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Harry Bateman*

29 May, 1882—21 January, 1946

In the sudden death (from coronary thrombosis) of Harry Bateman while en route to New York, near Milford, Utah, mathematics in the United States lost its outstanding representative of the British School of the generation just closing. Like his contemporaries and immediate predecessors among Cambridge mathematicians of the first decade of this century, before the new regulations for the Mathematical Tripos took effect, Bateman was thoroughly trained in both pure analysis and mathematical physics, and retained an equal interest in both throughout his scientific career. In bare outline the relevant details of his life are as follows:

Harry Bateman was born at Manchester, England, 29 May, 1882, a son of Samuel and Marnie Elizabeth (Bond) Bateman, and received his secondary education at the Manchester Grammar School. Bateman ascribed much of his subsequent success at Trinity College, Cambridge, to the excellent instruction he received at the school. In 1903 he was (bracketed) Senior Wrangler in the Tripos, and took his B.A. degree, proceeding to the M.A. in 1906, having been a Smith's Prizeman in 1905. From 1905 to 1911 he was a Fellow of Trinity College: the year 1905–06 was spent in study at Göttingen and Paris. From 1906 to 1907 he was a lecturer at the University of Liverpool, and from 1907 to 1910 a reader at the University of Manchester. He came to the United States in 1910 (he later became a naturalized U. S. citizen), as a lecturer at Bryn Mawr College, where another English mathematician, the late Charlotte Angas Scott, was the efficient and scholarly head of the mathematics department. In 1912, he went to the Johns Hopkins University as a Johnston scholar for three years, incidentally taking his Ph.D. (a curious proceeding for a mathematician of his eminence) in 1913. From 1915 to 1917 he was a lecturer at Johns Hopkins, and in 1917 he accepted the position which he held till his death, a professorship of mathematics, physics, and aeronautics at the then recently organized California Institute of Technology. He was a member of the American Mathematical Society (vice-President, 1935, Gibbs lecturer, 1943), the American Physical Society, the American Acoustical Society, the American Philosophical Society, the British Association for the Advancement of Science, the London Mathematical Society, the National (U. S.) Academy of Sciences, and a Fellow of the Royal Society (London). He is survived by his wife, Ethel (Horner Dodd) Bateman, and his daughter, Joan; a son died in childhood.

* Professor Bateman was a member of the Board of Collaborators of the Quarterly of Applied Mathematics from its foundation to his lamented decease.

Bateman was an almost unique combination of erudition and creativeness. It is most unusual for a mathematician to have the extraordinary range of exact knowledge that Bateman had, and not be crushed into sterility by the mere burden of an oppressive scholarship. But, as his numerous publications testify, Bateman retained his creative originality till his death. In pure mathematics, his dominating interest was in the analysis that has developed from classical mathematical physics. His technical skill in this broad field was unrivalled. His numerous contributions to mathematical physics are marked by a vivid, at times almost romantic, imagination. Students of the history of general relativity will find much of interest in some of his papers on electromagnetism.

A singularly modest and gentle man, Bateman was always ready to place his skill and his knowledge at the disposal of others, with no thought of personal credit. War work absorbed most of his time during the last four years of his life; and it is to be regretted that the incessant correspondence in connection with such work prevented him from putting the finishing touches to what he regarded as his most useful contributions to mathematical scholarship: an exhaustive work on definite integrals, and a critical census of all the special functions that have been considered in mathematics. If these works can be put into shape for publication, they will form a lasting memorial to Harry Bateman.

E. T. Bell

April, 1946.

LIST OF PUBLICATIONS BY HARRY BATEMAN*

1. Question 14943. Educational Times (2) **1**, 98–100 (1902).
2. Question 15119. Educational Times (2) **3**, 110–111 (1903).
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7. *The solution of partial differential equations by means of definite integrals.* Proc. L. M. S. (2) **1**, 451–458 (1904).
8. *Certain definite integrals and expansions connected with the Legendre and Bessel functions.* Mess. (2) **33**, 182–188 (1904).
9. *A generalisation of the Legendre polynomial.* Proc. L. M. S. (2) **3**, 111–123 (1905).
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* This bibliography was prepared by Joan Bateman. The following abbreviations are used: Bull. A. M. S. = Bulletin of the American Mathematical Society; Mess. = Messenger of Mathematics; Proc. C. P. S. = Proceedings of the Cambridge Philosophical Society; Proc. L. M. S. = Proceedings of the London Mathematical Society; Proc. N. A. S. = Proceedings of the National Academy of Sciences; Trans. A. M. S. = Transactions of the American Mathematical Society; Trans. C. P. S. = Transactions of the Cambridge Philosophical Society.

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CONTRIBUTIONS TO THE PROBLEM OF APPROXIMATION OF EQUIDISTANT DATA BY ANALYTIC FUNCTIONS*

PART B—ON THE PROBLEM OF OSCULATORY INTERPOLATION. A SECOND CLASS OF ANALYTIC APPROXIMATION FORMULAE

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Introduction. The present second part of the paper has two objectives. Firstly, we wish to carry further the important actuarial work on the subject of osculatory interpolation (Chapters I and II). Secondly, we construct even analytic functions $L(x)$, of extremely fast damping rate, such that the interpolation formula of cardinal type

$$F(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L(x - \nu) \tag{1}$$

reproduces polynomials of a certain degree and reduces to a smoothing formula for integral values of the variable x (Chapter III). This second problem is found to be intimately connected with the subject of osculatory interpolation.

A preliminary remark concerning our notation is necessary. In Part A, Section 2.21 we described various characteristic properties (or "type characteristics") of a polynomial interpolation formula of the form (1), such as: (i) The degree m of the composite polynomial function $L(x)$; (ii) its class C^μ, i.e., order of contact is μ; (iii) the highest degree k of polynomials for which the formula (1) is exact; (iv) the span s of the even polynomial function $L(x)$. For convenience we propose to summarize all these statements by saying that (1) is a formula of type[1]

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[1] The connection of these types characteristics with the notation as used by Greville in his paper *The general theory of osculatory interpolation,* Trans. Actuar. Soc. Amer., 45, pp. 202–265 (1944), especially pp. 210–211, is as follows: The first three symbols D^m, C^μ, E^k, require no further comment since they are identical respectively with the characteristics 4, 1, and 6 of Greville's classification, pp. 210–211. There remain three further characteristics to be discussed: (i) Whether the formula (1) is an "end-point" or "mid-point" formula. This point is of importance if (1) is written in terms of central differences, since it, then reduces to either the Everett or else the Steffensen form. The following statement is obvious: *The formula* (1) *is an "end point" or "mid-point" formula depending on whether the span s is even or odd.*

(ii) Greville's adjectives "ordinary" and "modified" agree respectively with our "ordinary" and "smoothing."

(iii) The highest order d of differences involved (explicitly or implicitly). We start with the following question: *Let x be given. How many ordinates y_n enter into the computation of $F(x)$ by* (1)? Assuming that $L(x)$ is continuous, hence $=0$, at the end point $x=s/2$ of its span, we have $L(x-n)\neq 0$, as long as n in such that

$$|x - n| < s/2.$$

This inequality is found to be equivalent to

$$-\frac{s}{2} + x < n < \frac{s}{2} + x. \tag{*}$$

Let $s=2\sigma$ be even (end-point formula) and let now x be anywhere within $0\leq x\leq 1$. By (*) $F(x)$ then re-

$$D^m, \quad C^\mu, \quad E^k, \quad s. \tag{2}$$

As an instance we may describe the k-point central interpolation formula of Part A, Section 2.121 as a formula of type[2]

$$D^{k-1}, \quad \left\{\begin{matrix} C^0 & \text{if } k \text{ is even} \\ C^{-1} & \text{if } k \text{ is odd} \end{matrix}\right\}, \quad E^{k-1}, \quad s = k.$$

In Chapter II we construct a class of ordinary interpolation formulae and two classes of smoothing interpolation formulae. These classes by no means describe all possible osculatory interpolation formulae. Furthermore, a number of interesting problems concerning remainder terms and orders of approximations await solution. No attempt has been made to see which of the numerous formulae tabulated by Greville, loc. cit., are contained in the three classes of formulae developed in Chapter II. The essential progress made in this direction may perhaps be briefly described as follows. The construction of an interpolation formula usually requires the solution of a more or less complicated system of linear equations, unless, as in Lagrange's formula, the basic interpolating functions are obvious from the start. These systems of equations are especially troublesome if one wishes to construct an osculatory interpolation formula of any *general* class. As Greville correctly points out, loc. cit., pp. 255–256, the mere agreement between the number of unknowns with the number of equations which they should satisfy, will, by itself, never prove the existence of a solution. Basically, our parametric representation of spline curves of order k (Part A, Section 3.15, Theorem 5) circumvents this difficulty.

An example which illustrates the operation of this representation is as follows. Let $F(x)$ be defined as equal to 0 for $x \leqq 0$, as well as for $x \geqq 4$. We propose to complete the definition of $F(x)$ in the range $0 \leqq x \leqq 4$ by four cubic arcs joining at $x = 1, 2, 3$, in such a way that $F(x)$ be of class C'' for all real x. Of course, we are not interested in the obvious but trivial solution $F(x) \equiv 0$. Let us now count the available parameters and the number of conditions. The 4 cubic arcs furnish $4 \cdot 4 = 16$ parameters. The second order contact requirements at $x = 0, 1, 2, 3, 4$ lead to a system of $3 \cdot 5 = 15$ homogeneous equations. The solution of a homogeneous system of 15 equations in 16 unknowns depends on anything from 1 to 16 arbitrary parameters, depending on the rank of the system. Our Theorem 5, for $k = 4$, furnishes immediately the one-parameter solution

$$F(x) = c \cdot M_4(x - 2) \tag{3}$$

the graph of which is given in Part A, Section 3.13. Again Theorem 5 will easily show that this is the most general solution of the problem. We see how this complicated system of 15 equations in 16 unknowns is explicitly solved by (3). As a variation of the problem, let us now define $F(x)$ to be equal to 0 for $x \leqq 0$, as well as for $x \geqq 3$, and let us propose now to bridge this gap by 3 cubic arcs giving a $F(x)$ of class C''. Now we find that the problem amounts to a system of 12 homogeneous equations in 12 unknowns. This tells us precisely nothing. Again by Theorem 5 we can readily show that

quires all y_n such that $-\sigma < n < \sigma + 1$ that is $s = 2\sigma$ consecutive ordinates. Let $s = 2\sigma + 1$ be odd (mid-point formula) and let x be anywhere within $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$. Again by (*) $F(x)$ now requires all y_n such that $-\sigma - 1 < n < \sigma + 1$, hence again $s = 2\sigma + 1$ consecutive ordinates. We have therefore proved the following: *The highest order d of differences involved is always related with the span s by the relation $s = d + 1$.*

[2] The symbol C^{-1} is to indicate the class of piecewise continuous functions.

the trivial solution $F(x)\equiv 0$ is the only solution. These considerations generalize and allow to characterize our basic functions

$$M_k(x) = \frac{1}{(k-1)!}\,\delta^k x_+^{k-1},$$

up to a multiplicative constant and a shift along the x-axis, as follows:[3] *Let $F(x)$ be $=0$ for $x\leqq 0$, as well as for $x\geqq n$, where n is a positive integer. We wish to complete the definition of $F(x)$ by a succession of n arcs, of degree $k-1$, joining at $x=1, 2, \cdots, n-1$ such as to furnish a $F(x)$ of C^{k-2}. Then $n=k$ is the smallest value of n for which this can be done in a non-trivial way and for this minimal value $n=k$ the gap is bridged by*

$$F(x) = c\cdot M_k(x - k/2)$$

and in no other way.

The reader who is mainly interested in the numerical applications may pass directly from here to the Appendix where the use of the tables is fully explained and one example is worked out.

I. THE COSINE POLYNOMIALS $\phi_k(u)$ AND CERTAIN RELATED SETS OF POLYNOMIALS

In the present chapter we propose to study further properties of the cosine polynomials

$$\phi_k(u) = \sum_{n=-\infty}^{\infty} M_k(n)\cos nu \tag{1}$$

which were mentioned in Part A, sections 3.14 and 4.1 (for $t=0$). By Part A, section 4.1, formula (6) (for $t=0$) we may also write

$$\phi_k(u) = \sum_{\nu=-\infty}^{\infty} \psi_k(u + 2\pi\nu)$$

and therefore

$$\phi_k(u) = (2\sin u/2)^k\cdot \sum_{\nu=-\infty}^{\infty} \frac{(-1)^{\nu k}}{(u+2\pi\nu)^k}\cdot \tag{2}$$

1.1. **Expression of $\phi_k(u)$ in terms of rational polynomials.** We introduce two new sets of periodic functions defined by

$$\rho_k(u) = (2\sin u/2)^k\cdot \sum_{\nu=-\infty}^{\infty} \frac{1}{(u+2\pi\nu)^k}, \tag{3}$$

$$\sigma_k(u) = (2\sin u/2)^k\cdot \sum_{\nu=-\infty}^{\infty} \frac{(-1)^{\nu}}{(u+2\pi\nu)^k}\cdot \tag{3'}$$

A comparison with (2) shows that

$$\phi_k(u) = \begin{cases}\rho_k(u) & \text{if } k \text{ is even,}\\ \sigma_k(u) & \text{if } k \text{ is odd.}\end{cases} \tag{4}$$

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[3] Problems of this kind concerning polygonal lines of a certain degree and class are of importance for the theory of formulae of mechanical quadratures. The author expects to discuss this connection elsewhere.

By differentiation of (3) and also (3′) we readily find the recurrence relations

$$\rho_{k+1}(u) = \cos\frac{u}{2}\,\rho_k(u) - \frac{2}{k}\sin\frac{u}{2}\,\rho_k'(u), \tag{5}$$

$$\sigma_{k+1}(u) = \cos\frac{u}{2}\,\sigma_k(u) - \frac{2}{k}\sin\frac{u}{2}\,\sigma_k'(u). \tag{5′}$$

These may be used in a progressive computation of ρ_k and σ_k if we start with

$$\rho_2(u) = 1, \qquad \sigma_1(u) = 1. \tag{6}$$

We prefer, however, to express $\rho_k(u)$ and $\sigma_k(u)$ as polynomials in the variable

$$x = \cos(u/2) \tag{7}$$

by means of

$$\rho_k(u) = U_{k-2}(\cos u/2), \qquad \sigma_k(u) = V_{k-1}(\cos u/2). \tag{8}$$

Substituting into (5), and (5′) respectively we find that the two sequences of polynomials $U_n(x)$, $V_n(x)$, both of exact degree n, satisfy the recurrence relations

$$U_{k+1}(x) = xU_k(x) + \frac{1}{k+2}(1-x^2)U_k'(x), \tag{9}$$

$$V_{k+1}(x) = xV_k(x) + \frac{1}{k+1}(1-x^2)V_k'(x), \tag{9′}$$

with initial values which by (6) and (8) are

$$U_0(x) = 1, \qquad V_0(x) = 1. \tag{10}$$

A simple calculation now gives

$$\left.\begin{aligned}
U_1(x) &= x, & V_1(x) &= x,\\
U_2(x) &= \frac{1}{3}(1+2x^2), & V_2(x) &= \frac{1}{2}(1+x^2),\\
U_3(x) &= \frac{1}{3}(2x+x^3), & V_3(x) &= \frac{1}{6}(5x+x^3)\\
U_4(x) &= \frac{1}{15}(2+11x^2+2x^4), & V_4(x) &= \frac{1}{24}(5+18x^2+x^4)\\
U_5(x) &= \frac{1}{45}(17x+26x^3+2x^5), & V_5(x) &= \frac{1}{120}(61x+58x^3+x^5).
\end{aligned}\right\} \tag{11}$$

We record as a lemma the following properties which are readily established by induction.

Lemma 1. *$U_k(x)$ and $V_k(x)$ are polynomials of exact degree k which are even or odd according as k is even or odd. The coefficients of their highest terms are positive. Also*

$$U_k(1) = V_k(1) = 1, \qquad U_k(-1) = V_k(-1) = (-1)^k. \tag{12}$$

In view of (4) and (8) we find the following expression of $\phi_k(u)$ in terms of our new polynomials:

$$\phi_k(u) = \begin{cases} U_{k-2}(x) & \text{if } k \text{ is even,} \\ V_{k-1}(x) & \text{if } k \text{ is odd.} \end{cases} \tag{13}$$

This shows that the even polynomials $U_{2\nu}$, $V_{2\nu}$ are of special interest.

1.2. **The zeros of the polynomials U_k and V_k.** We propose to prove the following proposition:

LEMMA 2. *The zeros of the even polynomials $U_{2\nu}(x)$ and $V_{2\nu}(x)$ are all simple and purely imaginary.*

We carry through the proof for $U_{2\nu}(x)$ only since the proof for $V_{2\nu}(x)$ is entirely similar. In order to deal with real zeros, we define a new sequence of polynomials $u_k(x)$ by

$$u_k(x) = i^{-k}U_k(xi), \qquad (k = 0, 1, \cdots). \tag{14}$$

These new polynomials are also real and satisfy a recurrence relation which in view of (9) is readily found to be

$$u_{k+1}(x) = xu_k(x) - \frac{1}{k+2}(1 + x^2)u_k'(x). \tag{15}$$

From (11) we find

$$u_0(x) = 1, \qquad u_1(x) = x, \qquad u_2(x) = \tfrac{1}{3}(2x^2 - 1), \qquad u_3(x) = \tfrac{1}{3}(x^3 - 2x), \cdots .$$

In view of (14) it obviously suffices to show that the zeros of $u_k(x)$ ar real and simple, while those of $u_{2\nu}(x)$ are also different from zero. This is readily done by induction as follows. Let $k = 2\nu$ be even and let us assume that the k zeros of $u_k(x)$ are

$$-\xi_\nu, -\xi_{\nu-1}, \cdots, -\xi_1, \xi_1, \xi_2, \cdots, \xi_\nu \qquad (0 < \xi_1 < \cdots < \xi_\nu), \tag{16}$$

and therefore simple. This, and the fact that $u_k(x)$ has a highest term of positive coefficient (Lemma 1 and (14)), imply that

$$u_k'(\xi_\nu) > 0,$$

and that the sequence of values of $u_k'(x)$, at the k roots (16), alternate in sign. By (15) we therefore find

$$u_{k+1}(\xi_\nu) < 0$$

and that the values of $u_{k+1}(x)$,at the k roots (16), alternate in sign. Since $u_{k+1}(0) = 0$, we conclude that $u_{k+1}(x)$ has ν positive and ν negative zeros which must therefore be simple.

Let now $k = 2\nu + 1$ be odd and let u_k have the simple zeros

$$-\xi_\nu, \cdots, -\xi_1, 0, \xi_1, \cdots, \xi_\nu \qquad (0 < \xi_1 < \cdots < \xi_\nu). \tag{17}$$

Now we conclude as before that $u_{k+1}(\xi_\nu) < 0$ and that the values of $u_{k+1}(x)$, at the k roots (17), alternate in sign. Again the conclusion is that $u_{k+1}(x)$ has simple real roots none of which vanishes. This proves the theorem by complete induction.

1.3. **A few corollaries.** In this last section of the present chapter we prove several auxiliary propositions which will be used in the next chapter in the derivation of interpolation formulae of various kinds. These propositions represent the solutions of the algebraic problems arising by the Fourier integral transformation of the problems of the construction of those interpolation formulae.

LEMMA 3. *Let $k=2\nu$ be even. We can determine uniquely an even polynomial $P_k(x)$, of degree k, and an odd polynomial $P_{k-1}(x)$, of degree $k-1$, satisfying the identity*

$$U_k(x)P_k(x) + U_{k+1}(x)P_{k-1}(x) \equiv 1. \tag{18}$$

Likewise polynomials $Q_k(x)$ and $Q_{k-1}(x)$, even and odd respectively, exist uniquely such as to satisfy

$$V_k(x)Q_k(x) + V_{k+1}(x)Q_{k-1}(x) \equiv 1. \tag{19}$$

We wish to show first that U_k and U_{k+1} have no common zeros. Indeed a common zero x of U_k and U_{k+1} would, by (9), be a zero of

$$(1 - x^2)U_k'(x).$$

Since by (12) $x \neq \pm 1$, x must be a zero of $U_k'(x)$. But this contradicts our Lemma 2 to the effect that $U_k(x)$ has only simple zeros. The polynomials $U_k(x)$, $U_{k+1}(x)$ having no common divisors, the identity (18) is assured by the elementary theory of the greatest common divisor of two polynomials. We now show that $P_k(x)$ is even and $P_{k-1}(x)$ is odd as follows. Replacing x by $-x$ in (18) we find

$$U_k(x)P_k(-x) - U_{k+1}(x)P_{k-1}(-x) \equiv 1.$$

Since our polynomials P_k, P_{k-1} are uniquely defined by (18) we find

$$P_k(x) = P_k(-x), \qquad P_{k-1}(x) = -P_{k-1}(-x),$$

which prove our statement. An identical reasoning proves the existence of the polynomials Q_k and Q_{k+1} satisfying (19).

The polynomials P_k and P_{k+1} are easily determined for low values of k by the method of indeterminate coefficients. Thus for $k=2$ by (11),

$$U_2(x) = (1 + 2x^2)/3, \qquad U_3(x) = (2x + x^3)/3,$$

from which we find

$$P_2(x) = 2x^2 + 3, \qquad P_1(x) = -4x,$$

satisfying the identity

$$U_2(x)P_2(x) + U_3(x)P_1(x) \equiv 1. \tag{20}$$

Likewise for $k=4$ we have by (11)

$$V_4(x) = (5 + 18x^2 + x^4)/24, \qquad V_5(x) = (61x + 58x^3 + x^5)/120.$$

The corresponding polynomials Q_4, Q_3 are found to be

$$Q_4(x) = (3648 + 4789x^2 + 83x^4)/760, \qquad Q_3(x) = -(1469x + 83x^3)/152.$$

They satisfy the identity

$$V_4(x)Q_4(x) + V_5(x)Q_3(x) \equiv 1. \tag{21}$$

The identities (18), (19) will later be used in the following form. Again for an even k, but replacing k by $k-2$, we get by (18) and (8)

$$\rho_k(u)P_{k-2}(x) + \rho_{k+1}(u)P_{k-3}(x) \equiv 1, \qquad (k \text{ even}, x = \cos u/2).$$

Likewise for an odd k, but replacing k by $k-1$, we obtain from (19) and (8) the identity

$$\sigma_k(u)Q_{k-1}(u) + \sigma_{k+1}(u)Q_{k-2}(k) \equiv 1, \qquad (k \text{ odd}, x = \cos u/2).$$

The even polynomials P_{k-2}, $x^{-1}P_{k-3}$, and Q_{k-1}, $x^{-1}Q_{k-2}$, may now be expressed in powers of

$$1 - x^2 = (\sin u/2)^2.$$

We have therefore proved the following:

LEMMA 4. *We can find constants a_ν, a_ν', b_ν, b_ν' such as to satisfy the following two identities*:

For an even k

$$\rho_k(u)\{a_0 - a_2(2\sin u/2)^2 + a_4(2\sin u/2)^4 - \cdots \pm a_{k-2}(2\sin u/2)^{k-2}\}$$
$$+ \rho_{k+1}(u)\{a_0' - a_2'(2\sin u/2)^2 + a_4'(2\sin u/2)^4 - \cdots$$
$$\mp a_{k-4}'(2\sin u/2)^{k-4}\}(2\cos u/2) \equiv 1, \qquad (22)$$

and for k odd

$$\sigma_k(u)\{b_0 - b_2(2\sin u/2)^2 + b_4(2\sin u/2)^4 - \cdots \pm b_{k-1}(2\sin u/2)^{k-1}\}$$
$$+ \sigma_{k+1}(u)\{b_0' - b_2'(2\sin u/2)^2 + b_4'(2\sin u/2)^4 - \cdots$$
$$\mp b_{k-3}'(2\sin u/2)^{k-3}\}(2\cos u/2) \equiv 1. \qquad (22')$$

As examples we mention that the identities (20) and (21) become on passing to the variable u

$$\rho_4(u)\{5 - \tfrac{1}{2}(2\sin u/2)^2\} + \rho_5(u)\{-2\}(2\cos u/2) \equiv 1 \qquad (23)$$

and

$$\sigma_5(u)\left\{\frac{213}{19} - \frac{991}{760}(2\sin u/2)^2 + \frac{83}{12160}(2\sin u/2)^4\right\}$$
$$+ \sigma_6(u)\left\{-\frac{194}{38} + \frac{83}{1216}\left(2\sin\frac{u}{2}\right)^2\right\}\left(2\cos\frac{u}{2}\right) \equiv 1. \qquad (24)$$

The last proposition which we wish to derive here concerns the expansion of $1/\phi_k(u)$ in ascending powers of the variable

$$s = \sin^2 u/2 = 1 - \cos^2 u/2 = 1 - x^2. \qquad (25)$$

Let us assume for the moment that k is even. Then by (13)

$$\phi_k(u) = U_{k-2}(x), \qquad (k \text{ even}). \qquad (26)$$

Now $U_{k-2}(x)$ is an even polynomial which, by Lemma 2, has purely imaginary zeros. Being an even polynomial, $U_{k-2}(x)$ may be expressed as a polynomial $U^*(s)$ in the variable

$$s = 1 - x^2 \qquad (25')$$

of degree $\kappa=(k-2)/2$. This change of variable transforms the purely imaginary zeros of $U_{k-2}(x)$ into the zeros

$$\alpha_1, \alpha_2, \cdots, \alpha_\kappa \qquad (\kappa = (k-2)/2)$$

of $U^*(s)$ which, by (25′), must all be positive and greater than 1. Finally, since $U_{k-2}(1)=U^*(0)=1$, we have the identity

$$\phi_k(u) = U_{k-2}(x) = \left(1-\frac{s}{\alpha_1}\right)\left(1-\frac{s}{\alpha_2}\right)\cdots\left(1-\frac{s}{\alpha_\kappa}\right). \tag{27}$$

An entirely similar identity is derived for an *odd k* by repeating our arguments for

$$\phi_k(u) = V_{k-1}(x),$$

instead of (26).

This establishes the following

LEMMA 5. *The reciprocal of the cosine polynomial $\phi_k(u)$ admits of an expansion*

$$\frac{1}{\phi_k(u)} = \sum_{n=0}^{\infty} c_{2n}^{(k)}(2\sin u/2)^{2n} \tag{28}$$

which converges for all real values of u and where the coefficients are positive rational numbers

$$c_{2n}^{(k)} > 0, \qquad (n = 0, 1, 2, \cdots). \tag{29}$$

Indeed, in view of (27), the expansion (28) may be obtained as

$$\frac{1}{\phi_k(u)} = \prod_{\nu=1}^{\kappa}\left(1+\frac{s}{\alpha_\nu}+\frac{s^2}{\alpha_\nu^2}+\cdots\right) = \sum_{n=0}^{\infty} c_{2n}^{(k)}(4s)^n$$

which reduces to (28), in view of (25). In conclusion we notice the following consequences of the identity (28). Since

$$\phi_k(u) = \sum_{\nu=-\infty}^{\infty} \psi_k(u+2\pi\nu),$$

where

$$\psi_k(u) = \left(\frac{2\sin u/2}{u}\right)^k,$$

we have in the neighborhood of the origin $u=0$

$$\phi_k(u) = \left(\frac{2\sin u/2}{u}\right)^k + u^k\cdot(\text{regular function of } u). \tag{30}$$

On multiplying (28) by $\phi_k(u)$ we therefore have

$$\left(\frac{2\sin u/2}{u}\right)^k\cdot\sum_{n=0}^{\infty} c_{2n}^{(k)}(2\sin u/2)^{2n} = 1+u^k\cdot(\text{regular function})$$

and also

$$\left(\frac{2\sin u/2}{u}\right)^k\cdot\sum_{0\leq 2n<k} c_{2n}^{(k)}(2\sin u/2)^{2n} = 1+u^k\cdot(\text{regular function}). \tag{31}$$

It is of special interest to point out that if

$$g_{k,m}(u) = \left(\frac{2 \sin u/2}{u}\right)^k \cdot \sum_{n=0}^{m-1} c_{2n}^{(k)} (2 \sin u/2)^{2n}$$

then

$$g_{k,m}(u) = \begin{cases} 1 + u^{2m} \cdot (\text{regular function}) & \text{if} \quad 2m < k \\ 1 + u^k \cdot (\text{regular function}) & \text{if} \quad 2m - 2 < k \leqq 2m. \end{cases} \tag{32}$$

As an illustration we find for $k=6$ by (13), and (11), and (25)

$$\frac{1}{\phi_6(u)} = \frac{1}{U_4(x)} = \frac{15}{2 + 11x^2 + 2x^4} = \frac{30}{30 - 30s + 4s^2} = \frac{1}{1 - s + \dfrac{2}{15} s^2}$$

whence

$$\frac{1}{\phi_6(u)} = 1 + s + \frac{13}{15} s^2 + \cdots$$

or

$$\frac{1}{\phi_6(u)} = 1 + \frac{1}{4} (2 \sin u/2)^2 + \frac{13}{240} (2 \sin u/2)^4 + \cdots . \tag{33}$$

The relations (32) now become (for $k=6$, $m=1, 2, 3$)

$$\left(\frac{2 \sin u/2}{u}\right)^6 = 1 + u^2 \cdot (\text{regular function}),$$

$$\left(\frac{2 \sin u/2}{u}\right)^6 \left\{1 + \frac{1}{4} (2 \sin u/2)^2\right\} = 1 + u^4 \cdot (\text{regular function}),$$

$$\left(\frac{2 \sin u/2}{u}\right)^6 \left\{1 + \frac{1}{4} (2 \sin u/2)^2 + \frac{13}{240} (2 \sin u/2)^4\right\} = 1 + u^6 \cdot (\text{regular function}).$$

In the next chapter we shall need the numerical values of the coefficients

$$c_{2n}^{(k)} \quad \text{for} \quad 2n < k. \tag{34}$$

It is of interest then to point out that these coefficients (34) may also be otherwise computed as coefficients of a simple generating function. Indeed, from (30) it is clear that the coefficients (34) will not change if on the left-hand side of (28) we replace $\phi_k(u)$ by the first term on the right-hand side of (30). That means, if

$$\left(\frac{u}{2 \sin u/2}\right)^k = \sum_{n=0}^{\infty} d_{2n}^{(k)} (2 \sin u/2)^{2n} \tag{35}$$

then

$$c_{2n}^{(k)} = d_{2n}^{(k)} \qquad (2n < k). \tag{36}$$

However, the coefficients of (35) are readily determined. Indeed, if we set

$$v = 2 \sin u/2 \quad \text{or} \quad u = 2 \arcsin v/2, \tag{37}$$

then (35) becomes

$$\left(\frac{2 \arcsin v/2}{v}\right)^k = \sum_{n=0}^{\infty} d_{2n}^{(k)} v^{2n}. \tag{38}$$

Since

$$\frac{2 \arcsin v/2}{v} = 1 + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{v^2}{4} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}\cdot\frac{v^4}{16} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7}\cdot\frac{v^6}{64} + \cdots, \tag{39}$$

we find the expansion (38) by raising (39) to the k-th power. Thus

$$d_0^{(k)} = 1, \quad d_2^{(k)} = \frac{k}{24}, \quad d_4^{(k)} = \frac{5k^2 + 22k}{5760}, \cdots. \tag{40}$$

Since all coefficients of (39) are positive, it is clear that the coefficients of (38) are likewise positive. This however does not imply the positivity of the coefficients of (28) beyond the kth term. For $k=6$ the values (40) agree with the coefficients of (33).

