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The Quarterly prints original papers in applied mathematics which have an intimate connection with application in industry or practical science. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

[^0][^1]
# ON THE PROPAGATION OF SMALL DISTURBANCES IN A MOVING COMPRESSIBLE FLUID* 

BY<br>G. F. CARRIER AND F. D. CARLSON<br>Harvard University

1. Introduction. Although the propagation of sound waves in moving media has received considerable attention $[1, \cdots, 9],{ }^{1}$ little information is available concerning the propagation of such disturbances in rotational streams or concerning the propagation of transient rotational phenomena. It is shown in the present paper that the wave fronts associated with those parts of a disturbance which are derivable from a potential propagate in a rotational stream according to those laws which they are already known to obey in an irrotational stream. It is further shown that the rotational disturbances drift with the stream rather than propagate relative to the moving fluid.

The analysis consists of an application of conventional perturbation procedures to the Navier-Stokes and continuity equations. The equations so derived are treated according to the theory of characteristics. The results obtained lead to a general expression for the Mach lines of an arbitrary supersonic flow and also suggest a new method of wind tunnel calibration which eliminates the necessity of placing an obstacle in that portion of the stream being calibrated. Finally, predictions are carried out as to the nature of pulses which are formed at a surface and then propagate through a boundary layer into a uniform stream.
2. The equations of motion. In this analysis, we shall consider the propagation of small disturbances in fluid streams which are characterized by three functions of the space coordinates and the time, namely: $\rho_{0}$ (the density), $p_{0}$ (the pressure), and $v_{0}$ (the velocity). No restrictions will be applied to these functions except that they obey the differential equations implying the conservation of momentum, mass, and energy. These equations, known familiarly as the Navier-Stokes, continuity, and energy equations, may be written in the forms:

$$
\begin{align*}
(v \cdot \operatorname{grad}) v+\partial v / \partial t+\frac{1}{\rho} \operatorname{grad} p & =\frac{\mu}{\rho} L(v)  \tag{1}\\
\operatorname{div} v+\partial \ln \rho / \partial t+v \cdot \operatorname{grad} \ln \rho & =0 .  \tag{2}\\
d U / d t+p d\left(\rho^{-1}\right) / d t & =Q+\frac{\mu}{\rho} \chi . \tag{3}
\end{align*}
$$

[^2]In the foregoing equations, $\mu$ is the viscosity of the fluid, $L$ symbolizes ( $\Delta+\frac{1}{3}$ grad div) where $\Delta$ is the Laplacian operator; $Q$ is the rate of heat accumulation, $U$ is the internal energy, and $\chi$ abbreviates the viscous dissipation terms. Discussions of these equations are conveniently found in [7] and [8]. The necessity of manipulating the energy equation in the investigation may be eliminated by using the following assumption. The changes in pressure and density accompanying the disturbance are taken to obey the law

$$
p / p_{0}=\left(\rho / \rho_{0}\right)^{\gamma}
$$

where $\rho, p, \nabla$, characterize the disturbed stream; that is, the disturbance is a phenomenon such that the changes in state from undisturbed to disturbed stream are isentropic. Note that this in no way restricts $p_{0}$ and $\rho_{0}$. The appendix indicates briefly the fact that while this assumption is by no means rigorously justified, it leads to valid results.

It is convenient at this point to introduce the small parameter e. Although this may be done in a fairly arbitrary manner, we shall define it in the following way in order to avoid any possible ambiguitics. Let the initial conditions of any particular problem be such that at time zero, $\rho=\rho_{0}+\epsilon \rho_{1}$, where the maximum value of $\rho_{1} / \rho_{0}$ over the region under consideration is unity. Thus, since we are considering small disturbances, $\epsilon$ is a small number compared to unity. Consistent with this notion, we shall write $\rho=\rho_{0}+\epsilon \rho_{1}++\epsilon^{2} \rho_{2}+\cdots, p=p_{0}+\epsilon p_{1}+\cdots$, and $v=\nabla_{0}+\epsilon \mathrm{V}_{1}+\cdots$, at time $t$; and shall require that the series be valid over a range of $\sigma$ Since disturbances can usually be expected to attenuate, it is certainly reasonable to expect that the series will converge for sufficiently small values of this parameter.

If we now substitute the foregoing forms of $\rho, p$, and $\nabla$, into Eqs. (1) and (2), eliminate the $p_{i}$ (except for $p_{0}$ ) by using Eq. ( $3^{\prime}$ ), and collect the coefficients of each power of $\epsilon$, we obtain:

$$
\begin{align*}
\left(v_{0} \cdot \operatorname{grad}\right) v_{0} & +\partial v_{0} / \partial \iota+\frac{1}{\rho_{0}} \operatorname{grad} p_{0}-\frac{\mu}{\rho_{0}} L\left(v_{0}\right) \\
+ & \epsilon\left\{\left(v_{0} \cdot \operatorname{grad}\right) v_{1}+\left(v_{1} \cdot \operatorname{grad}\right) v_{0}+\partial v_{1} / \partial t+\operatorname{grad}\left(a_{0}^{2} \frac{\rho_{1}}{\rho_{0}}\right)\right. \\
& \left.-\frac{\rho_{1}}{\rho^{2}} \operatorname{grad} \rho_{0}+a_{0}^{2} \frac{\rho_{1}}{\rho_{0}} \operatorname{grad} \ln \rho_{0}-\frac{\mu}{\rho_{0}}\left[L\left(v_{1}\right)+\frac{\rho_{1}}{\rho_{0}} L\left(v_{0}\right)\right]\right\}+\cdots=0 \tag{4}
\end{align*}
$$

and
$\partial \ln \rho_{0} / \partial t+\operatorname{div} v_{0}+v_{0} \cdot \operatorname{grad} \ln \rho_{0}$

$$
\begin{equation*}
+\epsilon\left\{\operatorname{div} \nabla_{1}+v_{1} \cdot \operatorname{grad} \ln \rho_{0}+v_{0} \cdot \operatorname{grad} \frac{\rho_{1}}{\rho_{0}}+\frac{\partial}{\partial t}\left(\frac{\rho_{1}}{\rho_{0}}\right)\right\}+\cdots=0 \tag{5}
\end{equation*}
$$

where, $a_{0}=\gamma p_{0} / \rho_{0}$.
When, in equations (4) and (5), the cocfficients of $\epsilon^{0}$ are equated to zero, we find two of the necessary conditions that the functions $\rho_{0}, p_{0}, v_{0}$, characterize a possible fluid stream. Since these quantities must vanish identically, we may omit them from Eqs. (4) and (5), and divide the remaining equalities through by $\epsilon$. When we allow $\epsilon$ to approach zero, we see that the mathematically exact solution to the problem is found by setting the coefficients of $\epsilon$ in Eqs. (4) and (5) to zero. Hence, we may expect
that the functions $\rho_{\mathrm{t}}, \mathrm{v}_{1}$, so determined will provide a good first approximation to the behaviour of small amplitude disturbances. This, of course, is the conventional perturbation reasoning.

If we had been willing to assume at the outset a functional relationship $p=p(\rho)$ applicable both to the stream and the disturbance, the perturbation procedure would have been unnecessary. The forthcoming techniques could have been applied directly to Eqs. (1) and (2). However, the solution possesses the desired generality only when we refrain from such restrictions on the nature of the stream. This leads to a choice between working with the energy equation or using the foregoing procedure; the latter seems more convenient. As a matter of fact, some of the results of this analysis differ from those of previous investigators only in that they are obtained for any stream wherein the medium behaves as a continuum rather than one of a very restricted character.

Recalling now that any vector may be expressed as the gradient of a scalar plus the curl of a vector and that

$$
(\mathbf{B} \cdot \operatorname{grad}) \mathbf{C}+(\mathbf{C} \cdot \operatorname{grad}) \mathbf{B}=\operatorname{grad}(\mathbf{B} \cdot \mathbf{C})+(\text { curl } \mathbf{B} \times \mathbf{C})+(\text { curl } \mathbf{C} \times \mathbf{B}),
$$

one may write

$$
\mathrm{V}_{1}=\operatorname{grad} \phi+\operatorname{curl} \mathbf{A},
$$

and the differential equations defining $\rho_{1}$ and $\mathbf{v}_{1}$ become

$$
\begin{align*}
& \operatorname{grad}\left[\left(\mathbf{v}_{0} \cdot \mathbf{v}_{1}\right)+a_{0}^{2} \frac{\rho_{1}}{\rho_{0}}+\frac{\partial \phi}{\partial t}\right]+\frac{\partial}{\partial t} \operatorname{curl} \mathbf{A}+a_{0}^{2} \frac{\rho_{1}}{\rho_{0}} \operatorname{grad} \ln \rho_{0} \\
& -\frac{\rho_{1}}{\rho_{0}^{2}} \operatorname{grad} p_{0}+\omega_{1} \times \mathrm{v}_{0}+\omega_{0} \times \mathrm{v}_{1}=\frac{\mu}{\rho_{0}}\left[L\left(\mathrm{v}_{1}\right)+\frac{\rho_{1}}{\rho_{0}} L\left(\nabla_{0}\right)\right] \tag{6}
\end{align*}
$$

and

$$
\begin{equation*}
\Delta \phi+\frac{d}{d t}\left(\frac{\rho_{1}}{\rho_{0}}\right)+\mathrm{v}_{1} \cdot \operatorname{grad} \ln \rho_{0}=0 \tag{7}
\end{equation*}
$$

where $\omega_{i}=$ curl $\mathbf{v}_{i}$, and $d / d t=\left[\mathrm{v}_{0} \cdot \operatorname{grad}+\partial / \partial t\right]$. When the operation "curl" is performed on Eq. (6), the following equality arises:

$$
\begin{align*}
\frac{\partial \omega_{1}}{\partial t}= & \operatorname{curl}\left\{\frac{\mu}{\rho_{0}}\left[L\left(\mathrm{v}_{1}\right)+\frac{\rho_{1}}{\rho_{0}} L\left(\mathrm{v}_{0}\right)\right]-a_{0}^{2} \frac{\rho_{1}}{\rho_{0}} \operatorname{grad} \ln \rho_{0}\right. \\
& \left.+\frac{\rho_{1}}{\rho_{0}^{2}} \operatorname{grad} p_{0}-\omega_{1} \times \mathrm{v}_{0}-\omega_{0} \times \mathrm{v}_{1}\right\} . \tag{8}
\end{align*}
$$

It is evident by inspection of Eq. (8) that an identically vanishing initial choice of $\omega_{1}$ does not imply that this function will vanish for all time, as for example, is the case in an irrotational stream. Thus, we cannot omit $\omega_{1}$ in this investigation.

It will prove useful to define an (artificial) auxiliary potential $\psi$ in the following manner ${ }^{2}$ (Eq. (6) implies the existance of this quantity).

[^3]\[

$$
\begin{align*}
-\operatorname{grad} \frac{\partial \psi}{\partial t} & +\operatorname{curl} \frac{\partial A}{\partial t}=\frac{\mu}{\rho_{0}}\left[L\left(\mathrm{v}_{1}\right)+\frac{\rho_{1}}{\rho_{0}} L\left(\mathrm{v}_{0}\right)\right]-a_{0}^{2} \frac{\rho_{1}}{\rho_{0}} \operatorname{grad} \ln \rho_{0} \\
& +\frac{\rho_{1}}{\rho_{0}^{2}} \operatorname{grad} p_{0}-\omega_{1} \times \nabla_{0}-\omega_{0} \times \nabla_{1}=0 \tag{9}
\end{align*}
$$
\]

Upon substitution of Eq. (9) into Eq. (6), the latter becomes

$$
\operatorname{grad}\left[\nabla_{0} \cdot \mathbf{V}_{1}+a_{0}^{2} \frac{\rho_{1}}{\rho_{0}}+\frac{\partial \phi}{\partial t}+\frac{\partial \psi}{\partial t}\right]=0
$$

This however, we may solve for $\rho_{1} / \rho_{0}$ arbitrarily choosing the "constant of integration" to be zero. ${ }^{3}$ We obtain

$$
\begin{equation*}
\rho_{1} / \rho_{0}=-a_{0}^{-2}\left[\nabla_{0} \cdot \operatorname{curl} \mathbf{A}+\frac{d \phi}{d t}+\frac{\partial \psi}{\partial t}\right] . \tag{10}
\end{equation*}
$$

This may be combined with Eq. (7) to give

$$
\begin{equation*}
\Delta \phi-\frac{d}{d t}\left(\frac{1}{a_{0}^{2}} \frac{d \phi}{d t}\right)+\mathrm{v}_{1} \cdot \operatorname{grad} \ln \rho_{0}=\frac{d}{d t}\left[\frac{1}{a_{0}^{2}}\left(\frac{\partial \psi}{\partial t}+\mathrm{v}_{0} \cdot \operatorname{curl} \mathbf{A}\right)\right] \tag{11}
\end{equation*}
$$

It will be shown directly that the wave front propagation can be derived from Eqs. (8), (9), and (11), provided we can justify the omission of the term $\left(\mu / \rho_{0}\right) L\left(\mathrm{v}_{1}\right)$ from Eqs. (8) and (9). We note, considering Eq. (8), that if only the terms $\partial \omega_{1} / \partial t$ and curl $\left(\mu / \rho_{0}\right) L\left(v_{1}\right)$ were non-vanishing, we would have virtually the equation for the conduction of heat, that is $\partial \omega_{1} / \partial t \cong\left(\mu / \rho_{0}\right) \Delta \omega_{1}$. The "conduction coefficient" is very small (in air the spreading of vorticity is known to be very slow) so that the term in question may be thought of as one which causes a small dispersive effect. It is to be understood, then, that this effect is to be superimposed on any results which are obtained by treating the equations from which this term has been omitted. With this omission we are now ready to apply the method of characteristics.

Hadamard [1] has shown the following facts concerning second order differential. equations which will be useful in the analysis of the foregoing equations. He considers the equation

$$
\sum_{i, k=1}^{n} a_{i k} p_{i k}+h=0
$$

in the $n$ independent variables $x_{1}, \cdots, x_{n}$, where the $a_{i k}$ and $h$ are functions of the unknown quantity $z$, the $x_{i}$, and the first partial derivatives of $z$ with regard to the $x_{i}$; $p_{i k}=\partial^{2} z / \partial x_{i} \partial x_{k}$. The differential equation which defines the characteristic surfaces (wave fronts) of this equation is given by

$$
\begin{equation*}
B=\sum_{i, k=1}^{n-1} a_{i k} P_{i} P_{k}-\sum_{i=1}^{n-1} a_{i n} P_{i}+a_{n n}=0 \tag{12}
\end{equation*}
$$

where $P_{i}=\partial x_{n} / \partial x_{i}$ when the "surface" is written in the form

$$
\begin{equation*}
x_{n}=x_{n}\left(x_{1}, \cdots, x_{n-1}\right) \tag{12a}
\end{equation*}
$$

[^4]Furthermore, let there be $s$ unknown functions $z_{1}, \cdots, z_{2}$, and $s$ equations of the form

$$
\sum_{i, k} a_{i k} p_{i k}+b_{i k} q_{i k}+\cdots+c_{i k} g_{i k}+h=0
$$

Here the $a_{i k}, b_{i k}, \cdots$ are respectively the coefficients of the second derivatives of $z_{1}, z_{2}, \ldots$. The characteristic surfaces of this system of equations are determined by the relation

$$
\left|\begin{array}{ccc}
B_{11}, & B_{12}, & \cdots,  \tag{13}\\
B_{21} & B_{1 z} \\
\vdots & & \\
B_{s 1} & & B_{s s}
\end{array}\right|=0
$$

The $B_{\alpha \beta}$ are analogous to the quantity $B$ of Eq. (12). In fact, when $\alpha$ takes the values $1,2, \cdots, s, B$ is derived respectively from the first, second, $\cdot \cdots$ sth, equations. When $\beta$ takes these values, the $B_{a \beta}$ are obtained from the $a_{i k}, b_{i k}, \cdots, c_{i k}$, respectively. The present problem deals with the five unknown quantities $\phi, \psi$, and the three components of A. Eq. (9) is equivalent to four scalar equations ${ }^{4}$ if we specify (for example) that $\psi$ and $\mathbf{A}$ are to be those solutions for which $\operatorname{div} \mathbf{A}=0$. This is no restriction since only curl $A$ appears in $v_{1}$. We may, then, apply the foregoing type of analysis to Eqs. (9) and (11) (with $\rho_{0} / \rho_{1}$ replaced by the expression given in Eq. (10)). In fact, in order to deternine the characteristic surfaces which define a motion involving the function $\phi$, we need only a brief inspection of Eq. (9). It is evident that no second derivatives of $\phi$ appear in this equation. Thus, when it is split into its four subdivisions, we find that the four quantities $B_{21}, \cdots, B_{51}$, which appear in the left column of Eq. (13), vanish. This implies that Eq. (13) is satisfied when either $B_{\mu}$ or the minor associated with this quantity vanishes. Since the vanishing of the former involves only the coefficients of derivatives of $\phi$, we may assume that this surface will be associated with the potential type of disturbance. The vanishing of the minor will correspond to the propagation of disturbances of the rotational type.

If we now compute $B_{11}$ using, of course, the $a_{i k}$ of Eq. (11), we find the same wave front equation which was found by Hadamard for the isentropic stream. That is, the time-position correlation of a wave front does not depend on the character of the stream but only (as the following equation will show) on the local values of the quantities $u_{0}, v_{0}, w_{0}$, and $a_{0}$. The first three of these are the components of $\nabla_{0}$. This wave front equation, in a form somewhat more convenient for our purposes than Hadamard's, is shown below.

$$
\begin{equation*}
\partial y / \partial \imath+u_{0} \partial y / \partial x+w_{0} \partial y / \partial z-v_{0} \pm a_{0}\left[1+(\partial y / \partial x)^{2}+(\partial y / \partial z)^{2}\right]^{1 / 2}=0 \tag{14}
\end{equation*}
$$

Eq. (8) indicates that whenever $\omega_{0}$ and $\phi$ are each non-vanishing in a given region, a rotational motion $\omega_{1}$, is generated continuously. This being so, there is always a possible "vorticity wave front" coincident with the wave front associated with $\phi$. Hence, if we treat Eq. (8) according to the foregoing method, using the components of curl A as the unknown function, we find that the determinant vanishes identically. This is to be expected since the operation which led to Eq. (8) eliminated the higher derivatives of $\phi$ while retaining the higher derivatives of $\mathbf{A}$. Hence, formally, the char-

[^5]acteristics method fails to give the desired information. We note that in this method, however, the only terms which affect the positional nature of the propagation are those containing second derivatives of the unknown functions. If, in Eq. (8), we segragate these terms of the required order we find that they comprise exactly the single term $d \omega_{1} / d t$. Therefore, in so far as the position of the disturbance is concerned, we have $d \omega_{1} / d l=0$; that is, the time rate of change of vorticity, relative to an observer moving with the particle, vanishes. In other words, the rotational disturbance drifts with the stream instead of propagating relative to it. This statement must be modified, of course, by the results of the diffusion-implying viscous term which was omitted in this analysis.
3. The two-dimensional problem. Since, in general, the functions $\rho_{0}, p_{0}, \nabla_{0}$ associated with any given stream are not known (even approximately in many cases), it scems of interest to describe a method of wind tunnel calibration based on the foregoing analysis (in particular on Eq. 14). This proposed procedure will be seen to have the advantage that it does not require the insertion of an obstacle into that portion of the stream being calibrated. Let us consider only tunnels which are bounded by the side walls $z= \pm b$, where $b$ is some constant. In this two-dimensional wind tunnel, the flow in the neighborhood of $z=0$ is essentially independent of $z$. Let us also restrict our consideration to disturbances having reflective symmetry about the plane $z=0$. Then at $z=0$, Eq. (14) reduces to
\[

$$
\begin{equation*}
\partial y / \partial t \pm a_{0}\left[1+(\partial y / \partial x)^{2}\right]^{1 / 2}-v_{0}+u_{0} \partial y / \partial x=0 . \tag{15}
\end{equation*}
$$

\]

We now have an equation, linear in the three quantities which we wish to determine; $u_{0}, v_{0}$, and $a_{0}$. Suppose we generate pulses at several points along some boundary of the stream, say by the use of an electric spark. The wave fronts of these pulses may be observed (photographed) at successive time intervals. The values of $\partial y / \partial x$ and $\partial y / \partial t$ can be determined from the photographs for each pulse throughout the region it traverses. For each point traversed by at least three pulses, we may form three simultaneous equations in the unknown quantities by using these experimentally determined values as coefficients in Eq. (15). Figures 1 to 4 illustrate such photographs of sound pulses in a fairly uniform stream of air. The development of the techniques used in obtaining these Schlieren photographs should be credited to the authors of [9]. In [9] the details of the experimental procedure are explained quite fully.

For an isentropic region of the stream (where the stagnation condition is known) only two pulses are needed since (see [11])

$$
a_{0}^{2}=\left(a_{s t}\right)^{2}-(\gamma-1)\left(u_{0}^{2}+\frac{v_{0}^{2}}{v_{0}}\right) / 2
$$

Finally, for an essentially one-dimensional stream (e.g., a jet or a slowly converging channel) Eq. (15) becomes

$$
\begin{equation*}
v_{0} / a_{\mathrm{st}}=\frac{\left[2(\gamma+1)-2(\gamma-1) \mu^{2}\right]^{1 / 2}-2 \mu}{\gamma+1} \tag{15a}
\end{equation*}
$$

where $\mu=a_{\mathrm{et}}^{-1} \partial y / \partial t$.
When the stream is supersonic, we may generate stationary disturbances (Mach
lines) by placing very small irregularities in the boundaries of the passage. For this case Eq. (15) reduces to

$$
\begin{equation*}
\partial y / \partial x=\frac{-u_{0} v_{0} \pm a_{0}^{2}\left[M^{2}-1\right]^{1 / 2}}{a_{0}^{2}-u_{0}^{2}} \tag{16}
\end{equation*}
$$

where $M$, the Mach number, is given by $M^{2}=\left(u_{0}^{2}+v_{0}^{2}\right) / a_{0}^{2}$. A mesh of Mach lines proceeding from both edges of the passage give sufficient information to calibrate any stream known to be isentropic with known stagnation condition. For a one-dimensional stream, we have the familiar formula for the Mach angle $\theta$

$$
\begin{equation*}
\partial y / \partial x=\tan \theta=\left(M^{2}-1\right)^{1 / 2} . \tag{17}
\end{equation*}
$$

Note that when one wishes to find the characteristics of a given supersonic stream, the classical Charpit procedure will always provide solutions for Eq. (16). The equation analogous to Eq. (16) in three dimensions follows directly from Eq. (14) by merely dropping the time dependent term.
4. The effect of boundary layers. A problem of considerable interest arises in connection with the ideas of the foregoing section when we inquire into the effect of the boundary layer on the form of the wave front when the pulse is generated at the surface of a boundary (or obstacle). We shall use the Charpit procedure to solve Eq. (15) for this case, using, of course, an idealized group of values for $u_{0}, v_{0}$, and $a_{0}$. The justification of the steps of this procedure are given in [10] and need not be given here, so we shall proceed formally with this method. The solution obtained will be in closed form and is readily verified (as a solution of Eq. 15) by mere substitution.

We characterize the stream by the functions

$$
u_{0} \equiv 0, v_{0}=v x / \delta, \quad \text { for } x \leqq \delta, v_{0}=v \text { for } x \geqq \delta, a_{0}=a=\text { const. }
$$

This simplification of an actual situation is somewhat drastic but useful information results. We first set $\xi=x / \delta, \eta=y / \delta, \tau=a t / \delta, M=v / a, p=\partial \eta / \partial \xi, q=\partial \eta / \partial \tau$, and Eq. (15) in the notation of [10] becomes

$$
F(\xi, \eta, \tau, p, q)=q \pm\left[1+p^{2}\right]^{1 / 2}-M \xi=0 \quad \text { for } \quad 0 \leqq \xi \leqq 1
$$

or

$$
\begin{equation*}
F=q \pm\left[1+p^{2}\right]^{1 / 2}-M=0 \text { for } 1 \leqq \xi . \tag{18}
\end{equation*}
$$

We proceed by considering the associated ordinary differential equations

$$
\begin{equation*}
\frac{d p}{F_{\xi}+p F_{\eta}}=\frac{d q}{F_{\tau}+q F_{\eta}}=\frac{d \eta}{-p F_{p}-q F_{q}}=\frac{d \xi}{-F_{p}}=\frac{d \tau}{-F_{q}} \tag{19}
\end{equation*}
$$

and choose any solution which expresses $p$ or $q$ in terms of a parameter $\alpha$. In our case, (formally)

$$
\begin{equation*}
\frac{-d p}{M}=d q / 0=-\left(\frac{p^{2}}{\left[1+p^{2}\right]^{1 / 2}}+q\right)^{-1} d \eta=\cdots \tag{19a}
\end{equation*}
$$

and $q=\alpha$ is the required solution. When this is substituted into Eq. (18), we obtain for $p$ in the respective regions

$$
\begin{equation*}
p=\left[(\alpha-M \xi)^{2}-1\right]^{1 / 2} \text { and } p=\left[(\alpha-M)^{2}-1\right]^{1 / 2} \tag{20}
\end{equation*}
$$

We now determine $\eta$ by the following integration

$$
\begin{equation*}
\eta-\beta=\int p d \xi+q d \tau \tag{21}
\end{equation*}
$$

$\beta$ is chosen to suit the initial and boundary conditions; $\alpha$ is determined by the relation $\partial \eta / \partial \alpha=0$, and the sign of $p$ must be taken consistent with that portion of the wave front under consideration.

For the initial condition $\eta=0$ at $\xi=\tau=0$ (a point source), and for $\xi \leqq 1$, we have for that downstream portion of the wave for which $p \leqq 0$,

$$
\begin{equation*}
\eta=\alpha \tau+\beta(\alpha-M \xi)-\beta(\alpha) \tag{22}
\end{equation*}
$$

where

$$
\begin{align*}
\beta(\alpha) & =\alpha\left[\alpha^{2}-1\right]^{1 / 2}-\operatorname{arccosh} \alpha  \tag{23}\\
\alpha & =\frac{M \xi}{2}+\frac{M \tau}{2}\left[1+\frac{4}{\tau^{2}-\xi^{2}}\right]^{1 / 2} . \tag{24}
\end{align*}
$$

This solution is valid when $\xi_{0} \leqq \xi \leqq \min (\tau, 1)$;

$$
\xi_{0}=\left[\tau^{2}+M^{-2}\right]^{1 / 2}-1 / M .
$$

When $\xi \leqq \xi_{0}, p \geqq 0$, and we must find $\eta$ by writing

$$
\eta(\tau, \xi)=\eta\left(\tau, \xi_{0}\right)+\int_{\xi_{0}}^{\xi} p d \xi
$$

which results in the formula

$$
\begin{equation*}
\eta(\tau, \xi)=\alpha \tau-\beta(\alpha)-\beta(\alpha-M \xi) \tag{25}
\end{equation*}
$$

where $\alpha$ and $\beta$ are still the quantities defined by Eqs. (23) and (24).
When we consider the upstream portion of the wave front, $\alpha$ must be negative. By the foregoing procedure we obtain

$$
\begin{align*}
& \eta_{1}=\alpha_{1} \tau-\beta\left(-\alpha_{1}\right)+\beta\left(M \xi-\alpha_{1}\right)  \tag{26}\\
& \alpha_{1}=\frac{M \xi}{2}-\frac{M \tau}{2}\left[1+\frac{4}{r^{2}-\xi^{2}}\right]^{1 / 2} \tag{27}
\end{align*}
$$

or

$$
\begin{equation*}
\eta_{1}(\xi, \tau)=-\eta(\xi, \tau)+M \xi \tau . \tag{28}
\end{equation*}
$$

This equation is again valid only when $\xi \geqq \xi_{0}$.
We now consider that portion of the wave exterior to the boundary layer (i.e. $\xi \geqq 1$ ). We must again extend the integration of Eq. (21), this time into the uniform stream. Using now $p=-\left[(\alpha-M)^{2}\right]^{1 / 2}$, we obtain for the downstream portion

$$
\begin{align*}
\eta & =\alpha \tau+\beta(\alpha-M)-\beta(\alpha)-\int_{1}^{\xi}\left[(\alpha-M)^{2}-1\right]^{1 / 2} d \xi \\
& =\alpha \tau+\beta(\alpha-M)-\beta(\alpha)-\left[(\alpha-M)^{2}-1\right]^{1 / 2}(\xi-1) \tag{29}
\end{align*}
$$

When $\partial \eta / \partial \alpha$ is equated to zero we find

$$
\begin{equation*}
\xi-1=\frac{\left[(\alpha-M)^{2}-1\right]^{1 / 2}}{\alpha-M}\left\{\tau+\frac{1}{M}\left[(\alpha-M)^{2}-1\right]^{1 / 2}-\frac{1}{M}\left[\alpha^{2}-1\right]^{1 / 2}\right\} . \tag{30}
\end{equation*}
$$

Fig. 1. Sound waves proceeding from point source in nose of vanc. Time delay between pulse generation and photography are 88 and 263 microseconds. The flow is directed to the left at a Mach number of . 423.


Fig. 3. Sound waves proceeding from flush sources. Bright spots indicate spark gap locations. $M=.33$.


Eqs. (29) and (30) may be considered as parametric equations for $\xi$ and $\eta$ in terms of the parameter $\alpha$ for each value of $\tau$. In a similar manner we obtain for the upstream portion

$$
\begin{equation*}
\eta_{1}=\alpha_{1} \tau-\beta\left(-\alpha_{1}\right)+\beta\left(M-\alpha_{1}\right)+\left[\left(\alpha_{1}-M\right)^{2}-1\right]^{1 / 2}\left(\xi_{1}-1\right) \tag{31}
\end{equation*}
$$

and

$$
\begin{equation*}
\xi_{1}-1=\frac{\left[\left(M-\alpha_{1}\right)^{2}-1\right]^{1 / 2}}{M-\alpha_{1}}\left\{\tau+\frac{1}{M}\left[\alpha_{1}^{2}-1\right]^{1 / 2}-\frac{1}{M}\left[\left(M-\alpha_{1}\right)^{2}-1\right]^{1 / 2}\right\} \tag{32}
\end{equation*}
$$

In Eqs. (29) to (32), $1+M \leqq \alpha<\infty$ and $-1 \geqq \alpha_{1}>-\infty$.
The solution is now complete except for surface reflections. Note that the velocity at which the point of contact between boundary and wave front moves is

$$
\begin{gather*}
\left.\frac{\partial \eta}{\partial \tau}\right|_{\xi=0}=\alpha(\tau, 0)=\left[1+\frac{M^{2} \tau^{2}}{4}\right]^{1 / 2} \\
\frac{\partial y}{\partial t}=a_{0}\left[1+\frac{v^{2} t^{2}}{4 \delta^{2}}\right]^{1 / 2} . \tag{33}
\end{gather*}
$$

Figure 5 illustrates the results of the foregoing section for a stream with Mach number .50 . The peculiar behavior of the solution for $\tau>\sqrt{5}$ leads us to investigate the rays of the propagation. It has been shown $[1,2,3]$ that Eq. (14) implies that the rays be defined by $d x=\left(l a_{0}+u_{0}\right) d t, d y=\left(m a_{0}+v_{0}\right) d t$, and $d z=\left(n a_{0}+w_{0}\right) d t$, where $l, m$, $n$, are the direction cosines of the outwardly directed wave front normal with the coordinates axes. In particular, it has been shown that for the conditions prevailing in the boundary layer specified above, the rays are given by (sec [9])

$$
\begin{equation*}
\eta=-\frac{1}{2 M}\left[\operatorname{arc} \cosh \left(m_{0}-M \xi\right)+\left(m_{0}+M \xi\right)\left\{\left(m_{0}-M \xi\right)^{2}-1\right\}^{1 / 2}\right]_{0}^{\xi} \tag{34}
\end{equation*}
$$

where $m_{0}$ is the value of $m^{-1}$ at the origin. The value of $m_{0}$ for which the ray becomes tangent to the line $\xi=1$ is given by $m_{0}=1+M$. However, any ray associated with a uniform stream which is directed parallel to that stream will maintain this orientation. Hence, this ray which just becomes tangent to the uniform stream bifurcates into the curves shown in Fig. 5. Note that no ray which is once reflected back into the boundary layer will ever leave this region. This implies that a sizeable portion of the energy of such pulses never leaves the boundary layer. Furthermore, as one can readily see from the few rays plotted in the figure, the reflections occur in such a manner that interference as well as the extremely turbulent conditions in such a region make the observation of waves in such regions improbable. This is borne out in Figs. 1 to 4. One large discrepancy between these pictures and the theory is easily noticed. The limiting upstream ray given by the theory does not agree too well with the evidence of Fig. 4. This is due to the fact that the pulse used in the experiment started as one of finite amplitude (probably traveling initially at a speed of about $2 a_{0}$ ). This means that at first the wave travels as though it were a small amplitude wave in a slower stream; hence, less distortion from the shape which would be expected with no boundary layer (a family of semi-circles) is the logical result to expect. This, of course, is consistent with the observations. One other remark is essential in view of
the initial idealization of the boundary layer. In the actually occurring physical situation, the boundary layer thickens in the direction of flow and thus the sharp break in wave front predicted in this theory is not valid. However, the transition from large velocity gradient to uniform stream occurs in a sufficiently small region so that little energy transfer from the stream part of the wave to the boundary layer is to be expected.


Fig. 5. Predicted wave fronts for the stream defined in section (4). Wave fronts are illustrated for $\tau=1$ $2, \sqrt{5}, 3,5$. The dotted curves are rays of the propagation. The sound source is at the origin.

If one wishes to account for the large pulse velocity in a mathematical manner, he can replace the constant value of $a_{0}$ by a function of the time, large at time zero but rapidly approaching the steady value. This, of course, makes the calculations tedious.
5. Intensity distribution. In general, it is difficult to obtain a solution for $\phi$ from Eq. (11). However, one very interesting solution for the isentropic uniform stream has recently appeared. Rott [6] has shown that when $v_{0}=w_{0}=0, u_{0}=$ const $=-M a_{0}$, Eq. (11) has a solution of the form

$$
\begin{equation*}
\phi_{1}=\frac{C}{R} \cos \omega\left(t-\frac{M x+R}{a_{0}\left(1-M^{2}\right)}\right), \tag{35}
\end{equation*}
$$

where $R=\left[x^{2}+\left(1-M^{2}\right)\left(y^{2}+z^{2}\right)\right]^{1 / 2}$. This solution implies a continuous point source at the origin and, of course, assumes no boundary layer if the plane (say) $y=0$ is to be a boundary.

The surfaces of constant phase (characteristic surfaces) are given by

$$
\begin{equation*}
t-\frac{M x+R}{a_{0}\left(1-M^{2}\right)}=\text { const. }=\lambda \tag{36}
\end{equation*}
$$

a relationship which can be shown to satisfy Eq. (14) for the given stream.

In connection with our pulse problem, we see immediately that we may form a new solution as the integral

$$
\begin{equation*}
\phi=\int_{0}^{\infty} \frac{C(\omega)}{R} \cos \omega\left(t-\frac{M x+R}{a_{0}\left(1-M^{2}\right)}\right) d \omega \tag{37}
\end{equation*}
$$

If $C$ is properly chosen, this solution corresponds to a pulse of any desired wave form originating at the origin. The surfaces of constant phase are again given by Eq. (36) and, for a given value of $t$, are circles with centers at the points $x=M a_{0} t, y=z=0$, and radii $a_{0}$ l.

Suppose we now choose two values of $\lambda$ (say $\lambda_{1}$ and $\lambda_{2}$ ) to represent two characteristic surfaces whose separation (along a radial line from the origin) we can call the wave length of the pulse. Then that upstream portion of the pulse at $y=z=0$ will have a wave length $\left(\lambda_{2}-\lambda_{1}\right) a_{0}(1-M)$ and the downstream section at $y=z=0$ will have a wave length $\left(\lambda_{2}-\lambda_{1}\right) a_{0}(1+M)$. That is, the ratio of the thicknesses of these two extreme portions of the pulse is $(1+M) /(1-M)$.

Inspection of Eq. (35) also leads to the conclusion that at these portions of the pulse the amplitudes vary in the ratio $(1-M) /(1+M)$. Thus the amplitude gradients at these two sections are in the ratio $(1-M)^{2} /(1+M)^{2}$. Since the density gradient is essentially the quantity observed in the Schlieren optical system, the foregoing constitutes an explanation of the far superior clarity of the wave front definition in the upstream portions of the photographs. This argument has assumed that the pulse started at time zero and has a small thickness to radius ratio at the time of observation.

Specifically, the amplitude ratio (i.e., the ratio of amplitude at any point on the wave front divided into the amplitude at the upstream extremum) is given by

$$
\begin{equation*}
\text { Amp. ratio }=1+M-\frac{M x}{a_{0} t(1-M)} \tag{38}
\end{equation*}
$$

## APPENDIX

We wish to justify here the use of Eq. ( $3^{\prime}$ ). We write the energy equation in the form

$$
\begin{equation*}
\frac{R}{\gamma-1} \frac{d T}{d t}+p \frac{d \rho^{-1}}{d l}=\frac{\alpha}{\rho} \Delta T+\frac{\mu}{\rho} \chi \tag{3}
\end{equation*}
$$

where $T$ is the temperature ( $p=\rho R T$ ) and $\chi$ involves a sum of products of the form $(\partial u / \partial y)(\partial v / \partial x),(\partial u / \partial y)^{2}, \cdots$. When, as in the foregoing work, the perturbation procedure is applied, the terms with coefficient $\epsilon$ can be written

$$
\begin{equation*}
\frac{1}{\gamma-1} \frac{d\left(T / T_{0}\right)}{d t}-\frac{d\left(\rho / \rho_{0}\right)}{d t}=\frac{\alpha}{\rho_{0} R T_{0}} \Delta T_{1}+\frac{\mu}{\rho_{0}} \chi_{1}+K \tag{3a}
\end{equation*}
$$

where we have grouped the terms not containing derivatives of the unknown functions $\rho_{1}, p_{1}, v_{1}, T_{1}$ in $K$. Such terms can obviously contribute nothing when the characteristics method is applied. In Eq. (3a) $(\gamma-1) \alpha / R$ is a ratio of specific heat to thermal conductivity, and $\chi_{1}$ involves products of the form $\left(\partial u_{0} / \partial y\right)\left(\partial v_{1} / \partial x\right)$, $\left(\partial u_{0} / \partial y\right)\left(\partial u_{1} / \partial y\right), \cdots$.

If we can show the third and fourth terms of this equation to be negligible, integration of Eq. (3a) leads to the terms with coefficient $\epsilon$ in the expansion of Eq. (3'). Using a dimensional treatment analogous to Prandtl's boundary layer analysis, we define a typical length $l$ for the disturbance (say the wave length, if a continuous wave is considered, or the breadth of a pulse, etc.) and compare first the terms $(\gamma-1)^{-1} \nabla_{0} \cdot \operatorname{grad}\left(T_{1} / T_{0}\right)$ and $\left(\mu / \rho_{0}\right)\left(\partial u_{0} / \partial y\right)\left(\partial u_{1} / \partial y\right)$. We note that $(\gamma-1)^{-1} \nabla_{0} \cdot \operatorname{grad}$ $\left(T_{1} / T_{0}\right) \sim(\gamma-1)^{-1}\left(T_{1} / T_{0}\right)\left(\left|v_{0}\right| / l\right)$ and

$$
\frac{\mu}{\rho_{0}} \frac{\partial u_{0}}{\partial y} \frac{\partial u_{1}}{\partial y} \approx \frac{\mu}{a_{0} \rho_{0}} \frac{\left|\mathrm{v}_{0}\right|}{\delta} \frac{\left|\mathrm{v}_{1}\right|}{l},
$$

so we have the requirement, if the latter term is to be onnitted, that $\rho_{0} \delta a_{0} / \mu\left|\mathbf{v}_{1}\right|$ $\gg T_{0} / T_{1}$. Since $T_{1} / T_{0}$ and $\left|v_{1}\right| / a_{0}$ are of appreciable magnitude (of order unity) in the same region, this inequality is essentially $\rho_{0} \delta a_{0} / \mu \gg 1$. Here, $\delta$ is the boundary layer thickness or other typical dimension of the stream. Thus we see that except for very restricted regions the above inequality will hold and the term in. question may be omitted. The other terms in $\chi_{1}$ admit a similar treatment.

We now compare the terms $\partial\left(T_{1} / T_{0}\right) / \partial t$ and $\alpha \Delta T_{1} / R T_{0} \rho_{0}$. Note that these are exactly the terms which would need to be compared in the still air case; that is, no functions characterizing the stream appear except the slowly varying $a_{0}^{2}$. Using the typical length $l$ and the fact that the propagation occurs at essentially velocity $a_{0}$, we obtain $\alpha \Delta T_{1} / R T_{0} \rho_{0} \sim \alpha T_{1} / R T_{0} l^{2} \rho_{0} \ll \partial\left(T_{1} / T_{0}\right) / \partial t \sim a_{0} T / l T_{0}$ as a necessary condition. That is, the number $\alpha / a_{0} l R \rho_{0} \ll 1$. This, of course, is the actual situation* and hence both terms in question may be omitted. This number is essentially the reciprocal of a Prandtl number-Reynolds number product.

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[^6]
# THIN CYLINDRICAL SHELLS SUBJECTED TO CONCENTRATED LOADS* 

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#### Abstract

A single differential equation of the eighth order in the radial displacement is given for the equilibrium of an element of a cylindrical shell undergoing small displacements due to a laterally distributed external load. The radial deflection of thin cylindrical shells subjected to concentrated, equal and opposite forces, acting at the ends of a vertical diameter, is analyzed by the Fourier method. Applications of the solution of the problem of the infinitely long cylinder to the problems of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference are also discussed.


1. Introduction. The bending problem of an infinitely long cylinder loaded with concentrated, equal and opposite forces, acting at the ends of a vertical diameter, is discussed first. The equations of equilibrium of an element of a cylindrical shell undergoing small displacements due to a laterally distributed external load are reduced to a single differential equation of the eighth order in the radial displacement. In this equation the various terms are compared as to the order of magnitude and it is found that some of the terms are negligible.

The specified loading function is represented by a Fourier integral in the longitudinal direction, and by a Fourier series in the circumferential direction. The integral representation has the advantage that the boundary conditions are automatically taken care of, and no subsequent cletermination of Fourier coefficients is necessary. The Fourier coefficients and the undetermined function in the Fourier integral in this case are determined simply from the loading condition. The radial displacement is represented in a like manner with the aid of an undetermined function which is obtained by substituting both radial displacement and loading expressions in the differential equation. The definite integrals involved in the expression for radial deflection are evaluated by means of Cauchy's theorem of residues.

The problem of the inextensional deformation of cylindrical and spherical shells was treated in detail by Lord Rayleigh in his "Theory of sound." The assumption of this type of deformation underlies the solution of many problems of practical importance, such as the determination of stresses in thin cylindrical shells subjected to two equal and opposite forces acting at the ends of a diameter or to internal hydrostatic pressure. It is found that the results obtained in the case of inextensional deformations correspond only to a first approximation of the solution in this paper, and the stresses in the proximity of the points of application of the forces are not given with sufficient accuracy.

The expression for the radial deflection of a thin cylinder of finite length is obtained from the corresponding solution for an infinitely long cylinder by using the method of images. It is seen that the difference of these two radial deflections can be given by a correction factor included in the expression for a cylinder of finite length.

[^7]The difference is believed to result from restraining the edges at the two ends of the finite cylinder. The results indicate that the radial deflection of an infinitely long cylinder has a very long wave length along the generatrix; however, the wave length decreases as the ratio of radius over thickness decreases. It is believed that the long wave length phenomenon is due to the elastic reaction along the circumference of the shell which can be explained by the radial deflection along the circumference.

Deflection curves of cylindrical shells with various lengths are calculated and the results show that the maximum radial deflection occurs at length over radius ratio $l / a \sim 20$. The radial deflection of an infinitely long cylinder with the radius over thickness ratio $a / h=100$, becomes zero at about $x / a=15$ and then reverses its sign. The edges of the corresponding cylinder with finite length are so restrained that the nega-


Fig. 1. Forces and moments on element of wall.
tive deflection portion of the infinite cylinder is brought to zero at the edges of the cylinder with finite length. Hence, the maximum deflection of a cylinder with $l / a \simeq 20$ is greater than that of the corresponding infinitely long cylinder.

