## QUARTERLY

OF

## APPLIED MATHEMATICS

EDITED BY
H. L. DRYDEN
J. M. LESSELLS

T. C. FRY<br>W. PRAGER<br>J. L. SYNGE

TH. V. KiRMAN

1. S. SOKOLNIKOFF

WITH THE COLLABORATION OF
H. BATEMAN
J. P. DEN HARTOG
K. O. FRIEDRICH
G. E. HAY
S. A. SCHELKUNOFF

SIR GEOFFREY TAYLOR
M. A. BLOT
H. W. EMMONS
J. A. GOFF
P. LE CORBEILLER
W. R. SEARS
S. P. TIMOSHENKO
L. N. BRILLOUIN
W. FELLER
J. N. GOODIER
F. D. MURNAGHAN
R. V. SOUTHWELL
H. S. TSIEN

## QUARTERLY

## OF

## APPLIED MATHEMATICS

This periodical is published quarterly under the sponsorship of Brown University, Providence, R.I. For its support, an operational fund is being set up to which industrial organizations may contribute. To date, contributions of the following industrial companies are gratefully acknowledged:
Bell Telephone Laboratories, Inc.; New York, N. Y.,
The Bristol Company; Waterbury, Conn.,
Curtiss Wright Corporation; Airplane Division; Buffalo, N. Y.,
Eastman Kodat Company; Rochester, N. Y.,
General Electric Company; Schenectady, N. Y., Gulf Researci and Development Company; Pittsburgh, Pa., Leeds \& Northrup Company; Philadelphia, Pa.,
Pratt \& Whitnex, Division Niles-Bement-Pond Company; West Hartford, Conn.,
Republic Aviation Corporation; Farmingale, Long Island, N. Y., Unted Aircraft Corporation; East Hartrord, Cons.,
Westinghouse Electric and Manufacturing Company; Pittsburge, Pa.
The Quarterly prints original papers in applied mathematics which have an intimate connection with application in industry or practical science. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

Manuscripts submitted for publication in the Quarterly of Applibd Mathematics should be sent to the Managing Editor, Professor W. Prager, Quartexly of Applied Mathematics, Brown University. Providence 12, R. I., either directly or through any one of the Editors or Collaborators. In accordance with their general policy, the Editors welcome particularly contributions which will be of interest both to mathematicians and to engineers. Authors will receive galley proofs only Seventy-five reprints without covers will be furnished frec; additional reprints and covers widt be supplied at cost.

The subscription price for the Quarteriy is $\$ 6.00$ per volume (April-January), single copies \$2.00. Subscriptions and orders for single copies may be addressed to: Quarteriy of Applied Mathe= matics, Brawn University, Providence 12, R.I, or to 450 Ahnaip St:, Menasha, Wisconsin.

[^0]
## QUARTERLY

## OF

## APPLIED MATHEMATICS

EDITED BY
H. L. DRYDEN
J. M. LESSELLS
T. C. FRY
W. PRAGER
J. L. SYNGE

TH. v. KARMAN
I. S. SOKOLNIKOFF

WITH THE COLIABORATION OF
H. BATEMAN
J. P. DEN HARTOG
K. O. FRIEDRICHS
G. E. HAY
S. A. SCHELKUNOFF

SIR GEOFFREY TAYLOR
M. A. BIOT
H. W. EMMONS
J. A. GOFF
P. I.E CORBEILLER
W. R. SEARS
S. P. TIMOSHENKO
L. N. BRILLOUIN
W. FEILER
J. N. GOODIER
F. D. MURNAGHAN
R. V. SOUTHWELL
H. S. TSIEN


## CONTENTS

T. Alfrey: Methods of representing the properties of viscoclastic materials ..... 143
H. J. Barten: On the deflection of a cantilever beam ..... 275
K. E. Bisshopp: The inverse of a stiffness matrix ..... 82
K. E. Bisshopp and D. C. Drucker: Large deflection of cantilever beams ..... 272
D. R. Blaskett and H. Schwerdtfeger : A formula for the solution of an arbitrary analytic equation ..... 266
M. H. Blewett: (See H. Poritsky)
E. Bromberg and J. J. Stoker: Non-linear theory of curved elastic sheets . ..... 246
G. F. Carrier: On the non-linear vibration problem of the elastic string ..... 157
G. F. Carrier: On the vibrations of the rotating ring ..... 235
P. Y. Chou: On velocity correlations and the solutions of the equations of turbulent fluctuation ..... 38
P. Y. Chou: Pressure flow of a turbulent fluid between two infinite parallel planes ..... 198
N. Coburn: The Kármán-Tsien pressure-volume relation in the two-dimen- sional supersonic flow of compressible fluids ..... 106
J. W. Craggs and C. J. Tranter: The capacity of twin cable ..... 268, 380
C. H. Dix, C. Y. Fu and E. W. McLemore: Rayleigh waves and free surface reflections ..... 151
D. C. Drucker: (See K. E. Bisshopp)
C. M. Fowler: Analysis of numerical solutions of transient heat-flow problems ..... 361
T. C. Fry: Some numerical methods for locating roots of polynomials ..... 89
C. Y. Fu: (See C. H. Dix)
W. Horenstein: On certain integrals in the theory of heat conductions ..... 183
J. C. Jaeger: Diffusion in turbulent flow between parallel planes ..... 210
Th. v. Kármán and H. S. Tsien: Lifting-line theory for a wing in non-uniform flow ..... 1
R. King and D. Middleton: The cylindrical antenna; current and impedance ..... 302
E. G. Kogbetliantz: Quantitative interpretation of maps of magnetic and gravitational anomalies by mathematical methods ..... 55
W. Kohn: The spherical gyrocompass ..... 87
C. C. Lin: On the stability of two-dimensional parallel flows.
I. General theory ..... 117
11. Stability in an inviscid fluid ..... 218
III. Stability in a viscous fluid ..... 27
A. N. Lowan: On the problem of heat conduction in a semi-infinite radiating wire ..... 84
E. W. McLemore: (See C. II. Dix)
D. Middleton: (See R. King)
I. Opatowski: Cantilever beams of uniform strength ..... 76
A. A. Popoff : A new method of integration by means of orthogonality foci ..... 166
H. Poritsky and M. H. Blewett: A method of solution of field problems by means of overlapping regions ..... 336
W. Prager: On plane elastic strain in doubly-connected domains ..... 377
A. Preisman: Graphical analyses of nonlinear circuits ..... 185
M. Richardson: The pressure distribution on a body in shear flow ..... 175
H. A. Robinson: On A. A. Popoff's method of integration by orthogonality foci ..... 383
S. A. Schaaf: A cylinder cooling problem ..... 356
S. A. Schelkunoff: Solution of linear and slightly nonlinear differential equa- tions ..... 348
H. Schwerdtfeger: (See D. R. Blasketi)
C. H. W. Sedgewick: On plastic bodies with rotational symmetry ..... 178
J. J. Stoker: (See E. Bromberg)
S. Timoshenko: On the treatment of discontinuities in beam deflection prob- lems ..... 182
C. J. Tranter: (See J. W. Craggs)
H. S. Tsien: (See Th. v. Karman)
A. Vazsonyi: On rotational gas flows ..... 29
S. E. Warschawski: On Theodorsen's method of conformal mapping of nearly circular regions ..... 12
Book Reviews ..... 384
Bibliographical lists ..... $184,276,386$

## QUARTERLY OF APPLIED MATHEMATICS

# ON THE STABILITY OF TWO-DIMENSIONAL PARALLEL FLOWS <br> PART III.-STABILITY IN A VISCOUS FLUID* 

BX<br>C. C. LIN**<br>Guggenheim Laboratory, California Institute of Technology

11. General considerations. The investigations in Part II* of the stability characteristics of a parallel flow in an inviscid fluid led to very useful information. In the first place, they enable us to visualize the effect of pressure forces very clearly. In the second place, the results can be used as a guide for studying the stability problem in a real fluid, since instability is expected to occur only for sufficiently large Reynolds numbers. Thus, if we know the general characteristics of "inviscid stability" for a given velocity distribution, some stability characteristics in a real fluid can be obtained by considering a modification of these results by the effect of viscosity. Such a consideration was first made by Heisenberg, [14] $\dagger$ who demonstrated that the effect of viscosity is generally destabilizing at very large Reynolds numbers. There are, however, a few points to be supplemented in his discussion. We shall therefore study this problem in some detail in $\$ 12$.

To get a good understanding of the stability problem, we want to know the following points for any given class of velocity distribution. First of all, we want to know whether this class of flows is stable for all Reynolds numbers. Secondly, if it is stable for certain Reynolds numbers with respect to disturbances of certain wavelengths, but unstable under other conditions, we want to know the general nature of the curve $c_{i}(\alpha, R)=0$ which separates regions of stability and instability in the $\alpha, R$ plane. Thirdly, such a curve will be expected to show a minimum in $R$, below which all small disturbances are damped out. It is therefore desirable to be able to calculate this minimum critical Reynolds number rapidly.

In the next section, we shall solve these problems for two classes of flows; namely, (a) velocity distributions of the symmetrical type, and (b) velocity distributions of the boundary-layer type. Indeed, it will be shown that in these cases the flow is always unstable for sufficiently large Reynolds numbers, whether the velocity curve has a point of inflection or not. The curve of neutral stability $c_{i}(\alpha, R)=0$ will be shown to belong to either of the types shown in Fig. 9. When the velocity curve has no point of inflection, the two asymptotic branches of the curve have the common asymptote $\alpha=0$

[^1](Fig. 9a). When there is a point of inflection, one branch has the asymptote $\alpha=0$, while the other has the asymptote $\alpha=\alpha_{0}>0$ (Fig. 9b). In either case, it will be shown that the region inside the loop is the region of instability. It should be observed how these results fit in with those of Rayleigh and Tollmien for the inviscid case, and with the destabilizing effect of viscosity noted by Heisenberg for large Reynolds numbers. Simple formulae will be derived to express the asymptotic branches of the curves in Fig. 9. In fact, it is by means of these asymptotic behaviours and a criterion of stability of Synge [63], that the results mentioned above are established. Simple rules will also be given, by which the minimum critical Reynolds number can be very easily obtained from quantities involving very simple integral and differential expressions of the velocity distribution $w(y)$. Very little numerical labor is involved for the calculation in any particular case.

Heisenberg also discussed the general shape of the curve $c_{i}(\alpha, R)=0$. However, his argument does not appear to be very decisive, and some of his results are not well stated. He did not obtain the asymptotic forms of the $\alpha(R)$ curve as given below. He also tried to estimate the order of magnitude of the critical Reynolds number, but did not try to make an approximate calculation in terms of simple differential and integral expressions of $w(y)$ (loc. cit., p. 600 ).

In order to obtain definite numerical results, which may be subjected to experimental verification, we shall apply our theory to the stability problem of special velocity distributions. In $\S 13$, we shall give the results of calculation of the neutral curves in (a) the Blasius case and (b) the Poiseuille case. The method of calculation and its numerical accuracy will be given in the Appendix. Frequent reference to the equations in it will therefore be made in the following sections.
12. Heisenberg's criterion and the general characteristics of the curve of neutral stability. We shall now proceed to study the general stability characteristics in a viscous fluid as indicated above. In the first place, we shall develop Heisenberg's criterion in a slightly improved form. We shall then restrict ourselves to velocity distributions of the symmetrical type and of the boundary-layer type. For these cases, we shall prove the results summarized in the next section
a) Heisenberg's criterion. Along the neutral curve

$$
c_{i}(\alpha, R)=0
$$

(if it exists), all the parameters $\alpha, c$, and $R$ are functions of one of them, say, $R$. Let us restrict ourselves to cases where $\alpha$ and $c$ do not approach zero along the neutral curve as $R$ becomes infinite. Then, the approximations (6.27) are valid for sufficiently large values of $R$. By using (6.26), (6.24) and (6.27), we can transform (6.13) into the following:

$$
\begin{aligned}
\sum_{n=0}^{\infty} \alpha^{2 n} K_{2 n+1}(c)= & \frac{e^{-\mathrm{ri} / 4}}{\sqrt{\left(\alpha R(1-c)^{5}\right.}}\left\{\sum_{n=0}^{\infty} \alpha^{2 n} K_{2 n}(c)+w_{2}^{\prime}(1-c) \sum_{n=0}^{\infty} \alpha^{2 n} K_{2 n+1}(c)\right\} \\
& +\frac{e^{\mathrm{ri} / 4}}{\sqrt{(\alpha R c)^{5}}}\left\{\sum_{n=0}^{\infty} \alpha^{2 n} H_{2 n}(c)+w_{1} c \sum_{n=0}^{\infty} \alpha^{2 n} K_{2 n+1}(c)\right\} .
\end{aligned}
$$

If there is an inviscid neutral disturbance with $c=c_{s}, \alpha=\alpha_{s}$, we have

$$
\sum_{n=0}^{\infty} \alpha_{s}^{2 n} K_{2 n+1}\left(\epsilon_{0}\right)=0 .
$$

Hence, when $\alpha=\alpha_{s}$, and $c=c_{s}+\Delta c$, differing but slightly from $c_{s}$, we have

$$
\begin{equation*}
\Delta c=\frac{e^{-\pi i / 4}}{\sqrt{\alpha_{s} R\left(1-c_{s}\right)^{5}}} \frac{\sum_{n=0}^{\infty} \alpha_{s}^{2 n} K_{2 n}\left(c_{s}\right)}{\sum_{n=0}^{\infty} \alpha_{s}^{2 n} K_{2 n+1}^{\prime}\left(c_{s}\right)}+\frac{e^{\pi i / 4}}{\sqrt{\alpha_{s} R c_{\theta}^{5}}} \frac{\sum_{n=0}^{\infty} \alpha_{s}^{2 n} H_{2 n}\left(c_{s}\right)}{\sum_{n=0}^{\infty} \alpha_{s}^{2 n} K_{2 n+1}^{\prime}\left(c_{s}\right)} \tag{12.1}
\end{equation*}
$$

Similar considerations of (6.14), (6.15), and (6.17) give respectively

$$
\begin{align*}
& \Delta c=\frac{e^{\pi i / 4}}{\sqrt{\alpha_{s} R c_{s}^{5}}} \frac{\sum_{n=0}^{\infty} \alpha_{s}^{2 n} H_{2 n+1}\left(c_{s}\right)}{\sum_{n=0}^{\infty} \alpha_{s}^{2 n} K_{2 n+2}^{\prime}\left(c_{s}\right)},  \tag{12.2}\\
& \Delta c=\frac{e^{x i / 4}}{\sqrt{\alpha_{s} R c_{s}^{5}}} \frac{\sum_{n=0}^{\infty} \alpha_{s}^{2 n} H_{2 n}\left(c_{s}\right)}{\sum_{n=0}^{\infty} \alpha_{s}^{2 n} K_{2 n+1}^{\prime}\left(c_{s}\right)},  \tag{12.3}\\
& \Delta c=\frac{e^{\pi i / 4}}{\sqrt{\alpha_{s} R c_{s}^{5}}} \frac{\sum_{n=1}^{\infty} \alpha_{s}^{2 n} H_{2 n-1}\left(c_{s}\right)+\left(1-c_{s}\right)^{2} \sum_{n=0}^{\infty} \alpha_{s}^{2 n+1} H_{2 n}\left(c_{s}\right)}{\sum_{n=1}^{\infty} \alpha_{s}^{2 n} K_{2 n}^{\prime}\left(c_{s}\right)+\left(1-c_{s}\right)^{2} \sum_{n=0}^{\infty} \alpha_{s}^{2 n+1} K_{2 n+1}^{\prime}\left(c_{s}\right)} . \tag{12.4}
\end{align*}
$$

In general, it is not very easy to determine whether $\Delta c_{i}>0$ or $<0$. However, when $c_{s}$ and $\alpha_{s}$ are both small, but not zero, all the above equations will reduce to

$$
\begin{equation*}
\Delta c=e^{\pi i / 4} / \sqrt{\alpha_{s} R c_{s}^{5}} K_{i}^{\prime}\left(c_{s}\right), \tag{12.5}
\end{equation*}
$$

after we make use of the reductions corresponding to (1) and (2) of the Appendix. Now, when $c$ is small,

$$
K_{1}(c)=\int_{y_{1}}^{y_{2}} d y(w-c)^{-2}=-\frac{1}{w_{1}^{\prime} c}-\frac{w_{0}^{\prime \prime}}{w_{1}^{\prime 3}}(\log c+i \pi)+O(1)
$$

hence, it can be easily verified that $K_{1}^{\prime}(c)$ is approximately real and positive for small real values of $c$. Hence, (12.5) shows that $\Delta c_{i}>0$ in every case. The disturbance with wave length $2 \pi / \alpha_{s}$, neutrally stable in the inviscid case, is unstable when viscous forces are considered. This result was first obtained by Heisenberg, and may be formulated as follows:

Heisenberg's criterion. If a velocity profile has an "inviscid" neutral disturbance with non-vanishing wave number and phase velocity, the disturbance with the same wave number is unstable in the real fluid when the Reynolds number is sufficiently large.

In Heisenberg's original discussion, only the first type of motion is considered. The last equation on p. 597 of his paper is essentially our equation (12.1) with all the terms in $\alpha^{2}$ dropped. Evidently, the above arguments hold only for $\alpha_{s}, c_{s} \neq 0$. It will be seen later from Fig. 9 that one neutral disturbance with $\alpha=0$ and $c=0$ for infinite $R$, is actually stable for finite values of $R$.
b) Asymptotic behavior of the neutral $\alpha(R)$ curve. We shall now study the general asymptotic behavior of the neutral curve, assuming that it exists. The answer to the existence problem will be evident during the course of the investigation. For large values of $\alpha R$, we would generally expect $z$ of (6.28) to be much greater than 1 , but it is also possible for $z$ to approach a finite value or zero. We shall therefore discuss both possibilities.

For large values of $z$, we have approximately,

$$
\begin{equation*}
\mathcal{F}_{r}=1, \quad \mathcal{F}_{i}=1 / \sqrt{2 z^{3}}=w_{1}^{\prime} / \sqrt{2 \alpha R c^{3}}, \tag{12.6}
\end{equation*}
$$

where $\mathcal{F}_{i}$ is small. If we refer to (5) and (7) of Appendix, we see that the imaginary part $v$ of the right-hand side member of those equations depends on that of $w_{1}^{\prime} c K_{1}(c)$ and those of the integrals $H_{1}, H_{2}, M$ 's and $N$ 's. If $\alpha$ and $c$ are small, which will be verified a posteriori, we have only the contribution from the first term; thus,*

$$
\begin{equation*}
v=-\pi w_{1} \frac{w w^{\prime \prime}}{w^{\prime 3}} \text { for } w=c \tag{12.7}
\end{equation*}
$$

By using (12.6) and (12.7), we see that the equations (8) in the Appendix can be approximated by

$$
\begin{equation*}
u=1 \tag{12.8}
\end{equation*}
$$

and

$$
\begin{equation*}
v=-\frac{\pi}{w_{1}^{\prime}} c w_{0}=w_{1}^{\prime} / \sqrt{2 \alpha R c^{3}} \tag{12.9}
\end{equation*}
$$

These are the equations for determining a relation $\alpha(R)$, if we eliminate $c$ between them.

In the case where $\alpha R c^{3}$ approaches a finite limit as $\alpha R \rightarrow \infty, c$ must approach zero. Hence, $v$ must approach zero, and from (8) of Appendix, $\mathcal{F}_{i}$ must also approach zero. From the curve for $\mathcal{F}_{i}(z)$, we see that this happens for $z=2.294$, for which $\mathcal{F}_{r}=2.292$. Then, using (9) of Appendix, (12.6), and (12.7), we have

$$
\begin{equation*}
\alpha R=w_{1}^{\prime 2} z^{3} / c^{3}, \quad z=2.294 \tag{12.10}
\end{equation*}
$$

and

$$
\begin{equation*}
u=f_{r}=2.292 . \tag{12.11}
\end{equation*}
$$

The two types of relations (12.8), (12.9) and (12.10), (12.11) evidently correspond to two different branches of the $\alpha(R)$ curve (cf. Fig. 9). These conditions can be satisfied in cases (2a) and (3) of section 6 (cf. (11.5), (11.7) of Appendix), but it appears to be difficult in case (2b) (cf. (11.6) of Appendix).

CASE (2a). Symmetrical velocity distribution with symmetrical $\phi(y)$. We consider the cases where both $\alpha$ and $c$ are small. By using (12.5) and noting that $u$ takes on a finite value in either (12.8) or (12.11), we see that we must have

$$
\begin{equation*}
u=\frac{w_{1}^{\prime} c}{H_{10} \alpha^{2}}, \text { where } H_{10}=H_{1}(0)=\int_{y_{1}}^{y_{2}} w^{2} d y ; \tag{12.12}
\end{equation*}
$$

[^2]i.e., $c$ must approach zero as fast as $\alpha^{2}$. The asymptotic behavior of the $\alpha(R)$ curves as given by (12.8)-(12.11) are as follows:
\[

$$
\begin{array}{lll}
R=\left(w_{1}^{\prime 11} / 2 \pi^{2} H_{10}^{5} w_{0}^{\prime \prime 2}\right) \alpha^{-11}, & c=\left(H_{10} / w_{1}^{\prime}\right) \alpha^{2}, & \text { (first branch), } \\
R=w_{1}^{\prime 6}\left(z / \mathcal{F}_{r} H_{10}\right)^{3} \alpha^{-7}, & c=\left(H_{10} \mathcal{F}_{r} / w_{1}^{\prime}\right) \alpha^{2}, & \text { (second branch), } \tag{12.14}
\end{array}
$$
\]

where $\mathcal{F}_{r}=2.292, z=2.294$ approximately.
Case (3). Boundary-layer profile. Here, the equation corresponding to (12.12) is (cf. (7) of Appendix)

$$
\begin{equation*}
u=w_{1} c / \alpha, \tag{12.15}
\end{equation*}
$$

i.e., $c$ must approach zero as fast as $\alpha$. Note that in the previous case, the relation (12.12) depends both on $w_{1}^{\prime}$ and on $\int_{y_{1}^{y}}^{y_{2}^{2}} w^{2} d y$. Here, it depends only on the initial slope of the velocity curve $w_{1}^{\prime}$. The two branches of the $\alpha(R)$ curve for large values of $R$ are

$$
\begin{equation*}
R=\left(w_{1}^{\prime 11} / 2 \pi^{2} w_{0}^{\prime \prime 2}\right) \alpha^{-6}, \quad c=\alpha / w_{1}^{\prime}, \quad \text { (first branch), } \tag{12.16}
\end{equation*}
$$

and

$$
\begin{equation*}
R=w_{1}^{\prime}\left(z / \mathcal{Y}_{r}\right)^{3} \alpha^{-4}, \quad c=\alpha \mathcal{F}_{r} / w_{1}^{\prime} \quad \text { (second branch), } \tag{12.17}
\end{equation*}
$$

where $\mathcal{F}_{r}=2.292, z=2.294$ approximately.
Effect of varying curvature in the curve of velocity distribution. In either case, the second branch of our asymptotic curve depends very little upon the shape of the velocity profile, while the first branch depends very much upon it through the term $w_{0}^{\prime \prime}$. This fact will enable us to correlate our present results with the inviscid investigations of Rayleigh and Tollmien, as discussed in Part II.

In all the cases considered, we have $w^{\prime \prime}<0$ for $y<y_{2}$ but sufficiently near to it. If $w^{\prime \prime}(y)$ never vanishes for $y_{1}<y<y_{2}$, we can replace $w_{0}^{\prime \prime}$ by $w_{1}^{\prime \prime}$ in the expressions (12.13) and (12.16). In general,

$$
w_{0}^{\prime \prime}=w_{1}^{\prime}+\frac{w_{1}^{\prime \prime}}{w_{1}} c+\left(\frac{w^{i v}}{2 w_{1}^{2}}-\frac{w_{1}^{\prime \prime} w_{1}^{\prime \prime \prime}}{2 w_{1}^{3}}\right) c^{2}+\cdots .
$$

Now, for a flow which is essentially parallel, the boundary condition $\Delta \Delta \psi=0$, which holds on the solid wall for all two-dimensional laminar flows, can be easily verified to be equivalent to $w_{1}^{\prime \prime \prime}=0$. Hence, we have

$$
w_{0}^{\prime}=w_{1}^{\prime}+\frac{w^{\mathrm{iv}}}{2 w_{1}^{\prime 2}} c^{2}+\cdots
$$

Thus, if $w_{1}^{\prime \prime}=0$, but $w^{\prime \prime}$ does not vanish for $y_{1}<y<y_{2}$, we have

$$
\begin{equation*}
R=\left\{2 w_{1}^{19} / \pi^{2} H_{10}^{9}\left(w^{\mathrm{iv}}\right)^{2}\right\} \alpha^{-19}, \quad \text { for case (2a) } \tag{12.18}
\end{equation*}
$$

and

$$
\begin{equation*}
R=\left\{2 w_{1}^{19} / \pi^{2}\left(w^{\mathrm{i}}\right)^{2}\right\} \alpha^{-10}, \quad \text { for case (3) } \tag{12.19}
\end{equation*}
$$

In case the velocity profile shows a point of inflection,

$$
w_{0}^{\prime}=0 \text { for } c=c_{s}
$$

Then, we have approximately

$$
\begin{equation*}
w_{0}^{\prime \prime}=\left(w_{1}^{\mathrm{iv}} / 2 w_{1}^{\prime 2}\right)\left(c^{2}-c_{s}^{2}\right) . \tag{12.20}
\end{equation*}
$$

It can now be seen from (12.13) and (12.16) that $R$ becomes infinite as $c$ approaches $c_{s}$. Let the corresponding value of $\alpha$ be denoted by $\alpha_{s}$. Then instead of (12.18) and (12.19), the following relations hold:

$$
\begin{array}{lll}
R=\left\{2 w_{1}^{\prime 19} / \pi^{2} H_{10}^{9}\left(w_{1}^{\text {iv }}\right)^{2}\right\} \alpha^{-11}\left(\alpha^{4}-\alpha_{s}^{4}\right)^{-2}, & \alpha_{s}^{2}=w_{1}^{\prime} c_{s} / H_{10}, & \text { for case (2a) }, \\
R=\left\{2 w_{1}^{\prime 1} / \pi^{2}\left(w_{1}^{\text {iv }}\right)^{2}\right\} \alpha^{-6}\left(\alpha^{2}-\alpha_{s}^{2}\right)^{-2}, & \alpha_{s}=w_{1}^{\prime} c_{s}, & \text { for case (3) } . \tag{12.22}
\end{array}
$$

Thus, for either a symmetrical or a boundary-layer distribution with a flex, we have

$$
\begin{equation*}
R \sim\left(\alpha-\alpha_{s}\right)^{-2} \tag{12.23}
\end{equation*}
$$

as $R \rightarrow \infty, \alpha \rightarrow \alpha_{s}, c \rightarrow c_{8}$. In all these approximations, we assume $\alpha_{s}$ and $c_{s}$ to be so small that the previous approximations still hold, but the qualitative nature of our conclusions cannot be changed for moderate values of $\alpha_{s}$ and $c_{s}$.

The general characteristics obtained from our foregoing discussions are summarized in Table II, and are indicated by the asymptotic branches of the solid lines in Fig. 9. Let us proceed to discuss their significance.
i) It may be expected that the region between the two asymptotic branches of the curves represents a region of instability. Thus, every symmetrical or boundary-layer profile is unstable for sufficiently large values of the Reynolds number. This point will be substantiated below.
ii) In the cases where $w_{1}^{\prime \prime}>0$, the two branches of curves approach the two different asymptotes $\alpha=0$ and $\alpha=\alpha_{s}$, leaving a finite inslable region for infinite Reynolds number. In the other cases, the two branches approach the same asymptote $\alpha=0$, leaving only the possibility of a neutral disturbance of infinite wave-length at infinite Reynolds number. These results agree with Heisenberg's criterion and the results obtained from the considerations of an inviscid fluid in Part II of this work. It thus appears that the inviscid disturbance with $\alpha=0, c=0$ is actually not as trivial as it may first appear to be, for it is actually the limiting case of neutral disturbances in a real fluid.
c) Existence of self-excited disturbances.* To establish the actual existence of selfexcited disturbances, we try to show that $c_{i}>0$ at least in the neighborhood of the neutral curve. Indeed, we may regard $c$ as a function of the two independent parameters $\alpha$ and $R^{\prime}=\alpha R$, and show that $\left(\partial c_{i} / \partial R^{\prime}\right)_{\alpha}<0$ for the first branch of the curve. This is analogous to Heisenberg's criterion, and demonstrates the same general conclusion that the effect of viscosity is destabilizing at large Reynolds numbers. To fix our ideas, we shall carry out the analysis for the case of symmetrical profiles. The other case can be carried out in a similar manner.

We begin with the equation

$$
\begin{equation*}
\frac{f_{2}(\alpha, c)}{f_{4}(\alpha, c)}=\frac{\phi_{31}}{\phi_{31}^{\prime}} \tag{6.14}
\end{equation*}
$$

where $f_{2}(\alpha, c)$ and $f_{4}(\alpha, c)$ are given by (6.26) and $\phi_{31} / \phi_{31}^{\prime}$ is given by (6.28). By using those relations, we can transform (6.14) into

[^3]$$
\mathcal{F}(z)=\phi_{22}^{\prime} /\left(\phi_{22}^{\prime}+\frac{1}{w_{1}^{\prime} c} \phi_{12}^{\prime}\right)
$$
or, by further using (6.30) and (6.24),
$$
\mathcal{F}(z)-1=w_{1}^{\prime} c \sum_{n=1}^{\infty} \alpha^{2 n} K_{2 n}(c) / \sum_{n=1}^{\infty} \alpha^{2 n} H_{2 n-1}(c) .
$$

We now regard $\alpha$ as fixed and consider the variation of $c$ with $R^{\prime}$ or $z$, which is a known function of $c$ and $R^{\prime}$. Taking logarithmic derivatives on both sides, we have

$$
\begin{equation*}
\frac{\mathcal{F}^{\prime}(z)}{\mathcal{F}(z)-1}=\left\{\frac{1}{c}+\frac{\sum_{n=1}^{\infty} \alpha^{2 n} K_{2 n}^{\prime}(c)}{\sum_{n=1}^{\infty} \alpha^{2 n} K_{2 n}(c)}-\frac{\sum_{n=1}^{\infty} \alpha^{2 n} H_{2 n-1}^{\prime}(c)}{\sum_{n=1}^{\infty} \alpha^{2 n} H_{2 n-1}(c)}\right\} \frac{d c}{d z} \tag{12.24}
\end{equation*}
$$

So far, $\alpha, c$, and $z$ are arbitrary. On the neutral curve, $c$ and $z$ are real, and we may use the relation (12.13) if we are on the first branch of the neutral curve with large values of $R^{\prime}$. Indeed, for large values of $z,(12.6)$ gives

$$
\frac{\mathcal{F}^{\prime}(z)}{\mathcal{F}(z)-1}=-\frac{3}{2 z}
$$

and

$$
-\frac{3 d z}{2 z}=-\frac{1}{2} \frac{d R^{\prime}}{R^{\prime}}-\frac{3}{2} \frac{d c}{c}
$$

By using these relations, it can be easily verified that (12.24) leads to

$$
\frac{d R^{\prime}}{2 R^{\prime}}-\frac{3 d c}{2 c}=d c\left\{\frac{1}{c}+\frac{\sum_{n=1}^{\infty} \alpha^{2 n} K_{2 n}^{\prime}(c)}{\sum_{n=1}^{\infty} \alpha^{2 n} K_{2 n}(c)}-\frac{\sum_{n=1}^{\infty} \alpha^{2 n} H_{2 n-1}^{\prime}(c)}{\sum_{n=1}^{\infty} \alpha^{2 n} H_{2 n-1}(c)}\right\}
$$

Remembering that $c=O\left(\alpha^{2}\right)$ and noting that

$$
K_{1}(c)=-\frac{1}{w_{1} c}-\frac{w_{0}^{\prime \prime}}{w_{0}^{\prime 3}}(\log c+i \pi)+O(1)
$$

where $O(1)$ is real, we can reduce the above equation to

$$
\frac{d R^{\prime}}{2 R^{\prime}}+\frac{3 d c}{2 c}=d c\left\{1+w_{1}^{\prime} c \frac{w_{0}^{\prime \prime}}{w_{0}^{\prime 3}}(\log c+i \pi)\right\}^{-1} \frac{d}{d c}\left\{\frac{w_{1}^{\prime} c w_{0}^{\prime \prime}}{w_{0}^{3}}(\log c+i \pi)\right\},
$$

or

$$
\begin{equation*}
\frac{1}{2} \frac{d R^{\prime}}{R^{\prime}}=-\frac{3 d c}{2 c}\left\{1+c \frac{d}{d c}\left[\frac{w_{1}^{\prime} c w_{0}^{\prime \prime}}{w_{0}^{\prime 3}}(\log c+i \pi)\right]\right\} \tag{12.25}
\end{equation*}
$$

For small valucs of $c$, the expression in the brackets has a positive real part and a negative imaginary part. Hence, $\left(\partial c / \partial R^{\prime}\right)_{\alpha}$ has a negative imaginary part. This completes the proof. Indeed, if $u_{1}^{\prime \prime}$ does not vanish, we have

$$
\left.\begin{array}{l}
\left(\frac{\partial c_{r}}{\partial R^{\prime}}\right)_{\alpha}=-\frac{c}{3 R^{\prime}} \sim R^{\prime-6 / 5},  \tag{12.26}\\
\left(\frac{\partial c_{i}}{\partial R^{\prime}}\right)_{\alpha}=\frac{2 \pi w_{1}^{\prime \prime}}{9 w_{1}^{\prime}} \frac{c^{2}}{R^{\prime}} \sim R^{\prime-7 / 5}
\end{array}\right\}
$$

In the above derivation, it should be noted that all approximations are made by neglecting small terms of higher orders in comparison with some terms which have been retained. Thus, the conclusion of stability or instability would not be altered by those terms of higher orders.
d) The minimum critical Reynolds number and the minimum critical wave-length. Having demonstrated the instability of the symmetrical and the boundary-layer profiles, we want to answer the following questions. First, does there exist a minimum critical Reynolds number, below which the flow is stable for disturbances of all wavelengths? If so, can we get an approximate estimate of its magnitude? Secondly, does there exist a minimum wave-length of the disturbance (maximum $\alpha$ ) below which the flow is stable at all Reynolds numbers? If so, can we get an approximate estimate of its magnitude? We shall see that in trying to answer these questions, we can also roughly depict the complete $\alpha(R)$ curve, which separates stability from instability.

The existence of these minimum values can be most conveniently inferred from a condition of stability derived by Synge* from energy considerations. His condition reads

$$
\begin{equation*}
(q R)^{2}<\left(2 \alpha^{2}+1\right)\left(4 \alpha^{4}+1\right) / \alpha^{2}, \quad q=\max \left|w^{\prime}\right| \tag{12.27}
\end{equation*}
$$

This condition insures the existence of a minimum critical Reynolds number. It permits $\alpha$ to become infinite only for $R \rightarrow \infty$. But we know from our previous considerations that $\alpha \rightarrow \alpha$, or 0 as $R \rightarrow \infty$. Hence, we would expect that there exists a maximum value of $\alpha$, above which there is always stability. The neutral curve must therefore take the general shape shown in Fig. 9. The asymptotic behaviors of the solid curves are drawn in qualitative accordance with (12.6), (12.7), and (12.23); the other parts of the solid curves are arbitrarily drawn to indicate the general shape of the curve. It is evident that the region outside the curve is the region of stability, and the enclosed region is the region of instability. Similar conclusions have been reached by Heisenberg $\dagger$ but his arguments and results appear to be somewhat obscure.

Having established the existence of the minimum critical value of $R$ and the maximum critical value of $\alpha$, we proceed to make an estimate of their magnitude. We shall see that our theory permits us to give a quite good approximation to the minimum value of $\alpha R$ (cf. (12.30)). Since this roughly corresponds to the minimum value of $R$ and also to the maximum value of $\alpha$ (as will be clear from the individual examples given below), we can get a rough estimate of these values by making a rough estimate of $\alpha$ corresponding to the minimum value of $\alpha R$.

Using the second equation of (8) of Appendix and the approximation (12.7) for $v$, we have approximately

$$
\begin{equation*}
\mathcal{F}_{i}(z)=v(c)=-\pi w_{1}^{\prime} \frac{w w^{\prime \prime}}{w^{\prime 3}} \tag{12.28}
\end{equation*}
$$

[^4]

Fig. 9. General nature of the curve of neutral stability. The dotted curve is curve of stability given by Synge.

If we recall that $z$ is proportional to $c(\alpha R)^{1 / 3}$, this equation determines $(\alpha R)^{1 / 3}$ as a function of $c$. It can then be easily verified that the minimum value of $(\alpha R)^{1 / 3}$ occurs when

$$
\begin{equation*}
z f_{i}^{\prime}(z)=c v^{\prime}(c) . \tag{12.29}
\end{equation*}
$$

If the point where this holds is denoted by $z=z_{0}$, we have approximately from(11.28)*

$$
\begin{equation*}
\alpha R=w_{1}^{\prime 2}\left(\frac{z_{0}}{c}\right)^{3} \tag{12.30}
\end{equation*}
$$

* Cf. Heisenberg, loc. cit., eq. (29a), p. 602 . He put $z_{0} \sim 1$.

The point $z_{0}$ is roughly the value where $\mathcal{F}_{i}(z)$ takes its maximum value, because (12.29) is approximately $\mathscr{f}_{!}^{\prime}(z)=0$, when $c$ is small. The corresponding value of $\alpha$ can be approximately obtained by taking

$$
\begin{equation*}
u=\mathcal{F}_{r}\left(z_{0}\right), \tag{12.31}
\end{equation*}
$$

in accordance with the first equation of (8) of Appendix, where $u$ is given by the real part of (5) or (7) of Appendix, as the case may be.
e) Approximate rules. We now procced to make some rough approximations in order to obtain simple rules, which are convenient for estimating the minimum critical Reynolds number. The condition $\mathcal{F}^{\prime}\left(z_{0}\right)=0$ gives $z_{0}=3.21$, where $\mathcal{F}_{r}\left(z_{0}\right)=1.49$ and $\mathcal{F}_{i}\left(z_{0}\right)=0.58$. The corresponding value of $c$ can be obtained from the second equation of (8) of Appendix. Putting $u=1.5$ in accordance with (12.31) and neglecting second order terms of $\lambda$, we have

$$
v(1-2 \lambda)=\mathcal{F}_{i}\left(z_{0}\right)=0.58
$$

i.e.,

$$
\begin{equation*}
-\pi w_{1}^{\prime}(1-2 \lambda)\left(w w^{\prime \prime} / w^{\prime 3}\right)=0.58 \tag{12.32}
\end{equation*}
$$

where $\lambda$ is defined by (4) of the Appendix. To find the value of $c$ from this equation, it is convenient to plot its left-hand side together with $w(y)$ against $y$, and read the value off the latter curve where the former curve gives the value 0.58 . The value of c so obtained turns out to be very close to its maximum value along the neutral curve, and is approximately the value where $R$ is a minimum.

The values of $\alpha$ and $R$ must be obtained from more rough approximations. With a consultation of the values of the integrals $H(c), K(c), M(c)$ and $N(c)$ given in the Appendix, we may derive the following reasonable estimates of $\alpha$ :

$$
\begin{array}{lll}
\alpha^{2}=w_{1}^{\prime} c / H_{10}, & H_{10}=\int_{y_{1}}^{y_{2}} w^{2} d y, & \text { for symmetrical profiles } \\
\alpha=w_{1}^{\prime} c_{1} & & \text { for boundary layer profiles. } \tag{12.34}
\end{array}
$$

These values turn out to be somewhat lower than the accurate values. With an approximate allowance for these inaccuracies and taking round numbers, we get the following approximate rules for the minimum critical Reynolds number:

$$
\begin{array}{ll}
R=\frac{30 w_{1}^{\prime}}{c^{3}} \sqrt{\frac{H_{10} w_{1}}{c}}, & \text { for symmetrical profiles } \\
R=\frac{25 w_{1}^{\prime}}{c^{4}}, & \text { for boundary layer profiles } \tag{12.36}
\end{array}
$$

The calculations have been carried out for the Blasius case and the plane Poiseuille case. In the first case, the thickness of the boundary layer is taken so that the initial slope is $w_{1}^{\prime}=2$. It is found that

$$
\left.\begin{array}{rl}
R & =5906  \tag{12.37}\\
& \text { for Poiseuille case } \\
R \delta_{1} & =502
\end{array}\right\}
$$

The quantity $\delta_{1}$ is the displacement thickness

$$
\delta_{1}=\int_{0}^{\infty}(1-w) d y=0.28673
$$

where $y$ is measured from the solid wall. These values for the minimum critical Reynolds number agree fairly well with those obtained below from more elaborate numerical calculations.

When these estimates of the minimum values of $R$ and the corresponding values of $\alpha$ (eqs. (12.32)-(12.36)) are combined with the asymptotic behavior of the $\alpha(R)$ curves (eqs. (12.16)-(12.17)), the curve of neutral stability in any case can be sketched with fair accuracy with very little labor,

The maximum value of $\alpha$ on the neutral curve cannot be very well estimated. It is usually somewhat higher than the values of $\alpha$ given by (12.33) and (12.34).
13. Stability characteristics of special velocity distributions. We shall now apply our theory to some special cases in order to obtain numerical results comparable with experiments. We take (a) the Blasius case as a typical boundary-layer profile, and (b) the plane Poiseuille motion as a typical symmetrical profile. In any case, the resultant curve of stability limit should have the general shape discussed in the last two sections. Only the results will be given here; the method of calculation and its accuracy will be discussed in the Appendix.
a) Stability of plane Poiseuille flow. The velocity distribution of plane Poiseuille motion is given by

$$
\begin{equation*}
w(y)=2 y-y^{2}, \quad \text { with } \quad w_{1}^{\prime}=2, \quad H_{1}(0)=8 / 15 \tag{13.1}
\end{equation*}
$$

Table II. Behavior of $R(\alpha)$ Curve for Large Values of $R$ for Velocity Distributions with $w^{\prime \prime}<0$ for the Main Part of the Profile.

|  | First branch |  |  | Second branch |
| :--- | :---: | :---: | :---: | :---: |
|  | $w_{1}{ }^{\prime \prime}<0$ | $w_{1}{ }^{\prime \prime}=0$ | $w_{1}{ }^{\prime \prime}>0$ |  |
| Symmetrical profile | $\alpha^{-11}$ | $\alpha^{-19}$ | $\left(\alpha-\alpha_{8}\right)^{-2}$ | $\alpha^{-7}$ |
| Boundary-layer profile | $\alpha^{-6}$ | $\alpha^{-10}$ | $\left(\alpha-\alpha_{8}\right)^{-2}$ | $\alpha^{-4}$ |

The two branches of the $\alpha(R)$ curve are given by (cf. (12.13), (12.14))

$$
\left.\begin{array}{ll}
R^{1 / 3}=8.44\left(\alpha^{2}\right)^{-11 / 6}, & c=4 \alpha^{2} / 15 ;  \tag{13.2}\\
R^{1 / 3}=5.96\left(\alpha^{2}\right)^{-7 / 6}, & c=0.611 \alpha^{2} .
\end{array}\right\}
$$

The numerical results are shown in Table III and Fig. 10. The significance of the column $s$ in the table will be explained in the next section. From the figure, we see that the minimum critical Reynolds number occurs at $R^{1 / 3}=17.45$, or $R=5314$, agreeing very well with our previous estimation.*

Earlier results. The stability of plane Poiseuille flow has been attempted by many authors. Comparatively recent papers are those of Heisenberg, [14], Noether [36], Goldstein [6], Pekeris [39, 40], Synge [64], and Langer [25]. The papers of Goldstein and Synge and one of the papers of Pekeris [39] give definite indication of stability at sufficiently low Reynolds numbers. Heisenberg's paper is in general agreement

[^5]with the present investigations. He gave only a very rough calculation, whose result is reproduced in the figure. It seems that his curve is
$$
R^{1 / 3}=13.4\left(\alpha^{2}\right)^{-11 / 6}
$$

This is different from our present result (13.2) by a numerical factor. It may be noted
Table III. Stability of Plane Poiseuille Flow.

| $c$ | $z$ | $\alpha$ | $R$ | $s$ | $\alpha^{2}$ | $R^{1 / 3}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 2.294 | 0 | $\infty$ |  |  |  |
| 0.05 | 2.363 | 0.3056 | $13.64 \times 10^{5}$ | .7214 | 0 | 0 |
| 0.10 | 2.448 | 0.4603 | $1.243 \times 10^{5}$ | .6544 | 0.0934 | 110.91 |
| 0.15 | 2.540 | 0.6024 | 31048 | .6192 | 0.3119 | 49.90 |
| 0.20 | 2.668 | 0.7506 | 12024 | .5752 | 0.5634 | 31.43 |
| 0.25 | 2.868 | 0.9263 | 6108 | .5161 | 0.8580 | 18.91 |
| 0.266 | 3.012 | 1.0101 | 5369 | .4795 | 1.0203 | 17.51 |
| 0.270 | 3.080 | 1.0414 | 5314 | .4637 | 1.0845 | 17.45 |
| 0.272 | 3.21 | 1.0836 | 5659 | .4358 | 1.1741 | 17.82 |
| 0.270 | 3.240 | 1.0888 | 5920 | .4298 | 1.1854 | 18.09 |
| 0.266 | 3.320 | 1.1007 | 6602 | .4144 | 1.2115 | 18.76 |
| 0.25 | 3.495 | 1.1033 | 9287 | .3836 | 1.2173 | 21.02 |
| 0.20 | 3.857 | 1.0254 | 26597 | .3309 | 1.0514 | 29.85 |
| 0.15 | 4.152 | 0.8824 | 92529 | .2963 | 0.7787 | 45.23 |
| 0.10 | 4.458 | 0.6990 | $4.9435 \times 10^{5}$ | .2663 | 0.4886 | 79.07 |



Fig. 10. Curve of neutral stability for the plane Poiscuille case.
that for the values of $\alpha$ for which his curve is drawn, the approximation used in deriving (13.2) is no longer legitimate. Noether's work is based upon a very good mathematical approach, which seems to promise further developments. However, in apply-
ing his method to particular examples, he neglected the terms in $\alpha^{2}$ in the inviscid solutions. As is evident from previous discussions, this is bound to lead to the wrong conclusion that the plane Poiseuille flow is stable (as Noether actually did). The mathematical analysis in Langer's work shows that the region of the c-plane for which $c_{i}>0$ must go to zero as $\alpha R$ becomes infinite. This is in agreement with present results. Langer, however, concluded from his analysis that the motion is probably stable in general. This would be a natural deduction if the effect of viscosity were only stabilizing. The instability of the plane Poiseuille flow must therefore be attributed to the destabilizing effect of viscosity. This is a very significant fact and will be discussed in greater detail in $\$ 14$.