II. POLYNOMIAL INTERPOLATION FORMULAE

In this chapter we wish to apply our general Theorem 2 of Part A, section 2.23 and our Lemmas 4 and 5 of the last section in deriving three distinct classes of polynomial interpolation formulae for each value of the positive integer k. The formulae of the first class (Theorem 1 below) are of the ordinary kind (see Part A, section 2.21 a and b), and of the type

$$D^k, \quad C^{k-2}, \quad E^{k-1}, \quad s = \begin{cases} 2k-2 & \text{if } k \text{ is even} \\ 2k-1 & \text{if } k \text{ is odd.} \end{cases}$$

The existence of ordinary interpolation formulae of degree k and class $k-2$ was previously conjectured by Mr. Greville who verified their existence up to and including $k=6$. (See Greville, loc. cit., pp. 212–213.) The formulae of the second class are smoothing interpolation formulae (Theorem 2 below). For a given integral k and each integral m, such that $0 \leqq 2m-2 < k$, a formula is derived which is of the type

$$D^{k-1}, \quad C^{k-2}, \quad E^{\min(2m-1,k-1)}, \quad s = k + 2m - 2.$$

These formulae are derived from an *ordinary* interpolation formula of type

$$D^{k-1}, \quad C^{k-2}, \quad E^{k-1}, \quad s = \infty,$$

discussed in Part A.

The formulae of the third and last class are again *smoothing* interpolation formulae (Theorem 3 below). While in the second class the degree D^{k-1} and the "order of contact" C^{k-2} were fixed, while the degree of exactness E^{2m-1} and the span $s=k+2m-2$ increased apace, in the present class the span $s=k$ is constant. More precisely, a formula is derived for each m such that $0 \leqq 2m \leqq k-1$ which is of type

$$D^{k-1}, \quad C^{k-2m}, \quad E^{2m-1}, \quad s = k.$$

These formulae are derived from the formula of ordinary k-point central interpolation in a manner somewhat reminiscent of Mr. Jenkins' original procedure.

2.1. Ordinary polynomial interpolation formulae of the Jenkins-Greville type. We are returning to our basic functions

$$M_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2 \sin u/2}{u}\right)^k e^{iux}du \tag{1}$$

and wish to show that the osculatory interpolation formulae of the type investigated by Jenkins and Greville may be readily derived in terms of these functions. We shall use the operational symbol σ to mean

$$\sigma f(x) = f(x + \tfrac{1}{2}) + f(x - \tfrac{1}{2}).$$

THEOREM 1. *We define the basic polynomial function $L(x)$ by the following two formulae according to the parity of k:*

$$L(x) = a_0 M_k(x) + a_2\delta^2 M_k(x) + \cdots + a_{k-2}\delta^{k-2}M_k(x)$$
$$+ a_0'\sigma M_{k+1}(x) + a_2'\sigma\delta^2 M_{k+1}(x) + \cdots + a_{k-4}'\sigma\delta^{k-4}M_{k+1}(x) \quad (k \text{ even}) \tag{2}$$
$$L(x) = b_0 M_k(x) + b_2\delta^2 M_k(x) + \cdots + b_{k-1}\delta^{k-1}M_k(x)$$
$$+ b_0'\sigma M_{k+1}(x) + b_2'\sigma\delta^2 M_{k+1}(x) + \cdots + b_{k-3}'\sigma\delta^{k-3}M_{k+1}(x), \quad (k \text{ odd}), \tag{3}$$

where the numerical constants a_ν, a_ν', b_ν, b_ν' are those defined in Lemma 4. Then

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L(x-n) \tag{4}$$

is an ordinary polynomial interpolation formula of type

$$D^k, \quad C^{k-2}, \quad E^{k-1}, \quad s = \begin{cases} 2k-2 & \text{if } \quad k \text{ is even}, \\ 2k-1 & \text{if } \quad k \text{ is odd}. \end{cases} \tag{5}$$

Indeed, we notice that

$$\delta e^{iux} = (e^{iu/2} - e^{-iu/2})e^{iux} = 2i \sin u/2 e^{iux},$$
$$\sigma e^{iux} = (e^{iu/2} + e^{-iu/2})e^{iux} = 2\cos u/2 e^{iux}. \tag{6}$$

Let k now be even, hence $L(x)$ defined by (2), and let

$$L(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)e^{iux}du.$$

We evidently obtain this integral representation by performing the operation

$$a_0 + a_2\delta^2 + \cdots + a_{k-2}\delta^{k-2}$$

on the relation (1) and add to it the result of performing the operation

$$a_0'\sigma + a_2'\sigma\delta^2 + \cdots + a_{k-4}'\sigma\delta^{k-4}$$

on (1), with k replaced by $k+1$. In view of (6) we have

$$\delta^\nu e^{iux} = (2i \sin u/2)^\nu e^{iux}$$
$$\sigma\delta^\nu e^{iux} = (2i\sin u/2)^\nu(2\cos u/2)e^{iux},$$

and therefore

$$g(u) = \left(\frac{2\sin u/2}{u}\right)^k \{a_0 - a_2(2\sin u/2)^2 + \cdots \pm a_{k-2}(2\sin u/2)^{k-2}\}$$
$$+ \left(\frac{2\sin u/2}{u}\right)^{k+1}\{a_0' - a_2'(2\sin u/2)^2 + \cdots$$
$$\mp a_{k-4}'(2\sin u/2)^{k-4}\}(2\cos u/2). \tag{7}$$

We now turn to Theorem 2 of Part A, section 2.23, which states that (4) is an *ordinary* interpolation formula if and only if the following identity holds:

$$\sum_{\nu=-\infty}^{\infty} g(u + 2\pi\nu) = 1. \tag{8}$$

It should be noticed now that both expressions in (7) contained within braces are periodic functions of period 2π. Since k is even we find that $\sum g(u+2\pi\nu)$ is identical with the left-hand side of our relation I (22). This proves (8) for even k. A precisely similar reasoning for odd k will show that $\sum g(u+2\pi\nu)$ is identical with the left-hand side of I (22′).

There remains the problem of showing that (4) is of the type as stated in the Theorem. Since $L(x)$ is by (2), (3), a linear combination of functions of the form

$$M_k(x + n), \qquad M_{k+1}(x + \tfrac{1}{2} + n),$$

it is clear that $L(x)$ is a polynomial line of degree k, of class C^{k-2}, with discontinuities at $x=n$, or $x=n+1/2$, according to whether k is even or odd. Finally (4) is exact for the degree $k-1$, again by Theorem 2 of Part A, section 2.23. There remains the discussion of the span s of $L(x)$. Now the span of $M_k(x)$ is $=k$ (see Part A, section 3.13) and therefore the span of $\delta^\nu M_k(x)$ is equal to $k+\nu$, while the span of $\sigma\delta^\nu M_k(x)$ is equal to $k+\nu+1$. Now it is immediately verified that the two terms of (2) involving δ^{k-2} and $\sigma\delta^{k-4}$ are both of span $2k-2$, while the similar two terms of (3) are both of span $2k-1$. This completes a proof of the Theorem.

As illustrations we mention that the identities I(23) and I(24) corresponding to the cases $k=4$ and $k=5$ give rise to the basic functions

$$L(x) = 5M_4(x) + \tfrac{1}{2}\delta^2 M_4(x) - 2\sigma M_5(x) \tag{9}$$

and

$$L(x) = \frac{213}{19} M_5(x) + \frac{991}{760}\delta^2 M_5(x) + \frac{83}{12160}\delta^4 M_5(x) - \frac{194}{38}\sigma M_6(x) - \frac{83}{1216}\sigma\delta^2 M_6(x). \tag{10}$$

These two basic functions give rise to ordinary interpolation formulae (4) which are of the types

$$D^4,\ C^2,\ E^3,\ s = 6 \quad\text{and}\quad D^5,\ C^3,\ E^4,\ s = 9, \tag{11}$$

respectively.

Incidentally, the characteristic function of (9) is, by I(23),

$$g(u) = \left(\frac{2\sin u/2}{u}\right)^4 \left(5 - 2\sin^2\frac{u}{2}\right) - 4\left(\frac{2\sin u/2}{u}\right)^5 \cos u/2$$

or

$$g(u) = \left(\frac{2\sin u/2}{u}\right)^4 \left(4 + \cos u - 4\frac{\sin u}{u}\right).$$

This agrees with our formula (11″) of Part A, section 2.122, as in fact the basic function (9) is identical with Jenkins' function there described by formula (11).

In concluding we wish to mention the numerical results for $k=6$. In this case we need the identity

$$U_4(x)P_4(x) + U_5(x)P_3(x) \equiv 1. \tag{12}$$

By (11) we have

$$U_4(x) = \frac{1}{15}(2 + 11x^2 + 2x^4), \qquad U_5(x) = \frac{1}{45}(17x + 26x^3 + 2x^5).$$

By indeterminate coefficients we find

$$P_4(x) = \frac{15}{2} + \frac{115}{7}x^2 + \frac{27}{21}x^4, \qquad P_3(x) = -\frac{285}{14}x - \frac{27}{7}x^3,$$

from which, on passing to the variable I(25), the identity (12) becomes

$$\rho_6(u)\left\{\frac{353}{14} - \frac{133}{28}(2\sin u/2)^2 + \frac{9}{448}(2\sin u/2)^4\right\}$$
$$+ \rho_7(u)\left\{-\frac{339}{28} + \frac{27}{56}(2\sin u/2)^2\right\}(2\cos u/2) \equiv 1.$$

The basic function corresponding to $k=6$ is therefore

$$L(x) = \frac{353}{14}M_6(x) + \frac{133}{28}\delta^2 M_6(x) + \frac{9}{448}\delta^4 M_6(x)$$
$$- \frac{339}{28}\sigma M_7(x) - \frac{27}{56}\sigma\delta^2 M_7(x), \tag{13}$$

giving rise to an ordinary interpolation formula of type

$$D^6,\ C^4,\ E^5 \ \text{ and } \ s = 10.$$

2.2. A first class of smoothing interpolation formulae derived from an ordinary interpolation formula of type D^{k-1}, C^{k-2}. We start by recalling an ordinary polynomial interpolation formula derived in Part A, section 4.2. Indeed, the formula (9) of that section furnishes, for $t=0$, the following polynomial basic function

$$L_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\psi_k(u)}{\phi_k(u)} e^{iux} du. \tag{14}$$

The corresponding formula

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L_k(x-n) \tag{15}$$

is, as we know, an ordinary polynomial interpolation formula of type

$$D^{k-1}, \qquad C^{k-2}, \qquad E^{k-1}, \qquad s = \begin{cases} \infty & \text{if } k \geqq 3, \\ k & \text{if } k = 1, 2. \end{cases} \tag{16}$$

We turn now to the expansion I(28) of Lemma 5, substituting I(28) into the integral (14) and integrating term-wise we obtain the expansion

$$L_k(x) = M_k(x) - c_2^{(k)} \delta^2 M_k(x) + c_4^{(k)} \delta^4 M_k(x) - \cdots, \qquad (k \geqq 3). \tag{17}$$

On comparing the present interpolation formula (15) with the formula (4) of the Jenkins-Greville type, we notice, by (5) and (16), that they are both of class C^{k-2} and that they are both exact for the degree $k-1$. The degree of (15) is lower by 1 than the degree of (4). This reduction of the degree to the lowest possible value $k-1$, for a formula of class C^{k-2}, was achieved at the price of having an infinite span. The infinite span of (15) clearly disqualifies this interpolation formula as far as numerical purposes are concerned.

We now turn to the *partial sums* of the series (17). They will yield smoothing interpolation formulae of considerable practical importance. Indeed, let

$$L_{k,m}(x) = M_k(x) - c_2^{(k)}\delta^2 M_k(x) + \cdots + (-1)^{m-1} c_{2m-2}^{(k)}\delta^{2m-2} M_k(x), \quad (2m-2<k). \tag{18}$$

The characteristic function of this basic function is identical with the left-hand side of the identity I(32). In view of our Theorem 2 of Part A, this identity I(32) proves that (18) is the basic function of a smoothing interpolation formula which is exact for the degree equal to $\min(2m-1, k-1)$. It is, moreover, visibly of degree $k-1$, of class C^{k-2}, and of span $s=k+2m-2$. One further important point is in need of proof, namely that the formulae based on (18) actually do smooth any given sequence (see Definition b of Part A, section 2.2). This will readily follow from Lemma 5. Indeed the characteristic function $\phi_{m,k}(u)$ of the formula

$$F(n) = \sum_{\nu} y_\nu L_{k,m}(n-\nu) \tag{19}$$

is, by Theorem 2, Part A, given by

$$\phi_{k,m}(u) = \sum_{\nu=-\infty}^{\infty} g_{k,m}(u+2\pi\nu).$$

By I(32) and I(28) we now have

$$\phi_{k,m}(u) = \sum_{\nu=-\infty}^{\infty} g_{k,m}(u+2\pi\nu) = \phi_k(u)\sum_{n=0}^{m-1} c_{2n}^{(k)}(2\sin u/2)^{2n}$$

$$< \phi_k(u)\sum_{n=0}^{\infty} c_{2n}^{(k)}(2\sin u/2)^{2n} = 1, \qquad (0<u<2\pi). \tag{20}$$

Since obviously $\phi_{m,k}(u)>0$, for all u, we see that (19) is indeed a smoothing formula according to our definition. Recalling the relations I(36), I(38), we may therefore state the following Theorem:

THEOREM 2. *Let k be a positive integer and m an integer such that $0<2m<k+2$. Let the positive rational numbers $d_{2n}^{(k)}$ be defined by the expansion*

$$\left(\frac{2\arcsin v/2}{v}\right)^k = \sum_{n=0}^{\infty} d_{2n}^{(k)} v^{2n}. \tag{21}$$

Then

$$L_{k,m}(x) = M_k(x) - d_2^{(k)}\delta^2 M_k(x) + d_4^{(k)}\delta^4 M_k(x) - \cdots + (-1)^{m-1} d_{2m-2}^{(k)}\delta^{2m-2} M_k(x) \tag{22}$$

gives rise to a smoothing interpolation formula

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L_{k,m}(x-n) \tag{23}$$

of type

$$D^{k-1}, \quad C^{k-2}, \quad E^{\min(2m-1,k-1)}, \quad s = k + 2m - 2. \tag{24}$$

Moreover, the formula (23) *preserves the degree* $k-1$. (*See Part* A, *section* 2.21, *Definition* d.)

If k *is fixed and* m *increases then the smoothing power of our formula* (23) *decreases according to our definition of Part* A, *section* 1.12, *Definition* 2.

The last statement concerning the decreasing smoothing power of (23) follows from (20), since $\phi_{k,m}(u)$ increases strictly as m increases while u is constant $(0<u<2\pi)$.

Notice, by (24), how on increasing m by one unit both the degree of exactness, as well as the span, increase by two units.

As illustration we find from I(40) that

$$L_{k,2}(x) = M_k(x) - \frac{k}{24}\delta^2 M_k(x), \quad (k \geqq 4), \tag{25}$$

yields a smoothing formula of type

$$D^{k-1}, \quad C^{k-2}, \quad E^3, \quad s = k + 2. \tag{26}$$

The characteristic function of (25) is

$$g_{k,2}(u) = \left(\frac{2 \sin u/2}{u}\right)^k \left(1 + \frac{k}{6}\sin^2\frac{u}{2}\right) \tag{27}$$

or

$$g_{k,2}(u) = \left(\frac{2 \sin u/2}{u}\right)^k \left\{1 + \frac{k}{12} - \frac{k}{12}\cos u\right\}.$$

For $k=4$ this function $g_{4,2}(u)$ agrees with the integrand of our formula (12″) of Part A, section 2.123. Also Mr. Jenkins' basic function, as given by formula (12) of Part A, section 2.123, may be derived by working out the various polynomial expressions of

$$L_{4,2}(x) = M_4(x) - \tfrac{1}{6}\delta^2 M_4(x) \tag{28}$$

from the explicit expressions of $M_4(x)$ (see Part A, section 3.13, (14)).

Likewise, by I(40)

$$L_{k,3}(x) = M_k(x) - \frac{k}{24}\delta^2 M_k(x) + \frac{k(5k+22)}{5760}\delta^4 M_k(x), \quad (k \geqq 6), \tag{29}$$

yields a smoothing formula of type

$$D^{k-1}, \quad C^{k-2}, \quad E^5, \quad s = k + 4. \tag{30}$$

2.3. A second class of smoothing interpolation formulae derived from the ordinary k-point central interpolation formula. Among the smoothing interpolation formulae (23) described by Theorem 2 the one of most interest is obtained by letting m

assume its largest value. If k is even, m is maximal if $2m-2=k-2$ or $m=k/2$. If k is odd, m is maximal if $2m-2=k-1$ or $m=(k+1)/2$. In either case

$$\max m = \mu = \left[\frac{k+1}{2}\right], \tag{31}$$

where $[x]$ represents the largest integer not exceeding x. The corresponding basic function (22) is

$$L_{k,\mu}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k \{1 + d_2^{(k)}(2\sin u/2)^2 + \cdots + d_{2\mu-2}^{(k)}(2\sin u/2)^{2\mu-2}\} e^{iux}du. \tag{32}$$

We recall that the smoothing interpolation formula based on this function is by (24) of the type

$$D^{k-1}, \qquad C^{k-2}, \qquad E^{k-1}, \qquad s = k + 2\mu - 2. \tag{33}$$

Indeed, the formula is exact for the degree $k-1$ because of

$$\left(\frac{2\sin u/2}{u}\right)^k \{1 + d_2^{(k)}(2\sin u/2)^2 + \cdots + d_{2\mu-2}^{(k)}(2\sin u/2)^{2\mu-2}\} = 1 + u^k\cdot(\text{regular function}). \tag{34}$$

An interesting counterpart to (32) is obtained as follows. An identity of the type (34) may also be obtained if in the expression within braces we replace $2\sin u/2$ by u. Indeed, rational constants $\gamma_{2\nu}^{(k)}$ may be determined such that

$$\left(\frac{2\sin u/2}{u}\right)^k \{1 + \gamma_2^{(k)}u^2 + \gamma_4^{(k)}u^4 + \cdots + \gamma_{2\mu-2}^{(k)}u^{2\mu-2}\} = 1 + u^k\cdot(\text{regular function}). \tag{35}$$

LEMMA 6. *The basic function*

$$\Gamma_{k,\mu}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k \{1 + \gamma_2^{(k)}u^2 + \cdots + \gamma_{2\mu-2}^{(k)}u^{2\mu-2}\} e^{iux}du \tag{36}$$

is identical, for all real values of x, with the basic function $C_k(x)$ of the k-point central interpolation method (*see Part* A, *section* 2.121).

Notice first that by differentiation of

$$M_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k e^{iux}du$$

we obtain

$$M_k^{(2\nu)}(x) = (-1)^\nu \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k u^{2\nu}e^{iux}du, \qquad (2\nu < k). \tag{37}$$

Therefore the integral (36) may also be written as[4]

[4] Assuming (41) already established, we see by (38) and the relations

$$M_k^{(\nu)}(x) = \delta^\nu M_{k-\nu}(x), \qquad (0 \le \nu \le k-1), \tag{*}$$

(see Part A, section 3.15, formula (23)) that we may express $C_k(x)$ as follows

$$\Gamma_{k,\mu}(x) = M_k(x) - \gamma_2^{(k)} M_k''(x) + \gamma_4^{(k)} M_k^{(4)}(x) - \cdots + (-1)^{\mu-1} \gamma_{2\mu-2}^{(k)} M_k^{(2\mu-2)}(x). \quad (38)$$

Now $M_k(x)$ and all its derivatives are functions of span $s=k$. Therefore also $\Gamma_{k,\mu}(x)$ has the span $s=k$. Furthermore by (35), and Theorem 2 of Part A, we conclude that

$$F(x) = \sum_{n=-\infty}^{\infty} y_n \Gamma_{k,\mu}(x-n) \quad (39)$$

is an interpolation formula of the following type characteristics:

$$E^{k-1}, \qquad s = k. \quad (40)$$

These last two properties (40) allow us to show readily that

$$\Gamma_{k,\mu}(x) = C_k(x) \quad \text{for all real } x. \quad (41)$$

Indeed, let k be even, $k=2\kappa$. Let $P_0(x)$ be the polynomial of degree $k-1$ defined by the following k conditions

$$P_0(-\kappa+1) = P_0(-\kappa+2) = \cdots = P_0(-1) = 0, \qquad P_0(0) = 1,$$
$$P_0(1) = P_0(2) = \cdots = P_0(\kappa) = 0. \quad (42)$$

Since (39) is exact for the degree $k-1$ we have the identity

$$P_0(x) = \sum_{n=-\infty}^{\infty} P_0(n)\Gamma_{k,\mu}(x-n), \quad \text{for all real } x. \quad (43)$$

We now restrict x to the range

$$0 \leqq x \leqq 1. \quad (44)$$

Then we may write (43) as

$$P_0(x) = \sum_{n=-\kappa+1}^{n=\kappa} P_0(n)\Gamma_{k,\mu}(x-n)$$

since $\Gamma_{k,\mu}(x-n)=0$ if $|x-n| \geqq \kappa$. In view of (42) this identity reduces to the single term, for $n=0$:

$$P_0(x) = \Gamma_{k,\mu}(x), \qquad (0 \leqq x \leqq 1),$$

and therefore (41) holds for the range (44). Likewise, applying the formula (39) to the polynomial $P_1(x)$, of degree k, defined by

$$P_1(-\kappa+2) = \cdots = P_1(-1) = 0, \qquad P_1(0) = 1, \qquad P_1(1) = \cdots = P_1(\kappa+1) = 0,$$

we find that (41) holds in the range $1 \leqq x \leqq 2$, and so forth. Similar arguments obviously apply, with obvious modifications, to the case of an odd k.

The coefficients $\gamma_{2\nu}^{(k)}$ are the expansion coefficients of

$$\left(\frac{u}{2\sin u/2}\right)^k = \sum_{\nu=0}^{\infty} \gamma_{2\nu}^{(k)} u^{2\nu}. \quad (45)$$

$$C_k(x) = M_k(x) - \gamma_2^{(k)}\delta^2 M_{k-2}(x) + \gamma_4^{(k)}\delta^4 M_{k-4}(x) + \cdots + (-1)^{\mu-1}\gamma_{2\mu-2}^{(k)}\delta^{2\mu-2} M_{k-2\mu+2}(x). \quad (**)$$

This formula reveals at a glance the following fact: *If k is even, then $C_k^{(2\nu)}(x)$ ($\nu=0, 1, 2, \cdots$) are continuous. If k is odd then $C_k^{(2\nu+1)}(x)$ ($\nu=0, 1, 2, \cdots$) are continuous.* The author learned this property from Kingsland Camp, *Notes on Interpolation*, Trans. Actuar. Soc. Amer., 38, p. 22 (1937).

In N. E. Nörlund's *Differenzenrechnung*, page 143, we find the expansion

$$\left(\frac{t}{\sin t}\right)^k = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{t^{2\nu}}{(2\nu)!} D_{2\nu}^{(k)}.$$

The coefficient $D_{2\nu}^{(k)}$ is a polynomial in k of degree ν. Nörlund's Table 6 on page 460, loc. cit., lists these polynomials for $\nu=0, 1, \cdots, 6$. We therefore have

$$\left(\frac{u}{2 \sin u/2}\right)^k = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{u^{2\nu}}{(2\nu)!} \cdot \frac{D_{2\nu}^{(k)}}{2^{2\nu}} \tag{46}$$

whence

$$\gamma_{2\nu}^{(k)} = (-1)^\nu \frac{D_{2\nu}^{(k)}}{(2\nu)!2^{2\nu}}. \tag{47}$$

The first few values are

$$\gamma_0^{(k)} = 1, \quad \gamma_2^{(k)} = \frac{k}{24}, \quad \gamma_4^{(k)} = \frac{k(5k+2)}{5760}. \tag{48}$$

The expansion coefficients of $t/\sin t$ are positive (see Nörlund, loc. cit., Chapter II, sections 2, 3). Therefore *the coefficients* $\gamma_{2\nu}^{(k)}$ *are all positive*. We shall use this fact later.

In view of the results of our last section it seems natural to consider the partial sums

$$\Gamma_{k,m}(x) = M_k(x) - \gamma_2^{(k)} M_k''(x) + \cdots + (-1)^{m-1} \gamma_{2m-2}^{(k)} M_k^{(2m-2)}(x),$$
$$\left(1 \leqq m < \mu = \left[\frac{k+1}{2}\right]\right), \tag{49}$$

of the sum (38). The properties of the interpolation formulae based on these functions are described by the following theorem:

THEOREM 3. *The formula*

$$F(x) = \sum y_\nu \Gamma_{k,m}(x-\nu), \qquad \left(1 \leqq m < \mu = \left[\frac{k+1}{2}\right]\right), \tag{50}$$

is a smoothing interpolation formula of the type

$$D^{k-1}, \qquad C^{k-2m}, \qquad E^{2m-1}, \qquad s = k. \tag{51}$$

The smoothing power of (50) *decreases, as m increases from* $m=1$ *to* $m=\mu-1$, *until for* $m=\mu$ (50) *reduces to the* (*ordinary*) *k-point central interpolation formula.*

Indeed, the characteristic function of (49) is

$$g(u) = \left(\frac{2 \sin u/2}{u}\right)^k \left\{1 + \gamma_2^{(k)} u^2 + \cdots + \gamma_{2m-2}^{(k)} u^{2m-2}\right\}. \tag{52}$$

From (45) we conclude that

$$g(u) = 1 + u^{2m} \cdot (\text{regular function})$$

and therefore (50) is exact for the degree $2m-1$, by Theorem 2 of Part A. The remaining three characteristics

$$D^{k-1}, \qquad C^{k-2m}, \qquad s = k,$$

are evident on inspection of (49). There remains the investigation of the characteristic function of the corresponding smoothing formula

$$F(n) = \sum_{\nu} y_\nu \Gamma_{k,m}(n-\nu). \tag{53}$$

By Theorem 2, Part A, this characteristic function is

$$\chi_m(u) = \sum_{\nu=-\infty}^{\infty} g(u+2\pi\nu).$$

From (52) and I(3), we obtain

$$\chi_m(u) = \phi_k(u) + \gamma_2^{(k)}(2\sin u/2)^2\phi_{k-2}(u) + \cdots + \gamma_{2m-2}^{(k)}(2\sin u/2)^{2m-2}\phi_{k-2m+2}(u). \tag{54}$$

This expression is obviously positive for all values of u. For $m=\mu$, however, we obtain the identity

$$\chi_\mu(u) \equiv 1 \tag{55}$$

since, by Lemma 6, we have before us an *ordinary* interpolation formula. Since (54) is a partial sum of the left-hand side of (55) we have therefore proved the inequalities

$$0 < \chi_m(u) < 1 \qquad (1 \leqq m < \mu,\ 0 < u < 2\pi). \tag{56}$$

Then (53) is indeed a smoothing formula. The final statement of the Theorem is evident from (54), since $\chi_m(u)$ increases with m.

As illustrations we mention the following four special cases, two from each end of the range of values of m.

(i) $m=2$. The formula based on

$$\Gamma_{k,2}(x) = M_k(x) - \frac{k}{24}M_k''(x) \qquad (k \geqq 4) \tag{57}$$

has the type

$$D^{k-1}, \qquad C^{k-4}, \qquad E^3, \qquad s = k. \tag{57'}$$

(ii) $m=3$. The formula based on

$$\Gamma_{k,3}(x) = M_k(x) - \frac{k}{24}M_k''(x) + \frac{k(5k+2)}{5760}M_k^{(4)}(x), \qquad (k \geqq 6), \tag{58}$$

has the type

$$D^{k-1}, \qquad C^{k-6}, \qquad E^5, \qquad s = k. \tag{58'}$$

The values (48) were used.

(iii) $m=\mu-1$. The formula based on[5]

[5] The formula (59) is especially instructive because we can observe very clearly how the addition to $C_k(x)$ of the extra term removes the crudest discontinuities of $C_k(x)$. Indeed, by the formula (*) of our preceding footnote we may write (59) as

$$\Gamma_{k,\mu-1}(x) = C_k(x) + (-1)^{\mu}\gamma_{2\mu-2}^{(k)} M_k^{(2\mu-2)}(x) \qquad (k \geqq 4) \tag{59}$$

has the type

$$D^{k-1}, \qquad \begin{Bmatrix} C^2 & \text{if} & k \text{ is even} \\ C^1 & \text{if} & k \text{ is odd} \end{Bmatrix}, \qquad E^{k-3}, \qquad s = k. \tag{59'}$$

(iv) $m = \mu - 2$. The formula based on

$$\Gamma_{k,\mu-2}(x) = C_k(x) + (-1)^{\mu}\gamma_{2\mu-2}^{(k)} M_k^{(2\mu-2)}(x) + (-1)^{\mu-1}\gamma_{2\mu-4}^{(k)} M_k^{(2\mu-4)}(x) \quad (k \geqq 6) \tag{60}$$

has the type

$$D^{k-1}, \qquad \begin{Bmatrix} C^4 & \text{if} & k \text{ is even} \\ C^3 & \text{if} & k \text{ is odd} \end{Bmatrix}, \qquad E^{k-5}, \qquad s = k. \tag{60'}$$

These formulae show clearly how an increase in "order of contact" is compensated by a corresponding loss in "reproductive power" and vice versa.