The problems of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference are also analyzed by using the corresponding solution for the radial deflection under a concentrated load. The action of the couple is equivalent to that of two equal and opposite forces acting at an infinitely small distance apart.
2. Fundamental equations. The fundamental equations of a cylindrical shell under the specified loading are obtained from considering the equilibrium of an element cut out by two diametrical sections and two cross sections perpendicular to the axis of the cylindrical shell as shown in Fig. 1.

In this discussion the usual assumptions are made; namely, that the material is isotropic and follows Hooke's law, the undeformed tube is cylindrical, the wall thickness is uniform and small compared to the radius, the deflections are small compared to this thickness so that second order strains can be neglected, and that straight lines in the cylinder wall and perpendicular to the middle surface remain straight after distortion.

The notation used for resultant forces and moments per unit length of wall section are indicated in Fig. 1. After simplification, the following equations of equilibrium are obtained:*

$$
\begin{align*}
a \frac{\partial N_{x}}{\partial x}+\frac{\partial N_{\phi x}}{\partial \phi} & =0, & a \frac{\partial M_{x \phi}}{\partial x}-\frac{\partial M_{\phi}}{\partial \phi}+a Q_{\phi} & =0 \\
\frac{\partial N_{\phi}}{\partial \phi}+a \frac{\partial N_{x \phi}}{\partial x}-Q_{\phi} & =0, & a \frac{\partial M_{x}}{\partial x}+\frac{\partial M_{\phi x}}{\partial \phi}-a Q_{x} & =0  \tag{1}\\
a \frac{\partial Q_{x}}{\partial x}+\frac{\partial Q_{\phi}}{\partial \phi}+N_{\phi}+q a & =0, & \left(N_{x \phi}-N_{\phi x}\right) a & =0
\end{align*}
$$

in which $q$ is the normal pressure on the element.
If $Q_{x}$ and $Q_{\phi}$ are eliminated from Eqs. (1) and the relations

$$
N_{x \phi}=N_{\phi x} ; \quad M_{x \phi}=-M_{\phi x}
$$

are used, the six equations in (1) can be reduced to the following three:

$$
\begin{array}{r}
a \frac{\partial N_{z}}{\partial x}+\frac{\partial N_{x \phi}}{\partial \phi}=0 \\
\frac{\partial N_{\phi}}{\partial \phi}+a \frac{\partial N_{x \phi}}{\partial x}+\frac{\partial M_{x \phi}}{\partial x}-\frac{1}{a} \frac{\partial M_{\phi}}{\partial \phi}=0  \tag{2}\\
-\frac{2}{a} \frac{\partial^{2} M_{x \phi}}{\partial \phi \partial x}+\frac{\partial^{2} M_{x}}{\partial x^{2}}+\frac{1}{a^{2}} \frac{\partial^{2} M_{\phi}}{\partial \phi^{2}}+\frac{N_{\phi}}{a}+q=0
\end{array}
$$

The relation between the resultant forces and moments and the strains of the middle surface will be taken the same as in the case of a flat plate:
$N_{x}=\frac{E h}{1-\nu^{2}}\left(\epsilon_{x}+\nu \epsilon_{\phi}\right), \quad N_{\phi}=\frac{E h}{1-\nu^{2}}\left(\epsilon_{\phi}+\nu \epsilon_{x}\right), \quad N_{x \phi}=N_{\phi x}=\frac{\gamma E h}{2(1+\nu)}$,
$M_{x}=-D\left(X_{x}+\nu X_{\phi}\right), \quad M_{\phi}=-D\left(X_{\phi}+\nu X_{x}\right), \quad M_{x \phi}=-M_{\phi x}=D(1-\nu) X_{x \phi}$,
where $D=E h^{3} / 12\left(1-\nu^{2}\right)$ is the flexural rigidity of the shell and $h$ is the thickness.
Resolving the displacement at an arbitrary point in the middle surface during deformation into three components- $u$ along the generator, $v$ along the tangent to the circular section, and $w$ along the normal to the surface drawn inwards-one finds that the extensional strains and changes of curvature in the middle surface are

$$
\begin{array}{rlr}
\epsilon_{x}=\frac{\partial u}{\partial x}, & \epsilon_{\phi}=\frac{1}{a} \frac{\partial v}{\partial \phi}-\frac{w}{a}, & \gamma_{x \phi}=\frac{\partial v}{\partial x}+\frac{\partial u}{a \partial \phi} \\
X_{x}=\frac{\partial^{2} w}{\partial x^{2}}, & X_{\phi}=\frac{1}{a^{2}} \frac{\partial^{2} w}{\partial \phi^{2}}+\frac{1}{a^{2}} \frac{\partial v}{\partial \phi}, & X_{x \phi}=\frac{1}{a} \frac{\partial^{2} w}{\partial x \partial \phi}+\frac{1}{a} \frac{\partial v}{\partial x} .
\end{array}
$$

Hence, Eqs. (2) can be put int the form of three equations with three unknowns $u, v, w$ :
$\frac{\partial^{2} u}{\partial x^{2}}+\frac{1+\nu}{2} \frac{\partial^{2} v}{\partial s \partial x}-\frac{\nu}{a} \frac{\partial w}{\partial x}+\frac{1-\nu}{2} \frac{\partial^{2} u}{\partial s^{2}}=0$,

[^8]\[

$$
\begin{align*}
& \begin{array}{l}
\frac{\partial^{2} v}{\partial s^{2}}+\frac{1+\nu}{2} \frac{\partial^{2} u}{\partial s \partial x}+\frac{1-\nu}{2} \frac{\partial^{2} v}{\partial x^{2}}-\frac{1}{a} \frac{\partial w}{\partial s} \\
\\
\quad+\frac{h^{2}}{12 a}\left(\frac{\partial^{3} w}{\partial x^{2} \partial s}+\frac{\partial^{3} w}{\partial s^{3}}\right)+\frac{h^{2}}{12 a^{2}}\left((1-\nu) \frac{\partial^{2} v}{\partial x^{2}}+\frac{\partial^{2} v}{\partial s^{2}}\right)=0
\end{array}  \tag{3}\\
& \frac{h^{2}}{12} \nabla^{4} w-\frac{1}{a}\left(\frac{\partial v}{\partial s}-\frac{w}{a}+\nu \frac{\partial u}{\partial x}\right)+\frac{h^{2}}{12 a}\left((2-\nu) \frac{\partial^{3} v}{\partial x^{2} \partial s}+\frac{\partial^{3} v}{\partial s^{2}}\right)-\frac{1-\nu^{2}}{E h} q=0
\end{align*}
$$
\]

where

$$
s=a \phi
$$

In the problem under investigation, the quantities $u$ and $v$ are of the order of magnitude of $\sqrt{h w / a}$. Consequently the last term and the third term in the second and the third equations, respectively, in (3) can be neglected safely.

In order to solve the simultaneous equations (3), one can apply first the operation $\partial^{2} / \partial x^{2}$, and then $\partial^{2} / \partial s^{2}$ to the first Eq. (3). Solving in each case for the term containing $v$, and substituting these expressions in the equation obtained by applying $\partial^{2} / \partial x \partial s$ to the second Eq. (3), one obtains an equation from which $v$ has been eliminated:

$$
\begin{equation*}
a \nabla^{4} u=\nu \frac{\partial^{3} w}{\partial x^{3}}-\frac{\partial^{3} w}{\partial x \partial s^{2}}+\frac{1+\nu}{1-\nu} \frac{h^{2}}{12}\left(\frac{\partial^{5} w}{\partial x^{3} \partial s^{2}}+\frac{\partial^{5} w}{\partial x \partial s^{4}}\right) . \tag{4}
\end{equation*}
$$

Similarly, applying $\partial^{2} / \partial x^{2}$ and $\partial^{2} / \partial s^{2}$ to the second Eq. (3) and solving for the terms containing $u$, and substituting in the first Eq. (3) after applying $\partial^{2} / \partial x \partial s$ to it, one obtains an equation from which $u$ has been eliminated:

$$
\begin{equation*}
a \nabla^{4} v=(2+\nu) \frac{\partial^{3} w}{\partial x^{2} \partial s}+\frac{\partial^{3} w}{\partial s^{3}}-\frac{h^{2}}{12}\left(\frac{2}{1-\nu} \frac{\partial^{5} w}{\partial x^{4} \partial s}+\frac{3-\nu}{1-\nu} \frac{\partial^{5} w}{\partial x^{2} \partial s^{3}}+\frac{\partial^{5} w}{\partial s^{5}}\right) \tag{5}
\end{equation*}
$$

Applying $\partial / \partial x$ to Eq. (4) and $\partial / \partial s$ to Eq. (5) and substituting these two equations into the third Eq. (3), after applying $\nabla^{4}$ to it, one obtains an equation from which both $u$ and $v$ are absent:
$\nabla^{8} w+\frac{12\left(1-\nu^{2}\right)}{a^{2} h^{2}} \frac{\partial^{4} w}{\partial x^{4}}+\frac{1}{a^{2}}\left(\frac{\partial^{6} w}{\partial s^{6}}+(2+\nu) \frac{\partial^{6} w}{\partial x^{4} \partial s^{2}}+(3+\nu) \frac{\partial^{6} w}{\partial x^{2} \partial s^{4}}\right)-\frac{1}{D} \nabla^{4} q=0$.
It is cvident that the third term in Eq. (6) is neligible in comparison with the other terms. Equation (6) is reduced to

$$
\begin{equation*}
\nabla^{8} w+\frac{12\left(1-\nu^{2}\right)}{a^{2} / h^{2}} \frac{\partial^{4} w}{\partial x^{4}}-\frac{1}{D} \nabla^{4} a=0 \tag{6a}
\end{equation*}
$$

Equation (6a) differs from the differential equation of the flat plate only by the second term. The flat plate equation can be obtained from equation (6a) by the substitution of $a=\infty$. Consequently, this second term represents the effect of curvature in the problem of the cylindrical shell.
3. Infinitely long cylinder loaded with two equal and opposite forces. The above equation will now be applied to an infinitely long thin cylinder loaded, as shown in

Fig. 2, by two equal and opposite compressive forces $P$ acting at the ends of a vertical diameter.


Fig. 2. Loads and components of displacements of an infinitely long cylinder.
The difficulties of integrating Eq. (6a) for this type of loading can be circumvented by replacing the concentrated force $P$ by a distributed load $q$ expressed as function of the longitudinal and circumferential coordinates, and applied to a small area which subsequently is reduced to an infinitesimal. This is made possible by representing the function in the longitudinal direction, by a Fourier integral and in the circumferential direction by a Fourier series. Since $q$ is an even function of both $x$ and $s$, it can be expressed by

$$
\begin{equation*}
q(x, s)=\left[\frac{q_{0}}{2}+\sum_{n=2,4}^{\infty} q_{n} \cos \frac{n s}{a}\right] \int_{0}^{\infty} f(\lambda) \cos \frac{x \lambda}{a} d \lambda \tag{7}
\end{equation*}
$$

The displacement $w$ can be expanded in a similar manner in terms of a function $w(\lambda)$ as yet undetermined:

$$
\begin{equation*}
w=\sum_{n=0,2 \ldots}^{\infty} \cos \frac{n s}{a} \int_{0}^{\infty} w(\lambda) \cos \frac{\lambda x}{a} d \lambda . \tag{8}
\end{equation*}
$$

It can be shown that the above expression for $w$ satisfies the following requirements: at the point where the load is applied, the deflection and moment are continuous, and the slope of the deflection curve vanishes. Furthermore, the deflection vanishes at infinity. Substituting Eqs. (7) and (8) in the differential equation (6a) one obtains the following relations. For $n=0$,

$$
\int_{0}^{\infty}\left\{w(\lambda)\left[\left(\frac{\lambda}{a}\right)^{8}+\frac{E h}{a^{2} D}\left(\frac{\lambda}{a}\right)^{4}\right]-\frac{q_{0}}{2 D} f(\lambda)\left(\frac{\lambda}{a}\right)^{4}\right\} \cos \frac{\lambda x}{a} d \lambda \equiv 0 ;
$$

therefore,

$$
w(\lambda)=\frac{\left(q_{0} / 2 D\right) f(\lambda)}{(\lambda / a)^{4}+E h / a^{2} D} .
$$

Similarly for $n=2,4, \cdots$,

$$
w(\lambda)=\frac{\left(q_{n} f(\lambda) / D\right)\left[(\lambda / a)^{2}+(n / a)^{2}\right]^{2}}{\left[(\lambda / a)^{2}+(n / a)^{2}\right]^{4}+\left(E h / a^{2} D\right)(\lambda / D)^{4}} .
$$

Hence, the solution of Eq. (6a) is


$$
\begin{align*}
w= & \frac{1}{2 D} \int_{0}^{\infty} \frac{q_{0} f(\lambda)}{(\lambda / a)^{4}+\left(E h / a^{2} D\right)} \cos \frac{\lambda x}{a} d \lambda \\
& +\frac{1}{D} \sum_{n=2,4}^{\infty} \cos \frac{n s}{a} \int_{0}^{\infty} \frac{q_{n} f(\lambda)\left[(\lambda / a)^{2}+(n / a)^{2}\right]^{2}}{\left[(\lambda / a)^{2}+(n / a)^{2}\right]^{4}+\left(E h / a^{2} D\right)(\lambda / a)^{4}} \cos \frac{\lambda x}{a} d \lambda . \tag{9}
\end{align*}
$$

It is next desired to find $q_{n}$ and $f(\lambda)$. In order to accomplish this the functions $q_{n}$ and $f(\lambda)$ must be determined from the loading condition. This is shown in Fig. 3. Since the cylinder is loaded symmetrically with respect to the generatrix and with respect to the circle passing through the origin, only the positive direction need be considered.


Fig. 3. Loading of the cylinder.
From Eq. (7),

$$
q\left(\frac{x}{a}\right)=\int_{0}^{\infty} f(\lambda) \cos \frac{\lambda x}{a} d \lambda, \quad f(\lambda)=\frac{1}{\pi} \int_{-\infty}^{\infty} q\left(\frac{x}{a}\right) \cos \frac{\lambda x}{a} d\left(\frac{x}{a}\right)
$$

and

$$
q\left(\frac{x}{a}\right)=1 \text { when }-\delta \leqq x \leqq \delta, \quad q\left(\frac{x}{a}\right)=0 \text { when } x>\delta \text { and } x<-\delta
$$

Therefore

$$
f(\lambda)=\frac{2}{\pi} \int_{0}^{b} \cos \frac{\lambda x}{a} d\left(\frac{x}{a}\right)=\frac{2}{\pi \lambda} \sin \lambda \frac{\delta}{a}
$$

Similarly $q_{n}$ can be determined from the expansion of the loading function along the circumference in a Fourier series. With

$$
q_{0}=\frac{2}{\pi} \int_{-\pi / 2}^{\pi / 2} q(z) d z,
$$

where $z=s / a$, and if $z=\pi / 2, s=\pi a / 2$, one obtains

$$
q_{0}=\frac{2}{\pi a} \int_{-c}^{c} q d s=\frac{4 c}{\pi a} q, \quad q_{n}=\frac{2}{\pi a} \int_{-c}^{c} q \cos n \frac{s}{a} d s=\frac{4 q}{\pi n} \sin n \frac{c}{a},
$$

where $c$ is as shown in Fig. 3. Substituting $q_{n}$ and $f(\lambda)$ in Eq. (9), one finds that

$$
\begin{aligned}
w= & \frac{1}{2 D} \int_{0}^{\infty} \frac{\left(8 q c / \pi^{2} a \lambda\right) \sin \lambda \delta / a}{(\lambda / a)^{4}+E h / a^{2} D} \cos \frac{\lambda x}{a} d \lambda \\
& +\frac{1}{D} \sum_{n=2,4}^{\infty} \cos \frac{n s}{a} \int_{0}^{\infty} \frac{\left(8 q / \pi^{2} n \lambda\right) \sin n c / a \sin \lambda \delta / a\left[(\lambda / a)^{2}+(n / a)^{2}\right]}{\left[(\lambda / a)^{2}+(n / a)^{2}\right]^{4}+E h / a^{2} D(\lambda / a)^{4}} \cos \frac{\lambda x}{a} d \lambda .
\end{aligned}
$$

Next, the case of a concentrated load applied at the origin may be considered. Such a load can be obtained by making the lengths $2 \delta$ and $2 c$ of the loaded portion infinitely small. Substituting

$$
P=4 q c \delta, \quad \sin \frac{\lambda \delta}{a} \approx \frac{\lambda \delta}{a}, \quad \sin \frac{n c}{a} \approx \frac{n c}{a}
$$

in the above equation, one obtains

$$
\begin{align*}
w= & \frac{P a^{2}}{D \pi^{2}} \int_{0}^{\infty} \frac{\cos \lambda(x / a) d \lambda}{\lambda^{4}+J^{2}} \\
& +\frac{2 P a^{2}}{\pi^{2} D} \sum_{n=2,4}^{\infty} \cos \frac{n s}{a} \int_{0}^{\infty} \frac{\left[\lambda^{2}+n^{2}\right]^{2} \cos (\lambda x / a) d \lambda}{\left[\lambda^{2}+n^{2}\right]^{4}+J^{2} \lambda^{4}} \tag{10}
\end{align*}
$$

where

$$
J^{2}=\frac{E h a^{2}}{D}=12\left(1-\nu^{2}\right)\left(\frac{a}{h}\right)^{2}
$$

In order to evaluate the definite integrals in Eq. (10) Cauchy's theorem of residues will be applied. Let us consider the integral

$$
\int_{0}^{\infty} \frac{\cos \lambda(x / a) d \lambda}{\lambda^{4}+J^{2}}
$$

where the characteristic equation $\lambda^{4}+J^{2}=0$ has four complex roots

$$
\lambda=J^{1 / 2}(-1)^{1 / 4}
$$

Cauchy's theorem yields

$$
\begin{equation*}
\int_{0}^{\infty} \frac{\cos \lambda(x / a) d \lambda}{\lambda^{4}+J^{2}}=\frac{\pi}{2 \sqrt{2}} J^{-3 / 2} e^{-\sqrt{J / 2}(x / a)}\left(\cos \sqrt{\frac{J}{2}} \frac{x}{a}+\sin \sqrt{\frac{J}{2}} \frac{x}{a}\right) \tag{11}
\end{equation*}
$$

The rational function in the integrand of second definite integral in Eq. (10) can be expressed in the form of partial fractions,

$$
\begin{align*}
\frac{\left(\lambda^{2}+n^{2}\right)^{2}}{\left[\lambda^{2}+n^{2}\right]^{4}+J^{2} \lambda^{4}}= & \frac{1}{2}\left\{\frac{1}{\left(\lambda^{2}+n^{2}\right)^{2}+i J \lambda^{2}}+\frac{1}{\left(\lambda^{2}+n^{2}\right)^{2}-i J \lambda^{2}}\right\} \\
= & \frac{1}{\alpha_{1}^{2}-\alpha_{2}^{2}} \frac{1}{\lambda^{2}-\alpha_{1}^{2}}+\frac{1}{\alpha_{2}^{2}-\alpha_{1}^{2}} \frac{1}{\lambda^{2}-\alpha_{2}^{2}} \\
& +\frac{1}{\alpha_{3}^{2}-\alpha_{4}^{2}} \frac{1}{\lambda^{2}-\alpha_{3}^{2}}+\frac{1}{\alpha_{4}^{2}-\alpha_{3}^{2}} \frac{1}{\lambda^{2}-\alpha_{4}^{2}} \tag{12}
\end{align*}
$$

where $\pm \alpha_{1}, \pm \alpha_{2}, \pm \alpha_{3}$ and $\pm \alpha_{4}$ are the roots of the denominator,

$$
\begin{align*}
\pm \alpha_{1}= & \pm \alpha_{4}^{*}= \pm A \pm i B \\
= & \pm \frac{1}{\sqrt{2}}\left[\sqrt{\left(-n^{2}+\eta\right)^{2}+\left(-\frac{J}{2}+\phi\right)^{2}-\left(n^{2}-\eta\right)}\right]^{\frac{1}{2}} \\
& \pm \frac{i}{\sqrt{2}}\left[\sqrt{\left(-n^{2}+\eta\right)^{2}+\left(-\frac{J}{2}+\phi\right)^{2}}+\left(n^{2}-\eta\right)\right]^{1} \\
\pm \alpha_{2}= & \pm \alpha_{3}^{*}= \pm C \mp i G  \tag{13}\\
= & \pm \frac{1}{\sqrt{2}}\left[\sqrt{\left(n^{2}+\eta\right)^{2}+\left(\frac{J}{2}+\phi\right)^{2}}-\left(n^{2}+\eta\right)\right]^{1} \\
& \pm \frac{i}{\sqrt{2}}\left[\sqrt{\left(n^{2}+\eta\right)^{2}+\left(\frac{J}{2}+\phi\right)+\left(n^{2}+\eta\right)}\right]^{1}
\end{align*}
$$

where the asterisk denotes the complex conjugate, and

$$
\begin{equation*}
\phi=\sqrt{\frac{1}{2}\left(R^{2}+\frac{1}{4} J^{2}\right),} \quad \eta=\sqrt{\frac{1}{2}\left(R_{2}-\frac{1}{4} J^{2}\right)}, \quad R_{2}=n^{2} J \sqrt{1+\left(J / 4 n^{2}\right)^{2}} . \tag{13a}
\end{equation*}
$$

Hence

$$
\begin{align*}
I= & \int_{-\infty}^{\infty} \frac{\left(\lambda^{2}+n^{2}\right)^{2} e^{i \lambda z / a} d \lambda}{\left(\lambda^{2}+n^{2}\right)^{4}+J^{2} \lambda^{4}} \\
= & \frac{2 \pi i}{8 R_{2}}\left\{\frac{\alpha_{2}}{\alpha_{1} \alpha_{2}}(\eta-i \phi) e^{i x \alpha_{1} / a}-\frac{\alpha_{1}}{\alpha_{1} \alpha_{2}}(\eta-i \phi) e^{i x \alpha_{2} / a}\right. \\
& \left.-\frac{\alpha_{4}}{\alpha_{3} \alpha_{4}}(\eta+i \phi) e^{i x \alpha_{3} / a}+\frac{\alpha_{3}}{\alpha_{3} \alpha_{4}}(\eta+i \phi) e^{i x \alpha_{4} / a}\right\} \tag{14}
\end{align*}
$$

Since

$$
\alpha_{1} \alpha_{2}=-n^{2}=\alpha_{3} \alpha_{4}
$$

Eq. (14) can be simplified as follows. Now

$$
\begin{align*}
& \int_{0}^{\infty} \frac{\left(\lambda^{2}+n^{2}\right)^{2} \cos x \lambda / a d \lambda}{\left(\lambda^{2}+n^{2}\right)^{4}+J^{2} \lambda^{4}} \\
&= \frac{\pi}{4 R_{2} n^{2}}\left\{\left[(\phi C+\eta G) \cos \frac{A x}{a}+(\phi G-\eta C) \sin \frac{A x}{a}\right] e^{-B x / a}\right. \\
&\left.\quad+\left[(\phi A-\eta B) \cos \frac{C x}{a}+(\eta A+\phi B) \sin \frac{C x}{a}\right] e^{-G x / a}\right\} \tag{15}
\end{align*}
$$

Simplifying the integrals (15) and (11) in Eq. (10), one obtains

$$
\begin{aligned}
\frac{w / h}{P / E h^{2}}= & \frac{\sqrt[4]{3\left(1-\nu^{2}\right)}}{2 \pi}\left(\frac{a}{h}\right)^{1 / 2}\left(\cos \sqrt{\frac{J}{2}} \frac{x}{a}+\sin \sqrt{\frac{J}{2}} \frac{x}{a}\right) e^{-\sqrt{J / 2}(x / a)} \\
& +\frac{\sigma\left(1-\nu^{2}\right)}{\pi}\left(\frac{a}{h}\right)^{2} \sum_{n=2,4}^{\infty} \frac{\cos n s / a}{R_{2} n^{2}}\left\{\left[(\phi C+\eta G) \cos \frac{A x}{a}\right.\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.+(\phi G-\eta C) \sin \frac{A x}{a}\right] e^{-B x / a}+\left[(\phi A-\eta B) \cos \frac{C x}{a}\right. \\
& \left.\left.+(\eta A+\phi B) \sin \frac{C x}{a}\right] e^{-G x / a}\right\} \tag{16}
\end{align*}
$$

It is seen that the first term of the above expression is very small as compared to the second term, and therefore can be neglected without appreciable error. For a


Fig. 4
certain value of the $a / h$ ratio, $G$ is found to be very large as compared to $B$. The terms containing $e^{-a x / a}$ can then be completely neglected, provided that $x / a$ is not near zero.

In the case when $x / a=0 \mathrm{Eq}$. (16) can be simplified as follows:

$$
\begin{equation*}
\left[\frac{w / h}{P / E h^{2}}\right]_{x / a=0}=\frac{3 \sqrt{2}\left(1-\nu^{2}\right)}{\pi}\left(\frac{a}{h}\right)^{2} \sum_{n=2,4}^{\infty} \frac{\cos n s / a}{n^{3}} \frac{\sqrt{1+\Xi}}{\Xi} \tag{17}
\end{equation*}
$$

where

$$
z^{2}=1+\frac{3\left(1-\nu^{2}\right) a^{2}}{4 n^{4} h^{2}}
$$

The left side of (17) has a maximum at $s=0$. Figure 4 shows the variation of this maximum with the ratio $a / h$. Figure 5 shows the variation of $w$ along the generatrix through a point of loading, and Fig. 6 shows the projection of lines of constant $w$ on
the plane through the axis of the cylinders and perpendicular to the line of action of the two forces $P$.
4. A cylinder of finite length loaded with two equal and opposite forces. The expression for the radial deflection in a thin cylinder of finite length can be obtained from Eq. (16) by using the method of images.* If one imagines the cylinder of finite length prolonged in both the positive and the negative $x$-directions, and loaded with a series of forces, $P$, of alternating sense, applied along the generatrix ( $s / a=0$ ) at a distance $l$ from one another (see Fig. 7), then the deflections of the infinite cylinder


Fig. 5
are evidently equal to zero at a distance $l / 2$ from the applied loads $P$. Hence one may consider the given cylinder of length $l$ and radius $a$ as a portion of the infinitely long cylinder loaded as shown in Fig. 7. From Eq. (16) one finds that the deflection of any point, $\beta$, (at a distance $\zeta$ from the $s$-axis) on the shell due to the load $P$ acting at the center is

$$
\left.\begin{array}{c}
w_{a}=\frac{P a^{2}}{2 \pi D} \sum_{\pi=2,4}^{\infty} \frac{\cos n s / a}{R_{2} n}\left\{\left[(\phi C+\eta G) \cos A \frac{\zeta}{a}+(\phi G-\eta C) \sin A \frac{\zeta}{a}\right] e^{-B \zeta / a}\right. \\
+ \tag{18a}
\end{array}\left[(\phi A-\eta B) \cos C \frac{\zeta}{a}+(\eta A+\phi B) \sin C \frac{\zeta}{a}\right] e^{-G \zeta / a}\right\} .
$$

The deflection produced by two adjacent forces a distance $l$ apart is

[^9]\[

$$
\begin{align*}
w_{b}= & -\frac{P a^{2}}{2 \pi D} \sum_{n=2,4}^{\infty} \frac{\cos n \zeta / a}{R_{2} n}\left\{\left[(\phi C+\eta G) \cos A \frac{l-\zeta}{a}\right.\right. \\
& \left.+(\phi G-\eta C) \sin A \frac{l-\zeta}{a}\right] e^{-B(l-\zeta) / a}+\left[(\phi C+\eta G) \cos A \frac{l+. \zeta}{a}\right. \\
& \left.\left.+(\phi G-\eta C) \sin A \frac{l+\zeta}{a}\right] e^{-B(l+b) / a}\right\} \tag{18b}
\end{align*}
$$
\]

Since the terms containing $e^{-G(1+5) / a}$ are all small compared to the other terms, they


Fig. 6


Frg. 7. Series of equidistant opposite forces acting on an infinitely long cylinder.
can be neglected without causing appreciable error. One obtains similarly $w_{c}, w_{d}, \cdots$. The total radial deflection at any point $\beta$ is given by the sum

$$
\begin{aligned}
w= & w_{a}+w_{b}+w_{c}+\cdots \\
= & \frac{P_{a} a^{2}}{2 \pi D} \sum_{n=2,4}^{\infty} \frac{\cos n s / a}{R_{2} n}\left\{\left[(\phi A-\eta B) \cos C \frac{\zeta}{a}+(\eta A+\phi B) \sin C \frac{\zeta}{a}\right] e^{-(\sigma \zeta / a}\right. \\
& +(\phi C+\eta G)\left[\cos A \frac{\zeta}{a} e^{-B \zeta / a}-2 \cos A \frac{\zeta}{a} \cosh B \frac{\zeta}{a}\left(\cos A \frac{l}{a} e^{-B l / a}\right.\right. \\
& \left.-\cos \frac{2 A l}{a} e^{-2 B l / a}+\cos 3 A \frac{l}{a} e^{-3 B l / a}-\cdots\right)
\end{aligned}
$$

the plane through the axis of the cylinders and perpendicular to the line of action of the two forces $P$.
4. A cylinder of finite length loaded with two equal and opposite forces. The expression for the radial deflection in a thin cylinder of finite length can be obtained from Eq. (16) by using the method of images.* If one imagines the cylinder of finite length prolonged in both the positive and the negative $x$-directions, and loaded with a series of forces, $P$, of alternating sense, applied along the generatrix $(s / a=0)$ at a distance $l$ from one another (see Fig. 7), then the deflections of the infinite cylinder


Fig. 5
are evidently equal to zero at a distance $l / 2$ from the applied loads $P$. Hence one may consider the given cylinder of length $l$ and radius $a$ as a portion of the infinitely long cylinder loaded as shown in Fig. 7. From Eq. (16) one finds that the deflection of any point, $\beta$, (at a distance $\zeta$ from the $s$-axis) on the shell due to the load $P$ acting at the center is

$$
\begin{gather*}
w_{a}=\frac{P a^{2}}{2 \pi D} \sum_{n=2,4}^{\infty} \frac{\cos n \zeta / a}{R_{2} n}\left\{\left[(\phi C+\eta G) \cos A \frac{\zeta}{a}+(\phi G-\eta C) \sin A \frac{\zeta}{a}\right] e^{-B \zeta / a}\right. \\
+  \tag{18a}\\
\left.+\left[(\phi A-\eta B) \cos C \frac{\zeta}{a}+(\eta A+\phi B) \sin C \frac{\zeta}{a}\right] e^{-G \zeta / a}\right\} .
\end{gather*}
$$

The deflection produced by two adjacent forces a distance $l$ apart is

[^10]\[

$$
\begin{align*}
w_{b}= & -\frac{P a^{2}}{2 \pi D} \sum_{n=2,4}^{\infty} \frac{\cos n s / a}{R_{2} n}\left\{\left[(\phi C+\eta G) \cos A \frac{l-\zeta}{a}\right.\right. \\
& \left.+(\phi G-\eta C) \sin A \frac{l-\zeta}{a}\right] e^{-B(l-\delta) / a}+\left[(\phi C+\eta G) \cos A \frac{l+. \zeta}{a}\right. \\
& \left.\left.+(\phi G-\eta C) \sin A \frac{l+\zeta}{a}\right] e^{-B(l+b) / a}\right\} \tag{18b}
\end{align*}
$$
\]

Since the terms containing $e^{-a(1+5) / a}$ are all small compared to the other terms, they


Fig. 6


Fig. 7. Series of equidistant opposite forces acting on an infinitely long cylinder.
can be neglected without causing appreciable error. One obtains similarly $w_{c}, w_{d}, \cdots$. The total radial deflection at any point $\beta$ is given by the sum

$$
\begin{aligned}
w= & w_{a}+w_{b}+w_{c}+\cdots \\
= & \frac{P_{a}^{2}}{2 \pi D} \sum_{n=2,4}^{\infty} \frac{\cos n \zeta / a}{R_{2} n}\left\{\left[(\phi A-\eta B) \cos C \frac{\zeta}{a}+(\eta A+\phi B) \sin C \frac{\zeta}{a}\right] e^{-a \zeta / a}\right. \\
& +(\phi C+\eta G)\left[\cos A \frac{\zeta}{a} e^{-B \zeta / a}-2 \cos A \frac{\zeta}{a} \cosh B \frac{\zeta}{a}\left(\cos A \frac{l}{a} e^{-B l / a}\right.\right. \\
& \left.-\cos \frac{2 A l}{a} e^{-2 B l / a}+\cos 3 A \frac{l}{a} e^{-3 B l / a}-\cdots\right)
\end{aligned}
$$

$\left.-2 \sin A \frac{\zeta}{a} \sinh B \frac{\zeta}{a}\left(\sin A \frac{l}{a} e^{-B L / a}-\sin 2 A \frac{l}{a} e^{-2 A / / a}+\cdots\right)\right]$
$+(\phi G-\eta C)\left[\sin A \frac{\zeta}{a} e^{-B \zeta / a}-2 \cos A \frac{\zeta}{a} \cosh B \frac{\zeta}{a}\left(\sin A \frac{l}{a} e^{-B \hbar / a}\right.\right.$
$\left.-\sin 2 A \frac{l}{a} e^{-2 B l / a}+\sin 3 A \frac{l}{a} e^{-3 B U / a}-\cdots\right)$
$\left.\left.+2 \sin A \frac{\zeta}{a} \sinh B \frac{\zeta}{a}\left(\cos A \frac{l}{a} e^{-B l / a}-\cos 2 A \frac{l}{a} e^{-2 B l / a}+\cdots\right)\right]\right\}$.
We sum the series in the above expression, obtaining
$\sum_{m=1,3}^{\infty} e^{-m B l / a} \cos m A \frac{l}{a}=\frac{1}{2} \sum_{m=1,3}^{\infty}\left[e^{-m(B-i A) l / a}+e^{-m(B+i A) l / a}\right]$

$$
=\frac{1}{2}\left[\frac{\sinh (B l / a) \cos (A l / a)}{\sinh ^{2}(B l / a) \cos ^{2}(A l / a)+\cosh ^{2}(B l / a) \sin ^{2}(A l / a)}\right]
$$

$\sum_{m=2,4}^{\infty} e^{-m B l / a} \cos m A \frac{l}{a}=\frac{e^{-l B / a}}{2} \frac{\left[\sin (B l / a) \cos ^{2}(A l / a)-\cosh (B l / a) \sin ^{2}(A l / a)\right]}{\left[\sin ^{2}(A l / a) \cosh ^{2}(B l / a)+\cos ^{2}(A l / a) \sinh ^{2}(B l / a)\right]}$,
$\sum_{m=1,3}^{\infty} e^{-m B l / a} \sin m A \frac{l}{a}=\frac{1}{2 i} \sum_{m=1,3}^{\infty}\left[e^{-m(B-i A) l / a}-e^{-m(B+i A) l / a}\right]$

$$
=\frac{1}{2}\left[\frac{\cosh (B l / a) \sin (A l / a)}{\sinh ^{2}(B l / a) \cos ^{2}(A l / a)+\cosh ^{2}(B l / a) \sin ^{2}(A l / a)}\right],
$$

$\sum_{m=2,4}^{\infty} e^{-m B l / a} \sin m A \frac{l}{a}=\frac{e^{-B l / a} \cos (A l / a) \sin (A l / a)[\sinh (B l / a)+\cosh (B l / a)]}{2\left[\sinh ^{2}(B l / a) \cos ^{2}(A l / a)+\cosh ^{2}(B l / a) \sin ^{2}(A l / a)\right]}$.
Thus Eq. (19) is reduced to
$\frac{w / h}{P / E h^{2}}=\frac{6\left(1-\nu^{2}\right)}{\pi}\left(\frac{a}{h}\right)^{2} \sum_{n=2,4}^{\infty} \frac{\cos n s / a}{R_{2} n^{2}}\left\{\left[(\phi C+\eta G) \cos \frac{\zeta}{a} \Lambda\right.\right.$
$\left.+(\phi G-\eta C) \sin A \frac{\zeta}{a}\right] e^{-B \zeta / a}$
$+\left[(\phi A-\eta B) \cos C \frac{\zeta}{a}+(\eta A+\phi B) \sin C \frac{\zeta}{a}\right] e^{-0 \zeta / a}$
$+\frac{\sinh (B l / a) \cos (A l / a)-e^{-B l / a}\left[\sinh (B l / a) \cos ^{2}(A l / a)-\cosh (B l / a) \sin ^{2}(A l / a)\right]}{\sinh ^{2}(B l / a) \cos ^{2}(A l / a)+\cosh ^{2}(B l / a) \sin ^{2}(A l / a)}$
$\times\left[(\phi G-\eta C) \sin A \frac{\zeta}{a} \sinh B \frac{\zeta}{a}-(\phi C+\eta G) \cos A \frac{\zeta}{a} \cosh B \frac{\zeta}{a}\right]$
$-\cosh (B l / a) \sin (A l / a)-e^{-B l / a} \cos (A l / a) \sin (A l / a)[\sinh (B l / a)+\cosh (B l / a)]$
$\sinh ^{2}(B l / a) \cos ^{2}(A l / a)+\cosh ^{2}(B l / a) \sin ^{2}(A l / a)$
$\left.\times\left[(\phi G-\eta C) \cos A \frac{\zeta}{a} \cosh B \frac{\zeta}{a}+(\phi C+\eta G) \sin A \frac{\zeta}{a} \sinh B \frac{\zeta}{a}\right]\right\}$.

It is obvious that the first two terms of Eq. (20) are equivalent to the solution of the infinitely long cylinder given by Eq. (16). The remaining terms are evidently the correction factors due to the restrained edges at the two ends of the cylinder of finite length. The radial deflection under the applied force can be obtained by putting $\zeta / a=0$,

$$
\begin{align*}
\frac{w / h}{\left(P / E h^{2}\right)}= & \frac{6\left(1-\nu^{2}\right)}{\pi}\left(\frac{a}{h}\right)^{2} \sum_{n-2,4 \ldots}^{\infty} \frac{\cos n s / a}{R_{2} \eta^{2}}\{(\phi C+\eta G)+(\phi A-\eta B) \\
& -(\phi C+\eta G) \frac{\sinh (B l / a) \cos (A l / a)-e^{-B l / a\left[\sinh (B l / a) \cos ^{2}(A l / a)-\cosh (B l / a) \sin ^{2}(A l / a)\right]}}{\sinh ^{2}(B l / a) \cos ^{2}(A l / a)+\cosh ^{2}(B l / a) \sin ^{2}(A l / a)} \\
& -(\phi C-\eta C) \frac{\cosh (B l / a) \sin (A l / a)-e^{-B l / a} \cos (A l / a) \sin (A l / a)[\sinh (B l / a)+\cosh (B l / a)]}{\sinh ^{2}(B l / a) \cos ^{2}(A l / a)+\cosh ^{2}(B l / a) \sin ^{2}(A l / a)} \\
& -(\phi A-\eta B) \frac{\sinh (G l / a) \cos \left(C l / a-e^{-a l l a}\left[\sinh (G l / a) \cos ^{2}(C l / a)-\cosh (G l / a) \sin ^{2}(C l / a)\right]\right.}{\sinh ^{2}(G l / a) \cos ^{2}(C l / a)+\cosh ^{2}(G l / a) \sin ^{2}(C l / a)} \\
& \left.-(\eta A+\phi B) \frac{\cosh (G l / a) \sin (C l / a)-e^{-G l / a} \cos (C l / a) \sin (C l / a)[\sinh (G l / a)+\cosh (D l / a)]}{\sinh ^{2}(G l / a) \cos ^{2}(C l / a)+\cosh ^{2}(G l / a) \sin ^{2}(C l / a)}\right\} . \tag{21}
\end{align*}
$$

Some applications of the solution of the problem of the infinitely long cylinder. The problems of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference can be analyzed by using the solution given by Eq. (16) for a single load. The action of the couple is equivalent to that of the two forces $P$ shown in Fig. 8, where $\lim _{\Delta x \rightarrow 0} P \Delta x=T_{c}$.


Fig. 8. Two couples acting on an infinitely long cylinder.
It is easy to see that the deflection for the case when the force $P$ is applied at the point $O_{1}$, at a distance $\Delta x$ from the origin, can be obtained from the deflection $w$, given in Eq. (16), by writing $x-\Delta x$ instead of $x$ and also $-P$ instead of $P$. This and the original $w$ are then added. The radial deflection due to the two equal and opposite forces applied at $O$ and $O_{1}$ is now obtained in the form

$$
-w_{T}=w(x, s)-w(x-\Delta x, s)
$$

When $\Delta x$ is very small, this approaches the value

$$
w_{T}=\frac{d w(x, s)}{d x} \Delta x .
$$

As $T_{c}$ is the moment of the applied torque and is equal to $P \Delta x$, the radial deflection due to this torque is

$$
\begin{equation*}
w_{T_{1}}=\frac{T_{c}}{P} \frac{d w}{d x}, \tag{22}
\end{equation*}
$$

where $w$ is the radial deflection due to the concentrated load $P$.

For the radial deflection due to the couple acting along the circumferential direction one finds similarly (Fig. 8) that

$$
\begin{equation*}
w_{r_{z}}=\frac{T_{c}}{P} \frac{d w}{d s} . \tag{23}
\end{equation*}
$$

Substituting $w$ from Eq. (16) in Eqs. (22) and (23) one obtains for the couple acting along the circumferential direction,

$$
\begin{align*}
\frac{w_{T, ~} / h}{T_{c} / E h^{2}}= & \frac{6\left(1-\nu^{2}\right)}{\pi}\left(\frac{a}{h}\right)^{2} \sum_{n=2,4}^{\infty} \frac{\cos \eta s / a}{R_{2} n^{2}}\left\{e^{-B x / a} \cos A x / a[A(\phi G-\eta C)-B(\phi C+\eta G)]\right. \\
& -e^{-B x / a} \sin (A x / a)[(\phi C+\eta G) A+B(\phi G-\eta C)] \\
& +e^{-G x / a} \cos (C x / a)[C(\eta A+\phi B)-G(\phi A-\eta B)] \\
& \left.-e^{-G x / a} \sin (C x / a)[C(\phi A-\eta B)+G(\eta A+\phi B)]\right\} \tag{24}
\end{align*}
$$

while for the couple acting along the generatrix direction,

$$
\begin{align*}
\frac{w_{T_{2}} / h}{T_{c} / E h^{2}}= & -\frac{6\left(1-\nu^{2}\right)}{\pi}\left(\frac{a}{h}\right)^{2} \sum_{n=2,4 \ldots}^{\infty} \frac{\sin n s / a}{R_{2} n}\{[(\phi C+\eta G) \cos (A x / a) \\
& +(\phi G-\eta C) \sin (A x / a)] e^{-B z / a}+[(\phi A-\eta B) \cos (C x / a) \\
& \left.+(\eta A+\phi B) \sin (C x / a)] e^{-G x / a}\right\} . \tag{25}
\end{align*}
$$

In the case when $x / a=0$,

$$
\frac{w_{r_{1}} / h}{T_{c} / E h^{2}}=0, \quad \text { at any } s / a
$$

Hence the condition that the slope of the deflection curve $d w / d x$ must vanish under the concentrated load ( $x / a=0$ ) is satisfied.

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## References

[^11]
## THERMAL DEFLECTIONS OF ANISOTROPIC THIN PLATES*

BY<br>WILLIAM H. PELL**<br>Bell Aircrafl Corporation

1. Introduction. Equations governing the deflection of an isotropic thin elastic plate subjected to a temperature distribution of the form

$$
\begin{equation*}
T(x, y, z)=T_{0}(x, y)+z T_{1}(x, y), \tag{1.1}
\end{equation*}
$$

where the neutral plane of the plate is taken to lie in the $x y$-plane, have been derived by Nádai. ${ }^{1}$ He did not consider the solution of these equations, and they have not been treated to any considerable extent by subsequent writers in thermo-elasticity. ${ }^{2}$ The isothermal theory of anisotropic thin elastic plates has been developed principally by Boussinesq, ${ }^{3}$ Voigt, ${ }^{4}$ and Lechnitzky. ${ }^{5}$ It appears that the only treatment of thermal effects for the anisotropic plate is due to Voigt, ${ }^{6}$ who considers a simple case in which no bending of the plate occurs.