Pekeris' second paper [40] is a numerical treatment of (4.1), replacing a derivative by a ratio of two finite differences. Unfortunately, his method is not suitable for the purpose, because he has virtually neglected the inner friction layer. In his approximation, he divided the half-width of the channel into (at most) four equal parts corresponding to $w=0,7 / 16,3 / 4,15 / 16,1$. From the present work, we know that the inner friction layer occurs definitely for $c<6 / 16$. We know also from our previous investigations that the function $\phi$ varies very rapidly in the neighborhood of the inner friction layer. Hence, it is not legitimate to replace $\phi^{\prime}$ by $\Delta \phi / \Delta y$ for the interval ( $0,1 / 4$ ) , y being measured from the solid wall here. Also, most of the combinations of values $(\alpha, R)$ he selected do not correspond to a strong instability. These values are marked with crosses in Fig. 10.
b) Stability of Blasius flow. For this case, we choose the boundary-layer thickness to be defined by

$$
\begin{equation*}
\tilde{y}=\tilde{\delta}=6 \tilde{x} / \sqrt{R_{x}}, \quad R_{x}=\tilde{u}_{1} \tilde{x} / \nu, \tag{13.3}
\end{equation*}
$$

where $\tilde{x}, \tilde{y}$ are the dimensional distances from the leading edge and the wall respectively, and $\bar{u}_{1}, \nu$ are the dimensional free stream velocity and the kinematic viscosity respectively.* With this definition, the dimensional displacement thickness is

$$
\begin{equation*}
\tilde{\delta}_{1}=0.28673 \tilde{\delta} . \tag{13.4}
\end{equation*}
$$

Such a choice has the convenience that the initial part of the velocity curve can be very accurately represented by

$$
\begin{equation*}
w(y)=2 y-3 y^{4}, \tag{13.5}
\end{equation*}
$$

$y$ being measured from the wall. Also, since the edge of the boundary layer is farther from the solid wall than that set by Tollmien and Schlichting, greater accuracy can be expected. To make it easy to compare with other results, all final values are presented in terms of

$$
\begin{equation*}
\alpha_{1}=\alpha \delta_{1}, \quad R_{1}=R \delta_{1} \tag{13.6}
\end{equation*}
$$

The two asymptotic branches of the $\alpha(R)$ curve are given by the following formulae (cf. (12.16) and (12.17)):

$$
\begin{array}{ll}
R_{1}=2.21(10)^{-5} \alpha_{1}^{-10}, & c=1.74 \alpha_{1} \\
R_{1}=0.0622 \alpha_{1}^{-4}, & c=4.00 \alpha_{1} \tag{13.8}
\end{array}
$$

[^6]These formulae may be compared with those given by Tollmien.* The complete numerical result is shown in Table IV and Fig. 11. The minimum critical Reynolds number occurs at $R_{1}=420$, agreeing fairly well with our previous estimation and the earlier results of Tollmien and Schlichting.

Table IV. Stability of Blasius Flow.

| $c$ | $z$ | $\alpha$ | $R$ | $s$ | $\alpha_{1}$ | $R_{1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 2.294 | 0 | $\infty$ | $\infty$ | .7214 | 0 |
| 0.05 | 2.294 | 0.0473 | $81.45 \times 10^{5}$ | .7214 | 0.0136 | $23.353 \times 10^{3}$ |
| 0.10 | 2.296 | 0.1040 | $4.655 \times 10^{5}$ | .7205 | 0.0298 | $1.335 \times 10^{5}$ |
| 0.15 | 2.311 | 0.1730 | 84555 | .7135 | 0.0496 | 24244 |
| 0.20 | 2.341 | 0.2588 | 24783 | .6998 | 0.0742 | 7106 |
| 0.25 | 2.396 | 0.3693 | 9536 | .6759 | 0.1059 | 2734 |
| 0.30 | 2.481 | 0.5156 | 4388 | .6414 | 0.1478 | 1258 |
| 0.35 | 2.624 | 0.7149 | 2358 | .5897 | 0.2050 | 676 |
| 0.40 | 2.942 | 1.0778 | 1477 | .4967 | 0.3090 | 423 |
| 0.411 | 3.21 | 1.2968 | 1470 | .3459 | 0.3718 | 421 |
| 0.40 | 3.540 | 1.4264 | 1944 | .3763 | 0.4090 | 557 |
| 0.35 | 4.219 | 1.2992 | 5392 | .2893 | 0.3725 | 1546 |
| 0.30 | 4.382 | 1.0055 | 12399 | .2733 | 0.2883 | 3555 |
| 0.25 | 4.685 | 0.7578 | 34739 | .2472 | 0.2173 | 9961 |



Fig. 11. Curve of neutral stability for the Blasius case:

- present calculation, --- Schlichting's calculation.

Earlier results. The stability of the boundary layer has been calculated by Tollmien and later by Schlichting, approximating the velocity distribution by linear and parabolic distributions. For the evaluation of the imaginary part corresponding to the inviscid solutions, they used the exact profile. The calculation of Schlichting is shown dotted in the figure. Tollmien's curve agrees fairly well with the present calculations,

[^7]except for a somewhat lower peak. Schlichting also calculated the amplification of the unstable disturbances [52], and the amplitude distribution and energy balance of the neutral disturbances [54]. Since the neutral curve in his calculation is inexact, it might be desirable to repeat some of his work if experimental results were available. For those calculations, the present scheme promises less numerical labor than Schlichting's original work.
14. Physical significance of the results. Prospect of further developments. Let us now summarize all the results which have been obtained and discuss their physical significance. In the first place, we may conclude that all the inertia forces controlling the stability of two-dimensional parallel flows can be considered in terms of the distribution of vorticity. If the gradient of vorticity of the main flow does not vanish inside the fluid, then self-excited disturbances cannot exist except through the effect of viscosity.

In fact, the effect of viscosity is in general destabilizing for very large Reynolds numbers. Thus, if a wavy disturbance of finite wave-length can exist neutrally for an inviscid fluid, it will be amplified through the effect of viscosity. Indeed,* if the Reynolds number of a flow is continually decreased, a disturbance of finite wave-length, which is damped at very large Reynolds numbers, becomes amplified, unless the wave-length is so small as to cause excessive dissipation at any Reynolds number. For still smaller Reynolds numbers, the damping effect becomes predominant, and we have again a decay of the disturbance. However, for the particular disturbance of infinite wavelength (essentially a steady deviation), the effect of viscosity may be said to be always of the nature of a damping.

The effect of viscosity is essentially one diffusion of vorticity. It can be seen more clearly from the following considerations. Let us imagine a disturbance originating from the inner friction layer where the phase velocity of the disturance is equal to the velocity of the main flow. During one period $2 \pi l / \alpha c U$ of the disturbance, the viscous forces will propagate it side-wise through a distance of the order $\sqrt{2 \pi \nu l / \alpha c U}$ $=l \sqrt{2 \pi / \alpha R c}$. It is significant to compare this distance with the distance between the inner friction layer and the solid boundary. For if they are nearly equal, it means that the effect of viscosity is dominant at least from the solid surface to that layer. This ratio is approximately

$$
\begin{equation*}
s=\sqrt{2 \pi / z^{3}} \tag{14.1}
\end{equation*}
$$

where $z$ is defined by (6.28). This quantity may be regarded as a measure of the effect of viscosity. Its value is included in Tables III and IV. We notice that the value of $s$ decreases from 0.7 to 0.5 as we follow the lower branch of the neutral curve of stability from infinite Reynolds number to the minimum critical Reynolds number. Then, as $s$ decreases to zero, we are following the other branch of the neutral curve to infinite Reynolds numbers. Thus (see Figs. 9, 10, 11), the lower branch is essentially controlled by the effect of viscosity. The effect here is stabilizing, since an increase of Reynolds number gives instability. On the other branch, the effect of viscosity on diffusion of vorticity is predominant in comparison with the effect of dissipation. Here, an increase of Reynolds number gives stability; i.e., the effect of viscosity is destabilizing. This destabilizing mechanism is essentially to shift the phase difference between the $u$ and $v$ components of the disturbance. It has been ex-

[^8]plained in some detail in Prandtl's article [42] from the point of view of energy balance.

If we consider disturbances from the wall and from the inner friction layer, we may regard the region in between to be wholly governed by the effect of viscosity, if these disturbances meet after a period. Thus, it is not without significance that the minimum critical Reynolds number occurs for $s=\frac{1}{2}$ approximately, which may be regarded as marking the passage from stabilizing effect to destabilizing effect of the viscous forces.

These discussions hold both for symmetrical velocity distributions and for bound-ary-layer distributions. In both cases, it has been demonstrated that instability is essentially caused by the effect of viscosity. These velocity distributions are unstable whether a point of inflection occurs in the velocity profile or not. Thus, although the gradient of vorticity plays a part in controlling the stability of the flow, it is by no means the dominant factor, particularly at low Reynolds numbers. There is thus no reason to associate a point of inflection in the profile directly with instability. This removes Taylor's objection of instability theories based on von Doenhoff's experiments.* Even if the point of inflection in the velocity profile occurs in the leading part of the plate in his experiments, the flow there is definitely stable.

There is another objection raised by Taylor against Tollmien's work on the stability of the boundary layer. He questions whether the change of boundary-layer thickness should not have a drastic influence. This point can best be settled experimentally. So far as mathematical considerations are concerned, it seems justifiable to consider a boundary layer as a parallel flow; the fractional variation of thickness is very small over a distance of one wave-length of the disturbance, and the error incurred is only a few per cent. A fuller discussion of all the errors involved in the theory will be given in the Appendix.

Another point should be settled by experimental investigations. Since the general impression had been that the plane Poiseuille flow was stable, Prandtl suggested that instability occurred at the entrance flow where the velocity distribution is not yet parabolic. The present work certainly concludes that such entrance flows are unstable, if they can be considered as approximately parallel. It is hard to decide theoretically whether a well-developed turbulence has already been reached before the parabolic profile is established. This presumably depends upon the conditions of disturbance at the inlet. The question can be best settled experimentally.

Of the six types of problems mentioned in section 3, Part I, the three types (1), (2), and (5) seem to be quite settled. The present work on the boundary layer checks Tollmien's result approximately, with a minimum critical Reynolds number $R_{1}=R \delta_{1}=420$. The minimum critical Reynolds number for plane Poiseuille flow is found to be 10600 based upon the width of the channel. These values are at least not in disagreement with the existing experimental results. It would be very interesting if experiments could be carried out to check the theoretical results so far obtained.

Since plane Couette motion is concluded to be stable while plane Poiseuille motion is concluded to be unstable, it seems interesting to investigate a combination of them to find out when does the instability begin as one changes both the pressure gradient and the relative motion of the plates.

[^9]The stability of two-dimensional jets and wakes has never been investigated with the effect of viscosity included. It seems that a study of the stability of the twodimensional wake might give us valuable information regarding the von Kármán vortex street,--particularly regarding the minimum Reynolds number of its occurrence and the width of the street as compared with the size of the body.*

Transition to turbulence. The success of Taylor's theory of transition [68, 70] to turbulence in the boundary-layer as caused by external turbulence seems to throw the instability theories at a disadvantage. However, it seems that Taylor's work should be regarded as only one phase of the problem, i.e., concerning cases where the external turbulence plays the dominant role. In fact, it is not impossible to construct an instability theory, taking account of the free turbulence outside the bound-ary-layer if this is the main cause of transition. The boundary condition $\phi^{\prime}+\alpha \phi=0$ at the edge of the boundary layer signifies that the disturbance there has equal magnitudes in directions parallel and perpendicular to the wall. This can be easily reconciled with the nearly isotropic turbulence in the free stream. Of course, the theory can only be pushed to the point where non-linear effects begin to appear. Otherwise, we have to deal with a very difficult mathematical problem. It is possible that the beginning of non-linear effect is not far from the actual point of transition. Then the instability theory should give useful results regarding transition, which might be expected to check with experiment.

## Appendix

In the following paragraphs, we shall describe the methods by which the numerical calculations are carried out. We shall then give a discussion of the numerical accuracy involved in the calculations. Special emphasis will be placed on the case of Blasius flow.
a) Transformation of equations. The basic equations for the determination of the stability characteristics are given at the end of Part I. To carry out the numerical calculation in any particular case, we have to evaluate the functions (6.24) which occur in the equations (6.14), (6.15) and (6.17) through the relations (6.26). It is found convenient to transform (6.24) into

$$
\begin{align*}
& \phi_{12}=(1-c)\left(1-\alpha^{2} H_{2}\right)^{-1}\left(1-\sum_{n=2}^{\infty} \alpha^{2 n} M_{2 n}\right), \\
& \phi_{22}=K_{1} \phi_{12}-(1-c) \sum_{n=0}^{\infty} \alpha^{2 n} N_{2 n+1}, \\
& \phi_{12}^{\prime}=(1-c)\left(1-\alpha^{2} H_{2}\right)^{-1}\left(\alpha^{2} H_{1}-\sum_{n=2}^{\infty} \alpha^{2 n} M_{2 n-1}\right)+(1-c)^{-1} w_{2}^{\prime} \phi_{12},  \tag{1}\\
& \phi_{22}^{\prime}=K_{1} \phi_{12}^{\prime}+(1-c)^{-1}\left(1-\sum_{n=1}^{\infty} \alpha^{2 n} N_{2 n}\right)+(1-c)^{-1} w_{2}^{\prime} \phi_{22},
\end{align*}
$$

where the functions $M_{n}(c)$ and $N_{n}(c)$ are defined by

$$
\left.\begin{array}{cc}
M_{n}=H_{n}-H_{2} H_{n-2}, & n \geqq 3,  \tag{2}\\
N_{n}=K_{n}-K_{1} H_{n-1}, & n \geqq 2 .
\end{array}\right\}
$$

[^10]The principal advantages of such transformations is to bring out the dominant terms in the functions $\phi_{12}, \phi_{12}^{\prime}, \phi_{22}$ and $\phi_{22}$. For the terms in $M^{\prime}$ 's and $N^{\prime}$ s are usually very small (particularly for small values of $c$, with which we are usually concerned), while all the terms in the series of ( 6.24 ) are of considerable importance. This point will be fully discussed below.

The calculation of (6.13) (Case 1) is still quite complicated; it is found necessary to take ( 6.15 ) (Case 2b) as a first approximation. Since we are not going to make any actual calculation for this case, we shall not go into further details with (6.13). All the other equations (6.14), (6.15), and (6.17) (Cases 2a, 2b,3) can be transformed into the form

$$
\begin{equation*}
f(z)=\frac{(1+\lambda)(u+i v)}{1+\lambda(u+i v)} \tag{3}
\end{equation*}
$$

with $\mathcal{F}(z)$ defined by (6.31) and $\lambda=\lambda(c)$ defined by

$$
\begin{equation*}
w_{1}^{\prime}\left(y_{1}-y_{0}\right)=-c(1+\lambda) \tag{4}
\end{equation*}
$$

Thus, $\lambda$ is usually very small. The quantities $u$ and $v$ are real functions of $\alpha$ and $c$, different for different cases. For Case 2 a we have

$$
\begin{align*}
u+i v= & 1+w_{1}^{\prime} c \phi_{22}^{\prime} / \phi_{12}^{\prime} \\
= & w_{1}^{\prime} c\left(K_{1}+\frac{1}{w_{1}^{\prime} c}\right) \\
& +\frac{w_{1}^{\prime} c}{\alpha^{2}}\left(1-\alpha^{2} H_{2}\right)\left(1-\alpha^{2} H_{2}-\alpha^{4} N_{4}-\cdots\right)\left(H_{1}-\alpha^{2} M_{3}-\alpha^{4} M_{5}-\cdots\right)^{-1} \tag{5}
\end{align*}
$$

where the second form of the right-hand side is derived by using (1). Similarly, for Cases 2 b and 3 , we have, respectively,

$$
\begin{align*}
u+i v= & 1+w_{1}^{\prime} c \phi_{22} / \phi_{12} \\
= & w_{1}^{\prime} c\left(K_{1}+\frac{1}{w_{1}^{\prime} c}\right) \\
& +w_{1}^{\prime} c \alpha^{2}\left(1-\alpha^{2} H_{2}\right)\left(1-\alpha^{4} M_{4}-\alpha^{6} M_{6}-\cdots\right)^{-1}\left(N_{3}+\alpha^{2} N_{5}+\cdots\right) \tag{6}
\end{align*}
$$

and

$$
\begin{align*}
u+i v= & 1+w_{1}^{\prime} c\left(\phi_{22}^{\prime}+\alpha \phi_{22}\right) /\left(\phi_{12}^{\prime}+\alpha \phi_{12}\right) \\
= & w_{1}^{\prime} c\left(K_{1}+\frac{1}{w_{1}^{\prime} c}\right) \\
& +\frac{w_{1}^{\prime} c}{\alpha^{2}} \frac{\left(1-\alpha^{2} H_{2}\right)\left\{\left(1-\alpha^{2} H_{2}-\alpha^{4} N_{4}-\cdots\right)-(1-c)^{2}\left(\alpha^{3} N_{3}+\alpha^{5} N_{5}+\cdots\right)\right\}}{(1-c)^{2}\left(1-\alpha^{4} M_{4}-\alpha^{6} M_{6}-\cdots\right)+\alpha\left(H_{1}-\alpha^{2} M_{3}-\cdots\right)} . \tag{7}
\end{align*}
$$

Equation (3) contains the two real equations

$$
\left.\begin{array}{l}
\mathcal{F}_{r}(z)=(1+\lambda)\left\{u(1+\lambda u)-\lambda v^{2}\right\}\left\{(1+\lambda u)^{2}+(\lambda v)^{2}\right\}^{-1}  \tag{8}\\
\mathcal{F}_{i}(z)=(1+\lambda) v\left\{(1+\lambda u)^{2}+(\lambda v)^{2}\right\}^{-1}
\end{array}\right\}
$$

for $z, \alpha, c$. Thus, for each value of $z$, we can determine corresponding values of $\alpha$ and $c$. Finally, the Reynolds number is given by (cf. (6.28))

$$
\begin{equation*}
\alpha R=\frac{1}{w_{0}^{\prime}(1+\lambda)^{3}}\left(\frac{w_{1}^{\prime} z}{c}\right)^{3} . \tag{9}
\end{equation*}
$$

The actual procedure of calculation will be described presently.
b) Procedure of calculation. The calculations required in $\S 13$ are as follows: (a) to find the values of $\alpha$ and $z$ corresponding to each value of $c$ by using equations (8), with $u$ and $v$ defined by (7) and (9); and (b) to calculate $R$ from (9). To do this, we may take the following procedure. We first plot $\mathcal{F}_{i}$ against $\mathcal{F}_{r}$; then plot the corresponding right-hand side members of (8) in a similar manner in the same diagram. Noting that the latter are functions of $\alpha$ and $c$ only, we may plot by drawing curves of constant $\alpha$ (or constant $c$ ). The intersections of this set of curves with the ( $\mathcal{F}_{\mathrm{r}}, \mathcal{F}_{i}$ ) curve give the desired results.

This procedure is however, very laborious. A simpler method is as follows: As will be seen below, the imaginary parts of $H$ 's, $M$ 's and $N$ 's appearing in (5) and (7) are very small compared with that of $K_{1}(c)$, we can therefore use the approximation

$$
\begin{equation*}
v=v(c)=-\pi w_{1}^{\prime}\left(w w^{\prime \prime} / w^{\prime 3}\right) \text { for } w=c . \tag{10}
\end{equation*}
$$

The following steps are then taken:
i) Calculation of $\alpha R$. In this step, the auxiliary functions

$$
\lambda(c), w_{0}^{\prime}(c), v(c)
$$

are required. These are tabulated in Tables V and VI for the cases considered. Having calculated these functions, we can determine $z$ and $u$ for each value of $c ; \alpha R$ is then

Table V. Auxiliary Functions for Calculating the Stability of the Plane Poiseuille Flow.

| $c$ | $w_{0}^{\prime}$ | $\lambda$ | $v$ |  | $w_{1}^{\prime} c R l$ | $H_{1}$ | $H_{2}$ | $M_{3}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :---: |
| 0 | 2 | 0 | 0 | 0 | $N_{2}$ |  |  |  |
| 0.05 | 1.94936 | 0.01282 | 0.08482 | 0.06499 | 0.53333 | 0.21817 | 0.06038 | 0.19340 |
| 0.10 | 1.89737 | 0.02633 | 0.18397 | 0.10187 | 0.41000 | 0.20696 | 0.05047 | 0.20401 |
| 0.15 | 1.84391 | 0.04061 | 0.30066 | 0.13015 | 0.35583 | 0.18276 | 0.04183 | 0.21636 |
| 0.20 | 1.78885 | 0.05573 | 0.43905 | 0.15351 | 0.30667 | 0.16982 | 0.02813 | 0.22777 |
| 0.25 | 1.73205 | 0.07180 | 0.60460 | 0.17356 | 0.26250 | 0.15626 | 0.02293 | 0.25918 |
| 0.30 | 1.67332 | 0.08893 | 0.80463 | 0.19121 | 0.22333 | 0.14209 | 0.01872 | 0.26191 |
| 0.35 | 1.61245 | 0.10728 | 1.04910 | 0.20699 | 0.18917 | 0.12729 | 0.01540 | 0.27000 |
| 0.40 | 1.54919 | 0.12701 | 1.35193 | 0.22130 | 0.16000 | 0.11182 | 0.01282 | 0.27062 |
| 0.45 | 1.48324 | 0.14836 | 1.73296 | 0.23438 | 0.13583 | 0.09563 | 0.01087 | 0.26232 |
| 0.50 | 1.41421 | 0.17157 | 2.22144 | 0.24645 | 0.11667 | 0.07875 | 0.00937 | 0.23366 |

determined from (9). In actual practice, a procedure of successive approximations is used. By taking

$$
u^{(0)}=\mathcal{F}_{r}\left(z^{(0)}\right), \quad z=z^{(0)}, \quad \mathcal{f}_{r}^{(0)}=\mathcal{F}_{r}\left(z_{r}^{(0)}\right), \quad \mathcal{f}_{i}^{(0)}=\mathcal{f}_{i}\left(z^{(0)}\right)=0
$$

as the initial approximation for $u, z, \mathcal{F}_{r}, \mathcal{F}_{i}$, we can obtain the successive approximations by the formulae

$$
\left\{\begin{array}{l}
\mathcal{F}_{i}^{(n+1)}=v\{1+\lambda\}\left\{\left(1+\lambda u^{(n)}\right)^{2}+(\lambda v)^{2}\right\}^{-1} \\
u^{(n+1)}=\mathcal{F}_{r}^{(n+1)}\left\{\left(1+\lambda u^{(n)}\right)^{2}+(\lambda v)^{2}\right\}\left\{(1+\lambda)\left(1+\lambda u^{(n)}\right)\right\}^{-1}+\lambda v\left(1+\lambda u^{(n)}\right)^{-1} .
\end{array}\right.
$$

Table VI. Auxiliary Functions for Calculating the Stability of Blasius Flow.

| $c$ | $\nu$ | $w_{1}^{\prime} c R l$ | $H_{l}$ | $H_{2}$ | $M_{3}$ | $N_{3}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0.60260 | 0.23513 | 0.07966 | 0.17615 |
| 0.05 | 0.00088 | 0.02194 | 0.53378 | 0.22392 | 0.06975 | 0.18676 |
| 0.10 | 0.00799 | 0.06121 | 0.46995 | 0.21212 | 0.06111 | 0.19911 |
| 0.15 | 0.02406 | 0.12108 | 0.41112 | 0.19972 | 0.05369 | 0.21052 |
| 0.20 | 0.05773 | 0.20469 | 0.35730 | 0.18678 | 0.04741 | 0.22193 |
| 0.25 | 0.11498 | 0.31568 | 0.30848 | 0.17322 | 0.04221 | 0.23275 |
| 0.30 | 0.20438 | 0.45835 | 0.26465 | 0.15905 | 0.03800 | 0.24466 |
| 0.35 | 0.33718 | 0.63751 | 0.22583 | 0.14425 | 0.03468 | 0.25275 |
| 0.40 | 0.52839 | 0.85766 | 0.19200 | 0.12878 | 0.03210 | 0.25337 |
| 0.45 | 0.79820 | 1.12133 | 0.16318 | 0.11259 | 0.03015 | 0.24507 |
| 0.50 | 1.17350 | 1.42576 | 0.13935 | 0.09571 | 0.02865 | 0.21641 |

In each approximation, $\mathcal{F}_{7}^{(k)}$ and $z^{(k)}$ are determined graphically from $f_{i}^{(k)}$.
ii) Calculation of $\alpha$. For this purpose, the additional auxiliary functions

$$
w_{1}^{\prime} c R l=w_{1}^{\prime} c R l\left\{K_{1}(c)+\frac{1}{w_{1}^{\prime} c}\right\}, H_{1}(c), H_{2}(c), M_{3}(c), N_{3}(c), \ldots
$$

are required. These are tabulated in Tables V and VI for the cases considered. The methods of evaluating these functions and their accuracy will be discussed below. For sufficient accuracy in the final results, only the real parts of $H_{2}, M_{3}, N_{3}$ are required, besides $w_{1}^{\prime} c R l$ and $H_{1}$. Having calculated these functions, we can determine the value of $\alpha$ from the real part of the equations (5) or (7). A similar method of successive approximations may be used by writing those equations in the forms

$$
\begin{aligned}
& \alpha^{2}=\frac{w_{1}^{\prime} c}{H_{1}\left(u-w_{1}^{\prime} c R l\right)}\left(1-\alpha^{2} H_{2}\right) \frac{1-\alpha^{2} H_{2}-\alpha^{4} N_{4}-\cdots}{1-\alpha^{2} P_{2}-\alpha^{4} P_{4}-\cdots}, \quad P_{2 n}=M_{2 n+1} / H_{1}, \\
& \alpha=\frac{w_{1}^{\prime} c}{u-w_{1}^{\prime} c R l}\left(1-\alpha^{2} H_{2}\right) \frac{\left(1-\alpha^{2} H_{2}-\alpha^{4} N_{4}-\cdots\right)-(1-c)^{2}\left(\alpha^{3} N_{3}+\alpha^{5} N_{5}+\cdots\right)}{(1-c)^{2}\left(1-\alpha^{4} M_{4}-\alpha^{6} M_{6}-\cdots\right)+\alpha\left(H_{1}-\alpha^{2} M_{3}-\alpha^{4} M_{5} \cdots\right)} .
\end{aligned}
$$

An approximate value of $\alpha$ is put into the right-hand side to obtain an approximation of the higher order on the left-hand side. For the initial approximation, take $\alpha=0$.
c) Numerical accuracy of the calculations. The numerical accuracy of our calculation as based upon the final equations given in section 6 are limited by several factors:
i) by neglecting quantities of the orders $e^{-P}$ and $(\alpha R)^{-1}$ in the reduction of the determinantal equations of the boundary-value problems,
ii) by using the inviscid solutions for $\phi_{1}$ and $\phi_{2}$ (error of the order $\left(\alpha R^{-1}\right)$ ),
iii) by the approximations of the rapidly varying solutions $\phi_{3}$ and $\phi_{4}$ as discussed at the end of $\S 6$,
iv) by the boundary-layer approximation used in setting up the equation of stability (except in the cases of plane Couette and Poiseuille flows).

Finally, certain numerical approximations have to be used in the actual evaluation of the quantities $u$ and $v$ in equations (11.11). We shall now discuss these factors one by one.

The inaccuracy due to (i) and (ii) is negligible in all the cases considered, because $\alpha R$ is always sufficiently large. In connection with (iii), the situation is more compli-
cated. The first approximation of the asymptotic solution should give an error of the order of $(\alpha R)^{-1 / 2}$; while the first approximation using Hankel functions should give an error of the order $(\alpha R)^{-1 / 3}$. It might therefore be thought that the asymptotic method should always give a better approximation. However, this is not the case. For the order of accuracy of the first method is based upon a fixed value of $y$, while that of the second is based upon a fixed value of $\eta$. Thus, if $\alpha R$ may be allowed to become very large while $y-y_{0}$ remains to be of the order of unity, the first method is definitely better. This is the case with the quantities $\phi_{42}$ and $\phi_{42}^{\prime}$. With $\phi_{31}$ and $\phi_{31}^{\prime}$, the situation is different. Here, $y_{1}-y_{0}$ is always small. Except for one branch of the neutral curve for profiles with a flex, $y_{1}-y_{0}$ goes to zero as $\alpha R$ becomes infinite. Because of the smallness of $y_{1}-y_{0}$, the asymptotic solution (which fails to be accurate in the neighborhood of $y_{0}$ ) never gives a good approximation. This is why the other method has to be used in most of the calculations, and we are limited to an accuracy of $(\alpha R)^{-1 / 3}$. We note that the curvature of the velocity distribution does not come into this approximation. Thus, for better accuracy, a second approximation should be used, the error being then reduced to the order of $(\alpha R)^{-2 / 3}$. However, since the error in the method used is only a few per cent, and an improvement in accuracy would not alter the general conclusions, it does not seem worth while to improve the accuracy in the light of general interest. Indeed, the inaccuracy due to the other causes (to be discussed) is also of the same order of magnitude. Another support to the method used is that it does agree with the asymptotic method when $z$ is large; there is only a negligible difference (cf. eq. (6.31)).

The effect of the change of the thickness of the boundary layer might be taken to be more serious than a mere numerical inaccuracy. Taylor regarded this as invalidating the existing instability theory of the boundary layer. This question can best be settled experimentally. For the present, we only want to discuss its effect upon our boundary value problem. An approximate estimate of this effect may be obtained by considering the change of the thickness of the boundary layer for one wavelength of the disturbance. This can be easily verified* to be $\pi(1.72)^{2} / \alpha_{1} R_{1}$. For the lowest value of $\alpha_{1} R_{1}$ involved in the calculations of $\S 13$, this is about 6 per cent. Thus, the error is not large. Hence, in the physical interpretation of the results, we need only consider a change of Reynolds number as we pass down stream. One interesting point is the following: as the Reynolds number keeps on increasing, all disturbances finally become stable, if the linear theory holds throughout. Thus, the transition to turbulence depends upon the occurrence of the non-linear effect and hence must depend upon the amount of initial disturbance.
d) Calculation of $\phi_{12}^{(0)}, \phi_{22}^{(0)}$, etc. We shall now discuss the method by which these quantities are evaluated for the calculation of $u$ and $v$ in the equations (5) and (7). A discussion of the accuracy of the present method and of Tollmien's method of evaluating these quantitics will also be made.

The original question is to evaluate the integrals $H_{m}(c)$ and $K_{m}(c)$ as occurring in (6.24). Various methods are possible for carrying out the calculation, including straightforward numerical integration. The method to be described is an attempt at a simple one. With the transformations (2), we hope to bring out the dominant terms of the series (6.24), and the calculations of $u$ and $v$ according to (7) and (9) are based upon the use of the transformed series (1). We make the following approximations.

[^11]i) The imaginary part $v$ is chiefly given by that of the first term, namely, $w_{1}^{\prime} c\left(K_{1}+1 / w_{1}^{\prime} c\right)$; this implies that the imaginary part due to $H_{2}(c), M_{3}(c), N_{3}(c)$, etc. is negligible.
ii) The real part receives also little contribution from those of $H_{2}(c), M_{3}(c), N_{3}(c)$, etc., and hence these need be calculated only approximately.
iii) The series are cut short; terms like $N_{4}, M_{4}, \cdots$ are entirely neglected.

Let us proceed to justify these approximations.
The justification of (ii) and (iii) is based upon the following two facts.
a) The quantities in the series involved decrease roughly like $1 / m!, m$ being the number of integrations involved in defining a certain term.
b) For $\alpha<1$, the terms also decrease as $\alpha^{m}$. Thus, the accuracy is not very good for $\alpha>1$, namely for low Reynolds numbers.

But from a consultation of Tables V and VI, and the manner in which the integrals $H_{2}(c), M_{3}(c), H_{3}(c)$, etc., enter (5) and (7), we see that an error of ten per cent in these integrals will cause a negligible error in the final results.

The justification of (i) needs more explanation. For definiteness, let us take $N_{3}(c)$ as an example. Now,

$$
N_{3}(c)=\int_{y_{1}}^{y_{2}} d y(w-c)^{-2} \int_{y_{1}}^{y} d y(w-c)^{2} \int_{y_{1}}^{y} d y(w-c)^{-2} .
$$

This can be expressed as the sum of the following three integrals:

$$
\begin{aligned}
& N_{31}(c)=K_{1}(c) \cdot \int_{y_{1}}^{\nu_{0}} d y(w-c)^{2} \int_{y}^{y_{2}} d y(w-c)^{-2}, \\
& N_{32}(c)=\int_{y_{1}}^{\nu_{2}} d y(w-c)^{-2} \int_{y_{0}}^{y} d y(w-c)^{2} \int_{\nu}^{y_{2}} d y(w-c)^{-2}, \\
& N_{33}(c)=\int_{\nu_{0}}^{y_{2}} d y(w-c)^{-2} \int_{y_{0}}^{y} d y(w-c)^{2} \int_{y}^{y_{2}} d y(w-c)^{-2} .
\end{aligned}
$$

The third integral is real, because $y>y_{0}$. A further transformation of the last integration in $N_{31}$ and $N_{32}$ like

$$
\int_{y}^{y_{2}} d y(w-c)^{-2}=K_{1}(c)-\int_{y_{1}}^{y} d y(w-c)^{-2}
$$

gives

$$
\begin{aligned}
N_{31}= & \left\{K_{1}(c)\right\}^{2} \int_{y_{1}}^{y_{0}} d y(w-c)^{2}-K_{1}(c) \int_{y_{1}}^{y_{0}} d y(w-c)^{2} \int_{y_{1}}^{y} d y(w-c)^{-2} \\
N_{32}= & K_{1}(c) \int_{\nu_{1}}^{y_{0}} d y(w-c)^{-2} \int_{y_{0}}^{y} d y(w-c)^{2} \\
& -\int_{y_{1}}^{y_{0}} d y(w-c)^{-2} \int_{y_{0}}^{v} d y(w-c)^{2} \int_{\nu_{1}}^{y} d y(w-c)^{-2} .
\end{aligned}
$$

Now, the last integral is real because $y<y_{0}$. Further, it can be easily verified that

$$
\int_{v_{1}}^{v_{0}} d y(w-c)^{2} \int_{y_{1}}^{y} d y(w-c)^{-2}=\int_{y_{1}}^{y_{0}} d y(w-c)^{-2} \int_{y_{0}}^{y} d y(w-c)^{2}
$$

Hence, the only term which can contribute to the imaginary part of $N_{3}(c)$ is $\left\{K_{1}(c)\right\}^{2} \int_{v_{1}}^{\nu^{\circ}} d y(w-c)^{2}$. Now, $c$ is usually small so that we may put

$$
\int_{y_{1}}^{y_{0}} d y(w-c)^{2}=\frac{w_{1}^{\prime 2}}{3}\left(y_{1}-y_{0}\right)^{3}=\frac{1}{3} \frac{\left(w_{1}^{\prime} c\right)^{3}}{w_{1}^{\prime 4}} .
$$

Hence,

$$
\left\{K_{1}(c)\right\}^{2} \int_{y_{1}}^{y_{0}} d y(w-c)^{2}=\left\{w_{1}^{\prime} c K_{1}(c)\right\}^{2} \frac{1}{3} \frac{w_{1}^{\prime} c}{w_{1}^{\prime 4}} .
$$

Now we have approximately

$$
w_{1}^{\prime} c K_{1}(c)=1-v i .
$$

Substituting into the above expression, we obtain the imaginary part of $N_{3}(c)$ as $-2 w_{1}^{\prime} c v / 3 w_{1}^{\prime 4}$. This will give a contribution of approximately $-(2 / 3)\left\{\alpha c(1-c) / w_{1}^{\prime}\right\}^{2} v$ to the imaginary part of $v$. This is negligible, because the factor preceding $v$ is at most of the order of 0.02 in our calculations. Hence, it is justifiable to neglect the contribution of $N_{3}$ to $v$. With the other terms, the approximation is even better; thus, the imaginary part of $H_{2}(c)$ is of the order of $c^{3}$ times that of $K_{1}(c)$, and that of $M_{3}(c)$ is of the order of $c^{6}$ times that of $K_{1}(c)$.

Having thus justified the approximations described above, the task is to evaluate $K_{1}(c), H_{1}(c), H_{2}(c), M_{3}(c)$, and $N_{3}(c)$ with proper degree of accuracy. For parabolic distribution, these integrals can be evaluated exactly; the approximation (ii) is not necessary. Thus,

$$
\begin{align*}
H_{1}(c)= & A  \tag{11}\\
K_{1}(c)= & -\frac{1}{2 a^{2}\left(1-a^{2}\right)}+\frac{A}{2 a^{3}}\left\{\log \frac{1+a}{1-a}+i \pi\right\}  \tag{12}\\
H_{2}(c)= & \frac{1}{30} \frac{4 a^{2}-3}{a^{2}}+\frac{A}{2 a^{3}} \log (1+a)-\frac{2 a^{2}}{15} \log a^{2}-\frac{B}{4 a^{3}}\left\{\log \left(1-a^{2}\right)-i \pi\right\}  \tag{13}\\
M_{3}(c)= & \frac{B^{2}}{2 a^{2}} K_{1}+\frac{8 a^{2} A}{15}\left\{\frac{1}{a+1}+\log \frac{a+1}{a}\right\} \\
& +\frac{1}{225}\left\{-\frac{54}{7}+\frac{108}{5} a^{2}-\frac{38}{3} a^{4}-64 a^{6}\right\} \tag{14}
\end{align*}
$$

$R l N_{3}(c)=\frac{1}{24 a^{2}}\left(1+3 a^{2}\right)-\frac{\left(1-a^{2}\right)^{2}}{16 a^{5}} \log \frac{1+a}{1-a}$

$$
\begin{equation*}
+\frac{1}{16 a^{6}}\left\{\int_{0}^{1} d x\left\{\left(a^{2}-x^{2}\right) \log \left\lvert\, \frac{a+x}{a-x}\right.\right\}^{2}-B \pi^{2}\right\}, \tag{15}
\end{equation*}
$$

$$
\begin{align*}
I_{m} N_{3}(c)= & \frac{\pi}{16 a^{B}}\left\{2 B \log \frac{1+a}{1-a}+\frac{32 a}{15} \log \frac{1+a}{2 a}\right. \\
& \left.-\frac{2 a}{15}(1-a)\left(2+3 a+18 a^{2}+20 a^{3}\right)\right\} \tag{16}
\end{align*}
$$

where

$$
\begin{equation*}
a^{2}=1-c, \quad A=a^{4}-\frac{2 a^{2}}{3}+\frac{1}{5}, \quad B=A-\frac{8 a^{5}}{15} \tag{17}
\end{equation*}
$$

These are the equations on which Table $V$ is based, where only the real parts are given. For any other profile, the rule is as follows:
i) Evaluate $K_{1}(c)$ with as much accuracy as possible. Usually, it is broken up into two parts. Thus,

$$
\begin{gather*}
K_{1}(c)=K_{11}(c)+K_{12}(c) ; \quad K_{11}(c)=\int_{y_{1}}^{y_{j}} d y(w-c)^{-2} \\
K_{12}(c)=\int_{y_{j}}^{y_{2}} d y(w-c)^{-2} \tag{18}
\end{gather*}
$$

where $y_{1}<y_{i}<y_{2}$. The value of $y_{j}$ is chosen so that $K_{11}(c)$ can be calculated with sufficient accuracy by developing $w$ as a power series of $\left(y-y_{0}\right)$, while $K_{12}(c)$ can be evaluated by developing the integrand as a power series of $c / w$.
ii) Evaluate by numerical integration the quantities

$$
\begin{align*}
& H_{1}(0)=\int_{y_{1}}^{y_{2}} d y w^{2},  \tag{19}\\
& H_{1}^{\prime}(0)=-2 \int_{y_{1}}^{y_{2}} w d y  \tag{20}\\
& H_{2}(0)=\int_{y_{1}}^{y_{2}} w^{-2} d y \int_{y_{1}}^{y} w^{2} d y,  \tag{21}\\
& M_{3}(0)=\int_{y_{1}}^{y_{2}} w^{2} d y \int_{y}^{y_{2}} w^{-2} d y \int_{y_{1}}^{y} w^{2} d y,  \tag{22}\\
& N_{3}(0)=\int_{y_{1}}^{y_{2}} w^{-2} d y \int_{y_{1}}^{y} w^{2} d y \int_{y}^{y} w^{-2} d y . \tag{23}
\end{align*}
$$

iii) The integral $H_{1}(c)$ is then given by

$$
\begin{equation*}
H_{1}(c)=H_{1}(0)+H_{1}^{\prime}(0) c+c^{2} \tag{24}
\end{equation*}
$$

iv) The real part of the integrals $H_{2}(c), M_{3}(c), N_{3}(c)$ are obtained by comparison with the corresponding quantities for parabolic distribution (Table V). For example,

$$
\begin{equation*}
M_{3}(c)-M_{3}(0)=\left(\frac{w^{\prime}}{2}\right)^{2} \times \text { corresponding quantity for parabolic distribution. } \tag{25}
\end{equation*}
$$

The idea of the last step is essentially to approximate the given profile with a parabolic one.

For the Blasius profile, (with $w_{1}^{\prime}=2, y_{j}-y_{1}=0.4$ ). We obtain

$$
\left.\begin{array}{rl}
K_{11}(c)+\frac{1}{w_{1}^{\prime} c}= & -0.5615-0.3937 c-1.543 c^{2}-1.803 c^{3}-1.368 c^{4}-5.022 c^{5} \\
& +\cdots+\frac{9}{8}\left(c^{2}+\frac{21}{8} c^{5}+\cdots\right)\left(\log \frac{0.8-c}{c}+i \pi\right) \\
= & 0.7080+1.3546 c+2.588 c^{2}+3.860 c^{3}+5.446 c^{4}+7.455 c^{5}  \tag{26}\\
& +\cdots, \\
K_{12}(c) \\
K_{1}(c)+\frac{1}{w_{1}^{\prime} c}= & 0.1465+1.2467 c+1.045 c^{2}+2.039 c^{3}+4.078 c^{4}+2.423 c^{5} \\
& +\cdots+\frac{9}{8}\left(c^{2}+\frac{21}{8} c^{5}+\cdots\right)\left(\log \frac{0.8-c}{c}+i \pi\right)
\end{array}\right\}
$$

In evaluating these integrals, we take

$$
\left.\begin{array}{ll}
w=2\left(y-y_{1}\right)-3\left(y-y_{1}\right)^{4}, & 0 \leqq y-y_{1} \leqq 0.4  \tag{27}\\
w=1-\left\{0.9-\left(y-y_{1}\right)\right\}^{2}, & 0.4<y-y_{1} \leqq 0.9 \\
w=1 & 0.9 \leqq y-y_{1} \leqq 1
\end{array}\right\}
$$

For the integrals $H_{1}(0)$ and $H_{1}^{\prime}(0)$, we make use of the known values of the displacement thickness and the momentum thickness $\delta_{2}$.

$$
\left.\begin{array}{l}
\delta_{1}=\frac{1}{6}(1.7208)=0.28673  \tag{28}\\
\delta_{2}=\frac{1}{12}(1.32824)=0.11067
\end{array}\right\}
$$

Thus,

$$
\begin{equation*}
H_{1}(0)=1-\delta_{1}-\delta_{2}=0.6026, \quad H_{1}^{\prime}(0)=-2\left(1-\delta_{1}\right)=-1.4265 \tag{29}
\end{equation*}
$$

The values of $H_{2}(0), M_{3}(0), N_{3}(0)$, as evaluated by numerical integration, are given in the first row of Table VI. The rest of the table is constructed by following the procedure described above.