III. A SECOND CLASS OF ANALYTIC INTERPOLATION FORMULAE

In Part A, section 4.2, we described a class of *ordinary* analytic interpolation formulae of basic function

$$L_k(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\psi_k(u, t)}{\phi_k(u, t)} e^{iux} du \qquad (k = 1, 2, \cdots ; t > 0). \tag{1}$$

These interpolation formulae are exact for the degree $k-1$. The basic function (1) as well as those of the smoothing interpolation formulae derived from it in Part A, section 4.3, dampen out like a descending exponential function. In the present last chapter we wish to construct smoothing analytic interpolation formulae of basic functions dampening out like

$$\exp(-c^2x^2),$$

hence much more rapidly. In view of the development of section 2.2 it would seem fairly obvious how such formulae may be derived. We clearly need an analogue of Lemma 5 which we state as a conjecture: *The reciprocal of $\phi_k(u, t)$ admits of an expansion*

$$\Gamma_{k,\mu-1}(x) = C_k(x) + (-1)^{\mu}\gamma_{2\mu-2}^{(k)}\delta^{2\mu-2} M_{k-2\mu+2}(x). \tag{59'}$$

Let k be *odd*, hence $2\mu-2 = k-1$ and therefore

$$\Gamma_{k,\mu-1}(x) = C_k(x) + (-1)^{\mu}\gamma_{k-1}^{(k)}\delta^{k-1} M_1(x). \tag{59''}$$

As seen from the graph of $M_1(x)$, the corrective term is a step-function with discontinuities at $x = n + 1/2$ whose values are proportional with the binomial coefficients of order $k-1$: $\binom{k-1}{\nu}$. Their addition to $C_k(x)$ offsets the discontinuities of $C_k(x)$ and turn it into a function (59'') of class C^1. If k is *even*, hence $2\mu-2 = k-2$, we have

$$\Gamma_{k,\mu-1}(x) = C_k(x) + (-1)^{\mu}\gamma_{k-2}^{(k)}\delta^{k-2} M_2(x). \tag{59'''}$$

As seen from the graph of $M_2(x)$, the corrective term is now an ordinary polygonal line with vertices at $x = n$, whose ordinates (at these vertices) are proportional to the binomial coefficients of order $k-2$: $\binom{k-2}{\nu}$. Again, the superposition of this polygonal line on $C_k(x)$ offsets the corners of $C_k(x)$ and turns it into a function (59''') of class C^2. The formulae (59'''), (59'') are especially convenient for constructing tables of these functions from existing tables of $C_k(x)$, i.e., tables of Lagrange interpolation coefficients.

$$\frac{1}{\phi_k(u, t)} = \sum_{n=0}^{\infty} c_{2n}^{(k)}(t)(2 \sin u/2)^{2n}, \tag{2}$$

which converges for all real values of u and where the coefficients are all positive

$$c_{2n}^{(k)}(t) > 0, \qquad (n = 0, 1, 2, \cdots). \tag{3}$$

A proof of this conjecture would require a closer function-theoretic study of the entire periodic function $\phi_k(u, t)$ which has not been carried through as yet.

Since

$$\phi_k(u, t) = \sum_{\nu=-\infty}^{\infty} \psi_k(u + 2\pi\nu, t)$$

(see Part A, section 4.1, formula (6)) we have

$$\phi_k(u, t) = \psi_k(u, t) + u^k \cdot (\text{regular function}).$$

Therefore the expansion (2) agrees in its terms of order less than k with the similar terms of the expansion

$$\frac{1}{\psi_k(u, t)} = e^{t(u/2)^2}\left(\frac{u}{2 \sin u/2}\right)^k = \sum_{n=0}^{\infty} d_{2n}^{(k)}(t)(2 \sin u/2)^{2n}, \qquad (-\pi \leqq u \leqq \pi). \tag{4}$$

Hence

$$c_{2n}^{(k)}(t) = d_{2n}^{(k)}(t), \qquad (0 \leqq 2n < k). \tag{5}$$

The expansion (4) is readily determined and its coefficients are found to be positive as follows. We turn back to section 1.3 where in terms of the variable

$$v = 2 \sin u/2 \tag{6}$$

we have by I(35)

$$\left(\frac{u}{2 \sin u/2}\right)^k = \sum_{n=0}^{\infty} d_{2n}^{(k)} v^{2n} \qquad (-\pi \leqq u \leqq \pi). \tag{7}$$

Also by I(39)

$$u/2 = \arcsin v/2 = \frac{v}{2} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{v^3}{8} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}\cdot\frac{v^5}{32} + \cdots \qquad (-2 \leqq v \leqq 2). \tag{8}$$

On substituting (8) into the exponential series we find the expansion

$$e^{t(u/2)^2} = \sum_{n=0}^{\infty} e_{2n}(t) v^{2n} \qquad (-2 \leqq v \leqq 2) \tag{9}$$

with positive coefficients, the first three of which are found to be

$$e_0(t) = 1, \qquad e_2(t) = \frac{t}{4}, \qquad e_4(t) = \frac{t^2}{32} + \frac{t}{48}. \tag{10}$$

On multiplying the series (7) and (9) we obtain the expansion (4). From the values I(40) and (10) we readily find

$$d_0^{(k)}(t) = 1, \quad d_2^{(k)}(t) = \frac{k}{24} + \frac{t}{4}, \quad d_4^{(k)}(t) = \frac{5k^2 + 22k}{5760} + \frac{tk}{96} + \frac{t}{48} + \frac{t^2}{32}. \tag{11}$$

Our arguments of section 2.2 may now be repeated leading to the following theorem:

THEOREM 4. *Let k be a positive integer and m an integer such that $0 < 2m < k+2$. Then*

$$L_{k,m}(x, t) = M_k(x, t) - d_2^{(k)}(t)\delta^2 M_k(x, t) + \cdots + (-1)^{m-1} d_{2m-2}^{(k)}(t)\delta^{2m-2} M_k(x) \tag{12}$$

gives rise to a "smoothing" analytic interpolation formula

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L_{k,m}(x - n, t) \tag{13}$$

which is exact for the degree $\min(2m-1, k-1)$. *Moreover* (13) *always preserves the degree* $k-1$.

The adjective "smoothing" was purposely written in quotation marks in order to indicate that there is no general proof as yet that (13) always reduces, for integral values of the variable x, to a smoothing formula in the sense of our Definition 1 of Part A, section 1.1. For indeed, (2) and (3), which imply such a proof, were only conjectured. In the Appendix we give 8-place tables of the three basic functions

$$\left.\begin{aligned} L_1(x) &= L_{4,2}(x, 1/8), \\ L_2(x) &= L_{4,2}(x, 1/2), \\ L_3(x) &= L_{6,3}(x, 1/2), \end{aligned}\right\} \tag{14}$$

as well as 7-place tables of their first and second derivatives. For these three sets of values of the parameters k, t, and m, the interpolation formula (13) is indeed a smoothing formula. This point is verified by an inspection of the corresponding characteristic functions

$$\phi_i(u) = L_i(0) + 2L_i(1) \cos u + 2L_i(2) \cos 2u + \cdots, \qquad (i = 1, 2, 3,). \tag{15}$$

From the values of $L_i(n)$, as given by our tables, we computed the following table for these characteristic functions:

u	$\phi_1(u)$	$\phi_2(u)$	$\phi_3(u)$
0°	1.00000	1.00000	1.00000
30°	.99734	.99519	.99952
60°	.96332	.93655	.97760
90°	.85492	.76500	.84693
120°	.67727	.51297	.56702
150°	.50474	.29296	.27879
180°	.43283	.20728	.16123

Since $0 < \phi_i(u) < 1$ for $0 < u \leqq 180°$, all three formulae (13) are smoothing formulae according to Part A, section 1.1. Also $\phi_2(u) < \phi_1(u)$ implies that L_2 gives a stronger smoothing formula as compared to L_1.

Our set of tables is intended mainly for the purpose of illustrating the method. A more complete set of tables would be needed in order to furnish smoothing of a desired strength, as required by the needs of the numerical data at hand.

APPENDIX

Description of the tables and their use for the analytic approximation of equidistant data. In the Tables I, II, and III, we have tabulated the following three functions

$$L_1(x) = M_4(x, 1/8) - \frac{19}{96}\delta^2 M_4(x, 1/8) \tag{1}$$

$$L_2(x) = M_4(x, 1/2) - \frac{7}{24}\delta^2 M_4(x, 1/2) \tag{2}$$

$$L_3(x) = M_6(x, 1/2) - \frac{3}{8}\delta^2 M_6(x, 1/2) + \frac{199}{1920}\delta^4 M_6(x, 1/2) \tag{3}$$

and their first two derivatives. The function $M_k(x, t)$ occurring in these definitions may be defined by the integral

$$M_k(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t(u/2)^2}\left(\frac{2\sin u/2}{u}\right)^k \cos uxdu.$$

A given sequence of equidistant ordinates

$$\{y_n\} \tag{4}$$

is approximated by either one of the three analytic functions

$$F_i(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_i(x-\nu), \qquad (i = 1, 2, 3). \tag{5}$$

The choice among these approximations depends on the amount of smoothing desired. The formula (5), for $i=1$ and $i=2$, is exact for (i.e., reproduces) cubic polynomials. For $i=3$ the formula (5) is exact for quintic polynomials. For the same data (4), the sequence $\{F_2(n)\}$ is always smoother than the sequence $\{F_1(n)\}$. Generally, the sequence $\{F_3(n)\}$ should be smoother than the sequence $\{F_1(n)\}$.

The first and second derivatives of the approximation (5) may be computed by the similar formulae

$$F_i'(x) = \sum y_\nu L_i'(x-\nu), \tag{6}$$

$$F_i''(x) = \sum y_\nu L_i''(x-\nu). \tag{7}$$

The arrangement of our tables is such as to facilitate the computation of $F_i(x)$ by (5), as explained in the Appendix to Part A.

An example of smoothing with subtabulation to tenths. We propose to compute a table of the approximation $F_2(x)$, in the range $31 \leqq x \leqq 34$, for the same ordinates $\{y_n\}$ as were used in our example of Part A (Appendix). The ordinates which we now require are given by the following table:

n	y_n	Δ	Δ^2	Δ^3	Δ^4	Δ^5
26	32840					
27	34790	1950				
28	37260	2470	520			
29	40440	3180	710	190		
30	44750	4310	1130	420	230	
31	51120	6370	2060	930	510	280
32	59390	8270	1900	− 160	−1090	−1600
33	67550	8160	− 110	−2010	−1850	− 760
34	73820	6270	−1890	−1780	230	2080
35	77830	4010	−2260	− 370	1410	1180
36	80240	2410	−1600	660	1030	− 380
37	81660	1420	− 990	610	− 50	−1080
38	82330	670	− 758	240	− 370	− 320
39	82680	350	− 320	430	190	560

From these values and the Table II of $L_2(x)$ and $L_2''(x)$ we obtain the following tables of the approximation $F_2(x)$ and its second derivative $F_2''(x)$ shown with their differences.

x	$F_2(x)$	Δ	Δ^2	Δ^3	Δ^4	$F_2''(x)$	Δ	Δ^2	Δ^3	Δ^4
31.0	51232.76					1901.77				
31.1	51989.86	75710				1767.20	−13457			
31.2	52764.62	77476	1766			1611.60	−15560	−2103		
31.3	53555.48	79086	1610	−156		1436.84	−17476	−1916	187	
31.4	54360.69	80521	1435	−175	−19	1245.42	−19142	−1666	250	63
31.5	55178.34	81765	1244	−191	−16	1040.29	−20513	−1371	295	45
31.6	56006.39	82805	1040	−204	−13	824.64	−21565	−1052	319	24
31.7	56842.68	83629	824	−216	−12	601.66	−22298	− 733	319	0
31.8	57684.98	84230	601	−223	− 7	374.38	−22728	− 430	303	−16
31.9	58531.03	84605	375	−226	− 3	145.59	−22879	− 151	279	−24
32.0	59378.53	84750	145	−230	− 4	− 82.29	−22788	91	242	−37
32.1	60225.21	84668	− 82	−227	3	− 307.15	−22486	302	211	−31
32.2	61068.82	84361	− 307	−225	2	− 527.12	−21997	489	187	−24
32.3	61907.17	83835	− 526	−219	6	− 740.51	−21339	658	169	−18
32.4	62738.12	83095	− 740	−214	5	− 945.74	−20523	816	158	−11
32.5	63559.62	82150	− 945	−205	9	−1141.23	−19549	974	158	0
32.6	64369.71	81009	−1141	−196	9	−1325.45	−18422	1127	153	− 5
32.7	65166.57	79686	−1323	−182	14	−1496.90	−17145	1277	150	− 3
32.8	65948.46	78189	−1497	−174	8	−1654.18	−15728	1417	140	−10
32.9	66713.83	76537	−1652	−155	19	−1796.14	−14196	1532	115	−25
33.0	67461.25	74742	−1795	−143	12	−1921.91	−12577	1619	87	−28
33.1	68189.46	72821	−1921	−126	17	−2030.98	−10907	1670	51	−36
33.2	68897.38	70792	−2029	−108	18	−2123.19	− 9221	1686	16	−35
33.3	69584.08	68670	−2122	− 93	15	−2198.74	− 7555	1666	− 20	−36
33.4	70248.81	66473	−2197	− 75	18	−2258.04	− 5930	1625	− 41	−21
33.5	70890.96	64215	−2258	− 61	14	−2301.69	− 4365	1565	− 60	−19
33.6	71510.12	61916	−2299	− 41	20	−2330.31	− 2862	1503	− 62	− 2
33.7	72105.98	59586	−2330	− 31	10	−2344.57	− 1426	1436	− 67	− 5
33.8	72678.41	57243	−2343	− 13	18	−2345.07	− 50	1376	− 60	7
33.9	73227.40	54899	−2344	− 1	12	−2332.47	1260	1310	− 66	− 6
34.0	73753.07	52567	−2332	12	13	−2307.47	2500	1240	− 70	− 4

A comparison of the approximation $F_2(x)$ with the strictly interpolating function $F(x)$, obtained in the Appendix of Part A, is of interest. The function $F(x)$ was obtained by the formula

$$F(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_k(x - \nu, t), \qquad (k = 4, t = 1/2),$$

where, in view of III(1) and III(2), we may define $L_k(x, t)$ by the expansion

$$L_k(x, t) = M_k(x, t) - c_2^{(k)}(t)\delta^2 M_k(x, t) + c_4^{(k)}(t)\delta^4 M_k(x, t) - \cdots . \qquad (8)$$

Our present approximation $F_2(x)$ was computed by the formula (5), for $i=2$, where $L_2(x)$, by (2), III(5) and III(11), happens to be identical with the sum

$$L_2(x) = M_k(x, t) - c_2^{(k)}(t)\delta^2 M_k(x, t)$$

of the first two terms of the series (8). A comparison of the tables of $F(x)$ and $F_2(x)$ shows that their difference in the range $31 \leqq x \leqq 34$ nowhere exceeds 0.23% of the value of $F(x)$.

TABLE I. $L_1(x) = L_{4,2}(x, 1/8)$, $L_1'(x)$, $L_1'\ (x)$

$L_1(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
3	−.00041123	−.00018550	−.00007621	−.00002830	−.00000943
2	−.03462580	−.02756299	−.02099452	−.01533104	−.01073240
1	.14220425	.08277004	.03516336	−.00066601	−.02556532
0	.78566556	.77475976	.74281594	.69204028	.62577328
−1	.14220425	.21251290	.29166041	.37661030	.46353840
−2	−.03462580	−.04145730	−.04699226	−.04985257	−.04841746
−3	−.00041123	−.00083691	−.00157669	−.00277249	−.00458632
−4			−.00000003	−.00000017	−.00000074

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
3	−.00000280	−.00000074	−.00000017	−.00000003	
2	−.00718751	−.00458632	−.00277249	−.00157669	−.00083691
1	−.04092087	−.04841746	−.04985257	−.04699226	−.04145730
0	.54811118	.46353840	.37661030	.29166041	.21251290
−1	.54811118	.62577328	.69204028	.74281594	.77475976
−2	−.04092087	−.02556532	−.00066601	.03516336	.08277004
−3	−.00718751	−.01073240	−.01533104	−.02099452	−.02756299
−4	−.00000280	−.00000943	−.00002830	−.00007621	−.00018550

$L_1'(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
3	.0030922	.0015611	.0007155	.0002953	.0001089
2	.0709642	.0690870	.0616092	.0514079	.0405956
1	−.6512052	−.5359047	−.4163489	−.3016954	−.1986416
0	.0000000	−.2168063	−.4183406	−.5915396	−.7269119
−1	.6512052	.7515608	.8262912	.8662852	.8650057
−2	−.0709642	−.0638857	−.0445045	−.0099839	.0416488
−3	−.0030922	−.0056121	−.0094213	−.0147669	−.0217952
−4		−.0000001	−.0000006	−.0000026	−.0000103

$L_1'(x)$

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
3	.0000357	.0000103	.0000026	.0000006	.0000001
2	.0304945	.0217952	.0147669	.0094213	.0056121
1	−.1113051	−.0416488	.0099839	.0445045	.0638857
0	−.8188067	−.8650057	−.8662852	−.8262912	−.7515608
−1	.8188067	.7269119	.5915396	.4183406	.2168063
−2	.1113051	.1986416	.3016954	.4163489	.5359047
−3	−.0304945	−.0405956	−.0514079	−.0616092	−.0690870
−4	−.0000357	−.0001089	−.0002953	−.0007155	−.0015611

$L_1''(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
4	−.0000004	−.0000001			
3	−.0197354	−.0114013	−.0059477	−.0027759	−.0011500
2	.0202421	−.0521077	−.0926044	−.1078983	−.1061742
1	1.0966561	1.1912649	1.1844282	1.0974670	.9568292
0	−2.1943249	−2.1159568	−1.8927238	−1.5554043	−1.1427100
−1	1.0966561	.8923803	.5868711	.2020705	−.2337191
−2	.0202421	.1269947	.2654156	.4283134	.6058876
−3	−.0197354	−.0311722	−.0454298	−.0617347	−.0788289
−4	−.0000004	−.0000019	−.0000092	−.0000377	−.0001346

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
3	−.0004201	−.0001346	−.0000377	−.0000092	−.0000019
2	−.0947351	−.0788289	−.0617347	−.0454298	−.0311722
1	.7867260	.6058876	.4283134	.2654156	.1269947
0	−.6915708	−.2337191	.2020705	.5868711	.8923803
−1	−.6915708	−1.1427100	−1.5554043	−1.8927238	−2.1159568
−2	.7867260	.9568292	1.0974670	1.1844282	1.1912649
−3	−.0947351	−.1061742	−.1078983	−.0926044	−.0521077
−4	−.0004201	−.0011500	−.0027759	−.0059477	−.0114013
−5					−.0000001

TABLE II. $L_2(x)=L_{4,2}(x, 1/2)$, $L_2'(x)$, $L_2''(x)$

$L_2(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
5	−.00000001				
4	−.00003297	−.00001721	−.00000871	−.00000428	−.00000203
3	−.00453670	−.00313034	−.00210474	−.00137859	−.00087930
2	−.04033977	−.03775949	−.03379714	−.02913417	−.02429574
1	.20271717	.14753253	.09914645	.05829935	.02523450
0	.68438455	.67711287	.65567464	.62117134	.57534514
−1	.20271717	.26343912	.32793367	.39399436	.45907812
−2	−.04033977	−.04070831	−.03791256	−.03092183	−.01869149
−3	−.00453670	−.00640787	−.00882100	−.01183236	−.01545906
−4	−.00003297	−.00006127	−.00011052	−.00019364	−.00032972
−5	−.00000001	−.00000003	−.00000007	−.00000018	−.00000042
x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
4	−.00000094	−.00000042	−.00000018	−.00000007	−.00000003
3	−.00054590	−.00032972	−.00019364	−.00011052	−.00006127
2	−.01965692	−.01545906	−.01183236	−.00882100	−.00640787
1	−.00024449	−.01869149	−.03092183	−.03791256	−.04070831
0	.52044824	.45907812	.39399436	.32793367	.26343912
−1	.52044824	.57534514	.62117134	.65567464	.67711287
−2	−.00024449	.02523450	.05829935	.09914645	.14753253
−3	−.01965692	−.02429574	−.02913417	−.03379714	−.03775949
−4	−.00054590	−.00087930	−.00137859	−.00210474	−.00313034
−5	−.00000094	−.00000203	−.00000428	−.00000871	−.00001721

$L_2'(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
5	.0000001				
4	.0002093	.0001145	.0000607	.0000311	.0000154
3	.0162494	.0120195	.0086291	.0060153	.0040721
2	.0162409	.0339777	.0441321	.0482532	.0478931
1	−.5820676	−.5195203	−.4469715	−.3695748	−.2920635
0	.0000000	−.1448007	−.2821139	−.4050264	−.5077160
−1	.5820676	.6294198	.6567760	.6601752	.6369100
−2	−.0162409	.0104651	.0471784	.0943909	.1518602
−3	−.0162494	−.0213049	−.0270545	−.0332049	−.0392596
−4	−.0002093	−.0003705	−.0006358	−.0010581	−.0017083
−5	−.0000001	−.0000002	−.0000006	−.0000015	−.0000034

$L_2'(x)$

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
4	.0000074	.0000034	.0000015	.0000006	.0000002
3	.0026769	.0017083	.0010581	.0006358	.0003705
2	.0444848	.0392596	.0332049	.0270545	.0213049
1	−.1876748	−.1518602	−.0943909	−.0471784	−.0104651
0	−.5858619	−.6369100	−.6601752	−.6567760	−.6294198
−1	.5858619	.5077160	.4050264	.2821139	.1448007
−2	.1876748	.2920635	.3695748	.4469715	.5195203
−3	−.0444848	−.0478931	−.0482532	−.0441321	−.0339777
−4	−.0026769	−.0040721	−.0060153	−.0086291	−.0120195
−5	−.0000074	−.0000154	−.0000311	−.0000607	−.0001145

$L_2''(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
5	−.0000010	−.0000004	−.0000001		
4	−.0012284	−.0007087	−.0003952	−.0002129	−.0001108
3	−.0465343	−.0380479	−.0298698	−.0225871	−.0164844
2	.2201282	.1369510	.0687454	.0162834	−.0210828
1	.5579563	.6842412	.7580806	.7819152	.7615818
0	−1.4606815	−1.4227560	−1.3118991	−1.1365778	− .9099683
−1	.5579763	.3809905	.1594624	−.0960473	.3711575
−2	.2201282	.3157558	.4193623	.5245097	.6230076
−3	−.0465343	−.0543645	−.0601451	−.0620423	−.0578316
−4	−.0012284	−.0020585	−.0033357	−.0052283	−.0079270
−5	−.0000010	−.0000025	−.0000057	−.0000126	−.0000270
x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
4	−.0000557	−.0000270	−.0000126	−.0000057	−.0000025
3	−.0116249	−.0079270	−.0052283	−.0033357	−.0020585
2	−.0450247	−.0578316	−.0620423	−.0601451	−.0543645
1	.7053804	.6230076	.5245097	.4193623	.3157558
0	−.6486751	−.3711575	−.0960473	.1594624	.3809905
−1	−.6486751	−.9099683	−1.1365778	−1.3118991	−1.4227560
−2	.7053804	.7615818	.7819152	.7580806	.6842412
−3	−.0450247	−.0210828	.0162834	.0687454	.1369510
−4	−.0116249	−.0164844	−.0225871	−.0298698	−.0380479
−5	−.0000557	−.0001108	−.0002129	−.0003952	−.0007087
−6				−.0000001	−.0000004

TABLE III. $L_3(x)=L_{6,3}(x, 1/2)$, $L_3'(x)$, $L_3''(x)$

$L_3(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
6	.00000024	.00000012	.00000005	.00000002	.00000001
5	.00008286	.00005169	.00003150	.00001874	.00001088
4	.00268003	.00209760	.00160285	.00119677	.00087361
3	.00146844	.00384345	.00530620	.00602933	.00618622
2	−.06657958	−.06134625	−.05394260	−.04532872	−.03631488
1	.20814167	.14737265	.09315196	.04655844	.00818720
0	.70841267	.70114512	.67967926	.64500315	.59869152
−1	.20814167	.27390708	.34268907	.41215975	.47975381
−2	−.06657958	−.06856746	−.06617125	−.05825575	−.04376882
−3	.00146844	−.00197317	−.00659556	−.01244745	−.01948472
−4	.00268003	.00333890	.00404886	.00476644	.00542627
−5	.00008286	.00012978	.00019870	.00029745	.00043550
−6	.00000024	.00000049	.00000096	.00000183	.00000340

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
5	.00000616	.00000340	.00000183	.00000096	.00000049
4	.00062367	.00043550	.00029745	.00019870	.00012978
3	.00593823	.00542627	.00476644	.00404886	.00333890
2	−.02754330	−.01948472	−.01244745	−.00659556	−.00197317
1	−.02182865	−.04376882	−.05825575	−.06617125	−.06856746
0	.54280388	.47975381	.41215975	.34268907	.27390708
−1	.54280388	.59869152	.64500315	.67967926	.70114512
−2	−.02182865	.00818720	.04655844	.09315196	.14737265
−3	−.02754330	−.03631488	−.04532872	−.05394260	−.06134625
−4	.00593823	.00618622	.00602933	.00530620	.00384345
−5	.00062367	.00087361	.00119677	.00160285	.00209760
−6	.00000616	.00001088	.00001874	.00003150	.00005169
−7		.00000001	.00000002	.00000005	.00000012

$L_3'(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
6	−.0000017	−.0000009	−.0000004	−.0000002	−.0000001
5	−.0003813	−.0002500	−.0001598	−.0000996	−.0000605
4	−.0062356	−.0053949	−.0044992	−.0036324	−.0028472
3	.0288514	.0189098	.0106360	.0041176	−.0007095
2	.0379708	.0648953	.0815717	.0893524	.0898535
1	−.6356363	−.5771501	−.5054638	−.4254184	−.3417929
0	.0000000	−.1447858	−.2828738	−.4080067	−.5147714
−1	.6356363	.6763588	.6953825	.6897327	.6576776
−2	−.0379708	.0001221	.0497076	.1103706	.1808527
−3	−.0288514	−.0401754	−.0523583	−.0646105	−.0758656
−4	.0062356	.0069002	.0072273	.0070158	.0060307
−5	.0003813	.0005675	.0008239	.0011670	.0016121
−6	.0000017	.0000033	.0000063	.0000116	.0000206

$L_3'(x)$

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
5	−.0000358	−.0000206	−.0000116	−.0000063	−.0000033
4	−.0021709	−.0016121	−.0011670	−.0008239	−.0005675
3	−.0040148	−.0060307	−.0070158	−.0072273	−.0069002
2	.0847953	.0758656	.0646105	.0523583	.0401754
1	−.2590022	−.1808527	−.1103706	−.0497076	−.0001221
0	−.5989336	−.6576776	−.6897327	−.6953825	−.6763588
−1	.5989336	.5147714	.4080067	.2828738	.1447858
−2	.2590022	.3417929	.4254184	.5054638	.5771501
−3	−.0847953	−.0898535	−.0893524	−.0815717	−.0648953
−4	.0040148	.0007095	−.0041176	−.0106360	−.0189098
−5	.0021709	.0028472	.0036324	.0044992	.0053949
−6	.0000358	.0000605	.0000996	.0001598	.0002500
−7		.0000001	.0000002	.0000004	.0000009

$L_3''(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
6	.0000117	.0000061	.0000031	.0000015	.0000007
5	.0015631	.0010868	.0007350	.0004835	.0003094
4	.0077612	.0088518	.0089242	.0083240	.0073357
3	−.1069900	−.0913730	−.0739648	−.0565175	−.0403140
2	.3239975	.2160536	.1197566	.0385786	−.0256675
1	.5032414	.6588169	.7667843	.8261081	.8390021
0	−1.4591699	−1.4253081	−1.3259553	−1.1676245	−.9605716
−1	.5032414	.3045939	.0708748	−.1867923	−.4548011
−2	.3239975	.4384255	.5526106	.6585611	.7474910
−3	−.1069900	−.1186224	−.1237228	−.1195410	−.1033391
−4	.0077612	.0052612	.0009447	−.0055676	−.0145672
−5	.0015631	.0021860	.0029705	.0039174	.0050049
−6	.0000117	.0000217	.0000392	.0000685	.0001165
−7				.0000001	.0000001

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
6	.0000003	.0000001	.0000001		
5	.0001926	.0001165	.0000685	.0000392	.0000217
4	.0061776	.0050049	.0039174	.0029705	.0021860
3	−.0261859	−.0145672	−.0055676	.0009447	.0052612
2	−.0726654	−.1033391	−.1195410	−.1237228	−.1186224
1	.8104398	.7474910	.6585611	.5526106	.4384255
0	−.7179590	−.4548011	−.1867923	−.0708748	.3045939
−1	−.7179590	−.9605716	−1.1676245	−1.3259553	−1.4253081
−2	.8104398	.8390021	.8261081	.7667843	.6588169
−3	−.0726654	−.0256675	.0385786	.1197566	.2160536
−4	−.0261859	−.0403140	−.0565175	−.0739648	−.0913730
−5	.0061776	.0073357	.0083240	.0089242	.0088518
−6	.0001926	.0003094	.0004835	.0007350	.0010868
−7	.0000003	.0000007	.0000015	.0000031	.0000061

AN ITERATION METHOD FOR CALCULATION WITH LAURENT SERIES*

BY

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Introduction. The power series is a basic concept of Analysis which is of fundamental importance from the theoretical as well as from the computational point of view. The theoretical importance of power series springs from the fact that it represents any analytic function in the neighborhood of a regular point. The reason for its practical importance is the ease with which implicitly defined functions, by finite relations or differential equations, may be expanded in power series by the so-called method of undetermined coefficients, known and used since the dawn of mathematical analysis.

Laurent series play a definitely minor role as compared to power series. One reason is the more complicated nature of the connection between the sum of the series and its coefficients. Another reason, dependent on the first, is the difficulty of calculations with Laurent series.

The purpose of this paper is to describe a method whereby rational or algebraic operations with Laurent series may be performed with high accuracy at the expense of a reasonable amount of labor. A general approximation method to empirical data, developed by one of us,[1] required the very accurate reciprocation of certain Laurent series. This problem of reciprocation of Laurent series was the starting point of our investigation. Our method for solving this particular problem turned out to be identical with a method of reciprocation of finite matrices already investigated by H. Hotelling.[2] We finally point out that our method of computation with Laurent series extends to computations with trigonometric series provided these series converge absolutely.

1. Newton's algorithm and statement of the problem. Let

$$f(x) \equiv a_0x^m + a_1x^{m-1} + \cdots + a_m = 0, \qquad (a_0 \neq 0), \tag{1}$$

be an algebraic equation with numerical real or complex coefficients. If x is a simple root of this equation, then very close approximations to x may be readily computed by Newton's iterative algorithm represented by the recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\cdot \tag{2}$$

The reason for the fast convergence of x_n towards x is as follows: Expanding the right-

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[1] I. J. Schoenberg, *Contributions to the problem of approximation of equidistant data by analytic functions*, Part A, Quart. Appl. Math.