The first part of this paper is concerned with the derivation of two partial differential equations governing the deflection of a thin elastic plate possessing one plane of elastic symmetry parallel to the faces of the plate, and subjected to a temperature distribution described by a function of the form (1.1). One of these equations, with suitable boundary conditions, defines a stress function $F$; the other, the deflection function $w$. In the second part, recent results in anisotropic plate theory are used to solve the equation for the stress function with rather general boundary conditions for the case where $T_{0}(x, y)$ is a polynomial in $x$ and $y$. The problem of solving the equation of the deflection is a difficult one, and a solution valid throughout the region enclosed by the plate is not available. Since the thermal deflection problem for the isotropic plate is of interest in itself, however, the case of the isotropic circular plate with radial temperature distribution is considered in the concluding portion, and the solution is obtained.
2. The thermo-elastic equations for anisotropic plates. Let us consider a thin elastic plate composed of a medium possessing at each point at least one plane of elastic symmetry parallel to the middle plane of the plate, which is chosen to lie in the $x y$-plane. Let the plate be subjected in its interior to a temperature distribution given by (1.1). The plate is supposed acted upon by forces on its edge lying in the middle plane, but to be frec of lateral load and body forces.

[^12]The Kirchhoff assumptions of the thin plate theory lead to the well-known fundamental relations

$$
\begin{gather*}
u=-z \frac{\partial w}{\partial x}, \quad v=-z \frac{\partial w}{\partial y},  \tag{2.1}\\
\left|\tau_{z z}\right| \ll \max \left\{\left|\tau_{x x}\right|,\left|\tau_{y y}\right|,\left|\tau_{x y}\right|\right\}, \tag{2.2}
\end{gather*}
$$

valid throughout the thickness, $2 h$, of the plate, where $w$ is the deflection of the middle surface of the plate, and $u$ and $v$ are the displacements of a point $(x, y, z)$ of the plate in the $x$ - and $y$-directions, respectively. To the assumptions (2.1) and (2.2) is adjoined the following one: the stress tensor $\tau$ at any point in the plate is the sum

$$
\begin{equation*}
\tau=\tau^{0}+\tau^{1} \tag{2.3}
\end{equation*}
$$

where $\tau^{0}$ is a plane stress tensor generated by $T_{0}(x, y)$ and reactions at the plate edge. and $\tau^{1}$ arises from the bending of the plate, i.e., from the action of $z T_{1}$. It should be remarked that this supposition is fundamental in the thermo-elastic theory of thin plates presented here. Together with (2.1) and (2.2) it serves here the same purpose that (2.1) and (2.2) do alone in the isothermal theory, i.e., they reduce the thermoelastic plate problem to one two-dimensional in character.

The generalized Hooke's law ${ }^{7}$ taking into account thermal effects is expressed by

$$
\begin{align*}
\frac{\partial u}{\partial x} & =a_{11} \tau_{x x}+a_{12} \tau_{y y}+a_{13} \tau_{z z}+a_{16} \tau_{x y}+a_{1} T \\
\frac{\partial v}{\partial y} & =a_{12} \tau_{x x}+a_{22} \tau_{y y}+a_{23} \tau_{z z}+a_{26} \tau_{x y}+a_{2} T \\
\frac{\partial w}{\partial z} & =a_{13} \tau_{x x}+a_{23} \tau_{y y}+a_{33} \tau_{z z}+a_{36} \tau_{x y}+a_{3} T  \tag{2.4}\\
\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} & =a_{44} \tau_{y z}+a_{45} \tau_{x z}+2 a_{4} T, \\
\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} & =a_{45 \tau_{y z}}+a_{55} \tau_{x z}+2 a_{5} T, \\
\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & =a_{16} \tau_{x x}+a_{26} \tau_{y y}+a_{36} \tau_{z z}+a_{86} \tau_{x y}+2 a_{6} T,
\end{align*}
$$

where the $a_{i j}$ are elastic constants and $a_{i}$ are the coefficients of thermal expansion of thermal expansion of the medium. For the plane stress system ${ }^{8}$ (2.4) reduces to

$$
\begin{align*}
\frac{\partial u^{0}}{\partial x} & =a_{11} \tau_{x x}^{0}+a_{12} \tau_{y y}^{0}+a_{16} \tau_{x y}^{0}+a_{1} T_{0} \\
\frac{\partial v^{0}}{\partial y} & =a_{12} \tau_{x x}^{0}+a_{22} \tau_{y y}^{0}+a_{26} \tau_{x y}^{0}+a_{2} T_{0}  \tag{2.5}\\
\frac{\partial u^{0}}{\partial y}+\frac{\partial v^{0}}{\partial x} & =a_{16} \tau_{x x}^{0}+a_{26} \tau_{y y}^{0}+a_{66} \tau_{x y}^{0}+2 a_{6} T_{0}
\end{align*}
$$

[^13]With the introduction of a stress function $F(x, y)$, one has

$$
\begin{equation*}
\tau_{x x}^{0}=\frac{\partial^{2} F}{\partial y^{2}}, \quad \tau_{x y}^{0}=-\frac{\partial^{2} F}{\partial x \partial y}, \quad \tau_{y y}^{0}=\frac{\partial^{2} F}{\partial x^{2}} . \tag{2.6}
\end{equation*}
$$

The resultants of these stresses acting across the thickness of the plate are, respectively,

$$
N_{x x}=2 h \tau_{x x}^{0}, \quad N_{x y}=2 h \tau_{x y}^{0}, \quad N_{y v}=2 h \tau_{y v}^{0} .
$$

- Since the displacements must satisfy a compatibility condition, it follows from (2.5) and (2.6) that $F$ must satisfy the equation

$$
\begin{align*}
a_{22} \frac{\partial^{4} F}{\partial x^{4}}-2 a_{26} \frac{\partial^{4} F}{\partial x^{3} \partial y}+\left(2 a_{12}+a_{66}\right) & \frac{\partial^{4} F}{\partial x^{2} \partial y^{2}}-2 a_{16} \frac{\partial^{4} F}{\partial x \partial y^{3}}+a_{11} \frac{\partial^{4} F}{\partial y^{4}} \\
& =-\left\{a_{2} \frac{\partial^{2} T_{0}}{\partial x^{2}}-2 a_{0} \frac{\partial^{2} T_{0}}{\partial x \partial y}+a_{1} \frac{\partial^{2} T_{0}}{\partial y^{2}}\right\} . \tag{2.7}
\end{align*}
$$

It is assumed that the derivatives appearing in (2.7) are continuous.
The generalized Hooke's law for the strains and stresses associated with the temperature function $z T_{1}$ permits one to write

$$
\begin{align*}
\tau_{z x=}^{1}= & c_{11}\left(\frac{\partial u^{1}}{\partial x}-a_{1 z} T_{1}\right)+c_{12}\left(\frac{\partial v_{1}}{\partial y}-a_{2 z} T_{1}\right)+c_{13}\left(\frac{\partial w^{1}}{\partial z}-a_{3} z T_{1}\right) \\
& +c_{10}\left(\frac{1}{2}\left[\frac{\partial u^{1}}{\partial y}+\frac{\partial v_{1}}{\partial x}\right]-a_{6} T_{1}\right), \\
\tau_{y y}^{1}= & c_{12}\left(\frac{\partial u^{1}}{\partial x}-a_{1 z} T_{1}\right)+c_{22}\left(\frac{\partial v^{1}}{\partial y}-a_{2 z} T_{1}\right)+c_{23}\left(\frac{\partial w^{1}}{\partial z}-a_{3 z} T_{1}\right) \\
& +c_{26}\left(\frac{1}{2}\left[\frac{\partial u^{1}}{\partial y}+\frac{\partial v^{1}}{\partial x}\right]-a_{6 z} T_{1}\right),  \tag{2.8}\\
\tau_{z z}^{1}= & c_{13}\left(\frac{\partial u^{1}}{\partial x}-a_{1} z T_{1}\right)+c_{23}\left(\frac{\partial v^{1}}{\partial y}-a_{2 z} T_{1}\right)+c_{33}\left(\frac{\partial w^{1}}{\partial z}-a_{3 z} T_{1}\right) \\
& +c_{36}\left(\frac{1}{2}\left[\frac{\partial u^{1}}{\partial y}+\frac{\partial v^{1}}{\partial x}\right]-a_{6 z} T_{1}\right), \\
\tau_{x y}^{1}= & c_{18}\left(\frac{\partial u^{1}}{\partial x}-a_{1 z} T_{1}\right)+c_{26}\left(\frac{\partial v^{1}}{\partial y}-a_{2 z} T_{1}\right)+c_{36}\left(\frac{\partial w^{1}}{\partial z}-a_{3 z} T_{1}\right) \\
& +c_{68}\left(\frac{1}{2}\left[\frac{\partial u^{1}}{\partial y}+\frac{\partial v^{1}}{\partial x}\right]-a_{6 z} T_{1}\right) .
\end{align*}
$$

The assumption (2.1) gives

$$
\begin{equation*}
\frac{\partial u^{1}}{\partial x}=-z \frac{\partial^{2} w}{\partial x^{2}}, \quad \frac{\partial v^{1}}{\partial y}=-z \frac{\partial^{2} w}{\partial y^{2}}, \quad \frac{1}{2}\left(\frac{\partial u^{1}}{\partial y}+\frac{\partial v^{1}}{\partial x}\right)=-z \frac{\partial^{2} w}{\partial x \partial y}, \tag{2.9}
\end{equation*}
$$

and (2.9) applied to the third equation of (2.8) yields

$$
\begin{align*}
\frac{\partial w}{\partial z}-a_{3} z T_{1}= & -\frac{z}{c_{33}}\left\{c_{13}\left(\frac{\partial^{2} w}{\partial x^{2}}-a_{1} T_{1}\right)+c_{23}\left(\frac{\partial^{2} w}{\partial y^{2}}-a_{2} T_{1}\right)\right. \\
& \left.+c_{36}\left(\frac{\partial^{2} w}{\partial x \partial y}-a_{6} T_{1}\right)\right\} . \tag{2.10}
\end{align*}
$$

Now (2.9) and (2.10) are inserted in (2.8) with the result

$$
\begin{align*}
& \tau_{x x}^{1}=-z\left(b_{11} \frac{\partial^{2} w}{\partial x^{2}}+b_{12} \frac{\partial^{2} w}{\partial y^{2}}+b_{16} \frac{\partial^{2} w}{\partial x \partial y}+\alpha_{1} T_{1}\right), \\
& \tau_{y y}^{1}=-z\left(b_{12} \frac{\partial^{2} w}{\partial x^{2}}+b_{22} \frac{\partial^{2} w}{\partial y^{2}}+b_{26} \frac{\partial^{2} w}{\partial x \partial y}+\alpha_{2} T_{1}\right),  \tag{2.11}\\
& \tau_{x y}^{1}=-z\left(b_{16} \frac{\partial^{2} w}{\partial x^{2}}+b_{26} \frac{\partial^{2} w}{\partial y^{2}}+b_{66} \frac{\partial^{2} w}{\partial x \partial y}+\alpha_{6} T_{1}\right),
\end{align*}
$$

where

$$
\begin{aligned}
b_{i k}=c_{i k}-\frac{c_{i 3} c_{k 3}}{c_{33}}, & i, k=1,2, \text { o } \\
\alpha_{i}=b_{1 i} a_{1}+b_{2 i} a_{2}+b_{\mathrm{ci}} a_{6}, & i=1,2,6
\end{aligned}
$$

Noting $\left.\left.\tau_{v z}^{1}\right]_{z= \pm h}=\tau_{x z}^{1}\right]_{z= \pm h}=0$, we may now integrate, with respect to $z$, the equations of equilibrium

$$
\frac{\partial \tau_{x x}^{1}}{\partial x}+\frac{\partial \tau_{x y}^{1}}{\partial y}+\frac{\partial \tau_{x z}^{1}}{\partial z}=0, \quad \frac{\partial \tau_{x y}^{1}}{\partial x}+\frac{\partial \tau_{y v}^{1}}{\partial y}+\frac{\partial \tau_{y z}^{1}}{\partial z}=0
$$

obtaining

$$
\begin{align*}
\tau_{x z}^{1}= & \frac{z^{2}-h^{2}}{2}\left[b_{11} \frac{\partial^{3} w}{\partial x^{3}}+2 b_{16} \frac{\partial^{3} w}{\partial x^{2} \partial y}+\left(b_{12}+b_{66}\right) \frac{\partial^{3} w}{\partial x \partial y^{2}}\right. \\
& \left.+b_{26} \frac{\partial^{3} w}{\partial y^{3}}+\alpha_{1} \frac{\partial T_{1}}{\partial x}+\alpha_{6} \frac{\partial T_{1}}{\partial y}\right], \\
\tau_{y z}^{1}= & \frac{z^{2}-h^{2}}{2}\left[b_{16} \frac{\partial^{3} w}{\partial x^{3}}+\left(b_{12}+b_{66}\right) \frac{\partial^{3} w}{\partial x^{2} \partial y}+2 b_{26} \frac{\partial^{3} w}{\partial x \partial y^{2}}\right.  \tag{2.12}\\
& \left.+b_{22} \frac{\partial^{3} w}{\partial y^{3}}+\alpha_{6} \frac{\partial T_{1}}{\partial x}+\alpha_{2} \frac{\partial T_{1}}{\partial y}\right] .
\end{align*}
$$

The resultants of these stresses acting across the thickness of the plate are, respectively,

$$
Q_{x}=\int_{-h}^{h} \tau_{x z} d z, \quad Q_{y}=\int_{-h}^{h} \tau_{y z} d z .
$$

The following equation expressing the condition of statical equilibrium of an arbi-
trary element of the plate may be obtained in the usual way ${ }^{9}$ since the derivation does not depend on the material composing the plate:

$$
\frac{\partial Q_{x}}{\partial x}+\frac{\partial Q_{y}}{\partial y}+N_{x x} \frac{\partial^{2} w}{\partial x^{2}}+2 N_{x y} \frac{\partial^{2} w}{\partial x \partial y}+N_{y y} \frac{\partial^{2} w}{\partial y^{2}}=0 .
$$

The values of $Q_{x}$ and $Q_{y}$ obtained from (2.12) are now introduced, and the result is the following differential equation for the deflection:

$$
\begin{align*}
b_{11} \frac{\partial^{4} w}{\partial x^{4}}+3 b_{18} \frac{\partial^{4} w}{\partial x^{3} \partial y}+2\left(b_{12}\right. & \left.+b_{66}\right) \frac{\partial^{4} w}{\partial x^{2} \partial y^{2}}+3 b_{26} \frac{\partial^{4} w}{\partial x \partial y^{4}}+b_{22} \frac{\partial^{4} w}{\partial y^{4}} \\
= & -\left\{\alpha_{1} \frac{\partial^{2} T_{1}}{\partial x^{2}}+2 \alpha_{6} \frac{\partial^{2} T_{1}}{\partial x \partial y}+\alpha_{2} \frac{\partial^{2} T_{1}}{\partial y^{2}}\right\} \\
& +\frac{3}{2 h^{3}}\left\{N_{x x} \frac{\partial^{2} w}{\partial x^{2}}+2 N_{x y} \frac{\partial^{2} w}{\partial x \partial y}+N_{y y} \frac{\partial^{2} w}{\partial y^{2}}\right\} \tag{2.13}
\end{align*}
$$

Again, the continuity of the derivatives appearing is assumed.
The portion of the $x y$-plane occupied by the middle plane of the plate will be called $S_{0}$; the boundary of $S_{0}$ will be called $C_{0}$, and it will be assumed that $C_{0}$ is an analytic curve. The problem of thermo-elastic deflection is solved if a solution for each of (2.7) and (2.13) valid throughout $S_{0}$ can be found which satisfies appropriate boundary conditions on $C_{0}$.
3. The stress function. Since derivatives of $F$ appear in (2.13), the solution of (2.7) will be considered first. Lechnitzky ${ }^{10}$ has shown that the roots of the characteristic equation

$$
\begin{equation*}
a_{11} \mu^{4}-2 a_{1 \beta} \mu^{3}+\left(2 a_{12}+a_{66}\right) \mu^{2}-2 a_{26} \mu+a_{22}=0 \tag{3.1}
\end{equation*}
$$

are necessarily complex. These roots will be denoted by $\mu_{k}=\alpha_{k} \pm i \beta_{k}, k=1,2$, where $\beta_{k} \neq 0$. Two cases must be distinguished: $\mu_{1} \neq \mu_{2}$ and $\mu_{1}=\mu_{2}$.

If $F_{p}$ denotes a particular solution, and $\mu_{1}$ and $\mu_{2}$ are distinct, then the most general solution of (2.7) is given by

$$
\begin{equation*}
F(x, y)=F_{1}\left(z_{1}\right)+\bar{F}_{1}\left(\bar{z}_{1}\right)+F_{2}\left(z_{2}\right)+\bar{F}_{2}\left(\bar{z}_{2}\right)+F_{p}(x, y), \tag{3.2a}
\end{equation*}
$$

where $F_{k}\left(z_{k}\right)$ is in each case an arbitrary function of $z_{k}=x+\mu_{k} y$. The $F_{k}\left(z_{k}\right)$ is analytic in the region $S_{k}$ of the $z_{k}$-plane which corresponds to the region $S_{0}$ of the $z$-plane under the transformation ${ }^{11}$

$$
\begin{equation*}
z_{k}=p_{k} z+\bar{q}_{k} \bar{z}, \quad k=1,2 \tag{3.3}
\end{equation*}
$$

where

$$
\begin{equation*}
p_{k}=\frac{1}{2}\left(1-i \mu_{k}\right), \quad q_{k}=\frac{1}{2}\left(1-i \bar{\mu}_{k}\right), \quad k=1,2 . \tag{3.4}
\end{equation*}
$$

If $\mu_{1}=\mu_{2}$, then

[^14]\[

$$
\begin{equation*}
F(x, y)=\bar{z}_{1} F_{1}\left(z_{1}\right)+z_{1} \bar{F}_{1}\left(\bar{z}_{1}\right)+G_{1}\left(z_{1}\right)+\bar{G}_{1}\left(\bar{z}_{1}\right)+F_{p}(x, y) . \tag{3.2b}
\end{equation*}
$$

\]

For the plate without holes, $S_{k}$ is a simply-connected domain, and $F_{k}\left(z_{k}\right)$ and $G_{1}\left(z_{1}\right)$ are single-valued analytic functions in $S_{k}$ and $S_{1}$, respectively.
4. The boundary conditions for $F$. The plane stresses must satisfy the wellknown conditions

$$
\begin{align*}
& X_{n}^{0}=\tau_{x x}^{0} \cos (x, n)+\tau_{x y}^{0} \cos (n, y), \\
& Y_{n}^{0}=\tau_{x y}^{0} \cos (x, n)+\tau_{y y}^{0} \cos (y, n), \tag{4.1}
\end{align*}
$$

where $X_{n}^{u}, Y_{n}^{0}$ are the $x$ - and $y$-components, respectively, of the force acting on the edge of the plate, and $n$ is the exterior unit normal to $C_{0}$. If $F$ is introduced through (2.6), the boundary conditions on $F$ may be written

$$
\begin{equation*}
\frac{\partial F}{\partial x}+i \frac{\partial F}{\partial y}=i \int_{0}^{s}\left(X_{n}^{0}+i Y_{n}^{0}\right) d s+c \equiv f_{1}(s)+i f_{2}(s)+c \tag{4.2}
\end{equation*}
$$

where $s$ is the arc-length along $C_{0}$, measured from an arbitrary point with the usual convention as to positive $s$, and $c=c^{\prime}+i c^{\prime \prime}$ is an arbitrary, complex constant. A familiar alternative form of (4.2) is

$$
\left.\begin{array}{rl}
\frac{\partial F}{\partial n} & =f(s),  \tag{4.3a}\\
F & =g(s)
\end{array}\right\} \text { on } C_{0}
$$

where $f$ and $g$ are prescribed functions along $C_{0}$, except for the arbitrary constants $c^{\prime}$ and $c^{\prime \prime}$ appearing in them.

The solutions considered here are assumed to be such that $F_{k}^{\prime}\left(z_{k}\right), F_{1}\left(z_{1}\right)$ and $G_{1}^{\prime}\left(z_{1}\right)$ are continuous in $S_{0}+C_{0}$. In this case (4.2) and (4.3) hold. To ensure the existence of such a solution, it is necessary to demand that

$$
\begin{gather*}
\int_{C_{0}} X_{n}^{0} d s=0, \quad \int_{C_{0}} Y_{n}^{0} d s=0  \tag{4.4a}\\
\int_{C_{0}}\left\{f_{1}(s) \cos (x, s)+f_{2}(s) \cos (y, s)\right\} d s=0 \tag{4.4b}
\end{gather*}
$$

The physical significance of these conditions is of interest. Equation (4.4a) expresses mathematically the fact that the resultant of the external forces acting on the plate must vanish, and (4.4b) that the resultant of the external moments must vanish.

The analytical similarity between the boundary value problem presented by (2.7) and (4.3) and that of the clamped plate under lateral load makes it possible to use recent results on the lattter problem obtained by Morkovin. ${ }^{12} \mathrm{H}$ is treatment depends essentially on the handling of the boundary conditions.

The solution (3.2a) leads to boundary conditions expressed in terms of both $z_{1}$ and $z_{2}$, along either $C_{1}$ or $C_{2}$ (the boundaries of $S_{1}$ and $S_{2}$, respectively). Boundary conditions in terms of a single variable are obtained by mapping conformally a band of the $z_{k}$-plane containing $C_{k}$ in its interior onto an annular region of a $\zeta_{k}$-plane con-

[^15]taining in its interior the circumference of the unit circle $\gamma_{k}$ in a way such that $C_{k}$ is mapped into $\gamma_{k}$. By a proper choice ${ }^{13}$ of the functions effecting this mapping, the transforms $z_{1}$ and $z_{2}$ of any given value of $z$ on $C_{0}$ can be mapped onto $\gamma_{1}$ and $\gamma_{2}$ in such a way that $\zeta_{1}=\zeta_{2}$. This common value is denoted by $\sigma=e^{i \phi}$. Thus the boundary conditions contain $\sigma$ alone.

Let

$$
\begin{equation*}
z_{k}=\omega_{k}\left(\zeta_{k}\right), \quad k=1,2 \tag{4.5}
\end{equation*}
$$

be the functions achieving the desired mapping. Then $F$ has one of the forms

$$
\begin{align*}
& F=\phi_{1}\left(\zeta_{1}\right)+\phi_{1}\left(\bar{\zeta}_{1}\right)+\phi_{2}\left(\zeta_{2}\right)+\phi_{2}\left(\zeta_{2}\right)+F_{p}  \tag{4.6a}\\
& F=\bar{\omega}_{1}\left(\bar{\zeta}_{1}\right) \phi_{1}\left(\zeta_{1}\right)+\omega_{1}\left(\zeta_{1}\right) \bar{\phi}_{1}\left(\zeta_{1}\right)+\psi_{1}\left(\zeta_{1}\right)+\Psi_{1}\left(\bar{\zeta}_{1}\right)+F_{p} \tag{4.6b}
\end{align*}
$$

where $\phi_{k}\left(\zeta_{k}\right) \equiv F_{k}\left(\omega_{k}\left(\zeta_{k}\right)\right)$ and $\psi_{1}\left(\zeta_{1}\right) \equiv G_{1}\left(\omega_{1}\left(\zeta_{1}\right)\right)$. These functions are analytic and single-valued in some neighborhoods of $\gamma_{k}$, and hence possess Laurent expansions

$$
\begin{equation*}
\phi_{k}\left(\zeta_{k}\right)=\sum_{n=-\infty}^{\infty} \gamma_{n k} \zeta_{k}^{n}, \quad \psi_{1}\left(\zeta_{1}\right)=\sum_{n=-\infty}^{\infty} \mu_{n 1} \zeta_{1}^{n} . \tag{4.7}
\end{equation*}
$$

The coefficients $\gamma_{n k}$ and $\mu_{n 1}$ are to be determined from the boundary conditions on $F$, expressed now on $\gamma_{1}$ or $\gamma_{2}$, as desired. If one defines

$$
\begin{align*}
I_{k}(\sigma) & =\bar{p}_{k} \omega_{k}^{\prime}(\sigma) \sigma+\bar{q}_{k} \bar{\omega}_{k}^{\prime}(\bar{\sigma}) \bar{\sigma}, \\
J_{k}(\sigma) & =\sigma\left(p_{k} \bar{p}_{k}+q_{k} \bar{q}_{k}\right)+\bar{\sigma} \frac{\bar{\omega}_{k}^{\prime}(\bar{\sigma})}{\omega_{k}^{\prime}(\sigma)} 2 p_{k} \bar{q}_{k} \tag{4.8}
\end{align*}
$$

then boundary conditions on $\gamma_{1}$ or $\gamma_{2}(k=1$ or 2 , respectively) equivalent to (4.3) are given by

$$
\begin{align*}
& \left.i \sigma \frac{d s}{d \sigma} \frac{\partial F}{\partial u}\right]_{C_{0}}=\frac{1}{2 \beta_{k}}\left[H_{k}(\sigma)\left(\frac{\partial F}{\partial x}-i \frac{\partial F}{\partial y}\right)_{C_{0}}+\bar{H}_{k}(\bar{\sigma})\left(\frac{\partial F}{\partial x}+i \frac{\partial F}{\partial y}\right)_{C_{0}}\right]  \tag{4.9a}\\
& F]_{C_{0}}=\frac{1}{2 \beta_{k}} \int_{1}^{\sigma}\left[H_{k}(\sigma)\left(\frac{\partial F}{\partial x}-i \frac{\partial F}{\partial y}\right)_{C_{0}}-\bar{H}_{k}(\bar{\sigma})\left(\frac{\partial F}{\partial x}+i \frac{\partial F}{\partial y}\right)_{c_{0}}\right] \frac{d \sigma}{\sigma}+c^{\prime \prime \prime}, \tag{4.9b}
\end{align*}
$$

where ${ }^{14}$ the left-hand member of (4.9a) is defined by

$$
\begin{equation*}
\left.\left.\left.i \sigma_{k} \frac{d s}{d \sigma_{k}} \frac{\partial F}{\partial n}\right]_{C_{0}}=\frac{1}{\beta_{k}}\left\{J_{k}\left(\sigma_{k}\right) \frac{\partial F}{\partial \sigma_{k}}\right]_{C_{0}}+\bar{J}_{k}\left(\bar{\sigma}_{k}\right) \frac{\partial F}{\partial \bar{\sigma}_{k}}\right]_{C_{0}}\right\} \tag{4.10}
\end{equation*}
$$

and $c^{\prime \prime \prime}$ is an arbitrary real constant.
5. The determination of $F$. The function $F$ will be determined under the assumption that the edge force vector (4.2) has the form ${ }^{15}$

$$
\begin{equation*}
\left(\frac{\partial F}{\partial x}+i \frac{\partial F}{\partial y}\right)_{c_{0}}=\sum_{j=k_{1}}^{k_{1}} c_{j} \sigma^{i}+c \tag{5.1}
\end{equation*}
$$

[^16]where the $c_{i}$ are complex constants. The temperature function $T_{0}(x, y)$ is taken to be the polynomial
\[

$$
\begin{equation*}
T_{0}(x, y)=\sum_{i=0}^{k} \sum_{j=0}^{i} t_{j, i-j}^{0} x^{j} y^{i-j} \tag{5.2}
\end{equation*}
$$

\]

where the $t_{j, i-j}$ are real constants, and $k$ is an arbitrary integer greater than or equal to 0 . Then a particular integral

$$
\begin{equation*}
F_{p}=\sum_{i=2}^{k+2} \sum_{j=0}^{i} h_{j, i-j} x^{i} y^{i-j} \tag{5.3}
\end{equation*}
$$

of (2.7) can be found without difficulty. The Laurent expansions of $\omega_{k}\left(\zeta_{k}\right)$, along with (5.1) and the $\zeta_{k}$-plane transform ${ }^{10}$ of $\left.F_{p}\right]_{C_{0}}$, are now substituted in (4.9). The resulting equations yield recurrence formulas for the coefficients $\gamma_{n k}$ (or $\gamma_{n 1}$ and $\mu_{n 1}$ ).

For the case of the circular plate, the contour $C_{0}$ is a circle and the mapping functions are easily found to be

$$
\begin{equation*}
z_{k}=\omega_{k}\left(\zeta_{k}\right)=a p_{k}\left(\zeta_{k}+\frac{\Gamma_{k}}{\zeta_{k}}\right), \quad \Gamma_{k}=\frac{\bar{q}_{k}}{p_{k}}, \quad k=1,2 \tag{5.4}
\end{equation*}
$$

Moreover, if $\omega_{k}\left(\zeta_{k}\right)$ are given by (5.4), then a consequence of the single-valuedness of $F_{k}\left(z_{k}\right)$ and $G_{1}\left(z_{1}\right)$ is that (4.7) may be written

$$
\begin{equation*}
\phi_{k}\left(\zeta_{k}\right)=\sum_{n=0}^{\infty} \gamma_{n k}\left(\zeta_{k}^{n}+\frac{\Gamma_{k}^{n}}{\zeta_{k}^{n}}\right), \quad \psi_{1}\left(\zeta_{1}\right)=\sum_{n=0}^{\infty} \mu_{n 1}\left(\zeta_{1}^{n}+\frac{\Gamma_{1}^{n}}{\zeta_{1}^{n}}\right) \tag{5.5}
\end{equation*}
$$

The direct mapping function corresponding to (5.4) is

$$
\begin{equation*}
z=a \zeta_{0} \tag{5.6}
\end{equation*}
$$

and hence it follows that

$$
\begin{equation*}
\left.F_{p}\right]_{C_{0}}=\sum_{n=0}^{k+2}\left(C_{n} \sigma^{n}+\bar{C}_{n} \bar{\sigma}^{n}\right) \equiv C(\sigma) \tag{5.7}
\end{equation*}
$$

For $k=0$ the operator (4.10) becomes $\sigma(\partial / \partial \sigma)+\bar{\sigma}(\partial / \partial \bar{\sigma})$, and one then obtains

$$
\begin{equation*}
\left.i \sigma \frac{d s}{d \sigma} \frac{\partial F_{p}}{\partial n}\right]_{C_{0}}=\sum_{n=0}^{k+2}\left(D_{n} \sigma^{n}+\bar{D}_{n} \bar{\sigma}^{n}\right) \equiv D(\sigma) \tag{5.8}
\end{equation*}
$$

The coefficients $C_{n}$ and $D_{n}$ are in general complex constants.
If $A(\sigma)$ and $B(\sigma)$ denote the right-hand members of (4.9a) and (4.9b) respectively, then the insertion of (5.1) and (5.6) in $A$ and $B$ yields

$$
\begin{equation*}
A(\sigma)=\frac{a}{2} \sum_{n=0}^{k_{1}-1}\left(A_{n} \sigma^{n}+\bar{A}_{n} \bar{\sigma}^{n}\right), \quad B(\sigma)=\frac{a}{2} \sum_{n=0}^{k_{1}-1}\left(B_{n} \sigma^{n}+\bar{B}_{n} \bar{\sigma}^{n}\right) \tag{5.9}
\end{equation*}
$$

where

[^17]\[

$$
\begin{align*}
& A_{n}=\left\{\begin{array}{ll}
c_{1}, & n=0, \\
\bar{c}_{0}+c_{2}+c, & n=1, \\
c_{n+1}+\bar{c}_{-n+1}, & n=2,3, \cdots, k_{1}-1 ; \\
B_{n}= \begin{cases}\sum_{j=-k_{1}}^{k_{1}}, \frac{c_{j}+\bar{c}_{j}}{j-1}-c^{\prime}+\frac{1}{a} c^{\prime \prime \prime}, & n=0, \\
\bar{c}_{0}-c_{2}+\bar{c}, & n=1, \\
\frac{\bar{c}_{-n+1}-c_{n+1}}{n}, & n=2,3, \cdots, k_{1}-1\end{cases}
\end{array} . \begin{array}{l}
\frac{1}{n},
\end{array}\right. \\
& \hline \tag{5.10}
\end{align*}
$$
\]

and the prime on $\sum$ indicates that the term corresponding to $j=1$ is to be omitted. For convenience in writing, let

$$
\Pi(\sigma)=\sum_{n=0}^{k_{2}}\left(\Pi_{n} \sigma^{n}+\bar{\Pi}_{n} \bar{\sigma}^{n}\right), \quad \Lambda(\sigma)=\sum_{n=0}^{k_{2}}\left(\Lambda_{n} \sigma^{n}+\bar{\Lambda}_{n} \bar{\sigma}^{n}\right), \quad k_{2}=\max \left[k+2, k_{1}-1\right],
$$

where

$$
\begin{equation*}
\Pi_{n}=A_{n}-D_{n}, \quad \Lambda_{n}=B_{n}-C_{n}, \tag{5.11}
\end{equation*}
$$

with $A_{n}, B_{n}=0$ if $n \geqq k_{1}$, and $C_{n}, D_{n}=0$ if $n \geqq k+3$.
If $X_{n}^{0}$ and $Y_{n}^{0}$ are such that (5.1) is valid, then condition (4.4a) is satisfied, and (4.4b) demands that $c_{1}=\bar{c}_{1}$.

The determination of $\phi_{k}$ and $\psi_{1}$ is simplified by using not the boundary conditions (4.9), but an equivalent set. ${ }^{17}$ For the case of equal roots these are

$$
\begin{equation*}
\bar{\omega}(\bar{\sigma}) \phi(\sigma)+\omega(\sigma) \bar{\phi}(\bar{\sigma})+\psi(\sigma)+\bar{\psi}(\bar{\sigma})=\Lambda(\sigma), \tag{5.12a}
\end{equation*}
$$

$\bar{\omega}(\bar{\sigma}) \sigma \phi^{\prime}(\sigma)+\omega^{\prime}(\sigma) \sigma \bar{\phi}(\bar{\sigma})+\sigma \psi^{\prime}(\sigma)$

$$
\begin{equation*}
=\frac{1}{2}\left\{\frac{p}{\bar{H}(\bar{\sigma})}-\frac{\bar{q}}{H(\sigma)}\right\}\left\{\Pi(\sigma) \bar{\sigma} \psi(\bar{\sigma})+\frac{1}{\beta} \sigma \Lambda^{\prime}(\sigma) \bar{\omega}^{\prime}(\bar{\sigma}) \bar{J}(\bar{\sigma})\right\} . \tag{5.12b}
\end{equation*}
$$

Subscripts have been omitted in the above, since the distinction between the $z_{1}$ and $z_{2}$-planes is not involved in the discussion. Substitution in (5.12) of the series ${ }^{18}$ (6.4) for $\phi(\sigma)$ and $\psi(\sigma)$, together with $\omega(\sigma)$ from (5.4), then yields recurrence relations for $\gamma_{\mathrm{n}}$ and $\mu_{\mathrm{n}}$. For the explicit form of these rather lengthy formulas, the reader is referred to the author's thesis. ${ }^{19}$

It should be noted that equations (5.12) are valid for any given boundary of the admissible class, provided the functions $\Pi(\sigma)$ and $\Lambda(\sigma)$ corresponding to this boundary are found.

The treatment of the case $\mu_{1} \neq \mu_{2}$ does not depart materially from that of the case of equal roots, and hence will be omitted here. ${ }^{20}$
6. Extent of arbitrariness in solutions. Since a certain amount of arbitrariness is present in the functions occurring in the Muschelišvili solution of the plane stress problem for isotropic media, it is reasonable to ask if this phenomenon persists in the

[^18]case of anisotropic media. This question is answered affirmatively below, and the extent of the arbitrariness determined for the case of equal roots. This arbitrariness stems in part from the fact that the stress function is here, as in the Muschelišvili theory, the real part of an analytic function. In addition, it will be noted that the boundary conditions (5.12) contain three arbitrary constants, and one may expect this arbitrariness to be manifested in the solution.

If the prescribed functions $X_{n}^{0}(s)$ and $Y_{n}^{0}(s)$ satisfy the conditions (4.4) and are representable in the form (5.1), then the uniqueness theorem ${ }^{21}$ for the first boundary value problem of elasticity assures one that for a given distribution of temperature $T_{0}$, the state of stress in the interior of the plate is determinate. Since the stresses are given by (2.6), it is thus seen that the second derivatives of $F$ are determined in $S_{0}$.

If one lets $F_{h}$ be a solution of (2.7) and (4.3), lets $F_{h}^{(1)}=2 \operatorname{Re}\left\{\bar{z}_{1} F_{1}^{(1)}\left(z_{1}\right)+G_{1}^{(1)}\left(z_{1}\right)\right\}$ be another solution having the same second derivatives as $F_{h}$, and lets

$$
\begin{equation*}
F_{h}^{(2)}=F_{h}-F_{h}^{(1)}=2 \operatorname{Re}\left\{\bar{z}_{1} F_{1}^{(2)}\left(z_{1}\right)+G_{1}^{(2)}\left(z_{1}\right)\right\} \tag{6.1}
\end{equation*}
$$

then it follows that

$$
\begin{equation*}
\frac{\partial^{2} F_{h}^{(2)}}{\partial x^{2}}=\frac{\partial^{2} F_{h}^{(2)}}{\partial x \partial y}=\frac{\partial^{2} F_{h}^{(2)}}{\partial y^{2}}=0 \tag{6.2}
\end{equation*}
$$

throughout $S_{0}$. The real and imaginary parts of $F_{i}^{(2)}$ and $G_{1}^{(2)}$ must satisfy the CauchyRiemann equations in $S_{1}$, and these together with the three equations (6.2) enable one to show that

$$
\begin{aligned}
& F_{1}^{(2)}\left(z_{1}\right)=-i \eta_{3} z_{1}+\left(\eta_{1}+i \eta_{2}\right) \\
& G_{1}^{(2)}\left(z_{1}\right)=\left(\nu_{3}+i \nu_{4}\right) z_{1}+\left(\nu_{1}+i \nu_{2}\right)
\end{aligned}
$$

where $\nu_{i}$ and $\eta_{i}$ are arbitrary real constants. Thus the functions (4.7) and

$$
\begin{align*}
& \phi_{1}\left(\zeta_{1}\right)=\eta_{1}+i \eta_{2}-i \eta_{3} a p_{1}\left(\zeta_{1}+\frac{\Gamma_{1}}{\zeta_{1}}\right)+\phi_{1}^{(1)}\left(\zeta_{1}\right) \\
& \psi_{1}\left(\zeta_{1}\right)=\nu_{1}+i \nu_{2}+\left(\nu_{3}+i \nu_{4}\right) a p_{1}\left(\zeta_{1}+\frac{\Gamma_{1}}{\zeta_{1}}\right)+\psi_{1}^{(1)}\left(\zeta_{1}\right), \tag{6.3}
\end{align*}
$$

describe the same state of stress in $S_{0}$. The arbitrariness in (6.3) is removed by choosing $\nu_{i}$ and $\eta_{i}$ so as to simplify $\phi_{1}$ and $\psi_{1}$. The choice made here is such that

$$
\begin{equation*}
\phi_{1}\left(\zeta_{1}\right)=\sum_{n=1}^{\infty} \gamma_{n 1}\left(\zeta_{1}^{n}+\frac{\Gamma_{1}^{n}}{\zeta_{1}^{n}}\right), \quad \psi_{1}\left(\zeta_{1}\right)=\sum_{n=2}^{\infty} \mu_{n 1}\left(\zeta_{1}^{n}+\frac{\Gamma_{1}^{n}}{\zeta_{1}^{n}}\right), \tag{6.4}
\end{equation*}
$$

where $\gamma_{11}=\bar{\gamma}_{11}$.
For the case $\mu_{1} \neq \mu_{2}$, a more lengthy consideration ${ }^{22}$ shows that one may write

$$
\begin{equation*}
\phi_{1}\left(\zeta_{1}\right)=\sum_{n=2}^{\infty} \gamma_{n 1}\left(\zeta_{1}^{n}+\frac{\Gamma_{1}^{n}}{\zeta_{1}^{n}}\right), \quad \phi_{2}\left(\zeta_{2}\right)=\sum_{n=1}^{\infty} \gamma_{n 2}\left(\zeta_{2}^{n}+\frac{\Gamma_{2}^{n}}{\zeta_{2}^{n}}\right) \tag{6.5}
\end{equation*}
$$

where $\gamma_{22}=\bar{\gamma}_{22}$.
In both cases the arbitrariness is a reflection of that inherent in the values $f(s)$

[^19]and $g(s)$ which $F$ must assume along $C_{0}$. The choice of $\nu_{i}$ and $\eta_{i}$ implied by (6.4) (or (6.5)) is found to dictate the selection of the arbitrary constants in (4.3), or in (4.9), the modified form of these boundary conditions, and conversely. This somewhat inverted method of eliminating the arbitrariness in $\phi_{1}$ and $\psi_{1}$ (or $\phi_{1}$ and $\phi_{2}$ ) is adopted in order to have these functions assume the form usual in the Muschelišvili theory. ${ }^{23}$
7. The differential equation for the deflection; the associated boundary conditions. The coefficients $N_{x x}, N_{y y}$, and $N_{x y}$ in (2.13) may now be regarded as known, and the thermo-elastic deflection problem for the anisotropic thin plate is reduced to that of solving the partial differential equation (2.13) for $w$, subject to the appropriate boundary conditions to be satisfied on $C_{0}$. For the first and second boundary value problems of plate theory these are, respectively,
\[

$$
\begin{equation*}
M_{n}=m(s), \quad Q_{n}+\frac{\partial I_{n s}}{\partial s}=p(s), \quad \text { on } \quad C_{0} \tag{7.1a}
\end{equation*}
$$

\]

and

$$
\begin{equation*}
w=w(s), \quad \frac{\partial w}{\partial n}=w_{n}(s), \quad \text { on } \quad C_{0} \tag{7.1b}
\end{equation*}
$$

where $m, p, w$, and $w_{n}$ are prescribed functions along $C_{0}$. The quantities ${ }^{24} M_{n}, H_{n s}$ and $Q_{n}$ are the flexural couple, torsional couple, and the shearing force, respectively, which act on the edge of the plate.

The specialization of (2.7) and (2.13) to the isotropic case will now be given, since the resulting equations will be used in the sequel. For isotropic media

$$
\begin{align*}
a_{i i} & =\frac{1}{E}, \quad i=1,2,3, \quad a_{i i}=\frac{2(1+\sigma)}{E}, \quad i=4,5,6 \\
a_{i j} & =-\frac{\sigma}{E}, \quad i, j=1,2,3, i \neq j, \quad a_{i j}=0, i, j=4,5,6, i \neq j  \tag{7.2}\\
a_{i} & =\alpha, i=1,2,3, \quad a_{i}=0, i=4,5,6
\end{align*}
$$

where $E$ denotes Young's modulus, $\sigma$ is Poisson's ratio, and $\alpha$ is the coefficient of thermal expansion. One casily finds that

$$
\begin{array}{rlr}
b_{11}=b_{22}=\frac{E}{1-\sigma^{2}}, & b_{06}=\frac{E}{1+\sigma} \\
b_{12}=\frac{E \sigma}{1-\sigma^{2}}, & b_{10}=b_{26}=0  \tag{7.3}\\
\alpha_{1}=\alpha_{2}=\frac{\alpha E}{1-\sigma}, & \alpha_{6}=0
\end{array}
$$

Using these values of the elastic parameters, one obtains from (2.7) and (2.13) the equations

$$
\begin{gather*}
\nabla^{4} F=-\alpha E \nabla^{2} T_{0}  \tag{7.4}\\
\nabla^{4} w=-\alpha(1+\sigma) \nabla^{2} T_{1}+\frac{1}{D}\left\{N_{x x} \frac{\partial^{2} w}{\partial x^{2}}+2 N_{x y} \frac{\partial^{2} w}{\partial x \partial y}+N_{y y} \frac{\partial^{2} w}{\partial y^{2}}\right\} \tag{7.5}
\end{gather*}
$$

[^20]where $D=2 E h^{8} / 3\left(1-\sigma^{2}\right)$. Except for changes in notation, these are the same equations that Nádai ${ }^{25}$ obtains. The quantities (7.2) and (7.3) must also be inserted in (7.1) in order to obtain the boundary conditions for the isotropic plate.

