setting $f(\alpha) + g(\beta) = \alpha - \beta$, as before, and substituting $r = e^{\alpha+\beta}\phi$ where $\phi = \phi(\alpha - \beta)$ is a function yet to be determined. After making these substitutions and carrying out the differentiations, we get

$$[\phi^2 - \phi'^2] \cos 2(\alpha - \beta) - 2\phi\phi' \sin 2(\alpha - \beta) = 0$$

which is satisfied by

$$\phi = C[\cos (\alpha - \beta) + \sin (\alpha - \beta)].$$

We thus have

$$r = Ce^{\alpha+\beta}[\cos (\alpha - \beta) + \sin (\alpha - \beta)].$$

Using relations (1) and (2), we find that

$$z = Ce^{\alpha+\beta}[\sin (\alpha - \beta) - \cos (\alpha - \beta)].$$

The curves $\alpha = \text{const.}$ and $\beta = \text{const.}$ are logarithmic spirals which intersect the straight lines through the origin at an angle of $\pi/4$. This solution corresponds to the solution obtained in 1.

It is interesting to see that these networks of cycloids or logarithmic spirals, known in the case of plane strain, are also admissible in the case of rotational symmetry.

ON THE TREATMENT OF DISCONTINUITIES IN BEAM DEFLECTION PROBLEMS*

By S. TIMOSHENKO (*Stanford University*)

In a note on the treatment of discontinuities in beam deflection problems Mr. E. Kosko[1] attributes to R. Macaulay the method whereby the number of constants of integration can be always reduced to two, independently of the number of forces. This method was, however, originated by A. Clebsch, and is discussed in his book "Theorie der Elasticität Fester Körper," 1862, page 389. In Russia it was called the Clebsch method and was widely used in textbooks on strength of materials. It was also used in German books. See, for example, A. Föppl, Festigkeitslehre, 5th ed. 1914, page 124.

* Received Jan. 14, 1945.

[1] Quarterly of Appl. Math. 2, 271–272 (1944).

ON CERTAIN INTEGRALS IN THE THEORY OF HEAT CONDUCTIONS*

By WILLIAM HORENSTEIN (*Math. Tables Project, Nat. Bureau of Standards*)

The integrals

$$\phi = \int_0^t x^{-3/2} \exp\left(-\frac{a^2}{x} - b^2x\right) dx,$$
$$\psi = \int_0^t x^{-1/2} \exp\left(-\frac{a^2}{x} - b^2x\right) dx \tag{1}$$

frequently arise in the theory of heat conduction. It is the purpose of this note to express these integrals in terms of tabulated functions.

By simple transformations the above integrals may be written in the form

$$\phi = 2\int_{1/\sqrt{t}}^{\infty} \exp\left(-a^2\lambda^2 - \frac{b^2}{\lambda^2}\right) d\lambda, \qquad \psi = -\frac{1}{2b}\frac{d\phi}{db}, \tag{2}$$

Let us consider the integral

$$u = \int_c^{\infty} \exp\left(-a^2\lambda^2 - \frac{b^2}{\lambda^2}\right) d\lambda = \int_c^{\infty} F d\lambda. \tag{3}$$

By obvious transformations (3) may be written successively in the forms

$$u = \int_0^{\infty} F d\lambda - \int_0^c F d\lambda = \frac{\sqrt{\pi}}{2a} e^{-2ab} - e^{2ab}\int_0^c \exp\left[-a^2\left(\lambda + \frac{b}{a\lambda}\right)^2\right] d\lambda$$
$$= \frac{\sqrt{\pi}}{2a} e^{-2ab} - \frac{b}{a} e^{2ab}\int_{b/ac}^{\infty} \lambda^{-2} \exp\left[-a^2\left(\lambda + \frac{b}{a\lambda}\right)^2\right] d\lambda$$
$$= \frac{\sqrt{\pi}}{2a} e^{-2ab} + e^{2ab}\int_{b/ac}^{\infty} \left(1 - \frac{b}{a\lambda^2}\right) \exp\left[-a^2\left(\lambda + \frac{b}{a\lambda}\right)^2\right] d\lambda$$
$$- e^{2ab}\int_{b/ac}^{\infty} \exp\left[-a^2\left(\lambda + \frac{b}{a\lambda}\right)^2\right] d\lambda$$
$$u = \frac{\sqrt{\pi}}{a}\cosh 2ab - \frac{\sqrt{\pi}\, e^{2ab}}{2a} \operatorname{erf}\left(\frac{b}{c} + ac\right)$$
$$- e^{2ab}\int_{b/ac}^{\infty} \exp\left[-a^2\left(\lambda + \frac{b}{a\lambda}\right)^2\right] d\lambda, \tag{4}$$

where $\operatorname{erf}(x) = (2/\sqrt{\pi})\int_0^x \exp(-\xi^2) d\xi$.

In an entirely similar manner one finds

$$u = \frac{\sqrt{\pi}}{2a} e^{-2ab} \operatorname{erf}\left(\frac{b}{c} - ac\right) + e^{-2ab}\int_{b/ac}^{\infty} \exp\left[-a^2\left(\lambda - \frac{b}{a\lambda}\right)^2\right] d\lambda. \tag{5}$$

From (4) and (5) one obtains

* Received May 2, 1945.

$$u = \int_c^\infty \exp\left(-a^2\lambda^2 - \frac{b^2}{\lambda^2}\right) d\lambda$$

$$= \frac{\sqrt{\pi}}{2a}\cosh 2ab + \frac{\sqrt{\pi}}{4a}\left[e^{-2ab}\,\mathrm{erf}\left(\frac{b}{c} - ac\right) - e^{2ab}\,\mathrm{erf}\left(\frac{b}{c} + ac\right)\right]. \tag{6}$$

Accordingly,

$$\phi = 2\int_{1/\sqrt{t}}^\infty \exp\left(-a^2\lambda^2 - \frac{b^2}{\lambda^2}\right) d\lambda$$

$$= \frac{\sqrt{\pi}}{a}\cosh 2ab + \frac{\sqrt{\pi}}{2a}\left[e^{-2ab}\,\mathrm{erf}\left(b\sqrt{t} - \frac{a}{\sqrt{t}}\right) - e^{2ab}\,\mathrm{erf}\left(b\sqrt{t} + \frac{a}{\sqrt{t}}\right)\right]. \tag{7}$$

An expression for ψ is obtained by differentiating (7) with respect to b in accordance with the second Eq. (2).

BIBLIOGRAPHICAL LIST

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