[2] H. Hotelling, *Some new methods in matrix calculation*, Ann. Math. Statist. 14, 1–34 (1943), especially p. 14, and *Further points in matrix calculation and simultaneous equations*, Ann. Math. Statist. 14, 440–441 (1943).

hand side in a power series in $x_n - x$, in the neighborhood of the simple root x, we find that (2) may be written as

$$x_{r+1} - x = c_2(x_n - x)^2 + c_3(x_n - x)^3 + \cdots, \tag{3}$$

with coefficients c_ν depending only on the root x. Because there is no linear term in $x_n - x$ on the right-hand side we find that from a certain point on the error $x_{n+1} - x$ is of the order of magnitude of the square of the previous error $x_n - x$. From this stage on, each step will approximately double the number of correct decimal places of the previous approximation x_n. This type of rapid convergence is sometimes referred to as "quadratic convergence."

Let us notice that the iterative process (2) requires a division, by $f'(x_n)$, at each step of the process. This is a serious handicap in computing with machines which do not perform the operation of division, as for example the standard punch-card machines. This division is likewise a handicap if we wish to extend the process to the realm of matrices where division is a difficult numerical operation.

We propose to modify Newton's algorithm (2) so as to require only the operations of addition, subtraction and multiplication in its performance. It will then be shown how the modified Newton algorithm allows us to carry out numerically rational as well as algebraic operations on Laurent series. The most general numerical problem whose solution is facilitated by our method may be formulated as follows.

PROBLEM. *Let*

$$f(w, z) \equiv a_0(z)w^m + a_1(z)w^{m-1} + \cdots + a_m(z) = 0 \tag{4}$$

be an equation with the following properties:

1. *The coefficients $a_\nu(z)$ are all regular and uniform functions of z in the ring*

$$R: \quad r_1 < |z| < r_2. \tag{5}$$

2. *We have*

$$a_0(z) \neq 0 \quad \text{in} \quad R. \tag{6}$$

3. *The discriminant $D(z)$ of* (4) *satisfies*

$$D(z) \neq 0 \quad \text{in} \quad R \tag{7}$$

so that the equation (4) *has no critical point in R. Let now $w = w(z)$ be a branch of a solution of* (4) *which is necessarily regular in R but need not be uniform in R. Given the numerical values of the Laurent expansions of the coefficients $a_\nu(z)$, the problem is to find the values of the coefficients of the Laurent expansions of $w(z)$.*

Remarks. 1. The difficulty of this problem is due to its being concerned with Laurent series rather than ordinary power series. Indeed, if everything else is unchanged, we replace, in its formulation, the ring (5) by the circle $|z| < r_2$, then all Laurent series mentioned become power series, in which case the power series expansion of the branch $w = w(z)$ may be obtained by the method of undetermined coefficients (see the first paragraph of our Introduction).

2. We did not require the branch $w = w(z)$ to be uniform in R. However, we do not restrict our problem by assuming $w(z)$ to be uniform in R. Indeed, if $w(z)$ returns to its values after k turns in R, $k > 1$, we change variable by setting

$$z = \zeta^k.$$

Our equation (4) thereby becomes

$$f(w, \zeta^k) = 0$$

and the branch $w(z)$ becomes uniform in the corresponding ring in the ζ-plane. If we can determine its uniform Laurent series

$$w(z) = \sum_{-\infty}^{\infty} \omega_n \zeta^n$$

we also have its algebraic expansion

$$w(z) = \sum_{-\infty}^{\infty} \omega_n z^{n/k}.$$

3. Even the case $m=1$ is far from trivial. Thus

$$a_0(z)w - 1 = 0$$

amounts to the important problem of the reciprocation of a given Laurent series.

2. The modification of Newton's algorithm. We return in this section to the case of the ordinary algebraic equation (1). We now impose the restriction that

$$f(x) \text{ has only simple zeros.} \tag{8}$$

This condition implies that the polynomials $f(x)$, $f'(x)$ have no common divisors and that we can therefore determine uniquely, by rational operations alone, two polynomials $\phi(x)$ and $\psi(x)$ satisfying the identity

$$f(x)\phi(x) + f'(x)\psi(x) \equiv 1, \tag{9}$$

and such that the degrees of ϕ and ψ do not exceed $m-2$ and $m-1$, respectively. The coefficients of $\phi(x)$, $\psi(x)$ are rational functions of the coefficients a_ν. For later reference it is important to remark that the coefficients of $\psi(x)$ may be written as a quotient of polynomials in a_ν divided by the discriminant D of the polynomial $f(x)$. Indeed, the coefficients of ϕ and ψ are determined, in view of (9), by a system of linear equations whose determinant is precisely the discriminant D of $f(x)$. This procedure leads to explicit expressions of ψ and D in determinant form. Thus for $m=3$ we obtain

$$\psi(x) = \frac{1}{D}\begin{vmatrix} x^2 & x & 1 & 0 & 0 \\ 3a_0 & 0 & 0 & a_0 & 0 \\ 2a_1 & 3a_0 & 0 & a_1 & a_0 \\ a_2 & 2a_1 & 3a_0 & a_2 & a_1 \\ 0 & a_2 & 2a_1 & a_3 & a_2 \end{vmatrix}, \quad D = \begin{vmatrix} 3a_0 & 0 & 0 & a_0 & 0 \\ 2a_1 & 3a_0 & 0 & a_1 & a_0 \\ a_2 & 2a_1 & 3a_0 & a_2 & a_1 \\ 0 & a_2 & 2a_1 & a_3 & a_2 \\ 0 & 0 & a_2 & 0 & a_3 \end{vmatrix}. \tag{10}$$

This expression, which generalizes to any value of m, indeed shows that the coefficients of $\psi(x)$ have the common denominator D if regarded as rational functions of the a_ν's.

Now we modify Newton's algorithm (2) to its new form[3]

[3] We learn from a note by J. S. Frame, *Remarks on a variation of Newton's method*, Amer. Math. Monthly, 52, 212–214 (1945), that precisely the same modification of Newton's algorithm has already been used since 1942 by H. Schwerdtfeger, of the University of Adelaide, South Australia, for the numerical solution of ordinary algebraic and transcendental equations.

$$x_{n+1} = x_n - f(x_n)\psi(x_n). \tag{11}$$

Setting

$$F(x) \equiv x - f(x)\psi(x) \tag{12}$$

we may write (11) as

$$x_{n+1} = F(x_n). \tag{11'}$$

On comparing (2) and its modification (11) we see that the division required by (2), at each step of the process, is not present in (11). Now we want to show that the algorithm (11′) also enjoys the property of (2) of producing fast convergence towards the zeros of $f(x)$. Indeed, let x be a root of (1),

$$f(x) = 0, \tag{13}$$

and let us expand $F(x_n)$ about the point $x_n = x$. Writing for convenience $f^{(\nu)}(x) = f^{(\nu)}$, $\psi^{(\nu)}(x) = \psi^{(\nu)}$, we have by Taylor's formula

$$f(x_n) = f + f'(x_n - x) + \tfrac{1}{2}f''(x_n - x)^2 + \cdots,$$
$$\psi(x_n) = \psi + \psi'(x_n - x) + \tfrac{1}{2}\psi''(x_n - x)^2 + \cdots,$$

hence

$$f(x_n)\psi(x_n) = f\psi + (f\psi' + f'\psi)(x_n - x) + \tfrac{1}{2}(f\psi'' + 2f'\psi' + f''\psi)(x_n - x)^2 + \cdots.$$

By (9) and (13) we have $f = 0$, $f'\psi = 1$ and therefore

$$F(x_n) - x = -\tfrac{1}{2}(2f'\psi' + f''\psi)(x_n - x)^2 + \cdots.$$

This shows that *we may write our relation* (11) *in the form*

$$x_{r+1} - x = b_2(x)(x_n - x)^2 + b_3(x)(x_n - x)^3 + \cdots + b_{2m-1}(x)(x_n - x)^{2m-1}, \tag{14}$$

where the $b_\nu(x)$ are polynomials in x with coefficients which are polynomials in a_ν divided by the common denominator D.

Again the missing linear term in $x_n - x$, on the right-hand side of (14), shows that if x_0 is sufficiently close to x, then the algorithm (11) will insure that $x_n \to x$ with quadratic convergence.

An important special case of (1) is the equation

$$ax^m - 1 = 0, \qquad (a \neq 0). \tag{15}$$

The identity

$$(-1)(ax^m - 1) + \frac{x}{m}(max^{m-1}) = 1$$

shows that in this case

$$\psi(x) = \frac{1}{m}x.$$

The relation (11) now becomes

$$x_{n+1} = x_n + \frac{1}{m}x_n(1 - ax_n^m). \tag{16}$$

In particular if $m = 1$, (15) reduces to

$$ax - 1 = 0 \tag{17}$$

when (16) becomes

$$x_{n+1} = x_n + x_n(1 - ax_n). \tag{18}$$

3. **The reciprocation of matrices.** The advantage of the modified, or division-free, Newton algorithm (11) appears in connection with matrix calculations. In recent years H. Hotelling has recommended the following procedure of finding the reciprocal $X = A^{-1}$ of a given numerical non-singular matrix

$$A = \|\alpha_{ij}\| \qquad (i, j = 1, \cdots, m). \tag{19}$$

Obtain in some way, e.g. by the so-called Gauss, or Doolittle, process a good approximation X_0 to A^{-1}. Then improve this approximation by the recurrence relation

$$X_{n+1} = X_n + X_n(I - AX_n). \tag{20}$$

In the case of $m = 1$ this relation is identical with (18).

In studying the convergence of X_n towards $X = A^{-1}$ Hotelling metrizes the space of real $m \times m$ matrices by means of the absolute value or norm

$$N(A) = \sqrt{\sum_{i,j} \alpha_{ij}^2} \tag{21}$$

which enjoys the following properties

$$N(A + B) \leqq N(A) + N(B), \qquad N(AB) \leqq N(A)N(B). \tag{22}$$

By means of these inequalities Hotelling derived an estimate of $N(X_n - X)$ which was improved by A. T. Lonseth as follows:[4]

Inequality of Hotelling and Lonseth. *Let X_0 be an approximation to $X = A^{-1}$ such that*

$$N(I - AX_0) = k < 1. \tag{23}$$

Starting with X_0 we obtain the sequence X_n by (20). *Then*

$$N(X_n - X) \leqq N(X_0)\cdot k^{2^n}\cdot(1 - k)^{-1}. \tag{24}$$

This interesting result shows in particular that the inequality (23) is sufficient to insure the convergence of the process.

Our generalizations (16) and (11) of the recurrence relation (18) suggest similar iterative procedures for the solution of non-linear algebraic matrix equations. We prefer, however, to pass on to a discussion of calculations with Laurent series.

4. **Calculations with Laurent series.** Let

$$a(z) = \sum_{-\infty}^{\infty} \alpha_n z^n, \qquad (r_1 < |z| < r_2), \tag{25}$$

be a Laurent series converging in the ring (5). There is no inherent restriction of the generality of the Problem formulated in our Introduction if we assume that the ring R contains the unit-circle $|z| = 1$, i.e.

[4] See Hotelling's second note already mentioned.

$$r_1 < 1 < r_2. \tag{26}$$

An advantage of this normalization is that it implies that $\alpha_n \to 0$ exponentially as $n \to +\infty$ or $n \to -\infty$, insuring that the sequence $\{\alpha_n\}$ is "finite" to a fixed number of decimal places.

The relation (25) sets up a one-one correspondence

$$a(z) \sim \{\alpha_n\}$$

between functions $a(z)$ uniform and regular in R and sequences $\{\alpha_n\}$ for which the series (25) converges in R. To the function $a(z) \equiv 1$ corresponds the unit-sequence

$$I: \quad \alpha_0 = 1, \quad \alpha_n = 0 \quad \text{if} \quad n \neq 0.$$

This correspondence may be interpreted as an isomorphism concerning the operations of addition, subtraction, multiplication and multiplication by a scalar. Indeed, if

$$b(z) = \sum_{-\infty}^{\infty} \beta_n z^n, \qquad (r_1 < z < r_2), \tag{27}$$

is a second series then we find, on multiplying (25) and (27) that to the product

$$c(z) = a(z)b(z) \tag{28}$$

corresponds the series

$$c(z) = \sum_{-\infty}^{\infty} \gamma_n z^n \tag{29}$$

where

$$\gamma_n = \sum_{\nu=-\infty}^{\infty} \alpha_{n-\nu}\beta_\nu. \tag{30}$$

Thus to the operation (28) of multiplication of the functions a, b, corresponds the operation of *convolution* (30) of the two sequences $\{\alpha_n\}$, $\{\beta_n\}$, an operation which we write as

$$\gamma = \alpha\beta. \tag{31}$$

We mention incidentally a third interpretation of Laurent series isomorphic to the two already discussed. Indeed, consider the (4-way) infinite matrix

$$\|\alpha_{j-i}\| \tag{32}$$

in which α_{j-i} is the element in the ith row and jth column, both i and j assuming all integral values. Such matrices may be designated "striped" for the reason that all elements lying on a line, sloping down at a 45° angle, are identical. To every sequence $\{\alpha_n\}$ corresponds one such matrix, and conversely. The isomorphism between such matrices and sequences becomes evident if we remark that the multiplication of two striped matrices $\|\alpha_{j-i}\|$, $\|\beta_{j-i}\|$, is another striped matrix $\|\gamma_{j-i}\|$, where the sequence $\{\gamma_n\}$ is given by (30). This remark throws some light on the connection between Laurent series and the case of finite matrices of §3.

We finally define the *norm* of the function $a(z)$, or of the sequence $\{\alpha_n\}$, as the non-negative number

$$N(a) = N(\alpha) = \sum_{-\infty}^{\infty} | \alpha_n |. \tag{33}$$

This norm also enjoys the two properties (22) or

$$N(a + b) \leqq N(a) + N(b), \qquad N(ab) \leqq N(a)N(b). \tag{34}$$

Their verification is immediate in this case.

We are now able to attack the general Problem of section 1. However, it is essential to discuss first the important case $m = 1$ of reciprocation.

4.1. The reciprocation of Laurent series. With the norm of a Laurent series as defined by (33), the result of Hotelling and Lonseth (section 3) applies to Laurent series without any change. Assuming that the sum $a(z)$ of the given Laurent series (25) does not vanish in R, we are to find the expansion

$$w(z) = \frac{1}{a(z)} = \sum_{-\infty}^{\infty} \omega_n z^n. \tag{35}$$

Let

$$w_0(z) = \sum_{-\infty}^{\infty} \omega_n^{(0)} z^n \tag{36}$$

be an approximation to (35) such that

$$N(1 - aw_0) = N(I - \alpha\omega^{(0)}) = k < 1. \tag{37}$$

The very important problem of how such approximations may be obtained will be discussed later (section 4.3). This starting sequence is now to be improved by the relation

$$\omega^{(n+1)} = \omega^{(n)} + \omega^{(n)}(I - \alpha\omega^{(n)}). \tag{38}$$

The rapid convergence is assured by the Hotelling-Lonseth inequality

$$N(\omega^{(n)} - \omega) \leqq N(\omega^{(0)}) \cdot k^{2^n} \cdot (1 - k)^{-1}. \tag{39}$$

Pending a discussion of procedures for obtaining the first approximation, we may therefore regard the numerical problem of reciprocation as solved. This implies that we may perform all *four* rational operations on Laurent series and that we may thus find the Laurent expansion of any *rational* function of Laurent series.

4.2. The general algebraic case. We turn now to the general case of the equation (4) with the two additional, and as we have seen, unessential restrictions that our ring R contains the unit-circle $|z| = 1$ and that the solution $w = w(z)$, of (4), be uniform in R. The problem is to find the numerical values of the coefficients of the expansion

$$w(z) = \sum_{-\infty}^{\infty} \omega_n z^n. \tag{40}$$

We return to our discussion (section 2) of the division-free Newton algorithm (11), especially in its expanded form (14). This discussion remains valid if applied to (4) rather than (1). The algorithm is in this case

$$w_{n+1} = w_n - f(w_n, z)\psi(w_n), \tag{41}$$

the expanded form of which is

$$w_{n+1} - w = b_2(w)(w_n - w)^2 + \cdots + b_{2m-1}(w)(w_n - w)^{2m-1}. \tag{42}$$

We have to remember, however, that $\psi(w_n)$ is a polynomial in w_n with coefficients which are polynomials in a_ν divided by $D = D(z)$. Since D is a polynomial in the a_ν, we may first derive its Laurent expansion by additions and multiplications from the given Laurent series of the coefficients $a_\nu(z)$ of (4). Secondly, since

$$D(z) \neq 0 \quad \text{in} \quad R,$$

we may also find by the method of section 4.1 the Laurent expansion of $1/D(z)$. In this way we arrive at the Laurent expansions of the coefficients of $\psi(w_n)$. These preliminary Laurent series operations allow to put the relation (41) in the form

$$w_{n+1} = w_n + (c_0(z) + c_1(z)w_n + \cdots + c_{2m-1}(z)w_n^{2m-1}), \tag{43}$$

where

$$c_\mu(z) = \sum_\nu \gamma_{\mu\nu} z^\nu \tag{44}$$

are numerically known Laurent expansions.

Now let

$$w_0(z) = \sum_{-\infty}^{\infty} \omega_\nu^{(0)} z^\nu \tag{45}$$

be an approximation to (40), (See section 4.3.) Starting from this approximation we obtain the successive series

$$w_n(z) = \sum_{-\infty}^{\infty} \omega_\nu^{(n)} z^\nu \tag{46}$$

by means of (43). This operation of deriving w_{n+1} from w_n is of course to be performed on the corresponding sequences of coefficients. By (43), (44), (46), the operation takes the form

$$\omega^{(n+1)} = \omega^{(n)} + (\gamma_0 + \gamma_1\omega^{(n)} + \cdots + \gamma_{2m-1}(\omega^{(n)})^{2m-1}). \tag{47}$$

Will the expansion (46) converge towards the expansion (40) of the solution? To answer this question we return to the form (42) of our relation. Taking the norms of both sides of (42) and using the properties (34) of the norm, we obtain

$$N(w_{n+1} - w) \leqq N(b_2(w))[N(w_n - w)]^2 + \cdots + N(b_{2m-1}(w))[N(w_n - w)]^{2m-1}. \tag{48}$$

This relation shows that if

$$N(w_0 - w) = \sum_{-\infty}^{\infty} |\omega_\nu^{(0)} - \omega_\nu| \tag{49}$$

is sufficiently small then (48) will indeed imply

$$\lim_{n\to\infty} N(w_n - w) = 0 \tag{50}$$

with quadratic convergence.

4.3. Derivation of an approximate Laurent expansion. The method for computation with Laurent series described in the previous sections will now become effective provided we can solve the following problem.

INITIAL APPROXIMATION PROBLEM. *Let*

$$F(z) = \sum_{-\infty}^{\infty} c_\nu z^\nu \tag{51}$$

be regular in the ring R containing the circle $|z| = 1$. *This function* $F(z)$, *whose Laurent coefficients* c_ν *are unknown, is defined by an algebraic equation which allows us to compute the value of* $F(z)$ *for any given z of R, in particular for any root of unity. We are to describe a practical method whereby, given* $\epsilon > 0$, *we may compute the coefficients* c_ν^* *of a Laurent series*

$$F^*(z) = \sum_{-\infty}^{\infty} c_\nu^* z^\nu \tag{52}$$

regular in R, such that

$$N(F - F^*) = \sum_{-\infty}^{\infty} | c_\nu - c_\nu^* | < \epsilon. \tag{53}$$

We shall now solve this problem by the method of trigonometric interpolation.[5] Let m be a positive integer and let

$$z_\mu = e^{2\pi i\mu/m}, \qquad (\mu = 0, 1, \cdots, m-1), \tag{54}$$

be the mth roots of unity. These roots of unity satisfy the following orthogonality relations

$$\frac{1}{m}\sum_{\mu=0}^{m-1} z_\mu^\nu \bar{z}_\mu^s = \begin{cases} 1 & \text{if} \quad \nu \equiv s \pmod{m} \\ 0 & \text{if} \quad \nu \not\equiv s \pmod{m}. \end{cases} \tag{55}$$

If m is odd, $m = 2n+1$, we consider the Laurent polynomial

$$F_m(z) = \sum_{\nu=-n}^{n} c_{m,\nu} z^\nu \tag{56}$$

having m arbitrary coefficients.

If m is even, $m = 2n$, we define our polynomial so as to contain again only m arbitrary coefficients as

$$F_m(z) = \sum_{\nu=-(n-1)}^{n-1} c_{m,\nu} z^\nu + \tfrac{1}{2} c_{m,n}(z^n + z^{-n}). \tag{57}$$

Whether m is even or odd we may always write

$$F_m(z) = \sum_{\nu=-n}^{n}{}' c_{m,\nu} z^\nu, \qquad \left(n = \left[\frac{m}{2}\right]\right), \tag{58}$$

[5] Concerning the subject of trigonometric interpolation we refer to the classical memoir by Ch. J. de la Vallée Poussin, *Sur la convergence des formules d'interpolation entre ordonées equidistantes*, Bulletin de l'Academie royale de Belgique, 319–410 (1908), and to Dunham Jackson, *The theory of approximation*, American Mathematical Society Colloquium Publications, vol. 11, New York, 1930, chap. IV.

where the summation symbol $\sum'$ is to indicate that if m is even, then

$$c_{m,-n} = c_{m,n}, \tag{59}$$

and that the terms of (58) for $\nu = \pm n$ are to be taken with half their value. The relation $n = [m/2]$ is to indicate that n is the greatest integer not exceeding $m/2$.

We shall now require the Laurent polynomial (58) to interpolate the function (51) in the points (54). This gives the m equations

$$\sum_{\nu=-n}^{n}{}' c_{m,\nu} z_\mu^\nu = F(z_\mu), \qquad (\mu = 0, 1, \cdots, m-1). \tag{60}$$

On multiplying (60) by $\bar{z}_\mu^s/m$ (s fixed, $-n \leqq s \leqq n$) we find in all cases, in view of (55) after summation by μ, that

$$c_{m,s} = \frac{1}{m} \sum_{\mu=0}^{m-1} F(z_\mu)\bar{z}_\mu^s, \qquad (-n \leqq s \leqq n). \tag{61}$$

The construction of our approximate Laurent expansion (58), i.e. (52), has now been completed. The following theorem will now show that the condition (53) may also be realized by the present method of construction.

THEOREM. *We assume the Laurent series*

$$F(z) = \sum_{\nu=-\infty}^{\infty} c_\nu z^\nu \tag{62}$$

to converge absolutely on the unit circle $|z| = 1$, *i.e.*

$$\sum_{-\infty}^{\infty} |c_\nu| < \infty.\text{[6]} \tag{63}$$

Then our interpolating Laurent polynomial (58) *satisfies the condition*

$$\lim_{m\to\infty} N(F - F_m) = 0. \tag{64}$$

Remark. Notice that the regularity of $F(x)$ in a ring containing $|z| = 1$ implies our condition (63) but not conversely. This remark is of importance concerning calculations with absolutely convergent Fourier series. (See section 5.)

Proof. Let N be a positive integer. We shall restrict ourselves to values of $m \geqq 2N$, hence $n \geqq N$. We may then write

$$N(F - F_m) \leqq \sum_{\nu=-N+1}^{N-1} |c_\nu - c_{m,\nu}| + \sum_{N \leqq |s| \leqq n}{}' |c_{m,s}| + \sum_{|\nu| \geqq N} |c_\nu|. \tag{65}$$

We shall now estimate the three sums on the right-hand side.

[6] See Dunham Jackson, loc. cit., for other conditions insuring the convergence of trigonometric interpolation.

Let $\epsilon>0$ be given. In view of our assumption (63) we may choose N such that

$$\sum_{|\nu|\geqq N} |c_\nu| < \epsilon. \tag{66}$$

An upper bound for the second sum of (65) may now be obtained as follows. By (61) and (62) we have

$$c_{m,s} = \frac{1}{m}\sum_{\mu=0}^{m-1} \bar{z}_\mu^s \sum_{\nu=-\infty}^{\infty} c_\nu z_\mu^\nu = \sum_{\nu=-\infty}^{\infty} c_\nu \frac{1}{m}\sum_{\mu=0}^{m-1} z_\mu^\nu \bar{z}_\mu^s,$$

and finally by (55)

$$c_{m,s} = \sum_{\nu\equiv s \pmod m} c_\nu. \tag{67}$$

For all $m\geqq 2N$, whether m is even or odd, we now have

$$\sum_{N\leqq|s|\leqq n}' |c_{m,s}| \leqq \sum_{N\leqq|s|\leqq n}' \sum_{\nu\equiv s \pmod m} |c_\nu| \leqq \sum_{|\nu|\geqq N} |c_\nu| < \epsilon \tag{68}$$

by (66). We now return to (61). Since (62) converges uniformly on $|z|=1$, $F(z)$ is continuous on $|z|=1$ and (62) is its Fourier series. We therefore have the Fourier-Cauchy relations

$$c_\nu = \frac{1}{2\pi i}\int_{|z|=1} F(z)z^{-\nu-1}dz. \tag{69}$$

From the definition of this integral as a limit of Cauchy sums we may now write (with $z_m=1$)

$$c_\nu = \lim_{m\to\infty} \frac{1}{2\pi i}\sum_{\mu=0}^{m-1} F(z_\mu)z_\mu^{-\nu-1}(z_{\mu+1}-z_\mu),$$

$$c_\nu = \lim_{m\to\infty} \frac{1}{2\pi i}\sum_{\mu=0}^{m-1} F(z_\mu)z_\mu^{-\nu}(e^{2\pi i/m}-1),$$

and finally, by (61),

$$c_\nu = \lim_{m\to\infty} \frac{1}{m}\sum_{\mu=0}^{m-1} F(z_\mu)z_\mu^{-\nu} = \lim_{m\to\infty} c_{m\,\nu}. \tag{70}$$

We now have indeed

$$\sum_{\nu=-N+1}^{N-1} |c_\nu - c_{m,\nu}| < \epsilon, \quad \text{provided} \quad m > m_0(\epsilon). \tag{71}$$

By (65), (66), (68), and (71) we now have

$$N(F-F_m) < 3\epsilon, \quad \text{provided} \quad m > m_0(\epsilon), \tag{72}$$

and our theorem is established.

4.31. The 24-ordinate scheme of numerical harmonic analysis. The interpolation of our given function $F(z)$ in the 24th roots of unity will provide satisfactory approximation for most ordinary purposes. Let us assume for definiteness that $F(z)$ is *real* for real z. On the unit-circle $z=e^{i\theta}$ we then have

$$F(e^{i\theta}) = R(\theta) + iI(\theta), \tag{73}$$

where the real part $R(\theta)$ is an *even* function, while the imaginary part $I(\theta)$ is *odd*. Denote by

$$F_\mu = R_\mu + iI_\mu, \qquad (\mu = 0, 1, \cdots, 23), \tag{74}$$

the computed values of our function at 15°—intervals in θ, i.e. for the points (54) with $m=24$. We now interpolate the 13 ordinates R_μ ($\mu=0, 1, \cdots, 12$) by a cosine polynomial

$$A_0 + A_1 \cos\theta + \cdots + A_{11}\cos 11\theta + A_{12}\cos 12\theta, \tag{75}$$

and the 13 ordinates $I_0=0, I_1, \cdots, I_{11}, I_{12}=0$ by a sine polynomial

$$B_1 \sin\theta + \cdots + B_{11}\sin 11\theta. \tag{76}$$

These polynomials are readily obtained by the 24-ordinate scheme as described in E. T. Whittaker and G. Robinson, *The calculus of observations*, ed. 3, 1940, section 137, pp. 273–278. The complex function (73) is now interpolated in the 24 points by the trigonometric polynomial

$$F_{24}(e^{i\theta}) = A_0 + A_1\cos\theta + \cdots + A_{11}\cos 11\theta + A_{12}\cos 12\theta$$
$$+ iB_1 \sin\theta + \cdots + iB_{11}\sin 11\theta.$$

Setting

$$z = e^{i\theta}, \qquad \cos\nu\theta = \tfrac{1}{2}(z^\nu + z^{-\nu}), \qquad i\sin\nu\theta = \tfrac{1}{2}(z^\nu - z^{-\nu}), \tag{77}$$

we obtain the Laurent sum with real coefficients

$$F_{24}(z) = A_0 + \sum_{\nu=1}^{11} \tfrac{1}{2}(A_\nu + B_\nu)z^\nu + \tfrac{1}{2}A_{12}z^{12}$$
$$+ \sum_{\nu=1}^{11} \tfrac{1}{2}(A_\nu - B_\nu)z^{-\nu} + \tfrac{1}{2}A_{12}z^{-12}. \tag{78}$$

This initial approximate Laurent expansion will be used in section 6 in our example of reciprocation of a Laurent series.

5. Calculation with Fourier series. The method of calculation with Laurent series described in sections 4, 4.1, 4.2, 4.3 and 4.31, applies unchanged to the realm of absolutely convergent Fourier series written in the complex form

$$F(z) = \sum_{-\infty}^{\infty} c_\nu z^\nu, \quad \text{where} \quad z = e^{i\theta},$$

with the definition of the norm as

$$N(F) = \sum_{-\infty}^{\infty} |c_\nu|.$$

The general problem of section 1 may now be reformulated, replacing the ring R by the unit circle $|z|=1$. The coefficients $a_\nu(z)$ of the equation (4) are now defined by given absolutely convergent Fourier series. The conditions (6) and (7) remain unchanged. The important fact that a uniform, continuous solution $w=w(z)$ of (4) along the unit-circle admits of an *absolutely* convergent Fourier expansion is now assured by a general theorem of N. Wiener and P. Lévy.[7] The effectiveness of the interpolation method of section 4.3 for obtaining a satisfactory initial approximate Fourier series is secured by our theorem of section 4.3.