A solution of the deflection equation (2.13) valid throughout the domain $S_{0}$ and satisfying the given boundary conditions on $C_{0}$, is not available; the same is true of (7.5) and its associated boundary conditions. ${ }^{26}$ By further specialization, however, it is possible to obtain the solution of the thermo-elastic problem for a case which is of some interest. This will be done in the following sections.
8. The istropic circular plate with radial temperature distribution. Let us consider an isotropic circular plate of radius $a$, and let it be subjected to a temperature distribution given by

$$
\begin{equation*}
T(x, y, z)=T_{0}(r)+z T_{1}(r) \tag{8.1}
\end{equation*}
$$

where $r=\sqrt{x^{2}+y^{2}}$, and the origin is assumed to be at the center of the plate. As in section 2 , the second derivatives of $T_{0}$ and $T_{1}$ are assumed continuous on $0 \leqq r \leqq a$. The edge of the plate is taken to be subjected to a uniform force $P$ per unit length of the arc parameter $s$.

It has been seen that before the deflection $w$ can be found, it is necessary to solve (7.4) for $F$. In the case at hand, $F_{p}$ can be found easily whether or not radial symmetry exists, for if $F$ is a solution of

$$
\begin{equation*}
\nabla^{2} F=-\alpha E T_{0} \tag{8.2}
\end{equation*}
$$

then it is also a solution of (7.4). But (8.2) is the well-known Poisson's equation, and a solution is at once available from potential theory. Since $T_{0}$ has radial symmetry in the present case, however, the Laplacian operator becomes

$$
\begin{equation*}
\nabla^{2}=\frac{1}{r} \frac{d}{d r}\left(r \frac{d}{d r}\right) \tag{8.3}
\end{equation*}
$$

and successive integrations of (8.2) give

$$
\begin{equation*}
F_{p}(r)=-\alpha E \int_{0}^{r} \frac{d \xi}{\xi} \int_{0}^{\xi} T_{0}(\chi) \chi d \chi \tag{8.4}
\end{equation*}
$$

as the particular solution (7.4) which is needed in $F$ (see (3.2)). For the isotropic plate, the roots of the characteristic equation (3.1) are $\mu_{1}=\mu_{2}=i$ and their conjugates. The transformation (4.5) becomes

$$
\begin{equation*}
z=a \zeta \tag{8.5}
\end{equation*}
$$

and if $|\zeta|=\rho$, then $r=a \rho$. If one lets $f_{p}(\rho) \equiv F_{p}(a p)$, then from (5.7)

$$
\begin{equation*}
C(\sigma)=f_{p}(\sqrt{\sigma \bar{\sigma}})=f_{p}(1) \tag{8.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\left.D(\sigma)=i \sigma \frac{d s}{\partial \sigma} \frac{\partial F_{p}}{\partial n}\right]_{C_{0}}=\left(\sigma \frac{\partial}{\partial \sigma}+\bar{\sigma} \frac{\partial}{\partial \bar{\sigma}}\right) f_{p}(\rho)\right]_{p-1}=\sqrt{\sigma \bar{\sigma}} f_{p}^{\prime}(\sqrt{\sigma \bar{\sigma}})=f_{p}^{\prime}(1), \tag{8.7}
\end{equation*}
$$

[^21]i.e., $C(\sigma)$ and $D(\sigma)$ reduce to constants on $\gamma$. This result is clearly a consequence of the fact that $T_{0}$ is a function of $r$ alone. One finds that
\[

$$
\begin{array}{lll}
C_{0}=\frac{1}{2} f_{p}(1), & C_{j}=0, & j=1,2, \cdots, \\
D_{0}=\frac{1}{2} f_{p}^{\prime}(1), & D_{j}=0, & j=1,2, \cdots . \tag{8.8}
\end{array}
$$
\]

With the edge forces as specified above, it follows that one may write

$$
X_{n}^{0}+i Y_{n}^{0}=-P e^{i 0}
$$

and then (4.2) yields

$$
\left(\frac{\partial F}{\partial x}+i \frac{\partial F}{\partial y}\right)_{c_{0}}=a P(1-\sigma)+c .
$$

This is of the assumed form (5.1), and the constants defined in that expression have the values

$$
\begin{equation*}
c_{0}=a P, \quad c_{1}=-a P . \tag{8.9}
\end{equation*}
$$

These are now inserted in (5.10) and the result together with (8.8) substituted in (5.11). Proceeding as indicated in section 5 , one finds that the recurrence formulas simplify to

$$
\begin{equation*}
\gamma_{1}=-\frac{a}{4}\left\{P+\frac{1}{a^{2}} f_{p}^{\prime}(1)\right\}, \quad \gamma_{n}=0, n=2,3, \cdots, \quad \mu_{n}=0, n=2,3, \cdots \tag{8.10}
\end{equation*}
$$

The above results in conjunction with (3.2b) enable one to write

$$
\begin{equation*}
F(r)=\frac{1}{2}\left\{-P+\frac{\alpha E}{r^{2}} \int_{0}^{r} T_{0}(\chi) x d x\right\} r^{2}-\alpha E \int_{0}^{r} \frac{d \xi}{\xi} \int_{0}^{\xi} T_{0}(\chi) \chi d \chi . \tag{8.11}
\end{equation*}
$$

In view of the radial symmetry, it is expedient to write (7.5) in terms of polar coordinates in the form

$$
\begin{align*}
\nabla^{4} w=-\alpha(1+\sigma) \nabla^{2} T_{1}+\frac{1}{D}\left\{N_{r r} \frac{\partial^{2} w}{\partial r^{2}}+2 N_{r \theta}\right. & \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial w}{\partial \theta}\right) \\
& \left.+N_{\theta \theta}\left(\frac{1}{r} \frac{\partial w}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} w}{\partial \theta^{2}}\right)\right\}, \tag{8.12}
\end{align*}
$$

where $N_{r r}=2 h T_{r r}^{0}, N_{r \theta}=2 h \tau_{r \theta}^{0}, N_{\theta \theta}=2 h \tau_{e \theta}^{0}$, and

$$
\begin{equation*}
\tau_{\pi r}^{0}=\frac{1}{r} \frac{\partial F}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} F}{\partial \theta^{2}}, \quad \tau \tau_{\theta}^{0}=-\frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial F}{\partial \theta}\right), \quad \tau_{\theta \theta}^{0}=\frac{\partial^{2} F}{\partial r^{2}} . \tag{8.13}
\end{equation*}
$$

With $F$ of the form (8.11), these stresses become

$$
\begin{align*}
& \tau_{\mathrm{rr}}^{0}=-P+\frac{\alpha E}{a^{2}} \int_{0}^{a} T_{0}(x) x d \chi-\frac{\alpha E}{r^{2}} \int_{0}^{r} T_{0}(x) x d x, \\
& \tau_{\theta \theta}^{0}=-P+\frac{\alpha E}{a^{2}} \int_{0}^{a} T_{0}(x) x d x+\frac{\alpha E}{r^{2}} \int_{0}^{r} T_{0}(x) x d x-\alpha E T_{0}(r),  \tag{8.14}\\
& \tau_{r \theta}^{0}=0 .
\end{align*}
$$

Since $T_{1}, N_{r r}, N_{r \theta}$, and $N_{\theta \theta}$ are functions of $r$ alone, $\nabla^{2}$ has the form (8.3), and the equation for the deflection may therefore be written

$$
\begin{align*}
& \frac{d}{d r}\left(r \frac{d}{d r}\left[\frac{1}{r} \frac{d}{d r}\left\{r \frac{d w}{d r}\right\}\right]\right)=-\alpha(1+\sigma) \frac{d}{d r}\left(T \frac{d T_{1}}{d r}\right) \\
&+\frac{2 h}{D}\left[\left(P_{0}-\frac{\alpha E}{r} \int_{0}^{r} T_{0}(\chi) \chi d \chi\right) \frac{d^{2} w}{d r^{2}}\right. \\
&\left.+\left(P_{0}+\frac{\alpha E}{r^{2}} \int_{0}^{r} T_{0}(\chi) \chi d \chi-\alpha E T_{0}\right) \frac{d w}{d r}\right] \tag{8.15}
\end{align*}
$$

where $P_{0}$ is the constant

$$
\begin{equation*}
P_{0}=\frac{\alpha E}{a^{2}} \int_{0}^{a} T_{0}(\chi) x d \chi-P \tag{8.16}
\end{equation*}
$$

Equation (8.15) may be integrated with respect to $r$ and an equation obtained thereby which is not only of lower order than (8.15), but which also has simpler coefficients. Noting that

$$
\begin{aligned}
\frac{d}{d r}\left\{\left(P_{0} r\right.\right. & \left.\left.-\frac{\alpha E}{r} \int_{0}^{r} T_{0}(\chi) x d \chi\right) \frac{d w}{d r}\right\} \\
& =\left(P_{0} r-\frac{\alpha E}{r} \int_{0}^{r} T_{0}(\chi) x d x\right) \frac{d^{2} w}{d r^{2}}+\left(P_{0}+\frac{\alpha E}{r^{2}} \int_{0}^{r} T_{0}(\chi) x d \chi-\alpha E T_{0}\right) \frac{d w}{d r}
\end{aligned}
$$

we can carry out the desired integration immediately, obtaining

$$
\left.\begin{array}{rl}
\frac{d}{d r}\left\{\frac{1}{r} \frac{d}{d r}\left(r \frac{d w}{d r}\right)\right.
\end{array}\right\},
$$

where $k_{1}$ is a constant of integration. Thus for the thin circular plate under uniform compression on its edge, and with $T_{0}(r)$ and $T_{1}(r)$ arbitrary save for certain conditions of continuity, the problem of finding the thermo-elastic deflection is reduced to that of solving the third order differential equation (8.17) with appropriate boundary conditions.

A repetition of the above integration is impossible for $T_{0}$ and $T_{1}$ of the general nature assumed above. Therefore, in order to complete the integration of the equation, $T_{0}$ and $T_{1}$ will be supposed to have the form ${ }^{27}$

$$
\begin{equation*}
T_{0}(r)=\frac{D}{2 h \alpha E} \sum_{j=0}^{m} t_{0}, r^{i}, \quad T_{1}(r)=-\frac{1}{\alpha(1+\sigma)} \sum_{j=0}^{n} t_{1}, r^{i}, \tag{8.18}
\end{equation*}
$$

where $t_{0 i}, t_{1 j}$ are real constants and $m, n$ are arbitrary positive integers. The solution of (8.17) may now be sought in the form of a power series. The polynomials $T_{0}$ and $T_{1}$ are now inserted in the differential equation, and if one makes the abbreviations

[^22]$$
b_{j}=\frac{t_{0 j}}{j+2}, \quad d_{j}=j t_{1 j}
$$
the result is
\[

$$
\begin{equation*}
\frac{d^{3} w}{d r^{3}}+\frac{1}{r} \frac{d^{2} w}{d r^{2}}+\left\{\sum_{j=0}^{m} b_{j}\left(r^{j}-a^{j}\right)+\frac{2 h P}{D}-\frac{1}{r^{2}}\right\} \frac{d w}{d r}=\sum_{j=1}^{n} d_{j} r^{j-1} \tag{8.19}
\end{equation*}
$$

\]

The obvious replacement

$$
\begin{equation*}
u(r)=\frac{d w}{d r} \tag{8.20}
\end{equation*}
$$

gives an equation of order two, and the further substitution

$$
b_{0}^{\prime}=\frac{2 h P}{D}-\sum_{j=1}^{m} b_{j} a^{j}
$$

then yields

$$
\begin{equation*}
\frac{d^{2} u}{d r^{2}}+\frac{1}{r} \frac{d u}{d r}+\left(\sum_{j=1}^{m} b_{j} r^{j}+b_{0}^{\prime}-\frac{1}{r^{2}}\right) u=\sum_{j=0}^{n-1} d_{j+1} r^{j} \tag{8.21}
\end{equation*}
$$

as the equation to be solved for $u$. It will be observed that this equation has a regular singular point at $r=0$, and that the indices relative to this singular point are $\pm 1$. Only the solution of the form

$$
\begin{equation*}
u=r \sum_{i=0}^{\infty} \lambda_{i} r^{i} \tag{8.22}
\end{equation*}
$$

will be considered here. This represents the solution relative to $r=0$ which is bounded there. It is evident from physical considerations that this boundedness must obtain for the simply-connected plate, and hence it is sufficient to consider the solution of the above form. The series (8.22) is now inserted in (8.21) and the following recurrence relation defining the $\lambda_{j}$ is obtained:

$$
(j+4)(j+2) \lambda_{j+2}+b_{0}^{\prime} \lambda_{j}+\sum_{i=0}^{m} \lambda_{j-i} b_{j}= \begin{cases}d_{j+2}, & j=-1,0, \cdots, n-2  \tag{8.23}\\ 0, & j=n-1, n, \cdots\end{cases}
$$

where $\lambda_{j}=0$, if $j<0$. These equations permit $\lambda_{j}$ to be expressed in terms of the arbitrary quantity $\lambda_{0}$ and the known quantities $b_{0}^{\prime}, b_{i}$, and $d_{i}$. It is not expedient to give a formula for $\lambda_{i}$, since such an expression would be quite lengthy. The first several $\lambda_{i}$ may be computed with little trouble if $m$ and $n$ are not large, but the labor involved increases rapidly as these numbers become larger.

Up to now, the series defined by (8.23) is a formal solution of the differential equation (8.21). It is not difficult to show that this series converges for $0 \leqq r \leqq a$ and hence is actually a solution of the equation. For the discussion of the convergnece it suffices to consider the series

$$
\begin{equation*}
\sum_{i=h}^{\infty} \lambda_{i} r^{i} \tag{8.24}
\end{equation*}
$$

where $h=m+n+2$. If one defines

$$
\begin{aligned}
M & =\max \left\{1,\left|b_{0}^{\prime}\right|,\left|b_{1}\right|,\left|b_{2}\right|, \cdots,\left|b_{n}\right|\right\} \\
K & =\max \left\{\left|\lambda_{0}\right|,\left|\lambda_{1}\right|, \cdots,\left|\lambda_{h-1}\right|\right\}
\end{aligned}
$$

and sets $E(\theta)=(h+\theta)(h+2+\theta)$, repeated use of the recurrence relation (8.23) with $j>n-2$ enables one to deduce that

$$
\begin{equation*}
\left|\lambda_{h+k(m+2)+i}\right|<\frac{K\{M(m+1)\}\left[\frac{[k(+2)+i+2}{2}\right]}{E(0) E(m+2) E(2 m+4) \cdots E(k(m+2)+i)}=\mu_{k(m+2)+i} \tag{8.25}
\end{equation*}
$$

for $k=0,1,2, \cdots, i$ assuming the values $0,1, \cdots, m+1$ for each $k$. It may then be shown that the dominant series defined by the $\mu_{k(m+2)+i}$ converges uniformly for any finite $r$, and application of the well-known Weierstrass theorem yields the same convergence of (8.24) and (8.22). The desired solution $w$ of the deflection equation is then obtained by inserting (8.22) in (8.20) and integrating. We obtain

$$
\begin{equation*}
w(r)=r^{2} \sum_{i=0}^{\infty} \kappa_{i} r^{i}+k \tag{8.26}
\end{equation*}
$$

where $\kappa$ is a constant of integration and $\kappa_{i}=\lambda_{i} /(i+2)$. The uniform convergence of this integrated series on the interval $0 \leqq r \leqq a$ follows immediately from that of (8.22). Reference to the recursion formula (8.23) reveals that one may write

$$
\begin{equation*}
\kappa_{i}=\lambda_{0} \xi_{i}+\delta_{i} \tag{8.27}
\end{equation*}
$$

where $\xi_{i}$ contains the parameters $P, D, h, a$, and some or all of the $t_{0}$, while $\delta_{i}$ contains not only these but also some or all of the $t_{1 j}$. Thus the deflection may be written in the form

$$
\begin{equation*}
w(r)=\lambda_{0} w_{0}(r)+w_{1}(r)+\kappa, \tag{8.28}
\end{equation*}
$$

where

$$
\begin{equation*}
w_{0}(r)=\sum_{j=0}^{\infty} \xi_{j} r^{j}, \quad w_{1}(r)=\sum_{j=0}^{\infty} \delta_{j} r^{j} \tag{8.29}
\end{equation*}
$$

It is clear that $w_{1}(r)$ is a particular solution of (8.21) and $w_{0}(r)$ the solution of the homogeneous equation which is bounded at $r=0$.
9. The circular plate with clamped and simply supported edge. The constants $\lambda_{0}$ and $\kappa$ in (8.28) will be determined by the mode of support of the plate. The assumption of radially symmetric deflection limits one to the consideration of boundary conditions compatible with this type of deflection. The two methods of support most commonly encountered are those of the clamped edge and the simply supported edge. The boundary conditions for these are, respectively,

$$
\begin{equation*}
\left.\left.\frac{d w}{d r}\right]_{r=a}=0, \quad w\right]_{r=a}=0, \tag{9.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\left.M_{r}\right]_{r-a}=\left[\frac{d^{2} w}{d r^{2}}+\frac{\sigma}{r} \frac{d w}{d r}\right]_{r=a}=0, \quad w\right]_{r=a}=0 \tag{9.2}
\end{equation*}
$$

The substitution of (8.28) in (9.1) and (9.2) yields for the plate with clamped edge the constants

$$
\lambda_{0}=-\frac{a w_{1}^{\prime \prime}(a)+\sigma w_{1}^{\prime}(a)}{a w_{0}^{\prime \prime}(a)+\sigma w_{0}^{\prime}(a)}, \quad \kappa=\frac{a w_{1}^{\prime \prime}(a)+\sigma w_{1}^{\prime}(a)}{a w_{0}^{\prime \prime}(a)+\sigma w_{0}^{\prime}(a)} w_{0}(a)-w_{1}(a),
$$

and for the plate with a simply supported edge,

$$
\lambda_{0}=-\frac{w_{1}^{\prime}(a)}{w_{0}^{\prime}(a)}, \quad \kappa=\frac{w_{1}^{\prime}(a)}{w_{0}^{\prime}(a)} w_{0}(a)-w_{1}(a) .
$$

These quantities are now inserted in (8.28), giving for the plate with clamped edge the deflection

$$
\begin{equation*}
w(r)=\frac{w_{1}^{\prime}(a)}{w_{0}^{\prime}(a)}\left\{w_{0}(a)-w_{0}(r)\right\}+w_{1}(r)-w_{1}(a) \tag{9.3}
\end{equation*}
$$

and for the plate with simply supported edge

$$
\begin{equation*}
w(r)=\frac{a w_{1}^{\prime \prime}(a)+\sigma w_{1}^{\prime}(a)}{a w_{0}^{\prime \prime}(a)+\sigma w_{0}^{\prime}(a)}\left\{w_{0}(a)-w_{0}(r)\right\}+w_{1}(r)-w_{1}(a) . \tag{9.4}
\end{equation*}
$$

It will be assumed that $w_{0}^{\prime}(a)$ and $a w_{0}^{\prime \prime}(a)+\sigma w_{0}^{\prime}(a)$ do not vanish. It may be shown that an $a$ which causes the former (latter) to vanish is the radius of the clamped (simply supported) plate with given temperature distribution $T_{0}(r)$ for which $P$ is a critical (i.e., buckling) load. The question of stability is not under consideration here, hence the above assumption is made.

The case $m=0$ is of particular interest, for then $w$ contains the Bessel function of the first kind of order zero. In this case $T_{0}(r)$ reduces to a constant, and (8.21) becomes

$$
\frac{d^{2} u}{d r^{2}}+\frac{1}{r} \frac{d u}{d r}+\left(\frac{2 h P}{D}-\frac{1}{r^{2}}\right) u=\sum_{j=0}^{n-1} d_{j+1} r^{i} .
$$

If the recurrence relation is written in terms of the $\kappa_{i}$ rather than the $\lambda_{i}$, the result is

$$
(i+4)^{2} \kappa_{i+2}+\frac{2 h P}{D} \kappa_{i}=\left\{\begin{array}{cl}
l_{1, i+2}, & i=-1,0, \cdots, n-2 \\
0 & i=n-1, n, \cdots,
\end{array}\right.
$$

from which one obtains easily the deflection

$$
w(r)=\lambda_{0} J_{0}\left(r \sqrt{\frac{2 h P}{D}}\right)+w_{10}(r)+\kappa .
$$

Here $w_{10}(r)$ designates the form which $w_{1}(r)$ assumes for $m=0$. Proceeding as in the more general case, for the plate with clamped edge we find that the deflection is

$$
\begin{aligned}
w(r)= & \frac{w_{10}^{\prime}(a)}{\sqrt{\frac{2 h P}{D}} J_{0}\left(a \sqrt{\frac{2 h P}{D}}\right)}\left\{J_{0}\left(r \sqrt{\frac{2 h P}{D}}\right)-J_{0}\left(a \sqrt{\frac{2 h P}{D}}\right)\right\} \\
& +w_{10}(r)-w_{10}(a)
\end{aligned}
$$

and for the plate with simply supported edge,

$$
\begin{aligned}
w(r)=\frac{a w_{1}^{\prime \prime}(a)+\sigma w_{1}^{\prime}(a)}{\sqrt{\frac{2 h P}{D}}\left\{\sqrt{\frac{2 h P}{D}} J_{0}\left(a \sqrt{\frac{2 h P}{D}}\right)-(1-\sigma) J_{1}\left(a \sqrt{\frac{2 h P}{D}}\right)\right\}} \\
\cdot\left\{J_{0}\left(r \sqrt{\frac{2 h P}{D}}\right)-J_{0}\left(a \sqrt{\frac{2 h P}{D}}\right)\right\}+w_{10}(r)-w_{10}(a)
\end{aligned}
$$

10. Conclusion. The problem of flexure is considered for a thin anisotropic elastic plate, subjected in its interior to a temperature distribution of the form

$$
T(x, y, z)=T_{0}(x, y)+z T_{1}(x, y)
$$

The usual assumptions of thin plate theory, together with Hooke's law extended to encompass thermal effects, permit one to derive two partial differential equations governing the deflection of the plate. For the isotropic plate these equations specialize to those given by Nádai. If thermal effects are supposed absent, and $N_{x x}, N_{v y}, N_{x y}$ are interpreted as arising from edge forces alone, then (2.13) becomes the equation for the deflection of an anisotropic thin plate stressed in its own plane.

A method is given for determining the stress function $F$ for a plate with edge forces representable in the form of a trigonometric polynomial, and the determination of $F$ is carried out for the circular plate.

For the thermo-elastic problem formulated with the above generality, a suitable solution of the deflection equation is not available. Accordingly, the problem is specialized to the simpler case of the isotropic circular plate with radially symmetric temperature distribution and $T_{0}(r), T_{1}(r)$ in the form of polynomials. The solution of the resulting deflection equation may be found in the form of a power series, and convergence established. Boundary conditions for the clamped and simply supported plate are then considered, and the deflection for each of these modes of support determined.

# CONTRIBUTIONS TO THE PROBLEM OF APPROXIMATION OF EQUIDISTANT DATA BY ANALYTIC FUNCTIONS* 

# PART A.-ON THE PROBLEM OF SMOOTHING OR GRADUATION. A FIRST CLASS OF ANALYTIC APPROXIMATION FORMULAE 

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Introduction. Let there be given a sequence of ordinates

$$
\left\{y_{n}\right\} \quad(n=0, \pm 1, \pm 2, \cdots),
$$

corresponding to all integral values of the variable $x=n$. If these ordinates are the values of a known analytic function $F(x)$, then the problem of interpolation between these ordinates has an obvious and precise meaning: we are required to compute intermediate values $F(x)$ to the same accuracy to which the ordinates are known. Undoubtedly, the most convenient tool for the solution of this problem is the polynomial central interpolation method. It uses the polynomial of degree $k-1$, interpolating $k$ successive ordinates, as an approximation to $F(x)$ only within a unit interval in $x$, centrally located with respect to its $k$ defining ordinates. Assuming $k$ fixed, successive approximating arcs for $F(x)$ are thus obtained which present discontinuities on passing from one arc to the next if $k$ is odd, or discontinuities in their first derivatives if $k$ is even (see section 2.121). Actually these discontinuities are irrelevant in our present case of an analytic function $F(x)$. Indeed, if the interpolated values obtained are sufficiently accurate, these discontinuities will be apparent only if we force the computation beyond the intrinsic accuracy of the $y_{n}$.

The situation is quite different if $y_{n}$ are empirical data. In this case we are to determine an approximation $F(x)$ which, for $x=n$, may disagree with $y_{n}$ by amounts depending on the accuracy of the data, provided we thereby improve the smoothness of the resulting approximation $F(x)$. In various applied fields such as Ballistics and Actuarial mathematics it is at times desirable to compute very smooth approximations $F(x)$ to an accuracy surpassing by far the accuracy to which the physical or statistical function involved may be determined. This physically unjustified accuracy becomes desirable whenever the approximation $F(x)$ enters into numerical processes of some complexity, such as the numerical solution of differential equations. Modern electronic computing machines, especially, require a good amount of forced mathematical accuracy in such auxiliary tables in order to avoid the excessive accumulation of rounding errors in the computation of the solution. These remarks justify the desirability of approximation methods to empirical data furnishing easily computed approximations $F(x)$ which are very smooth functions of $x$. Approximations meeting these requirements are of two kinds: 1. Polynomial approximation, where $F(x)$ is composed of a succession of polynomial arcs meeting with a certain number of continuous derivatives. 2. Analytic approximations, where $F(x)$ is an analytic and regular function of $x$ for all real values of $x$.

[^23]Important work concerning polynomial approximations is to be found in the actuarial literature under the subject of osculatory interpolation. Of the extensive literature we mention especially the fundamental work of W. A. Jenkins and the valuable systematization of the subject by T. N. E. Greville. ${ }^{1}$ Especially important are those formulae derived by these authors which do not strictly interpolate the given ordinates, but rather combine the operation of smoothing the data and the operation of interpolation in one formula. Mr. Jenkins discusses interpolation formulac written in the convenient Everett (or Steffensen) form. Mr. Greville's starting point is his elegant expression of each polynomial arc in terms of the end point values of those derivatives which are to be continuous on passing from one arc to the next. Each of these two modes of attack has its peculiar advantages and one or the other seem indispensable for an algebraic treatment of the subject. The present writer has found the Lagrange form (explicitly in terms of the ordinates $y_{n}$ ) of such formulae preferable for two reasons: 1. The Lagrange form seems better adapted to computation with modern desk computing machines and undoubtedly superior for computation with punch-card machines. 2. The Lagrange form suggests a treatment of the subject by means of elementary concepts of Fourier analysis which, firstly, affords a more exhaustive treatment of the problem of polynomial approximations, secondly, shows how to extend these methods so as to furnish analytic approximations.

The explicit Lagrange form of the $k$-point central interpolation method, as well as of all the interpolation formulae of osculatory interpolation, is extremely simple in its formal appearance. Indeed, to every such formula corresponds an even function $L(x)$, defined for all real values of $x$, in terms of which the corresponding formula may be written as follows

$$
\begin{equation*}
F(x)=\sum_{n=-\infty}^{\infty} y_{n} L(x-n) . \tag{1}
\end{equation*}
$$

The simplicity of this formula springs from the fact that it depends on the single function $L(x)$ which describes the formula completely. Incidentally $F(x)=L(x)$ if

$$
\begin{equation*}
y_{0}=1, \quad y_{n}=0 \quad(n \neq 0) . \tag{2}
\end{equation*}
$$

Thus every interpolation method of this kind exhibits its corresponding $L(x)$ if we apply the method to the ordinates (2) (for an example see section 2.121).

The polynomial interpolation formulae arise from (1) if $L(x)$ is a composite polynomial function of arcs defined by various polynomials in successive unit intervals, such that $L(x)=0$ for sufficiently large values of $|x|$ (for an important example see chapter II, formula (11)). The number of continuous derivatives of $F(x)$ is, of course, equal to the number of continuous derivatives of $L(x)$ for all real $x$.

We obtain the formally simplest interpolation formula (1) if we choose

$$
\begin{equation*}
L(x)=\frac{\sin \pi x}{\pi x} \tag{3}
\end{equation*}
$$

[^24]in which case (1) becomes
\[

$$
\begin{equation*}
F(x)=\sum_{n=-\infty}^{\infty} y_{n} \frac{\sin \pi(x-n)}{\pi(x-n)} \tag{4}
\end{equation*}
$$

\]

This expression which interpolates the ordinates $y_{n}$, is known to mathematicians under the name of the cardinal series. ${ }^{2}$ For this reason we wish to call the general formula (1) a formula of the cardinal type, referring to $L(x)$ as the basic function of the formula.

The aim of the paper, of which the present Part A is the first, is twofold. Firstly, we propose to carry through to a certain stage of completion the important actuarial work concerning polynomial approximations. Incidentally, our work will answer Mr . Greville's conjecture (loc. cit. pp. 212-213) concerning the existence of an "ordinary" interpolation formula furnishing an approximation $F(x)$ composed of polynomial arcs of degree $m+2$, having $m$ continuous derivatives and such that if the data $y_{n}$ are the values of a polynomial of degree $m-1$ then $F(x)$ reduces identically to that polynomial. In Part B it will be shown how to obtain such formulae for every value of $m$. (The case of $m=2$ reduces to Jenkins' formula mentioned in section 2. 122.) Secondly, we shall derive formulae of the cardinal type (1) with basic functions $L(x)$ which are analytic and regular for all real or complex values of $x$. The classical basic function (3) is of course analytic; however, its excessively slow rate of damping, for increasing $x$, makes the classical cardinal series (4) inadequate for numerical purposes. Our analytic $L(x)$, derived in chapter IV, dampen out exponentially. In Part B we will derive similar $L(x)$ which will dampen out even faster: like $\exp \left(-C^{2} x^{2}\right)$.

The paper is divided into five chapters. In chapter I we discuss the general problem of smoothing by means of a linear compound formula. This discussion, by no means exhaustive, is to serve as a guide to what is likely to be useful among formulae of the cardinal type (1) which smooth and interpolate at the same time. It serves to restrict somewhat the arbitrariness of the problem. The rather obvious idea of using cosine polynomials (or series) in this connection affords the possibility of a brief exposition of this subject in the more scientific manner of E. de Forest, W. F. Sheppard, E. T. Whittaker, and others, and may be followed up elsewhere.

Chapters II and III form the common foundation of both parts A and B. In chapter II we describe the interpolatory properties of the formula (1) in terms of extremely simple properties of the Fourier-transform

$$
\begin{equation*}
g(u)=\int_{-\infty}^{\infty} L(x) \cos u x d x \tag{5}
\end{equation*}
$$

of the basic function $L(x)$ (Theorem 4). Thus we are assured that our formula (1) will be exact for (i.e., reproduce) polynomials of degree $k-1$, provided $g(u)-1$ has a zero of order $k$ at $u=0$ and $g(u)$ has zeros of order $k$ at all points $u=2 \pi n$ ( $n= \pm 1, \pm 2, \cdots$ ). This elementary fact is reminiscent of N . Wiener's fundamental

[^25]description of the closure properties of the family of translation functions $\{L(x-\lambda)\}$ in terms of the zeros of $g(u)$. Chapter III contains a somewhat general discussion of polygonal lines, the individual ares of which are polynomials of degree $k-1$, joined together with $k-2$ continuous derivatives. A general parametric representation of such curves is obtained (Theorem 5) which greatly facilitates their use for the purpose of approximation of data. For $k=4$ they represent approximately the curves drawn by means of a spline and for this reason we propose to call them spline curves of order $k$. These polynomial spline curves are finally smoothed out, by means of one-dimensional heat flow during the time interval $t$, into analytic spline curves of order $k$. An analytic spline curve of order $k$ is represented by a series of the cardinal type
\[

$$
\begin{equation*}
F(x)=\sum_{n=-\infty}^{\infty} f_{n} M_{k}(x-n, t) \tag{6}
\end{equation*}
$$

\]

where the basic function $M_{k}(x, t)$ is defined as

$$
\begin{equation*}
M_{k}(x, t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{-i(u / 2)^{2}}\left(\frac{2 \sin u / 2}{u}\right)^{k} \cos u x d x, \tag{7}
\end{equation*}
$$

while the coefficients $f_{n}$ may be thought of as arbitrary parameters.
The family of functions (6) forms the basis of our work. Its principal advantages for purposes of numerical approximation spring from two sources: 1) The basic function $M_{k}(x, t)$ dampens out like $\exp \left(-x^{2} t^{-1}\right)$ (see III, formula (39)). As seen from our Table I, for $k=4$ and $t=0.5$, we have $M_{4}(x, 1 / 2)=0$ to something like 10 decimal places for $|x| \geqq 5$. This causes the great flexibility of the graph of $F(x)$ on varying the parameters $f_{n}$ and the ease in computing $F(x)$.2) The family (6) contains (or represents) all polynomials of degree $k-1$. The simplest analytic family of this type is obtained for $k=0$ and $t>0$ when (6) becomes

$$
\begin{equation*}
F(x)=\sum_{n=-\infty}^{\infty} f_{n} \frac{1}{\sqrt{\pi t}} e^{-(x-n)^{2} / l} . \tag{8}
\end{equation*}
$$

This family obviously still enjoys the first property. However, (8) fails badly in its ability of representing even the simplest types of curves because of the low value of $k=0$. Indeed $F(x) \equiv 0$, for all $f_{n}=0$, is the only constant value (8) is capable of representing.

Chapter IV contains the chief results of the present Part A. We show how the family of curves (6) can be used to approximate given data. First we derive an analytic interpolation formula of the cardinal type (1) which leaves the given ordinates unchanged (Theorem 8). Secondly we extend the result to a family of formulae depending on a positive smoothing parameter $\epsilon$ such as to combine a certain variable amount of smoothing (depending on $\epsilon$ ) with the operation of interpolation (Theorem 9).

In collaboration with Lt. J. H. Levin, the author has had the opportunity of applying on a large scale this analytic approximation method at the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland. The computations were performed on punch-card machines. The given equidistant data $y_{n}$ were the values of the drag coefficient of a projectile as a function of its velocity. Since very accurately computed values of the derivatives $F^{\prime}(x), F^{\prime \prime}(x)$ of the approximation $F(x)$ were also desired, it seems doubtful if any of the existing osculatory interpolation formulae
would have furnished satisfactory results in view of the complicated trend of the data to be approximated.

In the last chapter we discuss procedures for the accurate computation of the functions and constants tabulated at the end of the paper. The most noteworthy problem encountered in this connection is the following: Let

$$
\begin{equation*}
F(z)=\sum_{-\infty}^{\infty} a_{n} z^{n} \tag{9}
\end{equation*}
$$

be a Laurent series which converges in a ring $\alpha<|z|<\beta$. We assume furthermore that $f(z)$ does not vanish in this ring:

$$
\begin{equation*}
F(z) \neq 0, \quad(\alpha<|z|<\beta) \tag{10}
\end{equation*}
$$

Under these circumstances we have an expansion of the reciprocal

$$
\begin{equation*}
\frac{1}{F(z)}=\sum_{-\infty}^{\infty} \omega_{n} z^{n} \tag{11}
\end{equation*}
$$

If the coefficients $a_{n}$ of the expansion (9) are given numerically the problem consists in finding very accurate numerical values of the $\omega_{n},{ }^{3}$ A very efficient iteration method solving this problem has been developed by H. A. Rademacher and the author. It solves the similar problem of finding the expansion of the $n$th root of $f(z)$ and generally of any algebraic function of Laurent series. This subject will be discussed elsewhere in a joint publication with Professor Rademacher.

In a sequel to these papers we expect to discuss the fitting of curves of the form (6) to data, in the sense of least squares. This will be accomplished by constructing series of the cardinal type (1) which also enjoy the orthogonality property

$$
\int_{-\infty}^{\infty} L(x) L(x-n) d x=\left\{\begin{array}{lll}
1 & \text { if } & n=0 \\
0 & \text { if } & n \neq 0
\end{array}\right.
$$

This construction reduces to the problem of computing the Laurent expansion of the square root $\sqrt{f(z)}$ of an expansion (9).

The author wishes to express his appreciation for the encouraging interest shown in his work by Dr. A. N. Lowan of the Mathematical Tables Project. He has benefited much by the helpful advice of Dr. L. S. Dederick, Major A. A. Bennett, Lt. J. H. Levin and others. Especially valuable were the author's frequent discussions with Dr. C. B. Morrey. The tables were computed by Mrs. Mildred Young. The author takes this opportunity of expressing his thanks to the officials of the Ballistic Research Laboratory for their permission to publish these tables.

The reader who is mainly interested in the numerical applications, may pass directly from this point to the Appendix, where the use of the tables is fully explained and one example is worked out.

[^26]
## I. DEFINITIONS OF SMOOTHING AND SMOOTHING FORMULAE

1.1. A definition of smoothing formulae. Let $\left\{y_{n}\right\}(n=\cdots-2,-1,0,1,2, \cdots)$ be a given sequence or "table" which we wish to smooth. This smoothing operation is ordinarily performed by means of a formula of the following type

$$
\begin{equation*}
F_{n}=y_{n-p} L_{p}+\cdots+y_{n-1} L_{1}+y_{n} L_{0}+y_{n+1} L_{-1}+\cdots+y_{n+p} L_{-p} \tag{1}
\end{equation*}
$$

where the numerical coefficients $L_{\text {, }}$ are symmetric about the middle term $L_{0}$, i.e., $L_{y}=L_{-p}$. The linear transformation (1) if applied to the original sequence $\left\{y_{n}\right\}$ will transform it into the smoothed sequence $\left\{F_{n}\right\}$. By extending the definition of $L_{\nu}=0$ for $|\nu|>p$ we may rewrite (1) as

$$
\begin{equation*}
F_{n}=\sum_{p=-\infty}^{\infty} y_{v} L_{n-p} . \tag{2}
\end{equation*}
$$

If $y_{i}=$ const. $=c$, we also wish that $F_{n}=c$; therefore

$$
\begin{equation*}
\sum_{p} L_{p}=1 \tag{3}
\end{equation*}
$$

is a natural requirement.
When does the formula (1) actually smooth? As an example let $p=1$ and let the coefficients $L_{v}$ be $(-1,3,-1)$. If we now apply the formula (1) to the periodic sequence

$$
\left\{y_{n}\right\}=\{\cdots, 0,1,0,1,0,1, \cdots\}
$$

we obtain

$$
\left\{F_{n}\right\}=\{\cdots,-2,3,-2,3,-2,3, \cdots\}
$$

which is a good deal rougher than the original sequence. Obviously this situation deserves some clarification.

There seems no doubt that the "smoothness" of a sequence $\left\{y_{n}\right\}$ depends in some way on its differences of higher order, especially on the sums of their square. We also notice that the formula (2) agrees with the rule of multiplication of Fourier series. This suggests the use of such series.

Let us assume for the moment that

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty}\left|y_{n}\right|<\infty . \tag{4}
\end{equation*}
$$

We now define a function $T(u)$ by

$$
\begin{equation*}
T(u)=\sum_{n=-\infty}^{\infty} y_{n} e^{i n u} \tag{5}
\end{equation*}
$$

and call it the characteristic function of our sequence $\left\{y_{n}\right\}$; it is a complex-valued continuous function of $u$ of period $2 \pi$.

Now (5) implies

$$
e^{-i u} T(u)=\sum y_{n} e^{i(n-1) u}=\sum y_{n+1} e^{i n u}
$$

and by subtracting (5) we get

$$
\begin{equation*}
\left(e^{-i u}-1\right) T(u)=\sum_{n=-\infty}^{\infty} \Delta y_{n} e^{i n u} \tag{6}
\end{equation*}
$$

This shows that we obtain the characteristic function of the sequence $\left\{\Delta y_{n}\right\}$ of first
differences of $\left\{y_{n}\right\}$ by multiplying the characteristic function $T(u)$ of $\left\{y_{n}\right\}$ by the factor $e^{-i u}-1$. Generally

$$
\begin{equation*}
\left(e^{-i u}-1\right)^{m} T(u)=\sum_{n=-\infty}^{\infty} \Delta^{m} y_{n} e^{i n u} \quad(m=0,1,2, \cdots) . \tag{7}
\end{equation*}
$$

Since $\left|e^{-i u}-1\right|=2|\sin (u / 2)|$, the Parseval relation furnishes the equation

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty}\left(\Delta^{m} y_{n}\right)^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}[2 \sin (u / 2)]^{2 m}|T(u)|^{2} d u \quad(m \geqq 0) \tag{8}
\end{equation*}
$$

These formulae furnish an expression for the sums of the squares of the differences of any order in terms of the characteristic function $T(u)$ of the sequence.

Let us now turn to the "smoothed" sequence $\left\{F_{n}\right\}$. Let

$$
\begin{equation*}
\phi(u)=\sum_{n=-\infty}^{\infty} L_{n} e^{i n u}=L_{0}+2 L_{1} \cos u+2 L_{2} \cos 2 u+\cdots \tag{9}
\end{equation*}
$$

be the characterstic function of the sequence $\left\{L_{n}\right\}$. We shall also refer to $\phi(u)$ as the characteristic function of the smoothing formula (2). Notice that $\phi(u)$ is always real and even. By multiplication of the two Fourier series (5) and (9) we obtain, in view of (2),

$$
\begin{equation*}
T(u) \phi(u)=\sum_{n=-\infty}^{\infty} F_{n} e^{i n u} . \tag{10}
\end{equation*}
$$

Hence the characteristic function of the "smoothed" sequence $\left\{F_{n}\right\}$ is obtained by multiplying the characteristic function $T(u)$ of $\left\{y_{n}\right\}$ by the characteristic function $\phi(u)$ of the smoothing formula (2). By now applying (8) to the sequence $\left\{F_{n}\right\}$ we obtain

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty}\left(\Delta^{m} F_{n}\right)^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}(2 \sin (u / 2))^{2 m}|T(u)|^{2}(\phi(u))^{2} d u, \quad(m \geqq 0) . \tag{11}
\end{equation*}
$$

A comparison of the relations (8) and (11) will readily furnish an answer to the question: what is a smoothing formula? Indeed, we notice that the integrands in (8) and (11) differ only, for each fixed value of $m$, by the factor $\phi(u)^{2}$ in (11). This justifies the following definition.

Definition 1. Let $L_{n}$ be a symmetric sequence of coefficients, i.e., $L_{-n}=L_{n}$. The formation of the weighted means

$$
\begin{equation*}
F_{n}=\sum_{n=-\infty}^{\infty} y_{n} L_{n-\prime} \quad(n=0, \pm 1, \pm 2, \cdots) \tag{12}
\end{equation*}
$$

is said to be a smoothing formula if

$$
\begin{align*}
& \sum_{n} L_{n}=1,  \tag{13}\\
& \sum_{n}\left|L_{n}\right|<\infty, \tag{14}
\end{align*}
$$

while the characteristic function

$$
\begin{equation*}
\phi(u)=\sum_{-\infty}^{\infty} L_{n} e^{i n u}=L_{0}+2 L_{1} \cos u+2 L_{2} \cos 2 u+\cdots \tag{15}
\end{equation*}
$$

satisfies the condition

$$
\begin{equation*}
-1 \leqq \phi(u) \leqq 1, \quad(0 \leqq u \leqq 2 \pi) \tag{16}
\end{equation*}
$$

The necessity of the condition (16) is justified as follows: By a comparison of (8) and (11), in view of (16), we obtain the inequalities

$$
\sum_{n=-\infty}^{\infty}\left(\Delta^{m} F_{n}\right)^{2} \leqq \sum_{n=-\infty}^{\infty}\left(\Delta^{m} y_{n}\right)^{2}, \quad(m=0,1,2, \cdots)
$$

Actually the equality sign in one of these relations will arise only under highly exceptional or else trivial conditions. This remark should make it clear why the smoothing quality of a formula violating (16) should be highly questionable.