We see that the method of approximation developed above is purely a numerical one, and the calculation can be done without excessive labor. In any case, even if the above method does not give satisfactory results, suitable approximations can always be devised for the evaluation of the necessary integrals. This is the advantage of using Heisenberg's form of the inviscid solutions. In the method used by Tollmien, it is necessary that the profile may be approximated by linear and parabolic parts; otherwise, the numerical labor is excessive. It is not clear at once what is the effect of such an approximation on the solutions $\phi(y)$. A more serious criticism of Tollmien's method is the joining of the inviscid solutions at the point of junction of the two approximate profiles. Mathematically speaking, such a junction presents an essential singularity in the coefficients of the differential equation (3.8) or (3.14). Numerically speaking, serious difficulty would be expected when $c$ is equal or even only very near to the velocity of junction, because the inviscid solution fails at the critical layer where $w=c$. Tollmien did not publish how he overcame this difficulty.*

* Tollmien [73], footnote, p. 37.


# THE CYLINDRICAL ANTENNA; CURRENT AND IMPEDANCE* 

BY<br>RONOLD KING and DAVID MIDDLETON<br>Cruft Laboratory and the Research Laboratory of Physics, Harvard Universily

1. Introduction. The definition and the determination of the impedance of a symmetrical, center-driven antenna of small, circular cross section involves three major problems. These are first the theoretical analysis including the formulation of boundary conditions; second the apparatus and the technique of experimental measurement; and third the coordination of experiment with theory. Of these


Fig. 1. Cylindrical antenna with hemispherical ends. problems only the first is the subject of this paper; the last two are considered in detail elsewhere. ${ }^{1}$ The present discussion is concerned specifically with an analytical improvement in the solution of the theoretical problem.

The boundary and driving conditions in this analysis are the same as implied in earlier analyses, ${ }^{2,3}$ and the same integral equation is obtained. However, the present paper introduces a new approach to the solution of Hallén's integral equation in that it replaces a function arbitrarily chosen for reasons of mathematical convenience in the approximate evaluation of the equation by a function actually fitted to the true distribution of current. As a consequence, new parameters are introduced to replace those used by Hallén (or equally those used by Gray ${ }^{12}$ ) in the successive approximations, and as would be expected the resulting development shows a more rapid convergence, in so far as this is indicated by a relatively small difference between first and second order solutions.

The antenna actually analyzed is a theoretical one in the sense that no exact experimental analogue can be constructed. Its properties are summarized as follows:
(1) The antenna is a highly conducting cylinder of small radius a extending unbroken from $z=-h$ to $z=+h$ as shown in Fig. 1. Postulated inequalities are

$$
\begin{equation*}
\beta a \ll 1, \quad a \ll h \tag{1}
\end{equation*}
$$

where $\beta=\omega / c$ is the phase constant and $c=3 \times 10^{8} \mathrm{~m} / \mathrm{sec}$.
(2) The ends of the antenna at $z= \pm h$ contribute nothing to the electrical problem so that it is correct to write

$$
\begin{equation*}
I_{h}=0 \quad \text { at } \quad z= \pm h . \tag{2}
\end{equation*}
$$

(3) The antenna is center-driven by a slice generator consisting of a disk of neg-

* Received Aug. 6, 1945.
${ }^{1}$ R. King and D. D. King. J. Appl. Phys. 16, 445 (1945).
${ }^{2}$ E. Hallén, Nova Acta, Royal Soc. Sciences, Upsala 11, 1 (1938).
${ }^{3}$ R. King and C. W. Harrison, Jr., Proc. I.R.E. 31, 548 (1943).
ligible thickness at the center, $z=0$, which is in all respects like any other piece of the antenna except that a scalar potential difference

$$
\begin{equation*}
V_{0}^{e}=\lim _{z \rightarrow 0}\left(\phi_{+z}-\phi_{-z}\right)=\phi_{+0}-\phi_{-0} \tag{3}
\end{equation*}
$$

is maintained between its faces.
(4) All other conductors and all dielectrics are sufficiently far away so that their individual effects are indistinguishable from the composite effect of the universe as a whole. If $R$ is the distance from the center of the antenna to the nearest point on any other conductor or on a dielectric the following inequalities must be satisfied

$$
\begin{equation*}
\beta R \gg 1 ; \quad R \gg h . \tag{4}
\end{equation*}
$$

The degree in which this theoretical antenna can be realized physically is summarized briefly below. Details are found elsewhere. ${ }^{1,4,5,6,7}$
(a) A metal wire or rod can be constructed to satisfy completely the properties assumed in conjunction with (1).
(b) If a solid cylinder with flat ends or a hollow cylinder is used (2) is not exactly true. A small current exists at the ends to charge the sharp edges, the end surfaces, or the inner surfaces of a tube near the ends. This leads to an error in $h$ of the order of magnitude of $a$, and a consequent hidden shift in the theoretical impedance curves. For particular values of $h$ near anti-resonance large errors in impedance are involved. A solid cylinder of length $2 h$ along the axis with hemispherical ends as shown in Fig. 1 is a satisfactory physical approximation that satisfies (1) and (2).
(c) It is physically impossible to provide a slice generator. At very low frequencies a two-wire drive is satisfactory to approximate (3) but in this case (4) can not be satisfied. At high frequencies where (4) is readily satisfied a two-wire drive involves adjacent end surfaces and a gap in the antenna which are not taken into account in the theory. The effects of gap and end surfaces are compensating and may be taken into account roughly in comparing theoretical and experimental results by including a lumped capacitance in parallel with the experimentally measured impedances if the gap is large and a similar capacitance in parallel with the theoretical impedances if the gap is very narrow and the adjacent end surfaces of the antenna are very close together. ${ }^{1}$ A vertical antenna of length $h$ over a conducting plane, driven from a coaxial line, may be a good approximation of a slice generator, but the unavailability of an infinite, perfectly conducting plane leads to other difficulties.
(d) The condition for the far zone (4) can not be fulfilled at low radio frequencies (where accurate measurements can be made easily) because it is not possible to get far enough away from the earth. At high frequencies where this is possible, accurate measurements are difficult and the dimensions of the antenna and its driving structure become undesirably small. ${ }^{1}$

The analysis of the theoretical antenna subject to the conditions (1) to (4) discussed above, can be reduced to one-dimensional form involving the total current $I_{z}$ if it is assumed that the cross-sectional distribution of the density of current is inde-

[^12]pendent of the axial distribution. In effect, this means that the cross-sectional distribution and the internal impedance per unit length may be obtained from the analysis of an infinitely long cylinder. This is an excellent approximation subject to (1). At high frequencies the internal impedance per unit length is given by
\[

$$
\begin{equation*}
z^{i}=\frac{1}{2 \pi a} \sqrt{\frac{\omega \mu \Pi}{2 \sigma}}(1+j) \tag{5}
\end{equation*}
$$

\]

with $\sigma$ the conductivity in mhos per meter, $\mu$ the relative permeability, $a$ the radius in meters, and $\Pi=4 \pi \times 10^{-7}$ henry per meter. Subject to this assumption in addition to (1)-(4) the vector potential at any point $z$ on the cylindrical surface of the antenna due to the axial current in the entire antenna is given by ${ }^{3,8,9}$

$$
\begin{equation*}
A_{z}=\frac{\Pi}{4 \pi} \int_{-h}^{h} I_{z}^{\prime} R_{1}^{-1} e^{-j \beta R_{1}} d z^{\prime} \tag{6}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{1}=\sqrt{\left(z-z^{\prime}\right)^{2}+a^{2}} \tag{7}
\end{equation*}
$$

and $I_{s}^{\prime} \equiv I_{z}\left(z^{\prime}\right)$ is the axial current at $z^{\prime}$.
The integral equation for the current, originally derived by Hallén, ${ }^{2}$ is ${ }^{10}$

$$
\begin{align*}
\frac{4 \pi}{\Pi} A_{z} & =\int_{-h}^{h} I_{2}^{\prime} R_{1}^{-1} e^{-i \beta R_{1}} d z^{\prime} \\
& =\frac{-j 4 \pi}{R_{c}}\left[C_{1} \cos \beta z+\frac{1}{2} V_{0}^{e} \sin \beta|z|-z^{i} \int_{0}^{z} I(s) \sin \beta(z-s) d s\right] \tag{8}
\end{align*}
$$

$V_{0}$ is the driving potential difference maintained by the slice generator at $z=0 ; C_{1}$ is a constant of integration which is later evaluated using (2); $R_{c}=c \Pi=376.7$ ohms $\doteq 120 \pi$ ohms. In practice the conductivity $\sigma$ is usually sufficiently high and therefore $z^{i}$ sufficiently small so that the last integral in (8) contributes negligibly to the final result. ${ }^{11}$ For simplicity it is omitted throughout the following analysis. If required it can be included readily at appropriate points with no change in the formulation.
2. Expansion of the integral equation. In the absence of an exact solution of the integral equation (8) in closed form, an approximate solution may be obtained by expanding the integral on the left in a converging power series in terms of an appropriately chosen parameter. If a converging series is obtained and a sufficient number of terms can be evaluated the choice of the parameter for expansion is unimportant. If only a few terms in the series can be evaluated readily it is of great importance to select the parameter in such a way that convergence is so rapid that the sum of two or three terms gives a satisfactory approximation. The several parameters which have been used, ${ }^{9,12}$ including that introduced below, will be discussed critically and results compared in another paper. The general definition of all such parameters is formulated below.

[^13]The solution of (8) may be formulated by expressing $I_{z}$ in terms of a convenient reference current such as the input current $I_{0}$ and a distribution function $f(z)$ that is unknown. Thus let

$$
\begin{equation*}
I_{z}=I_{0} f(z) ; \quad I_{z}^{\prime}=I_{0} f\left(z^{\prime}\right) \tag{9a}
\end{equation*}
$$

so that

$$
\begin{equation*}
I_{z}^{\prime}=I_{z} f\left(z^{\prime}\right) / f(z) \equiv I_{2} g\left(z, z^{\prime}\right) \tag{9b}
\end{equation*}
$$

The relative distribution function $g\left(z, z^{\prime}\right)$ is defined in (9b). Now let a function $\Psi(z)$ be defined by

$$
\begin{equation*}
\Psi(z)=\int_{-h}^{h} g\left(z, z^{\prime}\right) R_{1}^{-1} e^{-j \theta R_{1}} d z^{\prime} \tag{10}
\end{equation*}
$$

If the relative distribution function $g\left(z, z^{\prime}\right)$ were the actual one, it would be correct to write $I_{z} \Psi(z)$ for the integral on the left in (8). Whatever the form of $g\left(z, z^{\prime}\right)$, it is correct to write

$$
\begin{equation*}
\frac{4 \pi}{\mathrm{II}} A_{z}=\int_{-h}^{h} I_{z}^{\prime} R_{1}^{-1} e^{-j \beta R_{1} d z^{\prime}}=I_{z} \Psi(z)+\int_{-h}^{h}\left[I_{z}^{\prime}-I_{2} g\left(z, z^{\prime}\right)\right] R_{1}^{-1} e^{-j \beta R_{1} d z^{\prime}} \tag{11}
\end{equation*}
$$

The more nearly $g\left(z, z^{\prime}\right)$ approximates the true distribution the smaller will be the difference integral on the right in (11). If $g\left(z, z^{\prime}\right)$ can be chosen accurately enough so that the integral on the right in (11) is considerably smaller than the term $I_{z} \Psi(z)$ for all values of $z$, it is possible to treat this term as the principal part and the difference integral as a correction.

If $g\left(z, z^{\prime}\right)$ were the true relative distribution function so that the difference integral in (11) were zero, the function $\Psi(z)$ would be given by

$$
\begin{equation*}
\Psi(z)=\frac{4 \pi}{\Pi} \frac{A_{x}}{I_{z}} . \tag{12}
\end{equation*}
$$

That is, $\Psi(z)$ would be proportional to the ratio of the vector potential on the surface of the antenna at a point $z$ divided by the total axial current at $z$. It is clear from (6) that the vector potential at a point $z$ is determined largely by the current at and near $z$, except possibly at a few points where $I_{z}$ is very small compared with the currents elsewhere in the antenna. It may be assumed, therefore, that the ratio $A_{\pi} / I_{*}$ is reasonably constant and predominantly real at all points along the antenna except at and near very small or zero values of the current. Clearly, since $I_{z}=0$ at the ends and $A_{z}$ is not zero there, $\Psi(z)$ is infinite at $z= \pm h$. However, the product $I_{s} \Psi(z)$ must remain finite and relatively small at $z= \pm h$.

If $\Psi(z)$ is sensibly constant for most values of $z$, it must be exactly equal to $\Psi\left(z_{0}\right)$ at some point $z=z_{0}$, so chosen that $\Psi\left(z_{0}\right)$ is a good approximation of $\Psi(z)$ except where $I_{z}$ is small or zero. Let

$$
\begin{equation*}
\Psi=\left|\Psi\left(z_{0}\right)\right| \tag{13a}
\end{equation*}
$$

so that

$$
\begin{equation*}
\Psi(z)=\Psi e^{i \theta} \Psi \tag{13b}
\end{equation*}
$$

Also let a function $\gamma(z)$ be defined so that

$$
\begin{equation*}
\Psi(z)=\Psi+\gamma(z), \tag{14a}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma(z)=\Psi\left(e^{j \theta_{\Psi}}-1\right) \tag{14~b}
\end{equation*}
$$

If $\Psi(z)$ is predominantly real, $\gamma(z)$ is a small complex correction function except at values of $z$ where $I_{z}$ is small or zero. It is to be noted that $\gamma(z)$ is infinite at $z= \pm h$ but that $I_{z} \gamma(z)$ is finite and small there.

If $g\left(z, z^{\prime}\right)$ is not the true relative distribution function but only approximate, (12) is also approximate, and it is still possible to write ( $13 \mathrm{a}, \mathrm{b}$ ) and ( $14 \mathrm{a}, \mathrm{b}$ ) with $\Psi(z)$ defined as in (10). Substituting (14a) in (11) and using (11) in (8) solved for $I_{z}$ in the principal term $I_{z} \Psi$, one obtains

$$
\begin{align*}
I_{z}= & \frac{-j 4 \pi}{R_{c} \Psi}\left\{C_{1} \cos \beta z+\frac{1}{2} V_{0}^{e} \sin \beta|z|\right\} \\
& -\frac{1}{\Psi}\left\{I_{z} \gamma(z)+\int_{-h}^{h}\left[I_{z}^{\prime}-I_{z} g\left(z, z^{\prime}\right)\right] R_{1}^{-1} e^{-i \beta R_{1} d z^{\prime}}\right\} \tag{15}
\end{align*}
$$

This equation is exact. Like (8) it is an integral equation in the current, but the current appears in the integrand of a difference integral that is small. The term $I_{z} \gamma(z)$ is also small except near points where $I_{z}$ is small or vanishes, as at $z= \pm h$.

A more useful form of (15) is obtained as follows. Let (8) be written with $z=h$ in the form

$$
\begin{equation*}
0=\frac{-j 4 \pi}{R_{e} \Psi}\left\{C_{1} \cos \beta h+\frac{1}{2} V_{0}^{e} \sin \beta h\right\}-\frac{1}{\Psi} \int_{-h}^{h} I_{z}^{\prime} R_{1 h} e^{-j \beta R_{1 h} d z^{\prime}} \tag{16}
\end{equation*}
$$

The term in $z^{i}$ has been omitted in (16) just as in (15). Actually (16) is exactly equivalent to (15) when this is written with $z=h$. In (16)

$$
\begin{equation*}
R_{1 h}=\sqrt{\left(h-z^{\prime}\right)^{2}+a^{2}} \tag{17}
\end{equation*}
$$

The desired equation is obtained by subtracting (16) from (15). It is

$$
\begin{align*}
I_{z}= & \left.\frac{-j 4 \pi}{R_{\mathrm{c}} \Psi}\left\{\left.C_{1}[\cos \beta z-\cos \beta h]+\frac{1}{2} V_{0}^{\prime}|\sin \beta| z \right\rvert\,-\sin \beta h\right]\right\} \\
& -\frac{1}{\Psi}\left\{I_{z} \gamma(z)+\int_{-h}^{h}\left[I_{z}^{\prime}-I_{2} g\left(z, z^{\prime}\right)\right] R_{1}^{-1} e^{-i \beta R_{1}} d z^{\prime}-\int_{-h}^{h} I_{2}^{\prime} R_{1 h^{-}} e^{-i \beta R_{1 h} d z^{\prime}}\right\} \tag{18}
\end{align*}
$$

This is the final exact form of the integral equation. Its principal advantage over (8) lies in the fact that all terms on the right involving the current are small if the relative distribution function $g\left(z, z^{\prime}\right)$ is correctly chosen to make the difference terms small. The expression (18) must be used in preference to (15) because in (18) the right side vanishes for all values of $\beta h$ when $z= \pm h$ as required by (2), whereas the right side in (15) can not be made to vanish at $x= \pm h$ when $\cos \beta h=0$. In this case the arbitrary constant $C_{1}$ disappears from (15).

The integral equation (18) can be expressed as the sum of a principal current $\left(I_{z}\right)_{0}$ consisting of the trigonometric terms and a correction current $\left(I_{z}\right)_{c}$ given by the remaining terms. The correction term $\left(I_{z}\right)_{c}$ can then be expanded in a power series in $1 / \Psi$. Thus

$$
\begin{equation*}
I_{z}=\left(I_{z}\right)_{0}+\left(I_{z}\right)_{c}=\left(I_{z}\right)_{0}+\left(I_{z}\right)_{c_{1}}+\left(I_{2}\right)_{c_{2}}+\left(I_{z}\right)_{c_{3}}+\cdots . \tag{19}
\end{equation*}
$$

Here $\left(I_{z}\right)_{C_{1}}$ is obtained by substituting $\left(I_{z}\right)_{0}$ in $\left(I_{2}\right)_{c} ;\left(I_{z}\right)_{c_{1}}+\left(I_{z}\right)_{c_{2}}$ is obtained using $\left(I_{z}\right)_{0}+\left(I_{z}\right)_{c_{1}}$ in $\left(I_{z}\right)_{c}$, etc.

For convenience let

$$
\begin{equation*}
F_{n}(z)-F_{n}(h) \equiv F_{n z} ; \quad G_{n}(z)-G_{n}(h) \equiv G_{n z} \tag{2ab}
\end{equation*}
$$

where

$$
\begin{align*}
& \quad F_{0}(z) \equiv \cos \beta z ; \quad F_{0}(h) \equiv \cos \beta h ; \quad G_{0}(z) \equiv \sin \beta|z| ; \quad G_{0}(h) \equiv \sin \beta h  \tag{20b}\\
& F_{n}(z) \equiv F_{n-1, z} \int_{-h}^{h} g\left(z, z^{\prime}\right) R_{1}^{-1} e^{-i \beta R_{2} d z^{\prime}}-\int_{-h}^{h} F_{n-1, z^{\prime}} R_{1}^{-1} e^{-i \beta R_{1} d z^{\prime}-F_{n-1, z} \gamma(z)}  \tag{21a}\\
& F_{n}(h) \equiv-\int_{-h}^{h} F_{n-1, z^{\prime}} R_{1} \bar{h}^{-1} e^{-i \beta R_{1} h} d z^{\prime} . \tag{21b}
\end{align*}
$$

The first and last terms in (21a) may be combined into $F_{n-1, z} \Psi$ using (10) and (14a). Expressions for $G_{n}(z)$ and $G_{n}(h)$ are obtained from (21a) and (21b) by writing $G$ for $F$ throughout.

Using (19)-(21) in (18), the complete series solution for $I_{2}$ may be obtained. The constant $C_{1}$ may be evaluated from (16) using (19)-(21) as described in references 2 and 3 . The resulting $m$ th order current is ${ }^{13}$

$$
\begin{equation*}
\left(I_{z}\right)_{m}=\frac{j 2 \pi V_{0}^{0}}{R_{c} \Psi}\left\{\frac{\sum_{n=0}^{m} F_{n}(z) / \Psi^{n} \cdot \sum_{n=0}^{m} G_{n}(h) / \Psi^{n}-\sum_{n=0}^{m} G_{n}(z) / \Psi^{n} \cdot \sum_{n=0}^{m} F_{n}(h) / \Psi^{n}}{\sum_{n=0}^{m} F_{n}(h) / \Psi^{n}}\right\} \tag{22}
\end{equation*}
$$

This formula may be simplified using ( $20 \mathrm{a}, \mathrm{b}$ ). The result is

$$
\begin{equation*}
\left(I_{z}\right)_{m}=\frac{j 2 \pi V_{0}^{e}}{R_{c} \Psi}\left\{\frac{\sin \beta(h-|z|)+\sum_{n=1}^{m} M_{n}(z) / \Psi^{n}}{\cos \beta h+\sum_{n=1}^{m} F_{n}(h) / \Psi^{n}}\right\} \tag{23}
\end{equation*}
$$

where, in particular,

$$
\begin{align*}
M_{1}(z) \equiv & M_{1}^{I}(z)+j M_{1}^{I I}(z)=F_{1}(z) \sin \beta h-F_{1}(h) \sin \beta|z|+G_{1}(h) \cos \beta z \\
& -G_{1}(z) \cos \beta h,  \tag{24}\\
M_{2}(z) \equiv & M_{2}^{I}(z)+j M_{2}^{I I}(z)=F_{2}(z) \sin \beta h-F_{2}(h) \sin \beta \mid z \\
& +G_{1}(h) F_{1}(z)-G_{1}(z) F_{1}(h)+G_{2}(h) \cos \beta z-G_{2}(z) \cos \beta h, \tag{25}
\end{align*}
$$

and, as previously defined, ${ }^{2,4,5,6}$

$$
\begin{equation*}
F_{n}(h)=\alpha_{n} \equiv \alpha_{n}^{I}+j \alpha_{n}^{I I} . \tag{26}
\end{equation*}
$$

With

[^14]\[

$$
\begin{equation*}
\beta_{n}=\beta_{n}^{I}+j \beta_{n}^{I I} \equiv M_{n}(0)=M_{n}^{I}(0)+j M_{n}^{I I}(0), \tag{27}
\end{equation*}
$$

\]

the impedance of the antenna is defined by

$$
\begin{equation*}
Z_{0}=V_{0}^{\theta} / I_{0}, \tag{28}
\end{equation*}
$$

where $I_{0}$ is given by (23) with $z=0$. It is

$$
\begin{equation*}
\left(Z_{0}\right)_{m}=\left(R_{0}\right)_{m}+j\left(X_{0}\right)_{m}=\frac{-j R_{c} \Psi}{2 \pi}\left\{\frac{\cos \beta h+\sum_{n=1}^{m} \alpha_{n} / \Psi^{n}}{\sin \beta h+\sum_{n=1}^{m} \beta_{n} / \Psi^{n}}\right\} . \tag{29}
\end{equation*}
$$

This is a generalization of the formula obtained by Hallén ${ }^{2}$ and others ${ }^{3,13}$, as shown later.
3. Functions and parameters in the Hallén solution. The expressions for the current (23) and for the impedance (29) depend upon the constant parameter $\Psi$, and this in turn depends upon the relative distribution function $g\left(z, z^{\prime}\right)$. The definition of these quantities involves the following considerations: The relative distribution function $g\left(z, z^{\prime}\right)$ must be so chosen that it is a sufficiently good approximation of the actual distribution to make the difference integral in (11) small. Furthermore, it must be sufficiently simple in form that the integral (10) for $\Psi(z)$ can be evaluated and separated into a principal, constant, real part $\left|\Psi\left(z_{0}\right)\right| \equiv \Psi$ and a small correction term $\gamma(z)$ as in (14a, b).

The choice of distribution function made by Hallén depended upon the reasonable albeit implicit assumption that the vector potential $A_{z}$ at $z$ depends primarily upon the current at and near $z$. If contributions from all more distant elements of current are small, $A_{z}$ may be evaluated approximately by assuming the current at all points to be $I_{z}$ and neglecting retardation. This is equivalent to setting

$$
\begin{equation*}
g_{I I}\left(z, z^{\prime}\right)=e^{j \beta R_{1}} . \tag{30}
\end{equation*}
$$

The subscript $H$ will be used to designate parameters and functions in the Hallén analysis. With (30), (10) gives

$$
\begin{equation*}
\Psi_{H}(z)=\int_{-h}^{h} \frac{d z^{\prime}}{R_{1}}=\sinh ^{-1} \frac{h+z}{a}+\sinh ^{-1} \frac{h-z}{a} . \tag{31}
\end{equation*}
$$

Alternatively and equivalently

$$
\begin{equation*}
\Psi_{I I}(z)=\Omega+\ln \left(1-\frac{z^{2}}{h^{2}}\right)+\delta(z) \tag{32}
\end{equation*}
$$

where

$$
\begin{align*}
\Omega & =\Psi_{H}(z=0)=2 \ln \frac{2 h}{a}  \tag{33}\\
\delta(z) & =\ln \left\{\frac{1}{4}\left[\sqrt{1+\left(\frac{a}{h+z}\right)^{2}}+1\right]\left[\sqrt{1+\left(\frac{a}{h-z}\right)^{2}}+1\right]\right\} \tag{34}
\end{align*}
$$

If the function $\Psi_{H}(z)$ in (31) is plotted as a function of $z / a$ for a range of values of the ratio $h / a$, it is found to be moderately constant for large ratios $h / a$ except with
$z$ near $h$. Subject to the condition $a^{2} \ll h^{2}$, the maximum value of $\Psi_{I I}(z)$ is $\Omega$, the average value is $\Omega-2+2 \ln 2=\Omega-0.614$, the value at $z= \pm h$ is $\frac{1}{2} \Omega+\ln 2$. For $\Omega>15$ the average value or the maximum value are satisfactory approximations. In view of the fact that $\Psi_{H}(z)$ becomes smaller at $z= \pm h$ instead of becoming infinite as it would if the correct distribution function were used, it is clear that the curvature of $\Psi_{H}(z)$ is the reverse of what it should be. Therefore, the maximum value $\Omega$ is probably the best approximation of $\Psi_{H}(z)$ and this was Hallén's, although not explicitly for this reason. Thus the Hallén analysis sets

$$
\begin{align*}
\Psi_{H}\left(z_{0}\right) & \equiv \Psi_{H}=\Omega=2 \ln \frac{2 h}{a}  \tag{35a}\\
\gamma_{H}(z) & =\ln \left(1-\frac{z^{2}}{h^{2}}\right)+\delta(z) \tag{35b}
\end{align*}
$$

The Hallén expressions for the current and the impedance are given by (23) and (29) with $\Omega$ written for $\Psi$ and with appropriately modified functions $F_{n}(z), F_{n}(h), G_{n}(z)$, and $G_{n}(h), n>0$. The functions with $n=0$ are independent of the choice of $\Psi$. The Hallén functions are

$$
\begin{align*}
& F_{n H}(z)=\left(F_{n-1, z}\right)_{H} \Omega-\int_{-h}^{h}\left(F_{n-1, z^{\prime}}\right)_{H} \frac{e^{-j \beta R_{1}}}{R_{1}} d z^{\prime}  \tag{36a}\\
& F_{n H}(h)=-\int_{-h}^{h}\left(F_{n-1, z^{\prime}}\right)_{H} \frac{e^{-j \beta R_{1 h}}}{R_{1 h}} d z^{\prime} . \tag{36b}
\end{align*}
$$

$G_{n H}(z)$ and $G_{n H}(h)$ are obtained from the above by writing $G$ for $F$. These functions have been evaluated elsewhere ${ }^{3,13}$ for $n=1$ and $n=2$. The first order distribution of current and the first order impedance have been calculated and represented graphically ${ }^{3,11}$; the second order impedance has been evaluated by Bouwkamp. ${ }^{13}$ The Hallen formula for the $m$ th order current is

$$
\begin{align*}
& \left(I_{z}\right)_{m H}=\frac{j 2 \pi V_{0}^{2}}{R_{e} \Omega}\left\{\frac{\sin \beta(h-|z|)+\sum_{n=1}^{m} M_{n H}(z) / \Omega^{n}}{\cos \beta h+\sum_{n=1}^{m} F_{n H}(h) / \Omega^{n}}\right\},  \tag{37}\\
& \left(Z_{0}\right)_{m H}=\frac{-j R_{c} \Omega}{2 \pi}\left\{\frac{\cos \beta h+\sum_{n=1}^{m} \alpha_{n H} / \Omega^{n}}{\sin \beta h+\sum_{n=1}^{m} \beta_{n H} / \Omega^{n}}\right\} \tag{38a}
\end{align*}
$$

Here

$$
\begin{equation*}
\alpha_{n H}=\alpha_{n}^{I}+j \alpha_{n}^{I I}=F_{n H}(h), \quad(38 \mathrm{~b}) \quad \beta_{n H}=\beta_{n}^{I}+j \beta_{n}^{I I}=M_{n H}(0) \tag{38b}
\end{equation*}
$$

The functions $\alpha_{1}$ and $\beta_{1}$ are tabulated and represented graphically in references 2,3 , 11 ; the functions $\alpha_{2}$ and $\beta_{2}$ as calculated by Bouwkamp ${ }^{13}$ using graphical methods are listed in Table I and plotted in Figs. 2 and 3.
4. Functions and parameters in the improved solution. The relative distribution function $g\left(z, z^{\prime}\right)$ in (30) is the simplest and the most obvious one if an attempt is

Table I

made to solve the original integral equation (8) as was done by Hallén. On the other hand, if the formal solution is carried through to obtain (23) without previously selecting $g\left(z, z^{\prime}\right)$ as has been done in the present analysis, it is perfectly clear that the leading term in the distribution of current for any value of the distribution function must be of the form

$$
\begin{equation*}
I_{z}=K f_{1}(z) \tag{39a}
\end{equation*}
$$

with

$$
\begin{equation*}
f_{3}(z) \equiv \sin \beta(h-|z|) \tag{39b}
\end{equation*}
$$

$K$ is an amplitude factor independent of $z$. Accordingly, an approximate relative distribution function is

$$
\begin{equation*}
g_{K_{1}}\left(z, z^{\prime}\right)=\frac{\sin \beta\left(h-\left|z^{\prime}\right|\right)}{\sin \beta(h-|z|)}=\frac{f_{1}\left(z^{\prime}\right)}{f_{1}(z)} \tag{40}
\end{equation*}
$$

This function is known to be a very much better approximation of the actual current then the function assumed by Hallén, $g_{H}\left(z, z^{\prime}\right)=e^{i \beta R_{1}}$. The function $f_{1}(z)=\sin \beta(h-|z|)$ actually is proportional to the principal part of the current; the function $e^{\beta \beta R_{1}}$ is not. Using (40) and (10) we obtain


FIG. 2. The parameters $\alpha_{2}^{\prime}$ and $\alpha_{2}^{\prime \prime}$ as a function of $\beta h$.


FIg. 3. The parameters $\beta_{2}^{I}$ and $\beta_{2}^{I I}$ as a function of $\beta h$.

$$
\begin{equation*}
\Psi_{K 1}(z)=\int_{-h}^{h} g_{K 1}\left(z, z^{\prime}\right) R_{1}^{-1} e^{-j \beta R_{1} d z^{\prime}} \tag{41}
\end{equation*}
$$

This function involves the factor $f_{1}(z)=\sin \beta(h-|z|)$ in the denominator of the integrand. Since this is not a function of $z^{\prime}$ it is a constant in the integration. Therefore, it is convenient to introduce the function

$$
\begin{equation*}
\psi_{1}(z) \equiv \int_{-h}^{h} f_{1}\left(z^{\prime}\right) R_{1}^{-1} e^{-j \beta R_{1}} d z^{\prime} \tag{42a}
\end{equation*}
$$

so that

$$
\begin{equation*}
\Psi_{K 1}(z)=\frac{\psi_{1}(z)}{f_{1}(z)} \tag{42b}
\end{equation*}
$$

The function $\psi_{1}(z)$ can be written in the form

$$
\begin{equation*}
\Psi_{1}(z)=C(z) \sin \beta h-S(z) \cos \beta h \tag{43}
\end{equation*}
$$

where

$$
\begin{align*}
& C(z) \equiv \int_{-h}^{h} \cos \beta z^{\prime} R_{1}^{-1} e^{-j \beta R_{1}} d z^{\prime}  \tag{44}\\
& S(z) \equiv \int_{-h}^{h} \sin \beta\left|z^{\prime}\right| R_{1}^{-1} e^{-j \beta R_{1} d z^{\prime}} \tag{45}
\end{align*}
$$

These integrals are evaluated in the Appendix both in general and in a simpler approximate form. The latter is a good approximation if, as assumed throughout this analysis, $h^{2} \gg a^{2}$. Curves for $C(z)$ and $S(z)$ as calculated using the simpler forms which apply in this analysis are given in Figs. $4-7,20-23$ for $\beta h=\pi / 2$ and $\pi$ and for $\Omega$ $=2 \ln (2 h / a)=10$ and 20 . It is to be noted that

$$
\begin{array}{ll}
\psi_{1}(z)=C(z) ; & \beta h=\pi / 2 \\
\psi_{1}(z)=S(z) ; & \beta h=\pi \tag{47}
\end{array}
$$

It follows that the plots of $C(z)$ with $\beta h=\pi / 2$ are also plots of $\psi_{1}(z)$; these are given in Figs. 4 and 5 for $\Omega=10$ and 20. Similarly plots of $S(z)$ with $\beta h=\pi$ are also plots of $\psi_{1}(z)$; these are given in Figs. 6 and 7 for $\Omega=10$ and 20 . The function $\psi_{1}(z)$ is seen to have a very small imaginary part so that it and $\Psi_{K_{1}}(z)=\psi_{1}(z) / f_{1}(z)$ are predominantly real, in confirmation of the assumption made in conjunction with (14). Accordingly, the parameter $\Psi=\left|\Psi\left(z_{0}\right)\right|$ defined in (13a) may be chosen to be

$$
\begin{align*}
& \Psi_{K 1}=\left|\Psi_{K 1}(0)\right|=\left|\psi_{1}(0)\right| ; \quad \beta h=\pi / 2 ;  \tag{48}\\
& \Psi_{K 1}=\left|\Psi_{K 1}(h-\lambda / 4)\right|=\left|\psi_{1}(h-\lambda / 4)\right| ; \quad \beta h=\pi \tag{49}
\end{align*}
$$

The function

$$
\begin{equation*}
\left|\Psi_{K_{1}}(z)\right|=\frac{\left|\psi_{1}(z)\right|}{f_{1}(z)} \tag{50}
\end{equation*}
$$

is plotted in Figs. 4-7. For $\beta h=\pi / 2$ and both for $\Omega=10$ and 20 it is seen to be quite constant over the entire length of the antenna except near the ends where it becomes infinite, as it should. For $\beta h=\pi$ the function becomes infinite not only at the ends but also at the center. The infinity at the center is a result of approximating the


Fig. 4. The functions $C(z)=\psi_{1}(z) ;\left|\psi_{1}(0)\right| f_{1}(z)$, and $\left|\psi_{1}(z)\right| / f_{1}(z)$ near resonance, $\beta h=\pi / 2, \Omega=10$.


Fig. 5. The functions $C(z)=\psi_{1}(z),\left|\psi_{1}(0)\right| f_{1}(z)$, and $\left|\psi_{1}(z)\right| / f_{1}(z)$ near resonance, $\beta h=\pi / 2, \Omega=20$.


Fig. 6. The functions $\left.S(z)=\psi_{1}(z), \mid \psi_{1} h-\lambda / 4\right) \mid f_{1}(z)$, and $\left|\psi_{1}(z)\right| / f_{1}(z)$ near anti-resonance, $\beta h=\pi, \Omega=10$.


Fig. 7. The function $S(z)=\psi_{1}(z),\left|\psi_{1}(h-\lambda / 4)\right| f_{1}(z)$, and $\mid \psi_{1}(z) / f_{1}(z)$ near anti-resonance, $\beta h=\pi, \Omega=20$.
current by the distribution function $f_{1}(z)=\sin \beta(h-|z|)$. With $\beta h=\pi$ and $z=0$, this vanishes so that $\Psi_{\kappa_{1}}(z)$ necessarily diverges. Unlike the infinity at the ends, the infinity at $z=0$ is due to the fact that $f_{1}(z)$ and hence $g_{K}\left(z, z^{\prime}\right)$ are approximate and not exact distribution functions. Actually, the current does not vanish at $z=0$; it merely is small so that $\Psi(z)$ does not become infinite. The fact that $I_{z}$ is small at and near $z=0$ does not mean that $\Psi(z)$ necessarily becomes very large. $\Psi(z)$ is by definition proportional to the ratio $A_{z} / I_{z}$, and $A_{z}$ is determined largely by the current at $z$. Hence $A_{z=0}$ is small if $I_{z=0}$ is, and the ratio may remain moderately constant. Furtheremore, since $A_{z}$ at $z=h-\lambda / 4$ is determined principally by the large (near maximum) currents at and near $z=h-\lambda / 4$, it is affected only very slightly by a small current at $z=0$. Therefore $A_{z}$ at $z=h-\lambda / 4$ and $\Psi_{K 1}(h-\lambda / 4)$ will not be sensibly different if a fictitious zero current is assumed at $z=0$ or an actual small current. Accordingly the function $\mid \Psi_{K_{1}}\left(h-\lambda / 4 \mid\right.$ is a good approximation of $\Psi_{K_{1}(z)}$ for the actual current everywhere (including $z=0$ ) except near the ends, $z= \pm h$.

Although the qualitative argument to show that $\Psi_{K_{1}}(z)$ is sensibly constant and approximately equal to $\left|\Psi_{K 1}(h-\lambda / 4)\right|$ for all values of $z$ except the ends is sound, it can be verified directly using Hallén's first order distribution. It has been shown ${ }^{3}$ that a very satisfactory approximation of the Hallén first order current is given by

$$
\begin{equation*}
I_{z}=I_{0}^{\prime \prime}\left(\frac{\cos \beta z-\cos \beta h}{1-\cos \beta h}\right)+j_{t} I_{n_{2}^{\prime}} \sin \beta(h-|z|) ; \frac{\pi}{2} \leqq \beta h<2 \pi, \tag{51}
\end{equation*}
$$

where $I_{0}^{\prime \prime}$ is the component of current at $z=0$ in phase with the driving potential difference and $I_{m}^{\prime}$ is the maximum value of the component of current in phase quadrature with the driving potential difference. $I_{m}{ }^{\prime}$ occurs at $z=h-\lambda / 4$. With

$$
\begin{equation*}
k=I_{o}^{\prime \prime} / I_{m}^{\prime} ; \quad|k|<1 \tag{52}
\end{equation*}
$$

it is possible to write (51) in the form

$$
\begin{equation*}
I_{z}=j I_{m}^{\prime}\left\{\sin \beta(h-|z|)-j k\left(\frac{\cos \beta z-\cos \beta h}{1-\cos \beta h}\right)\right\} \equiv j I_{m}^{\prime} f_{2}(z) \tag{53}
\end{equation*}
$$

With this approximate current, an appropriate distribution function $g\left(z, z^{\prime}\right)$ is defined by

$$
\begin{align*}
g_{2}\left(z, z^{\prime}\right)=\frac{I_{z}^{\prime}}{I_{z}} & =\frac{\sin \beta\left(h-\left|z^{\prime}\right|\right)-j k\left(\cos \beta z^{\prime}-\cos \beta h\right) /(1-\cos \beta h)}{\sin \beta(h-|z|)-j k(\cos \beta z-\cos \beta h) /(1-\cos \beta h)} \\
& \equiv \frac{f_{2}\left(z^{\prime}\right)}{f_{2}(z)} ; \quad \frac{\pi}{2} \leqq \beta h<2 \pi \tag{54}
\end{align*}
$$

The ratio factor $k$ is negative and small compared with unity. It is plotted in Fig. 8 as a function of $\beta$ from the data of Figs. 9-11 in reference 3. Only values of $\beta$ hear $\pi$ are used because for $\beta h$ not near integral multiples of $\pi$ the distribution ( $\cos \beta z$ $-\cos \beta h)$ does not differ greatly from $\sin \beta(h-|z|)$. At $\beta h=\pi / 2$ they are identical.

Using the notation (42a, b),

$$
\begin{equation*}
\psi_{2}(z) \equiv \int_{-h}^{h} f_{2}\left(z^{\prime}\right) R_{1}^{-1} e^{-i \beta R_{1} d z^{\prime}} \tag{55a}
\end{equation*}
$$

so that

$$
\begin{equation*}
\Psi_{K 2}(z)=\frac{\psi_{2}(z)}{f_{2}(z)} \tag{55b}
\end{equation*}
$$

The function $\psi_{2}(z)$ can be written

$$
\begin{equation*}
\psi_{2}(z)=C(z) \sin \beta h-S(z) \cos \beta h-j k\left[\frac{C(z)-E(z) \cos \beta h}{1-\cos \beta h}\right] \tag{56}
\end{equation*}
$$

where $C(z)$ and $S(z)$ are defined in (44) and (45), and $E(z)$ is given by

$$
\begin{equation*}
E(z) \equiv \int_{-h}^{h} R_{1}^{-1} e^{-j \beta R_{1}} d z^{\prime} \tag{57}
\end{equation*}
$$



FIg. 8. The quantity $k=I_{0}^{\prime \prime} / I_{m}^{\prime}$ as a function of $\beta h$ near anti-resonance.
This function is evaluated in the Appendix both in general and in a simpler approximate form valid when it is possible to write $a^{2} \ll h^{2}$, as in the present analysis. $E(z)$ is plotted in Figs. 24 and 25 for $\beta h=\pi / 2$ and $\pi$ and with $\Omega=10$ and 20. The function $\psi_{2}(z)$ is necessarily predominantly real because it is known that the first two terms in (56)-these are identically $\psi_{1}(z)$-are predominantly real and that $k$ is small. The functions $\left|\psi_{2}(z)\right|$ and $\left|\Psi_{K_{2}}(z)\right|=\left|\psi_{2}(z)\right| /\left|f_{2}(z)\right|$ are shown in Figs. 9 and 10 for $\beta h=\pi$ and $\Omega=10$ and 20 . It is seen that $\left|\Psi_{K 2}(z)\right|$ does not become infinite at $z=0$, and is reasonably constant and equal to $\left|\Psi_{K_{2}}(h-\lambda / 4)\right| \doteq\left|\psi_{2}(h-\lambda / 4)\right|$ for all values of $z$ except at the ends where it becomes infinite, as it should. Comparison of Figs. 9 and 10 with 6 and 7 shown that $\left|\psi_{2}(h-\lambda / 4)\right|$ differs only slightly from $\left|\psi_{1}(h-\lambda / 4)\right|$. The difference is greater for the smaller value of $\Omega$. It follows that $\left|\psi_{1}(h-\lambda / 4)\right|$ is a satisfactory parameter even for $\beta h=\pi$. If desired $\left|\psi_{2}(h-\lambda / 4)\right|$ may be used especially for small values of $\Omega$, but the difference is not over about $3 \%$ for $\Omega \geqq 10$.


Fig. 9. The functions $\left|f_{2}(z)\right|,\left|\psi_{2}(h-\lambda / 4)\right|\left|f_{2}(z)\right|$, and $\left|\psi_{2}(z)\right| /\left|f_{2}(z)\right|$ near anti-resonance, $\beta h=\pi, \Omega=10$.


FIG. 10. The functions $\left|f_{2}(z)\right|,\left|\psi_{2}(h-\lambda / 4)\right|\left|f_{2}(z)\right|$, and $\left|\psi_{2}(z)\right| /\left|f_{2}(z)\right|$ near anti-resonance, $\beta h=\pi, \Omega=20$.