We finally mention briefly the special problem of the reciprocation of a non-vanishing absolutely convergent Fourier series

$$A(\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \tag{79}$$

with *real* coefficients a_ν, b_ν. In applying our method, we have to pass to the complex variable z, by means of the relations (77), obtaining the series

$$A(\theta) = a(z) = \sum_{-\infty}^{\infty} \alpha_n z^n, \tag{80}$$

which is now to be reciprocated. The coefficients α_n being complex, it would appear that computations with complex numbers are unavoidable. This, however, is not the case since we may proceed as follows. Working with the real series

$$f(z) = \sum_{-\infty}^{\infty} R(\alpha_n)z^n, \qquad g(z) = \sum_{-\infty}^{\infty} I(\alpha_n)z^n, \tag{81}$$

we have by (80)

$$\frac{1}{a(z)} = \frac{1}{f+ig} = \frac{f}{f^2+g^2} - i\,\frac{g}{f^2+g^2} = \sum_{-\infty}^{\infty} \omega_n z^n. \tag{82}$$

Starting with (81) and operating with *real* series only, we now form the expansions of f^2, g^2 and then f^2+g^2. The real "Laurent" series of f^2+g^2 is now reciprocated by the method of section 4.1 and finally the series for

$$f/(f^2+g^2), \qquad g/(f^2+g^2)$$

obtained by multiplications. This furnishes the complex values of the ω_n of (82). Returning to the variable θ, by (77) we finally obtain the ordinary real Fourier expansion of

$$1/A(\theta)$$

6. An example of reciprocation of a Laurent series. Our numerical example will benefit by the following general remark concerning the modified Newton algorithm (11). For simplicity we limit ourselves to the case of the equation (15) or

$$ax^m - 1 = 0 \tag{83}$$

which is solved by the recurrent relation

[7] See Antoni Zygmund, *Trigonometrical series*, Warszawa-Lwów, 1935, pp. 140–142.

$$x_{n+1} = x_n + \frac{1}{m} x_n(1 - a x_n^m). \tag{84}$$

Let us assume that our first approximation x_0 is of such accuracy that x_2 will have all the accuracy we want, while x_1 does not quite do. More precisely we assume the "residual"

$$r = 1 - a x_0^m \tag{85}$$

so small that we may neglect r^3 everywhere in our calculations. We may use this fact in eliminating x_1 between the two equations

$$\begin{aligned} x_1 &= x_0 + \frac{1}{m} x_0(1 - x_0^m), \\ x_2 &= x_1 + \frac{1}{m} x_1(1 - x_1^m). \end{aligned} \tag{86}$$

Indeed, by (85), (86), we have

$$x_1 = x_0\left(1 + \frac{1}{m} r\right)$$

and neglecting r^3 we find

$$x_1^m = x_0^m\left(1 + r + \frac{m-1}{2m} r^2\right).$$

If we then compute x_2 in this way, i.e., neglecting r^3 wherever it appears, we easily find the following approximation to x_2:

$$x_2' = x_0 + \frac{1}{m} x_0\left(r + \frac{m+1}{2m} r^2\right). \tag{87}$$

We may interpret both equations (85), (87) as a recurrence relation furnishing x_2' in terms of the first approximation x_0. This process converges "cubically." Indeed, a simple calculation will show that we may write (87) as

$$x_2' - x = \frac{(m+1)(2m+1)}{6x^2}(x_0 - x)^3 + (\text{terms of order} > 3).$$

We note especially the following special case: To solve

$$ax - 1 = 0 \tag{88}$$

we set

$$r = 1 - a x_0 \tag{89}$$

and compute

$$x_2' = x_0 + x_0(r + r^2). \tag{90}$$

We turn now to our example which consists in expanding the reciprocal of the Bessel function

$$J_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{(2\cdot 4)^2} - \frac{z^6}{(2\cdot 4\cdot 6)^2} + \cdots \tag{91}$$

into a Laurent series between the first two positive roots of this function, which are approximately $\xi_1 = 2.4$, $\xi_2 = 5.5$. In order to avoid even exponents we consider

$$J_0(\sqrt{z}) = 1 - \frac{z}{2^2} + \frac{z^2}{(2\cdot 4)^2} - \cdots \tag{92}$$

whose reciprocal is to be expanded in Laurent series between its zeros

$$\xi_1^2 = 5.76 \quad \text{and} \quad \xi_2^2 = 30.25.$$

Let us notice that 13 is near the geometric mean of these numbers. In order to realize the condition (26) we replace in (92) z by $13z$, also changing the sign of the function for formal reasons. Thus let

$$a(z) = -J_0(\sqrt{13z}) = \sum_{n=0}^{\infty} \alpha_n z^n \tag{93}$$

be the entire function whose reciprocal

$$w(z) = -\frac{1}{J_0(\sqrt{13z})} = \sum_{-\infty}^{\infty} \omega_n z^n \tag{94}$$

we are to expand in a Laurent series convergent on and near the unit circle $|z| = 1$. Below are the 10-place values of the coefficients α_n of (93) as computed by

$$\alpha_n = (-1)^{n+1}(13)^n/(2^n\cdot n!)^2.$$

n	α_n
0	−1.00000 00000
1	3.25000 00000
2	−2.64062 50000
3	.95355 90278
4	−.19369 16775
5	.02517 99181
6	−.00227 31870
7	.00015 07726
8	−.00000 76564
9	.00000 03072
10	−.00000 00100
11	.00000 00003
$\sum$	= .39229 24951

μ	R_μ	I_μ	A_μ	B_μ
0	2.549 122	.000 000	.601 975	
1	2.262 721	−.257 032	1.063 727	−.361 583
2	1.655 481	−.395 409	.489 993	−.144 072
3	1.081 379	−.427 381	.219 762	−.062 309
4	.661 333	−.405 462	.097 293	−.028 189
5	.379 302	−.361 625	.042 831	−.012 990
6	.193 500	−.309 968	.018 815	−.006 018
7	.070 603	−.256 308	.008 261	−.002 786
8	−.011 124	−.202 973	.003 630	−.001 285
9	−.065 027	−.150 752	.001 604	−.000 588
10	−.099 093	−.099 722	.000 727	−.000 260
11	−.117 947	−.049 615	.000 368	−.000 099
12	−.123 985	.000 000	.000 136	

From these values, rounded to 6 places, we computed to 6 places the values of $a(z_\mu)$ at the 24th roots of unity

$$z_\mu = \cos(15\mu)^\circ + i\sin(15\mu)^\circ, \qquad (\mu = 0, 1, \cdots, 12),$$

and from these the values of the reciprocal

$$w(z_\mu) = 1/a(z_\mu) = R_\mu + iI_\mu, \qquad (\mu = 0, 1, \cdots, 12),$$

which are tabulated above. The coefficients A_μ and B_μ of the interpolating cosine and sine polynomials (75), (76) where then found by the 24-ordinate scheme. They are tabulated above.

From these values we computed the coefficients $\omega_n^{(0)}$ of the approximation

$$w_0(z) = \sum_{-12}^{12} \omega_n^{(0)} z^n$$

according to (78) by the formulae

$$\begin{aligned} \omega_0^{(0)} &= A_0, \\ \omega_n^{(0)} &= \tfrac{1}{2}(A_n + B_n), \\ \omega_{-n}^{(0)} &= \tfrac{1}{2}(A_n - B_n), \\ \omega_{12}^{(0)} &= \omega_{-12}^{(0)} = \tfrac{1}{2}A_{12}, \qquad (n = 1, 2, \cdots, 11). \end{aligned}$$

These values rounded to 5 places are in the first column of the following table which contains the complete computation according to the relations (89) and (90). The last column headed $\omega = \omega^{(0)} + \omega^{(0)}(r + r^2)$ gives the 9-place values of the coefficients ω_n of (94).

Remarks. 1. The basic numerical process in this computation is obviously the convolution of sequences. Thus the second column $\alpha\omega^{(0)}$ is obtained by the convolution of the column α with the column $\omega^{(0)}$. According to the formula (30) this is done very simply rewriting the column α, say, in reverse order, then matching it with the column $\omega^{(0)}$ such that the zero term of one column corresponds to the nth term of the other. The accumulated products of matching elements gives the nth term of the product column $\alpha\omega^{(0)}$. This operation is very familiar from the process of smoothing by means of a linear compound formula.

2. The operation of convolution of sequences implies an important check by means of their sums, for it is clear that the sum of the product column should equal the product of the sums of the factor sequence, except for the accumulated rounding error. At the very bottom of each column we wrote the actual sum of the sequence in that column. Directly below it we wrote (in parentheses) the value of this sum in terms of the sums of the columns which enter into its composition.

3. The column of final residuals $I - \alpha\omega$ was also computed (values not recorded here) and its terms were found to be so small that a further repetition of the process, with our 10-place values of the α_n, would not alter our 9-place values of the ω_n. As final checks we found by (93)

$$a(1)^{-1} = 2.54911\ 8356, \qquad a(-1)^{-1} = -.12398\ 5065,$$
$$a(i)^{-1} = .19349\ 9936 - .30996\ 7383\ i.$$

The corresponding values of $w(z)$, computed by (94), were found to be

$$w(1) = 2.54911\ 8355, \qquad w(-1) = -.12398\ 5067,$$
$$w(i) = .19349\ 9940 - .30996\ 7383\ i.$$

n	$\omega^{(0)}$	$\alpha\omega^{(0)}$	$r=I-\alpha\omega^{(0)}$	r^2
−29				
−28				
−27				
−26				
−25				
−24				49
−23				4
−22				−102
−21				64
−20				18
−19				− 64
−18				85
−17				− 65
−16				9
−15				49
−14				− 80
−13				62
−12	.00007	−.00007 00000	.00007 00000	− 7
−11	.00023	−.00000 25000	.00000 25000	− 68
−10	.00049	.00007 26562	−.00007 26562	97
− 9	.00110	−.00004 80946	.00004 80946	− 42
− 8	.00246	.00002 68539	−.00002 68539	− 35
− 7	.00552	−.00000 52301	.00000 52301	61
− 6	.01242	−.00001 62992	.00001 62992	− 27
− 5	.02791	.00002 32702	−.00002 32702	− 17
− 4	.06274	−.00001 52800	.00001 52800	24
− 3	.14104	−.00000 13045	.00000 13045	− 14
− 2	.31703	.00001 89326	−.00001 89326	7
− 1	.71266	−.00002 53421	.00002 53421	− 13
0	.60198	1.00002 07580	−.00002 07580	−.00000 00056
1	.35107	.00000 39064	−.00000 39064	− 70
2	.17296	−.00001 57627	.00001 57627	237
3	.07873	.00000 43626	−.00000 43626	− 89
4	.03455	.00001 22969	−.00001 22969	−116
5	.01492	−.00001 61378	.00001 61378	174
6	.00640	.00000 57617	−.00000 57617	−145
7	.00274	.00000 04662	−.00000 04662	67
8	.00117	.00000 78363	−.00000 78363	30
9	.00051	−.00001 53488	.00001 53488	− 91
10	.00023	.00001 88876	−.00001 88876	91
11	.00013	−.00001 23781	.00001 23781	− 37
12	.00007	.00006 13042	−.00006 13042	− 29
13		.00002 88471	−.00002 88471	81
14		−.00009 48797	.00009 48797	− 75
15		.00004 63585	−.00004 63585	3
16		−.00001 07391	.00001 07391	53
17		.00000 14982	−.00000 14982	− 39
18		−.00000 01411	.00000 01411	− 13
19		.00000 00096	−.00000 00096	37
20		−.00000 00005	.00000 00005	− 11
21				− 11
22				0
23				32
24				− 21
25				80
26				−124
27				5
28				103
29				− 92
30				43
31				− 13
32				3
$\sum$	2.54913	1.00000 45679 (1.00000 45680)	−.00000 45679	.00000 00002 (.00000 00000)

n	$r+r^2$	$\omega^0(r+r^2)$	$\omega=\omega^{(0)}+\omega^{(0)}(r+r^2)$
−29		1	
−28		2	
−27		5⁺	.00000 0001
−26		12	.00000 0001
−25		26	.00000 0003
−24	49	58	.00000 0006
−23	4	130	.00000 0013
−22	−102	292	.00000 0029
−21	64	657	.00000 0066
−20	18	1476	.00000 0148
−19	− 64	3318	.00000 0332
−18	85	7459	.00000 0746
−17	− 65	16767	.00000 1677
−16	9	37691	.00000 3769
−15	49	84725⁻	.00000 8472
−14	− 80	1 90454	.00001 9045
−13	62	4 28120	.00004 2812
−12	6 99993	2 62369	.00009 6237
−11	24932	−1 36696	.00021 6330
−10	−7 26465	− 37118	.00048 6288
− 9	4 80904	− 68748	.00109 3125
− 8	−2 68574	− 27684	.00245 7232
− 7	52362	36008	.00552 3601
− 6	1 62965	− 35203	.01241 6480
− 5	−2 32719	9539	.02791 0954
− 4	1 52824	9186	.06274 0919
− 3	13031	− 49463	.14103 5054
− 2	−1 89319	21184	.31703 2118
− 1	2 53408	− 48016	.71265 5198
0	−.00002 07636	−.00000 53032	.60197 4697
1	− 39134	23489	.35107 2349
2	1 57864	7594	.17296 0759
3	− 43715	− 34341	.07872 6566
4	−1 23085	22715⁻	.03455 2271
5	1 61552	4594	.01492 0459
6	− 57762	− 20946	.00639 7905
7	− 4595	− 47102	.00273 5290
8	− 78333	− 20304	.00116 7970
9	1 53397	−1 15311	.00049 8469
10	−1 88785	−1 73068	.00021 2693
11	1 23744	−3 92531	.00009 0747
12	−6 13071	−3 12836	.00003 8716
13	−2 88390	1 65178	.00001 6518
14	9 48722	70470	.00000 7047
15	−4 63582	30065⁻	.00000 3006
16	1 07444	12827	.00000 1283
17	− 15021	5472	.00000 0547
18	1398	2335⁻	.00000 0233
19	− 59	996	.00000 0100
20	− 6	425⁺	.00000 0043
21	− 11	182	.00000 0018
22	0	78	.00000 0008
23	32	33	.00000 0003
24	− 21	14	.00000 0001
25	80	6	.00000 0001
26	−124	2	
27	5	1	
28	103	1	
29	− 92		
30	43		
31	− 13		
32	3		
$\sum$	−.00000 45677	−.00001 16443 (−.00001 14637)	2.54911 8355

THE PROPAGATION OF WAVES IN ORTHOTROPIC MEDIA*

BY

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1. Introduction. The present article is an extension of a previous paper dealing with the elasticity problems of orthotropic media.[1] Here, the displacement potentials which define the dynamic phenomena in such media are discussed.

2. The dynamic problem. Hooke's law for an orthotropic medium may be written in the form

$$\begin{aligned} \sigma_x &= b_{11}e_x + b_{12}e_y + b_{13}e_z, \cdots, \\ \tau_{xy} &= b_{66}\gamma_{xy}, \cdots, \end{aligned} \tag{1}$$

wherein we use the conventional notation for the stresses and the strains. If we limit ourselves to a consideration of those materials which are isotropic in the y, z plane[2] and for which

$$b_{66} = b_{55} = (b_{11}b_{22} - b_{12}^2)/(b_{11} + b_{22} + 2b_{12}), \tag{1a}$$

the number of independent elastic constants is reduced to four and the dynamic elasticity problem may be easily treated.[3] We utilize the familiar equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho X - \rho \frac{\partial^2 u}{\partial t^2} = 0, \cdots, \tag{2}$$

and define the displacements in terms of potentials as

$$\begin{aligned} e_x &= \frac{\partial u}{\partial x} = \frac{\partial^2 \phi_1}{\partial x^2}, \cdots, \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^2}{\partial x \partial y}(\phi_1 + \phi_2), \cdots. \end{aligned} \tag{3}$$

When Eqs. (1), (2), and (3) are combined, we obtain

$$\frac{\partial}{\partial x}\left\{\left[a\frac{\partial^2}{\partial x^2} + \alpha\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{\partial^2}{\partial t^2}\right]\phi_1 + \beta\left(\frac{\partial^2 \phi_2}{\partial y^2} + \frac{\partial^2 \phi_3}{\partial z^2}\right)\right\} + X = 0, \tag{4a}$$

$$\frac{\partial}{\partial y}\left\{\beta\frac{\partial^2 \phi_1}{\partial x^2} + \left(\alpha\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial y^2} + \gamma\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)\phi_2 + \delta\frac{\partial^2 \phi_3}{\partial z^2}\right\} + Y = 0, \tag{4b}$$

$$\frac{\partial}{\partial z}\left\{\beta\frac{\partial^2 \phi_1}{\partial x^2} + \delta\frac{\partial^2 \phi_2}{\partial y^2} + \left(\alpha\frac{\partial^2}{\partial x^2} + \gamma\frac{\partial^2}{\partial y^2} + b\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)\phi_3\right\} + Z = 0, \tag{4c}$$

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[1] G. F. Carrier, *The thermal stress and body force problems of the infinite orthotropic solid*, Quart. Appl. Math. **2**, 31–36 (1944).

[2] The isotropy implies that $b_{22}=b_{33}$, $b_{12}=b_{13}$, $b_{44}=(b_{22}-b_{23})/2$.

[3] These conditions are imposed in order that the roots of Eq. (8) will appear in a useful form. They include isotropic media as a special case.

where a, b, α, γ, β^2, and δ, are given respectively by b_{11}/ρ, b_{22}/ρ, b_{66}/ρ, $(b_{22}-b_{33})/2\rho$, $(a-\alpha)(b-\alpha)$, and $b-\gamma$.

If we now consider the homogeneous equations (i.e., vanishing body forces) and require the ϕ_i to vanish at infinity,[4] we may begin the integration by removing the leftmost derivative of each equation, multiplying each term in the remaining forms by $\exp[-i(x\xi+y\eta+z\zeta)]$, and integrating the equations so obtained over the infinite region. We find[5]

$$\left[a\xi^2 + \alpha(\eta^2+\zeta^2) + \frac{\partial^2}{\partial t^2}\right]\psi_1 + \beta\eta^2\psi_2 + \beta\zeta^2\psi_3 = 0 \tag{5}$$

and two similar equations. Here

$$\psi_j = \iiint_{-\infty}^{\infty} \phi_j(x, y, z, t) \exp\left[-i(x\xi + y\eta + z\zeta)\right]dxdydz. \tag{6}$$

We now have three ordinary linear differential equations (in t) the solutions to which are of the form

$$\psi_j = A_j(\xi, \eta, \zeta) \cos \omega t. \tag{7}$$

In order that these solutions be non-trivial, the determinant of coefficients of Eqs. (5) wherein $\partial^2/\partial t^2$ has been replaced by $-\omega^2$ must vanish, that is

$$\begin{vmatrix} a\xi^2 + \alpha(\eta^2+\zeta^2) - \omega^2, & \beta\eta^2, & \beta\zeta^2 \\ \beta\xi^2, & \alpha\xi^2 + b\eta^2 + \gamma\zeta^2 - \omega^2, & \delta\zeta^2 \\ \beta\xi^2, & \delta\eta^2, & \alpha\xi^2 + \gamma\eta^2 + b\zeta^2 - \omega^2 \end{vmatrix} = 0. \tag{8}$$

The three roots of this equation are easily shown to be

$$\omega_1^2 = a\xi^2 + b(\eta^2 + \zeta^2), \tag{9}$$

$$\omega_2^2 = \alpha\xi^2 + \gamma(\eta^2 + \zeta^2), \tag{10}$$

$$\omega_3^2 = \alpha(\xi^2 + \eta^2 + \zeta^2). \tag{11}$$

Corresponding to these roots, the A_{kj} must respectively obey the relations[6]

$$A_{11}:A_{12}:A_{13} = a - \alpha:\beta:\beta, \tag{9a}$$

$$A_{21}:A_{22}:A_{23} = 0:\beta\zeta^2:-\beta\eta^2, \tag{10a}$$

$$A_{31}:A_{32}:A_{33} = (b-\alpha)(\eta^2+\zeta^2):-\beta\xi^2:-\beta\xi^2. \tag{11a}$$

Because of Eq. (6) it is evident that

$$\begin{aligned}\phi_j &= \frac{1}{8\pi^3}\iiint_{-\infty}^{\infty} \psi_j(\xi, \eta, \zeta, t) \exp\left[i(x\xi + y\eta + z\zeta)\right]d\xi d\eta d\zeta \\ &= \sum_{k=1,2,3} \frac{1}{8\pi^3}\iiint_{-\infty}^{\infty} A_{jk}(\xi, \eta, \zeta) \cos \omega_k t \exp\left[i(x\xi + y\eta + z\zeta)\right]d\xi d\eta d\zeta. \end{aligned}\tag{12}$$

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[4] This condition may be justified by noting that functions describing a time dependent phenomenon which originated at t_0 in a defined region of the medium must vanish at infinity for all time t.

[5] The procedure thus far has been identically that of the reference in footnote 1.

[6] Here, A_{kj} is the A_j of Eq. (7) corresponding to ω_k.

The solutions of interest are those for which the A_{kj} are explicit functions of ω_k only. We consider first the solution associated with ω_1 and find it convenient to write A_{11} as

$$A_{11}(\omega_1) = (a - \alpha) \int\int\int_{-\infty}^{\infty} \frac{B_{11}(r_1)}{r_1} \exp [i\mathbf{r}_1 \cdot \omega_1] dx dy dz.$$

Here, ω_1 is defined as $\omega_1 = \mathbf{i}a^{1/2}\xi + \mathbf{j}b^{1/2}\eta + \mathbf{k}b^{1/2}\zeta$, and $\mathbf{r}_1 = \mathbf{i}a^{-1/2}x + \mathbf{j}b^{-1/2}y + \mathbf{k}b^{-1/2}z$; B_{11} is an arbitrary function of $r_1 = |\mathbf{r}_1|$.

If η is the angle between $\mathbf{r}_1$ and ω_1, and ν is the polar angle about ω_1, the value of A_{11} becomes

$$\begin{aligned} A_{11} &= (a - \alpha) a^{1/2} b \int\int\int_{-\infty}^{\infty} B_{11}(r_1) \exp [i r_1 \omega_1 \cos \mu] r_1 \sin \mu d\mu d\nu dr_1 \\ &= 4\pi (a - \alpha) a^{1/2} b \omega_1^{-1} \int_0^{\infty} B_{11}(r_1) \sin r_1 \omega_1 dr_1 \\ &= 4\pi (a - \alpha) a^{1/2} b \omega_1^{-1} C_{11}(\omega_1). \end{aligned} \tag{13}$$

Using the same notation[7] and procedure, we find that ϕ_{11} becomes[8]

$$\begin{aligned} \phi_{11} &= (8\pi^3 b a^{1/2})^{-1} \int\int\int_{-\infty}^{\infty} A_{11}(\omega_1) \cos \omega_1 t \exp [i r_{11} \omega_1 \cos \mu] \sin \mu d\mu d\nu d\omega_1 \\ &= [(a - \alpha)/\pi r_1] \int_0^{\infty} C_{11}(\omega_1) \{\sin \omega_1 (r_1 - t) + \sin \omega_1 (r_1 + t)\} d\omega_1 \\ &= (2\pi^2 r_1)^{-1} [B_{11}(r_1 - t) + B_{11}(r_1 + t)]. \end{aligned} \tag{14}$$

Thus we find that one of the possible propagation phenomena is described by a motion wherein the wave fronts are the ellipsoids $r_1 = \text{const.}$, and whose amplitude attenuates like $1/r_1$. Equation (9a) indicates that the other displacement potentials associated with ω_1 differ from ϕ_{11} by a constant factor. In fact we could now write the displacements for this motion as the gradient of a single potential in a distorted coordinate system. This fact will be useful later.

The determination of the motion associated with ω_2 requires only a slight variation on the foregoing procedure. As required by Eq. (9b), we define $A_{22} = -\zeta^2 C(\omega_2)/\omega_2$, $A_{23} = \eta^2 C(\omega_2)/\omega_2$, where $C(\omega_2)$ is derived as before from an arbitrary function $B(r_2)$ and ω_2 and $\mathbf{r}_2$ are defined in the same manner as were ω_1 and $\mathbf{r}_1$. The expression for ϕ_{22} becomes

$$\begin{aligned} \phi_{22} &= \frac{1}{8\pi^3} \int\int\int_{-\infty}^{\infty} \frac{-\zeta^2 C(\omega_2)}{\omega_2} \cos \omega_2 t \exp [i(x\xi + y\eta + z\zeta)] d\xi d\eta d\zeta \\ &= \frac{1}{8\pi^2} \frac{\partial^2}{\partial z^2} \int\int\int_{-\infty}^{\infty} \frac{C(\omega_2)}{\omega_2} \cos \omega_2 t \exp [i(x\xi + y\eta + z\zeta)] d\xi d\eta d\zeta \\ &= \frac{1}{2\pi^2} \frac{\partial^2}{\partial z^2} \left[\frac{B(r_2 - t)}{r_2} + \frac{B(r_2 + t)}{r_2} \right]. \end{aligned} \tag{15}$$

Similarly,

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[7] The angle ν should now be measured about $\mathbf{r}_1$.

[8] The final step is due to the inverse relationship implied by Eq. (13).

$$\phi_{32} = -\frac{1}{2\pi^2}\frac{\partial^2}{\partial y^2}\left[\frac{B(r_2 - t)}{r_2} + \frac{B(r_2 + t)}{r_2}\right]. \tag{16}$$

The displacement $\mathbf{v}_2$ associated with this solution can now be written in the form

$$\mathbf{v}_2 = \frac{1}{2\pi^2}\,\text{curl}\left\{\mathbf{i}\frac{\partial^2}{\partial y\partial z}\left[\frac{B(r_2 - t)}{r_2}\right]\right\}. \tag{17}$$

The fact that the equations governing our problem are linear implies that the $\partial^2/\partial y\partial z$ of the foregoing expression is superfluous,[9] that is, we may write

$$\mathbf{v}_2 = \frac{1}{2\pi^2}\,\text{curl}\left\{\mathbf{i}\frac{F(r_2 - t)}{r_2}\right\}. \tag{18}$$

The displacements corresponding to ω_3 arise in an entirely analogous manner and are given by

$$\mathbf{v}_3 = \begin{vmatrix} \mathbf{i}(b-\alpha), & \mathbf{j}\beta, & \mathbf{k}\beta \\ \dfrac{\partial}{\partial x}, & \dfrac{\partial}{\partial y}, & \dfrac{\partial}{\partial z}, \\ 0, & \dfrac{P(r_3 - t)}{r_3}, & \dfrac{Q(r_3 - t)}{r_3} \end{vmatrix}, \tag{19}$$

where it is required that $\partial(P/r_3)/\partial y + \partial(Q/r_3)/\partial z = 0$. The first of the three foregoing solutions corresponds mathematically to the potential solutions found for the isotropic solid and the latter two to the rotational motions; together they suggest a way of factoring the equations for the displacements. We write the body forces in the following manner, choosing the coefficients of the various derivatives in the operators to correspond to those found in the foregoing results; namely

$$(X, Y, Z) = \text{grad}^*\, \Phi(x, y, z, t) + \text{curl}^*\, \mathbf{M}(x, y, z, t), \tag{20}$$

where

$$\text{grad}^* = \mathbf{i}(a - \alpha)\partial/\partial x + \mathbf{j}\beta\partial/\partial y + \mathbf{k}\beta\partial/\partial z$$

and

$$\text{curl}^*\,\mathbf{M} = \begin{vmatrix} \mathbf{i}(b-\alpha), & \mathbf{j}\beta, & \mathbf{k}\beta \\ \dfrac{\partial}{\partial x}, & \dfrac{\partial}{\partial y}, & \dfrac{\partial}{\partial z} \\ M_x, & M_y, & M_z \end{vmatrix}. \tag{21}$$

We must require that $\partial M_y/\partial y + \partial M_z/\partial z = 0$. We now assume the displacement in the form

$$\mathbf{v} = \text{grad}^*\,\phi + \text{curl}^*\,\mathbf{G}, \tag{22}$$

substitute into equations (1), (2), and (3), and obtain a set of equations which are separable into the following:

[9] Actually, all derivatives of B/r_2 constitute solutions.

$$a\frac{\partial^2\phi}{\partial x^2}+b\left(\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right)-\frac{\partial^2\phi}{\partial t^2}=-\Phi, \tag{23a}$$

$$\alpha\frac{\partial^2 G_x}{\partial x^2}+\gamma\left(\frac{\partial^2 G_x}{\partial y^2}+\frac{\partial^2 G_x}{\partial z^2}\right)-\frac{\partial^2 G_x}{\partial t^2}=-M_x, \tag{23b}$$

$$\left[\alpha\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)-\frac{\partial^2}{\partial t^2}\right](G_y,G_z)=-(M_y,M_z). \tag{23c}$$

The relation $\partial G_y/\partial y+\partial G_z/\partial z=0$ will be automatically satisfied.

Although the above equations may be transformed into the familiar Poisson form by trivial transformations, it is interesting to extend the foregoing procedures to obtain a formal method for the solution of, say, Eq. (23a). The steps leading to Eq. (5) transform Eq. (23a) into

$$\left[\frac{\partial^2}{\partial t^2}+a\xi^2+b(\eta^2+\zeta^2)\right]\psi=T(\xi,\eta,\zeta,t), \tag{24}$$

where ψ is defined as before and

$$T=\iiint_{-\infty}^{\infty}\Phi(p,q,s,t)\exp\left[-i(p\xi+q\eta+s\zeta)\right]dp\,dq\,ds. \tag{24a}$$

Conventional operational procedures then give a particular integral of Eq. (24) as

$$\psi=\frac{1}{\omega_1}\int_0^t T(\xi,\eta,\zeta,\alpha)\sin\omega_1(t-\alpha)d\alpha \tag{25}$$

and ϕ becomes (according to Eq. (12)),

$$\begin{aligned}\phi&=\frac{1}{8\pi^3}\iiint_{-\infty}^{\infty}\left[\int_0^t\Phi(p,q,s,\alpha)\iiint_{-\infty}^{\infty}\frac{\sin\omega_1(t-\alpha)}{\omega_1}\right.\\&\qquad\left.\cdot\exp\left[i\{(x-p)\xi+(y-q)\eta+(z-s)\zeta\}\right]d\xi\,d\eta\,d\zeta\,d\alpha\right]dp\,dq\,ds \qquad (26)\\&=\frac{1}{8\pi^3}\iiint_{-\infty}^{\infty}\left[\int_0^t\Phi(p,q,s,\alpha)dI(\alpha)\right]dp\,dq\,ds,\end{aligned}$$

where[10]

$$\begin{aligned}I(\alpha)&=\frac{1}{ba^{1/2}}\int_0^\infty\int_0^{2\pi}\int_0^\pi\cos\omega_1(t-\alpha)\exp\left[i\omega_1\cdot\mathrm{R}_1\right]\sin\mu\,d\mu\,d\nu\,d\omega_1\\&=\frac{4\pi}{ba^{1/2}}\int_0^\infty\frac{\cos\omega_1(t-\alpha)}{R_1\omega_1}\sin\omega_1R_1d\omega_1\\&=\frac{2\pi}{ba^{1/2}R_1}\int_0^\infty\frac{\sin\omega_1(R_1+t-\alpha)+\sin\omega_1(R_1-t+\alpha)}{\omega_1}d\omega_1. \qquad (27)\end{aligned}$$

The foregoing integral is a step function which (since $0\leq\alpha\leq t$, and $R_1>0$) has a single

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[10] Again we use the type of coordinate transformation which led to Eq. (12).

step of magnitude π at $\alpha=t-R_1$. R_1 is defined by $R_1^2=(x-p)^2/a+(y-q)^2/b+(z-s)^2/b$. The evaluation of the Stieltjes integral of Eq. (26) now yields

$$\phi=\frac{1}{4\pi}\int\int\int_{-\infty}^{\infty}\frac{\Phi(a^{1/2}p,\, b^{1/2}q,\, b^{1/2}s,\, t-R_1)}{R_1}\,dpdqds \tag{28}$$

and we have the familiar retarded potential. The expressions for the components of G can be obtained in analogous fashion.