So far we were concerned merely with the ability of a formula (2) to smooth the sequence. However, the discrepancies between the two sequences also deserve attention. By subtracting (10) from (5) we obtain

$$
T(u)(1-\phi(u))=\sum\left(y_{n}-F_{n}\right) e^{i n u}
$$

and therefore

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty}\left(y_{n}-F_{n}\right)^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}|T(u)|^{2}(1-\phi(u))^{2} d u . \tag{17}
\end{equation*}
$$

A comparison of the integrands of (17) and (11) reveals the obvious fact that strong smoothing may be achieved only if we allow relatively large discrepancies between $F_{n}$ and $y_{n}$. Indeed, the integral of (17) will be small only if $\phi(u)$ differs but little from 1 , while strong smoothing requires as small a $\phi(u)$ as possible.
1.11. Examples of smoothing formulae. (a) Our trivial example $L_{0}=3, L_{1}=L_{-1}=-1$, $L_{n}=0(n>1)$ has the characteristic function $\phi(u)=3-2 \cos u$. We find $\phi(u) \geqq 1$, with $\phi(\pi)=5$, which rules it out as a smoothing formula.
(b) If $L_{n} \geqq 0$ for all $n$, and $\sum L_{1}=1$, then (12) is always a smoothing formula. Indeed

$$
|\phi(u)|=\left|\sum L_{n} e^{i n u}\right| \leqq \sum\left|L_{n}\right|=1
$$

Thus

$$
\begin{equation*}
F_{n}=\left(y_{n-1}+y_{n}+y_{n+1}\right) / 3 \tag{18}
\end{equation*}
$$

is a smoothing formula with

$$
\phi(u)=(1+2 \cos u) / 3
$$

Let

$$
\phi_{1}(u)=|\phi(u)|=|1+2 \cos u| / 3=\sum L_{n}^{(1)} \cos u u .
$$

Since $(\phi(u))^{2}=\left(\phi_{1}(u)\right)^{2}$ it is clear from (11) that the formula (18) and the formula of characteristic function $\phi_{1}(u)$ have identical smoothing powers. However, since $0<1-\phi_{1}(u)<1-\phi(u)$ for $2 \pi / 3<u<4 \pi / 3$, we see by (17) that the formula

$$
F_{n}^{(1)}=\sum y_{v} L_{n-\nu}^{(1)}
$$

will alter the sequence $\left\{y_{n}\right\}$ much less than (18) will.
(c) Generally, our formula (17) shows that it is desirable for an efficient smoothing formula to have its characteristic function satisfy the more restrictive condition

$$
\begin{equation*}
0 \leqq \phi(u) \leqq 1 . \tag{19}
\end{equation*}
$$

1.12. A comparison of smoothing formuiae. Again our relations (11) justify the following definition.

Definition 2. Let $\phi_{1}(u)$ and $\phi_{2}(u)$ be the characteristic functions of two smoothing formulae. We say that the first is stronger than the second if

$$
\begin{equation*}
\left|\phi_{1}(u)\right| \leqq\left|\phi_{2}(u)\right| . \tag{20}
\end{equation*}
$$

with the inequality sign holding for some value of $u$.
Later in this paper we shall set up a basic sequence of smoothing formulae of progressively greater strength according to this definition. Here we remark only that two smoothing formulae cannot in general be compared on the basis of this definition. However, the following remark seems obvious. Let

$$
\begin{equation*}
F_{n}=\sum_{v} y_{v} L_{n-v} \tag{21}
\end{equation*}
$$

be a smoothing formula of characteristic function $\phi(u)$. The iteration, or repetition, of (21) may be thought of as another smoothing formula and its characteristic function is found to be $(\phi(u))^{2}$. Since $|\phi(u)| \leqq 1$ obviously

$$
(\phi(u))^{2} \leqq|\phi(u)| .
$$

This shows that the formula (21) and the sequence of its successive iterates form a sequence of smoothing formula of progressively increasing strength.
1.13. Smoothing formulae which are exact for polynomial values of a given degree. The following definition is in common use.

Definition 3. A smoothing formula (2) is said to be exact for the degree $m$ if it reproduces exactly the values $\left\{y_{n}\right\}$ of a polynomial of degree not exceeding $m$.

If

$$
\begin{equation*}
F_{n}=\sum y_{v} L_{n-\eta} \tag{22}
\end{equation*}
$$

is to be exact for the degree $m$, it is obviously sufficient that it be exact for the basic monomials $1, x, \cdots, x^{m}$. Thus the exactness for the degree $m$ is equivalent to the relations

$$
\begin{equation*}
n^{*}=\sum_{n-\infty}^{\infty} \nu^{*} L_{n-v} \quad(s=0,1, \cdots, m) \tag{23}
\end{equation*}
$$

Let us now assume for simplicity that the sequence of coefficients $L_{n}$ tends to zero exponentially as $n \rightarrow \infty$, i.e., we assume the existence of two positive constants $A$ and $B$ such that

$$
\left|L_{n}\right| \leqq A e^{-B|n|}
$$

for all values of $n$. This implies that the function

$$
\phi(u)=\sum_{n} L_{n} e^{i n u}
$$

is regular in the strip

$$
|I u|<B
$$

of the complex $u$-plane. Now

$$
\phi(u)=\sum L_{v} e^{i v u}=\sum L_{r-n} e^{i v u} \cdot e^{-i n u}
$$

and

$$
e^{i n u} \phi(u)=\sum_{\nu} e^{i v u} L_{n-\gamma}
$$

We now expand both sides in ascending powers of $u$ and compare like coefficients. Since

$$
\phi(u)=1+\frac{u^{2}}{2!} \phi^{\prime \prime}+\frac{u^{4}}{4!} \phi^{(4)}+\cdots,
$$

we get the identities in $n$

$$
n^{s}-\binom{s}{2} \phi^{\prime \prime} n^{s-2}+\binom{s}{4} \phi^{(4)} n^{8-4}-\cdots=\sum_{v=-\infty}^{\infty} \nu^{s} L_{n-p} \quad(s=0,1,2, \cdots)
$$

A comparison with (23) will show that a smoothing formula is always exact for a highest degree which is always odd. It also proves the following proposition which may evidently be established under conditions less stringent than the ones we used.

Theorem 1. A smoothing formula (22) is exact for a degree $2 \nu+1$ if and only if $\phi(u)-1$ has at $u=0$ a zero of order $2 \nu+2$, i.e.,

$$
\begin{equation*}
\phi^{\prime \prime}(0)=\phi^{(4)}(0)=\cdots=\phi^{(2 \nu)}(0)=0 \tag{24}
\end{equation*}
$$

As an illustration we mention the formula

$$
\begin{equation*}
F_{n}=\frac{1}{32}\left(-y_{n-3}+9 y_{n-1}+16 y_{n}+9 y_{n+1}-y_{n+3}\right)=y_{n}-\frac{3}{16} \delta^{4} y_{n}-\frac{1}{32} \delta^{6} y_{n} \tag{25}
\end{equation*}
$$

of characteristic function

$$
\begin{equation*}
\phi(u)=(8+9 \cos u-\cos 3 u) / 16 \tag{26}
\end{equation*}
$$

We find that $\phi^{\prime \prime}(0)=0$, hence (25) is exact for cubics. The symmetry property $\phi(u)+\phi(\pi-u)=1$ shows that

$$
\phi(\pi)=\phi^{\prime}(\pi)=\phi^{\prime \prime}(\pi)=\phi^{\prime \prime \prime}(\pi)=0 .
$$

This results in rather strong smoothing power. The formula (25) is part of a sequence of formulae, the next one of this kind being

$$
\begin{equation*}
F_{n}=\frac{1}{512}\left(3 y_{n-5}-25 y_{n-3}+150 y_{n-1}+256 y_{n}+150 y_{n+1}-25 y_{n+3}+3 y_{n+5}\right) \tag{27}
\end{equation*}
$$

or

$$
F_{n}=y_{n}+\frac{5}{32} \delta^{6} y_{n}+\frac{15}{256} \delta^{8} y_{n}+\frac{3}{512} \delta^{10} y_{n}
$$

Its characteristic function

$$
\phi(u)=(128+150 \cos u-25 \cos 3 u+3 \cos 5 u) / 256
$$

again enjoys the symmetry property $\phi(u)+\phi(\pi-u)=1$. Also $\phi(u)-1$ has a zero of
order 6 at $u=0$, hence $(25)$ is exact for quintics, while $\phi(u)$ has a zero of order 6 at $u=\pi$ resulting in strong smoothing power.
1.2. Smoothing a finite table. In 1.1 we have discussed the smoothing of an infinite table $\left\{y_{n}\right\}$ which is such that the series of the absolute values of its entries converges. By (8), (11) and the inequalities (16) we have found that the sum of the squares of the differences of order $m$ is diminished by smoothing. This is true for $m=0,1,2, \ldots$. Now we shall discuss briefly the practically most important case of a given finite table

$$
\begin{equation*}
\left\{y_{n}\right\} \quad(n=0,1, \cdots, N) \tag{28}
\end{equation*}
$$

To fix the ideas we assume the following simplest concrete situation: the third differences $\Delta^{3} y_{n}$ are slowly varying and of slowly varying signs, while the $\Delta^{4} y_{n}$ are of random signs. In this situation we naturally wish to minimize the 4 th differences of the table. Now we form an average value of the $\Delta^{3} y_{n}$ at each of the two ends of the table and we extend the column of $\Delta^{3} y_{n}$ with the corresponding constant average value at each end. ${ }^{4}$ Thus the $\Delta^{3} y_{n}$ are defined for all $n$ having one constant value for $n>N-3$ and another constant value for $n<0$. Now we extend the definition of $y_{n}$ for all $n$ from the values of the third differences. Also, we compute the $\Delta^{4} y_{n}$ for the extended infinite table. Clearly $\Delta^{4} y_{n}=0$ for $n<-1$ or $n>N-3$. Let

$$
T_{4}(u)=\sum \Delta^{4} y_{n} c^{i n u}
$$

be the characteristic function of the sequence of 4 th differences, the series containing really a finite sum of terms only.

Let us now apply to the extended table $y_{n}$ a smoothing formula

$$
\begin{equation*}
F_{n}=\sum_{\nu} y_{v} L_{n-\nu} \tag{29}
\end{equation*}
$$

of characteristic function $\phi(u)$, which is exact for cubics. The result is the new sequence $\left\{F_{n}\right\}(-\infty<n<\infty)$. Evidently $y_{n}$ are the values of cubics for large $|n|$ and therefore $F_{n}=y_{n}$ for large $|n|$, hence also $\Delta^{3} F_{n}=\Delta^{3} y_{n}$ and $\Delta^{4} F_{n}=0$ for large $|n|$. Notice also that we may think of the sequence $\left\{\Delta^{4} F_{n}\right\}$ as arising from $\left\{\Delta^{4} y_{n}\right\}$ by the smoothing operation (29). Therefore

$$
\sum_{n=-\infty}^{\infty}\left(\Delta^{4} y_{n}\right)^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left|T_{4}(u)\right|^{2} d u
$$

and

$$
\sum_{n=-\infty}^{\infty}\left(\Delta^{4} F_{n}\right)^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left|T_{4}(u)\right|^{2} \phi(u)^{2} d u
$$

Generally

$$
\begin{align*}
& \sum_{n=-\infty}^{\infty}\left(\Delta^{4+m} y_{n}\right)^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}(2 \sin u / 2)^{2 m}\left|T_{4}(u)\right|^{2} d u, \quad(m \geqq 0)  \tag{30}\\
& \sum_{n=-\infty}^{\infty}\left(\Delta^{4+m} F_{n}\right)^{2}=\frac{1}{2 \pi} \int_{0}^{2 \pi}(2 \sin u / 2)^{2 m}\left|T_{4}(u)\right|^{2} \phi(u)^{2} d u, \quad(m \geqq 0) \tag{31}
\end{align*}
$$

[^27]A comparison of (30) and (31) shows that the sums of the squares of the fourths and subsequent differences have been decreased by the smoothing operation. No such statement can or should be inferred concerning the finite sums of relevant differences of orders $0,1,2$, and 3 .

## II. INTERPOLATION FORMULAE

2.1. Interpolation formulae of the cardinal type. Let

$$
\begin{equation*}
F_{n}=\sum_{n} y_{v} L_{n-\eta} \tag{1}
\end{equation*}
$$

be a smoothing formula. If we apply it to the "elementary" table

$$
\begin{equation*}
y_{0}=1, \quad y_{n}=0 \quad(n \neq 0) \tag{2}
\end{equation*}
$$

then $F_{n}=L_{n}$. The even sequence $\left\{L_{n}\right\}$ may therefore be regarded as the smoothed version by (1) of the elementary table. Now suppose that we are given not only the even sequence of ordinates $L_{n}$ but an even function $L(x)$ defined for all real $x$ and such that $L(n)=L_{n}$. Then we may replace the integral variable $n$ in (1) by the continuous variable $x$ and we obtain the formula

$$
\begin{equation*}
F(x)=\sum_{\nu=-\infty}^{\infty} y_{v} L(x-\nu) \tag{3}
\end{equation*}
$$

We call $L(x)$ the basic function of the formula (3). The chief aim of this paper is to point out that the subject of interpolation is truly dominated by the formulae of the type (3), the kind of approximation desired depending only on the choice of the basic function $L(x)$. The particular basic function

$$
\begin{equation*}
L(x)=\frac{\sin \pi x}{\pi x} \tag{4}
\end{equation*}
$$

gives rise to the series

$$
\begin{equation*}
F(x)=\sum_{v=-\infty}^{\infty} y_{r} \frac{\sin \pi(x-\nu)}{\pi(x-\nu)} \tag{5}
\end{equation*}
$$

which is well known to mathematicians and referred to as the cardinal series. For this reason we wish to call (3) a series, or formula of cardinal type.

We notice here for further reference that the basic function (4) may also be written as a Fourier integral as follows.

$$
\begin{equation*}
\frac{\sin \pi x}{\pi x}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} e^{i x u} d u \tag{6}
\end{equation*}
$$

2.11. The two kinds of interpolation formulae, ordinary or smoothing. For integral values of $x=n$ our formula (3) becomes

$$
\begin{equation*}
F(n)=\sum_{\nu=-\infty}^{\infty} y_{\nu} L(n-\nu) \tag{7}
\end{equation*}
$$

Equation (3) is an interpolation formula in the usual sense if $F(n)=y_{n}$, for all $n$, and this is the case if and only if $L(x)$ satisfies the conditions

$$
\begin{equation*}
L(0)=1, \quad L(n)=0 \quad(n ; \neq 0) \tag{8}
\end{equation*}
$$

Otherwise, (7) is a smoothing formula. We shall follow the accepted actuarial practice of referring to (3) as an ordinary interpolation formula if (3) reproduces exactly the given ordinates $\left\{y_{n}\right\}$. Otherwise we call (3) a smoothing interpolation formula.
2.12. Examples of interpolation formulae of the cardinal type. Later in this paper we shall discuss various classes of such interpolation formulas all arising from a common general theory. For purposes of orientation and illustration we mention here a few concrete examples.
2.121. The $k$-point central interpolation formula. Let $k$ be a fixed integer ( $=1,2,3, \cdots$ ). By $k$-point central interpolation we mean the interpolation method whereby the polynomial of degree at most $k-1$, defined by $k$ consecutive ordinates $y_{n}$, is utilized within an interval of unit length centrally located with respect to the set of defining ordinates $y_{n}$. This set of $k$ defining ordinates $y_{n}$ is shifted up by one unit in the subscript for interpolation in the next unit interval. It seems obvious that this kind of interpolation is performed for any real value of $x$ by a formula of the cardinal type

$$
\begin{equation*}
F(x)=\sum_{n=-\infty}^{\infty} y_{n} C_{k}(x-n) \tag{9}
\end{equation*}
$$

with a suitable function $C_{k}(x)$. To obtain this function, it is sufficient to interpolate the elementary table (2) by means of this method of $k$-point central interpolation. The graphs indicate the resulting $C_{k}(x)$ for $k=1,2,3$, and $4 .{ }^{5}$ It is found that $C_{k}(x)$


[^28]we define the corresponding truncated function
\[

x_{+}^{[k]-1}=\left\{$$
\begin{array}{lll}
x^{[k]-1} & \text { if } & x>0 \\
0 & \text { if } & x<0
\end{array}
$$ \quad(k=1,2,3, \cdots)\right.
\]

is a polygonal line composed of arcs of degree $k-1$. Also $C_{k}(x)$ itself is continuous with a discontinuous first derivative if $k$ is even. For an odd $k, C_{k}(x)$ itself is discontinuous, the value assigned at a point of discontinuity being the arithmetic mean of the two local limits. Evidently for $k=2$ our formula (9) is identical with the method of linear interpolation and the graph of $F(x)$, as given by (9), is identical with the polygonal line of vertices $\left(n, y_{n}\right)$.

For further reference we mention the following formulae, which are valid for all real values of $x$

$$
\begin{align*}
& C_{1}(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\sin u / 2}{u / 2} e^{i u x} d u, \\
& C_{2}(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\frac{\sin u / 2}{u / 2}\right)^{2} e^{i u x} d u, \\
& C_{3}(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\frac{\sin u / 2}{u / 2}\right)^{3}\left(1+\frac{1}{8} u^{2}\right) e^{i u x} d u, \\
& C_{4}(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\frac{\sin u / 2}{u / 2}\right)^{4}\left(1+\frac{1}{6} u^{2}\right) e^{i u x} d u . \tag{10}
\end{align*}
$$

The $k$-point central interpolation method is the most important method for the interpolation and the construction of tables of analytic and regular functions. However, for the construction of tables of empirical functions, the low order of continuity of $C_{k}(x)$ is at times a serious limitation of this method. It seems indeed evident that the continuity properties of the linear compound

$$
F(x)=\sum_{n} y_{n} L(x-n)
$$

are directly dependent on the continuity properties of the basic function $L(x)$. We turn now to an interesting example of an "osculatory" interpolation formula having a basic $L(x)$ enjoying stronger continuity properties.
2.122. An osculatory interpolation formula of W. A. Jenkins. We define a basic function $L(x)$ for $x \leqq 0$ by

$$
L(x)= \begin{cases}0 & \text { if } x \leqq-3  \tag{11}\\ -\frac{1}{12}(x+3)^{3}(x+2) & \text { if }-3 \leqq x \leqq-2 \\ \frac{1}{12}(x+1)(x+2)(x+3)(3 x+7) & \text { if }-2 \leqq x \leqq-1 \\ \frac{1}{6}(x+1)\left(6-6 x-9 x^{2}-x^{3}\right) & \text { if }-1 \leqq x \leqq 0\end{cases}
$$

The definition of this function is to be completed for $x=0$ by continuity, if $k$ is given, and by the arithmetic mean of the two limits, if $k$ is odd. Then

$$
C_{k}(x)=\frac{1}{(k-1)!} \delta^{k} x_{+}^{[k]-1} \quad(-\infty<x<\infty)
$$

where $\delta^{k}$ is the symbol of the $k$ th central difference of step unity (compare Theorem 3 of section 3.11). We will return to this subject in Part B where the extremely simple law of formation of the integrals (10) will also be given.
and extend its definition by the requirement $L(-x)=L(x)$ to all real values of $x$. Since the conditions (8) are visibly verified we see that

$$
F(x)=\sum y_{n} L(x-n)
$$

is an ordinary interpolation formula. A closer inspection of the composite polynomial function (11) will show that $L(x), L^{\prime}(x)$, and $L^{\prime \prime}(x)$ are all continuous for all real values of $x$. Using a customary mathematical terminology we may say that $L(x)$ is of class $C^{\prime \prime}$. Moreover, in various ways it may be shown that the formula (11) is exact if the $y_{n}$ are the ordinates of a polynomial of degree 3 or less, i.e., $F(x)$ becomes identical with that cubic polynomial.

It is of interest to compare Jenkins' formula ( $11^{\prime}$ ) with the 4 -point central interpolation formula (9) (for $k=4$ ). Both are exact for cubics. $C_{4}(x)$ is of class $C$, while the present $L(x)$ of class $C^{\prime \prime}$. This was achieved by increasing the complexity of the basic function in two ways: 1) The interval where $L(x)$ is non-vanishing was increased from $|x| \leqq 2$ to $|x| \leqq 3$. 2) The degree of the polynomial arcs has increased from 3 to 4. Later Jenkins' formula (11') will appear as a member of a sequence of interpolation formulae of similar characteristics. Here we mention that the basic function (11) may be expressed in the form

$$
L(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\frac{\sin u / 2}{u / 2}\right)^{4}\left(4+\cos u-4 \frac{\sin u}{u}\right) e^{i u x} d u
$$

for all real values of $x$.
2.123. A smoothing interpolation formula of $W$. A. Jenkins. We define a basic function $L(x)$ by

$$
L(x)= \begin{cases}0 & \text { if } x \leqq-3  \tag{12}\\ -\frac{1}{36}(x+3)^{3} & \text { if }-3 \leqq x \leqq-2 \\ \frac{1}{36}\left(69+117 x+63 x^{2}+11 x^{3}\right) & \text { if }-2 \leqq x \leqq-1 \\ \frac{1}{18}\left(15-27 x^{2}-14 x^{3}\right) & \text { if }-1 \leqq x \leqq 0 \\ L(-x)=L(x) . & \end{cases}
$$

This particular $L(x)$, composed of cubic ares, is of class $C^{\prime \prime}$. The formula ${ }^{6}$

- W. A. Jenkins writes his interpolation formula (11) in the following Everett form

$$
\begin{aligned}
F(n+x)= & y_{n} \xi+\dot{o}^{2} y_{n} \frac{\xi\left(\xi^{2}-1\right)}{6}-\delta^{4} y_{n} \frac{\xi^{3}(\xi-1)}{12} \\
& +y_{n+1} x+\delta^{2} y_{n+1} \frac{x\left(x^{2}-1\right)}{2}-\delta^{4} y_{n+1} \frac{x^{3}(x-1)}{12}, \quad(0 \leqq x \leqq 1, x+\xi=1)
\end{aligned}
$$

Likewise his formula ( 12 ') takes the form

$$
\begin{aligned}
F(n+x)= & y_{n} \xi+\delta^{2} y_{n} \frac{\xi\left(\xi^{2}-1\right)}{6}-\delta^{4} y_{n} \frac{\xi^{3}}{36} \\
& +y_{n+1} x+\delta^{2} y_{n+1} \frac{x\left(x^{2}-1\right)}{6}-\delta^{4} y_{n+1} \frac{x^{3}}{36} .
\end{aligned}
$$

$$
F(x)=\sum_{v} y_{\nu} L(x-\nu)
$$

corresponding to (12) is exact for polynomials of degree 3 or less. However, while (11) was an ordinary interpolation formula, the formula (12') is a smoothing interpolation formula. Since

$$
L(0)=15 / 18, \quad L(1)=2 / 18, \quad L(2)=-1 / 36
$$

while $L(n)=0$ for $n \geqq 3$, we see that for $x=n\left(12^{\prime}\right)$ reduces to a smoothing formula of characteristic function

$$
\phi(u)=\frac{1}{18}(15+4 \cos u-\cos 2 u)
$$

We readily verify that $\phi^{\prime \prime}(0)=0$ and $\phi(\pi)=5 / 9 \leqq \phi(u) \leqq 1$. Hence (12') reduces for integral $x=n$ to a smoothing formula, according to our Definition 1. On comparing Jenkins' two formulae ( $11^{\prime}$ ) and ( $12^{\prime}$ ) we notice that they are both exact for cubics, giving rise to curves of class $C^{\prime \prime}$. Since ( $12^{\prime}$ ) is only a smoothing interpolation formula while ( $11^{\prime}$ ) is an ordinary interpolation formula, it has been possible to lower the degree of $L(x)$ from 4 to 3 . We finally mention that the function (12) may be expressed as

$$
L(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\frac{\sin u / 2}{u / 2}\right)^{4}\left(\frac{4}{3}-\frac{1}{3} \cos u\right) e^{i u x} d u .
$$

2.2. A general theory of interpolation formulae of the cardinal type. In this section we shall discuss various characteristic properties of interpolation formulae of the cardinal type in terms of the Fourier transform of the basic function $L(x)$. This discussion will provide a sufficiently broad foundation for the subsequent development of specific formulae in the latter part of this paper.
2.21. Characteristic properties of interpolation formulae. Some of the following definitions have already occurred in the previous sections. For convenient reference we include them in our present enumeration of properties of an interpolation formula

$$
\begin{equation*}
F(x)=\sum_{\nu=-\infty}^{\infty} y_{p} L(x-\nu) \tag{13}
\end{equation*}
$$

a. We say that (13) is an ordinary interpolation formula if $F(x)$ interpolates exactly the given ordinates $y_{n}$, i.e., if

$$
\begin{equation*}
L(0)=1, \quad L(\nu)=0 \quad(\nu \neq 0) \tag{14}
\end{equation*}
$$

b. We say that (13) is a smoothing interpolation formula if for $x=n$ (13) turns into a smoothing formula

$$
\begin{equation*}
F(n)=\sum_{v} y_{v} L(n-\nu) \tag{15}
\end{equation*}
$$

The term "smoothing formula" is meant, of course, in the sense of our Definition 1, section 1.1.
c. We say that (13) is exact for the degree $k-1$ if the relation

$$
P(x)=\sum_{n=-\infty}^{\infty} P(n) L(x-n)
$$

is an identity for any polynomial $P(x)$ of degree at most $k-1$. The last condition is in turn equivalent with the $k$ identities

$$
\begin{equation*}
x^{\nu}=\sum_{n=-\infty}^{\infty} n^{\nu} L(x-n) \quad(\nu=0,1, \cdots, k-1) \tag{16}
\end{equation*}
$$

out of which it can always be recovered by means of suitable linear combinations.
d. We say that (13) preserves the degree $k-1,{ }^{7}$ if for any polynomial $P(x)$ of degree $\nu \leqq k-1$ we have an identity

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty} P(n) L(x-n)=P(x)+(\text { a polynomial of degree }<\nu) \tag{17}
\end{equation*}
$$

Notice that the leading term of $P(x)$ is not altered by (13). Again in terms of the monomials $x^{\nu}$ we may say: (13) preserves the degree $k-1$ whenever the $k$ functions

$$
\begin{equation*}
Q_{\nu}(x)=\sum_{n=-\infty}^{\infty} n^{\nu} L(x-n), \quad(\nu=0,1, \cdots, k-1) \tag{18}
\end{equation*}
$$

are polynomials of the form

$$
\begin{equation*}
Q_{\nu}(x)=x^{\nu}+a_{11} x^{\nu-1}+\cdots+a_{\nu v} \quad(\nu=0,1, \cdots, k-1) \tag{19}
\end{equation*}
$$

e. We say that (13) is of degree $m$ and of class $C^{\mu}$, if the basic function $L(x)$ is a polygonal line of polynomial arcs of degree at most $m$ joining in such a way as to result in a function $L(x)$ having $\mu$ continuous derivatives. In the sequel, the junction points will always be either for integral values $x=n$ or else for $x=n+1 / 2$. Consequently no "condensation" of discontinuities will result by the formation of the linear compound (13). Hence the interpolation curve $F(x)$ will again be of degree $m$ and of class $C^{\mu}$. As examples we recall the formulae (11') and ( $12^{\prime}$ ) of W. A. Jenkins, which are both of class $C^{\prime \prime}$ and of degree 4 and 3 , respectively.
f. A formula (13) whose basic function $L(x)$ is composed of polynomial arcs will also be referred to as a polynomial interpolation formula. We shall say that it has the span $s$ if the even function $L(x)$ vanishes identically for $x>s / 2$, but not for $x>s^{\prime}$ with $0<s^{\prime}<s / 2$. Thus the $k$-point central interpolation formula (9) is of span $k$, while both formulae (11'), (12') of Jenkins' have the span 6. For obvious practical reasons it is desirable to work with polynomial formulae having as small a span as possible.
g. We say that (13) is an analytic interpolation formula if the basic function $L(x)$ is analytic and regular for all real $x$. The original cardinal series (5) is an example of this type. Obviously no analytic formula can possibly have a finite span. The role of the span is taken over by the rate of damping of $L(x)$ as $x$ increases. For obvious practical reasons it is desirable to work with analytic $L(x)$ damping out as fast as possible.
2.22. The characteristic function of the basic function $L(x)$. It was shown in chapter

[^29]I that various properties of a smoothing formula

$$
F_{n}=\sum_{,} y_{\nu} L_{n \rightarrow}
$$

are readily expressible in terms of its characteristic function

$$
\phi(u)=\sum_{n} L_{n} e^{i n u} .
$$

Likewise, the properties of the interpolation formula

$$
\begin{equation*}
F(x)=\sum_{,} y_{\nu} L(x-\nu) \tag{20}
\end{equation*}
$$

will largely depend on the behaviour of the function

$$
\begin{equation*}
g(u)=\int_{-\infty}^{\infty} L(x) e^{i u x} d x=\int_{-\infty}^{\infty} L(x) \cos u x d x . \tag{21}
\end{equation*}
$$

This even function $g(u)$ is the Fourier transform of $L(x)$. Following a terminology used in probability theory we shall refer to $g(u)$ as the characteristic function of $L(x)$.

Under certain general assumptions which will always be verified in our application, the relation (21) may be inverted ${ }^{8}$ to

$$
\begin{equation*}
L(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} g(u) e^{i u x} d u . \tag{22}
\end{equation*}
$$

However, it should be remarked that at times our integrals are not absolutely convergent and that they then converge only as a principal value in the sense of Cauchy: $\lim _{A \rightarrow \infty} \int_{-A}^{1}$. An example of this kind is our first formula (10)

$$
C_{1}(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\sin u / 2}{u / 2} e^{i u x} d u .
$$

Changing $u$ and $x$ to $2 \pi u$ and $x / 2 \pi$ respectively we see

$$
C_{1}\left(\frac{x}{2 \pi}\right)=\int_{-\infty}^{\infty} \frac{\sin \pi u}{\pi u} e^{i u x d u .}
$$

Inverting this relation we obtain

$$
\begin{equation*}
\frac{\sin \pi x}{\pi x}=\frac{1}{2 \pi} \int_{-\infty}^{\infty} C_{1}\left(\frac{u}{2 \pi}\right) e^{i u x} d u=\frac{1}{2 \pi} \int_{-\pi}^{\pi} e^{i u x} d u \tag{23}
\end{equation*}
$$

which is identical with (6) and shows that

$$
g(u)=C_{1}\left(\frac{u}{2 \pi}\right)=\left\{\begin{array}{lll}
1 & \text { if } & |u|<\pi \\
\frac{1}{2} & \text { if } & |u|=\pi \\
0 & \text { if } & |u|>\pi
\end{array}\right.
$$

is the characteristic function of the original cardinal series (5). It is precisely the dis. continuity of its characteristic function which causes the extremely slow damping of the basic function (23). (See 2.21, g.)

[^30]Similar reasons of slow damping will rule out the following rather obvious method of turning a given smoothing formula

$$
F_{n}(x)=\sum_{p} y_{v} L_{n-r}
$$

into a smoothing interpolation formula

$$
F(x)=\sum_{\nu} y_{\nu} L(x-\nu) .
$$

From

$$
\phi(u)=\sum_{n} L_{n} e^{i n u}
$$

we derive

$$
L_{n}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \phi(u) e^{i n u} d u .
$$

Now we simply define a basic function $L(x)$ by

$$
L(x)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \phi(u) e^{i x u} d u .
$$

The corresponding characteristic function $g(u)$ is found to be

$$
g(u)=\left\{\begin{array}{ccc}
\phi(u) & \text { if } & |u|<\pi \\
0 & \text { if } & |u|>\pi
\end{array}\right.
$$

Again the discontinuities of $g(u)$, or of one of its higher derivatives, will imply that the damping of $L(x)$ is too slow for numerical purposes. Indeed, by partial integrations, $L(x)$ is found to tend to zero as a certain negative power of $x$ only, as $x$ tends to infinity. (Concerning the order of magnitude of Fourier integrals for large values of $x$, see the theorem on page 11 of Bochner's book quoted in our footnote 8.)
2.23. Fundamental criteria in terms of characteristic functions. We shall now restrict ourselves to basic functions $L(x)$ which are everywhere continuous with the exception of possible "discontinuities of the first kind" (such as were exhibited by the basic functions $L(x)$ of section 2.121). Moreover, we shall assume that $L(x)$ dampens out exponentially. This means that we assume the existence of two positive constants $A$ and $B$ such that the inequality

$$
\begin{equation*}
|L(x)|<A e^{-B|x|} \tag{24}
\end{equation*}
$$

holds for all real values of $x$. This clearly rules out the basic function (23) of the cardinal serics. The assumption (24) implies that the characteristic function

$$
\begin{equation*}
g(u)=\int_{-\infty}^{\infty} L(x) e^{i u x} d x \tag{25}
\end{equation*}
$$

is analytic and regular not only on the real $u$-axis but also in the infinite strip

$$
\begin{equation*}
|I u|<B \tag{26}
\end{equation*}
$$

of the complex $u$-plane. It also implies that the expression

$$
\begin{equation*}
\lim _{p \rightarrow \infty} \sum_{n=-p}^{n} g(u+2 \pi n) e^{2 \pi i n x} \tag{27}
\end{equation*}
$$

converges uniformly in a circular neighborhood of $u=0$ and for every real value of $x$.
The following theorem will demonstrate the usefulness of the characteristic function of an interpolation formula.

Theorem 2. Let the basic function $L(x)$ satisfy the condition (24). Let the corresponding interpolation formula be

$$
\begin{equation*}
F(x)=\sum_{p} y_{v} L(x-\nu) \tag{28}
\end{equation*}
$$

For integral $x=n$ (28) reduces to the smoothing formula

$$
\begin{equation*}
F(n)=\sum y_{p} L(n-\nu) \tag{29}
\end{equation*}
$$

A. The characteristic function $\phi(u)$, of the smoothing formula (29) is given by the relation

$$
\begin{equation*}
\phi(u)=\sum_{v=-\infty}^{\infty} g(u+2 \pi \nu) \tag{30}
\end{equation*}
$$

In particular (28) is an ordinary interpolation formula (see 2.21 , a) if and only if

$$
\begin{equation*}
\sum_{v=-\infty}^{\infty} g(u+2 \pi \nu) \equiv 1 \tag{31}
\end{equation*}
$$

B. The formula (28) is exact for the degree $k-1$ (see $2.21, c$ ) if the following two conditions hold simultaneously:
$g(u)-1$ has a zero of order $k$ at $u=0$,
$g(u)$ has zeros of order $k$ for all non-vanishing integral multiples of $2 \pi ; u=2 \pi n(n \neq 0)$.
C. The formula (28) preserves the degree $k-1$ (see 2.21, d) if the condition (33) holds, together with the additional condition

$$
\begin{equation*}
g(0)=1 \tag{34}
\end{equation*}
$$

Remark. For some applications it is important to notice that an ordinary interpolation formula which preserves the degree $k-1$ is automatically exact for the degree $k-1$. This seems evident a priori. It is also evident in terms of our criteria, for (31) implies

$$
g(u)-1=-\sum_{v=0} g(u+2 \pi v)
$$

and the right-hand side has a zero of order $k$ at $u=0$ by (33).
Proof of A. Our formula (25) implies

$$
\begin{aligned}
L(x) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} g(u) e^{i x u} d u=\lim _{p \rightarrow \infty} \frac{1}{2 \pi} \int_{-(2 p+1) \pi}^{(2 p+1) \pi} g(u) e^{i x u} d u \\
& =\lim _{p \rightarrow \infty} \frac{1}{2 \pi} \sum_{v=-p}^{p} \int_{-\pi}^{\pi} g(u+2 \pi \nu) e^{i u x} e^{2 \pi i u x} d u
\end{aligned}
$$

$$
\begin{aligned}
& =\lim _{p \rightarrow \infty} \frac{1}{2 \pi} \int_{-\pi}^{\pi}\left\{\sum_{\nu=-p}^{p} g(u+2 \pi \nu) e^{2 \pi i x \nu}\right\} e^{i u x} d u, \\
& =\frac{1}{2 \pi} \int_{-\pi}^{\pi}\left\{\sum_{v=-\infty}^{\infty} g(u+2 \pi \nu) e^{2 \pi i x \nu}\right\} e^{i u x} d u .
\end{aligned}
$$

In particular, if $x=n$ is an integer, we find

$$
\begin{equation*}
L(n)=\frac{1}{2 \pi} \int_{-\pi}^{\pi}\left\{\sum_{-\infty}^{\infty} g(u+2 \pi v)\right\} e^{i n u d u} . \tag{35}
\end{equation*}
$$

Since the characteristic function $\phi(u)$ of (29) is by definition the function of Fourier coefficients $L(n)$ (see 1.1, (9)), the relation (30) is established.

Proof of B. We wish to apply Poisson's summation formula ${ }^{9}$

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty} f(x-n)=\sum_{n=-\infty}^{\infty} e^{2 \pi i n x} \int_{-\infty}^{\infty} f(v) e^{-2 \pi i v n} d v \tag{36}
\end{equation*}
$$

to the function

$$
\begin{equation*}
f(x)=e^{-i x u} L(x) . \tag{37}
\end{equation*}
$$

By (37) and (25) we find

$$
\int_{-\infty}^{\infty} f(v) e^{-2 \pi i v n} d v=\int_{-\infty}^{\infty} L(v) e^{-i(u+2 \pi n) v} d v=g(u+2 \pi n)
$$

hence by (36)

$$
e^{-i x u} \sum_{n-\infty}^{\infty} e^{i u n} L(x-n)=\sum_{n=-\infty}^{\infty} g(u+2 \pi n) e^{2 \pi i n x}
$$

and finally

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty} e^{i u n} L(x-n)=e^{i u x} \sum_{n=-\infty}^{\infty} g(u+2 \pi n) e^{2 \pi i n x} . \tag{38}
\end{equation*}
$$

This identity actually holds for all real $x$ and all real or complex values of $u$ within the strip (26). It contains implicitly all the statements of Theorem 1 . Thus for $x=0$ it reduces to (30). To prove our statement B we assume $x$ fixed and regard both members of (38) as functions of $u$, which we expand in series of powers of $u$, then equating the respective coefficients on both sides. On the left-hand side we have the expansion

$$
\sum_{n=0}^{\infty} \frac{i^{v} u^{\nu}}{\nu!} \sum_{n=-\infty}^{\infty} n^{n} L(x-n) .
$$

On the right-hand side, our assumption (33) implies that the terms $g(u+2 \pi n) e^{2 \pi i n x}$ ( $n \neq 0$ ) do not contribute any terms in $u$ of order less than $k$. Thus our identity (38) becomes

$$
\begin{equation*}
\left.\sum_{v=0}^{\infty} \frac{i^{\nu} u^{\nu}}{\nu!} \sum_{n=-\infty}^{\infty} n^{\nu} L(x-n)=e^{i x u} g(u)+u^{k} \text { (regular function of } u\right) . \tag{39}
\end{equation*}
$$

On the other hand our assumption (32) amounts to

[^31]$$
g(u)=1+u^{k} \text { (regular function) }
$$

This and (39) imply

$$
\begin{equation*}
\sum_{v=0}^{\infty} \frac{i^{\nu} u^{\nu}}{\nu!} \sum_{n=-\infty}^{\infty} n^{\nu} L(x-n)=\sum_{i=0}^{\infty} \frac{i^{\nu} u^{\nu}}{\nu!} x^{\nu}+u^{k} \text { (regular function). } \tag{40}
\end{equation*}
$$

A comparison of the coefficients of the first $k$ terms on each side of (40) furnishes the identities (16). This concludes a proof of B .

Proof of C. Since $g(u)$ is regular at $u=0$, and even, it has in view of (34) an expansion of the form

$$
g(u)=1-\frac{a_{2}}{2!} u^{2}+\frac{a_{4}}{4!} u^{4}-\frac{a_{\mu}}{6!} u_{8}+\cdots
$$

We now define a sequence of polynomials by means of the generating function

$$
\begin{equation*}
e^{i x u} g(u)=\sum_{\nu=0}^{\infty} Q_{\nu}(x) \frac{(i u)^{\nu}}{\nu!} \tag{41}
\end{equation*}
$$

or

$$
\begin{equation*}
e^{x u} g(u / i)=\sum_{v=0}^{\infty} Q_{v}(x) \frac{u^{v}}{\nu!} \tag{42}
\end{equation*}
$$

where

$$
\begin{equation*}
g(u / i)=1+\frac{a_{2}}{2!} u^{2}+\frac{a_{4}}{4!} u^{4}+\cdots \tag{43}
\end{equation*}
$$

A comparison of terms on both sides of (42), using (43), shows that

$$
\begin{equation*}
Q_{\nu}(x)=x^{\nu}+\binom{\nu}{2} a_{2} x^{\nu-2}+\binom{\nu}{4} a_{4} x^{\nu-4}+\cdots \tag{44}
\end{equation*}
$$

On substituting the expansion (41) into the right-hand side of (39) and by comparison of the first $k$ terms on both sides we find that the identities (18) and (19) are established. This completes the proof of our theorem.

As a brief illustration of our criteria let us consider again Jenkins' smoothing interpolation formula ( $12^{\prime}$ ) of 2.123 . Its basic function is

$$
\begin{equation*}
L(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\frac{2 \sin u / 2}{u}\right)^{4}\left(\frac{4}{3}-\frac{1}{3} \cos u\right) e^{i u x} d u \tag{45}
\end{equation*}
$$

A simple method of evaluating explicitly such integrals in order to find the polynomial expressions (12) will be discussed later. An inspection of the characteristic function

$$
g(u)=\left(\frac{2 \sin u / 2}{u}\right)^{4}(4-\cos u) / 3
$$

immediately reveals that our condition (33) is verified for $k=4$. Direct expansion shows that $g(u)=1-(7 / 240) u^{4}+\cdots$ and (32) is also verified for $k=4$. The interpolation formula ( $12^{\prime}$ ) is therefore exact for cubics. Also the fact that $L(x)$ is of class $C^{\prime \prime}$ is revealed by an inspection of the integral (45). Indeed we notice that $g(u)$ vanishes for $u=\infty$ like $u^{-4}$. This implies that we may differentiate (45) twice under the integral sign and that the integral

$$
L^{\prime \prime}(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} g(u)(i u)^{2} e^{i u x} d u
$$

is also continuous since it converges absolutely.

## III. THE THEORY OF SPLINE CURVES

The previous chapter provides a formal theory of interpolation formulae in terms of a basic function $L(x)$ which, as yet, is largely arbitrary. The present chapter will furnish the foundation for the derivation of special basic functions which are readily computed with great accuracy and lead to interpolation formulae enjoying the properties described in the previous chapter.
3.1. Polynomial spline curves of order $k$. A spline is a simple mechanical device for drawing smooth curves. It is a slender flexible bar made of wood or some other elastic material. The spline is place on the sheet of graph paper and held in place at various points by means of certain heavy objects (called "dogs" or "rats") such as to take the shape of the curve we wish to draw. Let us assume that the spline is so placed and supported as to take the shape of a curve which is nearly parallel to the $x$-axis. If we denote by $y=y(x)$ the equation of this curve then we may neglect its small slope $y^{\prime}$, whereby its curvature becomes

$$
1 / R=y^{\prime \prime} /\left(1+y^{\prime 2}\right)^{3 / 2} \approx y^{\prime \prime} .
$$

The elementary theory of the beam will then show that the curve $y=y(x)$ is a polygonal line composed of cubic arcs which join continuously, with a continuous first and second derivative. ${ }^{10}$ These junction points are precisely the points where the heavy supporting objects are placed.
3.11. Description of spline curves of order $k$. Our last remark suggests the following definition.

Definition 4. A real function $F(x)$ defned for all real $x$ is called a spline curve of order $k$ and denoted by $\Pi_{k}(x)$ if it enjoys the following properties:

1) It is compressed of polynomial arcs of degree at most $k-1$.
2) It is of class $C^{k-2}$, i.e., $F(x)$ has $k-2$ continuous derivatives.
3) The only possible function points of the various polynomial arcs are the integer points $x=n$ if $k$ is even, or else the points $x=n+1 / 2$ if $k$ is odd.

Thus a $\Pi_{1}(x)$ is a step function with possible discontinuities at the points $x=n+1 / 2$. A $\Pi_{2}(x)$ has an ordinary polygonal graph with vertices only at the integer points $x=n$. A $\Pi_{4}(x)$ corresponds to the elementary mathematical description of an ordinary (infinite) spline with the "dogs" placed at all or only some of the points with $x=n$.