The conclusions of the above analysis may be generalized and summarized as follows: 1 . The relative distribution function

$$
g_{1 K}\left(z, z^{\prime}\right)=\frac{\sin \beta\left(h-\left|z^{\prime}\right|\right)}{\sin \beta(h-|z|)}
$$

is a good approximation for all values of $h 2$. 2 . Suitable parameters for expansion are


Fig. 11. The parameter $\psi$, Eqs. (48) and (49), as a function of $\beta h$, for $\Omega=10,15,20$.
$\left|\Psi_{K 1}(0)\right|=\left|\psi_{1}(0)\right|$ for $\beta h \leqq \pi / 2 ;\left|\Psi_{K_{1}}(h-\lambda / 4)\right|=\left|\psi_{1}(h-\lambda / 4)\right|$ for $\beta h \geqq \pi / 2$. The following notation will be used from here on

$$
\psi \equiv \Psi= \begin{cases}\left|\Psi_{K 1}(0)\right|=\left|\psi_{1}(0)\right| ; & \beta h \leqq \pi / 2  \tag{58}\\ \left|\Psi_{K_{1}}(h-\lambda / 4)\right|=\left|\psi_{1}(h-\lambda / 4)\right| ; & \beta h \geqq \pi / 2\end{cases}
$$

Since $\Psi_{K 1}(z)$ has such a small imaginary part and is so well represented by (58) except at the ends of the antenna, the correction function $\gamma(z)$ in (14a) is sufficiently small to be neglected.

The parameter $\psi$ as defined in (58) is plotted as a function of $\beta h$ for $\Omega=10,15,20$
in Fig. 11. For $\Omega=10$ the curve in solid line for $\beta h>\pi / 2$ is $\left|\psi_{2}(h-\lambda / 4)\right|$, the curve in broken line is $\left|\psi_{1}(h-\lambda / 4)\right|$. The curves for $\Omega=15$, and 20 with $\beta h>\pi / 2$ are $\left|\psi_{1}(h-\lambda / 4)\right|$.
5. Distribution of current. The distribution of current is given by (23), the impedance by (29) with $\Psi(=\psi)$ obtained from (58) or Fig. 11 for the value of $h / a$ in question. The functions $F_{0}(z), F_{0}(h), G_{0}(z)$, and $G_{0}(h)$ are unchanged; they are defined by (20b). Upon substituting (40) in (10) and using (10) in the form (14a) in (21a) with $n=1$, this becomes

$$
\begin{equation*}
F_{1 K}(z)=F_{0 z} \Psi_{K 1}(z)-\int_{-h}^{h} F_{0 z^{\prime}} R_{1}^{-1} e^{-i \beta R_{1} d z^{\prime}} \tag{59}
\end{equation*}
$$

Upon comparing (59) with (36a) written for $n=1$, it follows that

$$
\begin{equation*}
F_{1 K}(z)=F_{1 H}(z)+(\psi-\Omega)\left(F_{0 z}\right)_{H} \tag{60}
\end{equation*}
$$

Since $\Psi(z)$ is not involved in (21b),

$$
\begin{equation*}
F_{1 K}(h)=F_{1 H}(h) \tag{61}
\end{equation*}
$$

Upon substituting (60) in (21a) with $\gamma(z)=0$ and $n=2$, this becomes

$$
\begin{equation*}
F_{2 K}(z)=\left(F_{1 z}\right)_{K} \Psi_{K 1}(z)-\int_{-h}^{h}\left(F_{1 z^{\prime}}\right)_{K} R_{1}^{-1} e^{-i \beta R_{1} d z^{\prime} .} \tag{62}
\end{equation*}
$$

Using (60) and (61) in (62), this gives

$$
\begin{equation*}
F_{2 K}(z)=\left(F_{1 z}\right)_{K} \psi-\int_{-h}^{h}\left(F_{1 z}\right)_{I I} R_{1}^{-1} e^{-j \beta R_{1}} d z^{\prime}-(\psi-\Omega) \int_{-h}^{h} F_{0 z} \cdot R_{1}^{-1} e^{-j \beta R_{1} d z^{\prime}} \tag{63}
\end{equation*}
$$

Upon comparing (63) with (36a), this time written with $n=2$, and using (60) and (61) as well as (36a) written with $n=1$, we see that

$$
\begin{aligned}
F_{2 K}(z)= & \left(F_{1 z}\right)_{H} \psi+\psi(\psi-\Omega)\left(F_{0 z}\right)_{H}+F_{2 H}(z)-\left(F_{1 z}\right)_{H} \Omega \\
& -(\psi-\Omega)\left[\left(F_{0 z}\right)_{H} \psi-F_{1 H I}(z)-(\psi-\Omega)\left(F_{0_{z}}\right)_{H}\right] .
\end{aligned}
$$

When terms are collected there results,

$$
\begin{equation*}
F_{2 K}(z)=F_{2 H}(z)+(\psi-\Omega)\left(F_{1 z}\right)_{H}+(\psi-\Omega) F_{1 H}(z)+(\psi-\Omega)^{2}\left(F_{0 z}\right)_{H} . \tag{64}
\end{equation*}
$$

Using (21b) with (60) and (61), we find

$$
\begin{equation*}
F_{2 K}(h)=F_{2 H}(h)+(\psi-\Omega) F_{1 H}(h) \tag{65}
\end{equation*}
$$

Subtracting (65) from (64), we have

$$
\begin{equation*}
\left(F_{2 z}\right)_{K}=\left(F_{2 z}\right)_{H}+2(\psi-\Omega)\left(F_{1 z}\right)_{H}+(\psi-\Omega)^{2}\left(F_{0 z}\right)_{H} \tag{66}
\end{equation*}
$$

Repetition of the above procedure to evaluate $\left(F_{n_{z}}\right)_{K}$ leads to:

$$
\begin{equation*}
\left(F_{n 2}\right)_{K}=\sum_{i=0}^{n} \frac{n!}{(n-i)!i!}(\psi-\Omega)^{i}\left(F_{n-i, 2}\right)_{H} ; \quad n \geqq 0 \tag{67}
\end{equation*}
$$

Expressions for $G_{2 K}(z), G_{2 K}(h),\left(G_{2 \varepsilon}\right)_{K}$, and $\left(G_{n z}\right)_{K}$ are obtained from (64)-(67) by writing $G$ for $F$.

If the functions $F$ and $G$ are combined to form the functions $M$ as defined in (24) and (25) for the first two terms, the result is

$$
\begin{align*}
& M_{1 K}(z)=M_{1 H}(z)+(\psi-\Omega) \sin \beta(h-|z|)  \tag{68}\\
& M_{2 K}(z)=M_{2 H}(z)+2(\psi-\Omega) M_{1 H}(z)+(\psi-b)^{2} \sin \beta(h-|z|) . \tag{69}
\end{align*}
$$

In general,

$$
\begin{equation*}
M_{n K}(z)=\sum_{i=0}^{n} \frac{n!}{(n-i)!i!}(\psi-\mathrm{b})^{i} M_{n-i, H}(z) ; \quad n \geqq 0, \tag{70}
\end{equation*}
$$

where it is understood that

$$
\begin{equation*}
M_{O H}(z)=\sin \beta(h-|z|) \tag{71}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
F_{n K}(h)=\sum_{i=0}^{n-1} \frac{(n-1)!}{(n-i-1)!i!}(\psi-\Omega)^{i} F_{n-i, H}(h) ; \quad n \geqq 1 \tag{72}
\end{equation*}
$$

Upon substituting (70)-(72) in (23) the general expression for the $m$ th order current becomes

$$
\begin{equation*}
\left(I_{z}\right)_{m}=\frac{j 2 \pi V_{0}^{e}}{R_{c} \psi}\left\{\frac{\left(D_{1}\right)_{m} \sin \beta(h-|z|)+\sum_{n=1}^{m}\left(D_{n+1}\right)_{m} M_{n H}(z) / \psi^{n}}{\cos \beta h+\sum_{n=1}^{m}\left(D_{n}\right)_{m-1} F_{n H}(h) / \psi^{n}}\right\} \tag{73}
\end{equation*}
$$

where with

$$
\begin{equation*}
x \equiv 1-\frac{\Omega}{\psi} \tag{74}
\end{equation*}
$$

the $D$ 's have the following significance:

| Order $m=$ | 0 | 1 | 2 | 3 | 4 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\left(D_{1}\right)_{m}=$ | 1 | $+x$ | $+x^{2}$ | $+x^{3}$ | $+x^{4}$ | $+\cdots$ |
| $\left(D_{2}\right)_{m}=$ |  | 1 | $+2 x$ | $+3 x^{2}$ | $+4 x^{8}$ | $+\cdots$ |
| $\left(D_{3}\right)_{m}=$ |  |  | 1 | $+3 x$ | $+6 x^{2}$ | $+\cdots$ |
| $\left(D_{4}\right)_{m}=$ |  |  |  | 1 | $+4 x$ | $+\cdots$ |
| $\left(D_{5}\right)_{m}=$ |  |  |  |  | 1 | $+\cdots$ |

It is interesting and significant to note than when the series in (75) are summed for an infinite number of terms, i.e., $m \rightarrow \infty$, then

$$
\begin{equation*}
D_{n}=\frac{d^{n-1}}{d x^{n-1}}\left(\frac{1}{1-x}\right)=\frac{1}{(1-x)^{n}}=\left(\frac{\downarrow}{\Omega}\right)^{n} ; \quad x<1 ; \quad n \geqq 1 \tag{76}
\end{equation*}
$$

With these values of $\left(D_{n}\right)_{m}$ and an infinite number of terms, (73) is identical with the expression (37) obtained by Hallén. Furthermore, if $a \rightarrow 0, \psi \rightarrow \Omega$ for all values of $\beta h$, so that (73) approaches (37) as the radius $a$ approaches zero.

It is important to note that if a finite number of terms is used in (73), all terms belonging to a given order $m$ of solution must be retained and no others. That is, if


Fig．12．The quantitics $D_{1}$ and $D_{2}$ for the second order theory as a function of $\beta h, \Omega=10,15,20$.


Fig．13．The quantity $D_{1}$ for the first order theory as a function of $\beta h, \Omega=10,15,20$.


Fig. 14. First order current for $\beta h=\pi / 2$ in units of $V_{0} / 60 \Omega D_{H}$.


Fig. 15. First order current for $\beta h=3 \pi / 4$ in units of $V_{0}^{0} / 60 \Omega D_{H}$.
an $m$ th order solution is evaluated, only terms contributed by $M_{n K}(z)$ and $F_{n K}(h)$ with $n=0,1,2, \cdots, n$ are used. It is readily verified that this is equivalent to writing

$$
\begin{align*}
& \left(D_{1}\right)_{2}=1+\left(1-\frac{\Omega}{\psi}\right)+\left(1-\frac{\Omega}{\psi}\right)^{2} \\
& \left(D_{2}\right)_{2}=1+2\left(1-\frac{\Omega}{\psi}\right)  \tag{77a}\\
& \left(D_{3}\right)_{2}=1
\end{align*}
$$



Fig. 16. First order current for $\beta h=\pi$ in units of $V_{0}^{0} / 60 \Omega D_{H}$.
The expressions $\left(D_{1}\right)_{2}$ and $\left(D_{2}\right)_{2}$ are plotted as functions of $\beta h$ for $\Omega=10,15,20$, in Fig. 12. Similarly it is correct to set

$$
\begin{align*}
& \left(D_{1}\right)_{0} \equiv 1 \\
& \left(D_{1}\right)_{1}=1+\left(1-\frac{\Omega}{4}\right)  \tag{77b}\\
& \left(D_{2}\right)_{1}=1
\end{align*}
$$

The function $\left(D_{1}\right)_{1}$ is shown in Fig. 13 for $\Omega=10,15,20$.
The first order distribution of current as calculated on the one hand from the Hallén formula (38) in reference 3, and on the other hand, from (73), are shown in Figs. 14-16 for $\beta h=\pi / 2,3 \pi / 4, \pi$, and with $\Omega=10,20$. The function $f^{\prime \prime}$ and $f^{\prime}$ when (73) is written in the form

$$
\begin{equation*}
\left(I_{z}\right)_{1}=\frac{2 \pi V_{0}^{*}}{R_{c} \Omega D_{H}}\left(f^{\prime \prime}+j f^{\prime}\right) \tag{78}
\end{equation*}
$$

are plotted in the figures. Numerical values of $D_{H}$ are given in reference 3 where $D$ is written instead of $D_{H}$.

Apart from the change in the input current which is discussed below in terms of the impedance, the general shape of the two sets of curves is much the same. The new, more exact theory leads to a distribution with somewhat greater relative amplitudes nearer the outer parts of the antenna, and with a somewhat larger component in phase with the driving potential difference. For $\beta h=\pi$ the first order solution of the new theory is the same as the first order solution of Hallén's theory if $\psi$ is written for $\Omega$. Since $\psi$ is less than $\Omega$, this means the new first order distribution is the same as Hallén's first order distribution for an antenna of greater radius, but only for $\beta h=\pi$.
6. The impedance. The formula for the impedance according to the new, more exact theory is

$$
\begin{equation*}
\left(Z_{0}\right)_{m}=\frac{-j R_{c}}{2 \pi}\left\{\frac{\cos \beta h+\sum_{n=1}^{m}\left(D_{n}\right)_{m-1} \alpha_{n} / \psi^{n}}{\left(D_{1}\right)_{m} \sin \beta h+\sum_{n=1}^{m}\left(D_{n+1}\right)_{m} \beta_{n} / \psi^{n}}\right\} \tag{79}
\end{equation*}
$$

where $\alpha_{n}$ and $\beta_{n}$ are defined in (38b, c); $\alpha_{1}$ is tabulated in reference $11 ; \alpha_{2}$ in Table I. Curves for $\left(R_{0}\right)_{m}$ and $\left(X_{0}\right)_{m}$ as calculated from (79) are given in Figs. 17-19. Both second and first order solutions are shown for $\Omega=10,15,20$. These are calculated from

$$
\begin{equation*}
\left(Z_{0}\right)_{m}=60 \psi\left|\frac{A_{1}+j A_{2}}{B_{1}+j B_{2}}\right| e^{j\left(\tan ^{-1} A_{2} / A_{1}-\tan ^{-1} B_{2} / B_{1}\right)}, \tag{80}
\end{equation*}
$$

where for the second order solution

$$
\begin{align*}
& A_{1}=\left(D_{1}\right)_{1} \alpha_{1}^{I I} / \psi+\left(D_{2}\right)_{1} \alpha_{2}^{I I} / \psi^{2}, \\
& A_{2}=-\left[\cos \beta h+\left(D_{1}\right)_{1} \alpha_{1}^{I} / \psi+\left(D_{2}\right)_{1} \alpha_{2}^{I} / \psi^{2}\right], \\
& B_{1}=\left(D_{1}\right)_{2} \sin \beta h+\left(D_{2}\right)_{2} \beta_{1}^{I} / \psi+\beta_{2}^{I} / \psi^{2}, \\
& B_{2}=\left(D_{2}\right)_{2} \beta_{1}^{I I} / \psi+\beta_{2}^{I I} / \psi^{2}, \tag{81}
\end{align*}
$$

with $\left(D_{1}\right)_{2}$ and $\left(D_{2}\right)_{2}$ given by (77a) and $\left(D_{1}\right)_{1},\left(D_{2}\right)_{1}$ by (77b).
For the first order solution

$$
\begin{align*}
& A_{1}=\left(D_{1}\right)_{0 \alpha_{1}}^{I I}, \\
& A_{2}=-\left[\psi \cos \beta h-\left(D_{1}\right)_{0 \alpha_{1}}^{I}\right], \\
& B_{1}=\psi\left(D_{1}\right)_{1} \sin \beta h+\beta_{1}^{I}, \\
& B_{2}=\beta_{1}^{I I}, \tag{82}
\end{align*}
$$

where $\left(D_{1}\right)_{0}$ and $\left(D_{1}\right)_{1}$ are given by (77b).
The impedance calculated according to the new, more exact theory differ considerably in some details but not in major outline from that obtained from the Hallén theory as calculated by King and Blake, ${ }^{11}$ King and Harrison, ${ }^{7}$ and Bouwkamp. ${ }^{13}$ In general, resistances at antiresonance are smaller and occur at smaller values of $\beta h$; resistances at resonances are greater and likewise occur at smaller


Fig. 17. The input resistance and reactance of a moderately thin, cylindrical, center-driven antenna, $\Omega=10$, or $h / a=75$, for the first and second order theories.


Fig. 18. The input resistance and reactance of a thin, cylindrical, center-driven antenna, $\Omega=15$, or $h / a=9.0 \times 10^{2}$, for the first and second order theories.
values of $\beta h$. These differences are most significant for large values of the radius of the antenna. A critical discussion of impedance calculated from the theory here presented and comparison with experiment and with the theories of Hallén, ${ }^{2,3}$ Gray, ${ }^{12}$ Schelkunoff, ${ }^{6,1}$ and others is reserved for a sequel to this paper. Accuracy of the results and convergence of the series involved also will be discussed therein.


Fig. 19. The input resistance and reactance of a very thin, cylindrical, center-driven antenna, $\Omega=20, h / a=1.1 \times 10^{4}$, for the first and second order theories.

Using a specially constructed coaxial line and driving conditions that approximate as closely as possible the idealized slice generator D. D. King has measured the impedance of cylindrical antennas with hemispherical ends. A complete description of the apparatus, of the method, and of the results will be contained in a doctoral dissertation and in a paper to be published in another journal. A cross-section of the results involving all of the critical values for $\Omega=10$ are shown below together with the corresponding theoretical results of the theory outlined above. The agreement is seen to be good for all quantities in the second order theory, only approximate in the first order theory.

Table II

|  | Antiresonant $R_{0}$ | $\frac{\|X\|_{\text {min }}}{\|X\|_{\text {max }}}$ | $\pi-\beta h_{\text {ancti-reas }}$ | $\begin{gathered} \text { Resonant } \\ R_{0} \end{gathered}$ | $\frac{\pi}{2} \beta h_{\text {ron }}$. | $R_{0}$ at $\beta h=\frac{\pi}{2}$ | $X_{0}$ at $\beta h=\frac{\pi}{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Experimental Results by D. D. King | 800 | 1.95 | . 60 | 71.5 | . 098 | 85 | 47 |
| Theoretical Results Second Order | 860 | 1.80 | . 61 | 71.0 | . 094 | 88 | 42.5 |
| Theoretical <br> Results <br> First Order | 840 | 1.27 | 8 8.39 | 64.8 | . 065 | 67 | 30 |

Appendix: Integral Functions
The Functions $C(z)$ and $S(z)$.

$$
\begin{align*}
& C(z) \equiv \int_{-h}^{h} R_{1}^{-1} e^{-i \beta R_{1}} \cos \beta z^{\prime} d z^{\prime}=\int_{0}^{h}\left(R_{1}^{-1} e^{-j \beta R_{1}}+R_{2}^{-1} e^{-j \beta R_{2}}\right) \cos \beta z^{\prime} d z^{\prime}  \tag{1}\\
& S(z) \equiv \int_{-h}^{h} R_{1}^{-1} e^{-j \beta R_{1}} \sin \beta\left|z^{\prime}\right| d z^{\prime}=\int_{0}^{h}\left(R_{1}^{-1} e^{-j \beta R_{1}}+R_{2}^{-1} e^{-j \beta R_{z}}\right) \sin \beta z^{\prime} d z^{\prime} \tag{2}
\end{align*}
$$

where

$$
\begin{equation*}
R_{1}=\sqrt{\left(z-z^{\prime}\right)^{2}+a^{2}} \tag{3a}
\end{equation*}
$$

$$
\begin{equation*}
R_{2}=\sqrt{\left(z+z^{\prime}\right)^{2}+a^{2}} \tag{3b}
\end{equation*}
$$

These integrals can be written in the form

$$
\begin{align*}
& C(z)=\frac{1}{2}\left[I_{1}+I_{2}+I_{3}+I_{4}\right]  \tag{4}\\
& S(z)=\frac{-j}{2}\left[I_{1}-I_{2}+I_{3}-I_{4}\right]=\frac{j}{2}\left[I_{4}-I_{3}+I_{2}-I_{1}\right] \tag{5}
\end{align*}
$$

where

$$
\begin{align*}
& I_{1}=\int_{0}^{h} R_{1}^{-1} e^{-i \beta\left(R_{1}-z^{\prime}\right)} d z^{\prime}=e^{j \beta z} \int_{0}^{h} R_{1}^{-1} e^{-i \beta\left(R_{1}+z-z^{\prime}\right)} d z^{\prime},  \tag{6}\\
& I_{2}=\int_{0}^{h} R_{1}^{-1} e^{-j \beta\left(R_{1}+z^{\prime}\right)} d z^{\prime}=e^{-j \beta z} \int_{0}^{h} R_{1}^{-1} e^{-j \beta\left(R_{1}-z^{\prime} z^{\prime}\right)} d z^{\prime},  \tag{7}\\
& I_{3}=\int_{0}^{h} R_{2}^{-1} e^{-j \beta\left(R_{z}-z^{\prime}\right)} d z^{\prime}=e^{-j \beta z} \int_{0}^{h} R_{2}^{-1} e^{-i \beta\left(R_{z}-z-z^{\prime}\right)} d z^{\prime},  \tag{8}\\
& I_{4}=\int_{0}^{h} R_{2}^{-1} e^{-j \beta\left(R_{z}+z^{\prime}\right)} d z^{\prime}=e^{j \beta z} \int_{0}^{h} R_{2}^{-1} e^{-i \beta\left(R_{2}+z+z^{\prime}\right)} d z^{\prime} . \tag{9}
\end{align*}
$$

The four integrals (6)-(9) can all be reduced to the form

$$
\begin{equation*}
\int_{v}^{v_{\Lambda}} v^{-1} e^{-v} d v \tag{10}
\end{equation*}
$$

by making the changes in the variable and in the upper and lower limits listed in Table III.

Table III

| Integral | $v$ | $\frac{d v}{d z^{\prime}}$ | $v_{0}$ for $z^{\prime}=0$ | $v_{h}$ for $z^{\prime}=h$ |
| :---: | :---: | :---: | :---: | :---: |
| $I_{1}$ | $j \beta\left(R_{1}+z-z^{\prime}\right)=j \beta\left[\sqrt{\left(z^{\prime}-z\right)^{2}+a^{2}}-\left(z^{\prime}-z\right)\right]$ | $j \beta\left[\frac{z^{\prime}-z}{R_{1}}-1\right]=-\frac{v}{R_{1}}$ | $j \beta\left(R_{0}+z\right)$ | $j \beta\left(R_{1 h}-u_{1}\right)$ |
| $I_{2}$ | $j \beta\left(R_{1}-z+z^{\prime}\right)=j \beta\left[\sqrt{\left(z^{\prime}-z\right)^{2}+a^{2}}+\left(z^{\prime}-z\right)\right]$ | $j \beta\left[\frac{z^{\prime}-z}{R_{2}}+1\right]=\frac{v}{R_{1}}$ | $j \beta\left(R_{0}-z\right)$ | $j \beta\left(R_{1 h}+u_{1}\right)$ |
| $I_{3}$ | $j \beta\left(R_{2}-z-z^{\prime}\right)=j \beta\left[\sqrt{\left(z^{\prime}+z\right)^{2}+a^{2}}-\left(z^{\prime}+z\right)\right]$ | $j \beta\left[\frac{z^{\prime}+z}{R_{2}}-1\right]=-\frac{v}{R_{2}}$ | $j \beta\left(R_{0}-z\right)$ | $j \beta\left(R_{2 h}-u_{2}\right)$ |
| $I_{4}$ | $j \beta\left(R_{2}+z+z^{\prime}\right)=j \beta\left[\sqrt{\left(z^{\prime}+z\right)^{2}+a^{2}}+\left(z^{\prime}+z\right)\right]$ | $j \beta\left[\frac{z^{\prime}+z}{R_{2}}+1\right]=\frac{v}{R_{2}}$ | $j \beta\left(R_{0}+z\right)$ | $j \beta\left(R_{2 h}+u_{2}\right)$ |

$$
R_{0} \equiv \sqrt{z^{2}+a^{2}} ; \quad u_{2}=h+z ; \quad u_{1}=h-z ; \quad R_{2 h}=\sqrt{u_{2}^{2}+a^{2}} ; \quad R_{\mu h}=\sqrt{u_{1}^{R_{1}+a^{2}}}
$$

In terms of exponential, sine, and cosine integrals,

$$
\begin{equation*}
\int_{j a}^{i b} \frac{e^{-u}}{u} d u=\mathrm{Ei}(-j b)-\mathrm{Ei}(-j a)=\mathrm{Ci}(b)-\mathrm{Ci}(a)-j \mathrm{Si}(b)+j \mathrm{Si}(a) . \tag{11}
\end{equation*}
$$

With (11) the several integrals (6)-(9) may be expressed as follows in terms of the exponential integral and the sine and cosine integrals

$$
\begin{align*}
I_{1} & =-e^{j \beta z}\left\{\operatorname{Ei}\left[-j \beta\left(R_{1 h}-u_{1}\right)\right]-\operatorname{Ei}\left[-j \beta\left(R_{0}+z\right)\right]\right\}  \tag{12a}\\
& =-e^{j \beta z}\left\{\operatorname{Ci} \beta\left(R_{1 h}-u_{1}\right)-\operatorname{Ci} \beta\left(R_{0}+z\right)-j \operatorname{Si} \beta\left(R_{1 h}-u\right)+j \operatorname{Si} \beta\left(R_{0}+z\right)\right\} .  \tag{12b}\\
I_{2} & =e^{-j \beta z}\left\{\operatorname{Ei}\left[-j \beta\left(R_{1 h}+u_{1}\right)\right]-\operatorname{Ei}\left[-j \beta\left(R_{0}-z\right)\right]\right\}  \tag{13a}\\
& =e^{-j \beta z}\left\{\operatorname{Ci} \beta\left(R_{1 h}+u_{1}\right)-\operatorname{Ci} \beta\left(R_{0}-z\right)-j \operatorname{Si} \beta\left(R_{1 h}+u_{1}\right)+j \operatorname{Si} \beta\left(R_{0}-z\right)\right\} .  \tag{13b}\\
I_{3} & =e^{-j \beta z}\left\{\operatorname{Ei}\left[-j \beta\left(R_{2 h}-u_{2}\right)\right]-\operatorname{Ei}\left[-j \beta\left(R_{0}-z\right)\right]\right\}  \tag{14a}\\
& =e^{-j \beta z}\left\{\operatorname{Ci} \beta\left(R_{2 h}-u_{2}\right)-\operatorname{Ci} \beta\left(R_{0}-z\right)-j \operatorname{Si} \beta\left(R_{2 h}-u_{2}\right)+j \operatorname{Si} \beta\left(R_{0}-z\right)\right\} .  \tag{14b}\\
I_{1} & =e^{j \beta z}\left\{\operatorname{Ei}\left[-j \beta\left(R_{2 h}+u_{2}\right)\right]-\operatorname{Ei}\left[-j \beta\left(R_{0}+z\right)\right]\right\}  \tag{15a}\\
& =e^{j \beta z}\left\{\operatorname{Ci} \beta\left(R_{2 h}+u_{2}\right)-\operatorname{Ci} \beta\left(R_{0}+z\right)-j \operatorname{Si} \beta\left(R_{2 h}+u_{2}\right)+j \operatorname{Si} \beta\left(R_{0}+z\right)\right\} . \tag{15b}
\end{align*}
$$

Upon combining the several integrals according to (4) and (5),

$$
\begin{align*}
C(z)= & \frac{1}{2} e^{j \beta z}\left\{\operatorname{Ei}\left[-j \beta\left(R_{2 h}+u_{2}\right)\right]-\operatorname{Ei}\left[-j \beta\left(R_{1 h}-u_{1}\right)\right]\right\} \\
& +\frac{1}{2} e^{-i \beta z}\left\{\operatorname{Ei}\left[-j \beta\left(R_{1 h}+u_{1}\right)\right]-\operatorname{Ei}\left[-j \beta\left(R_{2 h}-u_{2}\right)\right]\right\} .  \tag{16a}\\
C(z)= & \frac{1}{2} e^{j \beta z}\left\{\operatorname{Ci} \beta\left(R_{2 h}+u_{2}\right)-\operatorname{Ci} \beta\left(R_{1 h}-u_{1}\right)-j \operatorname{Si} \beta\left(R_{2 h}+u_{2}\right)+j \operatorname{Si} \beta\left(R_{1 h}-u_{1}\right)\right\} \\
& +\frac{1}{2} e^{-j \beta z}\left\{\operatorname{Ci} \beta\left(R_{1 h}+u_{1}\right)-\operatorname{Ci} \beta\left(R_{2 h}-u_{2}\right)-j \operatorname{Si} \beta\left(R_{1 h}+u_{1}\right)\right. \\
& \left.+j \operatorname{Si} \beta\left(R_{2 h}-u_{2}\right)\right\} . \tag{16b}
\end{align*}
$$

$$
\begin{align*}
S(z)= & \frac{j}{2} e^{j \beta z}\left\{\operatorname{Ei}\left[-j \beta\left(R_{2 h}+u_{2}\right)\right]+\operatorname{Ei}\left[-j \beta\left(R_{1 h}-u_{1}\right)\right]-2 \operatorname{Ei}\left[-j \beta\left(R_{0}+z\right)\right]\right\} \\
& +\frac{j}{2} e^{-j \beta z}\left\{\operatorname{Ei}\left[j \beta\left(R_{1 h}+u_{1}\right)\right]+\operatorname{Ei}\left[-j \beta\left(R_{2 h}-u_{2}\right)\right]-2 \operatorname{Ei}\left[-j \beta\left(R_{0}-z\right)\right]\right\} .  \tag{17a}\\
S(z)= & \frac{j}{2} e^{j \beta z}\left[\operatorname{Ci} \beta\left(R_{2 h}+u_{2}\right)+\operatorname{Ci} \beta\left(R_{1 h}-u_{1}\right)-j \operatorname{Si} \beta\left(R_{2 h}+u_{2}\right)\right. \\
& \left.-j \operatorname{Si} \beta\left(R_{1 h}-u_{1}\right)-2 \operatorname{Ci} \beta\left(R_{0}+z\right)+j 2 \operatorname{Si} \beta\left(R_{0}+z\right)\right] \\
& +\frac{j}{2} e^{-j \beta z}\left[\operatorname{Ci} \beta\left(R_{1 h}+u_{1}\right)+\operatorname{Ci} \beta\left(R_{2 h}-u_{2}\right)-j \operatorname{Si} \beta\left(R_{1 h}+u_{1}\right)\right. \\
& \left.-j \operatorname{Si} \beta\left(R_{2 h}-u_{2}\right)-2 \operatorname{Ci} \beta\left(R_{0}-z\right)+j 2 \operatorname{Si} \beta\left(R_{0}-z\right)\right] . \tag{17b}
\end{align*}
$$

In trigonometric form

$$
\begin{align*}
C(z)= & \frac{1}{2} \cos \beta z\left[\operatorname{Ci} \beta\left(R_{2 h}+u_{2}\right)+\operatorname{Ci} \beta\left(R_{1 h}+u_{1}\right)-\operatorname{Ci} \beta\left(R_{2 h}-u_{2}\right)\right. \\
& -\operatorname{Ci} \beta\left(R_{1 h}-u_{1}\right)-j \operatorname{Si} \beta\left(R_{2 h}+u_{2}\right)-j \operatorname{Si} \beta\left(R_{1 h}+u_{1}\right)+j \operatorname{Si} \beta\left(R_{2 h}-u_{2}\right) \\
& +j \operatorname{Si} \beta\left(R_{1 h}-u_{1}\right] \\
& +\frac{1}{2} \sin \beta z\left[\operatorname{Si} \beta\left(R_{2 h}+u_{2}\right)-\operatorname{Si} \beta\left(R_{1 h}+u_{1}\right)+\operatorname{Si} \beta\left(R_{2 h}-u_{2}\right)\right. \\
& -\operatorname{Si} \beta\left(R_{1 h}-u_{1}\right)+j \operatorname{Ci} \beta\left(R_{2 h}+u_{2}\right)-j \operatorname{Ci} \beta\left(R_{1 h}+u_{1}\right)+j \operatorname{Ci} \beta\left(R_{2 h}-u_{2}\right) \\
& \left.-j \operatorname{Ci} \beta\left(R_{1 h}-u_{1}\right)\right] . \tag{18}
\end{align*}
$$

$$
\begin{align*}
S(z)= & \frac{1}{2} \cos \beta z\left[\operatorname{Si} \beta\left(R_{2 h}+u_{2}\right)+\operatorname{Si} \beta\left(R_{1 h}+u_{1}\right)+\operatorname{Si} \beta\left(R_{2 h}-u_{2}\right)\right. \\
& +\operatorname{Si} \beta\left(R_{1 h}-u_{1}\right)-2 \operatorname{Si} \beta\left(R_{0}+z\right)-2 \operatorname{Si} \beta\left(R_{0}-z\right)+j \operatorname{Ci} \beta\left(R_{2 h}+u_{2}\right) \\
& +j \operatorname{Ci} \beta\left(R_{1 h}+u_{1}\right)+j \operatorname{Ci} \beta\left(R_{2 h}+u_{2}\right)+j \operatorname{Ci} \beta\left(R_{1 h}-u_{1}\right) \\
& \left.-j 2 \operatorname{Ci} \beta\left(R_{0}+z\right)-j 2 \operatorname{Ci} \beta\left(R_{0}-z\right)\right] \\
& -\frac{1}{2} \sin \beta z\left[\operatorname{Ci} \beta\left(R_{2 h}+u_{2}\right)-\operatorname{Ci} \beta\left(R_{1 h}+u_{1}\right)-\operatorname{Ci} \beta\left(R_{2 h}-u_{2}\right)\right. \\
& +\operatorname{Ci} \beta\left(R_{1 h}-u_{1}\right)-2 \operatorname{Ci} \beta\left(R_{0}+z\right)+2 \operatorname{Ci} \beta\left(R_{0}-z\right) \\
& -j \operatorname{Si} \beta\left(R_{2 h}+u_{2}\right)+j \operatorname{Si} \beta\left(R_{1 h}+u_{1}\right)+j \operatorname{Si} \beta\left(R_{2 h}-u_{2}\right) \\
& \left.-j \operatorname{Si} \beta\left(R_{1 h}-u_{1}\right)+j 2 \operatorname{Si} \beta\left(R_{0}+z\right)-j 2 \operatorname{Si} \beta\left(R_{0}-z\right)\right] . \tag{19}
\end{align*}
$$

With

$$
R_{0}=\sqrt{z^{2}+a^{2}} ; \quad u_{2}=h+z ; \quad u_{1}=h-z ; \quad R_{2 h}=\sqrt{u_{2}^{2}+a^{2}} ; \quad R_{1 h}=\sqrt{u_{1}^{2}+a^{2}} .
$$

These are exact expressions for the integrals (1) and (2). They are valid for all values of the argument $z$ and the parameters $h$ and $a$.

The integral $E(z)$.

$$
\begin{equation*}
E(z)=\int_{-\hbar}^{\hbar} \frac{e^{-j \beta R_{1}}}{R_{1}} d z^{\prime} \tag{20}
\end{equation*}
$$

Let the variable be changed by setting

$$
\begin{equation*}
\beta R_{1}=\beta \sqrt{\left(z-z^{\prime}\right)^{2}+a^{2}}=\sqrt{U^{2}+V^{2}} \tag{21}
\end{equation*}
$$

where

$$
\begin{equation*}
U \equiv \beta\left(z-z^{\prime}\right) ; \quad V \equiv \beta a \tag{22}
\end{equation*}
$$

The integral then becomes

$$
\begin{align*}
E(z)= & \int_{\beta R_{1 h}}^{\beta R_{2 h}}\left(U^{2}+V^{2}\right)^{-1 / 2} e^{-j\left(U^{2}+V^{2}\right) 1 / 2} d U \\
= & \int_{\beta R_{1 h}}^{\beta R_{2 h}}\left(U^{2}+V^{2}\right)^{-1 / 2} \cos \left(U^{2}+V^{2}\right)^{1 / 2} d U \\
& \left.\left.-j \int_{\beta R_{1 h}}^{\beta R_{2 h}}\left(U^{2}+V^{2}\right)^{-1 / 2} \sin \right] U^{2}+V^{2}\right)^{1 / 2} d U \tag{23}
\end{align*}
$$

with

$$
\begin{equation*}
R_{2 h}=\sqrt{(h+z)^{2}+a^{2}} ; \quad R_{1 h}=\sqrt{(h-z)^{2}+a^{2}} . \tag{24}
\end{equation*}
$$

Let the following symbols be introduced:

$$
\begin{align*}
& \operatorname{Cuv} x=\int_{0}^{x}\left(U^{2}+V^{2}\right)^{-1 / 2} \cos \left(U^{2}+V^{2}\right)^{1 / 2} d U  \tag{25}\\
& \text { Suv } x=\int_{0}^{x}\left(U^{2}+V^{2}\right)^{-1 / 2} \sin \left(U^{2}+V^{2}\right)^{1 / 2} d U \tag{26}
\end{align*}
$$

These functions satisfy the conditions Cuv $(-x)=-\operatorname{Cuv} x$, Suv $(-x)=-\operatorname{Suv} x$. In terms of the notation (25) and (26), the integral (20) becomes

$$
\begin{equation*}
E(z)=\operatorname{Cuv} \beta R_{2 h}-\operatorname{Cuv} \beta R_{1 h}-j \operatorname{Suv} \beta R_{2 h}+j \operatorname{Suv} \beta R_{1 h} . \tag{27}
\end{equation*}
$$

This is an exact expression for the integral (20).
Approximate Expressions for $C(z), S(z)$, and $E(z)$.
If the parameter $a$ appearing in $R_{2 h}=\sqrt{(h+z)^{2}+a^{2}}, R_{1 h}=\sqrt{(h-z)^{2}+a^{2}}$ is small compared with $h$, useful approximate expressions for the integrals $C(z), S(z)$, and $E(z)$ may be derived as follows. Expanding the integral cosine using

$$
\begin{equation*}
\overline{\mathrm{Ci}} x=C+\ln x-\operatorname{Ci} x=\int_{0}^{x} u^{-1}(1-\cos u) d u \tag{28}
\end{equation*}
$$

where $C$ is Euler's constant, one obtains

$$
\begin{gather*}
\mathrm{Ci} \beta\left(R_{2 h}-u_{2}\right)=C+\ln \beta\left(R_{2 h}-u_{2}\right)-\overline{\mathrm{Ci}} \beta\left(R_{2 h}-u_{2}\right)  \tag{29}\\
\mathrm{Ci} \beta\left(R_{1 h}-u_{1}\right)=C+\ln \beta\left(R_{1 h}-u_{1}\right)-\overline{\mathrm{Ci}} \beta\left(R_{1 h}-u_{1}\right)  \tag{30}\\
\mathrm{Ci} \beta\left(R_{0}-z\right)=C+\ln \beta\left(R_{0}-z\right)-\overline{\mathrm{Ci}} \beta\left(R_{0}-z\right) \tag{31}
\end{gather*}
$$

However,

$$
\begin{align*}
R_{2 h}-u_{2} & =\sqrt{(h+z)^{2}+a^{2}}-(h+z)  \tag{32}\\
R_{1 h}-u_{1} & =\sqrt{(h-z)^{2}+a^{2}}-(h-z)  \tag{33}\\
R_{0}-z & =\sqrt{z^{2}+a^{2}}-z \tag{34}
\end{align*}
$$

are all of order of magnitude $a$, so that the arguments $\beta\left(R_{2 h}-u_{2}\right)$ and $\beta\left(R_{1 h}-u_{1}\right)$ are of magnitude $\beta a$. It follows that since the arguments of $\overline{\mathrm{Ci}} \beta\left(R_{2 h}-u_{2}\right), \overline{\mathrm{Ci}} \beta\left(R_{1 h}-u_{1}\right)$ and $\overline{\mathrm{C}_{1}^{1}} \beta\left(R_{0}-z\right)$ are small these functions may be expanded in series. The leading


Fig. 21. The function $C(z)$ near anti-resonance, $\beta h=\pi, \Omega=20$.

Fig. 20. The function $C(z)$ near anti-resonance, $\beta h=\pi, \Omega=10$.
term in each case is of the order of magnitude $\frac{1}{4} \beta^{2} a^{2}$ so that the $\overline{\mathrm{Ci}}$ terms in (29)-(31) are negligible compared with the logarithm. Hence,

$$
\begin{align*}
\mathrm{Ci} \beta\left(R_{2 h}-u_{2}\right) & \doteq C+\ln \beta\left(R_{2 h}-u_{2}\right)  \tag{35}\\
\mathrm{Ci} \beta\left(R_{1 h}-u_{1}\right) & \doteq C+\ln \beta\left(R_{1 h}-u_{1}\right),  \tag{36}\\
\mathrm{Ci} \beta\left(R_{0}-z\right) & \doteq C+\ln \beta\left(R_{0}-z\right) . \tag{37}
\end{align*}
$$

Since the functions Si $\beta\left(R_{1 h}-u_{1}\right)$ and $\mathrm{Si} \beta\left(R_{2 h}-u_{2}\right)$ are of order of magnitude $\beta a$, they are negligible compared with Si $\beta\left(R_{2 \hbar}\right.$ 士 $u_{2}$ ) and Si $\beta\left(R_{1 h}+u_{1}\right)$ except very near the ends $z= \pm h$. If Ci $\beta\left(R_{2 h}+u_{2}\right), \mathrm{Ci} \beta\left(R_{1 h}+u_{h}\right)$, and $\mathrm{Ci} \beta\left(R_{0}+z\right)$ are expanded using (28) and the relation

$$
\begin{equation*}
\ln \frac{u+\left(u^{2}+a^{2}\right)^{1 / 2}}{a}=\sinh ^{-1} \frac{u}{a}, \tag{38}
\end{equation*}
$$

and the approximations,

$$
\begin{align*}
\overline{\mathrm{Ci}} \beta\left(R_{2 h}+u_{2}\right) & \doteq \overline{\mathrm{Ci}} 2 \beta u_{2}=\overline{\mathrm{Ci}} 2 \beta(h+z),  \tag{39}\\
\overline{\mathrm{Ci}} \beta\left(R_{1 h}+u_{1}\right) & \doteq \overline{\mathrm{Ci}} 2 \beta u_{1}=\overline{\mathrm{Ci}} 2 \beta(h-z),  \tag{40}\\
\overline{\mathrm{Ci}} \beta\left(R_{0}+z\right) & \doteq \overline{\mathrm{Ci}} 2 \beta z, \tag{41}
\end{align*}
$$

(18) and (19) reduce to the following approximate forms:

$$
\begin{align*}
C(z) \doteq & -\frac{1}{2} \cos \beta z[\operatorname{Ci} 2 \beta(h+z)+\overline{\operatorname{Ci}} 2 \beta(h-z)+j \operatorname{Si} 2 \beta(h+z)+j \operatorname{Si} 2 \beta(h-z)] \\
& +\frac{1}{2} \sin \beta z[\operatorname{Si} 2 \beta(h+z)-\operatorname{Si} 2 \beta(h-z)-j \overline{\operatorname{Ci}} 2 \beta(h+z)+j \overline{\operatorname{Ci}} 2 \beta(h-z)] \\
& +\cos \beta z\left[\sinh ^{-1} \frac{h+z}{a}+\sinh ^{-1} \frac{h-z}{a}\right] .  \tag{42}\\
S(z)= & \frac{1}{2} \operatorname{Cos} \beta z[\operatorname{Si} 2 \beta(h+z)+\operatorname{Si} 2 \beta(h-z)-2 \operatorname{Si} 2 \beta|z| \\
& -j \operatorname{Ci} 2 \beta(h+z)-j \operatorname{Ci} 2 \beta(h-z)+2 j \operatorname{Ci} 2 \beta z] \\
& +\frac{1}{2} \sin \beta z[\operatorname{Ci} 2 \beta(h+z)-\overline{\operatorname{Ci}} 2 \beta(h-z)+j \operatorname{Si} 2 \beta(h+z) \\
& -j \operatorname{Si} 2 \beta(h-z)-2 j \operatorname{Si} 2 \beta z]-\sin \beta|z| \overline{\operatorname{Ci}} 2 \beta z \\
& +\sin \beta|z|\left[\sinh ^{-1} \frac{h+z}{a}+\sinh ^{-1} \frac{h-z}{a}\right] \\
& -2 \sin \beta|z|\left[\sinh ^{-1} \frac{h+|z|}{a}-\sinh ^{-1} \frac{|z|}{a}\right] . \tag{43}
\end{align*}
$$

The last factor in (43) is written in the expanded form shown in order to contain $\Psi_{H}(z)=\sinh ^{-1}(h+z / a)+\sinh ^{-1}(h-z / a)$. The remaining two terms may be written in the following approximate form if desired,

$$
\begin{align*}
-\left[\sinh ^{-1} \frac{h+|z|}{a}-\sinh ^{-1} \frac{|z|}{a}\right]= & -\ln \frac{h+|z|+\left[(h+z)^{2}+a^{2}\right]^{1 / 2}}{|z|+\left(z^{2}+a^{2}\right)^{1 / 2}} \\
& \doteq \ln \left(\frac{|z|}{h+|z|}\right) . \tag{44}
\end{align*}
$$

The function (42) is shown graphically in Figs. 20 and 21 for $\beta h=\pi$; in Figs. 4 and


Fig. 22. The function $S(z)$ near resonance, $\beta h=\pi / 2, \Omega=10$.