3. More general media. It is quite evident that one might start with the general linear law relating the stresses and strains in an aerolotropic material and by the same procedures arrive at three equations analogous to Eqs. (4), using either the displacements themselves or the potentials defined in the foregoing. One would then arrive at a determinantal equation of the same form as Eq. (8). The roots could be found and Eq. (12) would be valid. However, the ω_k^2 of this general problem would not appear in the concise polynomial form found in the foregoing considerations. In fact, the integrand of Eq. (12) becomes sufficiently complex in this general case that it does not seem worth while to present in more detail the procedure outlined here.

REFLECTION IN A CORNER FORMED BY THREE PLANE MIRRORS*

BY

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1. **Introduction.** If a plane mirror is attached to the base of a projectile and a parallel beam of light projected on it, the direction of the reflected beam at any instant will give us information about the angular position of the projectile at that instant. It will not, however, indicate the angular position completely, because a rotation of the projectile about the normal to the mirror leaves the direction of the reflected ray unaltered.

This difficulty may be overcome by using more than one mirror, and the possibility of using a reflecting corner formed by three plane mirrors suggests itself. If the three mirrors are mutually perpendicular, the direction of the reflected beam gives no indication of the angular position of the projectile, because such a corner reverses the direction of any parallel beam falling on it. But if the angles of the corner are not right angles, this is no longer the case; in general, there will be six reflected beams, and their directions will determine the angular position of the projectile completely. With three parameters at our disposal (the three angles of the corner), we can secure a variety of different effects. The purpose of the present paper is to make a systematic study of the optical behavior of all corners formed by three plane mirrors.

The method used is based on the fact that the transformation of ray-directions due to reflection in a plane mirror is equivalent to a rigid-body rotation about the mirror-normal through two right angles (i.e. a half-turn), combined with a reversal of sense. Consequently, three successive reflections in three plane mirrors produce a transformation equivalent to three half-turns, combined with a reversal of sense. But, by Euler's theorem, three successive half-turns are equivalent to a single rotation (not, in general, itself a half-turn). Thus the transformation due to reflection in a corner formed by three plane mirrors may be described by giving the axis of the single equivalent rotation (called the *optic axis* in the present connection), and the angle of the rotation.

It is found that, when different orders of reflection in the three mirrors are taken into consideration, there are in general *three* optic axes and a *unique* angle of rotation. The rotation occurs in both positive and negative senses, so that in general there are six reflected rays resulting from a given incident direction. This is, of course, to be expected, since we can form six permutations of three mirrors.

It is useful to represent directions by points on the surface of a unit sphere. There are then two fundamental spherical triangles. One has for vertices the normals to the three mirrors, and the other has for vertices three directed optic axes. Actually there are two spherical triangles formed by (undirected) optic axes, but one is the reflection of the other in the center of the sphere, and so we pick out one for definiteness. The

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vertices of the triangle formed by the mirror-normals lie at the middle points of the sides of the triangle formed by the directed optic axes, and the angle of the single equivalent rotation is the defect from four right angles of the sum of the angles of the triangle formed by the directed optic axes.

Explicit formulae are given for the construction of the optic axes (3.10) and the angle of the single equivalent rotation (4.15).

The fact that a ray, to be reflected, must strike the front of a mirror, and not the back, introduces awkward conditions. These conditions are removed in the mathematical theory by supposing that the mirrors are planes which reflect from either side. Further, in investigating the effect of a second reflection, we may find that after reflection in the first mirror the course of the ray does not bring it into incidence with the second mirror. In such cases we shall disregard the position of the ray, and apply to its direction the transformation corresponding to reflection in the second mirror. These artificialities (from the practical standpoint) are introduced to avoid encumbering the mathematical theory with conditions which have no bearing on the fundamental transformation problem. Once the general theory has been set up, the conditions mentioned above may be looked into in any particular case. To facilitate this, we shall continue to call one side of each mirror the *front*.

2. **Equivalence of reflections and rigid body rotations.** Let **N** be a unit vector normal to a plane mirror, drawn out from the front (Fig. 1). Let **I** be a unit vector along an incident ray, and **I**$'$ a unit vector along the reflected ray. From a point O let us drawn the unit vectors **N**, **I**, **I**$'$, $-$**I**, $-$**I**$'$ (Fig. 2).

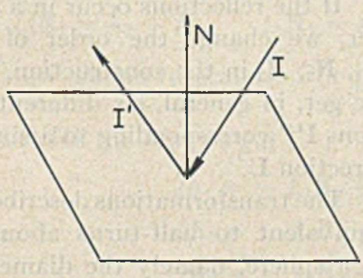

FIG. 1

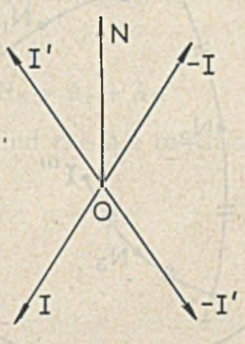

FIG. 2

It is clear from the law of reflection that **I**$'$ is obtained from $-$**I** by a half-turn about **N**; equivalently, $-$**I**$'$ is obtained from **I** by a half-turn about **N**.

The transformations may also be shown on the surface of the unit sphere with center O; the unit vectors are now represented by points on the surface of the unit sphere (Fig. 3). The law of reflection may be stated as follows. Join $-$**I** to **N** by a great circle, and produce it on to make the arc (**N**, **I**$'$) equal to the arc ($-$**I**, **N**). Equivalently, join **I** to **N** by a great circle, and produce it on to make the arc (**N**, $-$**I**$'$) equal to the arc (**I**, **N**).

In order that the incident ray may strike the front of the mirror, the arc ($-$**I**, **N**) must be less than $\frac{1}{2}\pi$; but in view of the remarks made in Section 1, this restriction will not be imposed.

Consider now a corner formed by three plane mirrors with unit normals $\mathbf{N}_1$, $\mathbf{N}_2$,

N_3, drawn out from the fronts. These are represented on the unit sphere in Fig. 4. Consider successive reflections of an incident ray I in the order N_1, N_2, N_3. We start by marking $-I$ on the unit sphere. We construct the first reflected ray I' by drawing

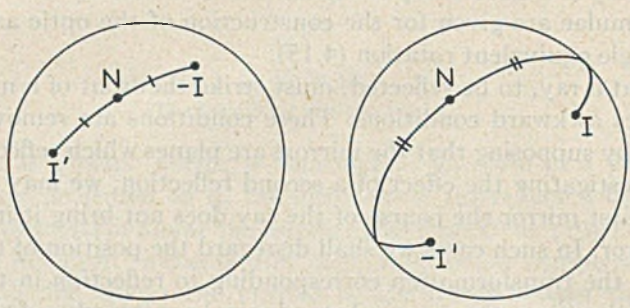

FIG. 3

a great circular arc from $-I$ through N_1, making N_1 the point of bisection. If the second reflected ray is denoted by I'', we construct $-I''$ by drawing a great circular arc from $-I'$ through N_2, making N_2 the point of bisection. We carry on from $-I''$ similarly through N_3 to form the final ray I'''.

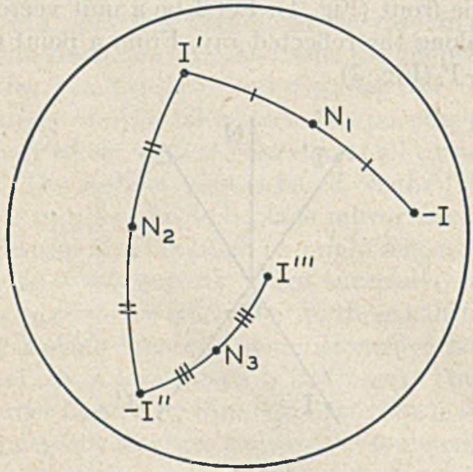

FIG. 4

We might also have started with I, instead of $-I$. The same rules of construction would have led us to $-I'''$.

If the reflections occur in a different order, we change the order of the points N_1, N_2, N_3 in the construction. In this way we get, in general, six different final directions I''' corresponding to a single incident direction I.

The transformations described above are equivalent to half-turns about diameters of a sphere, namely the diameters defined by N_1, N_2, N_3, We know by Euler's theorem that any succession of rotations about diameters of a sphere is equivalent either to no displacement at all or to a rotation through an angle less than 2π about a uniquely determined diameter. The former alternative means that I''' coincides with $-I$, no matter how I is chosen. On the other hand, if there is a unique axis of equivalent rotation, then rays incident along the axis of that rotation (and such rays alone) will undergo reversal as a result of the triple reflection. Every other ray will undergo a change of direction determined by application of the equivalent rotation.

We shall call the axis of the equivalent rotation the *undirected optic axis* of the reflecting corner for the order of reflections assigned. This definition of optic axis would be adequate if we were content to have the term denote a diameter, without sense of direction. It is, however, desirable to understand by optic axis one of the two unit vectors lying on the axis of the equivalent rotation. Accordingly, we shall proceed in the next section to define the *directed optic axis*. For the present, let us sum up our results as follows:

THEOREM I. *For three successive reflections in a given order in three plane mirrors, either every incident ray is reversed in direction, or there exists a unique undirected optic axis such that*

(a) *a ray incident along the undirected optic axis is reversed in direction;*

(b) *the directions of reflected rays are obtained from the directions of incidents rays reversed by a rigid body rotation through an angle less than 2π about the undirected optic axis.*

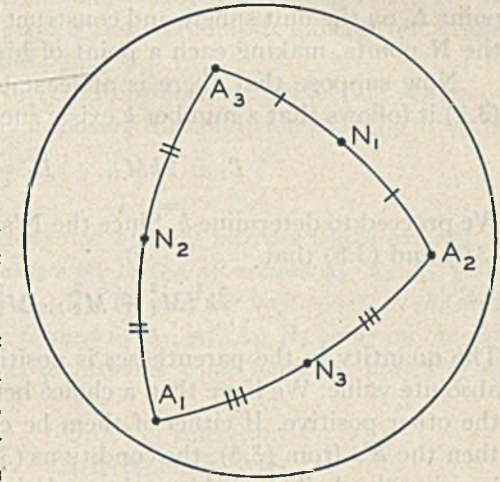

FIG. 5

3. Determination of the directed optic axes. Consider the following problem in spherical geometry: *Given three points* $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ *on a unit sphere, to construct a spherical triangle* $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, *such that* $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ *are the middle points of its sides* (Fig. 5).

In vector notation, we have

$$\mathbf{N}_1\cdot\mathbf{A}_2 = \mathbf{N}_1\cdot\mathbf{A}_3, \qquad \mathbf{N}_2\cdot\mathbf{A}_3 = \mathbf{N}_2\cdot\mathbf{A}_1, \qquad \mathbf{N}_3\cdot\mathbf{A}_1 = \mathbf{N}_3\cdot\mathbf{A}_2, \tag{3.1}$$

and also

$$L_1\mathbf{N}_1 = \mathbf{A}_2 + \mathbf{A}_3, \qquad L_2\mathbf{N}_2 = \mathbf{A}_3 + \mathbf{A}_1, \qquad L_3\mathbf{N}_3 = \mathbf{A}_1 + \mathbf{A}_2, \tag{3.2}$$

where the L's are unknown scalars. Our problem is to find the $\mathbf{A}$'s to satisfy (3.1) and (3.2).

Solving (3.2) for the $\mathbf{A}$'s, we get

$$\begin{aligned} 2\mathbf{A}_1 &= -L_1\mathbf{N}_1 + L_2\mathbf{N}_2 + L_3\mathbf{N}_3, \\ 2\mathbf{A}_2 &= L_1\mathbf{N}_1 - L_2\mathbf{N}_2 + L_3\mathbf{N}_3, \\ 2\mathbf{A}_3 &= L_1\mathbf{N}_1 + L_2\mathbf{N}_2 - L_3\mathbf{N}_3. \end{aligned} \tag{3.3}$$

Let us define M_1, M_2, M_3 by

$$M_1 = \mathbf{N}_2\cdot\mathbf{N}_3, \qquad M_2 = \mathbf{N}_3\cdot\mathbf{N}_1, \qquad M_3 = \mathbf{N}_1\cdot\mathbf{N}_2, \tag{3.4}$$

these being of course the cosines of the angles between the $\mathbf{N}$ vectors. Taking the scalar products of the first two of (3.3) and $\mathbf{N}_3$, we get

$$\begin{aligned} 2\mathbf{A}_1\cdot\mathbf{N}_3 &= -L_1M_2 + L_2M_1 + L_3, \\ 2\mathbf{A}_2\cdot\mathbf{N}_3 &= L_1M_2 - L_2M_1 + L_3. \end{aligned} \tag{3.5}$$

Hence, by the last of (3.1),

$$L_1M_2 - L_2M_1 = 0. \tag{3.6}$$

Similarly

$$L_2M_3 - L_3M_2 = 0, \qquad L_3M_1 - L_1M_3 = 0. \tag{3.7}$$

If all the M's vanish (i.e. if the **N** vectors are mutually perpendicular), we may choose the L's arbitrarily, save for the condition that the sum of their squares shall equal 4. For it is easy to verify that the **A**'s as given by (3.3) will then be unit vectors, and the conditions (3.1) and (3.2) will be satisfied. This means that we can take any point $\mathbf{A}_1$ on the unit sphere and construct the triangle by passing successively through the **N** points, making each a point of bisection. The triangle necessarily closes.

Now suppose that there is at least one M different from zero. From (3.6) and (3.7) it follows that a number k exists such that

$$L_1 = 2kM_1, \qquad L_2 = 2kM_2, \qquad L_3 = 2kM_3. \tag{3.8}$$

We proceed to determine k. Since the **N**'s and the **A**'s are unit vectors, if follows from (3.3) and (3.8) that

$$k^2(M_1^2 + M_2^2 + M_3^2 - 2M_1M_2M_3) = 1. \tag{3.9}$$

The quantity in the parentheses is positive definite, since no M can exceed unity in absolute value. We have then a choice between two real values of k, one negative and the other positive. If either of them be chosen, and the L's obtained from (3.8), and then the **A**'s from (3.3), the conditions (3.1) and (3.2) are satisfied, and so the problem is solved. The problem then admits of two (and only two) solutions; the two triangles are the diametrical opposites of one another.

To avoid confusion, let us pick out one of these two solutions for application to the optical problem. Let us decide to take the positive value for k. To sum up, the triangle required is given by

$$\begin{aligned} \mathbf{A}_1 &= k(-M_1\mathbf{N}_1 + M_2\mathbf{N}_2 + M_3\mathbf{N}_3), \\ \mathbf{A}_2 &= k(M_1\mathbf{N}_1 - M_2\mathbf{N}_2 + M_3\mathbf{N}_3), \\ \mathbf{A}_3 &= k(M_1\mathbf{N}_1 + M_2\mathbf{N}_2 - M_3\mathbf{N}_3), \end{aligned} \tag{3.10}$$

where the M's are given by (3.4) and

$$k = (M_1^2 + M_2^2 + M_3^2 - 2M_1M_2M_3)^{-1/2}, \tag{3.11}$$

the positive value being understood.

We have also

$$\begin{aligned} \mathbf{N}_1\cdot\mathbf{A}_2 = \mathbf{N}_1\cdot\mathbf{A}_3 &= kM_1, \\ \mathbf{N}_2\cdot\mathbf{A}_3 = \mathbf{N}_2\cdot\mathbf{A}_1 &= kM_2, \\ \mathbf{N}_3\cdot\mathbf{A}_1 = \mathbf{N}_3\cdot\mathbf{A}_2 &= kM_3, \end{aligned} \tag{3.12}$$

and

$$2kM_1\mathbf{N}_1 = \mathbf{A}_2 + \mathbf{A}_3, \qquad 2kM_2\mathbf{N}_2 = \mathbf{A}_3 + \mathbf{A}_1, \qquad 2kM_3\mathbf{N}_3 = \mathbf{A}_1 + \mathbf{A}_2. \tag{3.13}$$

We have now to show the connection of this problem in spherical geometry with our optical problem. We shall begin by proving that the diameter through $\mathbf{A}_2$ is the undirected optic axis for reflections in the order $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$.

We take an incident ray with $\mathbf{I} = \pm\mathbf{A}_2$. The first reflected ray is then given by $\mathbf{I}' = \mp\mathbf{A}_3$. The second reflected ray is given by $\mathbf{I}'' = \pm\mathbf{A}_1$. The third or final reflected ray is given by $\mathbf{I}''' = \mp\mathbf{A}_2$. Thus $\mathbf{I}''' = -\mathbf{I}$, which proves that the diameter through $\mathbf{A}_2$ is the undirected optic axis for reflections in the order $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$. In just the same way we may show that the diameter through A_2 is the undirected optic axis for reflections in the order N_3, N_2, N_1, also.

Further, if we start from A_3 or A_1, we can prove in the same way that the diameter through A_3 is the undirected optic axis for reflections in either of the orders N_2, N_3, N_1 or N_1, N_3, N_2, while the diameter through A_1 is the undirected optic axis for reflections in either of the orders N_3, N_1, N_2, or N_2, N_1, N_3.

We are now in a position to define the directed optic axes by selecting senses on the undirected optic axes. We shall do this by imposing the condition that the directed optic axes are given by (3.10) with k positive.

Let us sum up as follows:

THEOREM II. *For any reflecting corner composed of three mirrors which are not all mutually perpendicular, the directed optic axes are the three unit vectors* A_1, A_2, A_3, *given by* (3.10) *with positive k. The mirror-normals* N_1, N_2, N_3 *meet the unit sphere at the middle points of the sides of the spherical triangle formed by* A_1, A_2, A_3, N_1 *being on the side* A_2A_3, *and so on. The directed optic optic axes correspond to the following orders of reflection*:

Directed optic axis	Order of reflection
A_1	$N_3N_1N_2$ or $N_2N_1N_3$
A_2	$N_1N_2N_3$ or $N_3N_2N_1$
A_3	$N_2N_3N_1$ or $N_1N_3N_2$

We see from (3.10) that the directed optic axes are easily constructed in space by adding and subtracting the vectors kM_1N_1, kM_2N_2, kM_3N_3. This construction is shown

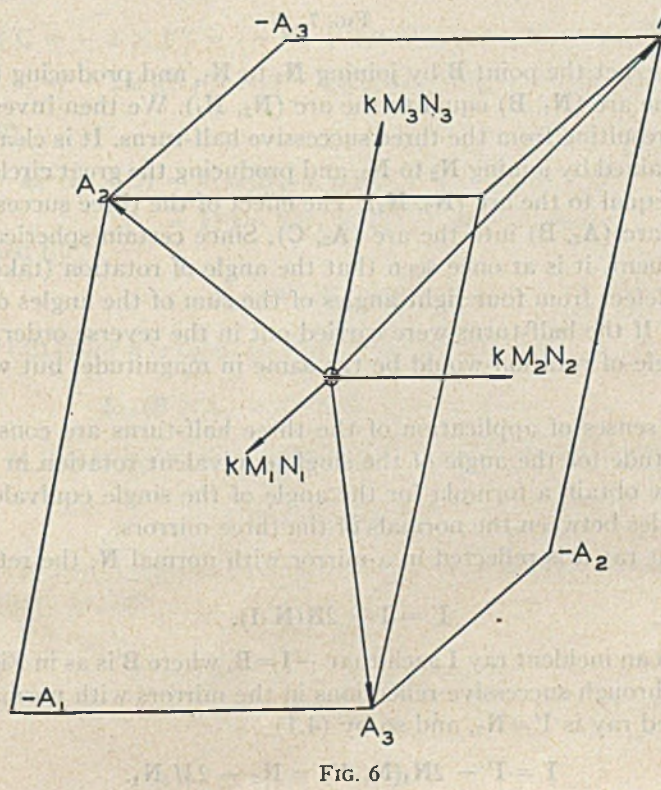

FIG. 6

in Fig. 6. It is interesting to note that six of the eight vertices of the parallelepiped are occupied by the vectors $\pm A_1$, $\pm A_2$, $\pm A_3$.

4. **The angle of the equivalent rigid-body rotation.** Fig. 7 shows the representations on the unit sphere of the three mirror-normals N_1, N_2, N_3 and the three directed optic axes A_1, A_2, A_3. We know that three successive half-turns about N_1, N_2, N_3 in order are equivalent to a rotation about A_2. We proceed to find the magnitude of this rotation.

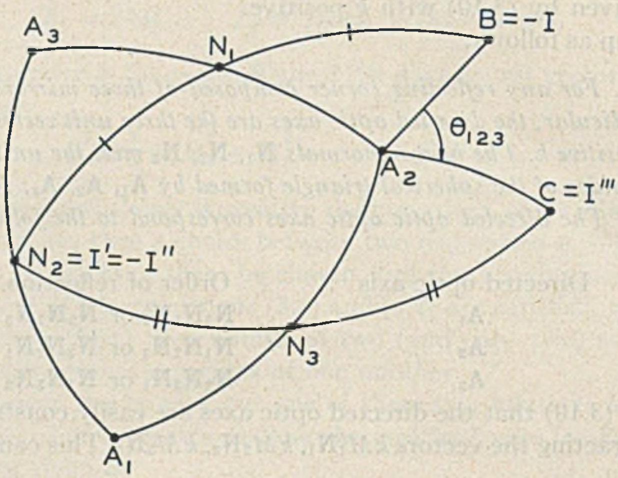

FIG. 7

We first construct the point B by joining N_2 to N_1, and producing the great circle so as to make the arc (N_1, B) equal to the arc (N_2, N_1). We then investigate the displacement of B resulting from the three successive half-turns. It is clear that the final position C is obtained by joining N_2 to N_3, and producing the great circle so as to make the arc (N_3, C) equal to the arc (N_2, N_3). The effect of the three successive half-turns is to rotate the arc (A_2, B) into the arc (A_2, C). Since certain spherical triangles are obviously congruent, it is at once seen that the angle of rotation (taken less than π) is equal to the defect from four right angles of the sum of the angles of the spherical triangle $A_1A_2A_3$. If the half-turns were carried out in the reverse order, then C would go to B; the angle of rotation would be the same in magnitude, but would have the opposite sense.

If the other senses of application of the three half-turns are considered, we get the same magnitude for the angle of the single equivalent rotation in all cases.

We shall now obtain a formula for the angle of the single equivalent rotation in terms of the angles between the normals of the three mirrors.

If an incident ray I is reflected in a mirror with normal N, the reflected ray I′ is given by

$$I' = I - 2N(N\cdot I). \tag{4.1}$$

Let us start with an incident ray I such that $-I = B$, where B is as in Fig. 7, and let us follow this ray through successive reflections in the mirrors with normals N_1, N_2, N_3. The first reflected ray is $I' = N_2$, and so by (4.1)

$$I = I' - 2N_1(N_1\cdot I') = N_2 - 2M_3N_1. \tag{4.2}$$

The second reflected ray is $\mathbf{I}''=-\mathbf{N}_2$. The third and final reflected ray is

$$\mathbf{I}''' = \mathbf{I}'' - 2\mathbf{N}_3(\mathbf{N}_3\cdot\mathbf{I}'') = -\mathbf{N}_2 + 2\mathbf{N}_3(\mathbf{N}_3\cdot\mathbf{N}_2) = -\mathbf{N}_2 + 2M_1\mathbf{N}_3, \tag{4.3}$$

the M's being as in (3.4).

We note that (3.10) give

$$-\mathbf{I}\cdot\mathbf{A}_2 = \mathbf{I}'''\cdot\mathbf{A}_2 = kM_2. \tag{4.4}$$

As already seen, the angle of rotation θ_{123} is the angle between the arcs $(\mathbf{A}_2, \mathbf{B})$ and $(\mathbf{A}_2, \mathbf{C})$, as shown in Fig. 7. This is the same as the angle between the vectors $(\mathbf{A}_2\times\mathbf{B})$ and $(\mathbf{A}_2\times\mathbf{C})$. If the rotation is considered positive when it is a right-handed rotation about $\mathbf{A}_2$, through an angle less than π, then the angle satisfies the equation

$$\sin\theta_{123}\cdot\mathbf{A}_2 = \frac{(\mathbf{A}_2\times\mathbf{B})\times(\mathbf{A}_2\times\mathbf{C})}{|\mathbf{A}_2\times\mathbf{B}|\,|\mathbf{A}_2\times\mathbf{C}|}, \tag{4.5}$$

where $\mathbf{B}=-\mathbf{I}$ and $\mathbf{C}=\mathbf{I}'''$. (The angle θ_{123} shown in Fig. 7 is negative in the defined sense.)

To evaluate the right-hand side of (4.5), we note that by (4.4) we have

$$|\mathbf{A}_2\times\mathbf{B}|\,|\mathbf{A}_2\times\mathbf{C}| = 1 - k^2M_2^2. \tag{4.6}$$

Also, identically,

$$(\mathbf{A}_2\times\mathbf{B})\times(\mathbf{A}_2\times\mathbf{C}) = \mathbf{A}_2[\mathbf{A}_2\cdot(\mathbf{B}\times\mathbf{C})]. \tag{4.7}$$

By (4.2) and (4.3),

$$\begin{aligned}\mathbf{B}\times\mathbf{C} &= -\mathbf{I}\times\mathbf{I}''' = (-\mathbf{N}_2 + 2M_3\mathbf{N}_1)\times(-\mathbf{N}_2 + 2M_1\mathbf{N}_3)\\ &= -2M_1(\mathbf{N}_2\times\mathbf{N}_3) - 2M_3(\mathbf{N}_1\times\mathbf{N}_2) - 4M_1M_3(\mathbf{N}_3\times\mathbf{N}_1),\end{aligned} \tag{4.8}$$

and so, by (3.10),

$$\begin{aligned}\mathbf{A}_2\cdot(\mathbf{B}\times\mathbf{C}) &= k(M_1\mathbf{N}_1 - M_2\mathbf{N}_2 + M_3\mathbf{N}_3)\cdot(\mathbf{B}\times\mathbf{C})\\ &= kP(-2M_1^2 + 4M_1M_2M_3 - 2M_3^2),\end{aligned} \tag{4.9}$$

where P is defined by

$$P = \mathbf{N}_1\cdot(\mathbf{N}_2\times\mathbf{N}_3). \tag{4.10}$$

When the value (3.11) for k is used, (4.9) may be written

$$\mathbf{A}_2\cdot(\mathbf{B}\times\mathbf{C}) = -2Pk^{-1}(1 - k^2M_2^2). \tag{4.11}$$

On substitution of this expression in (4.7), and then substitution from (4.6) and (4.7) in (4.5), we get

$$\sin\theta_{123} = -2Pk^{-1}. \tag{4.12}$$

We note that P is the determinant formed from the direction cosines of the three normals to the mirrors, and

$$P^2 = \begin{vmatrix} 1 & M_3 & M_2 \\ M_3 & 1 & M_1 \\ M_2 & M_1 & 1 \end{vmatrix} = 1 - k^{-2}. \tag{4.13}$$

Hence

$$P = \epsilon(1 - k^{-2})^{1/2}, \tag{4.14}$$

where ϵ is $+1$ or -1 according as the orientation of the triad $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ is positive or negative.

We may sum up as follows:

THEOREM III. *The magnitude (taken less than π) of the angle of the single equivalent rotation is the defect from four right angles of the angle-sum of the spherical triangle on the unit sphere whose vertices represent the directed optic axes. If a positive angle corresponds to a positive (right-handed) rotation about the directed optic axis involved, the angle is given both in magnitude and sign by*

$$\sin\theta = -2\epsilon k^{-1}(1-k^{-2})^{1/2}, \tag{4.15}$$

where k is given in terms of the cosines of the angles between the mirror-normals by (3.11), *and ϵ is $+1$ or -1 according as the triad of mirror-normals, in the order of the reflections, is positive or negative (i.e. right-handed or left-handed).*

Let us now see how the six reflected rays are to be constructed when the incident ray I is given and the three directed optic axes are known. We mark on the unit sphere (Fig. 8) the directed optic axes $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, and the incident ray reversed, $-\mathbf{I}$. Let us

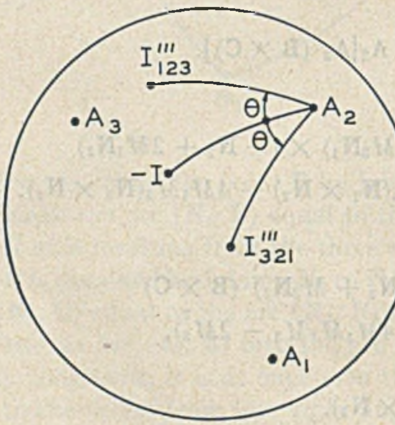

FIG. 8

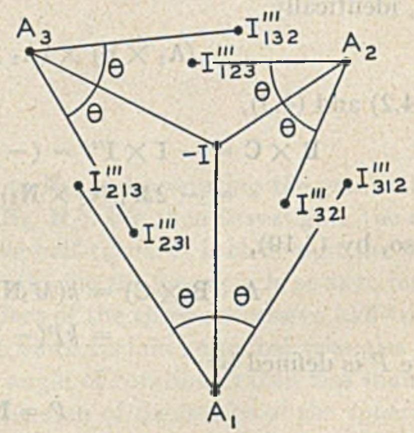

FIG. 9

suppose that the mirror-normals are so numbered that $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ form a positive triad. To obtain the reflected ray resulting from reflections in the order $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ we use the optic axis $\mathbf{A}_2$. We draw the arc ($\mathbf{A}_2$, $-\mathbf{I}$), and rotate it about $\mathbf{A}_2$ in the negative sense through an angle equal to the defect from four right angles of the angle-sum of the spherical triangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$, or equivalently through an angle $\sin^{-1} 2k^{-1}(1-k^{-2})^{1/2}$, as given in (4.15). This gives the reflected ray $\mathbf{I}'''_{123}$, the subscripts indicating the order of the reflections. To obtain $\mathbf{I}'''_{321}$, we rotate the arc ($\mathbf{A}_2$, $-\mathbf{I}$) about $\mathbf{A}_2$ in the opposite sense through the same angle.

Using the other optic axes, we obtain similarly the whole set of six reflected rays. These are shown in Fig. 9, the great circular arcs being shown as straight lines for simplicity. All the marked angles are equal.