It should be noticed that if a $\Pi_{k}(x)$ is of class $C^{k-1}$, then $\Pi^{(k-1)}(x)$ must necessarily be constant for all $x$. Thus such a $\Pi_{k}(x)$ reduces to a polynomial of degree $k-1$. It is just this relaxation of the requirement of the continuity of the $(k-1)$-order derivative of $\Pi_{k}(x)$ which turns the spline curve into a flexible and versatile instrument of approximation. Likewise, only the "dogs" (or "rats") enable the ordinary spline to trace curves differing from the graph of a cubic polynomial.

The special importance of spline curves will be due to the fact that by the addi-

[^32]tion of several spline curves of successive orders we may get any desired polygonal line of given degree $m$ and class $C^{\mu}$.
3.12. The evaluation of certain Fourier integrals. Our further work is based on the consideration of the functions
\[

$$
\begin{equation*}
M_{k}(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\frac{2 \sin u / 2}{u}\right)^{k} e^{i u x} d u \quad(k=1,2, \ldots ;-\infty<x<\infty) . \tag{1}
\end{equation*}
$$

\]

They have been evaluated explicitly for low values of $k$ by various authors. ${ }^{11}$ The following general explicit representation is essentially due to Laplace (see J. V. Uspensky, Introduction to mathematical probability, 1937, Example 3, pp. 277-278).

Theorem 3. Let $k$ be a positive integer. Define the function $x_{+}^{k-1}$ by

$$
x_{+}^{k-1}=\left\{\begin{array}{ccc}
x^{k-1} & \text { if } & x \geqq 0  \tag{2}\\
0 & \text { if } & x<0
\end{array}\right.
$$

For $k=1$ and $x=0$ this definition is modified to $0_{+}^{k-1}=1 / 2$.
The following identity holds for all real values of $x$

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\frac{2 \sin u / 2}{u}\right)^{k} e^{i u x} d u=\frac{1}{(k-1)!} \delta^{k} x_{+}^{k-1} \tag{3}
\end{equation*}
$$

where $\delta^{k}$ stands for the usual symbol of the kth order central difference of step equal to unity.

The identity (3) is correct for $k=1$. Indeed it is well known that

$$
\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} d u=\frac{1}{2}
$$

On replacing $u$ by $u x$ we get that

$$
\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\sin x u}{u} d u=\left\{\begin{array}{rll}
\frac{1}{2} & \text { if } & x>0 \\
0 & \text { if } & x=0 \\
-\frac{1}{2} & \text { if } & x<0
\end{array}\right.
$$

or

$$
\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\sin x u}{u} d u=x_{+}^{0}-\frac{1}{2}
$$

Therefore

$$
\begin{aligned}
\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{2 \sin u / 2}{u} \cos u x d u & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\sin \left(x+\frac{1}{2}\right) u}{u} d u-\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\sin \left(x-\frac{1}{2}\right) u}{u} d u \\
& =\frac{1}{2 \pi} \delta \int_{-\infty}^{\infty} \frac{\sin x u}{u} d u=\delta\left(x_{+}^{0}-\frac{1}{2}\right)=\delta x_{+}^{0}
\end{aligned}
$$

[^33]and (3) is established for $k=1$.
Let us consider for the moment the sequence of functions
\[

$$
\begin{equation*}
N_{k}(x)=\frac{1}{(k-1)!} \delta^{k} x_{+}^{k-1} \tag{4}
\end{equation*}
$$

\]

We have already shown that

$$
\begin{equation*}
M_{k}(x)=N_{k}(x) \tag{5}
\end{equation*}
$$

holds for $k=1$. Assume now that (5) holds. We wish to show that the similar identity for $k+1$, rather than $k$, arises from (5) by performing the operation

$$
\int_{x-1 / 2}^{x+1 / 2}
$$

on both sides of (5). This will be accomplished if we prove that

$$
\begin{equation*}
M_{k+1}(x)=\int_{x-1 / 2}^{x+1 / 2} M_{k}(x) d x \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
N_{k+1}(x)=\int_{x-1 / 2}^{x+1 / 2} N_{k}(x) d x \tag{7}
\end{equation*}
$$

In view of

$$
\int_{x-1 / 2}^{x+1 / 2} e^{i u x} d x=\frac{2 \sin u / 2}{u} e^{i u x}
$$

we obtain (6) by an integration under the integral sign as follows

$$
\int_{x-1 / 2}^{x+1 / 2} M_{k}(x) d x=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\frac{2 \sin u / 2}{u}\right)^{k}\left(\int_{x-1 / 2}^{x+1 / 2} e^{i u x} d x\right) d u=M_{k+1}(x)
$$

To prove (7) we notice that

$$
\begin{aligned}
\int_{x-1 / 2}^{x+1 / 2} N_{k}(x) d x & =\delta \int_{-\infty}^{x} N_{k}(x) d x=\delta \int_{-\infty}^{x} \delta^{k} \frac{x_{+}^{k-1}}{(k-1)!} d x \\
& =\delta \delta^{k} \int_{-\infty}^{x} \frac{x_{+}^{k-1}}{(k-1)!} d x=\delta^{k+1} \frac{x_{+}^{k}}{k!}=N_{k+1}(x)
\end{aligned}
$$

This concludes the proof of Theorem 3.
3.13. Explicit polynomial expressions for $M_{k}(x)$. The formula

$$
\begin{equation*}
M_{k}(x)=\frac{1}{(k-1)!} \delta^{k} x_{+}^{k-1} \tag{8}
\end{equation*}
$$

will readily show that $y=M_{k}(x)$ represents a spline curve of order $k$. Indeed, if $k$ is even, then

$$
(x+n)_{+}^{k-1}
$$

is a $\Pi_{k}$ and therefore also their linear combination (8). If $k$ is odd the same conclusion holds because

$$
\left(x+n-\frac{1}{2}\right)_{+}^{k-1}
$$

is a $\Pi_{k}$.
It also follows from (8) that the explicit polynomial expressions for

$$
(k-1)!M_{k}(x),
$$

in successive unit intervals, are identical with the successive partial sums of the expansion

$$
\delta^{k} x^{k-1}=\left(x+\frac{k}{2}\right)^{k-1}-\binom{k}{1}\left(x+\frac{k}{2}-1\right)^{k-1}+\cdots+(-1)^{k}\left(x-\frac{k}{2}\right)^{k-1}
$$

an expression which incidentally vanishes identically, being the $k$ th order difference of a polynomial of degree $k-1$. We thus get

For future reference we work out explicitly the cases $k=1,2,3$, and 4 . The expansions

$$
\begin{align*}
& \delta x^{0}=1-1 \\
& \delta^{2} x=(x+1)-2 x+(x-1)  \tag{10}\\
& \delta^{3} x^{2}=\left(x+\frac{3}{2}\right)^{2}-3\left(x+\frac{1}{2}\right)^{2}+3\left(x-\frac{1}{2}\right)^{2}-\left(x-\frac{3}{2}\right)^{2} \\
& \delta^{4} x^{3}=(x+2)^{3}-4(x+1)^{3}+6 x^{3}-4(x-1)^{3}+(x-2)^{3}
\end{align*}
$$

now furnish the following expressions

$$
M_{1}(x)=\left\{\begin{array}{ccc}
0 & \text { if } & x<-\frac{1}{2}  \tag{11}\\
1 & \text { if } & -\frac{1}{2}<x<\frac{1}{2} \\
0 & \text { if } & \frac{1}{2}<x,
\end{array}\right.
$$

to which must be added $M_{1}( \pm 1 / 2)=1 / 2$ as required by (1) for $k=1$.

$$
M_{2}(x)=\left\{\begin{array}{ccc}
0 & \text { if } & x \leqq-1  \tag{12}\\
x+1 & \text { if } & -1 \leqq x \leqq 0 \\
-x+1 & \text { if } & 0 \leqq x \leqq 1 \\
0 & \text { if } & 1 \leqq x
\end{array}\right.
$$

$$
\begin{align*}
& M_{3}(x)=\left\{\begin{array}{llr}
0 & \text { if } & x \leqq-\frac{3}{2} \\
(1 / 2)(x+3 / 2)^{2} & \text { if } & -\frac{3}{2} \leqq x \leqq-\frac{1}{2} \\
(1 / 2)(x+3 / 2)^{2}-(3 / 2)(x+1 / 2)^{2} & \text { if } & -\frac{1}{2} \leqq x \leqq \frac{1}{2} \\
(1 / 2)(-x+3 / 2)^{2} & \text { if } & \frac{1}{2} \leqq x \leqq \frac{3}{2} \\
0 & \text { if } & 3 \leqq x,
\end{array}\right.  \tag{13}\\
& M_{4}(x)=\left\{\begin{array}{llr}
0 & \text { if } & x \leqq-2 \\
(1 / 6)(x+2)^{3} & \text { if }-2 \leqq x \leqq-1 \\
(1 / 6)(x+2)^{3}-(4 / 6)(x+1)^{3} & \text { if }-1 \leqq x \leqq 0 \\
(1 / 6)(-x+2)^{3}-(4 / 6)(-x+1)^{3} & \text { if } & 0 \leqq x \leqq 1 \\
(1 / 6)(-x+2)^{3} & \text { if } & 1 \leqq x \leqq 2 \\
0 & \text { if } & 2 \leqq x .
\end{array}\right. \tag{14}
\end{align*}
$$

In deriving these expressions the expansions (10) were used up to the point from where the evenness of the functions $M_{k}(x)$ allowed us to complete their definition for all $x$ by symmetry.

3.14. Interpolation formula with $M_{k}(x)$ as basic function. The interpolation formula

$$
\begin{equation*}
F(x)=\sum_{\nu=-\infty}^{\infty} y_{v} M_{k}(x-\nu) \tag{15}
\end{equation*}
$$

will play an important role in our subsequent work. We mention it at this place because it contributes to our investigation of $k$-order spline curves.

To the basic function

$$
L(x)=M_{k}(x)
$$

corresponds the characteristic function

$$
\begin{equation*}
g(u)=\left(\frac{2 \sin u / 2}{u}\right)^{k} \tag{16}
\end{equation*}
$$

as seen by comparing III (1) with II (22). The characteristic function of the formula (15) for integral $x=n$ is

$$
\begin{equation*}
\phi_{k}(u)=M_{k}(0)+2 M_{k}(1) \cos u+2 M_{k}(2) \cos 2 u+\cdots . \tag{17}
\end{equation*}
$$

The $\left\{\phi_{k}(u)\right\}$ represent an interesting sequence of cosine polynomials which we will investigate more closely later in this paper. Here we mention without proof that

$$
\begin{equation*}
1=\phi_{1}(u)=\phi_{2}(u)>\cdots>\phi_{k}(u)>\cdots>0 \quad(0<u<2 \pi) \tag{18}
\end{equation*}
$$

while, of course, $\phi_{k}(0)=1$. Hence (15) is a smoothing interpolation formula of progressively increasing strength as $k$ increases. We assemble the various properties of (15) in the form of a theorem.

Theorem 4.

$$
\begin{equation*}
F(x)=\sum_{\nu=-\infty}^{\infty} y_{\nu} M_{k}(x-\nu) \tag{19}
\end{equation*}
$$

is a polynomial smoothing interpolation formula of degree $k-1$, class $C^{k-2}$ and span $2 s=k$ (see sections 2.21 and 3.13). It is exact for the degree 1 and preserves the degree $k-1$. The smoothing power of (19) increases progressively for increasing values of $k$.

The exactness of (19) for the degree 1 and the preservation of the degree $k-1$ follow by Theorem 2 (B and C). Indeed, by (16), $g(u)-1$ has a double zero for $u=0$ while $g(u)$ has zeros of order $k$ for $u=2 \pi n(n \neq 0)$. Since the preservation of the degree $k-1$ implies the identities II(18) and (19), the following corollary results.

Corollary. Any given polynomial $P_{k-1}(x)$ of degree at most $k-1$ may be represented in the form

$$
\begin{equation*}
P_{k-1}(x)=\sum_{n=-\infty}^{\infty} y_{n} M_{k}(x-n) \tag{20}
\end{equation*}
$$

where $\left\{y_{n}\right\}$ are the ordinates of some other suitably chosen polynomial of the same degree as $P_{k-1}$. This representation is unique.
3.15. The analytic representation of spline curves of order $k$. We know that if $\left\{y_{n}\right\}$ is an arbitrary sequence of ordinates, then our interpolation formula

$$
\begin{equation*}
F(x)=\sum_{n} y_{n} M_{k}(x-n) \tag{21}
\end{equation*}
$$

represents a spline curve of order $k$. This is true because all $M_{k}(x-n)$ are such curves. The following question arises: Let $F(x)$ be a given $\mathrm{\Pi}_{k}$; can we always represent it in the form (21) for an appropriate sequence $\left\{y_{n}\right\}$ ?

This question is answered affirmatively by the following theorem.
Theorem 5. Any spline curve $\Pi_{k}(x)$ may be represented in one and only one way in the form

$$
\begin{equation*}
\Pi_{k}(x)=\sum_{n=-\infty}^{\infty} y_{n} M_{k}(x-n) \tag{22}
\end{equation*}
$$

for appropriate values of the coefficients $y_{n}$. There are no convergence difficulties since $M_{k}(x)$ vanishes for $|x|>k / 2$. Thus (22) represents $a \Pi_{k}$ for arbitrary $\left\{y_{n}\right\}$ and represents the most general one.

In order to prove this theorem we return to the interpolation formula (21) and differentiate it repeatedly. By (8) we have

$$
M_{k}^{\prime}(x)=\delta^{k} \frac{d}{d x} \frac{1}{(k-1)!} x_{+}^{k-1}=\delta^{k} \frac{1}{(k-2)!} x_{+}^{k-2}=\delta M_{k-1}(x)
$$

and repeating we get

$$
\begin{equation*}
M_{k}^{(\nu)}(x)=\delta^{\prime} M_{k-p}(x) \quad(0 \leqq \nu \leqq k-1) . \tag{23}
\end{equation*}
$$

From (21) and (23) we obtain by partial summation

$$
F^{\prime}(x)=\sum_{n} y_{n} \delta M_{k-1}(x-n)=\sum_{n} \delta y_{n+1 / 2} M_{k-1}\left(x-n-\frac{1}{2}\right)
$$

or

$$
\begin{equation*}
F^{\prime}\left(x+\frac{1}{2}\right)=\sum_{n} \delta y_{n+1 / 2} M_{k-1}(x-n) \tag{24}
\end{equation*}
$$

If $k>2$, this formal rule of differentiation of a spline curve may now again be applied to (24) with the result

$$
F^{\prime \prime}(x+1)=\sum \delta^{2} y_{n+1} M_{k-2}(x-n)
$$

or

$$
F^{\prime \prime}(x)=\sum \delta^{2} y_{n} M_{k-2}(x-n)
$$

Generally for $0 \leqq \nu \leqq k-1$

$$
F^{(\nu)}(x)= \begin{cases}\sum_{n} \delta^{\nu} y_{n} M_{k-v}(x-n) & \text { if } \quad \nu \text { is even, }  \tag{25}\\ \sum_{n} \delta^{\nu} y_{n+1 / 2} M_{k-v}\left(x-n-\frac{1}{2}\right) & \text { if } \nu \text { is odd } .\end{cases}
$$

This result may be stated as follows: The vth derivative of the spline curve (21) may be obtained directly by applying the same interpolation formula (21) with $k-\nu$, rather than $k$, to the sequence of the $\nu$ th central differences $\delta^{v} y$ properly centered according to the parity of $\nu$. In particularly: $F^{(k-2)}(x)$ is obtained by interpolating linearly among the $\delta^{k-2} y$. $F^{(k-1)}(x)$ is a step function whose successive values agree with those of the corresponding $\delta^{k-1} y$.

Now let $F(x)$ be a given $\Pi_{k}$. We are to show the existence of a sequence $\left\{y_{n}\right\}$ such that (21) holds identically. Suppose for the moment that such $y_{n}$ have been found which do make (21) hold. Then by (25) for $\nu=k-1$ we have

$$
F^{(k-1)}(x)=\left\{\begin{array}{lll}
\delta^{k-1} y_{n} & \text { for } n-\frac{1}{2}<x<n+\frac{1}{2} & \text { if } k \text { is odd }  \tag{26}\\
\delta^{k-1} y_{n+1 / 2} & \text { for } n<x<n+1 & \text { if } k \text { is even }
\end{array}\right.
$$

In either case the successive constant values of the step function $F^{(k-1)}(x)$ determine uniquely the values of the differences of order $k-1$ of the sequence of the as yet unknown coefficients $y_{n}$. These differences in turn determine the coefficients $y_{n}$ uniquely up to an additive sequence of vanishing differences of order $k-1$. Let $\bar{y}_{n}$ be one sequence such that

$$
\delta^{k-1} \bar{y}_{n+1 / 2}=F^{(k-1)}\left(n+\frac{1}{2}\right) \quad \text { if } \quad k \text { is cven }
$$

or else

$$
\delta^{k-1} \bar{y}_{n}=F^{(k-1)}(n) \quad \text { if } \quad k \text { is odd }
$$

Consider the $k$-order spline curve

$$
\bar{F}(x)=\sum_{n} \bar{y}_{n} M_{k}(x-n)
$$

and let

$$
R(x)=F(x)-\bar{F}(x)
$$

From the way $\bar{F}(x)$ was defined it is clear that $\bar{F}^{(k-1)}(x)$ and $F^{(k-1)}(x)$ agree in their successive unit intervals of constancy. Hence $R(x)$ is a $\Pi_{k}$ whose various polynomial arcs are of degree $k-2$ or lower. Therefore $R(x)$ is identical with a polynomial of degree $k-2$. As such it allows of a representation of the form (21) in view of our Corollary of section 3.14. Therefore also

$$
F(x)=\bar{F}(x)+R(x)
$$

may be represented by our formula (21).
The unicity of the representation (21) is readily established. Indeed two different such representations would imply a representation of zero

$$
0=\sum_{n} y_{n} M_{k}(x-n)
$$

without all $y_{n}$ vanishing. However the $\delta^{k-1} y$ all vanish, and our conclusion would contradict the uniqueness of the representation (20) of polynomials.

A simple example might illustrate our proof of Theorem 5. Let us find the representation of the spline curve of order 4

$$
F(x)=\frac{1}{3!} x_{+}^{3}
$$

By (26) we have

$$
\delta^{3} y_{n+1 / 2}=F^{\prime \prime \prime}\left(n+\frac{1}{2}\right)=\left(n+\frac{1}{2}\right)_{+}^{0}=\left\{\begin{array}{lll}
1 & \text { if } & n \geqq 0 \\
0 & \text { if } & n<0
\end{array}\right.
$$

A sequence having these third differences is

$$
y_{n}= \begin{cases}0 & \text { if } n<0 \\ \binom{n+1}{3}=\frac{n\left(n^{2}-1\right)}{6} & \text { if } n \geqq 0\end{cases}
$$

Hence

$$
\begin{equation*}
x_{+}^{3}=\sum_{n=0}^{\infty} n\left(n^{2}-1\right) M_{1}(x-n) \tag{27}
\end{equation*}
$$

with the possibility still open that both member might differ by a third degree polynomial. However this possibility is excluded by the remark that both sides vanish identically for $x \leqq 0$. Incidentally, (27) implies

$$
-(-x)_{+}^{2}=\sum_{n=-\infty}^{0} n\left(n^{2}-1\right) M_{4}(x-n)
$$

and by addition and subtraction of the two relations we get the identities

$$
x^{3}=\sum_{n=-\infty}^{\infty} n\left(n^{2}-1\right) M_{4}(x-n), \quad|x|^{3}=\sum_{n=-\infty}^{\infty}\left|n\left(n^{2}-1\right)\right| M_{4}(x-n)
$$

In later applications we shall frequently operate with polygonal lines $F(x)$ of degree $k-1$ not having continuity properties as strong as a $\Pi_{k}$. Thus Jenkins' $L(x)$ defined by II (11) is of degree 4 and class $C^{\prime \prime}$. Let $F(x)$ be a polygonal line of degree $k-1$, having vertices at integral point $x=n$, and being for all real $x$ a function of class $C^{\mu}(-1 \leqq \mu \leqq k-2)$. We certainly obtain a curve of degree $k-1$ and class $C^{\mu}$ by addition of spline curves.

$$
\begin{equation*}
F(x)=\Pi_{\mu+2}+\Pi_{\mu+3}+\cdots+\Pi_{k} \tag{28}
\end{equation*}
$$

where $\Pi_{\nu}$ stands for $\Pi_{\nu}(x)$ or $\Pi_{\nu}(x+1 / 2)$, according to whether $\nu$ is even or odd.
Theorem 6. Any given polygonal line $F(x)$ of degree $k-1$ and class $C^{\mu}$ may be represented as a sum (28) of $k-\mu-1$ appropriate spline curves of orders $\mu+2, \mu+3, \cdots, k$.

This theorem is a corollary to Theorem 5. Indeed, $F^{(\mu+1)}(x)$ may have certain discontinuities. We determine a $\Pi_{\mu+2}$ having the same discontinuities in its $(\mu+1)$ st derivative. Then

$$
F(x)-\Pi_{\mu+2}
$$

is of degree $k-1$ and class $C^{\mu+1}$. Proceeding in this way the theorem is readily established.

Substituting for the $\Pi_{\nu}$ in (28) their expressions in terms of the $M_{\nu}$ we obtain an explicit (parametric) representation of such polygonal lines. Thus Jenkins' function II(11) may be represented as

$$
L(x)=4 M_{4}(x)+\frac{1}{2} M_{4}(x+1)+\frac{1}{2} M_{4}(x-1)-2 M_{6}\left(x+\frac{1}{2}\right)-2 M_{5}\left(x-\frac{1}{2}\right) .
$$

At a glance we recognize a curve of degree 4 , class $C^{\prime \prime}$, span $s=6$.
3.16. A summation property of spline curves. The degree of a polynomial is decreased or increased by one unit if we difference or else sum the polynomial. Not so with our spline curves $\Pi_{k}$. Indeed let

$$
\begin{equation*}
\mathrm{I}_{k}(x)=\sum_{n} y_{n} M_{k}(x-n) \tag{29}
\end{equation*}
$$

be a given spline curve. Then

$$
\mathrm{II}_{k}(x+1)=\sum y_{n} M_{k}(x+1-n)=\sum y_{n+1} M_{k}(x-n)
$$

and subtracting (29) we have

$$
\begin{equation*}
\Delta \Pi_{k}(x)=\Pi_{k}(x+1)-\Pi_{k}(x)=\sum \Delta y_{n} M_{k}(x-n) \tag{30}
\end{equation*}
$$

Hence $\Delta \Pi_{k}(x)$ will in general be also a $\Pi_{k}$. Now let the spline curve $\Pi_{k}^{*}(x)$ be given and we wish to find $\Pi_{k}(x)$ such that

$$
\begin{equation*}
\Delta \Pi_{k}(x)=\Pi_{k}^{*}(x) \tag{31}
\end{equation*}
$$

By using Theorem 5 the solution is immediately found. Indeed let

$$
\Pi I_{k}^{*}(x)=\sum y_{n}^{*} M_{k}(x-n)
$$

Now (29) will be a solution of (31) provided $\Delta y_{n}=y_{n}^{*}$, for all $n$. This relation defines the $y_{n}$ up to an additive constant which appears as an arbitrary additive constant in the solution $\Pi_{k}(x)$. It is thus seen that the operation of differencing or summing spline curves (29) reduces to performing the same operations on the sequence of the coefficients $\left\{y_{n}\right\}$.
3.17. An interpretation of $M_{k}(x)$ in probability theory. In concluding our discussion of polynomial spline curves we mention briefly the following interpretation of $M_{k}(x)$. As seen from its values as given by III(11) it is clear that $M_{1}(x)$ may be interpreted as the probability density function of the error comnitted on a random real variable $x$, if that variable is rounded off to its nearest integral value. Now III(1) shows that the characteristic function of $M_{k}(x)$ is the $k$ th power of the characteristic function of $M_{1}(x)$. From a known proposition in probability theory we may conclude that $M_{k}(x)$ is the density distribution function of the error committed on the sum

$$
x_{1}+x_{2}+\cdots+x_{k}
$$

of $k$ statistically independent real random variables $x_{1}, \cdots, x_{k}$, if each variable is replaced by its nearest integral value.

This interpretation, otherwise entirely irrelevant for our purpose, does make a few of the properties of $M_{k}(x)$ intuitively obvious, such as

$$
\begin{gathered}
M_{k}(x)\left\{\begin{array}{lll}
>0 & \text { for } & |x|>k / 2 \\
=0 & \text { for } & |x|<k / 2
\end{array}\right. \\
\int_{-\infty}^{\infty} M_{k}(x) d x=1
\end{gathered}
$$

In concluding we note the identity

$$
\int_{x-1 / 2}^{x+1 / 2} d x_{1} \int_{x_{1}-1 / 2}^{x_{1}+1 / 2} d x_{2} \cdots \int_{x_{k-1}-1 / 2}^{x_{k-1}+1 / 2} f\left(x_{k}\right) d x_{k}=\int_{-\infty}^{\infty} M_{k}(u-x) f(u) d u .
$$

3.2. Analytic spline curves of order $k$. The polynomial spline curves $\Pi_{k}(x)$ described in section 3.1 will be shown to be sufficient for the derivation of polynomial approximations to equidistant data enjoying various desirable properties. These polynomial approximations will have any a priori assigned number of continuous derivatives. However, in order to obtain analytic approximations we shall now proceed to derive from our spline curves $\Pi_{k}(x)$ an analoguous family of analytic functions.

To achieve this end we shall smooth out our $\Pi_{k}(x)$ by means of one-dimensional heat flow. Consider an infinite homogeneous bar (the $x$-axis) in which the temperature at the point $x$ at the time $t$ is denoted by $F(x, t)$. We assume the flow of heat to be governed by the equation

$$
\begin{equation*}
\frac{\partial F}{\partial t}=\frac{1}{4} \frac{\partial^{2} F}{\partial x^{2}} . \tag{32}
\end{equation*}
$$

If $F(x)=F(x, 0)$ is given, i.e., the temperature distribution at the time $t=0$ is known,
then $F(x, t)$ is determined by the following integral ${ }^{12}$

$$
\begin{equation*}
F(x, t)=\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(u-x)^{2} t^{-1}} F(u, 0) d u \tag{33}
\end{equation*}
$$

This result is easy to verify: by partial differentiations we find that $F(x, t)$ as defined by (33) indeed satisfies the differential equation (32) while familiar arguments originated by Weierstrass will show that (33) implies

$$
\lim _{t \rightarrow+0} F(x, t)=F(x, 0),
$$

provided $F(x, 0)$ is continuous and, e.g., bounded.
The solution of the problem of finding $F(x, t)$, if $F(x, 0)$ is given, is especially simple in the case when $F(x, 0)$ is defined by a Fourier integral

$$
\begin{equation*}
F(x, 0)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \psi(u) e^{i u x} d u \tag{34}
\end{equation*}
$$

Indeed, in this case we find

$$
\begin{equation*}
F(x, t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{-t(u / 2)^{2}} \psi(u) e^{i u x} d u . \tag{35}
\end{equation*}
$$

Notice that the temperature $t$ enters only in the additional exponential factor. This can be proved in two ways, either by substituting (34) into (33) or else by verifying directly that (35) satisfies the differential equation (32). Obviously (35) reduces to (34) for $t=0$, as it should.

We may (and wish to) think of $F(x, t)$, for a fixed $t>0$, as a smoothed version of $F(x)=F(x ; 0)$. In fact $F(x, t)$ is analytic and regular for all real or complex values of $x$ if $\psi(u)$ is, e.g., bounded.

If we now apply this heat-flow transformation to our basic $k$-order spline curve

$$
\begin{equation*}
M_{k}(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\frac{2 \sin u / 2}{u}\right)^{k} e^{i x \approx} d u \tag{36}
\end{equation*}
$$

we obtain by (35) its smoothed version

$$
\begin{equation*}
M_{k}(x, t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{-t(u / 2)^{2}}\left(\frac{2 \sin u / 2}{u}\right)^{k} e^{i \psi x} d u . \tag{37}
\end{equation*}
$$

Obviously

$$
\begin{equation*}
M_{k}(x, 0)=M_{k}(x) . \tag{38}
\end{equation*}
$$

[^34]The graph of $y=M_{k}(x, t)(\iota>0)$ is a bell-shaped curve which dampens out very fast. Later we shall learn how to compute its values very accurately. Here we mention that

$$
\begin{equation*}
0<M_{k}(x, t)<\frac{1}{\sqrt{\pi t}} e^{-(x-k / 2)^{2} \cdot t^{-1}} \text { if } x \geqq k / 2 \tag{39}
\end{equation*}
$$

Also, the recurrence relation III ( 6 ) generalizes so that $M_{k+1}(x, t)$ is obtained by the averaging operation

$$
\begin{equation*}
M_{k+1}(x, t)=\int_{x-1 / 2}^{x+1 / 2} M_{k}(x, t) d x \tag{40}
\end{equation*}
$$

If now

$$
\begin{equation*}
\Pi_{k}(x)=\sum_{n=-\infty}^{\infty} y_{n} M_{k}(x-n) \tag{41}
\end{equation*}
$$

is a spline curve of order $k$, then its heat-flow transform is

$$
\begin{equation*}
\Pi_{k}(x, t)=\sum_{n=-\infty}^{\infty} y_{n} M_{k}(x-n, t) \tag{42}
\end{equation*}
$$

The graph of this function may be called an analytic spline curve of order $k$. We notice that the series (42) fails to converge only if $y_{n}$ increases very fast with $|n|$.

Summarizing we see that the curve (42) arises from the step function

$$
\sum_{n} y_{n} M_{1}(x-n)
$$

by $k-1$ successive applications of the averaging operation

$$
\int_{x-1 / 2}^{x+1 / 2}() d x
$$

followed by the heat-flow transformation during a time interval $t$. The order of application of these $k$ operations is of course irrelevant.

The remaining parts of the paper are devoted to the problem of utilizing the two families of curves (41) and (42) for the purpose of approximating given equidistant data.
IV. A FIRST CLASS OF ANALYTIC INTERPOLATION FORMULAE

We shall now use the analytic spline curves III(42) to obtain

1. A smoothing interpolation formula which is exact for the degree 1 only.
2. An ordinary interpolation formula exact for the degree $k-1$.
3. A smoothing interpolation formula depending on a positive parameter $\epsilon$, of strictly increasing smoothing power as $\epsilon$ increases. For $\epsilon=0$ this formula reduces to 2, while for $\epsilon=\infty$ it is identical with 1 .
4.1. A smoothing interpolation formula exact for the degree 1. A comparison of the formulae III (15) and III(42) immediately suggests the following analytic interpolation formula

$$
\begin{equation*}
F(x)=\sum_{n=-\infty}^{\infty} y_{n} M_{k}(x-n, t) \tag{1}
\end{equation*}
$$

where $M_{k}(x, t)$ is defined by III(37). For the sequel we shall use the following notation

$$
\begin{align*}
\psi_{k}(u, t) & =e^{-t(u / 2)^{2}}\left(\frac{2 \sin u / 2}{u}\right)^{k}  \tag{2}\\
\psi_{k}(u) & =\psi_{k}(u, 0)=\left(\frac{2 \sin u / 2 \^{k}}{u}\right)^{k} \tag{3}
\end{align*}
$$

in terms of which $\operatorname{III}(37)$ becomes

$$
\begin{equation*}
M_{k}(x, t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \psi_{k}(u, t) e^{i u x} d u \tag{4}
\end{equation*}
$$

The characteristic function of the smoothing formula (1) for integral $x$ is

$$
\begin{equation*}
\phi_{k}(u, t)=\sum_{n=-\infty}^{\infty} M_{k}(n, t) \cos n u \tag{5}
\end{equation*}
$$

The general relation II (30) furnishes the following equivalent expression

$$
\begin{equation*}
\phi_{k}(u, t)=\sum_{r=-\infty}^{\infty} \psi_{k}(u+2 \pi \nu, t) \tag{6}
\end{equation*}
$$

The properties III(18) for the case $t=0$ generalize for $t>0$ as follows

$$
\begin{equation*}
1>\phi_{1}(u, t)>\phi_{2}(u, t)>\cdots>\phi_{k}(u, t)>\cdots>0, \quad(0<u<2 \pi, t>0) \tag{7}
\end{equation*}
$$

Moreover, for each fixed $u, 0<u<2 \pi, \phi_{k}(u, t)$ is strictly decreasing as $t$ increases. These last two properties we state without proving them here.

The arguments which lead to Theorem 4 now allow us to state the following theorem.

Theorem 7. For $t>0, k=1,2, \cdots$,

$$
\begin{equation*}
F(x)=\sum_{n=-\infty}^{\infty} y_{n} M_{k}(x-n, t) \tag{8}
\end{equation*}
$$

is an analytic smoothing interpolation formula which is exact for the degree 1 and preserves the degree $k-1$. The smoothing power of (8) increases whenever either $k$ or $t$ is increased.
4.2. An ordinary interpolation formula exact for the degree $k-1$. We shall now use the important property (7) to the effect that the periodic function $\phi_{k}(u, t)$ is positive for real $u$. This allows us to define the basic function

$$
\begin{equation*}
L_{k}(x, t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\psi_{k}(u, t)}{\phi_{k}(u, i)} e^{i u x} d u \tag{9}
\end{equation*}
$$

whose characteristic function is

$$
\begin{equation*}
g(u)=\frac{\psi_{k}(u, t)}{\phi_{k}(u, t)} \tag{10}
\end{equation*}
$$

Our Theorem 2 of Chapter II will readily yield the following result.
Theorem 8. For $t \geqq 0, k=1,2,3, \cdots$

$$
\begin{equation*}
F(x)=\sum_{n=-\infty}^{\infty} y_{n} L_{k}(x-n, t) \tag{11}
\end{equation*}
$$

is an ordinary interpolation formula which is exact for the degree $k-1$.
Firstly we realize by (10) and (2) that the conditions II (33), (34) of Theorem 2 are verified. Therefore (11) preserves the degree $k-1$. Secondly we have by (6), (10), and II (30)

$$
\phi(u)=\sum_{\nu} g(u+2 \pi \nu)=\sum_{\nu} \frac{\psi_{k}(u+2 \pi \nu, t)}{\phi_{k}(u+2 \pi \nu, t)}=\frac{1}{\phi_{k}(u, t)} \sum_{\nu} \psi_{k}(u+2 \pi \nu, t)=1 .
$$

Thus II (31) holds and (11) is therefore an ordinary interpolation formula. This concludes the proof of our theorem. Indeed, by the remark following Theorem 2 our formula (11) must also be exact for the degree $k-1$.

We also mention without further details the following two limiting relations

$$
\lim _{t \rightarrow \infty} \frac{\psi_{k}(u, t)}{\phi_{k}(u, t)}=\lim _{k \rightarrow \infty} \frac{\psi_{k}(u, t)}{\phi_{k}(u, t)}=\left\{\begin{array}{lll}
1 & \text { if } & |u|<\pi  \tag{12}\\
0 & \text { if } & |u|>\pi
\end{array}\right.
$$

They show, in view of the integral representation (9), that our present basic function $L_{k}(x, t)$ converges towards the basic function II (6) of the original cardinal series whenever either $k$ or else $t$ tends to infinity.
4.3. A family of smoothing interpolation formulae depending on a smoothing parameter $\epsilon$. In section 4.1 we have derived the smoothing interpolation formula (8) in the derivation of which no attempt was made to compromise between smoothness of results and goodness of fit. Such a compromise is afforded by the following basic function

$$
\begin{equation*}
L_{k}(x, t, \epsilon)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\epsilon+\phi_{k}(u, t)}{\epsilon+\phi_{k}(u, t)^{2}} \psi_{k}(u, t) e^{i u x} d u \quad(0 \leqq \epsilon \leqq \infty) \tag{13}
\end{equation*}
$$

which depends also on the smoothing parameter $\epsilon$. The corresponding interpolation formula

$$
\begin{equation*}
F(x)=\sum_{n=-\infty}^{\infty} y_{n} L_{k}(x-n, t, \epsilon) \tag{14}
\end{equation*}
$$

includes our previous formula (8) and (11) as special cases. Indeed by (13) ,(4) and (9) we find

$$
\begin{align*}
L_{k}(x, t, 0) & =L_{k}(x, t)  \tag{15}\\
L_{k}(x, t, \infty) & =M_{k}(x, t) \tag{16}
\end{align*}
$$

Let us now investigate the characteristic function of the smoothing formula (14) for integral $x$. By (13) and $\mathrm{II}(30)$ this characteristic function is $\left(\phi_{k}(u, t)\right.$ is periodic!):

$$
\phi_{k}(u, t, \epsilon)=\frac{\epsilon+\phi_{k}(u, t)}{\epsilon+\phi_{k}(u, t)^{2}} \sum_{v} \psi_{k}(u+2 / \pi \nu, t)
$$

This and (6) give

$$
\begin{equation*}
\phi_{k}(u, t, \epsilon)=\left(\epsilon \phi_{k}(u, t)+\phi_{k}(u, t)^{2}\right) /\left(\epsilon+\phi_{k}(u, t)^{2}\right) . \tag{17}
\end{equation*}
$$

On the other hand we have by (7) the inequalities

$$
\begin{equation*}
0<\phi_{k}(u, t)<1, \quad(0<u<2 \pi, t>0) \tag{18}
\end{equation*}
$$

Now (17), (18) imply

$$
\begin{equation*}
0<\phi_{k}(u, l, \epsilon)<1, \quad(0<u<2 \pi, t>0, \epsilon>0) \tag{19}
\end{equation*}
$$

and therefore (14) is a smoothing interpolation formula in the sense of section 2.21 b . Moreover, we see from (17) that for fixed $l$ and $u(0<u<2 \pi) \phi_{k}(u, t, \epsilon)$ decreases monotonically from

$$
\phi_{k}(u, t, 0)=1 \quad \text { to } \quad \phi_{k}(u, t, \infty)=\phi_{k}(u, t)
$$

as $\epsilon$ varies from $\epsilon=0$ to $\epsilon=\infty$. Finally (14) is exact for the degree 1 and preserves the degree $k-1$ for the same reason as mentioned in the case of (8).

We summarize these properties in the following theorem. ${ }^{13}$
Theorem 9. For $t \geqq 0, k=1,2, \cdots$,

$$
\begin{equation*}
F(x)=\sum_{n=-\infty}^{\infty} y_{n} L_{k}(x-u, t, \epsilon) \tag{20}
\end{equation*}
$$

of basic function (13), is a smoothing interpolation formula which is exact for the degree 1 and preserves the degree $k-1$. For $\epsilon=0(20)$ is identical with the ordinary interpolation formula (11). For increasing values of $\epsilon$ il increases in smoothing power until for $\epsilon=\infty$ (20) is identical with our smoothing formula (8).
4.31. A property of the derivatives of the approximation $F(x)$. Let the given data $\left\{y_{n}\right\}$ satisfy the additional condition

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty}\left|y_{n}\right|<\infty \tag{21}
\end{equation*}
$$

Assume also $t>0, \epsilon>0$. We know that the sequence $\{F(n)\}$ obtained by (20) is smoother than $\left\{y_{n}\right\}$ in the sense of our discussion in I Section 1.1. However, it appears
${ }^{13}$ The method used in deriving Theorems 8 and 9 obviously generalizes as follows. Let

$$
L(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} g(u) e^{i u x} d u
$$

be a basic function. Let the periodic function

$$
\begin{gathered}
\phi(u)=\sum_{\nu} g(u+2 \pi \nu) \\
0<\phi(u) \leqq 1
\end{gathered}
$$

Then

$$
L_{1}(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{g(u)}{\phi(u)} e^{i u x d u}
$$

is the basic function of an ordinary interpolation formula. Moreover

$$
L_{1}(x, \epsilon)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\epsilon+\phi(u)}{\epsilon+\phi(u)^{2}} g(u) e^{i u x} d u
$$

gives rise to a family of smoothing formulae of increasing strength, as $\epsilon$ increases.
to be of some interest to discuss here the smoothness of the function $F(x)$, rather than that of the sequence $\{F(n)\}$, as a function of the smoothing parameter $\epsilon$. In this connection we prove the following

Theorem 10. Denote by $F(x, \epsilon)$ the approximation (20) so as to indicate its dependence on $\epsilon$. The condition (21) insures the convergence of the integrals of the squares of the derivatives

$$
\begin{equation*}
\int_{-\infty}^{\infty}\left(F^{(m)}(x, \epsilon)\right)^{2} d x, \quad(m=0,1,2, \cdots) \tag{22}
\end{equation*}
$$

Also each of these integrals is a monotonically decreasing function of $\epsilon$ in the range $0<\epsilon<\infty$.

Indeed, let

$$
\begin{equation*}
T(u)=\sum_{n=-\infty}^{\infty} y_{n} e^{i n u} \tag{23}
\end{equation*}
$$

be the characteristic function of the sequence $\left\{y_{n}\right\}$. For convenience we define

$$
\begin{equation*}
\Omega_{k}(u, t, \epsilon)=\frac{\epsilon+\phi_{k}(u, t)}{\epsilon+\phi_{k}(u, t)^{2}} \tag{24}
\end{equation*}
$$

Then (13) becomes

$$
\begin{equation*}
L_{k}(x, t, \epsilon)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \Omega_{k}(u, t, \epsilon) \psi_{k}(u, t) e^{-i u x} d u \tag{25}
\end{equation*}
$$

By substitution of (25) into (20) we obtain

$$
\begin{equation*}
F(x, \epsilon)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \Omega_{k}(u, t, \epsilon) \psi_{k}(u, t) T(u) e^{-i u x} d u \tag{26}
\end{equation*}
$$

We may evidently differentiate under the integral sign obtaining

$$
F^{(m)}(x, \epsilon)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \Omega_{k}(u, t, \epsilon) \psi_{k}(u, t) T(u)(-i u)^{m} e^{-i u x} d u
$$

This formula exhibits the Fourier transform of $F^{(m)}(x, \epsilon)$. We now use the analogue of the Parseval relation for Fourier integrals finally obtaining

$$
\begin{equation*}
\int_{-\infty}^{\infty}\left(F^{(m)}(x, \epsilon)^{2} d x=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\Omega_{k}(u, t, \epsilon) \psi_{k}(u, t)\right)^{2}|T(u)|^{2} u^{2 m} d u\right. \tag{27}
\end{equation*}
$$

This relation establishes our theorem. Indeed, by (24) the behaviour of the function $\Omega_{k}(u, t, \epsilon)$ of period of $2 \pi$, is as follows: $\Omega_{k}(0, t, \epsilon)=1$, while for each fixed $u, 0<u<2 \pi$, it decreases from

$$
\Omega_{k}(u, t, 0)=1 / \phi_{k}(u, t) \text { to } \Omega_{k}(u, t, \infty)=1
$$

as $\epsilon$ increases from $\epsilon=0$ to $\epsilon=\infty$.
The discussion of I section 1.2 concerning the smoothing of a finite table also has an analogue concerning the derivatives of the approximations $F(x, \epsilon)$. We state the
result without further details. We assume the concrete situation of I section 1.2 where a finite table was extended to an infinite table by constant third differences at each end. To this extended table we apply our formula (20) with such a value of $k$ which will insure that the formula (20) preserves cubics, i.e., $k \geqq 4$. Then we can prove that the integrals (22) converge for $m=4,5,6, \cdots$ and represent decreasing functions of $\epsilon$.
4.32. Formula (20) as applied to subtabulation. Our formula (20) is excellently suited for the systematic interpolation, or subtabulation of given ordinates $y_{n}$. It is less suited for interpolation. The reason is obvious: For subtabulation to tenths we need only a table of the basic function $L_{k}(x, t, \epsilon)$ for the step $h=0.1$ only, while interpolation would call for a much more elaborate table of this function.