Fig. 23. The function $S(z)$ near resonance, $\beta h=\pi / 2, \Omega=20$


Fig. 24. The function $E(z), \beta h=\pi / 2, \Omega=10,20$.


FIg. 25. The function $E(z), \beta h=\pi, \Omega=10,20$.

5 for $\beta h=\pi / 2$. The function (43) is in Figs. 22 and 23 for $\beta h=\pi / 2$; in Figs. 6 and 7 for $\beta h=\pi$.

An approximate expression for $E(z)$ is obtained from (23) by adding and subtracting

$$
\begin{align*}
& \int_{\beta R_{1 h}}^{\beta R_{2 h}}\left(U^{2}+V^{2}\right)^{-1 / 2} d U \\
& E(z)=-\int_{\beta R_{1 h}}^{\beta R_{2 h}}\left(U^{2}+V^{2}\right)^{-1 / 2}\left[1-\cos \left(U^{2}+V^{2}\right)^{1 / 2}\right] d U \\
&-j \int_{\beta R_{1 h}}^{\beta R_{2 h}}\left(U^{2}+V^{2}\right)^{-1 / 2} \sin \left(U^{2}+V^{2}\right)^{1 / 2} d U \\
&+\int_{\beta R_{1 h}}^{\beta R_{2 h}}\left(U^{2}+V^{2}\right)^{-1 / 2} d U \tag{45}
\end{align*}
$$

If the small quantity $V=\beta a$ is neglected in the first two integrals in (45) these remain everywhere finite and vanish at $U=0$. This is not true in the last integral in which $V$ plays an important part. If $V$ is neglected in the first two integrals both in the integrand and in the limits, but retained in the last integral, the following approximate expression is obtained:

$$
\begin{align*}
E(z)= & -\int_{\beta(z-h)}^{\beta(z+h)}|U|^{-1}(1-\cos U) d U-j \int_{\beta(z-h)}^{\beta(z+h)} U^{-1} \sin U d U \\
& +\int_{\beta R_{1 h}}^{\beta R_{2 h}}\left(U^{2}+V^{2}\right)^{1 / 2} d U \tag{46}
\end{align*}
$$

Because the magnitude of $U$ and not $U$ itself appears in the denominator of the first integral, this must be evaluated in two steps for the ranges $z^{\prime}>z$ and $z^{\prime}<z$. It is not necessary to write $|U|$ in the second integral because the integrand does not change sign as $z^{\prime}$ passes through $z$. Hence with $\bar{U}=\beta\left(z^{\prime}-z\right)=-U$ the first integral in (46) becomes

$$
\begin{align*}
-\int_{0}^{\beta(z+h)} U^{-1}(1-\cos U) d U & -\int_{\beta(z-h)}^{0} \bar{U}^{-1}(1-\cos \bar{U}) d \bar{U} \\
= & -\int_{0}^{\beta(h+z)} U^{-1}(1-\cos U) d U \\
& -\int_{0}^{\beta(h-z)} \bar{U}^{-1}(1-\cos \bar{U}) d \bar{U} \tag{47}
\end{align*}
$$

Using (28) in (47), the sine integral in the second integral in (46), and evaluating the third integral directly, we write (46) in the form:

$$
\begin{align*}
E(z) \doteq & -\overline{\mathrm{Ci}} \beta(h+z)-\overline{\mathrm{Ci}} \beta(h-z)-j \operatorname{Si} \beta(h+z)-j \operatorname{Si} \beta(h-z) \\
& +\sinh ^{-1}\left(\frac{h+z}{a}\right)+\sinh ^{-1}\left(\frac{h-z}{a}\right) . \tag{48}
\end{align*}
$$

Use has been made of the fact that Si $\beta(z-h)=-\mathrm{Si} \beta(h-z)$. The function $E(z)$ in (48) is shown graphically in Figs. 24 and 25.

# A METHOD OF SOLUTION OF FIELD PROBLEMS BY MEANS OF OVERLAPPING REGIONS* 

BY<br>H. PORITSKY and M. H. BLEWETT<br>General Electric Company

1. Introduction. In problems involving the determination of fields, it often happens that the region $R$ for which the field is to be determined is difficult to handle directly, but can be broken up into several overlapping regions $R_{1}, R_{2}, \ldots$ for each of which the field can be determined by standard methods. We suppose that the breaking up is carried out in such a manner that every point of the region $R$ falls into at least one of the regions $R_{1}, R_{2}, \ldots$. In the following, Schwartz' "alternating procedure" is applied to the solution of field problems for such regions $R$.

To illustrate, let us consider the determination of a function $u$ harmonic over the region $R$ shown in Fig. 1, bounded by two circular arcs $A B C$ and $C D A$ with centers at $O$ and $O^{\prime}$. For simplicity we assume that the radii of the two circles are equal and the boundary values of $u$ are symmetric about the straight line through $A$ and $C$. It will be noticed that by completing the circular arcs by means of the dotted curves $A E C$ and $C F A$ one obtains the circular regions over which the determination of a harmonic function in terms of boundary values is well known. Here $R$ is the region bounded by the solid circular arcs $A B C$ and $C D A$,


Fig. 1. while the regions $R_{1}$ and $R_{2}$ are the circular regions bounded by the complete circles with centers at $O$ and $O^{\prime}$. The problem then is to utilize the relatively easy solution of the Dirichlet problem for the circular regions $R_{1}$ and $R_{2}$ in an efficient manner toward the solution of the Dirichlet problem over $R$.

This is done by assuming the values of $u$ over the $\operatorname{arc} A F C$; the solution of the Dirichlet problem for the circle $R_{1}$ with center $O$, based on these assumed values and on the known boundary values over $A B C$, is then utilized to furnish the values of $u$ over $A E C$. The procedure is then repeated by solving the Dirichlet problem for the circle $R_{2}$ with center at $O^{\prime}$, and the values over $A F C$ are recalculated. By alternating between $R$ and $R^{\prime}$ in this way, continual improvement of the values of $u$ over both arcs $A F C$ and $A E C$ results; in the limit this leads to a solution of the Dirichlet problem for the region of Fig. 1.

In the following we shall illustrate the procedure, not for the Laplace equation

$$
\begin{equation*}
\nabla^{2} u=0 \tag{1.1}
\end{equation*}
$$

but for the equation

$$
\begin{equation*}
\left(\nabla^{2}+k^{2}\right) u=0 \tag{1.2}
\end{equation*}
$$

which is encountered in wave motion under the assumption of sinusoidal oscillations, for the region shown in Fig. 2. Other cases of interest in connection with (1.2) which

[^15]can also be treated by the present method are given by the "open end correction of an organ pipe," wave passage through a change of width of a channel, $T$-sections, etc.
2. Wave propagation around a corner. We consider a solution of the differential equation (1.2) for the region shown in Fig. 2; this solution is to satisfy the boundary conditions
\[

$$
\begin{array}{r}
\frac{\partial u}{\partial u}=0 \text { on } D O G, E B F \\
u=A_{1} e^{i k x}+B_{1} e^{-i k x} \text { for large positive } x \\
u=A_{3} e^{i k y}+B_{3} e^{-i k y} \quad \text { for large positive } y \tag{2.3}
\end{array}
$$
\]

where $A_{1}, B_{1}, A_{3}$ and $B_{3}$ are proper constants. Equations (2.2) and (2.3) can be described physically by the statement that $u$ behaves as a plane wave at infinity.

The above problem is encountered in the propagation of a transverse electromagnetic wave around a corner or through a channel the section of which is shown in Fig. 2. Here the channel has an infinite depth in the $z$-direction; the field components are assumed to be independent of $z$, and the only non-vanishing magnetic field component is $H_{z}$. At the boundaries, which are assumed to be metallic and perfectly conducting, the electric field is normal; this


Fig. 2. leads to the vanishing of the normal derivative of $H_{z}$, i.e., $\partial H_{z} / \partial n=0$. Formulation of the field in terms of $H_{z}$ leads to the above problem.

On account of the vanishing of the normal derivative over the $y$-axis, reflection across it is possible, thus extending the region of


Fig. 3. Fig. 2 into the region shown in Fig. 3. This reflection is carried out in accordance with the relation

$$
\begin{equation*}
u(-x, y)=u(x, y) \tag{2.4}
\end{equation*}
$$

In view of this reflection the behavior of $u$ at $x=-\infty$ is given by the expression

$$
\begin{equation*}
u=B_{1} e^{i k z}+A_{1} e^{-i k x} . \tag{2.5}
\end{equation*}
$$

As a result of this reflection the semi-infinite strip DOCE of Fig. 2 can be replaced by the 2-way infinite strip of Fig. 3

$$
-\infty<x<\infty, \quad 0<y<b .
$$

Similar reflection can be carried out across the lower boundary $y=0$ of Fig. 3; this allows us to replace the semi-infinite vertical channel by a vertical channel extending to infinity both up and down. A proper behavior for $u$ at $y=-\infty$ can be obtained from (2.3).

The general procedure which was outlined in $\S 1$ is applied to the present case. First, we consider the strip $0<y<b$ of Fig. 3, and assume values for $\partial u / \partial n$ over the dotted part $B^{\prime} C B$ of its upper boundary. Since $\partial u / \partial n$ vanishes over the rest of its boundary and the behavior of $u$ at $\infty$ is described by (2.2) and (2.5), it is possible to determine $u$ at any point interior to this strip. This determination is carried out by means of a Green's function $G$. The derivation of $G$ will be described presently; for the present it will suffice to say that the value of $u$ at an interior point $P$ of the strip is given by the relation

$$
\begin{equation*}
u_{p}=u\left(x_{0}, y_{0}\right)=2 B_{1} \cos k x_{0}+\frac{1}{2 \pi} \int_{-b}^{b}\left(\frac{\partial u}{\partial y}\right)_{y=b} G d x . \tag{2.6}
\end{equation*}
$$

$G$ has a pole at $P=\left(x_{0}, y_{0}\right)$, and (2.6) requires that $G$ be evaluated on the dotted line $B^{\prime} C B$. After $u$ is determined in this way, differentiation of (2.6) with respect to $x$ allows one to compute $\partial u / \partial x$, and in particular to determine this derivative over $A B$. Turning now to the infinite vertical strip $0<x<b$, we repeat the same procedure and determine the function $u$ at any point interior to this strip; in particular, we evaluate $u$ and $\partial u / \partial y$ over $C B$. The process is then repeated.

The definition of the Green's function for the differential equation (1.2) and the boundary condition (2.1) for a finite region $R$ is specified by the following:
a) $G$ satisfies (1.2) everywhere in $R$ except at the pole $P$;
b) $\partial G / \partial n$ vanishes along the boundary of $R$;
c) at the pole $P, G$ becomes infinite like $-\ln r^{\prime}$, where $r^{\prime}$ is the distance from $P$.

We apply Green's theorem in the form

$$
\begin{equation*}
\int\left[u\left(\nabla^{2}+k^{2}\right) v-v\left(\nabla^{2}+k^{2}\right) u\right] d A=\int\left(u \frac{\partial v}{\partial n}-v \frac{\partial u}{\partial n}\right) d s \tag{2.8}
\end{equation*}
$$

to the region $R$, exclude the neighborhood of the point by means of a small circle of radius $\epsilon$ and let $\epsilon$ approach zero. This yields the equation

$$
\begin{equation*}
u_{p}=\frac{1}{2 \pi} \int \frac{\partial u}{\partial n} G d s, \tag{2.9}
\end{equation*}
$$

where the integration is carried out over the boundary of $R$. In the present case, for the infinite strip $0<y<b$ special additional considerations are required. It will be assumed that in addition to the requirements (2.7) the Green's function $G$ behaves at infinity like a divergent plane wave. Solutions of (1.2) which depend on $x$ only are

$$
\begin{equation*}
e^{ \pm i k x} . \tag{2.10}
\end{equation*}
$$

We consider the wave equation

$$
\begin{equation*}
\frac{\partial^{2} U}{\partial t^{2}}=c^{2} \nabla^{2} U \tag{2.11}
\end{equation*}
$$

and look for solutions of the form $u e^{ \pm i \omega t}$. If we set $k=\omega / c$, we find that $u$ satisfies Eq. (1.2), and that $e^{i k x}$ represents a plane wave traveling in the direction of positive $x$ while $e^{-i k x}$ represents a similar wave traveling in the direction of negative $x$. It will
be assumed that at $x= \pm \infty$ the Green's function $G$ behaves like a divergent plane wave of the same amplitude at $x=+\infty$ as at $x=-\infty$.

It will be assumed that the dimension $b$ satisfies the inequality

$$
\begin{equation*}
b<\pi / k . \tag{2.12}
\end{equation*}
$$

Physically this assumption means that the width $b$ of the strip is less than half the wave length $\lambda / 2=\pi / k$ of a plane wave at the frequency in question. The effect of this assumption and the features which arise when it is not satisfied will appear presently.

First, we place the pole $P$ on the $y$-axis. We shall obtain $G$ as a series in the form

$$
\begin{equation*}
G=\sum_{n=0}^{\infty} A_{n} u_{n} \tag{2.13}
\end{equation*}
$$

where $u_{n}$ are product solutions of the wave equation (1.2), i.e., $u_{n}$ consist of the product of a function of $x$ and a function of $y$; more explicitly,

$$
\begin{align*}
u_{0} & =\exp [i k \cdot|x|]  \tag{2.14}\\
u_{n} & =\cos \frac{n y}{b} \exp \left[-\sqrt{\left(\frac{n \pi}{b}\right)^{2}-k^{2} \cdot|x|}\right], \quad(n>0)
\end{align*}
$$

These product solutions $u_{n}(n>0)$ have been chosen so that they don't become infinite at $x= \pm \infty$, while $u_{0}$ represents a divergent plane wave. If the inequality (2.12) were not satisfied, several radicals in $u_{n}(n>0)$ would be imaginary, infinitely large values of $u_{n}$ could not be avoided, and additional stipulations regarding the behavior of $G$ at infinity would have to be made.

The functions $u_{n}$ are symmetric about the vertical line $x=0$ through the pole $P$, and continuous at $x=0$. However, $\partial u_{n} / \partial x$ is discontinuous at $x=0$, the discontinuity being

$$
\Delta\left(\frac{\partial u_{n}}{\partial x}\right)=\left\{\begin{array}{ll}
-2 i k & \text { for } n=0  \tag{2.15}\\
2 \sqrt{\left(\frac{n \pi}{k}\right)^{2}-k^{2} \cos \frac{n \pi y}{b}} & \text { for } n>0
\end{array}\right\}
$$

Thus each term $u_{n}$ may be considered as representing the wave function corresponding to a sinusoidal distribution of sources* over the line $x=0$. The density $\sigma$ of the sources is given by the familiar condition from potential theory

$$
\begin{equation*}
\text { discontinuity in normal derivative }=\Delta\left(\frac{\partial u}{\partial x}\right)=-2 \pi \sigma \text {, } \tag{2.16}
\end{equation*}
$$

and in the present case is given by

$$
\begin{equation*}
\sigma_{n}=\frac{1}{\pi} \sqrt{\frac{n^{2} \pi^{2}}{b^{2}}-k^{2} \cos \frac{n \pi y}{b}} . \tag{2.17}
\end{equation*}
$$

[^16]For (2.13) this yields

$$
\begin{equation*}
\sigma=\frac{1}{\pi} \sum_{n=0}^{\infty} A_{n} \sqrt{\frac{n^{2} \pi^{2}}{b^{2}}-k^{2} \cos \frac{n \pi y}{b}} \tag{2.18}
\end{equation*}
$$

Let us now consider the concentrated point source at the pole $P$, and express it as a Fourier series of cosines over the interval $x=0,0<y<b$, obtaining

$$
\begin{equation*}
\sigma=\frac{1}{b}+\frac{2}{b} \sum_{n=1}^{\infty} \cos \frac{n \pi y_{0}}{b} \cos \frac{n \pi y}{b} \tag{2.19}
\end{equation*}
$$

where $x=0, y=y_{0}$ are the coordinates of the pole $P$. Solving for $A_{n}$, we obtain for the Green's function $G$ the Fourier series

$$
\begin{align*}
G=\frac{2 \pi}{b}\{ & -\frac{1}{2 i k} \exp [i k \cdot|x|] \\
& \left.+\sum_{n=1}^{\infty} \frac{\cos \left(n \pi y_{0} / b\right)}{\sqrt{(n \pi / b)^{2}-k^{2}}} \cos \frac{n \pi y}{b} \exp \left[-\sqrt{\left(\frac{n \pi}{b}\right)^{2}-k^{2} \cdot|x|}\right]\right\} \tag{2.20}
\end{align*}
$$

Due to the behavior of $G$ at infinity it is found that after applying the Green's theorem (2.8) over the rectangular region $-l^{\prime}<x<l$ and letting $l$ and $l^{\prime}$ recede to infinity, certain additional terms $R^{\prime}$ and $R^{\prime \prime}$ arise from the boundaries $x=l$ and $x=l^{\prime}$. Equation (2.9) is now replaced by

$$
\begin{equation*}
u\left(x_{0}, y_{0}\right)=\left.\frac{1}{2 \pi} \int_{-l^{\prime}}^{t} \frac{\partial u(x, y)}{\partial y}\right|_{y=b} G d x+R^{\prime}+R^{\prime \prime} \tag{2.21}
\end{equation*}
$$

where

$$
\begin{align*}
R^{\prime} & =\left.\frac{1}{2 \pi} \int_{0}^{b}\left(\frac{\partial u}{\partial x} G-u \frac{\partial G}{\partial x}\right) d y\right|_{x=l},  \tag{2.22}\\
R^{\prime \prime} & =\left.\frac{1}{2 \pi} \int_{0}^{b}\left(-\frac{\partial u}{\partial x} G+u \frac{\partial G}{\partial x}\right) d y\right|_{x=-l^{\prime}} \tag{2.23}
\end{align*}
$$

In view of (2.4), (2.2), (2.20), (2.21) and (2.22), Eq. (2.21) can be given the form (2.6).
As explained above, in the present case not (2.6) but its $x$-derivative will be found useful. Differentiation of (2.6) yields

$$
\begin{equation*}
\frac{\partial u\left(x_{0}, y_{0}\right)}{\partial x_{0}}=-2 k B_{1} \sin k x_{0}+\frac{1}{2 \pi} \int_{-b}^{b}\left(\frac{\partial u}{\partial y}\right)_{y=b} \frac{\partial G}{\partial x_{0}} d x \tag{2.24}
\end{equation*}
$$

To obtain this equation, the integral in (2.6) has been differentiated under the integral sign; this is permissable since the limits of integration are independent of $x_{0}$. Since $(\partial u / \partial y)_{y=b}$ is also independent of $x_{0}$, only $G$ has to be differentiated. The explicit form of (2.24) is given by the relation

$$
\begin{equation*}
g\left(y_{0}\right)=\left(\frac{\partial u}{\partial x_{0}}\right)_{x_{0}=b}=-2 k B_{1} \sin k b-\frac{1}{b} \int_{-b}^{b} f(x)\left[K_{0}+\sum_{n=1}^{\infty} K_{n}\right] d x, \tag{2.25}
\end{equation*}
$$

where

$$
\begin{align*}
& f(x)=\left(\frac{\partial u}{\partial y}\right)_{y=b}, \quad K_{0}=\frac{1}{2} \exp [-i k(x-b)] \\
& K_{n}=(-1)^{n} \cos \frac{n \pi y_{0}}{b} \exp \left[\sqrt{\left(\frac{n \pi}{b}\right)^{2}-k^{2}(x-b)}\right] \tag{2.26}
\end{align*}
$$

A similar expression holds for $\left(\partial u / \partial y_{0}\right)$ along $C B$;

$$
\begin{equation*}
f\left(x_{0}\right)=\left(\frac{\partial u}{\partial y_{0}}\right)_{y_{0}-b}=-2 k B_{3} \sin k b-\frac{1}{b} \int_{-b}^{b} g(y)\left[K_{0}+\sum K_{n}\right] d y \tag{2.27}
\end{equation*}
$$

where $K_{0}, K_{n}$ are as in (2.26) but with the coordinates interchanged.
In applying (2.25) one must assume not only $f(x)$ but also $B_{1}$. Likewise in applying (2.27), $B_{3}$ is required along with $g(y)$. Furthermore, $A_{1}$ and $A_{3}$ are essential to the complete solution. In this connection it is advisable to keep the following relations between $f(x), g(y)$ and the constants $A_{1}, B_{1}, A_{3}$ and $B_{3}$ in mind:

$$
\begin{align*}
& A_{1}=B_{1}-\frac{1}{2 i k b} \int_{-b}^{b} f(x) e^{-i k x} d x  \tag{2.28}\\
& A_{3}=B_{3}-\frac{1}{2 i k b} \int_{-b}^{b} g(y) e^{-i k y d y}  \tag{2.29}\\
& A_{1} e^{i k b}-B_{1} e^{-i k b}=\frac{1}{2 i k b} \int_{-b}^{b} g(y) d y  \tag{2.30}\\
& A_{3} e^{i k b}-B_{3} e^{-i k b}=\frac{1}{2 i k b} \int_{-b}^{b} f(x) d x . \tag{2.31}
\end{align*}
$$

These relations enable one to express $A_{1}, B_{1}, A_{3}$ and $B_{3}$ in terms of $f(x)$ and $g(y)$.
The relation (2.28) is established by applying (2.6) to $u\left(x_{0}, y_{0}\right)$ for $x_{0}$ so large that $G$ reduces to its first term in (2 20), and comparing the result with (2.2). A similar derivation over $0<x<b$ yields (2.29). As regards (2.30) it is established by expanding $\partial u / \partial x$ in the horizontal strip $0<y<b$ in a series of cosines of $n \pi y / b$ and comparing for large positive $x$ this expansion with $\partial u / \partial x$ as derived from (2.2); a similar procedure applied over the vertical strip $0<x<b$ to $\partial u / \partial y$ leads to (2.31).

In the present example, in view of the geometric symmetry of the region of Fig. 2 about the diagonal $O B$, any function $u$ over the region can be expressed as the sum of a function which is odd about this diagonal, and one which is even about it. The calculations outlined are simplified considerably for even and odd functions $u$, are quite similar for the two cases and will be illustrated for the odd case.

In the odd case,

$$
\begin{gather*}
A_{3}=-A_{1}, \quad B_{3}=-B_{1}  \tag{2.32}\\
g(x)=-f(x) \tag{2.33}
\end{gather*}
$$

and the integral relations $(2.28)-(2.31)$ reduce to

$$
\begin{equation*}
g(y)=-2 k B_{1} \sin k b-\frac{1}{b} \int_{-b}^{b} f(x)\left[K_{0}+\sum_{n=1}^{\infty} K_{n}\right] d x \tag{2.34}
\end{equation*}
$$

$$
\begin{align*}
A_{1}-B_{1} & =-\frac{1}{2 i k b} \int_{-b}^{b} f(x) e^{-i k x} d x,  \tag{2.35}\\
A_{1} e^{i k b}-B_{1} e^{-i k b} & =-\frac{1}{2 i k b} \int_{-b}^{b} f(x) d x .
\end{align*}
$$

After (2.25) has been applied for an initially assumed $f(x)$ curve, the resulting $g(y)$ shape, changed in sign and plotted against $x$, can be considered as the next approximation to $f(x)$, in view of the relation (2.33) and the symmetry of the region about $x=y$.

From (2.35) the coefficients $A_{1}$ and $B_{1}$ may be determined in terms of $(\partial u / \partial y)_{y=b}=f(x)$. This yields

$$
\begin{align*}
& A_{1}=-\frac{1}{4 k b \sin k b} \int_{-b}^{b} f(x)\left[e^{-i k(x+b)}-1\right] d x \\
& B_{1}=-\frac{1}{4 k b \sin k b} \int_{-b}^{b} f(x)\left[e^{i k b(b-x)}-1\right] d x \tag{2.36}
\end{align*}
$$

The procedure used consisted in assuming $f(x)$, calculating $A_{1}$ and $B_{1}$ from (2.36), then applying (2.34) to calculate $g(y)$, and using the shape of the latter with the sign

Table 1

changed as the starting point of the next step. To prevent the solution from becoming infinite, at each step $f(x)$ is divided by $A_{1}$, thus yielding the case $A_{1}=1$. In the following numerical work the assumption $b=0.2 \lambda, k b=72^{\circ}$ is made.

Although from physical considerations one would be able to make a reasonably


Fig. 4.
good guess for the value of $f(x)$, it was felt that in order to test the method thoroughly, the assumption

$$
\begin{equation*}
\text { along } B C, f(x)=\left(\frac{\partial u}{\partial y}\right)_{y=b}=\text { constant }=1 \tag{2.37}
\end{equation*}
$$

would be more advisable. Making this assumption, solving for $A_{1}$ and $B_{1}$ from (2.36) (with $b=0.2 \lambda, k b=72^{\circ}$ ), and dividing by $A_{1}$, we obtain

$$
\begin{align*}
A_{1}=1, \quad B_{1} & =\frac{b k-e^{i k b} \sin k b}{b k-e^{-i k b} \sin k b}=.0634-.999 i \\
f(x) & =\left(\frac{\partial u}{\partial y}\right)_{y=b}=k(1.320-1.238 i) \tag{2.38}
\end{align*}
$$

The Green's function was evaluated for five positive and five negative values of $x$, and for five values of $y_{0}$, as shown in Table 1. The real part of this family of curves is shown in Fig. 4, the imaginary part being merely $(\pi / b) \sin k(x-b)$. These values, with
(2.38), were inserted in (2.34), the integration being made graphically with areas found by the trapezoidal rule, except near $y=b$.

As $y$ approaches $b$, the value of $g(y)$ increases so rapidly that extrapolation for the curve and the resulting graphical integration is difficult in this region. From physical considerations based on the fact that in a region which is small compared to a wavelength the function $u$ behaves like a harmonic function, it may be shown that a fairly accurate approximation is obtained by assuming $g(y)$ to vary as $(y-b)^{-1 / 3}$ as $y$ approaches $b$. By picking two points $y_{1}$ and $y_{2}$, two constants $A$ and $B$ can be found such that $g(y)=A+B(b-y)^{-1 / 3}$ is fitted to the curve already drawn in this neighborhood for $y<.9 b$; then the area is equal to

$$
\int_{y_{2}}^{b} g(y) d y=A\left(b-y_{2}\right)+\frac{3 B}{2}\left(b-y_{2}\right)^{2 / 3}
$$

The resulting first approximation for $\partial u / \partial x$ is shown in Table 2. By means of (2.36) the values of $A_{3}$ and $B_{3}$ (the negatives of $A_{1}$ and $B_{1}$ ) corresponding to these values were found to be $B_{3}=.1728-1.006 i, A_{3}=.998-.102 i$; thus $B_{1} / A_{1}=.2735-.980 i$.

In order to keep $A_{1}$ fixed at the value unity which we have assumed, we retain this value of the $B_{1} / A_{1}$ ratio and rename it $B_{1}$ as before. We must then divide the values of $\partial u / \partial y$ in Table 2 by $A_{1}$. Reinsertion now into (2.27) gives us the second approximation to $\partial u / \partial y$ shown in Table 3. The corresponding $A_{1}$ and $B_{1}$ yield the ratio $B_{1} / A_{1}=.2658-.960 i$. The third approximation is then carried out in similar fashion, with the the results shown in Table 4. In this case, we have $B_{1} / A_{1}=.266-.964 i$. The approximations to $\partial u / \partial y$ are shown in Fig. 5. Figure 6 shows the ratio $B_{1} / A_{1}$ and thus we see that this ratio is converging toward the value $.266-.962 i$, with the absolute value .997 .

Table 3.
The second approximation.

| $x$ | $f(x)=\partial u / \partial y$ |
| :--- | :--- |
| $.1 b$ | $k[1.119-.804 i]$ |
| $.3 b$ | $k[1.176-.844 i]$ |
| $.5 b$ | $k[1.309-.960 i]$ |
| $.7 b$ | $k[1.562-1.111 i]$ |
| $.9 b$ | $k[2.100-1.619 i]$ |

Table 4.
The third approximation.

| $x$ | $f(x)=\partial u / \partial y$ |
| :---: | :---: |
| $.1 b$ | $k[1.081-.785 i]$ |
| $.3 b$ | $k[1.132-.821 i]$ |
| $.5 b$ | $k[1.249-.911 i]$ |
| $.7 b$ | $k[1.515-1.095 i]$ |
| $.9 b$ | $k[2.12-1.557 i]$ |

A similar calculation could be carried out for a function which is even about the diagonal $O B$. The results of this, together with those already found for the odd function, would enable us to cover all cases involving a corner with these dimensions.


Fig. 5.


Fig. 6.
3. An alternative method of procedure. The procedure described and illustrated in the preceding sections can also be applied in a different way. Basically, the calculation was carried out by first assuming the field over the line $C B$ in Fig. 2, then calculating it over the line $B A$. It is possible to carry out the same calculation by assuming the field over $C B$ not as a function of $x$ or as a curve, but as a Fourier cosine series in $x$,

$$
\begin{equation*}
f(x)=\left.\frac{\partial u}{\partial y}\right|_{y=b}=\sum C_{n} \cos \frac{n \pi x}{b} \tag{3.1}
\end{equation*}
$$

Similarly, $g(y)=\partial u / \partial x$ over $A B$ can be converted into a similar Fourier cosine series in $y$,

$$
\begin{equation*}
g(y)=\left.\frac{\partial u}{\partial x}\right|_{x=b}=\sum D_{n} \cos \frac{n \pi y}{b} \tag{3.2}
\end{equation*}
$$

Applying (2.34), (2.35) and (2.26) to the calculation of $g(y)$ from $f(x)$, we obtain

$$
\begin{align*}
g\left(y_{0}\right) & =\sum D_{m} \cos \frac{m \pi y_{0}}{b} \\
& =-2 k B_{1} \sin k b-\frac{1}{2 b} \sum_{n} C_{n} \int_{-b}^{b} \exp [-i k(x-b)] \cos \frac{n \pi x}{b} d x \\
& \left.-\frac{1}{b} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty}(-1)^{m} C \cos \frac{n \pi y_{0}}{b} \int_{-b}^{b} \cos \frac{n \pi x}{b} \exp \sqrt{\left(\frac{m \pi}{b}-k^{2}(k-b)\right.}\right) d x \tag{3.3}
\end{align*}
$$

This leads to integrals involving a cosine and an exponential in z. After these integrations are carried out, each one of the coefficients $D_{n}$ of the expansion (3.2) turns out to be linearly dependent upon the coefficients $C_{n}$. Thus, instead of being given a curve $f(x)$ and computing from it the curve $g(y)$, one starts with $B_{1}$ and a series of coefficients $C_{n}$ represented by the Fourier expansion (3.1) and ends up with the coefficients $D_{n}$ by applying (3.4). The explicit relation between these two sets of coefficients is

$$
\begin{equation*}
D_{0}=-2 k B_{1} \sin k b+\sum_{n=0}^{\infty} P_{0 n} C_{n}, \quad D_{m}=\sum P_{m_{n}} C_{n} \text { for } m>0 \tag{3.4}
\end{equation*}
$$

where

$$
\begin{aligned}
& P_{0 n}=\frac{(-1)^{n} i k b\left(1-e^{2 i k b}\right)}{2\left(n^{2} \pi^{2}-k^{2} b^{2}\right)}, \\
& P_{m n}=\frac{(-1)^{n+m+1}\left(m^{2} \pi^{2}-k^{2} b^{2}\right)^{1 / 2}\left\{1-\exp \left[-2\left(m^{2} \pi^{2}-k^{2} b^{2}\right)^{1 / 2}\right]\right\}}{\left(m^{2}+n^{2}\right) \pi^{2}-k^{2} b^{2}}, \quad m>0
\end{aligned}
$$

The matrix $P_{m n}$ thus takes the place of the series of curves $\partial G / \partial x$ which were given in Table 1 and shown in Fig. 4. A similar set of equations expresses $C_{n}$ in terms of $D_{n}$ and $B_{3}$.

By proceeding as in $\S 2$ with the field which is odd about the $45^{\circ}$ diagonal $O B$, it is clear that for the final field the coefficients $D_{n}$ should be the negatives of the coefficients $C_{n}$. For the individual successive approximations, this of course is not necessarily the case. The calculation of the next improvement can be carried out by starting with $D_{n}$, changing their signs and putting them in place of $C_{n}$ in (3.4).

It is possible to replace $D_{m}$ by $C_{m}$ in (3.4). A solution of the resulting equations would lead to a complete determination of the field problem. However, the solution of the resulting equations itself involves some method of successive approximation; hence, this procedure is not advisable, and the successive calculation of $C$ 's and $D$ 's appears to be preferable, since it agrees in spirit with the method outlined above and constitutes just a variation of it.

# SOLUTION OF LINEAR AND SLIGHTLY NONLINEAR DIFFERENTIAL EQUATIONS* 

BY

S. A. SCHELKUNOFF<br>Bell Telephone Laboratories

Considering the practical importance of linear differential equations of the second order, or the equivalent systems of the first order equations, it is surprising that treatises give little attention to effective and sufficiently general methods for their solution. The treatises seem to be concerned primarily with power series expansions, Picard's method of successive approximations, numerical methods based on difference equations-methods which in theory are applicable to almost any differential equation and which are practically useless in the case of wave equations. On the positive side, in treatises on mathematical physics one finds a very effective asymptotic approximation which in this country is known as the Wentzel-Kramers-Brillouin approximation and in England as Jeffries' approximation and, of course, the RayleighSchrödinger perturbation method. The former has its obvious limitations and the latter is suitable only for a special class of boundary value problems.

Our purpose is to call. attention to another perturbation method which we developed several years ago in connection with the antenna problem. As time went on the virtues of the method became increasingly apparent. Searching for previous references to this method, we came across one by Bôcher ${ }^{1}$ to a paper by Liouville. ${ }^{2}$ In Liouville's paper we have found the Jeffries-Wentzel-Kramers-Brillouin approximation and a thorough discussion of the usual boundary value problem and associated orthogonal series but very little that has any direct bearing on the present paper.

The method is based on the idea that solutions of linear differential equations may be regarded as distorted or "perturbed" sinusoidal or exponential functions-the same idea which is back of the asymptotic approximation, of the Rayleigh-Schrödinger method, and of the Sturmian theory. It is hardly surprising that this method gives better results than Picard's method which regards the solutions as perturbed straight lines; but the difference is so remarkable that it deserves a special display in a separate note. In this paper, we restrict ourselves to an outline of the procedure and a statement of specific formulas reduced to a point where only simple integrations are needed in any special case. The exposition is based on the second order equation; the extension to higher order linear equations is simple enough. When it comes to nonlinear equations, excepting those which are only slightly nonlinear, $\dagger$ the virtues of the method are not quite clear at present. There is no question that the results

[^17]should be better when compared to those obtained by Picard's method; but the more complicated technique for numerical calculations may offset the advantages. This is something to be explored.

Suppose that our problem is to find the solutions of

$$
\begin{equation*}
\frac{d V}{d x}=-Z(x) I, \quad \frac{d I}{d x}=-Y(x) V, \tag{1}
\end{equation*}
$$

subject to the initial conditions

$$
\begin{equation*}
V=V(a), \quad I=I(a), \quad \text { if } \quad x=a . \tag{2}
\end{equation*}
$$

Picard simply integrates (1) and obtains a pair of integral equations

$$
\begin{equation*}
V(x)=V(a)-\int_{a}^{x} Z(\xi) I(\xi) d \xi, \quad I(x)=I(a)-\int_{a}^{x} Y(\xi) V(\xi) d \xi \tag{3}
\end{equation*}
$$

Thus the stage is set for successive approximations and the solution is obtained in the form of the infinite series

$$
\begin{equation*}
V(x)=V_{0}(x)+V_{1}(x)+V_{2}(x)+\cdots, \quad I(x)=I_{0}(x)+I_{1}(x)+I_{2}(x)+\cdots, \tag{4}
\end{equation*}
$$

where

$$
\begin{array}{ll}
V_{0}(x)=V(a), & V_{n}(x)=-\int_{a}^{x} Z(\xi) I_{n-1}(\xi) d \xi \\
I_{0}(x)=I(a), & I_{n}(x)=-\int_{a}^{x} Y(\xi) V_{n-1}(\xi) d \xi \tag{5}
\end{array}
$$

This procedure is so simple that it would be easy to overlook the fact that in substance we are regarding the solutions of (1) as perturbations of the solutions of

$$
\begin{equation*}
\frac{d V}{d x}=0, \quad \frac{d I}{d x}=0 \tag{6}
\end{equation*}
$$

and that we are dealing with a special application of a much more general perturbation method. Let*

$$
\begin{equation*}
Z(x)=Z_{0}(x)+\widehat{Z}(x), \quad Y(x)=Y_{0}(x)+\widehat{Y}(x), \tag{7}
\end{equation*}
$$

and suppose that the solutions of

$$
\begin{equation*}
\frac{d V_{0}}{d x}=-Z_{0}(x) I_{0}, \quad \frac{d I_{0}}{d x}=-Y_{0}(x) V_{0} \tag{8}
\end{equation*}
$$

subject to the initial conditions (2), are known. Then the solutions of (1) are identical with those of the following integral equations

$$
\begin{align*}
& V(x)=V_{0}(x)-\int_{a}^{x} \widehat{Z}(\xi) I(\xi) V_{1}(x, \xi) d \xi-\int_{a}^{x} \hat{Y}(\xi) V(\xi) V_{2}(x, \xi) d \xi, \\
& I(x)=I_{0}(x)-\int_{a}^{x} \widehat{Z}(\xi) I(\xi) I_{1}(x, \xi) d \xi-\int_{a}^{x} \widehat{Y}(\xi) V(\xi) I_{2}(x, \xi) d \xi, \tag{9}
\end{align*}
$$

[^18]where $V_{1}(x, \xi), I_{1}(x, \xi) ; V_{2}(x, \xi), I_{2}(x, \xi)$ satisfy (8) and are subject to the following conditions
\[

$$
\begin{equation*}
V_{1}(\xi, \xi)=1, \quad I_{1}(\xi, \xi)=0 ; \quad V_{2}(\xi, \xi)=0, \quad I_{2}(\xi, \xi)=1 \tag{10}
\end{equation*}
$$

\]

Essentially the procedure is to regard $-\widehat{Z}(x) I(x)$ and $-\widehat{Y}(x) V(x)$ as known functions and to write the general solution of the corresponding nonhomogeneous linear equation. The verification of the identity of the solutions of (1) and (9) is perfectly straightforward. If $I_{01}(x), I_{02}(x)$, are two linearly independent solutions of (8); then, as the reader can readily verify, $I_{01}^{\prime} I_{02}-I_{02}^{\prime} I_{01}$, differs from $Y_{0}$ only by a constant factor. Bearing this in mind, we have

$$
\begin{align*}
& V_{1}(x, \xi)=\frac{I_{01}^{\prime}(x) I_{02}(\xi)-I_{02}^{\prime}(x) I_{01}(\xi)}{I_{01}^{\prime}(x) I_{02}(x)-I_{02}^{\prime}(x) I_{01}(x)}  \tag{11}\\
& I_{1}(x, \xi)=-Y_{0}(x) \frac{I_{01}(x) I_{02}(\xi)-I_{02}(x) I_{01}(\xi)}{I_{01}^{\prime}(x) I_{02}(x)-I_{02}^{\prime}(x) I_{01}(x)}
\end{align*}
$$

Similarly,

$$
\begin{align*}
& V_{2}(x, \xi)=-Z_{0}(x) \frac{V_{01}(x) V_{02}(\xi)-V_{02}(x) V_{01}(\xi)}{V_{01}^{\prime}(x) V_{02}(x)-V_{02}^{\prime}(x) V_{01}(x)}  \tag{12}\\
& I_{2}(x, \xi)=\frac{V_{01}^{\prime}(x) V_{02}(\xi)-V_{02}^{\prime}(x) V_{01}(\xi)}{V_{01}^{\prime}(x) V_{02}(x)-V_{02}^{\prime}(x) V_{01}(x)}
\end{align*}
$$

Substituting $V_{0}(x), I_{0}(x)$ in the integrands of (9), we obtain $V_{1}(x), I_{1}(x)$; continuing the process we obtain solutions in the form (4).

In Picard's method $Z_{0}(x)=Y_{0}(x)=0$, which is the simplest possible choice. Naturally, the method will work well when $Z(x)$ and $Y(x)$ are small; otherwise it is far better to regard $Z_{0}(x)$ and $Y_{0}(x)$ merely as constants. If we are concerned with a finite interval, these constants may be chosen as some mean valucs* of $Z(x)$ and $Y(x)$-the average values, for example; then for $a=0$ (9) become

$$
\begin{align*}
& V(x)=V_{0}(x)-\int_{0}^{x} \hat{Z}(\xi) I(\xi) \cosh \Gamma_{0}(x-\xi) d \xi+K_{0} \int_{0}^{x} \hat{Y}(\xi) V(\xi) \sinh \Gamma_{0}(x-\xi) d \xi \\
& I(x)=I_{0}(x)+\frac{1}{K_{0}} \int_{0}^{x} \widehat{Z}(\xi) I(\xi) \sinh \Gamma_{0}(x-\xi) d \xi-\int_{0}^{x} \widehat{Y}(\xi) V(\xi) \cosh \Gamma_{0}(x-\xi) d \xi \tag{13}
\end{align*}
$$

where

$$
\begin{array}{lll}
V_{0}(x)=V_{0} \cosh \Gamma_{0} x-K_{0} I_{0} \sinh \Gamma_{0} x, & \Gamma_{0}=\sqrt{Z_{0} Y_{0}}, & K_{0}=\sqrt{Z_{0} / Y_{0}} \\
I_{0}(x)=-\frac{V_{0}}{K_{0}} \sinh \Gamma_{0} x+I_{0} \cosh \Gamma_{0} x, & V_{0}=V_{0}(0), & I_{0}=I_{0}(0) \tag{14}
\end{array}
$$

In practice it is found that these equations represent a great improvement on Picard's method and yet the integrations which have to be performed are not more difficult. If $Z(x)$ and $Y(x)$ are constants, Picard's method leads to power series-not

[^19]a satisfactory form for wave functions. John R. Carson ${ }^{3}$ employed Picard's method for approximate solution when $Z(x)$ and $Y(x)$ are slowly varying functions and succeeded in summing the series and obtaining the first order correction terms in a usable form; but any attempt to get the higher order terms by this method would seem to be out of the question. Theoretically, we should select $Z_{0}(x)$ and $Y_{0}(x)$ as near as possible to $Z(x)$ and $Y(x)$, subject to our ability to solve (8); but the integrations will be difficult to perform.* Thus we come back to (13) as the best compromise and it works very well.