5. Cases of perpendicular mirrors. Let us consider the case where two of the mirrors are perpendicular to one another. Let us take $\mathbf{N}_1$ perpendicular to $\mathbf{N}_2$, and write

$$M_1 \neq 0, \qquad M_2 \neq 0, \qquad M_3 = 0. \tag{5.1}$$

Then (3.10) give for the directed optic axes

$$\begin{aligned} \mathbf{A}_1 &= k(-\ M_1\mathbf{N}_1 + M_2\mathbf{N}_2), \\ \mathbf{A}_2 &= k(\ \ M_1\mathbf{N}_1 - M_2\mathbf{N}_2), \\ \mathbf{A}_3 &= k(\ \ M_1\mathbf{N}_1 + M_2\mathbf{N}_2), \end{aligned} \tag{5.2}$$

and from (3.11) we have

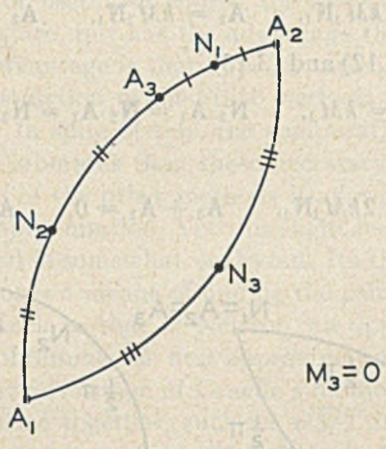

Fig. 10

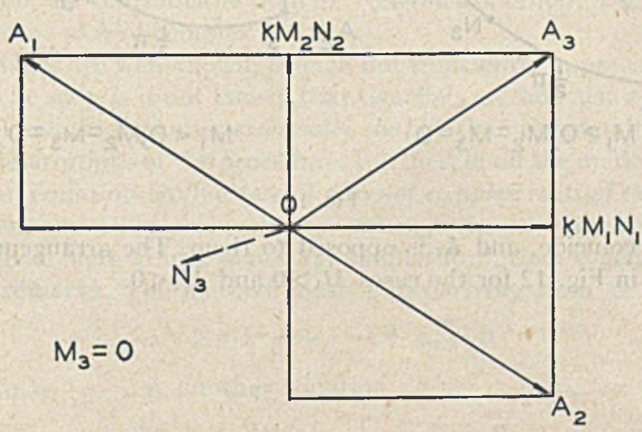

Fig. 11

$$k = (M_1^2 + M_2^2)^{-1/2}. \tag{5.3}$$

The equations (3.12) and (3.13) become

$$\mathbf{N}_1\cdot\mathbf{A}_2 = \mathbf{N}_1\cdot\mathbf{A}_3 = kM_1, \quad \mathbf{N}_2\cdot\mathbf{A}_3 = \mathbf{N}_2\cdot\mathbf{A}_1 = kM_2, \quad \mathbf{N}_3\cdot\mathbf{A}_1 = \mathbf{N}_3\cdot\mathbf{A}_2 = 0, \tag{5.4}$$

and

$$\mathbf{A}_2 + \mathbf{A}_3 = 2kM_1\mathbf{N}_1, \qquad \mathbf{A}_3 + \mathbf{A}_1 = 2kM_2\mathbf{N}_2, \qquad \mathbf{A}_1 + \mathbf{A}_2 = 0. \tag{5.5}$$

We note that $\mathbf{A}_1$ and $\mathbf{A}_2$ are opposed to one another, and $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{N}_1$, $\mathbf{N}_2$ are coplanar. The arrangement on the unit sphere is shown in Fig. 10. The construction in space for the directed optic axes is shown in Fig. 11.

Let us now consider the case where one of the mirrors is perpendicular to the two others. Let $\mathbf{N}_1$ be perpendicular to $\mathbf{N}_2$ and $\mathbf{N}_3$. We write

$$M_1 \neq 0, \qquad M_2 = 0, \qquad M_3 = 0. \tag{5.6}$$

From (3.10) we get for the directed optic axes

$$\mathbf{A}_1 = -kM_1\mathbf{N}_1, \qquad \mathbf{A}_2 = kM_1\mathbf{N}_1, \qquad \mathbf{A}_3 = kM_1\mathbf{N}_1, \tag{5.7}$$

where $k = |M_1|^{-1}$; from (3.12) and (3.13)

$$\mathbf{N}_1\cdot\mathbf{A}_2 = \mathbf{N}_1\cdot\mathbf{A}_3 = kM_1, \qquad \mathbf{N}_2\cdot\mathbf{A}_3 = \mathbf{N}_2\cdot\mathbf{A}_1 = \mathbf{N}_3\cdot\mathbf{A}_1 = \mathbf{N}_3\cdot\mathbf{A}_2 = 0, \tag{5.8}$$

and

$$\mathbf{A}_2 + \mathbf{A}_3 = 2kM_1\mathbf{N}_1, \qquad \mathbf{A}_3 + \mathbf{A}_1 = 0, \qquad \mathbf{A}_1 + \mathbf{A}_2 = 0. \tag{5.9}$$

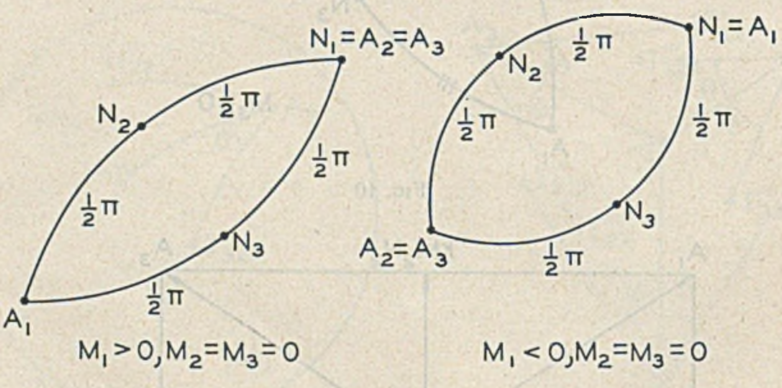

FIG. 12

Now $\mathbf{A}_2$ and $\mathbf{A}_3$ coincide, and $\mathbf{A}_1$ is opposed to them. The arrangement on the unit sphere is shown in Fig. 12 for the cases $M_1 > 0$ and $M_1 < 0$.

ON GRAEFFE'S METHOD FOR SOLVING ALGEBRAIC EQUATIONS*

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In the usual descriptions of the methods of solving numerical algebraic equations, Graeffe's method takes a minor place as compared with the methods of Newton, Horner, and others. It is not useful, of course, for correcting a single approximate value, as the other methods are, but has the advantage that no first approximation need be known. A second advantage is that approximations to *all* roots are obtained simultaneously, in contradistinction to the other methods which furnish approximations to *one* root at a time. In spite of this, the computations required by Graeffe's method are not much more laborious than those necessary to obtain an approximation to a single root by one of the other methods if allowance is made for the time necessary to find the first approximation. Yet this slight increase in labor may be the reason that Graeffe's method is somewhat neglected. Its third and perhaps its main advantage is that it also affords a means of finding the complex roots. It is true that by certain other methods, such as that of Newton, an approximation to a complex root can be improved, but obtaining the first approximation is rather difficult in the case of complex roots. A last advantage of Graeffe's method is that it automatically separates roots which are close together, such as $\sqrt{5/2}$ and $3/2$. It is known that Lagrange claimed this same advantage for this method of developing a root into a continued fraction, in contradistinction to Newton's method; Lagrange's method fails, however, in the case of complex roots.

These advantages are well known, though not sufficiently appreciated in practice. But so far as can be seen, it is not known that Graeffe's method also gives the *multiple* (*real or complex*) *roots*, in a manner *essentially simpler* than is generally pointed out in more elaborate descriptions of the procedure. Further, of all the methods it is the only one for solving an equation having *several pairs of complex roots of the same modulus*.

To derive these and other properties, for example the convergence of the process, we discuss the whole method in a somewhat simpler form than is generally used.

Preliminary remarks. The method consists in deriving from an equation

$$x^n + a_1x^{n-1} + \cdots + a_n = 0 \qquad (1)$$

with the roots $x_1, x_2, \cdots, x_n$ another equation

$$X^n + A_1X^{n-1} + \cdots + A_n = 0 \qquad (2)$$

having the roots $X_i = x_i^p$, where p is a large number (and for practical reasons a power of 2), so that the distinct roots of (2) are widely separated and can thus be easily calculated in the following manner.

Splitting of equation (2).

First case. All the roots of (2) are *positive* and *simple.*

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Then, from

$$X_1 \gg X_2 \gg \cdots \gg X_n,$$

we have

$$-A_1 = \sum X_i \approx X_1$$
$$A_2 = \sum X_i X_j \approx X_1 X_2$$
$$-A_3 = \sum X_i X_j X_k \approx X_1 X_2 X_3 \text{ etc.,}$$

since the first members of the sums predominate. Thus

$$X_1 \approx -A_1/1,\ X_2 \approx -A_2/A_1,\ X_3 \approx -A_3/A_2, \cdots,$$

that is, equation (2) is split into the n approximate linear equations

$$X + A_1 = 0,\ A_1 X + A_2 = 0,\ A_2 X + A_3 = 0, \cdots, A_{n-1} X + A_n = 0.$$

Second case. Several of the roots of (2) have the *same absolute value.*

(ia) There is one pair of complex roots, say

$$X_3 = Re^{iA}, \qquad X_4 = Re^{-iA},$$

so that

$$A_2 = \sum X_i X_j \approx X_1 X_2$$
$$-A_3 = X_1 X_2 X_3 + X_1 X_2 X_4 + \cdots \approx X_1 X_2 (X_3 + X_4) = X_1 X_2 \cdot 2R \cos A$$
$$A_4 \approx X_1 X_2 X_3 X_4 = X_1 X_2 \cdot R^2.$$

Thus X_3, X_4 are approximately the roots of the quadratic equation

$$A_2 X^2 + A_3 X + A_4 = 0, \tag{3}$$

so that equation (2) is split into the $n-2$ linear equations

$$X + A_1 = 0,\ A_1 X + A_2 = 0,\ A_4 X + A_5 = 0, \cdots, A_{n-1} X + A_n = 0$$

and the quadratic (3).

(ib) There are two pairs of complex roots, having the same absolute value R,

$$X_{3,4} = Re^{\pm iA}, \qquad X_{5,6} = Re^{\pm iB}.$$

Then

$$A_2 \approx X_1 X_2$$
$$-A_3 \approx X_1 X_2 (X_3 + X_4 + X_5 + X_6) = X_1 X_2 \cdot 2R(\cos A + \cos B)$$
$$A_4 \approx X_1 X_2 (X_3 X_4 + \cdots + X_5 X_6) = X_1 X_2 \cdot 2R^2 (1 + 2 \cos A \cos B)$$
$$-A_5 \approx X_1 X_2 (X_3 X_4 X_5 + \cdots + X_4 X_5 X_6) = X_1 X_2 \cdot 2R^3 (\cos A + \cos B)$$
$$A_6 \approx X_1 X_2 X_3 X_4 X_5 X_6 = X_1 X_2 \cdot R^4.$$

Therefore $X_3, \cdots, X_6$ satisfy the equation

$$A_2 X^4 + A_3 X^3 + A_4 X^2 + A_5 X + A_6 = 0. \tag{4}$$

To solve it, let us first compute R from

$$R^4 = A_6 / A_2. \tag{5}$$

With this R let

$$X = R \cdot Y, \quad \text{where} \quad Y = e^{i\phi}. \tag{6}$$

Then all the roots of Eq. (4) in Y have the modulus 1, and if Y is a root, then $1/Y$ is also a root, that is, Eq. (4) is "reciprocal":

$$Y^4 + BY^3 + CY^2 + BY + 1 = 0.$$

By

$$Y + Y^{-1} = 2 \cos \phi = Z \tag{7}$$

it becomes a quadratic equation in Z.

(ic) There are μ distinct pairs of complex roots having the same modulus R.

Then in a manner similar to (ib) above an equation M of degree 2μ is split off. The modulus R is obtained from the equation

$R^{2\mu}$ = quotient of the last coefficient to the first coefficient A_2 of the equation M. (8)

To compute the arguments A, B, $\cdots$ of the μ pairs of roots, we again set $X = RY$ as in (6), and divide by the leading coefficient $A_2R^{2\mu}$. The new equation in Y is again reciprocal:

$$(Y^{2\mu} + 1) + B(Y^{2\mu-1} + Y) + C(Y^{2\mu-2} + Y^2) + \cdots + NY^\mu = 0. \tag{9}$$

By substituting (7) into (9) it can be seen that (9) is an equation in Z of degree μ having μ real roots of $Z < 2$ which can be found by Graeffe's method. Complications cannot arise since all the roots are distinct. The transformed equation therefore will break up into a system of μ linear equations.

(iia) There are three roots with the same modulus R, one positive and the others complex, say

$$X_3 = R, \qquad X_{4,5} = Re^{\pm i\text{A}}.$$

Then

$$\begin{aligned} A_2 &\approx X_1X_2 \\ -A_3 &\approx X_1X_2(X_3 + X_4 + X_5 + X_6) = X_1X_2 \cdot R(1 + 2 \cos \text{A}) \\ A_4 &\approx X_1X_2(X_3X_4 + X_3X_5 + X_4X_5) = X_1X_2 \cdot R^2(1 + 2 \cos \text{A}) \\ -A_5 &\approx X_1X_2X_3X_4X_5 = X_1X_2 \cdot R^3, \end{aligned}$$

so that X_3, X_4, X_5 are approximately the roots of the cubic equation

$$A_2X^3 + A_3X^2 + A_4X + A_5 = 0. \tag{10}$$

The value of R is obtained from

$$R^3 = -A_5/A_2,$$

or more simply from

$$R = -A_4/A_3. \tag{11}$$

Thus equation (2) is split into $n-3$ linear equations and the cubic equation (10). The last equation can be broken up into the linear equation (11), that is $A_3X + A_4 = 0$, and a quadratic equation.

(iib) There are μ pairs of complex roots and one positive root, all of the modulus R.

An equation M of degree $2\mu+1$ results and from that a linear equation L is again split off. It consists of the two middle terms of M, that is

$$L: \ A_{\mu+2}X + A_{\mu+3} = 0 \tag{12}$$

with the solution R.

To find the arguments of the complex roots, we again set $X=RY$ and obtain a reciprocal equation in Y of degree $2\mu+1$. It has the solution $Y=1$, so that by dividing it by $Y-1$ there results a reciprocal equation of degree μ in Z that may be solved by Graeffe's method, yielding μ pairs of complex roots.

(iii) There are *multiple roots*.

Let X_2 have the multiplicity ν. Then equation M, mentioned above in (ic) and (iib), will be divisible by $(X-X_2)^\nu=(X-R)^\nu$, if X_2 is real, and by $[(X-X_2)(X-\overline{X}_2)]^\nu =(X^2-2R\cos AX+R^2)^\nu$, if X_2 is complex. The reciprocal equation in Y is then divisible by $(Y-1)^\nu$ or $(Y^2-2\cos AY+1)^\nu$, respectively.

Note. It is not always possible to eliminate the multiple roots of (2) by eliminating at once the multiple roots of (1), for distinct roots of (1) may, by the successive squarings, give the same roots of (2).

In summary we can say, if the absolute values of the roots of (2) are partly equal, partly different, then Eq. (2) is split up into several approximate equations M_i of lower degrees. The degree of an M is equal to the number of roots having the same modulus R. Thus *there are as many equations M as there are distinct moduli.* To every simple root there corresponds a linear equation.

Determination of the equations M_i. It is well known that by squaring the roots of (1) a series of equations $G_1, G_2, G_3, \cdots$ having the roots $x_i^2, x_i^4, x_i^8, \cdots$ results.

The *problem* now is to decide which equation G first breaks up into equations M_i, and what these M_i are.

We have seen that if the equation M has m roots with equal moduli R, then the absolute member of the normalized M (that is an M whose leading coefficient is 1) is equal to $\pm R^m$. In the following transformed equation it will be equal to $\pm R^{2m}$, that is, from a certain equation G_k the absolute member of a (normalized) M is (to the required degree of accuracy) squared when passing to the following transformed equation. This or a similar relation does not hold for the other coefficients of M, for they involve cos A. Since cos A changes to cos 2A in the following equations, the coefficients not only irregularly change their quantity, but often their signs too.

To find the various M_i into which an equation G_k is eventually split up, we must therefore seek only those coefficients that are squared when passing to the next equation G. That is, we begin with the leading coefficient 1 of all G_k and choose the first member A_i that by the last root-squaring is itself squared. The coefficients from 1 to A_i form the first equation M_1. The next equation M_2 has the leading coefficient A_i. Since A_i is itself squared when going a step further, it is unnecessary to normalize the supposed equation M_2, but we may choose immediately the first coefficient A_j after A_i that is squared by the root-squaring. Equation M_2 now has the coefficients lying between A_i and A_j. If A_k is the next coefficient after A_j that is squared by root-squaring, M_3 extends from A_j to A_k, and so on.

It is not necessary that the same equation G_k yield all the various M_i. The higher the degree of an M is, the later the mentioned quality of the extreme coefficients will generally appear. However, from the stage where an M is split off, it is no longer necessary to keep it during the further calculation. It is sufficient to treat only the rest of the G_k that remain after cancelling the M_i.

Resolution of an equation M. Every M of degree m must be solved separately, in the following manner.

(i) Normalize M to M'.

(ii) Find the modulus R of all roots of M from (8), that is, in the normalized M', from

$$R^m = -\text{absolute member of } M' \text{ if } m \text{ is even,}$$

and from the linear equation L (12), if m is odd

(iii) With this R set $X=RY$, where $Y=e^{i\phi}$, into M' and normalize again to M''. This new equation in Y is reciprocal.

(iv) If this is possible, divide M'' by $Y-1$ (that is, is $Y=1$ a root?), and repeat this division as often as possible. If this is possible s times, then M has the root R of multiplicity s. If m is odd, s is at least 1.

(v) Form the quotient $Q=M''/(Y-1)^s$; this is also a reciprocal polynomial.

(vi) In $Q=0$ set $Z=Y+Y^{-1}$. Then $Q=0$ is transformed into an equation $Q'=0$ in Z of degree $(m-s)/2$. The equation $Q'=0$ has all its roots real.

(vii) $\cos\phi=Z/2$ yields $(m-s)/2$ values of ϕ and therefore $m-s$ values of X: $X=Re^{\pm i\phi}$, and this together with the root $X=R$ of multiplicity s yields the complete system of roots of M.

Solution of equation (1). To every root X_i of M there corresponds one root x_i of (1). Since $X_i=x_i^p$, every X_i yields p tentative values of x_i from which the right root must be chosen. We do not, of course, calculate with the complex values of x_i, but take only the real component of Eq. (1) that is:

$$R^n \cos n\phi + R^{n-1}a_1 \cos (n-1)\phi + R^{n-2}a_2 \cos (n-2)\phi + \cdots + a_n = 0, \quad (13)$$

where ϕ is given by

$$\phi = (2q\pi + \phi_1)/p, \qquad q = 0, 1, 2, \cdots, p-1. \quad (14)$$

After having found the first ϕ satisfying (13), we stop the calculation with this ϕ_i.

If equation (1) is not too complicated, that is, if the M's have a low degree, the process can be abbreviated in the usual manner.

In all other cases the process of finding the complex roots can always be abbreviated in the following manner. In the chain of the transformed equations G_k we go back to the first equation $G_m(m\ll k)$, where M starts to split off, that is, where the double products have no more influence on the first (or second) decimal place of the corner-coefficients of M. This equation M is solved to one or two decimal places only. The root of Eq. (1) is now—to one or two decimal places—to be selected from a group of 2^m members, instead of from the 2^k members of (14). This decreases the work involved.

In addition, the process is abbreviated by the fact that not all 2^k equations of (13) have to be computed, since the members of the majority of them differ from those of the others only in the sign. An example will make this clearer.

Nevertheless this part of the labor is the most tedious of the whole process if the first transformed equation G_m, from which M is first split, has a large m, that is, if two or more roots of (1) are close together. It is good, however, to have a method that yields these roots at all.

In cases where there are *only one pair or two pairs of complex roots* this whole process is superfluous as will be seen in the example.

Convergence of the process. If the coefficients of the transformed equation G_k are determined exactly and the roots of G_k are calculated by splitting G_k into several M, then the roots of the M's are only approximately the roots of G_k. If, thereby, a root of G_k is found with the relative error ϵ_k, then the error of the same root in the following equation is

$$\epsilon_{k+1} \approx \epsilon_k^2,$$

as is easily seen. Let us suppose that we are dealing with the largest root X_1 and there are m complex pairs of roots of the same modulus R, then we see that R is calculated from the absolute member a of the equation M_k. Let now $X_2 = R \cdot 10^{-i}$ be the following root and let the m pairs of complex roots be

$$X' = Re^{\pm iA}, \quad X'' = Re^{\pm iB}, \cdots, \quad X^{(m)} = Re^{\pm i\Gamma}.$$

Then a, being the sum of the combinations of all roots of G_k by $2m$, is equal to

$$a = R^{2m} + 2R^{2m-1}(\cos A + \cos B + \cdots + \cos \Gamma) \cdot X_2 + \cdots$$
$$\approx R^{2m}(1 + 2S \cdot 10^{-i}),$$

where S is the sum of the cosines. Thus

$$R \approx \sqrt[2m]{a}\,(1 - (1/m)10^{-i}S),$$

that is, since $S < m$, the relative error ϵ_k of R is less than 10^{-i}.

From the following equation G_{k+1}, equation M_{k+1} is set up, and if b is the absolute member of M_{k+1}, then

$$b \approx R^{4m} + R^{4m-2} \cdot 2S'X_2^2, \quad \text{where} \quad S' < m.$$

Thus

$$R' \equiv R^2 \approx \sqrt[2m]{b}\,(1 - 10^{-2i}).$$

That is, the relative error of R' is

$$\epsilon_{k+1} < 10^{-2i} \approx \epsilon_k^2.$$

Now the roots $x_1, x_2, x_3, \cdots$ of (1) are the 2^kth roots of the roots X_i of G_k. Thus if ϵ is the relative error of X_1 of G_k, then

$$x_1 = \sqrt[2^k]{X_1(1 + \epsilon)} \approx \sqrt[2^k]{X_1}\,(1 + \epsilon/2^k).$$

That is, the relative error of x_1 is $\epsilon/2^k$. But from G_{k+1} we obtain, as we have seen

$$x_1 = \sqrt[2^{k+1}]{X_1^2(1 + \epsilon^2)} \approx \sqrt[2^k]{X_1} \cdot (1 + \epsilon^2/2^{k+1}),$$

that is, the relative error of x_1 is $\epsilon^2/2^{k+1}$ or the square of that of the equation G_k. Thus, we have:

The relative error of the roots x_i of (1) *as they are computed from the transformed equations G_k decreases quadratically with every following equation G_k, that is, if the roots x_i of* (1) *following from G_k are exact to r decimals, then equation G_{k+1} will yield them to $2r$ decimals exactly. Roughly, every following equation yields tiwce as many exact decimals of the roots x_i of* (1).

It must be taken into account that this property holds only if the roots are already sufficiently separated, for instance, if the difference of any two neighboring roots of G_k is at least equal to 100, or else the approximations above are invalid.

From this property it follows that Graeffe's method has its *greatest efficiency* if it is carried out to *many decimal* places. If a calculating machine is used this does not require more computational work than required for fewer decimal places. Now, it is true that in this case one must calculate more transformed equations at least if the number of decimals are to be fully used. But for this purpose it will be sufficient to have *one or two equations* more. If, for instance, the equation G_5 yields 5 exact decimals of the x_i of (1) then the next smaller root of G_5 has a modulus $r \approx 10^{-5}R$, where R is the modulus of the greatest root. In G_6 the relation of the two greatest roots will then be 10^{-10}, that is, only the 10th decimal place of the coefficients of G_6 will be influenced.

From this it follows that—apart from exceptional cases—*the same number of transformed equations will in general be necessary if a certain exactness is required.* For, suppose we have an equation with two roots having the ratio 1.1. Then by 3 or 4 transformations the ratio will become 3. Thus to have a certain exactness, it will be necessary to calculate 3 or 4 equations more than for an original equation with two roots having the ratio 3. The example is very unfavorable, for there will be few equations having roots of the ratio 1.1. At a ratio 1.5, there are only one or two more transformations required.

Since by raising to powers all roots with distinct moduli will be separated automatically by a quadratically convergent process, Graeffe's method is more powerful than other ones.

Influence of rounding off. These considerations of convergence are strictly valid only if the coefficients of the transformed equations are exact, but in reality the calculation is carried out to a fixed number ν of decimals. The errors of rounding off are, of course, increased by every squaring and multiplication. Now, these errors can be estimated by adding the proper inequality to every coefficient of the scheme. But in general this tedious supplement will be superfluous, for on the whole the error will be annulled by the process of extracting the 2^kth roots of the roots of G_k. That is, *if the calculation is carried out to ν decimals, the roots of* (1) *will on the whole be exact to ν decimals too.*

Moduli or roots lying close together. When the equations of the chain do not soon show signs of an approaching splitting up, this will signify that some moduli of even roots are close to each other.

Various procedures have been proposed for accelerating the convergence. But they are all unpractical. For they require much more work than does the Graeffe's method when carried on two or three steps further. Add to this that by these devices the calculation of the chain is interrupted which is very undesirable when at the end of the calculation the roots of (1) are to be computed from those of the M_i.

These procedures make no allowance for the quadratic convergence (or divergence) of Graeffe's method. For before it becomes obvious that several moduli or roots are close to each other several transformations are already effected. By then the roots will be separated so far that the greatest difficulties have been overcome, and the further calculation will proceed rapidly. It is not advisable, therefore, to disregard the entire calculation performed up to this point and instead, to apply Graeffe's method to a transform of the original equation.

For instance *Encke* (Gesammelte mathematische und astronomische Abhandlungen, vol. 1, Berlin, 1888, p. 185), when dealing with a certain equation states that "after 6 or 7 operations we have got the conviction that two trinomial factors lying close together are existing here." Then he abandons Graeffe's method and starts on a new calculation. Yet, if Graeffe's method is carried 2 or 3 steps further, all roots separate automatically.

Thus the usual procedure of Graeffe will be always the most suitable one. Indeed, one of the chief advantages of the method is, that no special devices are required.

On the other hand, if some knowledge of the position of the roots of $f(x)=0$ is not furnished by the first steps of Graeffe's method, but *by other sources*, then this knowledge may be used from the beginning to accelerate the separation of the roots by transforming the equation first.

If, for instance, several moduli are close to each other and their absolute value ρ is known approximately and if also several roots are close to each other and their values, too, are known approximately, we may proceed as follows.

We have a group G of roots lying near a circle C with radius ρ around the origin of the Gaussian plane. And in this group G there exist several places $u, v, w, \cdots$ where the roots "accumulate" so that G is divided into the subgroups $U, V, W, \cdots$.

Now the slow convergence of G when applying Graeffe's method arises from the fact that the quotient of two moduli of G is lying close to 1. This difficulty will be partly overcome by chosing the origin of the coordinates in the neighbourhood of one of the points $u, v, w, \cdots$, say u. The circle C will still be a circle, but it does no longer have the new origin as its center. Thus the quotients of the moduli of the group G have essentially changed. For the relations of the distances of the subgroup U from the origin to each other as well as to those of the subgroups $V, W, \cdots$ differ now widely from 1. The same holds for the relations of the distances of the group V to the distances of $W, \cdots$. However the relations of the distances of V to each other remain nearly the same as they were before, and the same will hold for W to each other.

By this transformation the separation of the group G will, therefore, be accelerated, that is, the equation of the group G will break up into the equations for the subgroups $U, V, W, \cdots$ faster than would have been the case without this transformation, and the subgroup U will even be split into its individual elements. The method is to be repeated eventually, as far as the subgroups $V, W, \cdots$ are concerned.

The value of this method is largely theoretical because equations with several points of accumulation, $u, v, w, \cdots$, near the same circle C do not occur frequently.

The details of the transformation mentioned above will depend on the nature of the roots:

i. Several roots near the circle C are real. We may suppose that all these real roots are positive. Otherwise we form the first transformed equation and get a new equation with the assumed property. The convergence will be most rapid if the origin of the coordinates is chosen in the neighborhood of the least of these positive roots, say a, i.e., if the following transformation is made:

$$x = y + a', \quad \text{where} \quad a' \approx a \approx \rho.$$

ii. The roots near C are all complex, but "simple," i.e., no two of them are close together. In this case, too, the transformation

$$x = y + \rho', \quad \text{where} \quad \rho' \approx \rho,$$

will be sufficient as is evident geometrically.

iii. All roots are complex, but several of them are lying near $u=a+ib$ (and $u'=a-ib$, of course). Then it will not be convenient in general to bring u into the origin immediately, for this would require a complex transformation. It will be more suitable to do this by two real transformations. First we make the transformation

$$x = y + a', \quad \text{where} \quad a' \approx a,$$

so that u will come near the y-axis. The equation in y will have several roots near $\pm ib$. We transform it by Graeffe's procedure and obtain an equation in Y having several roots near the point $(0;\ -b^2)$. Then by the second transformation

$$Y = U - b^2$$

we obtain an equation in U having several roots near the origin. The ratios of the moduli will now differ widely from 1.

This method will be particularly useful when *all* the roots of the equation are lying near the circle C, for instance, for the equations M_i. In this special case the method may be brought into a more convenient form by applying a procedure of Ostrowski. [Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent, Acta mathematica, 72, 245 (1940)].

In case *iii* above, i.e., if all roots of $f(x)=0$ are lying around $u=a+ib$ and $u'=a-ib$, the sum ma of all roots will be equal to the coefficient $-a_1$, so that the real part of all roots will be approximately $\bar{a}\approx -a_1/m$. After the transformation $x=y+a$, the coefficient of x^{m-1} will then vanish. The same holds for the equation in Y. That is:

If we know that $f(x)=0$ has all its roots near two conjugate complex numbers, we may apply the transformations of *iii* without knowing these points, by bringing $f(x)$ into the reduced form $\bar{f}(x)$, transforming $\bar{f}$ once by Graeffe's procedure into $F(x)$ and bringing the latter into its reduced form $\bar{F}(x)$. The roots of $\bar{F}(x)=0$ differ widely, and Graeffe's method will converge then rapidly.

Example. By the following example we show the efficiency of the method in the case of *roots lying close together* or having the *same moduli.*

$$8x^5 + 4x^4 + 18x^3 - 15x^2 - 18x - 81 = 0 \text{ or normalized:} \qquad \text{(A)}$$

$$x^5 + 0.5^4 + 2.25x^3 - 1.875x^2 - 2.25x - 10.125 = 0.$$

In the coefficients of the transformed equations there is always a power of 10 omitted; its exponent is given in italics on the left of the coefficient, so that, for instance, the last coefficient of G_2 is in reality $-10^4\cdot 1.0509\cdots$.