The following transformation of the formula (20), in terms of the function $M_{k}(x, t)$, is of importance for numerical applications. First of all we expand the even periodic function (24) in a Fourier (cosine series)

$$
\begin{equation*}
\Omega_{k}(u, t, \epsilon)=\frac{\epsilon+\phi_{k}(u, t)}{\epsilon+\phi_{k}(u, t)^{2}}=\sum_{\nu=-\infty}^{\infty} \omega_{\nu}^{(k)}(t, \epsilon) e^{i \nu u}=\sum_{\nu=-\infty}^{\infty} \omega_{\nu}^{(k)}(t, \epsilon) \cos \nu u . \tag{28}
\end{equation*}
$$

Substituting this expansion into (25) we get

$$
L_{k}(x, t, \epsilon)=\sum_{\nu=-\infty}^{\infty} \omega_{\nu}^{(k)}(t, \epsilon) \frac{1}{2 \pi} \int_{-\infty}^{\infty} \psi_{k}(u, t) e^{i(v-x) u} d u,
$$

which in view of (4) becomes

$$
\begin{equation*}
L_{k}(x, t, \epsilon)=\sum_{n=-\infty}^{\infty} \omega_{\nu}^{(k)}(t, \epsilon) M_{k}(x-\nu, t) . \tag{29}
\end{equation*}
$$

This formula expresses the basic function $L_{k}(x, t, \epsilon)$ in terms of the $M_{k}(x)$. If we substitute this expansion into our formula (20) we obtain

$$
F(x)=\sum_{n} y_{n} L_{k}(x-n, t, \epsilon)=\sum_{n, \eta} y_{n} \omega_{\nu}^{(k)}(t, \epsilon) M_{k}(x-n-\nu, t)
$$

and replacing $\nu$ by $\nu-n$

$$
F(x)=\sum_{n, \nu} y_{n} \omega_{\nu-n}^{(k)}(t, \epsilon) M_{k}(x-\nu, t) .
$$

A first summation by $n$ introduces the sums

$$
\begin{equation*}
f_{v}=\sum_{n=-\infty}^{\infty} y_{n} \omega_{r-n}^{(k)}(t, \xi) \tag{30}
\end{equation*}
$$

in terms of which our last expression becomes

$$
\begin{equation*}
F(x)=\sum_{r=-\infty}^{\infty} f_{\nu} M_{k}(x-\nu, t) \tag{31}
\end{equation*}
$$

The pair of relations (30) and (31) is equivalent to (20) and represents its practical form. The reason for this is that the basic function $M_{k}(x, t)$ dampens out like
$\exp \left(-x^{2}\right)$ (see III (39)) while $L_{k}(x, t, \epsilon), \epsilon<\infty$, dampens out like $\exp (-x)$ only. We notice incidentally that (31) is identical with our formula (8), to be applied to the new computed ordinates $\left\{f_{n}\right\}$ given by (30).

Frequently we require also tables of the derivatives $F^{\prime}(x)$ and $F^{\prime \prime}(x)$, of the approximation $F(x)$. These are then computed by the formulae

$$
\begin{align*}
F^{\prime}(x) & =\sum_{v} f_{v} M_{k}^{\prime}(x-\nu, t) \\
F^{\prime \prime}(x) & =\sum_{v} f_{v} M_{k}^{\prime \prime}(x-\nu, t)
\end{align*}
$$

from corresponding tables of $M_{k}^{\prime}$ and $M_{k}^{\prime \prime}$.
4.33. The least squares origin of formula (20). We want to sketch briefly the genesis of our formula (20). Let the sequence $\left\{y_{n}\right\}$ be given and consider the spline curve $F(x)$ given by (31), where the coefficients $\left\{f_{\nu}\right\}$ are as yet unknown. If we try to determine these unknowns by the requirement that $F(x)$ should interpolate strictly the given ordinates $y_{n}$, i.e.,

$$
\begin{equation*}
F(n)=y_{n} \quad(n=0, \pm 1, \pm 2, \cdots) \tag{32}
\end{equation*}
$$

we obtain an infinite system of linear equations in the unknown $f_{v}$, the solution of which was found to be given by

$$
f_{v}=\sum_{n=-\infty}^{\infty} y_{r} \omega_{r-n}^{(k)}(t, 0)
$$

This leads to our ordinary interpolation formula (11).
In view of Whittaker's well known smoothing method it seemed natural to proceed now as follows: Let $\in$ be a given positive number. To determine the unknown coefficients $f_{v}$ of (31) as solutions of the following minimal problem, we set

$$
\begin{equation*}
\sum_{n}\left(F(n)-y_{n}\right)^{2}+\epsilon \cdot \sum_{n}\left(f_{n}-y_{r}\right)^{2}=\text { minimum } \tag{33}
\end{equation*}
$$

For $\epsilon=0$ the solution is of course identical with the solution of the ordinary interpolation problem (32). For $\epsilon=\infty$ the solution is obviously $f_{n}=y_{n}$ in which case (31) reduces to (8). For $0 \leqq \epsilon \leqq \infty$ a system of "normal equations" arises whose solution is found to be given by (30). The explicit solution of these normal equations (matrix inversion) is performed by the numerical determination of the cosine coefficients $\omega_{p}$ of the expansion (28) for each given set of values of $k, t$ and $\epsilon$.

## V. THE COMPUTATION OF THE TABLES I, II, III

In this last chapter we shall discuss the methods used in the computation of our tables which allow us to use our formulae (30), (31) or

$$
\begin{align*}
f_{n} & =\sum_{r=-\infty}^{\infty} y_{j} \omega_{n-v}^{(k)}(t, \epsilon)  \tag{1}\\
F(x) & =\sum_{n=-\infty}^{\infty} f_{n} M_{k}(x-n, t) \tag{2}
\end{align*}
$$

for subtabulation to tenths.
5.1. The computation of the function $M_{k}(x, t)$ and its derivatives. This function is defined (see III (37)) by the integral

$$
\begin{equation*}
M_{k}(x, t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{-t(u / 2)^{2}}\left(\frac{2 \sin u / 2}{u}\right)^{k} e^{i u x} d u, \quad(t>0) \tag{3}
\end{equation*}
$$

For the special case of $t=0$ and $k=1,2,3, \cdots$ we found previously the explicit polynomial expressions III(3). It now so happens that for our present case of $t>0$ (3) allows us to define our function also for $k=0$ as

$$
M_{0}(x, t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{-l(u / 2)^{2}} e^{i u x} d u
$$

This last integral is known to be identical to

$$
\begin{equation*}
M_{0}(x, t)=\frac{1}{\sqrt{\pi t}} e^{-x^{2} / t} \tag{4}
\end{equation*}
$$

The recurrence relation $\operatorname{III}(40)$ shows that (3) is obtained from (4) by repeating $k$ times the averaging operation

$$
\int_{x-1 / 2}^{x+1 / 2} \text { or } \delta \int_{-\infty}^{x}
$$

The result, however, is not changed if we perform all $k$ integrations first to be followed by the operation $\delta^{k}$ of $k$ th central differencing. This proves the following result:

If we define a sequence of functions $g_{k}(x, l)$ by

$$
\begin{equation*}
g_{0}(x, t)=\frac{1}{\sqrt{\pi t}} e^{-x^{2} / t} \tag{5}
\end{equation*}
$$

and the recurrence relation

$$
\begin{equation*}
g_{k}(x, t)=\int_{-\infty}^{x} g_{k-1}(x, t) d x \quad(k=1,2,3, \cdots) \tag{6}
\end{equation*}
$$

then

$$
\begin{equation*}
M_{k}(x, t)=\delta^{k} g_{k}(x, t) \tag{7}
\end{equation*}
$$

This relation reduces the problem to the problem of computing the repeated integral $g_{k}(x, t)$ of the error function (5). This we do as follows. It is easy to prove by induction or otherwise that (5) and (6) imply

$$
\begin{equation*}
g_{k}(x, \ell)=\frac{1}{(k-1)!} \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{x}(x-u)^{k-1} e^{-u^{2} / t} d u \tag{8}
\end{equation*}
$$

With $x-u=v$ this becomes

$$
\begin{equation*}
g_{k}(x, t)=\frac{1}{(k-1)!} \frac{1}{\sqrt{\pi t}} \int_{0}^{\infty} e^{-(x-y)^{2} / t_{p}^{k-1} d v} \tag{9}
\end{equation*}
$$

By differentiating this with respect to $x$ we get

$$
\begin{aligned}
g_{k}^{\prime}(x, t) & =\frac{1}{(k-1)!} \frac{1}{\sqrt{\pi l}} \int_{0}^{\infty} e^{-(x-p)^{2} / t^{k-1}}\left(-2 x t^{-1}+2 v t^{-1}\right) d v \\
& =-\frac{2 x}{i} g_{k}(x, t)+\frac{2 k}{i} g_{k+1}(x, t)
\end{aligned}
$$

and therefore the recurrence relation

$$
\begin{equation*}
g_{k+1}(x, t)=\frac{t}{2 k} g_{k}^{\prime}(x, t)+\frac{x}{k} g_{k}(x, t), \quad(k=1,2, \cdots) \tag{10}
\end{equation*}
$$

which allows us to compute the successive values $g_{k}(x, t)$ by the operation of differentiation rather than integration. Indeed from (5) and (6) for $k=1$ we get

$$
\begin{equation*}
g_{1}(x, t)=\frac{1}{\sqrt{\pi t}} \int_{-\infty}^{x} e^{-x^{4} / t} d x \tag{11}
\end{equation*}
$$

Now (11) and (10) for $k=1$ will give

$$
g_{2}(x, t)=\frac{t}{2} \frac{1}{\sqrt{\pi t}} e^{-x^{2} / t}+x \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{x} e^{-x^{2} / t} d x
$$

from which $g_{3}(x, t)$ is readily determined.
This progressive computation is greatly simplified if we realize that $g_{k}(x, t)$ will appear as an expression of the form

$$
\begin{equation*}
g_{k}(x, t)=P_{k}(x, t) g_{0}(x, t)+Q_{k}(x, t) g_{1}(x, t) \tag{12}
\end{equation*}
$$

where $P_{k}, Q_{k}$ are polynomials in $x$ and $t$, while $g_{0}(x, t)$ is the error function (5) and $g_{1}(x, t)$ is the error integral (11). Substituting (12) into (10) we find

$$
g_{k+1}(x, t)=\frac{t}{2 k}\left(P_{k}^{\prime}+Q_{k}\right) g_{0}(x, t)+\left(\frac{t}{2 k} Q_{k}^{\prime}+\frac{x}{k} Q_{k}\right) g_{1}(x, t)
$$

On comparing with (12) for $k+1$, rather than $k$, we obtain the recurrence relations

$$
\begin{align*}
P_{k+1} & =\frac{t}{2 k}\left(P_{k}^{\prime}+Q_{k}\right) \\
Q_{k+1} & =\frac{t}{2 k} Q_{k}^{\prime}+\frac{x}{k} Q_{k} \quad(k=1,2, \cdots) \tag{13}
\end{align*}
$$

Since $P_{1}=0, Q_{1}=1$ we readily obtain the following explicit expressions

$$
\begin{array}{ll}
P_{2}=t / 2, & Q_{2}=x \\
P_{3}=t x / 4, & Q_{3}=t / 4+x^{2} / 2 \\
P_{4}=t\left(t+x^{2}\right) / 12, & Q_{4}=x\left(3 t+2 x^{2}\right) / 12 \\
P_{5}=t x\left(5 t+2 x^{2}\right) / 96, & Q_{5}=\left(3 t^{2}+12 t x^{2}+4 x^{4}\right) / 96 \\
P_{6}=t\left(4 t^{2}+9 t x^{2}+2 x^{4}\right) / 480, & Q_{8}=x\left(15 t^{2}+20 t x^{2}+4 x^{4}\right) / 480 \tag{14}
\end{array}
$$

Excellent tables of the probability function (5) and its integral (11) are now available.

By means of these tables the formulae (12) and (14) allow us to compute readily the function $g_{k}(x, t) .{ }^{14}$ It scems worth while to point out that the relation (7) goes over into III (8) if $t \rightarrow+0$. Indeed by an obvious change of variable we see that (11) becomes

$$
g_{1}(x, t)=\frac{1}{\sqrt{\pi}} \int_{-\infty}^{x / \sqrt{t}} e^{-u^{2}} d u
$$

and therefore

$$
\lim _{t \rightarrow+0} g_{1}(x, t)=x_{+}^{0}
$$

Now by induction we prove by (6), on letting $t \rightarrow+0$, that

$$
\lim _{t \rightarrow+0} g_{k}(x, t)=\frac{1}{(k-1)!} x_{+}^{k-1}
$$

which proves our last statement by continuity.
The computation of the derivatives of $M_{k}(x, t)$ is immediately settled by the relation

$$
\begin{equation*}
M_{k}^{(r)}(x, t)=\delta^{k} g_{k-r}(x, t), \tag{15}
\end{equation*}
$$

which is implied by (6).
5.2. The computation of the cosine coefficients $\omega_{n}^{(k)}(t, \epsilon)$. By $\operatorname{IV}(5)$ and $\operatorname{IV}(28)$ we can see that the problem consists in computing the values of the coefficients of the cosine expansion of the function

$$
\begin{equation*}
\Omega_{k}(u, t, \epsilon)=\frac{\epsilon+\phi_{k}(u, t)}{\epsilon+\phi_{k}(u, t)^{2}}=\sum_{n=-\infty}^{\infty} \omega_{n}{ }^{(k)}(t, \epsilon) \cos n u, \tag{16}
\end{equation*}
$$

where the even periodic function $\phi_{k}(u, t)$ is defined by its cosine expansion

$$
\begin{equation*}
\phi_{k}(u, t)=\sum_{n=-\infty}^{\infty} M_{k}(n, t) \cos n u . \tag{17}
\end{equation*}
$$

[^35]The coefficients $M_{k}(n, t)$ of this extremely fast convergent series are readily computed to 10 decimal places. The obvious procedure would be to compute from (17) a table of $\phi_{k}(u, t)$, then compute a similar table of $\Omega_{k}(u, t, \epsilon)$ which is then to be used in computing $\omega_{n}$ by some method of numerical harmonic analysis. It would be hard to achieve accuracy by this method and for this reason we proceeded differently. It should be born in mind that the cosine expansion of the denominator of (16) is readily obtained by the simple operation of multiplication of Fourier series. The only troublesome part is the computation of the expansion

$$
\begin{equation*}
\frac{1}{\epsilon+\phi_{k}(u, t)^{2}}=\sum_{-\infty}^{\infty} c_{n} \cos n u, \tag{18}
\end{equation*}
$$

i.e., the reciprocation of a given cosine series. This was done as follows. The abovementioned method of a 24 -ordinate harmonic analysis scheme was used for obtaining values of the $c_{n}$ 's accurate to $4-5$ decimal places. These values were then improved to values accurate to 8 or 9 places by an iteration method developed by H. A. Rademacher and the author. This method is closely related to the method recommended by H . Hotelling for the reciprocation of ordinary matrices and will be described elsewhere.

In concluding this paper we want to point out two special cases of our ordinary interpolation formula (11), or (20) for $\epsilon=0$, which are of mathematical interest. We mention first the case of $k=0, t>0$. This corresponds to interpolating our ordinates $y_{n}$ by means of a function $F(x)$ as described by the formula (8) of the Introduction. Although, as remarked there, the resulting interpolation formula is useless for practical purposes, it has the remarkable feature that the expansion coefficients $\omega_{n}^{(0)}(t, 0)$ of (16) may be obtained explicitly. Indeed the function $\phi_{0}(u, t)$ reduces to a Theta function which is a regular and uniform function of

$$
z=e^{i u}
$$

with singularities only at $z=0$ and $z=\infty$. The simple zeros of this function are real, negative and form a geometric progression. As a result we are able to find explicitly the decomposition in partial fractions of the reciprocal

$$
1 / \phi_{0}(u, l) .
$$

The expansion of these partial fractions into geometric power series furnishes explicitly the Laurent expansion in powers of $z$ and therefore also the cosine expansion (16).

The second special case of interest is $k>0, t=0$. In this case our formula (11) reduces to an ordinary polynomial interpolation formula of degree $k-1$ and class $k-2$. This does not contradict Mr. Greville's statement (loc. cit. page 212) to the effect that such formulae do not exist. Indecd Mr. Greville considers only basic polynomial functions $L(x)$ of finite span $s$ only, while our basic $L_{k}(x, 0)$ are of infinite span. This case, which is of considerable interest, requires a more detailed investigation of the cosine polynomials $\phi_{k}(u, 0)$. We postpone this discussion to the second part B of this paper.

## APPENDIX

## Description of the tables and their use for the analytic approximation of equidistant data.

Tables I and II. In Table I we find the 8-place values of the even function

$$
\begin{equation*}
M(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{-u^{2} / 8}\left(\frac{2 \sin u / 2}{u}\right)^{4} \cos u x d u \tag{1}
\end{equation*}
$$

and its derivatives $M^{\prime}(x), M^{\prime \prime}(x)$ for the step of $\Delta x=0.1$. The graph of $M(x)$ is a bellshaped curve and $M(x)$ vanishes to 8 places for $x \leqq-4.3$ and $x \geqq 4.3$. We now define a function of period $2 \pi$ by the cosine series

$$
\begin{equation*}
\phi(u)=M(0)+2 M(1) \cos u+2 M(2) \cos 2 u+\cdots \tag{2}
\end{equation*}
$$

and expand in cosine series the following functions

$$
\begin{equation*}
\frac{\epsilon+\phi(u)}{\epsilon+\phi(u)^{2}}=\omega_{0}(\epsilon)+2 \omega_{1}(\epsilon) \cos u+2 \omega_{2}(\epsilon) \cos 2 u+\cdots \tag{3}
\end{equation*}
$$

where $\epsilon$ is a non-negative parameter. Our Table II gives the 8 -plane values of these coefficients for $\epsilon=0,0.1,0.2, \cdots, 1.0$.

These tables may be used as follows to obtain an analytic approximation $F(x)$ to our ordinates $y_{n}$. We discuss first the case when $F(x)$ is to interpolate the ordinates, in the usual sense, i.e.,

$$
\begin{equation*}
F(n)=y_{n} \tag{4}
\end{equation*}
$$

For this end we compute first from the sequence $\left\{y_{n}\right\}$ a new sequence of coefficients $\left\{f_{n}\right\}$ by means of the formula
$f_{\tau}=\cdots+y_{n-2} \omega_{2}(0)+y_{n-1} \omega_{1}(0)+y_{n} \omega_{0}(0)+y_{n+1} \omega_{1}(0)+y_{n+2} \omega_{2}(0)+\cdots$
or

$$
f_{n}=\sum_{\nu} y_{\nu} \omega_{n-\nu}(0)
$$

where $\omega_{m}=\omega_{-m}$. The analytic approximation of the ordinates $y_{n}$ is then given by

$$
\begin{equation*}
F(x)=\sum_{n} f_{n} M(x-n) \tag{6}
\end{equation*}
$$

The values of $F(x), x$ to even tenths, are now readily computed. Thus

$$
\begin{aligned}
F(2.3)= & f_{-1} M(3.3)+f_{0} M(2.3)+f_{1} M(1.3)+f_{2} M(0.3) \\
& +f_{3} M(-1+.3)+f_{4} M(-2+.3)+f_{5} M(-3+.3)+f_{6} M(-4+.3)
\end{aligned}
$$

The tabular values of $M(x)$ are so arranged that all 8 values needed in this computation are found in the fourth column headed $x+3$. Generally, if the values of $f_{n}$ are written in a vertical column, we compute the values of $F\left(m+\nu \cdot 10^{-1}\right)(\nu=0,1, \cdots, 9)$ by matching the column of values of $f_{n}$ with the $\nu$ th column of the table of $M(x)$ in such a way that $f_{m}$ corresponds to the row for $x=0$. The products

$$
f_{n} M\left(m-n+\nu \cdot 10^{-1}\right)
$$

are then accumulated in the products counter of a desk computing machine. Also the values $f_{n}$ are best computed by ( $5^{\prime}$ ) in a similar way if the column of values of $\omega_{n}(0)$ is extended upwards by symmetry for negative values of $n$.

From the tables of $M^{\prime}(x)$ and $M^{\prime \prime}(x)$ we may likewise compute tables of the derivatives of $F(x)$ by

$$
\begin{equation*}
F^{(\nu)}(x)=\sum_{n} f_{n} M M^{(\nu)}(x-n) . \tag{7}
\end{equation*}
$$

A check of the computation of the coefficients $f_{n}$ is afforded by (4). Indeed the values $F(n)$ computed by (6) should agree with the $y_{n}$ to about eight significant figures.

The formula (6) is exact for cubics, i.e., if the $y_{n}$ are the ordinates of a polynomial of degree at most 3 , then $F(x)$ is identical with that polynomial.

If the conditions (4) of strict interpolation are not required, then we have the possibility of obtaining an approximation $F(x)$ which is such that the sequence $\{F(n)\}$ is smoother than the given $\left\{y_{n}\right\}$. The approximation $F(x)$ is then given by the pair of formulae

$$
\begin{align*}
f_{n} & =\sum_{\nu} y_{v} \omega_{n-}(\epsilon),  \tag{8}\\
F(x) & =\sum_{n} f_{n} M(x-n), \tag{9}
\end{align*}
$$

which are applied as above. The choice of the value of the smoothing parameter $\epsilon$ depends on the amount of smoothing desired. The strongest smoothing afforded by our table is obtained for $\epsilon=+\infty$. Then (3) shows that $\omega_{0}(\infty)=1, \omega_{1}(\infty)=\omega_{2}(\infty)$ $=\cdots=0$. Thus (8) becomes $f_{n}=y_{n}$ and (9) reduces to

$$
\begin{equation*}
F(x)=\sum_{n} y_{n} M(x-n) \tag{10}
\end{equation*}
$$

This formula is especially simple to apply. It should be remarked however that, if $\epsilon>0$, our formula (9) is exact only for lincar functions and the same is true of (10).

Table III. We may eliminate the coefficients $f_{n}$ between (8) and (9). In terms of the new even function

$$
\begin{equation*}
L(x, \epsilon)=\sum_{n=-\infty}^{\infty} \omega_{n}(\epsilon) M(x-n) \tag{11}
\end{equation*}
$$

our formulae (8), (9), then reduce to

$$
\begin{equation*}
F(x)=\sum_{n} y_{n} L(x-n, \epsilon) . \tag{12}
\end{equation*}
$$

Table III gives the values of $L(x, \epsilon)$ and $L^{\prime \prime}(x, \epsilon)$ for $\epsilon=0,0.1, \cdots, 1.0$ for the step $\Delta x=0.5$. These may be used for subtabulation to halves in preference to (5), (6) or (8), (9). For subtabulation to fifths or tens, the use of formulae (8), (9) is preferable because of the slower damping of the function $L(x, \epsilon)$. Even so, formula (12) and Table III allow us to estimate quickly how well $F(x)$ approximates the $y_{n}$. By (12) we have

$$
\begin{equation*}
F^{\prime \prime}(x)=\sum_{n} y_{n} L^{\prime \prime}(x-n, \epsilon) \tag{13}
\end{equation*}
$$

The table of $L^{\prime \prime}(x, \epsilon)$ then allows us to compute quickly a table of $F^{\prime \prime}(x)$ for the step $\Delta x=0.5$ or else only isolated values if such are needed.

Example of subtabulation to tenths. We consider the following fairly smooth sequence of 64 ordinates $y_{n}$ :

| $n$ | $y_{n}$ | $n$ | $y_{n}$ | $n$ | $y_{n}$ | $n$ | $y_{n}$ | $n$ | $y_{n}$ | $n$ | $y_{n}$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 24614 | 12 | 25370 | 23 | 29290 | 34 | 73820 | 45 | 82450 | 56 | 79962 |
| 2 | 24644 | 13 | 25504 | 24 | 30160 | 35 | 77830 | 46 | 82290 | 57 | 79698 |
| 3 | 24680 | 14 | 25660 | 25 | 31320 | 36 | 80240 | 47 | 82110 | 58 | 79431 |
| 4 | 24723 | 15 | 25850 | 26 | 32840 | 37 | 81660 | 48 | 81911 | 59 | 79161 |
| 5 | 24772 | 16 | 26080 | 27 | 34790 | 38 | 82330 | 49 | 81699 | 60 | 78889 |
| 6 | 24828 | 17 | 26350 | 28 | 37260 | 39 | 82680 | 50 | 81472 | 61 | 78614 |
| 7 | 24892 | 18 | 26660 | 29 | 40440 | 40 | 82840 | 51 | 81234 | 62 | 78338 |
| 8 | 24966 | 19 | 27040 | 30 | 44750 | 41 | 82830 | 52 | 80987 | 63 | 78060 |
| 9 | 25048 | 20 | 27490 | 31 | 51120 | 42 | 82780 | 53 | 80736 | 64 | 77780 |
| 10 | 25143 | 21 | 28010 | 32 | 59390 | 43 | 82700 | 54 | 80481 |  |  |
| 11 | 25250 | 22 | 28600 | 33 | 67550 | 44 | 82590 | 55 | 80223 |  |  |

The differences of the section of this table with which we will be concerned are as follows:

| $n$ | $y_{n}$ | $\Delta$ | $\Delta^{2}$ | $\Delta^{3}$ | $\Delta^{4}$ | $\Delta^{B}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 27 | 34790 |  |  |  |  |  |
| 28 | 37260 | 2470 |  |  |  |  |
| 29 | 40440 | 3180 | 710 |  |  |  |
| 30 | 44750 | 4310 | 1130 | 420 |  |  |
| 31 | 51120 | 6370 | 2060 | 930 | 510 |  |
| 32 | 59390 | 8270 | 1900 | -160 | -1090 | -1600 |
| 33 | 67550 | 8160 | -110 | -2010 | -1850 | -760 |
| 34 | 73820 | 6270 | -1890 | -1780 | 230 | 2080 |
| 35 | 77830 | 4010 | -2260 | -370 | 1410 | 1180 |
| 36 | 80240 | 2410 | -1600 | 660 | 1030 | -380 |
| 37 | 81660 | 1420 | -990 | 610 | -50 | -1080 |
| 38 | 82330 | 670 | -750 | 240 | -370 | -320 |

We illustrate the case of strict interpolation, i.e., we use our Tables II for $\epsilon=0$. From our formula (5) and the values of $\omega_{n}$ as given in the column of Table II, with the heading $\epsilon=0$, we obtain the following coefficients.

| $n$ | $f_{n}$ |
| :---: | :---: |
| 27 |  |
| 28 | 34662.222 |
| 29 |  |
| 30 | 40215.195 |
| 31 | 44060.182 |
| 32 | 50349.304 |
| 33 | 59490.524 |
| 34 | 68212.510 |
| 35 | 74566.216 |
| 36 | 78283.074 |
| 37 | 80460.234 |
| 38 | 81953.811 |
|  | 82356.888 |

From these values and our Table I of $M(x)$ and $M^{\prime \prime}(x)$, we obtain by the formulae (6) and (7) the following tables of $F(x)$ and $F^{\prime \prime}(x)$ with their differences.

Table of the function $F(x)$ and of its second derivative $F^{\prime \prime}(x)$.

| $x$ | $F(x)$ | $\Delta$ | $\Delta^{2}$ | $\Delta^{3}$ | $\Delta^{4}$ | $F^{\prime \prime}(x)$ | $\Delta$ | $\Delta^{2}$ | $\Delta^{3}$ | $\Delta^{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 31.0 | 51120.00 |  |  |  |  | 2117.97 |  |  |  |  |
| 31.1 | 51884.17 | 76417 |  |  |  | 1966.48 | -15149 |  |  |  |
| 31.2 | 52667.97 | 78380 | 1963 |  |  | 1787.44 | -17904 | -2755 |  |  |
| 31.3 | 53469.63 | 80166 | 1786 | $-177$ |  | 1583.71 | -20373 | -2469 | 286 |  |
| 31.4 | 54287.11 | 81748 | 1582 | -204 | -27 | 1359.15 | -22456 | -2083 | 386 | 100 |
| 31.5 | 55118.17 | 83106 | 1358 | -224 | -20 | 1118.30 | -24085 | -1629 | 454 | 68 |
| 31.6 | 55960.40 | 84223 | 1117 | -241 | -17 | 866.08 | -25222 | -1137 | 492 | 38 |
| 31.7 | 56811.29 | 85089 | 866 | -251 | $-10$ | 607.04 | -25868 | - 646 | 491 | $-1$ |
| 31.8 | 57668.25 | 85696 | 607 | -259 | $-8$ | 346.89 | -26051 | - 183 | 463 | $-28$ |
| 31.9 | 58528.68 | 86043 | 347 | -260 | $-1$ | 88.63 | -25826 | 225 | 408 | -55 |
| 32.0 | 59390.00 | 86132 | 89 | -258 | 2 | $-163.98$ | -25261 | 565 | 340 | -68 |
| 32.1 | 60249.69 | 85969 | $-163$ | -252 | 6 | - 408.22 | -24424 | 837 | 272 | -68 |
| 32.2 | 61105.30 | 85561 | - 408 | -245 | 7 | - 642.14 | -23392 | 1032 | 195 | $-77$ |
| 32.3 | 61954.51 | 84921 | - 640 | -232 | 13 | -864.26 | -22212 | 1180 | 148 | $-47$ |
| 32.4 | 62795.08 | 84057 | - 864 | -224 | 8 | -1073.51 | -20925 | 1287 | 107 | -41 |
| 32.5 | 63624.93 | 82985 | -1072 | -208 | 16 | -1269.11 | -19560 | 1365 | 78 | -29 |
| 32.6 | 64442.10 | 81717 | -1268 | -196 | 12 | -1450.39 | -18128 | 1432 | 67 | -11 |
| 32.7 | 65244.77 | 80267 | -1450 | $-182$ | 14 | -1616.76 | -16637 | 1491 | 59 | $-8$ |
| 32.8 | 66031.30 | 78653 | -1614 | -164 | 18 | -1767.70 | -15094 | 1543 | 52 | - |
| 32.9 | 66800.16 | 76886 | -1767 | -153 | 11 | -1902.77 | $-13507$ | 1587 | 44 | -8 |
| 33.0 | 67550.00 | 74984 | -1902 | -135 | 18 | -2021.68 | -11891 | 1616 | 29 | -15 |
| 33.1 | 68279.64 | 72964 | -2020 | -118 | 17 | -2124.30 | -10262 | 1629 | 13 | $-16$ |
| 33.2 | 68988.05 | 70841 | -2123 | $-103$ | 15 | -2210.71 | - 8641 | 1621 | - 8 | $-21$ |
| 33.3 | 69674.37 | 68632 | -2209 | -86 | 17 | -2281.13 | - 7042 | 1599 | -22 | -14 |
| 33.4 | 70337.91 | 65354 | -2278 | -69 | 17 | -2335.91 | - 5478 | 1564 | -35 | $-13$ |
| 33.5 | 70978.07 | 64016 | -2338 | $-60$ | 9 | -2375.46 | - 3955 | 1523 | -41 | $-6$ |
| 33.6 | 71594.50 | 61643 | -2373 | - 35 | 25 | $-2400.17$ | - 2471 | 1484 | -39 | 2 |
| 33.7 | 72186.94 | 59244 | -2399 | - 26 | 9 | $-2410.41$ | - 1024 | 1447 | -37 | 2 |
| 33.8 | 72755.29 | 56835 | -2409 | $-10$ | 16 | $-2406.55$ | 386 | 1410 | -37 | 0 |
| $33.9$ | 73299.58 | $54429$ | -2406 | 3 | 13 | -2389.01 | 1754 | 1368 | $-42$ | $-5$ |
| 34.0 | 73820.00 | 52042 | -2387 | 19 | 16 | $-2358.32$ | 3069 | 1315 | $-53$ | -11 |

An inspection of these tables shows that they are very smooth and that they define $F(x)$ and $F^{\prime \prime}(x)$ to 7 significant figures by 4 -point central interpolation. We have chosen on purpose an example for which it would be hard to obtain similar results by standard methods, if we are to maintain the forced accuracy requirement, and the same high degree of consistency between the function $F(x)$ and its second derivative $F^{\prime \prime}(x)$. For purposes of comparison we show also the interpolated values $F_{c}(x)$ for the range $x=31.6-32.5$ obtained by the 10 -point central interpolation method.
On comparing with our table of $F(x)$ we notice that

$$
F_{c}(x)<F(x)
$$

throughout this range, with the exception of the point $x=32.0$ where, of course, both values agree. The curve $F_{c}(x)$ has a corner at $x=32$. This is the typical discontinuity in the first derivative due to central interpolation methods (see the first paragraph of our Introduction).

| $x$ | $F_{e}(x)$ | $\Delta$ | $\Delta^{2}$ | $\Delta^{3}$ | $\Delta^{4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 31.6 | 55959.90 |  |  |  |  |
| 31.7 | 56810.60 | 85070 |  |  |  |
| 31.8 | 57667.55 | 85695 | 625 |  |  |
| 31.9 | 58528.22 | 86067 | 372 | -253 |  |
| 32.0 | 59390.00 | 86178 | 111 | -261 | -8 |
| 32.1 | 60248.72 | 85872 | -306 | -417 | -156 |
| 32.2 | 61103.61 | 85489 | -383 | -77 | 340 |
| 32.3 | 61952.37 | 84876 | -613 | -230 | -153 |
| 32.4 | 62792.77 | 84040 | -836 | -223 | 7 |
| 32.5 | 63622.68 | 82991 | -1049 | -213 | 10 |

Notice that we needed 12 coefficients $f_{n}$ for the subtabulation of three panels. Each additional coefficient $f_{n}(n=39,40, \cdots)$ allows the subtabulation of an additional panel.

It should be remarked that 53 ordinates $y_{n}$ enter into the computation of each coefficient $f_{n}$. This is due to the slow rate of damping of the $\omega_{n}(\epsilon)$ for $\epsilon=0$. Thus for $\epsilon=.1$ (very moderate smoothing) only 35 ordinates $y_{n}$ are needed, for $\epsilon=1.0$ only 23 , for $\epsilon=\infty$ only 1 . Concerning the important matter of dealing with the ends of a table see section 1.2 and the last paragraph of section 4.31 .

TAble I: $M_{k}(x, t), M_{k}^{\prime}(x, t), M_{k}^{\prime \prime}(x, \imath)$ for $k=4, \imath=0.5, \Delta x=0.1$. $M_{4}(x, 1 / 2)$

| $x$ | $x+0$ | $x+.1$ | $x+.2$ | $x+3$ | $x+.4$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | . 00000004 | . 00000002 | . 00000001 |  |  |
| 3 | . 00011325 | . 00005910 | . 00002991 | . 00001467 | . 00000697 |
| 2 | . 01616917 | . 01105340 | . 00737858 | . 00480621 | . 00305258 |
| 1 | . 22597004 | . 18940616 | . 15590118 | . 12596479 | . 09986387 |
| 0 | . 51549499 | . 51132566 | . 49901141 | . 47911917 | . 45254731 |
| -1 | . 22597004 | . 26483185 | . 30499058 | . 34523755 | . 39420963 |
| $-2$ | . 01616917 | . 02311310 | . 03230776 | . 04418973 | . 05917998 |
| -3 | . 00011325 | . 00021062 | . 00038032 | . 00066726 | . 00113822 |
| -4 | . 00000004 | . 00000010 | . 00000026 | . 00000062 | . 00000143 |
| $x$ | $x+.5$ | $x+.6$ | $x+7$ | $x+8$ | $x+.9$ |
| 3 | . 00000321 | . 00000143 | . 00000062 | . 00000026 | . 00000010 |
| 2 | . 00188907 | . 00113822 | . 00066726 | . 00038032 | . 00021062 |
| 1 | . 07764689 | . 05917998 | . 04418973 | . 03230776 | . 02311310 |
| 0 | . 42046084 | . 38420963 | . 34523755 | . 30499058 | . 26483185 |
| -1 | . 42046084 | . 45254731 | . 47911917 | . 49901141 | . 51132566 |
| -2 | . 07764689 | . 09986387 | . 12596479 | . 15590118 | . 18940616 |
| -3 | . 00188907 | . 00305258 | . 00480621 | . 00737858 | . 01105340 |
| -4 | . 00000321 | . 00000697 | . 00001467 | $.00002991$ | $.00005910$ |
| $-5$ |  |  |  | $.00000001$ | $.00000002$ |


| $M_{1}^{\prime}(x, 1 / 2)$ |  |  |  |  |  |
| ---: | :---: | :---: | :---: | :---: | :---: |
| $x$ | $x+0$ | $x+.1$ | $x+.2$ | $x+.3$ | $x+.4$ |
| 4 | -.00000039 | -.00000015 | -.00000006 | -.00000002 | -.00000001 |
| 3 | -.00071955 | -.00039340 | -.00020833 | -.00010680 | -.00005298 |
| 2 | -.05961795 | -.04334506 | -.03071644 | -.02120376 | -.01424920 |
| 1 | -.37860391 | -.35140346 | -.31784825 | -.28043885 | -.24150489 |
| 0 | .00000000 | -.08306134 | -.16227165 | -.23406492 | -.29541674 |
| -1 | .37860391 | .39695855 | .40419276 | .39846265 | .37855467 |
| -2 | .05961795 | .07996844 | .10465732 | .13368990 | .16673619 |
| -3 | .00071955 | .00127546 | .00219236 | .00365652 | .00592117 |
| -4 | .00000039 | .00000096 | .00000229 | .00000529 | .00001179 |
| $x$ | $x+.5$ | $x+.6$ | $x+.7$ | $x+.8$ | $x+.9$ |
| 3 | -.00002542 | -.00001179 | -.00000529 | -.00000229 | -.00000096 |
| 2 | -.00931577 | -.00592117 | -.00365652 | -.00219236 | -.00217546 |
| 1 | -.20306520 | -.16783619 | -.13368990 | -.10465732 | -.07996844 |
| 0 | -.34404758 | -.37855467 | -.39846265 | -.40419276 | -.39695855 |
| -1 | .34404758 | .29541674 | .23406492 | .16227165 | .08306134 |
| -2 | .20306520 | .24150489 | .28043885 | .31784825 | .35140346 |
| -3 | .00931577 | .01424920 | .02120376 | .03071644 | .04334506 |
| -4 | .00002542 | .00005298 | .00010680 | .00020833 | .00039340 |
| -5 |  | .00000001 | .00000002 | .00000006 | .00000015 |

$M_{1}^{\prime \prime}(x, 1 / 2)$

| $x$ | $x+.0$ | $x+.1$ |  | $x+.2$ | $x+.3$ |
| ---: | :---: | :---: | :---: | :---: | :---: |
| 4 | .00000357 | .00000145 | .00000056 | .00000021 | .00000008 |
| 3 | .00423106 | .00243772 | .00135797 | .00073109 | .00038024 |
| 2 | .18251117 | .14368197 | .10978191 | .08140988 | .05858190 |
| 1 | .23181861 | .30800376 | .35890239 | .38537940 | .38991971 |
| 0 | -.83712882 | -.81763132 | -.76058819 | -.67020231 | -.55301267 |
| -1 | .23181861 | .13144694 | .01013023 | -.12678241 | -.27209706 |
| -2 | .18251117 | .22494722 | .26885281 | .31126005 | .34845442 |
| -3 | .00423106 | .00710375 | .01154277 | .01816077 | .02768079 |
| -4 | .00000357 | .00000851 | .00001955 | .00004332 | .00009259 |
| -5 |  |  |  |  | .00000001 |
| $x$ | $x+.5$ | $x+.6$ |  | $x+.7$ |  |
| 4 | .00000003 | .00000001 |  | $x+.8$ | $x+.9$ |
| 3 | .00019097 | .00009259 | .00004332 | .00001955 | .00000851 |
| 2 | .04089359 | .02768079 | .01816077 | .01154277 | .00710375 |
| 1 | .37617315 | .34845442 | .31126005 | .26885281 | .22494722 |
| 0 | -.41725773 | -.27209706 | -.12678241 | .01013023 | .13144694 |
| -1 | -.41725773 | -.55301267 | -.67020231 | -.76058819 | -.81763132 |
| -2 | .37617315 | .38991971 | .38537940 | .35890239 | .30800376 |
| -3 | .04089359 | .05858190 | .08140988 | .10978191 | .14368197 |
| -4 | .00019097 | .00038024 | .00073109 | .00135797 | .00243772 |
| -5 | .00000003 | .00000008 | .00000021 | .00000056 | .00000145 |

Table II: $\omega_{n}^{(k)}(t, \epsilon)$ for $k=4, t=0.5, \epsilon=0(0.1) 1.0$.

| $n$ | $\epsilon=.0$ | $\epsilon=.1$ | $\epsilon=.2$ | $\epsilon=.3$ | $\epsilon=.4$ |
| :---: | ---: | ---: | ---: | ---: | ---: |
| 0 | 3.50637741 | 1.61378653 | 1.39953009 | 1.30308904 | 1.24631521 |
| 1 | -1.84900618 | -.26890929 | -.14132793 | -.09293505 | -.06784110 |
| 2 | .87238793 | -.08981772 | -.09332063 | -.08242675 | -.07223261 |
| 3 | -.40443570 | .07027891 | .04023694 | .02480160 | .01624479 |
| 4 | .18693997 | -.02078617 | -.00397484 | .00050538 | .00184671 |
| 5 | -.08636451 | .00133949 | -.00223908 | -.00188468 | -.00132999 |
| 6 | .03989615 | .00169234 | .00099749 | .00037463 | .00010684 |
| 7 | -.01842978 | -.00088114 | -.00010447 | .00005298 | .00006541 |
| 8 | .00851350 | .00019734 | -.00005372 | -.00003828 | -.00001761 |
| 9 | -.00393275 | .00001073 | .00002469 | .00000455 | -.00000116 |
| 10 | .00181670 | -.00002625 | -.00000273 | .00000174 | .00000130 |
| 11 | -.00083921 | .00001022 | -.00000129 | -.00000070 | -.00000014 |
| 12 | .00038767 | -.00000156 | .00000061 | .00000003 | -.00000006 |
| 13 | -.00017908 | -.00000044 | -.00000007 | .00000004 | .00000002 |
| 14 | .00008272 | .00000036 | -.00000003 | -.00000001 |  |
| 15 | -.00003821 | -.00000011 | .00000002 |  |  |
| 16 | .00001765 | .00000001 |  |  |  |
| 17 | -.00000815 | .00000001 |  |  |  |
| 18 | .00000377 |  |  |  |  |
| 19 | -.00000174 |  |  |  |  |
| 20 | .00000080 |  |  |  |  |
| 21 | -.00000037 |  |  |  |  |
| 22 | .00000017 |  |  |  |  |
| 23 | -.00000008 |  |  |  |  |
| 24 | .0000004 |  |  |  |  |
| 25 | -.00000002 |  |  |  |  |
| 26 | .00000001 |  |  |  |  |


| $n$ | $\epsilon=.5$ | $\epsilon=.6$ | $\epsilon=.7$ | $\epsilon=.8$ | $\epsilon=.9$ | $\epsilon=1.0$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1.20834767 | 1.18095463 | 1.16016154 | 1.14379093 | 1.13054096 | 1.11958158 |
| 1 | -. 05268720 | -. 04264921 | -. 03556878 | -. 03034057 | -. 02634263 | -. 02319971 |
| 2 | -. 06387537 | -. 05710538 | -. 05156939 | -. 04698035 | -. 04312407 | -. 03984269 |
| 3 | . 01107108 | . 00773445 | . 00547641 | . 00389078 | . 00274451 | . 00189634 |
| 4 | . 00219079 | . 00218246 | . 00204868 | . 00187682 | . 00170183 | . 00153738 |
| 5 | $-.00091344$ | -. 00062674 | $-.00043120$ | -. 00029659 | -. 00020265 | -. 00013620 |
| 6 | -. 00000351 | -. 00004704 | -. 00006163 | -. 00006358 | -. 00006018 | -. 00005476 |
| 7 | . 00005180 | . 00003700 | . 00002545 | . 00001716 | . 00001138 | . 00000739 |
| 8 | -. 00000656 | -. 00000140 | . 00000085 | . 00000171 | . 00000193 | . 00000187 |
| 9 | -. 00000206 | -. 00000175 | -. 00000127 | $-.00000087$ | -. 00000057 | $-.00000036$ |
| 10 | . 00000064 | . 00000025 | . 00000006 | -. 00000002 | -. 00000005 | -. 00000006 |
| 11 | . 00000003 | . 00000006 | . 00000006 | . 00000004 | . 00000002 | . 00000001 |
| 12 | -. 00000004 | $-.00000002$ | $-.00000001$ |  |  |  |