In the more explicit form the first order correction terms are

$$
\begin{align*}
V_{1}(x)= & V_{0}\left[B(x) \cosh \Gamma_{0} x-A(x) \sinh \Gamma_{0} x+C(x) \sinh \Gamma_{0} x\right] \\
& -K_{0} I_{0}\left[A(x) \cosh \Gamma_{0} x-B(x) \sinh \Gamma_{0} x+C(x) \cosh \Gamma_{0} x\right], \\
I_{1}(x)= & -\frac{V_{0}}{K_{0}}\left[B(x) \sinh \Gamma_{0} x-A(x) \cosh \Gamma_{0} x+C(x) \cosh \Gamma_{0} x\right]  \tag{15}\\
& +I_{0}\left[A(x) \sinh \Gamma_{0} x-B(x) \cosh \Gamma_{0} x+C(x) \sinh \Gamma_{0} x\right],
\end{align*}
$$

where

$$
\begin{align*}
& A(x)=\frac{1}{2} \int_{0}^{x}\left[\frac{\widehat{Z}}{K_{0}}-K_{0} \widehat{Y}\right] \cosh 2 \Gamma_{0} \xi d \xi, \\
& B(x)=\frac{1}{2} \int_{0}^{x}\left[\frac{\widehat{Z}}{K_{0}}-K_{0} \widehat{Y}\right] \sinh 2 \Gamma_{0} \xi d \xi,  \tag{16}\\
& C(x)=\frac{1}{2} \int_{0}^{x}\left[\frac{\widehat{Z}}{K_{0}}+K_{0} \widehat{Y}\right] d \xi .
\end{align*}
$$

In some instances it is preferable to express the results in terms of progressive waves; then $V(x)=V_{0}(x)+V_{1}(x)$ and $I(x)=I_{0}(x)+I_{1}(x)$ become

$$
\begin{align*}
V^{+}(x) & =K_{0} I_{0}^{+}\left[e^{-\Gamma_{0} x}-C(x) e^{-\Gamma_{0 x}}-E(x) e^{\Gamma_{0} x}\right] \\
I^{+}(x) & =I_{0}^{+}\left[e^{-\Gamma_{0} x}-C(x) e^{-1_{0} x}+E(x) e^{\Gamma_{0} x}\right] \\
V^{-}(x) & =-K_{0} I_{0}^{-}\left[e^{\Gamma_{0} x}+C(x) e^{j_{0} 0 x}+D(x) e^{-\Gamma_{0} x}\right]  \tag{17}\\
I^{-}(x) & =I_{0}^{-}\left[e^{\Gamma_{0} x}+C(x) e^{\Gamma_{0} x}-D(x) e^{-\Gamma_{0} x}\right]
\end{align*}
$$

where

$$
\begin{align*}
& D(x)=A(x)+B(x)=\frac{1}{2} \int_{0}^{x}\left[\frac{\widehat{Z}}{K_{0}}-K_{0} \widehat{Y}\right] e^{2 \Gamma 0 \xi} d \xi \\
& E(x)=A(x)-B(x)=\frac{1}{2} \int_{0}^{x}\left[\frac{\widehat{Z}}{K_{0}}-K_{0} \widehat{Y}\right] e^{-2 \operatorname{ro\xi } d \xi} \tag{18}
\end{align*}
$$

Equations (14) and (15) express the solutions in terms of $V$ and $I$ at the beginning of a finite interval $(0, l)$; one also often wants the corresponding expressions in terms of the final values. These are

[^20]\[

$$
\begin{align*}
& V_{0}(x)= V(l) \cosh \Gamma_{0}(l-x)+K_{0} I(l) \sinh \Gamma_{0}(l-x) \\
& I_{0}(x)= \frac{V(l)}{K_{0}} \sinh \Gamma_{0}(l-x)+I(l) \cosh \Gamma_{0}(l-x) \\
& V_{1}(x)= V(l)\left\{[B(x)-B(l)] \cosh \Gamma_{0}(l+x)-[A(x)-A(l)] \sinh \Gamma_{0}(l+x)\right. \\
&\left.-[C(x)-C(l)] \sinh \Gamma_{0}(l-x)\right\} \\
&+K_{0} I(l)\left\{[B(x)-B(l)] \sinh \Gamma_{0}(l+x)-[A(x)-A(l)] \cosh \Gamma_{0}(l+x)\right. \\
&\left.-[C(x)-C(l)] \cosh \Gamma_{0}(l-x)\right\}  \tag{19}\\
& \begin{aligned}
I_{1}(x)= & \frac{V(l)}{K_{0}}\left\{[A(x)-A(l)] \cosh \Gamma_{0}(l+x)-\right. \\
& -[B(x)-B(l)] \sinh \Gamma_{0}(l+x) \\
& +I(l)\left\{[A(x)-A(l)] \sinh \Gamma_{0}(l+x)-[B(x)-B(l)] \cosh \Gamma_{0}(l+x)\right. \\
& \left.-[C(x)-C(l)] \sinh \Gamma_{0}(l-x)\right\}
\end{aligned}
\end{align*}
$$
\]

Suppose now that the interval is infinite and that $Z(x)$ and $Y(x)$ are slowly varying functions. In this case, there exists the Liouville-Jeffries-Wentzel-KramersBrillouin approximation

$$
\begin{align*}
& V(x)= \pm A \sqrt{K(x) K\left(x_{0}\right)} \exp \left[\mp \int_{x_{0}}^{x} \Gamma(\xi) d \xi\right] \\
& I(x)=A \sqrt{K\left(x_{0}\right) / K(x)} \exp \left[\mp \int_{x_{0}}^{x} \Gamma(\xi) d \xi\right] \tag{20}
\end{align*}
$$

where

$$
\begin{equation*}
K(x)=\sqrt{Z(x) / Y(x)}, \quad \Gamma(x)=\sqrt{Z(x) Y(x)} \tag{21}
\end{equation*}
$$

To the communication engineer these approximations seem natural even without formal analysis. He would reason as follows. If the "characteristic impedance" $K(x)$ is independent of $x$, a progressive wave moving either to the left or to the right would suffer no reflection; it is only the sudden changes in the impedance that causes reflections. Hence the voltage $V(x)$ and current $I(x)$ associated with the progressive waves will be proportional to $\exp \mp\left[\int_{x_{\mathrm{c}}}^{x} \Gamma(x) d x\right]$. If $K(x)$ is a slowly varying function, we can ignore the reflections and in the first approximation consider the line as continuously "matched" and thus acting as a transformer. This means that the voltage will vary directly and the current inversely as the square root of the characteristic impedance: hence, equations (20).

There are several formal derivations; ${ }^{4}$ but the one which appeals to us most because it corresponds closely to the physical argument is also the one which permits further improvements in the approximation. Let us consider the "transfer parameter" $\Theta$

$$
\begin{equation*}
\Theta=\int^{x} \Gamma(\xi) d \xi, \quad \frac{d \Theta}{d x}=\Gamma(x) \tag{22}
\end{equation*}
$$

as the new independent variable. Substituting in (1), we obtain

[^21]\[

$$
\begin{equation*}
\frac{d V}{d \Theta}=-K(\Theta) I, \quad \frac{d I}{d \Theta}=-\frac{V}{K(\Theta)} \tag{23}
\end{equation*}
$$

\]

Eliminating first $I$ and then $V$ we have

$$
\begin{equation*}
V^{\prime \prime}(\Theta)-\frac{K^{\prime}(\Theta)}{K(\Theta)} V^{\prime}(\Theta)-V=0, \quad I^{\prime \prime}(\Theta)+\frac{K^{\prime}(\Theta)}{K(\Theta)} I^{\prime}(\Theta)-I=0 \tag{24}
\end{equation*}
$$

If $K(\Theta)$ is constant, we have simple progressive waves as anticipated; otherwise, we introduce new dependent variables in conformity with our idea of voltage and current transformation

$$
\begin{equation*}
V=[K(\Theta)]^{1 / 2} \bar{V}, \quad I=[K(\Theta)]^{-1 / 2} \bar{I} \tag{25}
\end{equation*}
$$

Incidentally, this is the transformation which should remove the first derivatives from (24). Substituting, we obtain

$$
\begin{equation*}
\bar{V}^{\prime \prime}(\Theta)=\left[1+\frac{3\left(K^{\prime}\right)^{2}}{4 K^{2}}-\frac{K^{\prime \prime}}{2 K}\right] \bar{V}, \quad \bar{I}^{\prime \prime}(\Theta)=\left[1-\frac{\left(K^{\prime}\right)^{2}}{4 K^{2}}+\frac{K^{\prime \prime}}{2 K}\right] \bar{I} \tag{26}
\end{equation*}
$$

We now have not only equations (20) but also the quantitative criterion of their goodness: $\left(K^{\prime} / K\right)^{2}$ and $K^{\prime \prime} / 2 K$ should be small compared with unity.

To improve on (20), we could repeat the process beginning with (22); but the analytical work is simpler if we turn to equations (13) and apply them to an infinite interval, assuming of course that in the entire interval the bracketed quantities in equations (26) differ but little from unity. Thus, the solutions of

$$
\begin{equation*}
\frac{d^{2} y}{d x^{2}}=[1+f(x)] y \tag{27}
\end{equation*}
$$

are also the solutions of

$$
\begin{equation*}
y(x)=y_{0}(x)+\int_{\infty}^{x} f(\xi) y(\xi) \sinh (x-\xi) d \xi \tag{28}
\end{equation*}
$$

provided the integral is convergent. The solutions asymptotic to $e^{\mp x}$ are

$$
\begin{equation*}
y(x) \simeq e^{\mp x}+\int_{\infty}^{x} f(\xi) e^{\mp x} \sinh (x-\xi) d \xi \tag{29}
\end{equation*}
$$

or

$$
\begin{align*}
& y^{+}(x) \simeq e^{-x}-\frac{1}{2} e^{-x} \int_{\infty}^{x} f(\xi) d \xi+\frac{1}{2} e^{x} \int_{\infty}^{x} e^{-2 \xi} f(\xi) d \xi \\
& y^{-}(x) \simeq e^{x}+\frac{1}{2} e^{x} \int_{i \infty}^{x} f(\xi) d \xi-\frac{1}{2} e^{-x} \int_{i \infty}^{x} e^{2 \xi} f(\xi) d \xi \tag{30}
\end{align*}
$$

From these equations we can obtain the well-known asymptotic expansions of Bessel functions as well as expansions of other types.

The case in which $\Theta=i \beta x$, where $\beta$ is a constant, occurs so frequently that a repetition is justified. Equations (26) become

$$
\begin{equation*}
\bar{V}^{\prime \prime}(x)=-\beta^{2} \bar{V}+\left[\frac{3\left(K^{\prime}\right)^{2}}{4 K^{2}}-\frac{K^{\prime \prime}}{2 K}\right] \bar{V}, \quad \bar{I}^{\prime \prime}(x)=-\beta^{2} \bar{I}+\left[\frac{K^{\prime \prime}}{2 K}-\frac{\left(K^{\prime}\right)^{2}}{4 K^{2}}\right] \bar{I} \tag{31}
\end{equation*}
$$

and the corresponding integral equations are

$$
\begin{align*}
& \bar{V}(x)=\bar{V}_{0}(x)+\frac{1}{\beta} \int_{\infty}^{x}\left[\frac{3\left[K^{\prime}(\xi)\right]^{2}}{4[K(\xi)]^{2}}-\frac{K^{\prime \prime}(\xi)}{2 K(\xi)}\right] \bar{V}(\xi) \sin \beta(x-\xi) d \xi \\
& \bar{I}(x)=\bar{I}_{0}(x)+\frac{1}{\beta} \int_{\infty}^{x}\left[\frac{K^{\prime \prime}(\xi)}{2 K(\xi)}-\frac{\left[K^{\prime}(\xi)\right]^{2}}{4[K(\xi)]^{2}}\right] \bar{I}(\xi) \sin \beta(x-\xi) d \xi \tag{32}
\end{align*}
$$

Suppose, for example, that $K(x)=K_{0}+k x$; then, asymptotically,

$$
\begin{align*}
& V(x)= \pm A \sqrt{K_{0}+k x}\left[1 \mp \frac{3 i k}{8 \beta\left(K_{0}+k x\right)}\right] e^{\mp i \beta x}, \\
& I(x)=\frac{A}{\sqrt{K_{0}+k x}}\left[1 \pm \frac{i k}{8 \beta\left(K_{0}+k x\right)}\right] e^{\mp i \beta x} . \tag{33}
\end{align*}
$$

In this case, however, the integrals in (32) can be evaluated in terms of sine and cosine integrals. Moreover, the complete result corresponds closely to the physical picture of reflection which invariably takes place when waves are traveling in transmission lines or media with variable characteristic impedance $K(x)$. Thus

$$
\begin{align*}
& V(x)=A\left[\sqrt{K_{0}+k x} e^{-i \beta x}+R_{V} \sqrt{K_{0}+k x} e^{i \beta x}\right], \\
& I(x)=A\left[\frac{e^{-i \beta x}}{\sqrt{K_{0}+k x}}+R_{I} \frac{e^{i \beta x}}{\sqrt{K_{0}+k x}}\right]
\end{align*}
$$

where $R_{V}$ and $R_{I}$ are the first order reflection coefficients given by

$$
\begin{aligned}
R_{V}=-3 R_{I}=-(3 / 4) \exp \left(2 i \beta k^{-1} K_{0}\right) & {\left[\mathrm{Ci}\left(2 \beta x+2 \beta k^{-1} K_{0}\right)\right.} \\
& \left.-i \operatorname{Si}\left(2 \beta x+2 \beta k^{-1} K_{0}\right)+\frac{i \pi}{2}\right]
\end{aligned}
$$

The succeeding correction terms represent successive reflections. The entire series resembles an asymptotic solution of the differential equation in question but it appears to be rapidly convergent.

An another example, take the case of principal waves on a thin cylindrical antenna when

$$
\begin{equation*}
K(x)=120 \log (2 x / a), \quad K^{\prime}(x)=\frac{120}{x}, \quad K^{\prime \prime}(x)=-\frac{120}{x^{2}} \tag{34}
\end{equation*}
$$

In this case we obtain

$$
\begin{align*}
& V(x)=A \sqrt{K(x)}\left[1-\frac{i}{2 \beta x}\left\{\frac{1}{4[\log (2 x / a)]^{2}}+\frac{1}{2 \log 2(x / a)}\right\}\right] e^{-i \Delta x} \\
& I(x)=\frac{A}{\sqrt{K(x)}}\left[1+\frac{i}{2 \beta x}\left\{\frac{-1}{4[\log (2 x / a)]^{2}}+\frac{1}{2 \log (2 x / a)}\right\}\right] e^{-i \beta x} \tag{35}
\end{align*}
$$

As the third example we shall take Rayleigh's equation for a nonlinear oscillator ${ }^{5}$

$$
\begin{equation*}
\ddot{q}+\left(R_{1} \dot{q}+R_{3} \dot{q}^{3}\right)+\omega^{2} q=0 \tag{36}
\end{equation*}
$$

By (13) we have

$$
\begin{equation*}
q(t)=q_{0}(t)-\frac{1}{\omega} \int_{0}^{t}\left[R_{1} \dot{q}(\tau)+R_{3} \dot{q}^{3}\right] \sin \omega(t-\tau) d \tau \tag{37}
\end{equation*}
$$

where $q_{0}(t)$ is a sinusoidal function. If $q=0$ up to $t=0$, then $q_{0}(t)=A \sin \omega t$. Substituting in (37) and integrating, we obtain

$$
\begin{equation*}
q(t)=A \sin \omega t-\frac{1}{2}\left(R_{1}+\frac{3}{4} \omega^{2} R_{3} A^{3}\right) t \sin \omega t-\frac{1}{32} \omega R_{3} A^{3}(\cos \omega t-\cos 3 \omega t) \tag{38}
\end{equation*}
$$

For a periodic solution we must have

$$
\begin{equation*}
R_{1}+\frac{3}{4} \omega^{2} R_{3} A^{2}=0 \tag{39}
\end{equation*}
$$

then

$$
\begin{equation*}
q(t)=A \sin \omega t-\frac{1}{32} \omega R_{3} A^{3}(\cos \omega t-\cos 3 \omega t) \tag{40}
\end{equation*}
$$

Equation (39) is precisely Rayleigh's equation for the amplitude of oscillations; equation (40) differs from his equation in that ours contains a term proportional to cos $\omega t$. Our approximation satisfies the initial condition $q(0)=0$ while Rayleigh's does not.

Originally this work was undertaken to obtain convenient analytic approximations to a number of problems in wave theory. It has since become apparent, however, that at least for a certain class of differential equations, the method would be suitable for numerical solution. The practicability of Picard's method for this purpose has already been explored by Thornton C. Fry; the present method should be quicker. The rapidity of convergence will be discussed in a separate paper.

[^22]
# A CYLINDER COOLING PROBLEM* 

BY

SAMUEL A. SCHAAF<br>University of California, Berkeley

1. Introduction. The linear cooling problem for non-homogeneous solids has been investigated extensively by Rust, ${ }^{1}$ Churchill, ${ }^{2}$ Carslaw, ${ }^{3}$ Mersman, ${ }^{4}$ and others. It is the purpose of this paper to obtain a solution for the corresponding cylindrical problem. The method used is that of the Laplace Transform.
2. The Problem. Let us consider an infinitely long circular cylinder of radius $a$ and initial temperature $T_{0}$, instantaneously immersed in an infinite medium initially at zero temperature. Let the heat conductivities and diffusivities of the cylinder and external medium be respectively $K_{\nu}$ and $h_{\nu}^{2}(\nu=1,2)$. Then if $r$ is the distance from the axis of the cylinder and $t$ is the time, the following differential system is satisfied ${ }^{5}$ by the temperature functions $T_{\nu}(r, t)$ :

$$
\begin{align*}
h_{1}^{2}\left\{\frac{\partial^{2} T_{1}}{\partial r^{2}}+\frac{1}{r} \frac{\partial T_{1}}{\partial r}\right\} & =\frac{\partial T_{1}}{\partial t} \quad 0 \leqq r<a, \quad t>0,  \tag{1}\\
h_{2}^{2}\left\{\frac{\partial^{2} T_{2}}{\partial r^{2}}+\frac{1}{r} \frac{\partial T_{2}}{\partial r}\right\} & =\frac{\partial T_{2}}{\partial t} \quad r>a, \quad t>0,  \tag{2}\\
\lim _{r \rightarrow a-} T_{1} & =\lim _{r \rightarrow a+} T_{2} \quad t>0,  \tag{3}\\
\lim _{r \rightarrow a-} K_{1} \frac{\partial T_{1}}{\partial r} & =\lim _{r \rightarrow a+} K_{2} \frac{\partial T_{2}}{\partial r} \quad t>0,  \tag{4}\\
\lim _{t \rightarrow 0} T_{1} & =T_{0} \quad 0 \leqq r<a,  \tag{5}\\
\lim _{t \rightarrow 0} T_{2} & =0 \quad r>a . \tag{6}
\end{align*}
$$

3. Solution. Let the Laplace transform of $T_{\nu}(r, t)$ be $T_{\nu}^{*}(r, s)$, i.e.,

$$
T_{r}^{*}(r, s)=\int_{0}^{\infty} e^{-s} t T_{v}(r, t) d t \quad s>0
$$

Applying this transform to (1)-(6), we obtain the corresponding set of ordinary differential equations containing $s$ as a parameter;

[^23]\[

$$
\begin{align*}
h_{1}^{2}\left\{\frac{d^{2} T_{1}^{*}}{d r^{2}}+\frac{1}{r} \frac{d T_{1}^{*}}{d r}\right\} & =-T_{0}+s T_{1}^{*} \quad 0 \leqq r<a  \tag{1*}\\
h_{2}^{2}\left\{\frac{d^{2} T_{2}^{*}}{d r^{2}}+\frac{1}{r} \frac{d T_{2}^{*}}{d r}\right\} & =s T_{2}^{*} \quad r>a  \tag{*}\\
\lim _{r+a-} T_{1}^{*} & =\lim _{r \rightarrow a+} T_{2}^{*}  \tag{*}\\
\lim _{r \rightarrow a-} K_{1} \frac{d T_{1}^{*}}{d r} & =\lim _{r \rightarrow a+} K_{2} \frac{d T_{2}^{*}}{d r} \tag{*}
\end{align*}
$$
\]

The solution ${ }^{6}$ of this system is

$$
\begin{align*}
& T_{1}^{*}(r, s)=\frac{T_{0}}{s}\left[1+\frac{K_{0}^{\prime}\left(\alpha_{2} \sqrt{s}\right) \cdot I_{0}\left(r \sqrt{s} / h_{1}\right)}{D(\sqrt{s})}\right]  \tag{7}\\
& T_{2}^{*}(r, s)=\frac{T_{0}}{s} \frac{\alpha_{3} I_{0}^{\prime}\left(\alpha_{1} \sqrt{s}\right) \cdot K_{0}\left(r \sqrt{s} / h_{2}\right)}{D(\sqrt{s})} \tag{8}
\end{align*}
$$

where

$$
\begin{align*}
\alpha_{1} & =a / h_{1}, \quad \alpha_{2}=a / h_{2}, \quad \alpha_{3}=K_{1} h_{2} / K_{2} h_{1},  \tag{9}\\
D(x) & =\alpha_{3} I_{0}^{\prime}\left(\alpha_{1} x\right) K_{0}\left(\alpha_{2} x\right)-I_{0}\left(\alpha_{1} x\right) K_{0}^{\prime}\left(\alpha_{2} x\right) \tag{10}
\end{align*}
$$

The functions $T_{\nu}(r, t)$ may now be obtained by use of the complex inversion formula ${ }^{7}$

$$
\begin{equation*}
T_{\nu}(r, t)=\lim _{\lambda \rightarrow \infty} \frac{1}{2 \pi i} \int_{c-i \lambda}^{c+i \lambda} e^{z t} T_{\nu}^{*}(r, z) d z, \quad c>0 . \tag{11}
\end{equation*}
$$

In order to reduce these contour integrals to real integrals we must first establish the following lemma.

Lemma. $D(z)$ does not vanish for $|\arg z| \leqq \frac{1}{2} \pi$.
Proof. We choose two numbers $A$ and $B$, arbitrary except that

$$
\begin{equation*}
0<A<1<B \tag{12}
\end{equation*}
$$

Then since $\alpha_{\nu}$ is positive $(\nu=1,2,3)$, it will be sufficient to show that, when $|\arg z|$ $\leqq \frac{1}{2} \pi, D(z)$ does not vanish for any values of $\alpha_{\nu}$ such that $A \leqq \alpha_{\nu} \leqq B(\nu=1,2,3)$. The proof now follows in four parts.
i) There is a number $R_{1}>0$, such that $D(z)$ does not vanish for

$$
A \leqq \alpha_{\nu} \leqq B, \quad|\arg z| \leqq \frac{1}{2} \pi, \quad|z|<R_{1} .
$$

This is true because we may use the ordinary series expansions of the Bessel function ${ }^{8}$ to write

$$
D(z)=\frac{1}{\alpha_{2} z}+\left(\alpha_{2}-\alpha_{1} \cdot \alpha_{3}\right) \frac{z}{2} \cdot \log \frac{\alpha_{2} z}{2 .}+B(z)
$$

[^24]where $B(z)$ is bounded in the neighborhood of $z=0$. The result is then evident.
ii) There is a number $R_{2}>0$, such that $D(z)$ does not vanish for
$$
A \leqq \alpha_{\nu} \leqq B, \quad|\arg z| \leqq \frac{1}{2} \pi, \quad|z|>R_{2}
$$

To see this we use the well-known asymptotic formulae for these Bessel functions ${ }^{9}$ to write

$$
D(z)=\frac{e^{\left(\alpha_{1}-\alpha_{2}\right) z}}{2 z \sqrt{\alpha_{1} \alpha_{2}}}\left\{\left(\alpha_{3}+1+\epsilon_{1}\right)-i\left(\alpha_{3}-1+\epsilon_{2}\right) e^{-2 \alpha_{1}}\right\}
$$

where $\epsilon_{k} \rightarrow 0$ as $z \rightarrow \infty$ for $k=1,2$, uniformly in the $\alpha_{\nu}$ providing $A \leqq \alpha_{s} \leqq B(\nu=1,2,3)$. Clearly, $D(z)$ can vanish only if

$$
e^{2 \alpha_{1} z}=\frac{\alpha_{3}-1+\epsilon_{2}}{\alpha_{3}+1+\epsilon_{1}} i
$$

But for sufficiently large $|z|$, say $|z|>R_{2}$, the right member is less than unity in absolute value. Hence for $|z|>R_{2}$, this relation cannot hold with $\mid$ arg $z \left\lvert\, \leqq \frac{1}{2} \pi\right.$.
iii) $D(z)$ does not vanish for $|\arg z|=\frac{1}{2} \pi$. To see this, we let $z=e^{\frac{1}{i r} y}$ ( $y$ real). Then ${ }^{10}$

$$
\begin{align*}
I_{0}\left(e^{\text {jix }} y\right) & =J_{0}(y)  \tag{13}\\
K_{0}\left(e^{1 i \pi} y\right) & =-\frac{1}{2} i_{\pi}\left[J_{0}(y)-i Y_{0}(y)\right] \tag{14}
\end{align*}
$$

Hence

$$
\begin{aligned}
D\left(y e^{\frac{1 i \pi}{i \pi}}\right)= & \left\{\alpha_{3} J_{0}^{\prime}\left(\alpha_{1} y\right) Y_{0}\left(\alpha_{2} y\right)-J_{0}\left(\alpha_{1} y\right) Y_{0}^{\prime}\left(\alpha_{2} y\right)\right\} \\
& +i\left\{\alpha_{3} J_{0}^{\prime}\left(\alpha_{1} y\right) J_{0}\left(\alpha_{2} y\right)-J_{0}\left(\alpha_{1} y\right) J_{0}^{\prime}\left(\alpha_{2} y\right)\right\}
\end{aligned}
$$

Therefore $D\left(y e^{\frac{3}{i r}}\right)$ can vanish only if

$$
\alpha_{3} J_{0}^{\prime}\left(\alpha_{1} y\right) Y_{0}\left(\alpha_{2} y\right)-J_{0}\left(\alpha_{1} y\right) Y_{0}^{\prime}\left(\alpha_{2} y\right)=\alpha_{3} J_{0}^{\prime}\left(\alpha_{1} y\right) J_{0}\left(\alpha_{2} y\right)-J_{0}\left(\alpha_{1} y\right) J_{0}^{\prime}\left(\alpha_{2} y\right)=0
$$

But this is impossible since it would imply either the existence of a common root for at least two of these Bessel functions, ${ }^{11}$ or the vanishing of the Wronskian

$$
W\left[J_{0}\left(\alpha_{2} y\right), Y_{0}\left(\alpha_{2} y\right)\right]=\frac{2}{\pi \alpha_{2} y}
$$

iv) We consider now the integral (see Fig. 1)

$$
f\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)=\frac{1}{2 \pi i} \int_{C} \frac{D^{\prime}(z)}{D(z)} d z
$$

From $i$, $i i$, and $i i i$ ) it follows that $D(z)$ does not vanish on $C$ for all $\alpha$, such that $A \leqq \alpha_{\nu} \leqq B(\nu=1,2,3)$. Now these Bessel functions are all analytic except possibly at $z=0$. Hence $f\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ is continuous.

[^25]Since $D(z)$ has no singularities inside $C, f\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ gives the number of zeros of $D(z)$ inside $C$. It can therefore take on only integral values; but this implies that $f\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ is constant.

Finally

$$
f(1,1,1)=\frac{1}{2 \pi i} \int_{C} \frac{W^{\prime}\left[I_{0}(z), K_{0}(z)\right]}{W\left[I_{0}(z), K_{0}(z)\right]} d z=0
$$

Hence $f\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right) \equiv 0$, for all $\alpha_{\nu}$ satisfying the relation $A \leqq \alpha_{\nu} \leqq B$. Therefore $D(z)$ has no roots inside $C$. Since the radii $R_{3}$ and $R_{4}$ (see Fig. 1) are arbitrary, except that


Fig. 1. The contour $C$, consisting of the circular arcs $|z|=R_{3}$ and $|z|=R_{4}$ and the line segments on the imaginary axis joining them. The only restriction is that $R_{3}<R_{1}$ and $R_{1}>R_{2}$.


Fig. 2. The contours $l_{1} \Gamma_{1}, \Gamma_{2}, \Gamma_{3}, L_{1}$ and $L_{2}$. The radius of $\Gamma_{z}$ is $\rho$.
$R_{3}<R_{1}$ and $R_{4}>R_{2}$, it follows that $D(z)$ has no zeros in the entire right-half plane, which concludes the proof of the lemma.

We now transform the contour integrals of (11) into real integrals. Let us consider $T_{1}(r, t)$ first. According to the lemma just established, $D(\sqrt{z})$ does not vanish for $|\arg \sqrt{z}| \leqq \frac{1}{2} \pi$, i.e., for $|\arg z| \leqq \pi$. Hence the integrand in (11) is analytic for $|\arg z| \leqq \pi$, and we may (see Fig. 2) replace the integral along $l$ by the sum of the integrals over $\Gamma_{1}, \Gamma_{2}, \Gamma_{3}, L_{2}$, and $L_{2}$. Using the asymptotic developments, we easily see that for large $z$,

$$
\frac{K_{0}^{\prime}\left(\alpha_{2} \sqrt{z}\right) I_{0}\left(r \sqrt{z} / h_{1}\right)}{D(\overline{\sqrt{z}})}=O\left\{\frac{1}{\sqrt{z}} \exp \left[\left(\frac{r}{h_{1}}-\alpha_{1}\right) \sqrt{z}\right]\right\} .
$$

Therefore as $\lambda \rightarrow \infty$, the integrals over $\Gamma_{1}$ and $\Gamma_{2}$ vanish, since $t>0$.

Near the origin, the term $K_{0}^{\prime}\left(\alpha_{2} \sqrt{z}\right) I_{0}\left(\alpha_{1} \sqrt{z}\right)$ dominates the denominator, and hence

$$
\lim _{z \rightarrow 0} \frac{K_{0}^{\prime}\left(\alpha_{2} \sqrt{z}\right) I_{0}\left(r \sqrt{z} / h_{1}\right)}{D(\sqrt{z})}=-1
$$

Therefore the integral over $\Gamma_{3}$ vanishes with $\rho$ (see Fig. 2).
On $L_{1}$, we set $z=\sigma^{2} e^{i z}, \sigma>0$. Then, using (13) and (14), we obtain

$$
\begin{align*}
& \frac{1}{2 \pi i} \int_{L_{1}} e^{z i} T_{1}^{*}(r, z) d z=\frac{T_{0}}{\pi i} \int_{v_{0}}^{\sqrt{R}} \frac{e^{-\sigma^{2}}}{\sigma} \\
& \left\{1+\frac{J_{0}\left(r \sigma / h_{1}\right) Y_{0}^{\prime}\left(\alpha_{2} \sigma\right)+i J_{0}\left(r \sigma / h_{1}\right) J_{0}^{\prime}\left(\alpha_{2} \sigma\right)}{\left[\alpha_{2} J_{0}^{\prime}\left(\alpha_{1} \sigma\right) Y_{0}\left(\alpha_{2} \sigma\right)-J_{0}\left(\alpha_{1} \sigma\right) Y_{0}^{\prime}\left(\alpha_{2} \sigma\right)\right]+i\left[\alpha_{3} J_{0}^{\prime}\left(\alpha_{1} \sigma\right) J_{0}\left(\alpha_{2} \sigma\right)-J_{0}\left(\alpha_{1} \sigma\right) J_{0}^{\prime}\left(\alpha_{2} \sigma\right)\right]}\right\} d \sigma . \tag{15}
\end{align*}
$$

On $L_{2}$, we set $z=\sigma^{2} e^{-i x}, \sigma>0$, and obtain the conjugate of (15). Adding these and taking the limit as $\lambda \rightarrow \infty$ and $\rho \rightarrow 0$, we obtain finally,

$$
\begin{equation*}
T_{1}(r, t)=\frac{-4 T_{0} \alpha_{3}}{\pi^{2} \alpha_{2}} \int_{0}^{\infty} \frac{e^{-\sigma^{2} t}}{\sigma^{2}} \frac{J_{0}\left(r \sigma / h_{1}\right) J_{0}^{\prime}\left(\alpha_{1} \sigma\right)}{\Delta(\sigma)} d \sigma \tag{16}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta(\sigma)= & {\left[\alpha_{3} J_{0}^{\prime}\left(\alpha_{1} \sigma\right) Y_{0}\left(\alpha_{2} \sigma\right)-J_{0}\left(\alpha_{1} \sigma\right) Y_{0}^{\prime}\left(\alpha_{2} \sigma\right)\right]^{2} } \\
& +\left[\alpha_{3} J_{0}^{\prime}\left(\alpha_{1} \sigma\right) J_{0}\left(\alpha_{2} \sigma\right)-J_{0}\left(\alpha_{1} \sigma\right) J_{0}^{\prime}\left(\alpha_{2} \sigma\right)\right]^{2} . \tag{17}
\end{align*}
$$

Similarly

$$
T_{2}(r, t)=\frac{2 T_{0} \alpha_{3}}{\pi} \int_{0}^{\infty} \frac{e^{-\sigma^{2} t}}{\sigma} \frac{J_{0}^{\prime}\left(\alpha_{1} \sigma\right)\left[J_{0}\left(\alpha_{1} \sigma\right) C_{1}\left(\alpha_{2} \sigma, r \sigma / h_{2}\right)-\alpha_{3} J_{0}^{\prime}\left(\alpha_{1} \sigma\right) C\left(\alpha_{2} \sigma, r \sigma / h_{2}\right)\right]}{\Delta(\sigma)} d \sigma,(18)
$$

where

$$
\begin{align*}
C(x, y) & =J_{0}(x) Y_{0}(y)-Y_{0}(x) J_{0}(y),  \tag{19}\\
C_{1}(x, y) & =J_{0}^{\prime}(x) Y_{0}(y)-Y_{0}^{\prime}(x) J_{0}(y) . \tag{20}
\end{align*}
$$

These formulae constitute the solution of the differential system (1)-(6).
4. Remarks. Since the Laplace Transform method is essentially a formal one, any solution obtained in this manner must always be verified. In the present case this is easily done. ${ }^{12}$

It may also be shown, under certain conditions as to boundedness and continuity necessarily satisfied by any physical temperature distribution, that expressions (16) and (18) constitute the unique solution of the system (1)-(6). ${ }^{13}$

In conclusion, the author would like to express his thanks to Professor G. C. Evans for his help in the preparation of this paper.

[^26]
# ANALYSIS OF NUMERICAL SOLUTIONS OF TRANSIENT HEAT-FLOW PROBLEMS* 

BY<br>CLARENCE M. FOWLER<br>U. S. Naval Academy

1. Introduction. The purpose of this paper is to present formal methods for establishing the convergence of numerical solutions of transient heat-flow problems, and to derive expressions for these solutions in terms of the initial temperatures and boundary values.

In general, heat-flow problems are classified under two groups, steady-state flow and transient flow. Steady-state problems are solved numerically by the relaxation method. Many papers dealing with the actual numerical work have been written, and Temple has established the validity of the relaxation method under various boundary conditions. Moskovitz ${ }^{2}$ has derived an expression in terms of the boundary temperatures for the steady-state numerical solution of a rectangular bar.

Although considerable work has been done on the actual application of numerical methods to transient heat-flow problems, very little has been written about the problems of convergence and the expression of solutions in terms of initial and boundary values. ${ }^{3}$ These last two considerations are the objects of this paper.

Two restrictions which simplify the analysis are placed on the examples considered here. First, only the one-dimensional slab is considered; secondly, the initial temperature distribution is assumed to be constant over the slab. However, by extensions of the methods used, solutions of problems concerning two- and three-dimensional rectangular objects with arbitrary initial temperature distributions are readily derived.

The various boundary conditions which have been studied include the following: the temperature at the boundary is given, and is either a constant or a function of time; the boundary is insulated; there is a constant energy input at the boundary; there is convection at the boundary. The author has made no attempt to consider all possible combinations of boundary conditions, but has tried to include enough representative cases to illustrate the methods.

The procedure followed throughout the paper has been to consider each example as a whole, and to derive solutions of the problem and take up a study of the convergence, before proceeding to the next example. In some cases solutions have been expressed in terms of a set of polynomials which are associated fundamentally with the difference equation; in other cases they have been expressed in finite Fourier series.

[^27]Properties of the polynomials needed throughout the paper have been demonstrated in an appendix.
2. The difference equation and boundary expressions. The basic one-dimensional difference equation satisfied by the numerical solutions is

$$
\begin{equation*}
T_{x, t}=\frac{T_{x-1, t-1}+a T_{x, l-1}+T_{x+1, l-1}}{a+2} \tag{2.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta t=\frac{(\Delta x)^{2}}{(a+2)^{\alpha}} \tag{2.2}
\end{equation*}
$$

Here, $\Delta t$ is the time interval between successive time values, $\Delta x$ is the distance between successive points across the slab, $t$ is the time (in units of $\Delta t$ ), $x$ is a space coordinate running across the slab (in units of $\Delta x$ ), $T_{x, t}$ is the temperature in the slab at time $t$ and position $x, \alpha$ is the thermal diffusivity of the material, and $a$ is the modulus of the equation.

As usually encountered, the difference equation has $a=0$ (Schmidt's equation). Dusinberre ${ }^{4}$ generalized Schmidt's equation by introducing the modulus. The value of using an arbitrary modulus lies in being able to select an arbitrary time increment as well as space increment. This is not possible in Schmidt's equation, since fixing $\Delta x$ determines $\Delta t$.

In dealing with convection, insulation, etc., at a boundary, it is always necessary to make some assumption to determine the numerical


Fig. 1. boundary expression. It should be emphasized that, for this reason there are several different expressions in use approximately representing the same boundary condition. However, it is possible to consider only one of them here, which is deduced as follows. Figure 1 shows the boundary ( $x=0$ ) and the first two interior points of the slab. The boundary expression is derived by making a heat balance over the shaded half-segment. The heat gain by conduction throughout the time increment $\Delta t$ referred to the initial time instant $t-1$ is

$$
-h A \Delta l\left(T_{0, t-1}-T_{a}\right)+k A \Delta t \frac{\left(T_{1, t-1}-T_{0, t-1}\right)}{\Delta x},
$$

where $h$ is the surface heat transfer coefficient, $k$ is the thermal conductivity, $T_{a}$ is the ambient temperature, and $A$ is a unit area perpendicular to the slab cross section. This quantity is equated to the heat capacity gain $\frac{1}{2}(\Delta x) c \rho A\left(T_{0, 九}-T_{0, \ell-1}\right)$, where $c$ and $\rho$ are the specific heat and density of the material, respectively. For rapid surface cooling, it is necessary to keep $\Delta x$ small, since it has been assumed above that $T_{0, t}$ will represent the temperature of the shaded segment.

After equating the two heat quantities above, and simplifying, we obtain the boundary condition

$$
\begin{equation*}
T_{0, \iota}=\frac{2 T_{1, t-1}+(a-2 N) T_{0, t-1}+2 N T_{0}}{a+2} \tag{2.3}
\end{equation*}
$$

[^28]where $N$, the equivalent numerical transfer coefficient, is given by the relation
\[

$$
\begin{equation*}
N=h \Delta x / k . \tag{2.4}
\end{equation*}
$$

\]

In the case where a boundary has a constant energy input per unit area, $q$, an analysis similar to that given above yields

$$
\begin{equation*}
T_{0,2}=\frac{2 T_{1, l-1}+a T_{0, \ell-1}+2 Q}{a+2} \tag{2.5}
\end{equation*}
$$

where

$$
\begin{equation*}
Q=q \Delta x / k . \tag{2.6}
\end{equation*}
$$

For an insulated boundary, $N=0$ and (2.3) becomes

$$
\begin{equation*}
T_{0, 九}=\frac{2 T_{1, t-1}+a T_{0, t-1}}{a+2} \tag{2.7}
\end{equation*}
$$

For simple boundary conditions such as temperature at $x=0$ held at $u_{0}$, or temperature at $x=l$ fixed as a function of time $f(t)$, the boundary conditions are simply $T_{0, t}=u_{0}$ or $T_{l, t}=f(t)$.
3. Convergence. There are two distinct types of convergence to be considered here. The first type deals with the convergence of numerical solutions as the time becomes large. The second type shows that as the time and space increments $\Delta t$ and $\Delta x$ are allowed to approach zero, the numerical solutions become identical with the corresponding analytic solutions.

From an inspection of (2.1) and (2.2), it is seen that in applying numerical solutions to any particular example, there are apparently two arbitrary quantities, the space increment $\Delta x$, and the modulus $a$, which in turn determines the time increment $\Delta t$. However, it is found from experience that if the value of $a$ is taken too small, the calculated numerical answers oscillate and ultimately diverge as the time becomes large. The first type of convergence is concerned with developing criteria which impose a lower limit on allowable values of $a$ which will then insure numerical convergence. Each example considered has such a criterion developed, since such criteria usually depend on the particular boundary conditions. Another related problem is that of determining the steady-state distribution given numerically. It is shown that numerical solutions converge to the same steady-state values as those determined analytically for the boundary conditions under consideration.

Both of the problems discussed above pertain to actual numerical solutions where the space and time increments are finite, non-zero quantities. The second type of convergence is treated in $\$ 11$, apart from the main body of the paper, since it does not deal with numerical solutions as applied, but rather to the limiting case where $\Delta x$ and $\Delta t$ approach zero. Under these conditions, the numerical solutions become identical with the analytic solutions for all values of time and position throughout the slab.
4. Particular solutions and contour integration. By substituting $T_{x, t}=F(t) \sin z x$ into (2.1), $F(t)$ is found to satisfy the subsidiary difference equation $F(t)$ $=[(a+2 \cos z) /(a+2)] F(t-1)$ and therefore $F(t)=[(a+2 \cos z) /(a+2)]$, from which we find as a particular solution to (2.1),

$$
\begin{equation*}
T_{x, t}=\left(\frac{a+2 \cos z}{a+2}\right)^{t} \sin z x . \tag{4.1}
\end{equation*}
$$

A similar analysis shows that $\sin z x$ may be replaced by $\cos z x$ or $e^{i z x}$.
Using (4.1) and extending it to the two or three-dimensional form, we can write down immediately solutions in Fourier series and integrals for rectangular objects, with arbitrary initial temperature distributions. However, such solutions are of little use as they converge too slowly. Instead of following the standard Fourier development, the author has found it expedient to consider only cases where the initial temperature over the slab is zero (which by a change in temperature origin includes any constant temperature distribution). This restriction, for the one-dimensional slab, allows the use of a method of contour integration which may be summarized as follows.
a) Having found that

$$
T_{x, z}=\left(\frac{a+2 \cos z}{a+z}\right)^{\prime}\left[\frac{A(z)}{z} \cos z x+\frac{B(z)}{z} \sin z x\right]
$$

is a particular solution as long as $A(z)$ and $B(z)$ are independent of $x$ and $t$, we formally integrate this solution with respect to $z$ over the prescribed path (Fig. 2) in the complex plane. The solution is (4.2) below, where the functions $A(z)$ and $B(z)$ are determined so that (4.2) satisfies the boundary conditions of the problem.

$$
\begin{equation*}
T_{x, t}=\frac{1}{\pi i} \int_{P}\left(\frac{a+2 \cos z}{a+2}\right)^{i}(A(z) \cos z x+B(z) \sin z x) \frac{d z}{z} . \tag{4.2}
\end{equation*}
$$

The path $P$ is chosen parallel to the real axis, extending from $+\infty$ to $-\infty$, and is located a finite distance $m$ above the real


Fig. 2. axis, $m$ being determined so that all poles of the integrand lie below $P$.
b) The integrand of (4.2) is shown to vanish over the arc, path $R$ of Fig. 2, except possibly at the slab boundary points $x=0$ or $x=l$, when $t$ is given the value zero. Then, as there are no poles enclosed by paths $P$ and $R$, it follows from Cauchy's theorem that at time zero $T_{x, 0}=0$, except possibly at the slab boundaries. It follows that (4.2) is the solution to the problem, for it satisfies the difference equation, the initial condition $T_{x, 0}=0$, and the boundary conditions.
c) The remaining step is the evaluation of the contour integral. This is accomplished in one of two ways. In either case, the integrand of (4.2) is shown to vanish over the paths $M$ and $N$ (Fig. 2) or over half these paths.

1) For semi-infinite slab problems, the integral over $P$ is evaluated in terms of an integral along the real axis and the residues of any poles lying between path $P$ and the real axis.
2) For finite slab problems, the functions $A(z)$ and $B(z)$ are generally such that the integrand is even-valucd, and therefore the solution (4.2) may equally well be integrated over the path $Q$ (Fig. 2) which is opposite path $P$. Thus, integration around the loop consisting of paths $P$ and $Q$, and the paths $M$ and $N$ shows that the required
integral is equal to half the sum of the residues at the poles enclosed by paths $P, Q$, $M$ and $N$, since the paths $M$ and $N$ contribute nothing to the integral over the loop.

All analysis of a purely rigorous nature has been omitted from the paper, but all doubtful cases have been tested for proper convergence and the vanishing of the integrals over the paths outlined above.
5. Semi-infinite slab. Boundary $x=0$ held at constant temperature $u_{0}$, initial temperature zero. Let us consider the following equation

$$
\begin{equation*}
T_{x, t}=\frac{u_{0}}{\pi i} \int_{P}\left(\frac{a+2 \cos z}{a+2}\right)^{t} e^{i x x} \frac{d z}{z} \tag{5.1}
\end{equation*}
$$

To prove that this expression is the solution of the problem it is necessary to show that $T_{x, 0}=0$ and that $T_{0, t}=u_{0}$.
: To show that $T_{z, 0}=0$, we set $t=0$ and make the substitution $z=R e^{i \phi}$. We then integrate $\left(u_{0} / \pi\right) \exp [i x R \exp (i \phi)] d \phi$ over the path $R, \phi$ varying from $\pi$ to 0 . It follows that as $R \rightarrow \infty$ this integral vanishes, except at $x=0$. Therefore, since there are no poles enclosed by paths $P$ and $R$, it follows from Cauchy's theorem that $T_{x, 0}=0$.