Table III: $L_{k}(x, t, \epsilon), L_{k}^{\prime \prime}(x, t, \epsilon)$ for $k=4, t=0.5, \varepsilon=0(0.1) 1.0, \Delta x=0.5$. $L_{4}(x, 1 / 2, \epsilon)$

| $x$ | $\epsilon=.0$ | $\epsilon=.1$ | $t=.2$ | $\epsilon=.3$ | $\epsilon=.4$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | 1.00000000 | . 70747935 | . 65457028 | . 62707488 | . 60947694 |
| 0.5 | . 62191163 | . 53757743 | . 51070485 | . 49509729 | . 48452022 |
| 1.0 | . 00000000 | . 20252568 | . 22066478 | . 22681461 | . 22949350 |
| 1.5 | -. 17291085 | -. 02061576 | . 01285810 | . 02919893 | . 03901344 |
| 2.0 | . 00000000 | -. 06545791 | -. 04840114 | -. 03681939 | -. 02872086 |
| 2.5 | . 07415615 | -. 02765903 | -. 03096280 | -. 02894823 | -. 02631330 |
| 3.0 | . 00000000 | . 00709183 | -. 00340667 | -. 00711221 | -. 00850828 |
| 3.5 | -. 03382251 | . 01344008 | . 00756627 | . 00392340 | . 00177090 |
| 4.0 | . 00000000 | . 00401304 | . 00502862 | . 00410188 | . 00314843 |
| 4.5 | . 01558996 | -. 00276042 | . 00041203 | . 00121901 | . 00135036 |
| 5.0 | . 00000000 | -. 00251219 | -. 00118870 | -. 00038015 | . 00001138 |
| 5.5 | -. 00719897 | -. 00027479 | -. 00076318 | -. 00054505 | -. 00033539 |
| 6.0 | . 00000000 | . 00065102 | -. 00007596 | -. 00021043 | -. 00019927 |
| 6.5 | . 00332530 | . 00042139 | . 00019012 | . 00003152 | -. 00002867 |
| 7.0 | . 00000000 | -. 00000773 | . 00012316 | . 00007297 | . 00003257 |
| 7.5 | -. 00153608 | -. 00015286 | . 00000861 | . 00003207 | . 00002580 |
| 8.0 | . 00000000 | -. 00006787 | -. 00002989 | -. 00000086 | . 00000704 |
| 8.5 | . 00070958 | . 00002025 | -. 00001865 | -. 00000923 | -. 00000251 |
| 9.0 | . 00000000 | . 00003031 | -. 00000162 | -. 00000502 | -. 00000321 |
| 9.5 | -. 00032779 | . 00000793 | . 00000477 | -. 00000028 | -. 00000120 |
| 10.0 | . 00000000 | -. 00000573 | . 00000301 | . 00000115 | . 00000010 |
| 10.5 | . 00015142 | -. 00000566 | . 00000017 | . 00000072 | . 00000036 |
| 11.0 | . 00000000 | -. 00000083 | -. 00000075 | . 00000011 | . 00000018 |
| 11.5 | -. 00006995 | . 00000159 | -. 00000045 | -. 00000014 | . 00000001 |
| 12.0 | . 00000000 | . 00000099 | -. 00000003 | -. 00000011 | -. 00000004 |
| 12.5 | . 00003231 | -. 00000007 | . 00000012 | -. 00000002 | -. 00000002 |
| 13.0 | . 00000000 | -. 00000034 | . 00000007 | . 00000001 | $-.00000001$ |
| 13.5 | -. 00001492 | -. 00000014 | . 00000000 | . 00000001 |  |
| 14.0 | . 00000000 | . 00000004 | -. 00000002 |  |  |
| 14.5 | . 00000690 | . 00000007 | -. 00000001 |  |  |
| 15.0 | . 00000000 | . 00000002 |  |  |  |
| 15.5 | -. 00000318 | -. 00000001 |  |  |  |
| 16.0 | . 00000000 | -. 00000001 |  |  |  |
| 16.5 | . 00000147 |  |  |  |  |
| 17.0 | . 00000000 |  |  |  |  |
| 17.5 | -. 000000068 |  |  |  |  |
| 18.0 | . 00000000 |  |  |  |  |
| 18.5 | . 000000031 |  | 9xatione |  |  |
| 19.0 19.5 | .00000000 -.00000014 |  |  |  |  |
| 20.0 | . 00000000 |  |  |  |  |
| 20.5 | . 00000007 |  |  |  |  |
| 21.0 | . 00000000 |  |  |  |  |
| 21.5 | $-.00000003$ |  |  |  |  |
| 22.0 | . 00000000 |  |  |  |  |
| 22.5 | . 00000001 |  |  |  |  |
| 23.0 | . 00000000 |  |  |  |  |
| 23.5 | -. 00000001 |  |  |  | . |


| $L_{4}(x, 1 / 2, \epsilon)$ |  |  |  |  |  |  |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $x$ | $\epsilon=.5$ | $\epsilon=.6$ | $\epsilon=.7$ | $\epsilon=.8$ | $\epsilon=.9$ | $\epsilon=1.0$ |
| 0.0 | .59702260 | .58765637 | .58031608 | .57438799 | .56948898 | .56536580 |
| 0.5 | .47675954 | .47077398 | .46599416 | .46207717 | .45880202 | .45601892 |
| 1.0 | .23077657 | .23140004 | .23168090 | .23177314 | .23175789 | .23168050 |
| 1.5 | .04557847 | .05027848 | .05380648 | .05654957 | .05874137 | .06053136 |
| 2.0 | -.02276409 | -.01820178 | -.01459584 | -.01167397 | -.00925829 | -.00722771 |
| 2.5 | -.0238427 | -.02167108 | -.01979233 | -.01816756 | -.01675611 | -.01552233 |
| 3.0 | -.00896161 | -.00898986 | -.00881784 | -.00855221 | -.00824659 | -.00792883 |
| 3.5 | .00044979 | -.00038991 | -.00093726 | -.00129962 | -.00154092 | -.00170083 |
| 4.0 | .0023858 | .00180229 | .00135734 | .00101562 | .00075048 | .00054256 |
| 4.5 | .00127570 | .00114308 | .00100305 | .00087278 | .00075725 | .00065681 |
| 5.0 | .00019600 | .00027865 | .00030985 | .00031471 | .00030617 | .00029109 |
| 5.5 | -.00019071 | -.00009654 | -.00003598 | .00000283 | .00002756 | .00004311 |
| 6.0 | -.00015995 | -.00012137 | -.00008970 | -.00006510 | -.00004639 | -.00003223 |
| 6.5 | -.00004696 | -.00004885 | -.00004474 | -.00003886 | -.00003288 | -.00002744 |
| 7.0 | .00000987 | -.00000179 | -.00000738 | -.00000973 | -.00001039 | -.00001018 |
| 7.5 | .00001687 | .00001000 | .00000536 | .00000238 | .00000050 | -.00000064 |
| 8.0 | .00000771 | .00000641 | .00000486 | .00000350 | .00000244 | .00000165 |
| 8.5 | .00000044 | .00000148 | .00000168 | .00000156 | .00000134 | .00000110 |
| 9.0 | -.00000156 | -.00000057 | -.00000004 | .00000021 | .00000031 | .00000034 |
| 9.5 | -.00000101 | -.00000067 | -.00000039 | -.00000020 | -.00000009 | -.00000002 |
| 10.0 | -.00000023 | -.00000027 | -.00000022 | -.00000017 | -.00000011 | -.00000008 |
| 10.5 | .00000011 | -.00000001 | -.00000005 | -.00000006 | -.00000005 | -.00000004 |
| 11.0 | .00000012 | .00000006 | .00000002 | .00000000 | -.00000001 | -.00000001 |
| 11.5 | .00000004 | .00000004 | .00000002 | .00000001 | .00000001 |  |
| 12.0 | .00000000 | .00000001 | .00000001 | .00000001 |  |  |
| 12.5 | -.00000001 |  |  |  |  |  |
| 13.0 | -.00000001 |  |  |  |  |  |


| $x$ | $\epsilon=.0$ | $\epsilon \pm .1$ | $\epsilon=.2$ | $\epsilon=.3$ | $\epsilon=.4$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | -3.47753764 | -1.50781449 | -1.27083552 | -1.16381926 | -1.10100908 |
| 0.5 | -1.03983382 | -. 69689346 | -. 61542693 | -. 57326418 | -. 54670549 |
| 1.0 | 2.15613767 | . 54167607 | . 40225158 | . 34798919 | . 31925128 |
| 1.5 | 1.50655095 | . 77131824 | . 63355043 | . 56889199 | . 53067521 |
| 2.0 | -. 58680458 | . 31875073 | . 30878327 | . 29072626 | . 27601758 |
| 2.5 | -. 68097024 | -. 03482567 | . 02460402 | . 04246822 | . 04943501 |
| 3.0 | . 24590776 | -. 12647279 | -. 07651577 | -. 05154399 | -. 03726679 |
| 3.5 | . 31311773 | -. 06455381 | -. 05654901 | -. 04581150 | -. 03775297 |
| 4.0 | -. 11165611 | . 01678358 | -. 00530735 | -. 01047426 | -. 01153447 |
| 4.5 | -. 14452585 | . 03142765 | . 01425664 | . 00665986 | . 00297134 |
| 5.0 | . 05142591 | . 00673809 | . 00811324 | . 00596871 | . 00423710 |
| 5.5 | . 06675341 | -. 00655085 | . 00060896 | . 00183704 | . 00187944 |
| 6.0 | -. 02374375 | -. 00477107 | -. 00194355 | -. 00054832 | . 00001970 |
| 6.5 | -. 03083551 | -. 00059653 | -. 00138896 | -. 00087620 | -. 00050023 |
| 7.0 | . 01096729 | . 00133431 | -. 00011460 | -. 00030729 | -. 00026914 |
| 7.5 | . 01424417 | . 00097571 | . 00035809 | . 00005960 | -. 00003417 |
| 8.0 | -. 00506618 | $-.00005726$ | . 00019854 | . 00010609 | . 00004375 |
| 8.5 | -. 00657998 | -. 00035828 | . 00001209 | . 00004959 | . 00003703 |
| 9.0 | . 00234028 | -. 00012112 | -. 00004884 | -. 00000116 | . 00000954 |
| 9.5 | . 00303957 | . 00004878 | -. 00003391 | -. 00001505 | -. 00000404 |
| 10.0 | -. 00108107 | . 00005884 | -. 00000242 | -. 00000732 | -. 00000433 |
| 10.5 | -. 00140411 | . 00001802 | . 00000898 | -. 00000026 | -. 00000164 |
| 11.0 | . 00049939 | -. 00001229 | . 00000485 | . 00000168 | $.00000013$ |
| 11.5 | . 00064862 | -. 00001316 | . 00000023 | . 00000113 | . 00000053 |
| 12.0 | -. 00023069 | -. 00000110 | -. 00000123 | . 00000016 | . 00000025 |
| 12.5 | -. 00029962 | . 00000374 | -. 00000083 | -. 00000023 | . 00000001 |
| 13.0 | $.00010657$ | . 00000182 | -. 00000005 | -. 00000015 | $-.00000005$ |
| 13.5 | . 00013841 | -. 00000018 | . 00000022 | -. 00000003 | $-.00000004$ |
| 14.0 | -. 00004923 | -. 00000067 | . 00000012 | . 00000002 |  |
| 14.5 | -. 00006394 | -. 00000033 | . 00000000 | . 00000002 | $\text { . } 00000001$ |
| $15.0$ | $.00002274$ | $.00000009$ | $-.00000003$ | . 00000001 |  |
| 15.5 16.0 | .00002954 -.00001050 | . 000000016 | $-.00000002$ |  |  |
| 16.0 16.5 | -. 000001050 | .00000003 -.0000003 |  |  |  |
| 17.0 | . 00000485 | -. 000000002 |  |  |  |
| 17.5 | . 00000630 | . 00000000 |  |  |  |
| 18.0 | -. 00000224 | . 00000001 |  |  |  |
| 18.5 | -. 00000291 |  |  |  |  |
| 19.0 | . 00000104 |  |  |  |  |
| 19.5 | . 00000134 |  |  |  |  |
| 20.0 | -. 00000048 |  |  |  |  |
| 20.5 | -. 00000062 |  |  |  |  |
| 21.0 | . 00000022 |  |  |  |  |
| 21.5 | . 00000029 |  |  |  |  |
| 22.0 | -. 00000010 |  |  |  |  |
| 22.5 | -. 00000013 |  |  |  |  |
| 23.0 | . 00000005 |  |  |  |  |
| 23.5 | . 00000006 |  |  |  |  |
| 24.0 | -. 00000002 |  |  |  |  |
| 24.5 | -. 00000003 |  |  |  |  |
| 25.0 | . 00000001 |  |  |  |  |
| 25.5 | . 00000001 |  |  |  |  |
| 26.0 | $-.00000001$ |  |  |  |  |


| $x$ | $\epsilon=.5$ | $\epsilon=.6$ | $\epsilon=.7$ | $\epsilon=.8$ | $\epsilon=.9$ | $\epsilon=1.0$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | -1.05919265 | -1.02916420 | -1.00647330 | -. 98868331 | -. 97433986 | -. 96251767 |
| 0.5 | -. 52821280 | -. 51450880 | -. 50390754 | -. 49544281 | -. 48851722 | -. 48273978 |
| 1.0 | . 30155959 | . 28962736 | . 28106624 | . 27464186 | . 26965348 | . 26567454 |
| 1.5 | . 50527188 | . 48711044 | . 47346009 | . 46281678 | . 45428116 | . 44728133 |
| 2.0 | . 26453429 | . 25546310 | . 24815810 | . 24216442 | . 23716440 | . 23293281 |
| 2.5 | . 05240385 | . 05363758 | . 05404202 | . 05402794 | . 05379802 | . 05345832 |
| 3.0 | -. 02823788 | -. 02210889 | -. 01772635 | -. 01446547 | -. 01196172 | -. 00998973 |
| 3.5 | -. 03182937 | -. 02737565 | -. 02393662 | -. 02121579 | -. 01901742 | -. 01720888 |
| 4.0 | -. 01135624 | -. 01078633 | -. 01011492 | -. 00944820 | -. 00882438 | -. 00825481 |
| 4.5 | . 00101034 | -. 00009918 | -. 00075259 | $-.00114593$ | -. 00138383 | -. 00152531 |
| 5.0 | . 00303129 | . 00219629 | . 00160744 | . 00118299 | . 00087064 | . 00063651 |
| 5.5 | . 00166647 | . 00142136 | . 00120000 | . 00101295 | . 00085821 | . 00073079 |
| 6.0 | . 00024844 | . 00033324 | . 00035457 | . 00034759 | . 00032845 | . 00030479 |
| 6.5 | -. 00027461 | -. 00014145 | -. 00006202 | -. 00001410 | . 00001495 | . 00003245 |
| 7.0 | -. 00020347 | -. 00014764 | $-.00010563$ | -. 00007505 | -. 00005291 | -. 00003686 |
| 7.5 | $-.00005788$ | -. 00005841 | -. 00005187 | $-.00004389$ | -. 00003637 | -. 00002987 |
| 8.0 | . 00001265 | -. 00000184 | -. 00000816 | -. 00001051 | -. 00001095 | -. 00001051 |
| 8.5 | . 00002301 | . 00001329 | . 00000717 | . 00000344 | . 00000121 | -. 00000014 |
| 9.0 | . 00000979 | . 00000777 | . 00000567 | . 00000399 | . 00000274 | . 00000184 |
| 9.5 | . 00000025 | . 00000163 | . 00000187 | . 00000172 | . 00000145 | . 00000117 |
| 10.0 | -. 00000199 | -. 00000071 | $-.00000008$ | . 00000021 | . 00000032 | . 00000034 |
| 10.5 | -. 00000133 | -. 00000085 | -. 00000049 | -. 00000026 | -. 00000012 | -. 00000004 |
| 11.0 | -. 00000029 | -. 000000033 | -. 00000026 | -. 00000019 | -. 00000013 | -. 00000008 |
| 11.5 | . 00000016 | . 00000000 | -. 00000005 | $-.00000006$ | -. 00000005 | $-.00000004$ |
| 12.0 | . 00000015 | $.00000007$ | $.00000002$ | $00000000$ | $-.00000001$ | -. 00000001 |
| 12.5 | . 00000005 | $.00000004$ | $.00000003$ | . 00000001 | . 00000001 |  |
| 13.0 | . 00000000 | . 00000001 | . 00000001 | . 00000001 |  |  |
| 13.5 14.0 | $\begin{aligned} & -.00000002 \\ & -.00000001 \end{aligned}$ |  |  |  |  |  |

## -NOTES-

## SOME APPLICATIONS OF THE REPEATED INTEGRALS OF THE ERROR FUNCTION*

By J. C. JAEGER (University of Tasmania)

1. Introductory. The repeated integrals of the error function

$$
\begin{equation*}
\mathrm{i}^{n} \operatorname{erfc} x=\int_{\pi}^{\infty} \mathrm{i}^{n-1} \operatorname{erfc} \xi d \xi, \quad n=1,2, \cdots \tag{1}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{i}^{0} \operatorname{erfc} x=\operatorname{erfc} x=\frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-\xi^{2}} d \xi \tag{2}
\end{equation*}
$$

have been studied by Hartree, ${ }^{1}$ who tabulates them for $n=1$ and $n=2$, and shows that they satisfy the recurrence relation

$$
\begin{equation*}
2 n \mathrm{i}^{n} \operatorname{erfc} x=\mathrm{i}^{n-2} \operatorname{erfc} x-2 x \mathrm{i}^{n-1} \operatorname{erfc} x . \tag{3}
\end{equation*}
$$

He also shows that

$$
\begin{equation*}
L\left\{(4 t)^{n / 2} \mathrm{j}^{n} \operatorname{erfc}\left(\frac{1}{2} a t^{-1 / 2}\right)\right\}=s^{-1-n / 2} e^{-a \sqrt{5}}, \tag{4}
\end{equation*}
$$

where $n=0,1,2, \cdots, a \geqq 0$, and $L\{v\}$ is written for the Laplace transform of a function $v(t)$ of $t$, that is

$$
\begin{equation*}
L\{v\}=\int_{0}^{\infty} e^{-s t} v(t) d t \tag{5}
\end{equation*}
$$

The functions (1) arise naturally in the theory of conduction of heat in the semiinfinite solid (or the sphere or slab) with prescribed surface temperature or flow of heat, since the Laplace transforms of the solutions of many such problems involve the functions on the right hand side of (4).

The objects of this note are, firstly, to indicate an extension of (4) which applies in the same way to problems with heat transfer at the surface, and, secondly, to give solutions in terms of the functions (1) of a number of problems of practical interest which involve heat generation in the solid.
2. Problems involving heat transfer at a surface at a rate proportional to its temperature difference from its surroundings. The required extension of (4) is that, if $n$ is a positive integer, and $\alpha, h$, and $x$ are positive,

$$
\begin{equation*}
L\left\{\alpha(-h)^{-n}\left[e^{2 H X+H^{2}} \operatorname{erfc}(H+X)-\sum_{r=0}^{n-1}(-2 H)^{r} \mathrm{i}^{r} \operatorname{erfc} X\right]\right\}=\frac{e^{-q x}}{q^{n+1}(q+h)} \tag{6}
\end{equation*}
$$

where

$$
\begin{equation*}
X=x / 2 \sqrt{\alpha t}, \quad H=h \sqrt{\alpha t}, \quad q=\sqrt{s / \alpha} \tag{7}
\end{equation*}
$$

[^36]To derive this result we notice that

$$
\begin{equation*}
\frac{e^{-q x}}{q^{n+1}(q+h)}=\frac{(-)^{n} e^{-q x}}{h^{n} q(q+h)}+\frac{e^{-q x}}{h^{n} q^{2}} \sum_{r=0}^{n-1}(-)^{n-r-1}\left(\frac{h}{q}\right)^{r} \tag{8}
\end{equation*}
$$

In the terms of the series we use (4); the result for the first term of the right-hand side of (8) is given in most tables of Laplace transforms; and (6) follows immediately.

Typical examples in which (6) arises are the following:
(i) The semi-infinite solid $x>0$. Zero initial temperature. The solid heated at $x=0$ for $t>0$ by heat transfer from a medium at at ${ }^{n / 2}, n=0,1$,

The temperature $v$ in the solid has to satisfy

$$
\begin{equation*}
\frac{\partial^{2} v}{\partial x^{2}}-\frac{1}{\alpha} \frac{\partial v}{\partial t}=0, \quad x>0, \quad t>0 \tag{9}
\end{equation*}
$$

with the boundary condition

$$
\frac{\partial v}{\partial x}=-h\left(a t^{n / 2}-v\right), \quad x=0, \quad t>0
$$

Also $v$ has to be bounded as $x \rightarrow \infty$. The Laplace transform of the solution is found to be

$$
L\{v\}=\frac{h a \Gamma(1+n / 2) e^{-q x}}{s^{1+n / 2}(q+h)},
$$

using the notation (7). Therefore, from (6)

$$
v=\frac{(-)^{n+1} a \Gamma(1+n / 2)}{h^{n} \alpha^{n / 2}}\left\{c^{2 H X+H^{2}} \operatorname{erfc}(H+X)-\sum_{r=0}^{n}(-2 H)^{r_{\mathrm{i}} r^{r}} \operatorname{erfc} X\right\}
$$

If heat transfer takes place from a medium whose temperature is

$$
\begin{equation*}
\sum_{n=0}^{N} a_{n} t^{n / 2} \tag{10}
\end{equation*}
$$

the solution follows at once. For the problem of the semi-infinite solid whose surface temperature is given by (10) the result is obtained in the same way by using (4). An empirical relation of the type (10) is often useful for representing observed surface temperatures, the term in $t^{1 / 2}$ being particularly valuable since it corresponds to constant flux of heat; for example the fall in temperature of the Earth's surface after sunset on a cloudless evening is approximately proportional to $t^{1 / 2}$ and may be represented very well by two or three terms of (10).
(ii) The semi-infinite solid $x>0$, of conductivity $K$ and diffusivity $\alpha$. At $x=0$ the solid is in contact with mass $M$ per unit area of well stirred fluid of specific heat $c^{\prime}$, whose temperature is equal to the surface temperature of the solid, i.e. to $\lim _{x \rightarrow+0}$ v. The initial temperatures of the solid and fluid are zero. Heat is supplied to the fuid at constant rate $Q$ per unit mass per unit time for $t>0$.

Here (9) has to be solved with boundary condition at $x=0$

$$
M c^{\prime} \frac{\partial v}{\partial l}-K \frac{\partial v}{\partial x}=Q M, \quad x=0, \quad t>0
$$

The Laplace transform of the solution is

$$
L\{v\}=\frac{Q^{-q x}}{\alpha^{2} c^{\prime} q^{3}(q+h)},
$$

where now $h=K / M c^{\prime} \alpha$. Thus from (6) we have

$$
v=\frac{Q}{h^{2} \alpha c^{\prime}}\left\{e^{2 H X+X^{2}} \operatorname{erfc}(H+X)-\operatorname{erfc} X+2 H i^{1} \operatorname{erfc} X\right\} .
$$

(iii) The semi-infinite solid $x>0$. Zero initial temperature. Heal is produced for $t>0$ in the solid at the rate $Q^{\prime n / 2}, n=-1,0,1, \cdots$, per unit time per unit volume. There is heat transfer at $x=0$ into a medium al zero temperature.

Here we have to solve the differential equation

$$
\frac{\partial^{2} v}{\partial x^{2}}-\frac{1}{\alpha} \frac{\partial v}{\partial t}=-\frac{Q}{K} t^{n / 2}, \quad x>0, \quad t>0,
$$

with boundary condition

$$
\frac{\partial v}{\partial x}-h v=0, \quad x=0, \quad t>0 .
$$

Here

$$
L\{v\}=\frac{Q \alpha \Gamma(1+n / 2)}{K}\left\{\frac{1}{s^{2+n / 2}}-\frac{h e^{-q x}}{\alpha^{2+n / 2} q^{n+4}(q+h)}\right\},
$$

and
$\left.v=\frac{Q \alpha l^{1+n / 2}}{K(1+n / 2)}+\frac{Q \Gamma(1+n / 2)}{K \alpha^{n / 2}(-h)^{n+2}}\left[e^{2 H X+H^{2}} \operatorname{erfc}(H+X)-\sum_{r=0}^{n+2}(-2 H)\right)^{\mathrm{i}} \mathrm{i} \operatorname{erfc} X\right]$.
3. Cases of generation of heat in a solid. The solutions of a number of problems of practical importance in which heat is generated for $t>0$ in a solid at the rate

$$
\begin{equation*}
\sum_{n=-1}^{N} a_{n} t^{n / 2} \tag{11}
\end{equation*}
$$

per unit time per unit volume can be expressed in terms of the functions (1). An expression of type (11) may be useful for representing an experimentally observed rate of generation of heat; the term in $t^{-1 / 2}$ is of value when the initial rate of heat production is high, as in the hydrating of cement. ${ }^{2}$

A problem involving (11) and radiation at the surface has already been given in §2(iii), here we give the solutions of some cases in which the surface temperature is zero.
(iv) The region $x>0 . x=0$ kept at zero for $t>0$. Heat production at the rate $t^{n / 2}$ in $0<x<a$, and zero in $x>a$. Zero initial temperature.

[^37]This is the fundamental practical problem of temperatures in hydrating concrete: a slab of concrete is poured on the surface of the semi-infinite solid, we assume here that the thermal constants of the concrete and the solid are the same. The solution is
$v=\frac{\alpha t^{1+n / 2}}{K(1+n / 2)}\left\{1-\Gamma(2+n / 2) 2^{n+1}\left[\mathrm{i}^{n+2} \operatorname{erfc} \frac{a-x}{2(\alpha t)^{1 / 2}}-\mathrm{i}^{n+2} \operatorname{erfc} \frac{a+x}{2(\alpha t)^{1 / 2}}\right.\right.$

$$
\left.\left.+2 \mathrm{i}^{n+2} \operatorname{erfc} \frac{x}{2(\alpha t)^{1 / 2}}\right]\right\}
$$

if $0<x<a$, and for $x>a$ it is
$v=\frac{\alpha \Gamma(1+n / 2)(4 l)^{1+n / 2}}{2 K}\left\{\mathrm{i}^{n+2} \operatorname{erfc} \frac{x-a}{2(\alpha l)^{1 / 2}}+\mathrm{i}^{n+2} \operatorname{erfc} \frac{x+a}{2(\alpha t)^{1 / 2}}\right.$
$\left.-2 \mathrm{i}^{n+2} \operatorname{erfc} \frac{x}{2(\alpha i)^{1 / 2}}\right\}$.
(v) The problem ${ }^{3}$ of (iv) except that heat is produced only in the region $a<x<b$. The temperature gradient at the surface is

$$
\frac{\alpha^{1 / 2}}{K} \Gamma(1+n / 2)(4 l)^{(1+n) / 2}\left[\mathrm{i}^{1+n} \operatorname{erfc} \frac{a}{2(\alpha t)^{1 / 2}}-\mathrm{i}^{1+n} \operatorname{erfc} \frac{b}{2(\alpha t)^{1 / 2}}\right]
$$

(vi) The slab $0<x<l$. The surfaces $x=0$ and $x=l$ kept at zero temperature. Heat generation at the rate $t^{n / 2}$. Zero initial temperature.

$$
\begin{aligned}
v=\frac{\alpha t^{1+n / 2}}{K(1+n / 2)}\left\{1-\Gamma(2+n / 2) 2^{n+2} \sum_{m=0}^{\infty}(-)^{m}\left[\mathrm{i}^{n+2} \operatorname{crfc}\right.\right. & \frac{m l+x}{2(\alpha t)^{1 / 2}} \\
& \left.\left.+\mathrm{i}^{n+2} \operatorname{erfc} \frac{(m+1) l-x}{2(\alpha t)^{1 / 2}}\right]\right\}
\end{aligned}
$$

(vii) The infinite region $r \geqq 0$. Zero initial temperature. Heat production at the rate $t^{n / 2}$ in the sphere $0 \leqq r<a$, zero elsewhere. ${ }^{4}$

$$
\begin{aligned}
v= & \frac{\alpha t^{1+n / 2}}{K(1+n / 2)}-\frac{\alpha a \Gamma(1+n / 2)(4 l)^{1+n / 2}}{2 K r}\left[\mathrm{i}^{n+2} \operatorname{erfc} \frac{a-r}{2(\alpha t)^{1 / 2}}-\mathrm{i}^{n+2} \operatorname{erfc} \frac{a+r}{2(\alpha t)^{1 / 2}}\right] \\
& -\frac{\alpha^{3 / 2} \Gamma(1+n / 2)(4 l)^{(n+3) / 2}}{2 K r}\left[\mathrm{i}^{n+3} \operatorname{erfc} \frac{a-r}{2(\alpha t)^{1 / 2}}-\mathrm{i}^{n+3} \operatorname{erfc} \frac{a+r}{2(\alpha t)^{1 / 2}}\right], 0 \leqq r<a . \\
v= & \frac{\alpha a \Gamma(1+n / 2)(4 t)^{1+n / 2}}{2 K r}\left[\mathrm{i}^{n+2} \operatorname{erfc} \frac{a+r}{2(\alpha t)^{1 / 2}}+\mathrm{i}^{n+2} \operatorname{erfc} \frac{r-a}{2(\alpha t)^{1 / 2}}\right] \\
& +\frac{\alpha^{3 / 2} \Gamma(1+n / 2)(4 t)^{(n+3) / 2}}{2 K r}\left[\mathrm{i}^{n+3} \operatorname{erfc} \frac{a+r}{2(\alpha t)^{1 / 2}}-\mathrm{i}^{n+3} \operatorname{erfc} \frac{r-a}{2(\alpha t)^{1 / 2}}\right], \quad r>a .
\end{aligned}
$$

[^38]
## BOOK REVIEWS

The development of mathematics. By E. T. Bell. Second edition. McGraw-Hill Book Company, Inc., New York and London, 1945. xiii + 637 pp. $\$ 5.00$.

The second edition of this well known book contains about fifty pages of new material. Listed in the order in which they occur in the book, the major additions deal with Moslem algebra, the development of symbolism, lattice theory, cubic surfaces, Levi-Civita's parallel displacement, definition of lengths, areas and volumes, the transition from intuitive to unintuitive thinking in modern mathematics, the development of mathematics in times of war, algebra of relations and consistency proofs. The index, too, has been considerably enlarged. Shorter additional paragraphs are concerned with applied mathematics in World War II, Egyptian algebra, number mysticism, Greek methods of computation, the Egyptian method of constructing a right angle, mathematical realism, the Greek treatment of loci, number theory, geometrical constructions, projective differential geometry, theory of quantics, intuitionism, Whitehead and Russell's Principia Mathematica, the Burali-Forti paradox, quantum theory, Skolem's theorem, Laplace's work on probability, R. A. Fisher's work on statistics, and matrix algebra in modern statistics.
S. Prager

Tables of associated Legendre functions. Prepared by the Mathematical Tables Project conducted under the sponsorship of the National Bureau of Standards. Official Sponsor: Lyman J. Briggs. Project Director: Arnold N. Lewan. Columbia University Press, New York, 1945. xxv +303 pp. $\$ 5.00$.
The main body of the present volume contains tables for $P_{n}^{m}(x), d P_{n}^{m}(x) / d x, i^{-n} P_{n}^{m}(i x), i^{-n} d P_{n}^{m}(i x) / d x$, $(-1)^{m} Q_{m}^{m}(x),(-1)^{m+1} d Q_{n}^{m}(x) / d x,-i^{n+2 m+1} Q_{n}^{m}(i x), i^{n+2 m-1} d Q_{m}^{m}(i x) / d x, P_{n+1 / 2}^{m}(x), d P_{n+1 / 2}^{m}(x) / d x,(-1)^{m} Q_{n+1 / 2}^{m}(x)$ and $(-1)^{m+1} d Q_{n+1 / 2}^{m}(x) / d x$ for integral values of $n$ (ranging from 0 to 4) and $n$ (ranging from 0 or 1 to 10 for the functions of integral degree and from -1 to 4 for the functions of half-integral degree). For the greater part, these functions are tabulated to six significant figures for values of the argument which increase from 0 or 1 to 10 by steps of 0.1 . The functions $P_{m}^{m}(\theta)$ and $d P_{m}^{m}(\theta) / d x$ are also tabulated for integral values of $m$ (ranging from 0 or 1 to 4) and $n$ (ranging from 1 to 10), the values of the argument increasing from $0^{\circ}$ to $90^{\circ}$ by steps of $1^{\circ}$. Auxiliary tables facilitate interpolation.
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[^1]:    Entered as second class matter March 14, 1944, at the post office at Providence, Rhode Island, ingder the act of March 3, 1879. Additional entry at Menasha, Wisconsin,

[^2]:    * Received Jan. 10, 1946.
    ${ }^{1}$ Numbers in brackets refer to the bibliography,

[^3]:    ${ }^{2}$ This will allow us to eliminate $\rho_{1} / \rho_{0}$ and thus obtain equations in which each unknown has the dimensions of a velocity potential.

[^4]:    ${ }^{3}$ Any such (actually time dependent) constant could be absorbed in $\partial \psi / \partial t$ and would contribute nothing to $\mathrm{V}_{1}$.

[^5]:    - Any equation of the form curl $M+\operatorname{grad} Q=\mathbf{C}$ can be reduced to the forms grad $Q=\mathbf{P}$ and curl $\mathbf{M}=\mathbf{N}$ if one is ingenious enough to separate $\mathbf{C}$ into the required parts.

[^6]:    * Numerical substitution indicates for air that $l \gg 10^{-8}$ inches is a sufficient condition.

[^7]:    * Received April 20, 1945.
    $\dagger$ Now at Polytechnic Institute of Brooklyn.

[^8]:    * See Ref. 5, p. 440.

[^9]:    * This method was used by A. Nádai, Z. angew. Math. Mech. 2, 1 (1922), and by M. T. Huber, Z. angew. Math. Mech. 6, 228 (1926).

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    ${ }^{6}$ W. Voigt, loc. cit., $\S 388$, ch. 7.

[^13]:    ${ }^{7}$ A. E. H. Love, Mathematical theory of elasticity, Cambridge University Press, Cambridge, ed. 4, 1927, pp. 151-160.
    ${ }^{8}$ Displacements, strains, and stresses associated with $T_{0}$ will be denoted by the superscript 0 ; those associated with $z T_{1}$, by the superscript 1 .

[^14]:    - A. Nádai, loc. cit., pp. 233-235.
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    ${ }^{11}$ A summary of anisotropic plate theory in English can be found in I. S. Sokolnikoff's Mathemalical theory of elaslicity, mimeograph lecture notes, Brown University, Providence, R. I., 1941, pp. 319-329.

[^15]:    I2 V. Morkovin, Quart. Appl. Math. 1, 116-129 (1943).

[^16]:    ${ }^{13}$ V. Morkovin, loc. cit.
    ${ }^{14}$ After the indicated differentiation of $\omega_{k}\left(\sigma_{k}\right), \sigma_{k}$ in $F$ and its derivative are to be replaced by the common value $\sigma$ of $\sigma_{1}$ and $\sigma_{2}$. Note also that $\bar{\sigma}=\sigma^{-1}$.
    ${ }^{25}$ Since $\sigma=e^{i \theta}$, it follows that $(\partial F / \partial x) c_{\theta}$ and $(\partial F / \partial y) c_{\theta}$ are trigonometric polynomials of order $k_{1}$.

[^17]:    ${ }^{16}$ This transform is most easily found by using the mapping function $z=\omega_{0}\left(5_{0}\right)$, obtained by setting $k=0$ in (4.5), and defining $\mu_{0}=i$. In this way one achieves a direct mapping from the $z$ - to the 50 -plane, and maps any point of $C_{0}$ into a point $\sigma_{0}$ with the same coordinates as that given by the successive mappings (3.3) and (4.5).

[^18]:    ${ }^{17}$ V. Morkovin, loc. cit.
    ${ }^{18}$ That $\phi$ and $\psi$ may be so written is clear after reading Section 6 .
    ${ }^{19}$ Thermal deflection of anisotropic thin plates, University of Wisconsin, 1943,
    ${ }^{20}$ For details see the reference of footnote 19.

[^19]:    ${ }^{21}$ For details see the reference of footnote 19.
    22 See the reference of footnote 19.

[^20]:    ${ }^{23}$ I. S. Sokolnikoff, loc. cit., pp. 243-251.
    ${ }^{24}$ These are linear functions of $M_{x}, M_{y}$ and $H_{x y}$. See I. S. Sokolnikoff, loc. cit., p. 326.

[^21]:    ${ }^{25}$ A. Nádai, loc. cit., pp. 264-268.
    ${ }^{26}$ Recent work by S. Bergman gives promise of extension to equations of the type (2.13) and (7.5). Bergman considers an equation a special case of which is

    $$
    \nabla^{\prime} u+a u_{x x}+2 b u_{x y}+c y_{y y}+d u_{x}+e u_{y}+f u=0,
    $$

    where $a, b, \cdots, f$ are analytir functions of $x$ and $y$. See Duke Math. J. 11, 617-649 (1944).

[^22]:    ${ }^{27}$ These polynomials may be regarded as approximations to power series representations of $T_{0}$ and $T_{1}$.

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[^24]:    ${ }^{1}$ W. A. Jenkins, Osculatory interpolation: New derivation and formulae, Record of the American Institute of Actuaries, 15, 87 (1926).

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    W. A. Jenkins wrote four papers on this subject of which the above paper is the first. References to the other three papers are found in the excellent bibliography in Greville's paper.

[^25]:    ${ }^{2}$ See J. M. Whittaker, Interpolatory function theory, Cambridge Tracts in Mathematics, 1935, pp. 62-64, for a discussion of the relation between the cardinal series and Stirling's interpolation series. The cardinal serics was probably first investigated in an important mémoire by Ch. J. de la Vallée Poussin, Sur la convergence des formules d'interpolation entre ordonnées équidistantes, Bull. Acad. Roy. Belgique, 1908, 319-410.

[^26]:    ${ }^{3}$ It should be remarked here that also A. C. Aitken's computation in 1925 of the coefficients of E. T. Whittaker's smoothing method amounted to the expansion in a Laurentseries of a certain simple rational function. Sce E. T. Whittaker and G. Robinson, Calculus of observations, London and Glasgow, 1940, pp. 308-312.

[^27]:    ${ }^{4}$ Compare G. J. Lidstone, Note on the computation of terminal values in graduation by Jenkins' modifed osculatory formula, Transactions of the Faculty of Actuaries (Scotland), 12, 277 (1930).

[^28]:    ${ }^{5}$ These graphs indicate geometrically the construction of the successive arcs of these curves. Thus $C_{3}(x)$ is defined in the interval $1 / 2<x<3 / 2$ by the parabola passing through the points $(0,1),(1,0)$, $(2,0)$. Similarly $C_{1}(x)$ is defined in $-1<x<0$ by the cubic which takes the values $0,0,1,0$ at $x=-2,-1$, 0,1 respectively. We mention incidentally the following general analytic expression of the basic function $C_{k}(x)$ of $k$-point central interpolation. In terms of the "central" factorial

    $$
    x^{[k]-1}= \begin{cases}x\left(x^{2}-1^{2}\right)\left(x^{2}-2^{2}\right) \cdots\left(x^{2}-(\nu-1)^{2}\right) & \text { if } \quad k=2 \nu \\ \left(x^{2}-\frac{1}{4}\right)\left(x^{2}-\frac{9}{4}\right) \cdots\left(x^{2}-\frac{(2 \nu-1)^{2}}{4}\right) & \text { if } \quad k=2 \nu+1\end{cases}
    $$

[^29]:    7 This property of interpolation formulac seems to have been neglected so far. It represents an important weaker form of the condition of exactness for the degree $k-1$. Compare T. N. E. Greville, loc. cit. pp. 210-211, for our slight departure from the standard terminology. Jenkins speaks of a modified interpolation formula in case the formula is not ordinary. The term "modified" seems natural in view of Jenkins' construction of such formulae by modifying certain terms of Everett's formula (the author is indebted for this last remark to Mr. Chalmers I. Weaver). It seems, however, less desirable if their construction is, as here, otherwise performed.

[^30]:    ${ }^{8}$ See e.g. S. Bochner, Vorlesungen iiber Fouriersche Integrale, Leipzig, 1932, Satz 11 b on p. 42.

[^31]:    ${ }^{9}$ See S. Bochner, loc. cit., theorem 10 on page 35.

[^32]:    ${ }^{10}$ The author is indebted for this suggestion to Professor L. H. Thomas of Ohio State University.

[^33]:    ${ }^{13}$ See S. Bochner, Fotrier analysis, Princeton University lectures, 1936-1937, where our $M_{k}(x)$ are worked out for $k=1,2,3$, and where the increasing smoothness qualitics of these functions are clearly noted. Bochner also considers the integrals

    $$
    M M_{k}^{*}(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left(\frac{2 \sin u / 2}{u}\right)^{k} \frac{\sin u x}{u} d u .
    $$

    Using Theorem 5 below, we readily obtain the identity $M_{k}^{*}(x)=-\frac{1}{2}+(1 / k!) \delta^{k} x_{+}^{k}$.

[^34]:    ${ }^{12}$ See H. S. Carslaw, Mathematical theory of the conduction of heat, Dover Publications, New York, 1945, Chapter III, Section 16. Certain smoothing properties of heat flow were already noticed by Ch. Sturm in 1886. See in this connection G. Polya, Qualitatives ibber Wärmeausgleich, Z. angew. Math. u. Mech. 13, 125-128 (1933). It should be mentioned here that Weierstrass derived his famous approximation theorem by means of the integral (33). Finally see E. Czuber, Wahrscheinlichkeitsrechnung, vol. I, Lcipzig-Berlin, 1924, pp. 417-418, for a brief sketch of a method of using (33) to derive analytic approximations to given data.

[^35]:    ${ }^{14}$ I owe to D. H. Lehmer the reference to the functions $H h_{n}(x)$ defined by

    $$
    \Pi h_{0}(x)=\int_{z}^{\infty} e^{-x^{2} / 2} d x, \quad \Pi h_{n}(x)=\int_{a}^{\infty}\left[h_{n-1}(x) d x\right.
    $$

    Tables of these functions were published by J. R. Airey as Tables XV, Group IV, of the Mathematical Tables of the British Association for the Advancement of Science. The relation between our $g_{k}(x, l)$ and these new functions is

    $$
    g_{k}(x, t)=\frac{1}{\sqrt{2 \pi}}(t / 2)^{(k-1) / 2} H h_{k-1}(-x \sqrt{2 / l} .
    $$

    This relation, for $k=4, t=\frac{1}{2}$, would readily allow us to compute our Table I by means of Airey's tables of $H h_{3}, H h_{2}$ and $H h_{1}$. However, for other sets of values of $k$ and $t_{1}$ such as $k=8, t=\frac{1}{4}$, which are needed for other purposes, the range of $x$ in Airey's table becomes insufficient. In this case tables of $M_{k}(x, t)$ and its derivatives are computed by our formula (12) and the excellent Tables of Probability Functions, vol. I (1941), vol. II (1942), prepared by the Mathematical Tables Project under the direction of A. N. Lowan.

[^36]:    * Received Sept. 19, 1945.
    ${ }^{1}$ D. R. Hartree, Some properties and applications of the repeated integrals of the error function, Proc. Manchester Lit. and Phil. Soc. 80, 85 (1935).

[^37]:    ${ }^{2}$ In this case two or three terms of (11) give quite a good representation of the observed heat of hydration over the first few days. An expression in terns of negative exponentials is more usual, but does not lead to solutions in terms of tabulated functions for the problems given here, except (vi), and in that case it does not give a solution which is useful for small values of the time.

[^38]:    ${ }^{3}$ Van Ostrand, On the flow of heat from a lock stratum in which heat is being generated, J. Wash. Acad. Sci. 22, 529 (1932), considers the case of a thin layer.
    ${ }^{4}$ This problem is of interest in connection with development of heat in wheat stacks.