To show that the integrand vanishes over the paths $M$ and $N$, let us substitute $z= \pm R+i y$ where $y$ varies from $-m$ to $+m$, and let $R \rightarrow \infty$; then it follows that these integrals vanish. $T_{0, t}$, the temperature at $x=0$, can equally well be integrated over path $Q$, since the resulting integrand is even-valued. Therefore, by Cauchy's theorem, since there are no contributions from the paths $M$ and $N, T_{0, t}=u_{0}$. It follows that (5.1) is the solution to the problem, for it satisfies the initial and boundary conditions.

To evaluate (5.1) it is convenient to integrate around the loop consisting of $P$, the half-path $M$, the real axis indented at the origin by a small semi-circle of radius $\epsilon$, and the half-path $N$. In the work that follows, the real axis will be denoted by $w$ to avoid confusion with the slab position variable.

Since the integrals over $M$ and $N$ vanish, and there are no poles enclosed in the loop,

$$
\begin{aligned}
T_{x, t} & =\frac{u_{0}}{\pi i} \int_{+\infty}^{+0}\left(\frac{a+2 \cos w}{a+2}\right)^{t} e^{-i x w} \frac{d w}{w} \\
& +\frac{u_{0}}{\pi} \int_{\pi}^{0}\left(\frac{a+2 \cos \epsilon e^{i \phi}}{a+2}\right)^{t} \exp [i x \epsilon \exp (i \phi)] d \phi \\
& +\frac{u_{0}}{\pi i} \int_{+\epsilon}^{+\infty}\left(\frac{a+2 \cos w}{a+2}\right)^{t} e^{+i x w} \frac{d w}{w}=0
\end{aligned}
$$

Regrouping terms and letting $\epsilon \rightarrow 0$, we have as the solution

$$
\begin{equation*}
T_{x, t}=u_{0}\left[1-\frac{2}{\pi} \int_{0}^{\infty}\left(\frac{a+2 \cos w}{a+2}\right)^{t} \sin w x \frac{d w}{w}\right] \tag{5.2}
\end{equation*}
$$

To express this solution in terms of the polynomials, we expand $(a+2 \cos w)^{\prime}$ according to (12.7) and substitute into (5.2),

$$
T_{x, t}=u_{0}\left[1-\frac{2}{\pi(a+2)^{t}} \int_{0}^{\infty}\left[P_{t}(t)+2 P_{t-1}(t) \cos w+\cdots\right.\right.
$$

$$
\left.\left.+2 P_{0}(t) \cos t w\right] \sin w x \frac{d w}{w}\right] .
$$

Combining the sine and cosine terms and making use of the identity (12.6a), $P_{t-r}(t)=P_{t+r}(l)$, we have

$$
T_{x, t}=u_{0}\left[1-\frac{2}{\pi(a+2)^{t}} \int_{0}^{\infty} \sum_{r=-t}^{r+t} P_{r+t}(t) \sin w(x+r) \frac{d w}{w}\right]
$$

or

$$
\begin{equation*}
T_{x, \iota}=u_{0}\left[1-\frac{1}{(a+2)^{\iota}} \sum_{r-t}^{r-t}\{x+r\} P_{r+\iota}(t)\right], \tag{5.3}
\end{equation*}
$$

where the symbols $\{x+r\}=1,0$, or -1 depending on whether $x+r$ is greater than, equal to or less than zero. This notation arises from the fact that $(2 / \pi) \int_{0}^{\infty}(\sin v v(x+r) / w) d w=\{x+r\}$ in accordance with the above convention.

From (5.2) it is easily shown that the solution converges as $t \rightarrow \infty$ when

$$
\begin{equation*}
a \geqq 0 . \tag{5.4}
\end{equation*}
$$

To show that the numerical solution converges to the analytic steady-state solution for the same boundary conditions when $t \rightarrow \infty$, the following device is used: $T_{x, t}$ may be equated to the integrand over the real axis, and added to the sum of all residues of poles enclosed by the real axis and the path $P$. The real axis is to be indented with small semi-circles at all poles lying on it. In the steady-state value of the solution, the only contributions which remain as $t \rightarrow \infty$, are those which occur where $z=2 n \pi$. All other contributions, including those along the real axis, drop out due to the rapidity with which the factor $[(a+2 \cos z) /(a+2)]^{2} \rightarrow 0$ when $z \neq 2 n \pi$, as $t \rightarrow \infty$

In the problem under consideration, there are no poles, and if we take into account the indentation at the origin, we find that the steady-state solution approaches $u_{0}$. This is also the value given by the analytic solution of the same problem.

Although simpler methods would have given the same result in this case, the method is very powerful, since it may be used on a solution with no further reduction from the contour integral form.
6. Semi-infinite slab, $T_{0,1}$ polynomial in time, initial temperature zero. Let us consider the following equation

$$
\begin{equation*}
T_{z, t}=(-1)^{n} \frac{C(2 n)!}{2 \pi i} \int_{P}\left(\frac{a+2 \cos z}{a+2}\right)^{t} e^{i x x_{z}-(2 n+1)} d z \tag{6.1}
\end{equation*}
$$

An analysis similar to that in $\S 5$ shows that $T_{x, 0}=0$ and also that the integrand vanishes over the paths $M$ and $N$.

When $x=0$, the integrand is even-valued in $z$, and therefore has the same value over path $Q$ as over $P$. $T_{0, t}$ may then be equated to half the sum of the residues at the poles enclosed by paths $P$ and $Q$. From (12.8) this becomes

$$
\begin{equation*}
T_{0, t}=\xi_{2 \mathrm{n}}(t), \tag{6.2}
\end{equation*}
$$

where $\xi_{2 n}(t)$ is a polynomial in $t$ of the $n$th degree. From proper combinations of these polynomials, contour integrals are readily derived for problems in which the boundary temperatures are arbitrary polynomials in time.

For the particular case where the boundary temperature is linear in time and
therefore given by $T_{0, t}=C \xi_{2}(l)=C t /(a+2)$, as follows from (12.10), the contour solution is given by,

$$
\begin{equation*}
T_{x, t}=-\frac{C}{\pi i} \int_{P}\left(\frac{a+2 \cos z}{a+2}\right)^{t} e^{i z x} \frac{d z}{z^{3}} \tag{6.3}
\end{equation*}
$$

We indent the origin, integrate as in $\S 5$ to obtain

$$
\frac{2 C}{\pi} \int_{\epsilon}^{\infty}\left(\frac{a+2 \cos w}{a+2}\right)^{i} \sin w x \frac{d w}{w^{3}}-\frac{C}{\pi} \int_{0}^{\pi}\left(\frac{a+2 \cos \epsilon e^{i \pi}}{a+2}\right)^{i} \frac{\exp \left[i x \epsilon e^{i \phi}\right]}{\epsilon^{2} e^{2 i \phi}} d \phi
$$

The first integral, after the expansion of $(a+2 \cos w)^{\ell}$ in terms of the polynomials and regrouping as in (5.2), becomes

$$
I_{1}=\frac{2 C}{\pi(a+2)^{\imath}} \sum_{r=-t}^{r=+t} \int_{e}^{\infty} \sin w(x+r) P_{r+t}(t) \frac{d w}{w^{3}}
$$

By integrating by parts and keeping all terms which do not drop out as $\epsilon \rightarrow 0$, the contribution of this integral is found to be

$$
I_{1} \cong \frac{2 C x}{\pi \epsilon}-\frac{C}{\pi(a+2)^{t}} \sum_{r=-t}^{r-t}(x+r)^{2} P_{r+t}(t) \int_{e}^{\infty} \sin w(x+r) \frac{d w}{w}
$$

The second integral is expanded in terms of $\epsilon$ and $\phi$, and all terms are retained which do not drop out at the integration limits or as $\epsilon \rightarrow 0$. We then have for this integral

$$
I_{2} \cong-\frac{2 C x}{\pi \epsilon}+\frac{C x^{2}}{2}+\frac{C t}{a+2}
$$

The sum of $I_{1}$ and $I_{2}$ with $\epsilon \rightarrow 0$ then gives for the solution

$$
\begin{equation*}
T_{x, t}=C\left[\frac{t}{a+2}+\frac{x^{2}}{2}-\frac{1}{2(a+2)^{t}} \sum_{r=-t}^{r+t}(x+r)^{2} P_{r+t}(t)\{x+r\}\right] \tag{6.4}
\end{equation*}
$$

where the symbols $\{x+r\}$ have the same meaning as in (5.3).
An analysis similar to that of $\S 5$ shows that if $a \geqq 0$, the numerical solution approaches $C t /(a+2)$ asymptotically as $t \rightarrow \infty$. This result also follows from the analytic theory.
7. Finite slab, length $l$, boundaries $x=0$ and $x=l$ held at constant temperatures $u_{0}$ and $u_{l}$ respectively, initial temperature zero. Let us consider the equation

$$
\begin{equation*}
T_{x, i}=\frac{1}{\pi i} \int_{P}\left(\frac{a+2 \cos z}{a+2}\right)^{i}[A(z) \cos z x+B(z) \sin z x] \frac{d z}{z} \tag{7.1}
\end{equation*}
$$

where the functions $A(z)$ and $B(z)$ must be determined to make (7.1) satisfy the boundary conditions $T_{0, t}=\mu_{0}$ and $T_{\imath, t}=\mu_{\imath}$. Referring to (5.1) and placing $x=0$, we see that

$$
\begin{equation*}
\frac{u_{0}}{\pi i} \int_{p}\left(\frac{a+2 \cos z}{a+2}\right)^{t} \frac{d z}{z}=u_{0} \tag{7.2}
\end{equation*}
$$

Therefore if $A(z)=u_{0}$ and $B(z)=\left(u_{l}-u_{0} \cos z l\right) / \sin z l,(7.1)$ reduces to $u_{0}$ at $x=0$ and to $u_{l}$ at $x=l$. The solution is therefore

$$
\begin{equation*}
T_{x, b}=\frac{1}{\pi i} \int_{P}\left(\frac{a+2 \cos z}{a+2}\right)^{i}\left(u_{0} \cos z x+\frac{u_{i}-u_{0} \cos z l}{\sin z l} \sin z x\right) \frac{d z}{z} . \tag{7.3}
\end{equation*}
$$

The integrand of (7.3) is an even-valued function of $z$ and therefore the solution may be equated to one-half the sum of the residues at poles enclosed by paths $P$ and $Q$ (there are no contributions from the paths $M$ and $N$ ). Poles occur at $z=n \pi / l$, $n$ any integer. After evaluating the residues and simplifying the result, we have

$$
\begin{align*}
T_{x, l}= & u_{0}
\end{align*}+\left(u_{l}-u_{0}\right) \frac{x}{l}, ~\left(\frac{2}{\pi} \sum_{1,2}^{\infty}(-1)^{n}\left(u_{l}-u_{0} \cos n \pi\right) \frac{\sin (n \pi x / l)}{n}\left(\frac{a+2 \cos (n \pi / l)}{a+2}\right)^{\prime} .\right.
$$

To express this solution in terms of the polynomials, it is necessary to expand $(a+2 \cos (n \pi) / l)^{t}$ as in (12.7),

$$
\begin{aligned}
& T_{x, l}= u_{0}+\left(u_{i}-u_{0}\right) \frac{x}{l} \\
&+\frac{2}{\pi(a+2)^{t}} \sum_{1,2}^{\infty}(-1)^{n}\left(u_{t}-u_{0} \cos n \pi\right) \frac{\sin (n \pi x / l)}{n}\left[P_{t}(t)+2 P_{t-1}(t) \cos \frac{n \pi}{l}\right. \\
&\left.+\cdots+2 P_{0}(t) \cos \frac{n \pi t}{l}\right]
\end{aligned}
$$

Combining the trigonometric terms and making use of the identity $P_{t-r}(t)=P_{t+r}(t)$, we obtain the solution in the form

$$
\begin{align*}
T_{x, l}= & u_{0}+\left(u_{l}-u_{0}\right)
\end{aligned} \begin{aligned}
& l \\
&+\frac{2}{\pi(a+2)^{l}} \sum_{r=-t}^{r-+t} P_{l+r}(l) \sum_{1,2}^{\infty} \frac{(-1)^{n}}{n}\left(u_{l}-u_{0} \cos n \pi\right) \sin \frac{n \pi(x+r)}{l} \tag{7.5}
\end{align*}
$$

However from the initial condition $T_{x, 0}=0$, it follows that at $t=0$, the resulting sine expansion of (7.4) must be equal to $-u_{0}-\left(u_{l}-u_{0}\right)(x / l)$. Therefore the infinite series in the double summation of (7.5) must equal $-F(x+r)$, where $F(x)$ is the periodic sine expansion of $u_{0}+\left(u_{l}-u_{0}\right)(x / l)$. The solution then becomes

$$
\begin{equation*}
T_{x, t}=u_{0}+\left(u_{l}-u_{0}\right) \frac{x}{l}-\frac{1}{(a+2)^{t}} \sum_{r=-t}^{r+t} P_{r+t}(t) F(x+r) . \tag{7.6}
\end{equation*}
$$

Figure 3 shows the graph of the periodic sine expansion of $F(x)=u_{0}+\left(u_{t}-u_{0}\right)(x / l)$ which applies to (7.6). In applying (7.6) it is


Fig. 3. usually simpler to plot the function and then pick off the different values of $F(x)$ required in the summation. It will be noticed that when $x+r$ is a multiple of $l$, the value of $F(x+r)$ is zero.

For finite slab problems, it is usually possible to get a solution in the form of a finite Fourier series, in addition to the poly-
nomial expansion. The derivation of such series is not difficult, but is too long to be given here in full. The finite series for $\S 7$ is

$$
\begin{equation*}
T_{x, t}=u_{0}+\left(u_{l}-u_{0}\right) \frac{x}{l}-\sum_{1}^{l-1} A_{n} \sin \frac{n \pi x}{l}\left(\frac{a+2 \cos (n \pi / l)}{a+2}\right)^{\prime}, \tag{7.7}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{n}=\frac{\sin (n \pi / l)\left(u_{l}-u_{0} \cos n \pi\right)}{l(1-\cos (n \pi / l))} \tag{7.8}
\end{equation*}
$$

The general method used for obtaining finite series, such as (7.7), consists of a replacement of the infinite Fourier series as in (7.4) by a finite number of terms of the same type, such that at $t=0$ the finite series reduces to the same function as given by the infinite series at $t=0$. Such expansions are possible since in numerical methods the initial temperature must be specified only at a finite number of points. The coefficients for the terms of such finite series are given by ${ }^{5}$

$$
A=\frac{2}{l} \sum_{x=1}^{x=l} F(x) \sin \frac{n \pi x}{l},
$$

where $F(x)$ is the function at $t=0$ over the interval 0 to $l$. The $A_{n}$ of (7.8) were calculated as above with $F(x)=u_{0}+\left(u_{l}-u_{0}\right)(x / l)$.

From an inspection of (7.7) we see that as $i \rightarrow \infty$ the solution converges provided $|\{a+2 \cos (n \pi / l)\} /(a+2)| \leqq 1$. A simple analysis yields the following criterion for convergence

$$
\begin{array}{ll}
\text { 1) } a+2 \geqq 2 \cos ^{2}(\pi / 2 l) & l \text { even, } \\
\text { 2) } a+2 \geqq 2 \cos ^{2}(\pi / l) & l \text { odd. } \tag{7.9}
\end{array}
$$

Equation (7.7) also shows that the numerical solution approaches $u_{0}+\left(u_{l}-u_{0}\right)(x / l)$ as $t \rightarrow \infty$, which is the analytic steady-state solution for the same boundary conditions.
8. Finite slab, length $l$, insulated at $x=0$, held at constant temperature $u_{l}$ at $x=l$, initial temperature zero. The boundary conditions are

$$
\begin{equation*}
T_{l, t}=u_{l}, \tag{8.1}
\end{equation*}
$$

and from Eq. (2.7),

$$
\begin{equation*}
T_{0, 九}=\frac{2 T_{1, t-1}+a T_{0, t 1}}{a+2} \tag{8.2}
\end{equation*}
$$

Imposing these conditions on the general contour integral (4.2), we find that $A(z)=u_{l} / \cos z l$ and $B(z)=0$. The solution is therefore

$$
\begin{equation*}
T_{x, l}=\frac{u_{l}}{\pi i} \int_{P}\left(\frac{a+2 \cos z}{a+2}\right)^{i} \frac{\cos z x}{\cos z l} \frac{d z}{z} . \tag{8.3}
\end{equation*}
$$

Poles occur at $z=0$ and $z=(2 n+1) \pi / 2 l$. Evaluating the integral as in $\S 7$ and regrouping terms, we obtain the solution

[^29]\[

$$
\begin{equation*}
T_{x, t}=u_{l}\left[1+\frac{4}{\pi} \sum_{1,3}^{\infty}(-1)^{(n+1) / 2}\left(\frac{a+2 \cos (n \pi / 2 l)}{a+2}\right)^{t} \frac{\cos (n \pi x / 2 l)}{n}\right] \tag{8.4}
\end{equation*}
$$

\]

An analysis similar to that of $\$ 7$ yields the polynomial expansion

$$
\begin{equation*}
T_{x, t}=u_{t}\left[1-\frac{1}{(a+2)^{i}} \sum_{r=-t}^{r=+t} P_{r+t}(t) F(x+r)\right] \tag{8.5}
\end{equation*}
$$

where $F(x)$ is the cosine Fourier expansion of unity from 0 to $l$ and minus unity from $l$ to $2 l$, as is evident from (8.4) and the initial condition $T_{x, 0}=0$. It will be noted that the value of $F(x+r)$ is zero when $x+r$ is an odd multiple of $l$.

The finite Fourier series solution is found to be

$$
\begin{equation*}
T_{x, t}=u_{l}\left[1+\sum_{1}^{2 l-1} A_{n} \sin \frac{n \pi(x-l)}{2 l}\left(\frac{a+2 \cos (n \pi / 2 l)}{a+2}\right)^{t}\right], \tag{8.6}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{n}=0 \quad n \text { even, } \quad A_{n}=\frac{\sin (n \pi / 2 l)}{l(1-\cos (n \pi / 2 l))} \quad n \text { odd } \tag{8.7}
\end{equation*}
$$

From (8.6) it follows that the convergence criterion is:

$$
\begin{array}{ll}
\text { 1) } a+2 \geqq 2 \cos ^{2}(\pi / 4 l) & l \text { even, } \\
\text { 2) } a+2 \geqq 2 \cos ^{2}(\pi / 2 l) & l \text { odd. } \tag{8.8}
\end{array}
$$

Also from (8.6) it follows that as $t \rightarrow \infty$, the numerical steady-state solution becomes $u_{l}$ which is also the analytic steady-state solution for the same problem.
9. Finite slab, length $l$, constant energy input $q$ at $x=l$, temperature kept at zero at $x=0$, initial temperature zero. The boundary conditions are (from Eq. (2.5))

$$
\begin{equation*}
T_{l, t}=\frac{2 T_{l-1, t-1}+a T_{l, l-1}+2 Q}{a+2} \tag{9.1}
\end{equation*}
$$

$$
\begin{equation*}
T_{0, \ell}=0 \tag{9.2}
\end{equation*}
$$

where $Q=q \Delta x / k$. Imposing these conditions on (4.2) we find that $A(z)=0$ and $B(z)=Q /(\sin z \cos z l)$. The solution is therefore

$$
\begin{equation*}
T_{x, t}=\frac{Q}{\pi i} \int_{P} \frac{\sin z x}{\sin z \cos z l}\left(\frac{a+2 \cos z}{a+2}\right)^{t} \frac{d z}{z} \tag{9.3}
\end{equation*}
$$

Poles occur at $z=0$ and $z=(2 n+1) \pi / 2 l$ (the set $z=n \pi, n \neq 0$ does not constitute poles, as the terms $\sin z x$ in the numerator also vanish at these points since $x$ is restricted to integral values). After evaluating the residues, regrouping terms and simplifying, we see that the solution becomes

$$
\begin{equation*}
T_{x, l}=Q\left[x+\frac{4}{\pi} \sum_{1,3}^{\infty}(-1)^{(n+1) / 2} \frac{\sin (n \pi x / 2 l)}{n \sin (n \pi / 2 l)}\left(\frac{a+2 \cos (n \pi / 2 l)}{a+2}\right)^{t}\right] \tag{9.4}
\end{equation*}
$$

The polynomial expansion is found to be,

$$
\begin{equation*}
T_{x, t}=Q\left[1-\frac{1}{(a+2)^{t}} \sum_{r=-t}^{r-+t} P_{r+t}(t) F(x+r)\right] \tag{9.5}
\end{equation*}
$$

where $F(x)$ is the sine expansion of $x$ from 0 to $l$ and of $2 l-x$ from $l$ to $2 l$, as follows from (9.4) and the initial condition $T_{x, 0}=0$.

The finite Fourier series is

$$
\begin{equation*}
T_{x, t}=Q\left[x-\sum_{1}^{2 l-1} A_{n} \sin \frac{n \pi x}{2 l}\left(\frac{a+2 \cos (n \pi / 2 l)}{a+2}\right)^{\cdot t}\right] \tag{9.6}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{n}=0 \text { i } n \text { even, } \quad A_{n}=\frac{(-1)^{(n-1) / 2}}{l(1-\cos (n \pi / 2 l))} \quad n \text { odd } \tag{9.7}
\end{equation*}
$$

From (9.6) the convergence criterion is found to be

$$
\begin{align*}
& \text { 1) } a+2 \geqq 2 \cos ^{2} \frac{\pi}{4 l} \quad l \text { even } \\
& \text { 2) } a+2 \geqq 2 \cos ^{2} \frac{\pi}{2 l} \quad l \text { odd } \tag{9.8}
\end{align*}
$$

From (9.6) it also follows that as $t \rightarrow \infty$ the numerical solution approaches $Q x$, which is equivalent to the analytic steady-state solution for the same boundary conditions.
10. Convection at a boundary. When the condition (2.3) is imposed on the general contour integral (4.2), $A(z)$ and $B(z)$ are generally such that the evaluation of the resulting integrals is difficult, due either to the uncertain nature of the poles, or to the evaluation of a complicated infinite integral.

For the semi-infinite slab with convection into temperature $T_{a}$, transfer coefficient $h$, the boundary condition is given by (2.3)

$$
\begin{equation*}
T_{0, t}=\frac{2 T_{1, t-1}+(a-2 N) T_{0, t-1}+2 N T_{0}}{a+2}, \quad N=h \Delta x / k \tag{10.1}
\end{equation*}
$$

We assume a contour integral solution of the form,

$$
\begin{equation*}
T_{x, i}=\frac{1}{\pi i} \int_{P} A(z)\left(\frac{a+2 \cos z}{a+2}\right)^{i} e^{i z x} \frac{d z}{z} \tag{10.2}
\end{equation*}
$$

Imposing the condition (10.1) on (10.2) we see that $A(z)=N T_{a} /(N-i \sin z)$, and the solution may therefore be written,

$$
\begin{equation*}
T_{x, t}=\frac{N T_{a}}{\pi i} \int_{P}\left(\frac{a+2 \cos z}{a+2}\right)^{i} \frac{e^{i z x}}{N-i \sin z} \frac{d z}{z} \tag{10.3}
\end{equation*}
$$

The integral (10.3) is to be evaluated in terms of an integral along the real axis indented at the origin, plus the sum of the residues at the poles enclosed by the path $P$ and the real axis. Aside from the root $z=0$, the denominator includes roots from the term $N-i \sin z$, which are found to be $z=-i \log \left(\sqrt{N^{2}+1}+N\right)$ and the infinite set $z=-i \log \left(\sqrt{N^{2}+1}-N\right)+(2 n+1) \pi, n$ any integer. However, in the loop of integration considered, only the residues at the poles $z=-i \log \left(\sqrt{N^{2}+1}-N\right)+(2 n+1) \pi$ are evaluated, since the other poles lie outside the loop. ( $N$ is taken greater than zero, otherwise the boundary would be insulated and no heat would flow, giving the trivial solution $T_{x, t} \equiv 0$.)

Following the analysis of (5.1), integrating along the indented real axis, and then letting $\epsilon \rightarrow 0$, we obtain the solution in the form

$$
\begin{align*}
T_{x, i}= & T_{a}\left[1-\frac{2 N}{\pi} \int_{0}^{\infty}\left(\frac{a+2 \cos w}{a+2}\right)^{\prime}\left(\frac{N \sin w x+\sin w \cos w x}{N^{2}+\sin ^{2} w}\right) \frac{d w}{w}\right. \\
& \left.+2 \pi i \sum \text { residues }\right] . \tag{10.4}
\end{align*}
$$

By a somewhat tedious but straightforward analysis, the residue term may be evaluated and simplified to yield,

$$
\begin{align*}
& 2 \pi i \sum \text { Res. }=\frac{4 N T_{a} \log \left(\sqrt{N^{2}+1}-N\right)}{\sqrt{N^{2}+1}}\left(\frac{a-2 \sqrt{N^{2}+1}}{a+2}\right)^{\prime}\left(\sqrt{N^{2}+1}-N\right)^{x} \\
& \sum_{1,3}^{\infty} \frac{\cos n \pi x}{n^{2} \pi^{2}+\log ^{2}\left(\sqrt{N^{2}+1}-N\right)} \tag{10.5}
\end{align*}
$$

By choosing a combination of known Fourier expansions for the hyperbolic sine and cosine, with further reduction, and recalling that in numerical analysis $x$ is always an integer, we can write (10.5) in the form

$$
2 \pi i \sum \text { Res. }=-T_{a}\left(\frac{a-2 \sqrt{N^{2}+1}}{a+2}\right)^{\prime}\left(N-\sqrt{N^{2}+1}\right)^{x}\left(\frac{\sqrt{N^{2}+1}-1}{\sqrt{N^{2}+1}}\right) .
$$

The final solution therefore becomes

$$
\begin{align*}
T_{x, t}=T_{a} & {\left[1-\frac{2 N}{\pi} \int_{0}^{\infty}\left(\frac{a+2 \cos w}{a+2}\right)^{\prime}\left(\frac{N \sin w x+\sin w \cos w x}{N^{2}+\sin ^{2} w}\right) \frac{d w}{w}\right.} \\
& \left.-\left(\frac{\sqrt{N^{2}+1}-1}{\sqrt{N^{2}+1}}\right)\left(\frac{a-2 \sqrt{N^{2}+1}}{a+2}\right)^{t}\left(N-\sqrt{N^{2}+1}\right)^{x}\right] . \tag{10.6}
\end{align*}
$$

Without evaluating the infinite integral (which is convergent when $t \rightarrow \infty$ provided $a \geqq 0)$ we see by inspection of the factor $\left[\left(a-2 \sqrt{N^{2}+1}\right) /(a+2)\right]^{t}$ that the criterion for convergence as $t \rightarrow \infty$ is,

$$
\begin{equation*}
a \geqq \sqrt{N^{2}+1}-1 . \tag{10.7}
\end{equation*}
$$

It will be noted that Schmidt's equation, where $a=0$, will not yield convergent answers if the boundary expression (2.3) is used, since by (10.7) a cannot be zero. It also follows from (10.6) that the numerical steady-state solution becomes $T_{a}$, which is also the solution predicted analytically for the same problem.

A polynomial expansion may be obtained from (10.6) by using (12.7) to expand the term $(a+2 \cos w)^{2}$. However, the analysis required to evaluate the resulting definite integrals is so involved that it is not worthwhile to include the expansion here.

Contour integral solutions are readily set up for finite slab problems involving convection at either or both boundaries, but it is generally difficult to evaluate these integrals. However, without evaluating the integrals, but using the method outlined
in (5.3), we can show that if the numerical work converges at all, it converges to the correct steady-state distribution as $t \rightarrow \infty$.
11. Convergence to analytic solution. The theory developed so far has been concerned only with numerical solutions as actually applied. The convergence problems already discussed have shown what values of $a$ are necessary to insure non-divergent numerical answers, and also that as the time increases, numerical solutions approach the same steady-state distribution as that given analytically for the same problem. These results hold when the space and time increments $\Delta x$ and $\Delta t$ are finite, non-zero quantities.

It is also possible to show that as the arbitrary increments $\Delta x$ and $\Delta t$ become very small, the numerical solutions approach the true analytic solution for all values of $x$ and $t$, and attain this solution in the limit. The formal procedure used to demonstrate this limiting convergence consists of a demonstration that the contour integrals derived for the numerical solutions transform into already known contour integral solutions for the corresponding analytic treatment, as $\Delta x$ and $\Delta t$ approach zero. Formal proofs of this convergence will be given for three of the numerical examples already discussed. The proofs for other examples are very similar to the ones given here.

The three examples considered with their analytic contour integral solutions are: ${ }^{6}$

1) Semi-infinite slab, end $x^{\prime}=0$ kept at temperature $u_{0}$, initial temperature zero,

$$
\begin{equation*}
T\left(x^{\prime}, t^{\prime}\right)=\frac{u_{0}}{\pi i} \int_{P} e^{i z x^{\prime}} e^{-\alpha z^{2} t^{\prime}} \frac{d z}{z} \tag{11.1}
\end{equation*}
$$

2) Semi-infinite slab, convection at $x^{\prime}=0$ into a medium of constant temperature $T_{a}$, initial temperature zero,

$$
\begin{equation*}
T\left(x^{\prime}, t^{\prime}\right)=\frac{h T_{a}}{k \pi i} \int_{P} \frac{e^{i z x^{\prime}} e^{-\alpha z^{2} t^{\prime}}}{(h / k)-i z} \frac{d z^{\prime}}{z^{\prime}} \tag{11.2}
\end{equation*}
$$

3) Finite slab, length $l^{\prime}$, end $x^{\prime}=0$ held at zero temperature, constant energy input $q$ at $x^{\prime}=l^{\prime}$, initial temperature zero,

$$
\begin{equation*}
T\left(x^{\prime}, t^{\prime}\right)=\frac{q}{k \pi i} \int_{P} \frac{\sin z x^{\prime}}{\cos z l^{\prime}} e^{-\alpha z t^{\prime}} \frac{d z}{z^{2}} \tag{11.3}
\end{equation*}
$$

Equation (11.3) is not given in Ref. 6 , but we can easily derive it by imposing the boundary condition $\partial T / \partial x-q / k=0$ on the general contour integral considered there.

The notation used in the above equations differs from that used in the present paper. In order to express these equations in our notation, it is necessary that the complex variable $\alpha$ be replaced by $z$, thermal diffusivity $K$ by $\alpha$, and convection coefficient $h$, by $h / k$.
$T\left(x^{\prime}, t^{\prime}\right)$ has been used to denote the analytic solution at the point $x^{\prime}$ and the time $t^{\prime}$, as distinguished from $T_{x, t}$, the numerical solution at point $x$ (in units of $\Delta x$ ) and time $t$ (in units of $\Delta t$ ). The path $P$ is the same for both numerical and analytic solutions, and is the limiting path allowable for the analytic integrals.

In the application of numerical solutions, position and time variables as well as slab lengths are expressed in terms of the arbitrary time and space increments $\Delta t$ and $\Delta x$. In order to discuss the convergence to analytic solutions, it is necessary to express these quantities in terms of the absolute units used analytically. If $M$ arbi-

[^30]trary space increments equal one absolute unit, then the absolute space position $x^{\prime}$ is given by,
\[

$$
\begin{equation*}
x^{\prime}=k / M \quad \text { or } x=M x^{\prime} . \tag{11.4}
\end{equation*}
$$

\]

From the relationship $\Delta t=(\Delta x)^{2} /(a+2) \alpha$ of (2.2), it follows that the time value in absolute units is

$$
\begin{equation*}
t^{\prime}=t / M^{2}(a+2) \alpha \text { or } t=M^{2}(a+2) \alpha t^{\prime} . \tag{11.5}
\end{equation*}
$$

To show that the numerical solution approaches the analytic solution for Case (1), when $\Delta x$ and $\Delta t$ approach zero, we substitute (11.4) and (11.5) into the numerical solution (5.1), obtaining

$$
\begin{equation*}
T_{x, l}=\frac{u_{0}}{\pi i} \int_{P}\left(\frac{a+2 \cos z}{a+2}\right)^{M / 2(a+2) \alpha t^{\prime}} e^{i z M x^{\prime}} \frac{d z}{z} . \tag{11.6}
\end{equation*}
$$

By replacing the variable $z$ by $z^{\prime} / M$, we can write (11.6) in the form

$$
\begin{equation*}
T_{x, t}=\frac{u_{0}}{\pi i} \int_{P}\left(\frac{a+2 \cos \left(z^{\prime} / M\right)}{a+2}\right)^{M 2(a+2) \alpha t} e^{i z^{\prime} x^{\prime}} \frac{d z^{\prime}}{z^{\prime}} . \tag{11.7}
\end{equation*}
$$

The factor $\left[\left(a+2 \cos z^{\prime} / M\right) /(a+2)\right]^{M^{2}(a+2) \alpha \ell^{\prime}}$ may be approximated by $\left[1-\left(z^{\prime}\right)^{2}\right.$ $\left./(a+2) M^{2}\right]^{M} M^{2}(a+2) \alpha t^{\prime}$ when $M$ is large, and in the limit becomes $\exp \left[-\left(z^{\prime}\right)^{2} \alpha t^{\prime}\right]$. The limit of the numerical expression (11.7) as $\Delta x$ and $\Delta t$ approach zero (or as $M \rightarrow \infty$ ) therefore becomes,

$$
\begin{equation*}
\underset{\lim \Delta x \rightarrow 0}{T_{x, t}}=\frac{u_{0}}{\pi i} \int_{P} e^{i z^{\prime} x^{\prime}} e^{-\alpha\left(z^{\prime}\right) t^{\prime}} \frac{d z^{\prime}}{z^{\prime}} \tag{11.8}
\end{equation*}
$$

which is identical with the analytic solution (11.1).
With convection at a boundary or constant energy input, the values $N=h \Delta x / k$ of (2.4) and $Q=q \Delta x / k$ of (2.6) become,

$$
\begin{equation*}
N=\frac{h}{k M} \tag{11.9}
\end{equation*}
$$

$$
\begin{equation*}
Q=\frac{q}{k M} . \tag{11.10}
\end{equation*}
$$

In Case (2) after substituting (11.4), (11.5) and (11.9) into the numerical solution (10.3), and then replacing the variable $z$ by $z^{\prime} / M$, we obtain

$$
\begin{equation*}
T_{x, t}=\frac{h T_{a}}{k \pi i M} \int_{P}\left(\frac{a+2 \cos \left(z^{\prime} / M\right)}{a+2}\right)^{M 2(a+2) \alpha \ell^{\prime}} \frac{e^{i z^{\prime} x^{\prime}}}{(h / k M)-i \sin \left(z^{\prime} / M\right)} \frac{d z^{\prime}}{z^{\prime}} \tag{11.11}
\end{equation*}
$$

As before, the term $\left[\left(a+2 \cos z^{\prime} / M\right) /(a+2)\right]^{M^{2}(a+2) \alpha t^{\prime}}$ becomes $\exp \left[-\left(z^{\prime}\right)^{2} \alpha t^{\prime}\right]$ in the limit. The term $i \sin \left(z^{\prime} / M\right)$ may be approximated by $i z^{\prime} / M$ when $M$ is large, We make these changes, cancel the $M$ outside the integral with those in the term $(h / k M)-\left(L z^{\prime} / M\right)$, and let $M$ approach infinity; then (11.11) becomes

$$
\begin{equation*}
\underset{\lim \Delta x \rightarrow 0}{T_{x, z}}=\frac{h T_{a}}{k \pi i} \int_{P} \frac{e^{i z^{\prime} x^{\prime}} e^{-\left(z^{\prime}\right) 2 a u^{\prime}}}{(h / k)-i z^{\prime}} \frac{d z^{\prime}}{z^{\prime}}, \tag{11.12}
\end{equation*}
$$

which is the analytic expression (11.2).
In Case (3), after substituting in (9.3) from (11.4), (11.5), (11.10), introducing the absolute length $l^{\prime}$ given by $l=M l^{\prime}$, and changing the variable $z$ to $z^{\prime} / M$, we obtain

$$
\begin{equation*}
T_{x, l}=\frac{q}{k \pi i M} \int_{P} \frac{\sin z^{\prime} x^{\prime}}{\sin \left(z^{\prime} / M\right) \cos z^{\prime} l^{\prime}}\left(\frac{a+2 \cos \left(z^{\prime} / M\right)}{a+2}\right)^{M 2(a+2) \alpha t^{\prime}} \frac{d z^{\prime}}{z^{\prime}} \tag{11.13}
\end{equation*}
$$

In the limit, the term $\sin \left(z^{\prime} / M\right)$ may be replaced by $z^{\prime} / M$ and as in (11.11), the solution may be written in the form

$$
\begin{equation*}
\underset{\lim \Delta x \rightarrow 0}{T_{x, l}}=\frac{q}{k \pi i} \int_{r} \frac{\sin z^{\prime} x^{\prime}}{\cos z^{\prime} l^{\prime}} e^{-\left(z^{\prime}\right) z \alpha t^{\prime}} \frac{d z^{\prime}}{\left(z^{\prime}\right)^{2}} \tag{11.14}
\end{equation*}
$$

which is identical with the analytic solution (11.3).
It will be noted that the restrictions on $a$ as given by (9.8) become $a \geqq 0$ when $l$ (the number of units of $\Delta x$ in the slab) becomes large. Hence the proof for this convergence to the analytic solution holds only when $a \geqq 0$. An analysis of the other examples treated in this paper shows that for this limiting convergence, all criteria reduce to $a \geqq 0$.
12. Appendix. Properties of the polynomials $P_{r}(t)$. The polynomials $P_{r}(t)$ are defined as the coefficients of $z^{r}$ in the expansion of the trinomial $\left(1+a z+z^{2}\right)^{2}$, and are therefore functions of $r, t$ and the modulus $a$. Two identities follow readily, the first by definition, the second by setting $z=1$ in (12.1),

$$
\begin{align*}
& \sum_{r=-t}^{r=+t} P_{r+t}(t) z^{r+t}=\left(1+a z+z^{2}\right)^{t}  \tag{12.1}\\
& \sum_{r=-t}^{r-t} P_{r+t}(t)=(a+2)^{t} \tag{12.2}
\end{align*}
$$

By expanding the trinomial in the form $\left[(1+a z)+z^{2}\right]^{\ell}$ and collecting coefficients, we obtain an explicit formula for $P_{r}(t)$,

$$
\begin{equation*}
P_{r}(t)=\binom{t}{r}\binom{t}{0} a^{r}+\binom{t-1}{r-2}\binom{t}{1} a^{r-2}+\binom{t-2}{r-4}\binom{t}{2} a^{r-4}+\cdots \tag{12.3}
\end{equation*}
$$

The polynomials may be expressed as definite integrals in the following way. By definition and Cauchy's theorem

$$
P_{r}(t)=\frac{1}{2 \pi i} \int_{C}\left(1+a z+z^{2}\right)^{t} \frac{d z}{z^{n+1}}
$$

where $C$ is a simple closed contour about the origin. By the choice of $C$ as a circle of unit radius, center at the origin, it follows that $z=e^{i \phi}$ and

$$
\begin{equation*}
P_{\mathrm{r}}(t)=\frac{1}{2 \pi} \int_{0}^{2 \pi}(a+2 \cos \phi)^{t} \cos (t-r) \phi d \phi . \tag{12.4}
\end{equation*}
$$

Writing the equality $\left(1+a z+z^{2}\right)^{t}=\left(1+a z+z^{2}\right)^{t-1}\left(1+a z+z^{2}\right)$ and collecting coefficients of $z^{r}$ on both sides of the equation, we have the following recursion formula, which may be used for rapid calculation of the polynomials,

$$
\begin{equation*}
P_{r}(t)=P_{r}(t-1)+a P_{r-1}(t-1)+P_{r-2}(t-1) . \tag{12.5}
\end{equation*}
$$

From (12.3), (12.4) and (12.5), it follows that

$$
\begin{array}{ll} 
& P_{t \rightarrow r}(t)=P_{t+r}(t) \\
P_{0}(0)=1, & P_{0}(t)=1, \quad P_{r}(0)=0, \quad r \neq 0 \tag{12.6b}
\end{array}
$$

As an example, the polynomials up to $t=3$ have been worked out for modulus $a=3$ using (12.5) and (12.6), and are shown in Table 1. Thus, $P_{4}(3)=30$ and $P_{2}(2)$ $=11$.

Table 1.


To construct a polynomial array, we start with $P_{0}(0)=1$. The polynomials following are calculated successively by use of the recursion formula (12.5). As a specific example, from the formula $P_{3}(3)=P_{3}(2)+a P_{2}(2)+P_{1}(2)$, we have on substituting the values presumably already calculated for $t=2, P_{3}(3)=6+3 \times 11+6=45$.

An important identity may be established as follows: we let $(a+2 \cos \theta)$ e $=\sum_{n=0}^{t} A_{n} \cos n \theta$. From (12.4),

$$
P_{r}(t)=\int_{0}^{2 \pi} \sum_{n=0}^{t} A_{n} \cos n \phi \cos (l-r) \phi d \phi
$$

from which it follows that $A_{t-r}=2 P_{r}(t), r \neq t$, and $A_{0}=P_{t}(t)$. Therefore

$$
\begin{align*}
(a+2 \cos \theta)^{\prime}= & P_{t}(t)+2 P_{t-1}(t) \cos \theta+\cdots+2 P_{t-r}(t) \cos r \theta+\cdots \\
& +2 P_{0}(t) \cos t \theta \tag{12.7}
\end{align*}
$$

The polynomials $\xi_{2 n}(t)$ are defined by,

$$
\begin{equation*}
\xi_{2 n}(t)=\frac{(-1)^{n}}{2} \frac{(2 n)!}{2 \pi i} \int_{c}\left(\frac{a+2 \cos z}{a+2}\right)^{t} \frac{d z}{z^{2 n+1}} \tag{12.8}
\end{equation*}
$$

where $C$ is a simple closed contour about the origin. By Cauchy's theorem, and evaluation of the residue at the pole $z=0,(12.8)$ becomes,

$$
\begin{equation*}
\xi_{2 n}(t)=\frac{(-1)^{n}}{2}\left[\frac{d^{n}}{d \theta^{n}}\left(\frac{a+2 \cos \theta}{a+2}\right)^{t}\right]_{\theta=0} \tag{12.9}
\end{equation*}
$$

The first three polynomials, evaluated from (12.9) are,

$$
\begin{align*}
& \xi_{2}(t)=\binom{t}{1}(a+2)^{-1} \\
& \xi_{4}(t)=\left(\frac{t}{1}\right)(a+2)^{-1}+12\binom{t}{2}(a+2)^{-2}  \tag{12.10}\\
& \xi_{6}(t)=\binom{t}{1}(a+2)^{-1}+60\binom{t}{2}(a+2)^{-2}+360\binom{t}{1}(a+2)^{-3}
\end{align*}
$$

Acknowledgments. The author is indebted to Cmdr. G. M. Dusinberre, of the Marine Engineering Department, U.S. Naval Academy, for suggesting the paper and for many valuable ideas concerning boundary conditions, and to Lt. H. G. Elrod of the same department for reading the paper. The author is also grateful to Prof. H. W. Emmons of the Harvard University Graduate School of Engineering for reading the paper carefully, and for suggesting the convergence problem of Section 11.

## -NOTES-

## ON PLANE ELASTIC STRAIN IN DOUBLY-CONNECTED DOMAINS*

By W. PRAGER (Brown University)

1. Introduction. The stresses associated with a state of plane elastic strain can be expressed in terms of the second derivatives of Airy's stress function. If $x_{1}, x_{2}, x_{3}$ are rectangular Cartesian coordinates, the axis of $x_{3}$ being normal to the plane of strain, the stress function $\phi\left(x_{1}, x_{2}\right)$ satisfics the differential equation $\Delta^{2} \phi=0$ ( $\Delta=\partial^{2} / \partial x^{2}+\partial^{2} / \partial y^{2}$ ), and the given stresses on the boundary determine the tangent planes of the stress surface $x_{3}=\phi\left(x_{1}, x_{2}\right)$ at all points of the boundary, when one such tangent plane is known for each bounding curve. In the case of a singly-connected domain only one such tangent plane must be known, and it can be chosen arbitrarily because the stresses define the stress function only to within an arbitrary linear function of $x_{1}$ and $x_{2}$. In the case of a doubly-connected domain, however, two such tangent planes must be known, and only one of them can be chosen arbitrarily. This paper is concerned with the determination of the second tangent plane in the case where one of the boundary curves is free from loads. Equations from which this tangent plane can be determined, were derived by J. H. Michell ${ }^{1}$ from the condition that the displacements must be single-valued. In the present paper it will be shown that Michell's equations are the natural boundary conditions of the variational problem for the stress function. This remark is of importance when the direct methods of the calculus of variations are used to determine the stress function for a doublyconnected domain. ${ }^{2}$
2. Notations. Basic relations. Throughout this paper Latin subscripts will have the range $1,2,3$, Greck subscripts the range 1,2 , and the summation convention for repeated subscripts will be used. The rectangular Cartesian coordinates $x_{i}$ are chosen so that the axis of $x_{3}$ is normal to the plane of strain; the position of the origin and the directions of $x_{1}$ and $x_{2}$ are arbitrary. Let $e_{i j}$ be the strain tensor and $s_{i j}$ the reduced stress tensor, i.e. the stress tensor divided by Young's modulus. The stressstrain relations can then be written in the form

$$
\begin{equation*}
e_{i j}=(1+\sigma) s_{i j}-\sigma s_{k k} \delta_{i j} \tag{1}
\end{equation*}
$$

where $\sigma$ denotes Poisson's ratio, and $\delta_{i j}$ is the Kronecker delta. For the state of plane strain under consideration the condition that $e_{33}=0$ gives

$$
\begin{equation*}
s_{33}=\sigma s_{\gamma \gamma} \tag{2}
\end{equation*}
$$

The equations of equilibrium in the plane of strain are

[^31]\[

$$
\begin{equation*}
s_{\gamma \alpha, \gamma}=0, \tag{3}
\end{equation*}
$$

\]

where the comma followed by the subscript $\gamma$ denotes partial differentiation with respect to $x_{\gamma}$. Equation (3) can be satisfied by setting

$$
\begin{equation*}
s_{\alpha \beta}=\epsilon_{\alpha \lambda} \epsilon_{\beta \mu} \phi_{, \lambda_{\mu}}, \tag{4}
\end{equation*}
$$

where $\phi=\phi\left(x_{1}, x_{2}\right)$ is Airy's stress function, and $\epsilon_{11}=\epsilon_{22}=0, \epsilon_{12}=-\epsilon_{21}=1$. Since $\epsilon_{\alpha \lambda \epsilon_{\alpha \mu}}=\delta_{\lambda \mu}$, the invariant $s_{\alpha \alpha}$ equals $\Delta \phi=\phi_{\text {. } \alpha \alpha}$. For a state of plane elastic strain $\Delta s_{\alpha \alpha}=0$, or $\Delta^{2} \phi=0$, i.e. the stress function is biharmonic.

On the boundary of the domain under consideration the surface stresses $f_{\alpha}$ are given. If $n_{\alpha}$ is the unit vector along the outward normal of the boundary, we have $f_{\alpha}=s_{\gamma \alpha} n_{\gamma}=\epsilon_{\gamma \lambda} \epsilon_{\alpha \mu} \phi_{, \lambda_{\mu}} n_{\gamma}$. Now $\epsilon_{\gamma \lambda} n_{\gamma}=t_{\lambda}$, the unit vector of the tangent of the boundary. Accordingly,

$$
\begin{equation*}
f_{\alpha}=\epsilon_{\alpha \mu} \phi_{, \lambda \mu} t_{\lambda}=\epsilon_{\alpha \mu} \partial \phi_{, \mu} / \partial s, \tag{5}
\end{equation*}
$$

where $\partial / \partial s$ denotes differentiation in the direction of the tangent vector $t_{\lambda}$. Multiplying both sides of (5) by $\epsilon_{\alpha \beta}$ and integrating along the boundary, we obtain

$$
\begin{equation*}
\phi_{\beta \beta}(s)=\epsilon_{\alpha \beta} \int_{0}^{1} f_{\alpha}(s) d s+\phi_{A \beta}(0) . \tag{6}
\end{equation*}
$$

The given surface forces $f_{\alpha}$ are thus seen to determine the gradient $\phi_{, \beta}(s)$ of the stress function along a bounding curve, when the gradient $\phi_{, \beta}(0)$ at one point of this curve is known. In other terms, the stress function $\phi$ and its normal derivative $\partial \phi / \partial n$ are defined at all points of a bounding curve, when $\phi$ and its gradient are known at a single point of this curve.

If, in particular, one of the bounding curves is free from loads, the stress function $\phi$ and its normal derivative along this curve equal the values of a linear function $a_{\alpha} x_{\alpha}+b$ and of its normal derivative. Establishing the boundary conditions for the stress function along a bounding curve which is free from loads is therefore equivalent to determining the three coefficients $a_{1}, a_{2}, b$ of this linear function.
3. The variational problem for the stress function. To the strain $e_{i j}$ and the reduced stress $s_{i j}$ corresponds the reduced elastic energy $U=\frac{1}{2} e_{i j} s_{i j}$. In the case of plane strain this energy equals

$$
\begin{equation*}
U=\frac{1}{2}\left[(1+\sigma) s_{i j} s_{i j}-\sigma s_{i i} s_{j i}\right]=\frac{1}{2}(1+\sigma)\left[s_{\alpha \beta} s_{\alpha \beta}-\sigma s_{\alpha \alpha} s_{\beta \beta}\right], \tag{7}
\end{equation*}
$$

in view of Eqs. (1) and (2). In terms of the stress function introduced in (4) the energy is

$$
\begin{equation*}
U=\frac{1}{2}(1+\sigma)\left[\phi_{, \alpha \beta} \phi_{, \alpha \beta}-\sigma \phi_{, \alpha \alpha} \phi_{, \beta \beta}\right] . \tag{8}
\end{equation*}
$$

According to the variational principle for the stresses the stress function corresponding to certain boundary conditions is then singled out from amongst all functions which fulfill these boundary conditions and admit continuous derivatives up to the fourth order, by the fact that it minimizes the integral

$$
\begin{equation*}
V=\int\left[\phi_{. \alpha \beta} \phi_{, \alpha \beta}-\sigma \phi_{, \alpha \alpha} \phi_{, \beta \beta}\right] d \omega, \tag{9}
\end{equation*}
$$

where $d \omega$ denotes the element of area, and the integration is extended over the entire domain. The condition $\delta V^{\prime}=0$ leads to

$$
\begin{equation*}
\int\left[\phi_{, \alpha \beta} \delta \phi_{, \alpha \beta}-\sigma \phi_{, \alpha \alpha} \delta \phi_{, \beta \beta}\right] d \omega=0 \tag{10}
\end{equation*}
$$

or

$$
\begin{align*}
(1-\sigma) \int \phi_{, \alpha \alpha \beta \beta} \delta \phi d \omega-(1-\sigma) & \int \phi_{, \alpha \beta \beta \delta \phi n_{\alpha} d s} \\
& +\int \phi_{, \alpha \beta} \delta \phi_{, \alpha} n_{\beta} d s-\sigma \int \phi_{, \beta \beta} \delta \phi_{, \alpha} n_{\alpha} d s=0 \tag{11}
\end{align*}
$$

In the case of a doubly-connected domain with loads on one bounding curve only, the stress function $\phi$ and its gradient can be considered as given on the loaded bounding curve. This curve does therefore not furnish any contribution to the line integrals in (11). On the other bounding curve, we have $\phi=a_{\alpha} x_{\alpha}+b$ and $\phi_{, \alpha}=a_{\alpha}$. In addition to the differential equation for the stress function, $\phi_{. \alpha \alpha \beta}=0$ or $\Delta^{2} \phi=0$, Eq. (11) thus gives the following equation which must be fulfilled on the load-free boundary:

$$
\begin{align*}
\delta a_{\gamma}\left[(1-\sigma) \int \phi_{, \alpha \beta \beta} x_{\gamma} n_{\alpha} d s-\int \phi_{, \gamma \beta} n_{\beta} d s\right. & \left.+\sigma \int \phi_{, \beta \beta} n_{\gamma} d s\right] \\
& +\delta b\left[(1-\sigma) \int \phi_{, \alpha \beta \beta} n_{\alpha} d s\right]=0 \tag{12}
\end{align*}
$$

Since $\delta a_{\gamma}$ and $\delta b$ are independent, the expressions in brackets must vanish separately. The second integral in the first bracket can be transformed as follows

$$
\int \phi_{, \gamma \beta} n_{\beta} d s=\int \phi_{, \beta \beta \gamma} d \omega=\int \phi_{, \beta \beta} n_{\gamma} d s
$$

The first bracket can therefore be written as

$$
(1-\sigma)\left[\int \phi_{, \alpha \beta \beta} x_{\gamma} n_{\alpha} d s-\int \phi_{, \beta \beta} n_{\gamma} d s\right]
$$

With the use of $n_{\gamma}=\epsilon_{\gamma \alpha} l_{\alpha}$, the second integral can be further transformed as follows

$$
\begin{aligned}
\int \phi_{, \beta \beta} n \imath_{\gamma} d s & =\epsilon_{\gamma \alpha} \int \phi_{, \beta \beta} t_{\alpha} d s=\epsilon_{\gamma \alpha} \int \phi_{, \beta \beta} d x_{\alpha} \\
& =-\epsilon_{\gamma \alpha} \int \phi_{, \beta \beta d} x_{\alpha} d x_{\delta}=-\epsilon_{\gamma \alpha} \int \phi_{, \beta \beta} x_{\alpha} t_{\delta} d s .
\end{aligned}
$$

Equation (12) is thus equivalent to

$$
\begin{equation*}
\int\left[x_{\gamma} \frac{\partial}{\partial n}(\Delta \phi)+\epsilon_{\gamma \alpha} x_{\alpha} \frac{\partial}{\partial s}(\Delta \phi)\right] d s=0 \tag{13}
\end{equation*}
$$

and

$$
\begin{equation*}
\int \frac{\partial}{\partial n}(\Delta \phi) d s=0 \tag{14}
\end{equation*}
$$

The scalar equivalents of (13) are

$$
\begin{equation*}
\int\left[x_{1} \frac{\partial}{\partial n}(\Delta \phi)+x_{2} \frac{\partial}{\partial s}(\Delta \phi)\right] d s=0 \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
\int\left[x_{2} \frac{\partial}{\partial n}(\Delta \phi)-x_{1} \frac{\partial}{\partial s}(\Delta \phi)\right] d s=0 \tag{16}
\end{equation*}
$$

Equations (14), (15) and (16) are Michell's conditions which are thus seen to be the natural boundary conditions of the variational problem for the stress function. The manner in which these equations are used in determining $\phi$ is obvious. Let $\phi_{0}, \phi_{1}, \phi_{2}, \phi_{3}$ be the biharmonic functions defined by the following boundary conditions:

1) $\phi_{0}$ and $\partial \phi_{0} / \partial n$ have the prescribed boundary values on the loaded boundary curve $C_{1}$ and vanish on the other boundary curve $C_{2}$;
2) $\phi_{1}=\partial \phi_{1} / \partial n=0$ on $C_{1}$,
$\phi_{1}=x_{1}$ and $\partial \phi_{1} / \partial n=n_{1}$ on $C_{2}$;
3) $\phi_{2}=\partial \phi_{2} / \partial n=0$ on $C_{1}$,
$\phi_{2}=x_{2}$ and $\partial \phi_{2} / \partial n=n_{2}$ on $C_{2} ;$
4) $\phi_{3}=\partial \phi_{3} / \partial n=0$ on $C_{1}$,
$\phi_{3}=1$ and $\partial \phi, / \partial n=0$ on $C_{2}$.
Substituting

$$
\phi=\phi_{0}+a_{1} \phi_{1}+a_{2} \phi_{2}+b \phi_{3}
$$

into Eqs. (14), (15) and (16), we obtain three linear equations from which $a_{1}, a_{2}$ and $b$ can be determined.

## THE CAPACITY OF TWIN CABLE-II*

## By J. W. CRAGGS and C. J. TRANTER (Military College of Science, Stoke-on-Trent, England)

1. Introduction. In a recent paper" (subsequently referred to as "I") we have given a method for determining the capacity of two circular wires surrounded by concentric touching dielectric sheaths. The present note gives the extension of the method to the case in which the dielectric sheaths are not in contact. The problem considered is the symmetrical one of two infinite parallel circular wires each of radius $R_{1}$ surrounded by concentric sheaths of radius $R_{2}$ and dielectric constant $K_{1}$, the distance between the centers of the wires being $2 L\left(L>R_{2}\right)$. The dielectric constant of the surrounding medium is taken as $K_{2}$.
2. The equations for solution. In line with the treatment in "I" we replace $R_{2}$ by unity, $R_{1} / R_{2}$ by $a$ and $L / R_{2}$ by $s$; we also write $K_{1} / K_{2}=K$. The potentials $V_{1}, V_{2}$ must therefore satisfy ( $i$ ) the differential equations

$$
\begin{array}{ll}
\nabla^{2} V_{1}=0, & a \leqq r \leqq 1 \\
\nabla^{2} V_{2}=0, & r \geqq 1, \quad x \geqq 0 \tag{2}
\end{array}
$$

and (ii) the boundary conditions

$$
\begin{equation*}
V_{1}=1 \tag{3}
\end{equation*}
$$

[^32]when $r=a$,
\[

$$
\begin{equation*}
V_{1}=V_{2}, \quad(4) \quad K \partial V_{1} / \partial r=\partial V_{2} / \partial r \tag{4}
\end{equation*}
$$

\]

when $r=1$,

$$
\begin{equation*}
V_{2}=0 \tag{6}
\end{equation*}
$$

when $x=0$. Here $\nabla^{2}$ is Laplace's operator in two dimensions and the coordinate systems are as shown in Fig. 1.


Fig. 1.
3. The analytical solution. As in "I" we write

$$
\begin{equation*}
V_{1}=1+B \log \frac{r}{a}+\sum_{n=1}^{\infty}\left\{\left(\frac{r}{a}\right)^{n}-\left(\frac{a}{r}\right)^{n}\right\} b_{n} \cos n \theta . \tag{7}
\end{equation*}
$$

The conformal transformation for the region $r>1, x>0$ can be written

$$
\begin{equation*}
\xi-i \eta=\log \frac{r e^{i \theta}+e^{\mu}}{r e^{i \theta}+e^{\mu}} \tag{8}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu=\log \left(s+\sqrt{s^{2}-1}\right) \tag{9}
\end{equation*}
$$

The boundaries $r=1, x=0$ then become $\xi=\mu, \xi=0$ respectively.
Since $V_{2}$ is odd in $\xi$ and even and periodic in $\eta$, we write

$$
\begin{equation*}
V_{2}=D \xi+\sum_{m=1}^{\infty} d_{m} \sinh m \xi \cos m \eta \tag{10}
\end{equation*}
$$

The constants $B, b_{n}$ of (7) and $D, d_{m}$ of (10) are now to be determined from the boundary conditions (4) and (5).

On the boundary $r=1(\xi=\mu)$, we find from (8) and (9)

$$
\begin{equation*}
\cos \eta=\frac{1+\cosh \mu \cos \theta}{\cos \theta+\cosh \mu} \tag{11}
\end{equation*}
$$

so that $0 \leqq \theta \leqq \pi$ corresponds to $0 \leqq \eta \leqq \pi$, and

$$
\begin{equation*}
\frac{\partial V}{\partial r}=\frac{\partial \xi}{\partial r} \cdot \frac{\partial V}{\partial \xi}=-\frac{\partial \eta}{\partial \theta} \cdot \frac{\partial V}{\partial \xi}=\frac{-\sinh \mu}{\cos \theta+\cosh \mu} \frac{\partial V}{\partial \xi} \tag{12}
\end{equation*}
$$

Thus (4) and (5) give

$$
\begin{equation*}
1-B \log a+\sum_{n=1}^{\infty} \frac{1-a^{2 n}}{a^{n}} b_{n} \cos n \theta=D_{\mu}+\sum_{m=1}^{\infty} d_{m} \sinh m \mu \cos m \eta \tag{13}
\end{equation*}
$$

$K B+K \sum_{n=1}^{\infty} \frac{1+a^{2 n}}{a^{n}} n b_{n} \cos n \theta$

$$
\begin{equation*}
=\frac{-\sinh \mu}{\cos \theta+\cosh \mu}\left\{D+\sum_{m=1}^{\infty} m d_{m} \cosh m \mu \cos m \eta\right\} . \tag{14}
\end{equation*}
$$

Multiplying (13) by $\cos m \eta(m=0,1,2, \cdots)$ and integrating with respect to $\eta$ from 0 to $\pi$, we have

$$
\begin{equation*}
D_{\mu}=1-B \log a+\sum_{n=1}^{\infty}(-1)^{n} e^{-n \mu} \frac{1-a^{2 n}}{a^{n}} b_{n} \tag{15}
\end{equation*}
$$

since

$$
\int_{0}^{\pi} \cos n \theta d \eta=(-1)^{n} \pi e^{-n \mu}
$$

and

$$
\begin{equation*}
d_{m} \sinh m \mu=\sum_{n=1}^{\infty} e^{-n \mu} \frac{1-a^{2 n}}{a^{n}} b_{n} I_{m}(n) \tag{16}
\end{equation*}
$$

where

$$
\begin{equation*}
I_{m}(n)=\frac{2}{\pi} e^{n \mu} \int_{0}^{\pi} \cos n \theta \cos m \eta d \eta \tag{17}
\end{equation*}
$$

Similar treatment of (14) gives for $B, b_{n}$

$$
\begin{equation*}
K B=-D \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
K \frac{1+a^{2 n}}{a^{n}} n b_{n}=2(-1)^{n+1} e^{-n \mu} D-e^{-n \mu} \sum_{m=1}^{\infty} m d_{m} \cosh m \mu I_{m}(n) \tag{19}
\end{equation*}
$$

Expansion of $\cos m \eta$ in (17) in terms of $u=\left(1+e^{-2 \mu}+2 e^{-\mu} \cos \theta\right)^{-1}$ leads to

$$
\begin{equation*}
I_{m}(n)=(-1)^{m+n} \sum_{p=0}^{m}(-1)^{p n} C_{m-p}^{n+p-1} C_{p} e^{(n-2 p) \mu} \tag{20}
\end{equation*}
$$

Eliminating $D, b_{n}$ from equations (15), (16), (18) and (19) we have

$$
\begin{align*}
B \log a-1-K B \mu & =2 B S+\frac{1}{2 K} \sum_{m=1}^{\infty} m d_{m} \alpha_{m} \cosh m \mu  \tag{21}\\
-d_{p} \sinh p \mu & =B \alpha_{p}+\frac{1}{K} \sum_{m=1}^{\infty} m d_{m} A_{m p} \cosh m \mu \tag{22}
\end{align*}
$$

where

$$
\begin{align*}
S & =\sum_{n=1}^{\infty}\left(\frac{1-a^{2 n}}{1+a^{2 n}}\right) \frac{e^{-2 n \mu}}{n} \\
\alpha_{p} & =2 \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1-a^{2 n}}{1+a^{2 n}}\right) I_{p}(n) \frac{e^{-2 n \mu}}{n}  \tag{23}\\
A_{m p} & =\sum_{n=1}^{\infty}\left(\frac{1-a^{2 n}}{1+a^{2 n}}\right) I_{m}(n) I_{p}(n) \frac{e^{-2 n \mu}}{n}
\end{align*}
$$

Following the procedure of " $I$ " we retain only a finite number $p$ of the coefficients $d_{m}$. Writing

$$
\begin{equation*}
\gamma_{m}=\frac{K}{m} \tanh m \mu \tag{24}
\end{equation*}
$$

and eliminating $m d_{m} \cosh m \mu$ between equations (21), (22) we find

The capacity is then given by $-\frac{1}{4} K_{1} B$.
4. Alternative method of solution. The above treatment provides a satisfactory basis of computation when $K \geqq 1$. For completeness it is interesting to notice that, when $K<1$, more rapid convergence to the true solution is obtained by climinating $D$ and $d_{m}$ from equations (13), (14) by treating (13) as a Fourier series in $\theta$ and (14) as one in $\eta$.

## ON A. A. POPOFF'S METHOD OF INTEGRATION BY MEANS OF ORTHOGONALITY FOCI*

By HOWARD A. ROBINSON (Research Laboralories, Armstrong Cork Company)
In a recently published paper ${ }^{1}$ a method is given which allows a marked reduction of the work necessary in computing the tristimulus values necessary in color specification work. The three tristimulus values are defined by the following relations:

$$
X=\int E_{L}(\lambda) \bar{x}(\lambda) R(\lambda) d \lambda, \quad Y=\int E_{L}(\lambda) \bar{y}(\lambda) R(\lambda) d \lambda, \quad Z=\int E_{L}(\lambda) \bar{z}(\lambda) R(\lambda) d \lambda
$$

where $E_{L}(\lambda)$ are tabulated relative energy functions of a known light source $L, \bar{x}(\lambda)$, $\bar{y}(\lambda), \bar{z}(\lambda)$ are tabulated luminosity functions and $R(\lambda)$ are the experimentally meas-

[^33]ured curves of reflecting power or percent transmission as a function of the wave length. The product functions $E_{L \bar{x}}, E_{L} \bar{y}$ and $E_{L} \bar{z}$ have also been tabulated and may be considered as the $\phi_{k}(x)$ in the original article. Work is now under way in setting up the necessary scales for the colormetric computations and will be published elsewhere.

It is pointed out in the introduction to Popoff's paper that the method requires the construction of certain diagrams, called scales, showing the abscissas of the centroids of certain areas associated with $\phi_{k}(x)$. Thus, operation (b) in Section 2 contains some unnecessary work, since it is unnecessary to find the centroids $\bar{a}_{r}, \bar{b}_{r}, \cdots$, only the abscissas of these centroids being required.

## BOOK REVIEWS

## Elementary electric-circuit theory. By Richard H. Frazier. McGraw-Hill Book Company, Inc., New York and London, 1945. ix +434 pp. $\$ 4.00$.

"This book is designed as a complete elementary exposition of electric-circuit theory requisite in the technical foundation of all students of electrical engineering regardless of their expected branch of specialization-electric power, communications, or electronics" (from Author's Preface). As such it may be recommended to readers of the Quarterly, experts in other than electrical fields, who may at times have difficulty in following the exposition of mathematical methods as applied to electrical problems. They will find in this book by Professor Frazier, of Massachusetts Institute of Technology, a modern presentation of the field of remarkably broad coverage in a relatively small volume. The power and generality of modern methods, such for instance as the various types of network transformations, are very well presented and thoroughly exemplified. The author has taken great pains to point out possible pitfalls, and if his reader will give equally great attention to details he will find himself amply repaid. Historical references and a selected bibliography enhance the value of this book.

## P. LeCorbeiller

Transmission lines, antennas and wave guides. By Ronold W. P. King, Harry Rowe Mimno and Alexander H. Wing. McGraw-Hill Book Company, Inc., New York and London, 1945. xv +347 pp. $\$ 3.50$.
The book is divided into four chapters. The first chapter, on transmission lines, is written by Alexander H . Wing; the second and third, respectively on antennas and on wave guides, is by Ronold W. P. King; the short concluding chapter is on wave propagation by Harry Rowe Mimno.

The chapter on transmission lines concentrates on those topics which in recent years have interested research workers in microwave laboratories. Those parts of the theory which are needed in problems of long line communication, such as crostalk and interference problems, are not considered; but ample attention is given to high frequency measurements, impedance matching, suppression of harmonics, etc. The emphasis is definitely on high frequencies and on relatively short lines. The exposition is good.

The chapter on antennas constitutes one-half of the book. For this reason it is particularly unfortunate that it should contain so much misinformation and misinterpretation. For the most part it would be difficult for an inexpert reader to recognize what is right and what is wrong. Throughout, the reader is given to understand that the conclusions are based on rigorous electromagnetic theory. Engineering approximations in common use are called "very crude" if they are in error by as much as twenty-five per cent and one is led to believe that those approximations which are called "good" by the author are really good. Apparently, however, the author has not set a uniform objective standard of quality of approximations. He declares that his theoretical impedance curves are in "good agreement" with measured impedances. He does not give the measured values; but measured values from three published sources, and one unpublished but made known to the author, agree among themselves and disagree with King's curves, in some regions by as much as twenty-five to seventy per cent. These measured values also agree with the theoretical results published by this reviewer and by Marion C. Gray. These facts are not mentioned in
the book. The author's attitude seems to be that expressed in one of his latest papers (Jour. App. Phys., August, 1945, p. 445): "In many instances disagreement between experimental and theoretical results may be a better check on the theory than close agreement."

On pp. 104 and 107 are shown curves relating to the length of the antenna at resonance (which is defined as the condition for which the input reactance vanishes) and the corresponding input resistance. On each curve, there is a point marked "sphere." The captions explain that the sphere is regarded as a cylinder whose height is equal to the diameter. One of these points is taken from a book by J. A. Stratton and the other from a paper by E. B. Moullin. The former was calculated for free and not forced oscillations; in fact, in the case of a transmitting spherical antenna the input susceptance is always capacitive and the input reactance does not vanish. E. B. Moullin calculated an approximate re-radiation resistance with reference to the maximum current of a sphere in a certain impressed field and not the input resistance of the spherical transmitting antenna. In fact, the latter resistance depends markedly on the separation between the two halves of the spherical antenna; if this separation is zero as implied by the author of the antenna chapter of the book, the input resistance becomes equal to zero automatically.

At times the author brands a correct conclusion as incorrect and then gives an incorrect result to replace it. For example, on p. 223 he purports to show that the effective area of a "half wave" self-tuned antenna depends considerably on its radius. He assumes that the effective length of the antenna is independent of the radius and takes into consideration only the variation of the effective area with input resistance. Actually, the effective length also varies with the radius and if this effect is included, the effective area of the half-wave antenna is found to be nearly independent of the radius-a conclusion well known in the art.

The chapter on wave guides occupies a relatively minor position in the book. It is confined primarily to detailed descriptions of various types and modes of propagation and the facts are substantially accurate. The inequality (10.1) on p. 251 is unduly restricted; but the fault is not particularly serious. On p. 269 we find: "The upper frequency limit of the $T M_{0,1}$ mode from the point of view of single mode operation is the cut-off for the $T E_{1,1}$ mode." The statement is not true; but it is clearly an over-sight and is not likely to cause serious trouble.

The book is concluded with an excellent thumb-nail sketch of factors affecting wave propagation over the earth. It is hard, however, to pass without comment the author's apparent approval of recent efforts to ascribe specific meaning to such general terms as "low, medium and high frequencies." If these recommendations are put into effect, the language will needlessly be robbed of valuable general terms.

S. A. Schelkunoff

Theory of fight. By Richard von Mises with the collaboration of W. Prager and Gustav Kuerti. (McGraw-Hill Publications in Aeronautical Science, Jerome C. Hunsaker, Consulting Editor.) McGraw-Hill Book Co., Inc. New York and London, 1945. XII + 629 pp. $\$ 6.00$.
This very comprehensive engineering text book is different from similar books in the same class; the author's extensive knowledge of the basic theories and the fundamental principles is everywhere evident. According to the preface the book is written primarily for new graduate students. However many of the chapters require a most thorough preparation in applied mechanics and considerable insight in fluid dynamics. The book will be of considerable interest to engineers who wish to familiarize themselves with particular aspects of the problems of engineering aerodynamics. The chapters on airplane performance control and stability are particularly complete with numerous useful references to experimental results.

Theodore Tueodorsen
The simple calculation of electrical transients. By G. W. Carter. Cambridge: At the University Press, New York: The Macmillan Company, 1945. viii + 120 pp. $\$ 1.75$.
In this little book Mr. Carter explains how to use Heaviside's operational method in the transient analysis of linear networks consisting of a finite number of meshes and does it very well. The method is explained step-by-step and each step is illustrated by practical examples.

The book is addressed to the engineer who wants to be able to use the operational method with confidence but is willing to accept some rules on faith. In the brief introductory chapter the reader learns the characteristics of the circuits to which he can apply the method. There a parallel is drawn between the

Steinmetz method of steady state analysis and the Heaviside method of transient analysis. In Chapter 2 the differential equation of a simple circuit is solved by successive integrations; this permits the author to introduce the operator $Q$, standing for $\int_{0}^{t}$ which is easier to understand than the Heaviside operator $p$. It is only after some experience with $Q$ that $p$ is brought into the picture. Gradually the method is developed and the reader learns to apply it to initially "dead" circuits and then to circuits in any initial state. The book begins with very simple examples, and it ends with complicated ones; thus, it should be easy for the student to gain confidence in the application of the method.
S. A. Schelkunoff

## BIBLIOGRAPHICAL LIST

The R.T.P. translations listed below are now available from the Durand Reprinting Committee, in care of California Institute of Technology, Pasadena 4, California.
R.T.P. Translation No. 2428 , The effect of rounding-off an originally sharp leading edge on the resistance of wings. By E. Wolff. 10 pages.
R.T.P. Translation No. 2429, Detonation and peroxides in the internal combustion engine. By A. Sokolik. 15 pages.
R.T.P. Translation No. 2432, New curves of heat-energy transformation. By Ch. Colombi. 8 pages.
R.T.P. Translation No. 2433 , On an oscillation phenomenon in fluids with stable density stratification. By H. Götler. 10 pages.
R.T.P. Translation No. 2439, The interaction of nitrogen with metals at high temperatures. By A. Gorvisy. 1 page.
R.T.P. Translation No. 2441, Welding. By CZ. 1 page.
R.T.P. Translation No. 2443, Photographic striation methods applied to the supersonic wind tunnel of the E.T.H.-Zurich. By P. de Haller. 4 pages.
R.T.P. Translation No. 2444, Optical multi-curve recorder of the Finnish state aircraft factory. 7 pages.
R.T.P. Translation No. 2451, New curves of heat-energy transformation. By Ch. Colombi. 8 pages.
R.T.P. Translation No. 2453, Practical applications of the diffraction of light on supersonic waves. By S. J. Sokoloff. 4 pages.
R.T.P. Translation No. 2454, Regulations for altitude flying and instructions on the use of respirators. 24 pages.
R.T.P. Translation No. 2455 , The "S.O.30N" and its design features. 3 pages.
R.T.P. Translation No. 2456, Fundamental problems of aeroengine design. By Alfred Gimm. 10 pages.
R.T.P. Translation No. 2459, Structure investigations in aluminium metallurgy. By A. Schrader. 16 pages.
R.T.P. Translation No. 2465 , On the development of play in bolted joints under fatigue. By B. Dirksen. 7 pages.
R.T.P. Translation No. 2466 , The S.O.30N. A very special prototype. By A. SaintArnaud. 1 page.
R.T.P. Translation No. 2467, Friction and leakage losses in piston rings. By M. Eweis. 1 page.
R.T.P. Translation No. 2468, The mechanics of the plastic deformation of mild steel. By K. Hohenemser and W. Prager. 17 pages.
R.T.P. Translation No. 2470 , The recent development of two-stroke engines. By J. Zeman. 12 pages.
R.T.P. Translation No. 2471, The principles of the magnetophone method. By H. Lubeck. 11 pages.
R.T.P. Translation Na. 2472, Buckling stresses on rectangular fixed plates. By Ferd. Schleicher. 14 pages.




 5. mo


## NOTEWORTHY DOVER PUBLICATIONS

## Vorlesungen über Differentialgeometrie

Vol. I: Elementare Differentialgeometrie
By W. Blaschke. Third revised edition, Text in German with English translation of Table of Contents and German-English Index-Glossary. $x+311$ pages. $51 / 2 \times 81 / 2$. Originally published at $\$ 9.00$

Pattial table of Contents: Introduction-Vectors-Theory of Curves-Extremal Properties of Curves-Surface StripsElements of the Theory of Surfaces - Invariant Derivatives on a Surface-Geometry on a Surface-Problems of Surface Theory in the Large-Extremal Properties of Surfaces-Line Geometry.
"The book is written in concise and 'snappy" style, but the sequences of logical steps are clear and the text is always interesting. References to original sources and historical remarks are frequent. The latter are sometimes more than mere paragraphs. At the ends of chapters are lists of problems and theorems to be proved by the reader. These are in many cases not elementary, but references are usually given." Bulletin of the American Mathematical Society.

## Available now:

## The phase rule and its applications

By A. Findlay. Text in English. Eighth revised edition. $x v+327$ pages. $55 / 2 \times 87 / 2$. $\$ 3.00$
"It has established itself as the standard work on the subject and still remains the best introduction to the phase rule and its applications. . . The book is assured of continued and well-deserved popularity." Nature.

## Polar molecules

By P. Debye Text in English. 167 pages.
$51 / 2 \times 81 / 2$. Originally published at $\$ 8.00$.
$\$ 3.50$
"It is thus of great value to all physicists and chemists who are interested is molecular structure, and, in suggesting new fields of work, is of the greatest possible value to research workers in this and allied subjects." Nature.

To appear in February 1946:
Aufgaben und Lehrsätze aus der Analysis

By G. Pólya and G. Szegö. Two volume set. Text in Germau with English translation of Table of Contents and GermanEnglish Index-Glossary, $51 / 2 \times 81 / 2$. Volume I: xxiv +340 pages. Volume II: xviii + 410 pages. Originally published at $\$ 14.40$ for both volumes.

Each valume $\$ 3.50$. The set $\$ 6.50$ "There are but few books which could be compared with this one as to the richness and charm of material, and amount of suggestions which an attentive reader is able to get out of it." Bulletin of the American Mathematical Society.

## To appear in Spring 1946:

## Cours d'analyse infinitésimale

By Ch.J.De La Vallée Poussin. Two volume set. Text in French. 5 $5 / 2 \times 81 / 2$. Two volume set- $\$ 7.50$
"The handling throughout is clear, elegant and concise; the various topics are illustrated by numerous carefully chosen examples selected with rare pedagogic skill to develop a real understanding of the text. ... In the compass of such a review it is impossible to point out all the merits of these volumes, so rich in varied topics, so lucid in exposition and elegant in presentation." Bulletin of the American Mathematical Society.

## To appear in Spring 1946:

## Applied elasticity

By J. Prescott. Text in English. 666 pages. $51 / 2 \times 81 / 2$. Originally published at $\$ 9.50$.
This well-known book provides a presentation of Elasticity lying midway between that given by Love in his classical treatise and that contained in the current textbooks on Theory of Structures and Strength of Materials.
"The author . . has undoubtedly produced an excellent and important contribution to the subject, not merely in the old matter which he has presented in new and refreshing form, but also in the many original investigations here published for the first time." Nature.

## CONT

C. C. Lin : On the stability of two-dimensic
ity in a viscous fluid.
R. King and D. Middeton : The cylindrice
H. Poritsky and M. H. Beewett : A metl means of overlapping regions336
S. A. SCHELKUNOFF: Solution of linear and slightly nonlinear differential equations ..... 348
S. A. SchaAf: A cylinder cooling problem ..... 356
C. M. Fowler: Analysis of numerical solutions of transient heat-flow problems ..... 361
Notes:
W. Prager: On plane elastic strain in doubly-connected domains ..... 377
J. W. Craggs and C. J. Tranter: The capacity of twin cable-II ..... 380
H. A. Robinson: On A. A. Popoff's method of integration by orthogonality foci ..... 383
Book Reviews ..... 384
Bibliographical List ..... 386

## New Books . . . McGraw-Hill

## MATHEMATICAL THEORY OF ELASTICITY

By I. S. Sokolnikofr, University of Wisconsin Ready in February

Provides a thorough foundation in the mathematical theory of elasticity, with application to problems on extension, torsion, and flexare of isotropic cylindrical bodies.

## X-RAYS IN PRACTICE

By Wayne T. Sprouli, Research Laboratories Division, General Motors Corporation. Ready in January
An authoritative and comprehensive treatment, giving the student, engineer, and technical man a broad understanding of X-rays, their nature, and the many purposes for which they may be used.

## THE DEVELOPMENT OF MATHEMATICS. Nen second edition

By E. T. Bell, California Institute of Technology. 618 pages, $\$ 5.00$
Tells the absorbing story of the role of mathematics in the evolution of civilization, from about 4000 b.C. to the present day. The revision contains new material covering recent trends in modern mathematics,

## ANALYTIC GEOMETRY. New third edition

By Fredertck S. Nowlan, University of British Columbia. Ready in January
Distinctive features: the study of plane geometry is basf 1 upon the use of direction cosines; the study of conics is based upon the defunition of the ger eral conic: and polar coordinates are treated from a new point of view.


[^0]:    Entered as second class matrer March 14, 1944, at the post office at Prowidence, Rhode Island, under the act of March 3, 1879. Additional entry at Menasha, Wisconsin.

[^1]:    * Received July 18, 1945. Parts I and II of this paper appeared in this Quarterly 3, 117-142, and 218-234 (1945).
    ** Now at Brown University.
    $\dagger$ The figures in brackets refer to titles in the Bibliography at the end of Part I.

[^2]:    * In fact, the other terms never give considerable contributions to the imaginary part even for only moderately small values of $\alpha$ and $c$. This point will be discussed in the Appendix. The approximation (12.7) will be used for all later calculations.

[^3]:    * This section was inserted late in 1944 after discussions with Prof. C. L. Pekeris. He mentioned the possibility that the neutral curve might be a curve of minimum damping with stable regions on both sides of it. See also Schlichting's calculations [52].

[^4]:    *Synge, [63], eq. (11.23), p. 258. His $\lambda$ is our $\alpha$. The condition is originally stated for plane Couette and plane Poiseuille motion; but it is easily seen that it holds for a general velocity distribution with $q=\max \left|w^{\prime}\right|$.
    $\dagger$ Heisenberg, loc. cit., p. 601.

[^5]:    * The values given here are somewhat different from those published before [27], because the computation of Tietjen's function has been revised.

[^6]:    * Cf. Goldstein [7], vol. 1, p. 135.

[^7]:    * Loc. cit. [73], first paper, p. 42.

[^8]:    *See Fig. 9.

[^9]:    *Taylor, loc. cit. [70], p. 308.

[^10]:    *This problem has been attempted by Heisenberg; see Goldstein's book [7].

[^11]:    * Cf. Goldstein [7], last column of table of p. 157.

[^12]:    (L. Brillouin, Quart. Appl. Math. 1, 201 (1943).
    ${ }^{5}$ L. Brillouin, El. Communication 22, 11 (1944).
    © S. A. Schelkunoff, J. Appl. Phys. 15, 54 (1944).
    ${ }^{7}$ R. King and C. W. Harrison, Jr., J. Appl. Phys. 15, 170 (1944).

[^13]:    ${ }^{8}$ R. King, Electromagnetic engineering Vol. 1, McGraw-Hill Book Co., New York, 1945, p. 241.

    - S. A. Schelkunoff, Electromagnetic waves, D. Van Nostrand Co., New York, 1943, pp. 140, 142 ff.
    ${ }^{10}$ Reference 3, equation (25). The complete derivation is given.
    ${ }^{11}$ R. King and F. G. Blake, Proc. I.R.E. 30, 335 (1942).
    ${ }^{12}$ M. C. Gray, J. Appl. Phys. 15, 61 (1944).

[^14]:    ${ }^{13}$ C. J. Bouwkamp, Physica 9, 609 (1942). In Bouwkamp's paper $G$ and $F$ are, respectively, the $F$ and $G$ functions in this analysis.

[^15]:    - Received June 8, 1945.

[^16]:    * By a "source" is meant here a solution of (1.1) which depends only upon the distance $\boldsymbol{r}$ from a fixed point, is singular at $r=0$ like $-\ln r$, and behaves at infinity like a divergent cylindrical wave. The distributions of such sources satisfy continuity-discontinuity relations similar to those in the case of logarithmic potentials.

[^17]:    * Received July 6, 1945.
    ${ }^{1}$ Maxime Bôcher, $A n$ introduction to the study of integral equations, Cambridge University Press, Cambridge, 1914.

    2 Joseph Liouville, Mémoires sur le dêveloppement des fonctions ou parties de fonctions en series donties divers termes sont assujettic d̀ satisfaire d̀ une même eqquation différentielle du second ordre, contenant un parametre variable, J. de Math., 2, 16-35, 418-436 (1837).
    $\dagger$ The meaning of "slightly" depends on the goodness of results expected from the process. Beyond that we shall not attempt to define it.

[^18]:    * In substance the theorem implied by equations (7), (8) and (9) is hardly new; but we have been unable to find its statement in just that form.

[^19]:    * Assuming that $Z$ and $Y$ do not change signs; if they do, it is best (although by no means necessary) to subdivide the interval.

[^20]:    ${ }^{3}$ John R. Carson, Propagation of periodic currents over nonuniform lines, Electrician, 86, 272-273 (1921).

    * This objection would not apply in strictly numerical handling of equations.

[^21]:    4John C. Slater and Nathaniel H. Frank, Introduction to theoretical physics, McGraw-Hill Book Co., Inc., New York, p. 148 (1933); John C. Slater, Microwave transmission, McGraw-Hill Book Co., Inc., New York, p. 73 (1942).

[^22]:    ${ }^{5}$ Ph. LeCorbeiller, The nonlinear theory of the maintenance of oscillations, I.E.E. Journal, 79, 361378 (1936).

    - Thornton C. Fry, The use of the integraph in the practical solution of differential equations by Picard's method of successive approximations, Proc. 2d Internat. Cong. Math. Toronto, 2, 405-428 (1924).

[^23]:    * Received June 18, 1945.
    ${ }^{1}$ W. M. Rust, Jr., Integral equations and the cooling problem, Amer. J. Math. 54, 190-212 (1932).
    ${ }^{2}$ R. V. Churchill, $A$ heat conduction problem, Philos. Mag. (7), 31, 81-87 (1941).
    ${ }^{3}$ H. S. Carslaw, A simple application of the Laplace transformation, Philos. Mag. (7), 30, 414-417 (1940).
    (W. A. Mersman, Heat conduction in an infinite composite solid, Bull. Amer. Math. Soc. 47, 956-964 (1941).
    ${ }^{5}$ H. S. Carslaw, Theory of heat conduction, Macmillan New York, ed. 2, 1921, Chapter I.

[^24]:    ${ }^{6}$ G. N. Watson, Theory of Bessel functions, Cambridge University Press, Cambridge, ed. 2, 1944, p. 79.
    ${ }^{7}$ D. V. Widder, The Laplace transformation, Princeton University Press, Princeton; Oxford University Press, London H. Milford, 1941, p. 66.
    ${ }^{8}$ G. N. Watson, loc. cit., pp. 77, 80.

[^25]:    ${ }^{9}$ G. N. Watson, loc. cit., pp. 202, 203.
    ${ }^{10}$ G. N. Watson, loc. cit., pp. 77, 78.
    ${ }^{11}$ It is a well-known result that these Bessel functions have no common roots. See G. N. Watson, loc. cit., pp. 479, 480, 481.

[^26]:    ${ }^{14}$ For an example of the method see H. S. Carslaw and J. C. Jaeger, A problem in conduction of heal, Proc. Cambridge Philos. Soc. 35, 394-404 (1939).
    ${ }^{13}$ For an example of the method see W. M. Rust, Jr., loc. cit., p. 196.

[^27]:    * Received April 11, 1945.
    ${ }^{1} \mathrm{G}$. Temple, The general theory of relaxation methods applied to linear systems, Proc. Roy. Soc. London (A), 169, 476-500 (1939).
    ${ }^{2}$ D. Moskovitz, The numerical solutions of Laplace's and Poisson's equations, Quart. Appl. Math. 2, 148-163 (1944).
    ${ }^{3}$ R. Courant, K. Friedrichs and H. Lewy, Über die partiellen Differenzgleichungen der mathematischen Physik, Math. Ann. 100, 22-74 (1928).

[^28]:    ' G. M. Dusinberre, Numerical methods for transient heat flow, Trans. A.S.M.E. 67, 703-709 (1945).

[^29]:    ${ }^{5}$ W. E. Byerly, Fourier's series and spherical, cylindrical and ellipsoidal harmonics, Ginn Co., Boston, 1895. pp. 30-35.

[^30]:    ${ }^{6}$ H. S. Carslaw, The conduction of heat, Macmillan Co., New York, ed. 2, 1921, pp. 97-99.

[^31]:    * Received Aug. 7, 1945.
    ${ }^{1}$ J. H. Michell, Proc. London Math. Soc. (1) 31, 100-146 (1899), Eqs. (13).
    2 The necessity of investigating the relations between the natural boundary conditions and Michell's equations arose in connection with work done under a contract in Applied Mechanics for Watertown Arsenal. The author is indebted to the authorities of Watertown Arsenal for the release of this note for publication.

[^32]:    * Received June 19, 1945.
    ${ }^{1}$ J. W. Craggs and C. J. Tranter, The capacity of twin cable, Quart. Appl. Math. 3, 268-272 (1945).

[^33]:    * Received August 9, 1945
    ${ }^{1}$ Quart. Appl. Math., 3, 166-174 (1945).