To avoid slips it is advisable to put under each transformed equation G_k the signs of the equations having the same roots, but of opposite signs. These signs are alternately equal or opposite to the signs of G_k.

As may be seen, it is not until G_5 that the pace of the coefficients begins to become more regular. This late start indicates that the moduli of the roots of (A) lie close together. Also from this point of view it is advisable to carry out the calculation to many decimals. Equation G_5 is the first equation where the approaching split is perceptible, for in the double products forming the coefficient of x^3 two zeros appear. From G_5 onwards the process goes rapidly, so that in G_8 the coefficients of x^5, x^3, x^0

1	*0*	+0,5000 00000 00000 0 −	*0*	+2,2500 00000 00000 0 +	*0*	−1,8750 00000 00000 0 −	*0*	−2,2500 00000 00000 0 +	*1*	−1,0125 00000 00000 0 +
1	*0*	−0,25 4,5	*0*	5,0625 1,875 −4,5	*0*	−3,5156 25 −10,1250 10,1250	*0*	+5,0625 −37,9687 5		*k* = 1
1	*0*	4,25 −	*0*	2,4375 +	*0*	−3,5156 25 +	*1*	−3,2906 25000 −	*2*	−1,0251 56250 +
1	*1*	−1,8062 5 0,4875	*1*	+0,5941 40625 2,9882 81250 −6,5812 5	*2*	−0,1235 96191 40625 0 −1,6041 79687 5 8,7138 28125	*3*	1,0828 21289 06250 0 −0,7208 12988 28125 0		*k* = 2
1	*1*	−1,3187 5 +	*1*	−2,9988 28125 −	*2*	+6,9860 52246 09375 0 −	*2*	3,6200 83007 81250 0 +	*4*	−1,0509 45336 91406 25 +
1	*2*	−1,7391 01562 5 −0,5997 65625	*2*	+8,9929 70123 29101 6 184,2571 27990 72265 6 7,2401 66015 625	*5*	−4,8804 92598 51515 3 −0,2171 20134 77325 4 −2,7718 68326 11084 0	*5*	1,3015 00098 34528 0 146,8391 80629 40478 3		*k* = 3
1	*2*	−2,3388 67187 5 +	*4*	2,0049 02641 29638 7 +	*5*	−7,8694 81059 39924 7 +	*7*	1,4814 96807 27750 1 +	*8*	−1,1044 86101 18141 2 +
1	*4*	−5,4702 99720 76416 0 4,0098 05282 59277 4	*8*	4,0196 34601 07722 90 −3,6811 34206 49632 75 0,2962 99361 45550 02	*11*	−6,1928 73214 42434 95 5,9405 13723 96565 23 −0,5166 49260 22060 19	*14*	2,1948 32789 97342 90 −1,7383 46490 72336 84		*k* = 4
1	*4*	−1,4604 94438 17138 6 +	*7*	6,3479 97560 36401 7 +	*10*	−7,6900 87506 79299 1 +	*13*	4,5648 62992 50060 6 +	*16*	−1,2198 89547 70291 6 +
1	*8*	−2,1330 44003 92955 24 1,2695 99512 07280 34	*15*	4,0297 07302 63875 1 −2,2462 66006 54448 5 0,0912 97259 85001 2	*21*	−5,9137 44586 21336 40 5,7955 47827 95796 66 −0,3563 28379 92070 33	*27*	2,0837 97414 03015 87 −1,8762 11474 09150 94		*k* = 5
1	*7*	−8,6344 44918 56749 0 +	*15*	1,8747 38555 94427 8 +	*20*	−4,7452 51381 76100 7 +	*26*	2,0758 59399 38649 3 +	*32*	−1,4881 30508 59482 5 +
1	*15*	−7,4553 63905 17759 5 3,7494 77111 88855 6	*30*	3,5146 44653 14403 7 −0,0819 45223 36114 3 0,0004 15171 87987 7	*41*	−2,2517 41067 61047 5 7,7833 87305 49838 0 −0,2569 83618 16203 7	*52*	4,3091 92246 02125 1 −14,1231 06704 30060 7		*k* = 6
1	*15*	−3,7058 86793 28903 9 +	*30*	3,4331 14601 66277 1 +	*41*	5,2746 62619 72586 8 −	*52*	−9,8139 14458 27935 6 −	*64*	−2,2145 32410 61069 3 +
1	*30*	−13,7335 96924 67411 6 6,8662 29203 32554 2	*61*	1,1786 27586 81501 3 0,0003 90946 05083 0 −0,0000 00019 62782 9	*83*	−2,7822 06575 19333 6 −6,7384 58605 23764 8 −0,0016 41361 28275 9	*105*	9,6312 91699 44245 8 23,3618 22652 83927 4		*k* = 7
1	*30*	−6,8673 67721 34857 4 +	*61*	1,1790 18513 23801 4 +	*83*	−9,5223 06541 71374 3 +	*106*	3,2993 11435 22817 3 +	*128*	−4,9041 53787 64520 5 +
1	*61*	−4,7160 73942 02203 1 2,3580 37026 47602 8	*122*	1,3900 84654 55797 7 −0,0000 00130 78636 1	*167*	−9,0674 32187 43643 4 7,7798 98526 14379 7 −0,0000 00067 35725 5	*212*	10,8854 55946 62738 7 −9,3397 71157 77744 6		*k* = 8
1	*61*	−2,3580 36915 54600 3	*122*	1,3900 84523 77161 6	*167*	−1,2875 33728 64989 2	*212*	1,5458 84788 84994 1	*257*	−2,4050 72447 09578 9

are fixed to 16 places since they are no longer influenced by the double products.

Thus G_8 is split into the two equations M:

$$M_1 \equiv X^2 - 2.358036915546003 \cdot 10^{61} X + 1.390084523771616 \cdot 10^{122} = 0$$

$$M_2 \equiv 10^{122} \cdot 1.390084523771616 X^3 - 10^{167} \cdot 1.287533728649992 X^2$$
$$+ 10^{212} \cdot 1.545884788849941 X - 10^{256} \cdot 2.405072447095789 = 0.$$

These two equations must now be solved. Since in G_8 the coefficient of X^4 is approximately twice as large as the coefficient of X^3 in G_7, this signifies that M_1 has a real double-root, so that M_1 may be put into the form

$$M_1 \equiv (X - m)^2 = X^2 - 2mX + m^2$$

as is approximately confirmed. Thus from the coefficient of X we have

$$X_1 = X_2 = 1.17901845777300 \cdot 10^{61},$$

whereas from the coefficient of X^0 we have

$$X_1 = X_2 = 1.17901845777393 \cdot 10^{61}.$$

The difference of the two values arises from the rounding off errors we mentioned above. To annul them slightly we could take the mid-value of the two values:

$$X_1 = X_2 = 1.17901845777346 \cdot 10^{61}.$$

Equation M_2 no longer splits into factors as can be seen from the course of its coefficients. Otherwise some of the double products should begin to converge to zero. Now, this is only the case with the product $2A_4A_0$, and there only because of the coefficient A_4. This follows from the fact that equation M_1 is already split off and has no influence on M_2. Thus M_2 has all its roots with the same modulus R. Since its degree is odd, we have according to p. 179, ii and according to (12):

$$R = 10^{212} \cdot 1.545884788849941 / 10^{167} \cdot 1.287533728649892$$
$$= 10^{45} \cdot 1.200500425311787.$$

We now set $X = RY$ in M_2 and get the reciprocal equation in Y

$$M'' \equiv BY^3 - 1.855595246409359 Y^2 + 1.855595246407256 Y - B = 0$$

or, to make the equation wholly reciprocal, instead of the two middle coefficients we put their arithmetic mean, and divide the resulting equation by $Y - 1$:

$$Q = M''/(Y - 1) = Y^2 + 0.2284659663166439 Y + 1 = 0.$$

Since this equation is no longer divisible by $Y - 1$, we put $Z = Y + Y^{-1}$ and obtain instead of Q:

$$Z = 2 \cos \phi = - 0.2284659663166439,$$
$$\phi = 292^g.71149255575355.$$

(The notation "g" refers to the division of the quadrant into 100 parts, instead of into 90.) Thus the roots of M_2 are

$$X_3 = R, \qquad X_{4,5} = Re^{\pm i\phi}.$$

The roots of the original equation arise from the X_i by extracting the 256th root. Now

$$\sqrt[256]{X_1} = \sqrt{3}, \qquad r = \sqrt[256]{R} = 1.5,$$
$$\phi/256 = 1^g.143405439670912305,$$

so that the roots x_i are to be found among the values

$$\sqrt{3}\cdot(\cos 2k\pi/256 + i \sin 2k\pi/256)$$

and

$$1.5(\cos\phi_k + i \sin \phi_k), \quad \text{where} \quad \phi_k = (2k\pi + \phi)/256, \qquad k = 1, 2, \cdots, 256.$$

There are two roots of the first kind, thus either two conjugate-complex ones or two real opposite ones or a real double root, and three roots of the second kind, thus there is always one real root.

Because of the simple character of the moduli of our roots, the procedure of finding their arguments could be very much abbreviated. However, to show the general method, we do not make use of this special property. We begin with the more difficult part, namely equation M_2. Thus we look back in the chain of transformed equations G_1 until we come to the first equation where our M_2 starts to split off. That is G_5, where the coefficient of X^3 is determined to two places. For the transition to G_6 makes two zeros in each of the double products. That is, from G_5 an equation M is split off

$$1.87X^3 - 10^5\cdot 4.75X^2 + 10^{11}\cdot 2.08X - 10^{17}\cdot 1.49 = 0$$

or, putting $X = 1.5Y$:

$$Y^3 - 0.596Y^2 + 0.596Y - 1 = (Y-1)(Y^2 + 0.404Y + 1).$$

This gives

$$\cos \mathrm{B} = -0.202 \quad \text{or} \quad \mathrm{B} = 287^g.\mathrm{B}/32 = 8^g.97.$$

Thus the argument of the roots equals

$$\phi_m = 8^g.97 + 2m\pi/32, \quad \text{where} \quad m = 1, 2, \cdots, 32;$$

by this the number of values to be tried has sunk from 256 to 32. We can *correct these values ϕ_m by comparing them with the former values ϕ_k*, that are nearly exact. For that we must determine the values of k from the equation $\phi_k \equiv (2k\pi + \phi)/256 \approx 8^g.97$, that is $k \approx 5$, thus $k = 5$ and $\phi_5 = 8^g.9559 \cdots \approx 8^g.956$.

With this value we try to verify the original equation, which on putting $x = 1.5\cdot y$, becomes

$$(y-1)\left(3y^4 + 4y^3 + 7y^2 + \frac{16}{3}y + 4\right) = 0$$

or according to (13):

$$3\cos 4\phi_m + 4\cos 3\phi_m + 7\cos 2\phi_m + \frac{16}{3}\cos\phi_m + 4 = 0.$$

Now the values ϕ_m are

$$\phi_1 = 8^g.956,\ \phi_2 = \phi_1 + 12.5^g = 21.456^g, \cdots, \phi_8 = 96^g.456$$
$$\phi_9 = 100^g + \phi_1, \cdots, \phi_{16} = 100^g + \phi_8,$$
$$\phi_{17} = 200^g + \phi_1, \cdots, \phi_{24} = 200^g + \phi_8,$$
$$\phi_{25} = 300^g + \phi_1, \cdots, \phi_{32} = 300^g + \phi_8.$$

These 32 values must be tried. The calculation is carried out to 4–5 decimals. We find the solution ϕ_{13}, that is

$$\phi = \phi_{13} = 146^g.4559054397.$$

(It is not necessary to compute all $32 \cdot 4 = 128$ values of cosines, since $\phi_{17}, \cdots, \phi_{32}$ yield values equal or opposite to those yielded by $\phi_1, \cdots, \phi_{16}$. Also, in the group belonging to ϕ_1 through ϕ_{16} not all values are different.) Thus all the roots of the modulus 1.5 are found.

For finding the roots of modulus $\sqrt{3}$, we do not need the somewhat tedious procedure above, but can abbreviate it in several ways. A method that is always applicable consists in dividing the original equation by the product of the three linear factors already found, thus by

$$(x - 1.5)(x^2 - 3x \cos \phi + 2.25).$$

In the quotient we put $x = y\sqrt{3}$ and get $y^2 + 1 = 0$, thus $y = i$; so we have as roots of the equation

$$x_1 = i\sqrt{3}, \qquad x_2 = -i\sqrt{3}, \qquad x_3 = 1.5, \qquad x_{4,5} = 1.5 \cdot e^{\pm i\phi}.$$

After this general and somewhat tedious way of finding the arguments of the roots x_4, x_5 having the modulus 1.5, we give in the following the simple *method appropriate to all cases where only two or four complex roots exist.*

From M_1, M_2 we determine the moduli of their respective roots, as we did above, that is $X_1 = X_2$ and R. Thus the moduli of the roots of the original equation are, as above,

$$\sqrt[256]{X_1} = \sqrt{3}, \qquad \sqrt[256]{R} = 1.5,$$

and the roots themselves are

$$x_{1,2} = \sqrt{3} \cdot e^{\pm i\psi}, \qquad x_3 = 1.5, \qquad x_{4,5} = 1.5 \cdot e^{\pm i\phi}.$$

Now we use the property of the coefficients of the original equation, namely that the sum of all the roots or their combinations at four are:

$$x_1 + x_2 + x_3 + x_4 + x_5 = -0.5 = 2\sqrt{3}\cos\psi + 1.5 + 3\cos\phi.$$
$$x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots = -18/8 = p/x_1 + p/x_2 + p/x_3 + p/x_4 + p/x_5,$$

where $p = x_1x_2x_3x_4x_5 = 81/8$. By inserting the above values we get the two equations

$$2\sqrt{3}\cos\psi + 3\cos\phi = -2,$$
$$2\sqrt{3}\cos\psi + 4\cos\phi = -8/3,$$

thus

$$\cos\phi = -2/3, \quad \text{or} \quad \phi = 146^g.4559054397.$$

Then

$$\cos\psi = 0 \qquad \text{or} \quad \psi = 100^g.$$

Conclusions. Graeffe's method has the following properties.

i. It yields not only one root at a time but *all roots* simultaneously, even the complex ones; this is accomplished by no other method.

ii. It is the only method that automatically discovers *roots lying close together* which easily escape attention. There is no method other than Graeffe's which solves, without special attention, an equation having, for instance, the roots $\sqrt{5/2}=1.581\cdots$ and $3/2=1.5$, not to mention theoretical cases such as $1.67324\cdots$ and $1.67331\cdots$ Lagrange's method is the only other one with the same advantage of separating those roots, but the process requires great attention and much computational work and fails entirely in the case of complex roots.

iii. An especially valuable property is that even *complex roots of the same modulus* are automatically obtained.

iv. These advantages are not due to special artifices. Any other method requires a first approximation which must then be corrected. But the finding of this first approximation is difficult, particularly in the case of complex roots. Only Bernoulli's method does not require a first approximation, but for that it yields only two real roots at a time, and the process of approximation may be very slow. Graeffe's method on the other hand does not require a first approximate value. Besides, it is *not necessary to use criteria of convergence* in order to determine if the approximate value is sufficiently close the actual root.

Therefore it seems to us that *Graeffe's method is by far the best for solving algebraic equations.* Only if one does not need all roots of the equation, but only a single one, will it be inferior to other methods.

—NOTES—

ON COMPRESSIBLE FLOW ABOUT BODIES OF REVOLUTION*

By W. R. SEARS** (*Northrop Aircraft, Inc.*)

The linear-perturbation theory of compressible fluid flow originated by Glauert and Prandtl has recently been presented in a revised and clarified form by Goldstein and Young.[1] These authors show three alternative procedures by which the compressible flow in an x, y, z-space can be deduced from a corresponding incompressible flow.

The linearized differential equation satisfied by the velocity potential ϕ is

$$\beta^2 \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{1}$$

where β^2 denotes $1-U^2/a^2$, U and a being the stream velocity and the velocity of sound, respectively, in the undisturbed parallel flow. If the solution of (1) for the case $\beta=1$ (incompressible) is $\phi=Ux+f(x, y, z)$, corresponding solutions for $\beta<1$ are given by the following alternative forms:

I) $$\phi = Ux + \frac{1}{\beta} f(x, \beta y, \beta z)$$

II) $$\phi = Ux + f(x, \beta y, \beta z)$$

III) $$\phi = Ux + f(x/\beta, y, z).$$

Each of these variants represents a somewhat different compressible flow, but all three are related to the given incompressible flow. The results determined by the theory are consistent, of course, as far as the linear theory is applicable, and the procedure used in any given problem is the one that provides the greatest ease of calculation. For example, in I the geometry of a slender body remains unaltered as β varies; in II the body is distorted but the pressures on its surface are unchanged; and so forth.

Method II is the one used by Tsien and Lees in a recent paper,[2] while both I and II are presented by Liepmann and Puckett in a new textbook.[3] Sauer[4] writes, in effect,

IV) $$\phi = Ux + \lambda f(x, \beta y, \beta z)$$

and selects the value of λ most convenient for any given problem; this includes both I and II. Finally, B. Göthert,[5] rejecting II because of a fancied discrepancy (actually

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** Now at Cornell University.

[1] S. Goldstein and A. D. Young, *The linear perturbation theory of compressible flow with applications to wind-tunnel interference*, British Aero. Res. Com. Reports and Memoranda No. 1909 (1943).

[2] H. S. Tsien and L. Lees, *The Glauert-Prandtl approximation for subsonic flow of a compressible fluid*, J. Aero. Sci. 12, 173–187, 202 (1945).

[3] H. W. Liepmann and A. Puckett, *Introduction to the aerodynamics of compressible fluids*, John Wiley and Sons, New York, 1946.

[4] R. Sauer, *Theoretische Einführung in die Gasdynamik*, Springer, Berlin, 1943. Reprinted by Edwards Bros., Inc., Ann Arbor, 1945.

[5] B. Göthert, *Ebene und räumliche Strömung bei hohen Unterschallgeschwindigkeiten*, Lilienthal Gesellschaft f. Luftfahrtforschung, Bericht 127, 97–101 (1940).

caused by an error in his application of the method), introduces still another variant by writing

V) $$\phi = \beta^2 Ux + f(x, \beta y, \beta z).$$

We are particularly concerned here with the application of these procedures to the flow about bodies of revolution. There is some confusion on this subject: Goldstein and Young,[1] using I, find that the superstream velocities at a streamline body in the absence of trailing vortices are $1/\beta$ times those at the same location in incompressible flow, while both Sauer[4] and Göthert[5] conclude that these velocities are unaffected by compressibility, at least for slender bodies.[6]

This confusion is partly due to the fact that, while the procedures are equivalent and must yield consistent results in the linear approximation, they may produce different results when they are applied outside this range. For example, let the maximum velocity at the surface of a slender body in incompressible flow be denoted by $U\cdot[1+F(n)]$ where n is the ratio of maximum diameter to length, so that $F(n)$ is a given function for any family of bodies. The several variants of the theory then yield the following results for the maximum surface velocity (divided by the stream velocity) in the compressible flow:

I) $$\frac{1}{\beta}F(n)$$

II) $$F(n/\beta)$$

III) $$\frac{1}{\beta}F(n)$$

IV) $$\lambda F(n/\lambda\beta)$$

V) $$\frac{1}{\beta^2}F(\beta n).$$

Obviously these results are all the same if $F(n)$ is proportional to n or can be approximated successfully in that form. But for a typical family of bodies, the ellipsoids of revolution, $F(n)$ actually has the form[7]

$$F(n) = \frac{n^2 \log p - 2n^2\sqrt{1-n^2}}{2\sqrt{1-n^2} - n^2 \log p} \quad \text{where} \quad p = \frac{1+\sqrt{1-n^2}}{1-\sqrt{1-n^2}} \tag{2}$$

or, neglecting terms of order n^2,

$$F(n) = -\,n^2 \log n. \tag{2a}$$

The absence of a linear term in this expression is what leads Göthert to the conclusion that there is no correction for compressibility. Sauer's similar conclusion apparently results somewhat analogously from the fact that he considers only the

[6] Göthert admits a correction "for greater thickness ratio" and proceeds to calculate it by means of Method V above.

[7] This is obtained from H. Lamb, *Hydrodynamics*, Cambridge, 1932, §105. Note that the ratio diameter/length, n, is equal to $\sqrt{1-\zeta_0^{-2}}$ in Lamb's notation.

limiting case nM0. Actually, in (2a) we have retained the leading term while neglecting $O(n^2)$, which is consistent with the linear-perturbation theory.

$F(n)$ according to (2) and (2a) is plotted in Figure 1. It is clear that Göthert's and Sauer's conclusion cannot be correct in the range of practical interest ($1/10<n<1/3$ and $0.6\leqq\beta<1$, say), since all of the various procedures listed above result in appreciable corrections to the velocity ratio. It seems more reasonable to conclude merely that the linear-perturbation theory cannot distinguish between the various results. In this situation the formula of Method I might well be adopted by reason of its simplicity.

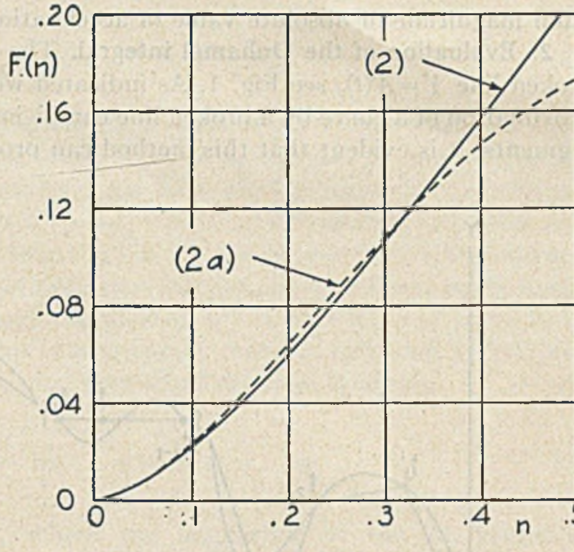

FIG. 1. The superstream velocity ratio for ellipsoids of revolution in incompressible flow.

ON THE NUMERICAL TREATMENT OF FORCED OSCILLATIONS*

BY ALVIN C. SUGAR** (*Northrop Aircraft*)

1. Introduction. The differential equation, with typical initial conditions, of an harmonic oscillator subject to the action of a general disturbing force $ma(t)$ is given by

$$\ddot{x} + \omega^2 x = a(t), \qquad x(0) = 0 = \dot{x}(0). \tag{1}$$

This equation occurs in problems involving from one to infinitely many degrees of freedom. Its solution can be expressed as follows:

$$x = \frac{D}{\omega}, \quad \text{where} \quad D = \int_0^t a(\tau) \sin \omega(t - \tau) d\tau \tag{2}$$

is the so-called Duhamel integral. If in (1) we replace only x by D/ω we obtain an expression for the acceleration of the body.

$$\ddot{x} = a(t) - \omega D. \tag{3}$$

In this note a simple expression which is an approximation of D is found. This expression provides a convenient process for evaluating x and related quantities. Using the resulting simplified form of the acceleration a quick and easy vector method of obtaining the maximum acceleration is explained. Rapid methods of finding the

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maximum displacement are also considered. By maximum acceleration is meant maximum magnitude or absolute value of acceleration and similarly for displacement.

2. Evaluation of the Duhamel integral. The curve $y=a(t)$ is approximated by a broken line $Y=A(t)$, see Fig. 1. As indicated we are taking $A(t_0)=0$. Since the approximation of a curve by a broken line can be improved by increasing the number of segments, it is evident that this method can produce a solution which is as accurate

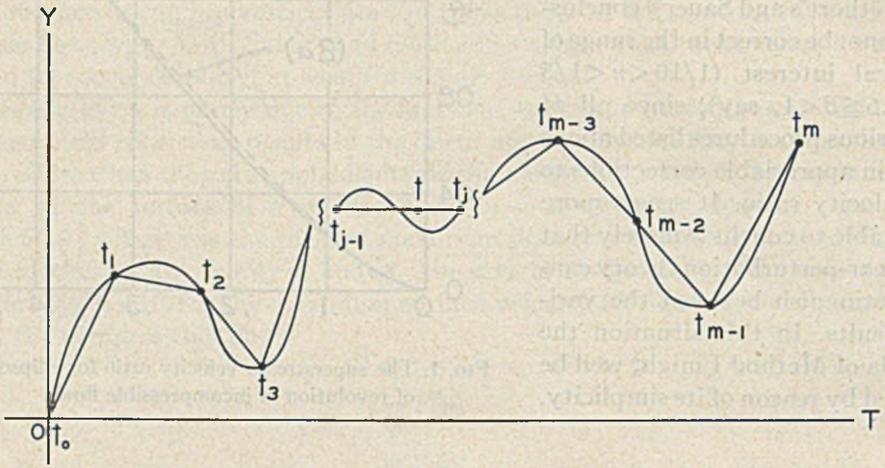

FIG. 1

as desired. Frequently in engineering problems the forcing function is known only approximately and the additional error introduced by a broken line consisting of relatively few segments is negligible.

Let $A(t)=A_i(t)$, $t_{i-1}\leqq t\leqq t_i$, where $y_i=A_i(t)$ is the equation of a straight line of slope μ_i. Since the lines $y_i=A_i(t)$ and $y_{i+1}=A_{i+1}(t)$ intersect at the point $(t_i, A_i(t_i))$, it follows that

$$A_{i+1}(t_i) = A_i(t_i). \tag{4}$$

With the notations of Fig. 1 the Duhamel integral can be approximated in the following manner:

$$D = \int_0^t a(\tau) \sin \omega(t-\tau)d\tau$$

$$\approx \sum_{i=1}^{j-1}\int_{t_{i-1}}^{t_i} A_i(\tau) \sin \omega(t-\tau)d\tau + \int_{t_{j-1}}^{t} A_j(\tau)\sin\omega(t-\tau)d\tau.$$

Integration by parts enables us to write this in the form

$$\omega D = A(t) - \frac{1}{\omega}\left[\sum_{i=1}^{j}\mu_i \sin\omega(t-t_{i-1}) - \sum_{i=0}^{j-1}\mu_i\sin\omega(t-t_i)\right]$$

where we have defined $\mu_0=0$ in order to obtain the following compact result:

$$\omega D = A - \frac{1}{\omega}\left[\sum_{i=0}^{j-1}(\mu_{i+1}-\mu_i)\sin\omega(t-t_i)\right]. \tag{5}$$

From (3) and (5) it is evident that

$$\ddot{x} \approx A - \omega D = \frac{1}{\omega} \sum_{i=0}^{j-1} (\mu_{i+1} - \mu_i) \sin \omega(t - t_i). \tag{6}$$

3. Maximum acceleration and displacement. It is, at times, of importance to know the maximum displacement or acceleration. In this paragraph we show a vector method of obtaining the maximum acceleration. Since the expression for $\ddot{x}$ for the jth time interval is, apart from a constant factor, a sum of sinusoids of the same frequency, ω, it is equal to a sinusoid of frequency ω. The amplitude of this sinusoid can be obtained by vector methods. It is evident that this amplitude is equal to the maximum of the absolute value of the resultant sinusoid over a time interval equal to or in excess of one half period. From this it is apparent that the following vector procedure can be used in determining $\max|\ddot{x}|$ over all of the time intervals.

If

$$\frac{\pi}{\omega} \leqq \min (t_{i+1} - t_i), \qquad i = 0, 1, \cdots, m, \tag{7}$$

then $\max|\ddot{x}| = \max OP_i$ (Fig. 2), where the magnitude of the ith vector is $|(\mu_i - \mu_{i-1})/\omega|$ and its argument with respect to OP_1 is equal to the phase angle $-\omega t_{i-1}$.

Without condition (7) $\max OP_i$ is an upper bound of $|\ddot{x}|$ and probably a pretty good approximation of $\max|\ddot{x}|$.

Fairly rapid methods of computing the maximum displacement[1] can be devised e.g. when the frequency is large, then, for the ith interval, $\max|x| \approx (1/\omega^2)[\max|A_i(t)| + \max|\ddot{x}|]$. For any frequency the problem of finding $\max|x|$ may be reduced to that of finding the maximum value of the curve obtained by the superposition of a sinusoid and a straight line. This can be handled by obvious methods involving use of elementary differential calculus.

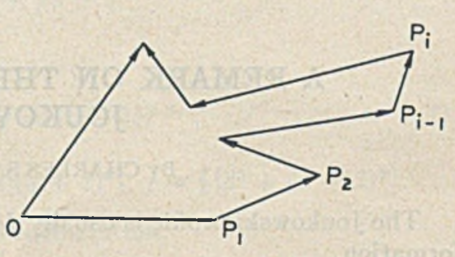

FIG. 2

A still better method of approximating the maximum displacement is available if s is known or can be quickly evaluated, where s is defined by the differential equation,

$$\ddot{s} = a(t), \qquad \dot{s}(0) = 0 = s(0). \tag{8}$$

This method permits direct use of the above vector procedure. This is easily shown by evaluating the Duhamel Integral by repeated integration by parts.[2] Thus,

$$x = \frac{1}{\omega}\int_0^t \ddot{s} \sin \omega(t-\tau)d\tau = s - \omega \int_0^t s \sin \omega(t - \tau)d\tau$$

$$\approx \frac{1}{\omega}\sum_{k=0}^{j-1} (\nu_{k+1} - \nu_k) \sin \omega(t - t_k), \tag{9}$$

[1] A mechanical analyzer was invented by M. A. Biot to obtain this maximum [Bulletin Seismological Soc. Amer. 31, 151–171 (1941)].

[2] As was done by G. W. Housner in obtaining his formulas (2) and (3), Bulletin Seismological Soc. Amer. 31, 143–149 (1941).

where ν_k and t_k are defined by a broken line approximation of s. If the initial conditions of (8) defining s are changed to $\dot{s}(0)=s_0$ and $s(0)=s_0$, then (9) takes the following form:

$$x = -\frac{\dot{s}_0}{\omega}\sin\omega t - s_0\cos\omega t + s - \omega\int_0^t s\sin\omega(t-\tau)d\tau, \tag{10}$$

which again can be expressed as a sum of sinusoids of the same frequency.

On the basis of limited experience, the following suggestions for computation seem good. If the curve is sufficiently smooth, then the term containing μ_1 will make a sizeable contribution; consequently the first time interval should be as small as convenient. It seems best to take $A(0)=a(0)$. The vector polygon will obviously be simplest if t_i is selected so that as many values as possible of ωt_i are multiples of π.

If we had assumed $A(0)\neq 0$, then it would follow that

$$\omega D = A - A(0)\cos\omega t - \frac{1}{\omega}\sum_{i=0}^{j-1}(\mu_{i+1}-\mu_i)\sin\omega(t-t_i).$$

If we let $\dot{s}_k=\dot{s}(t_k)$ then it is clear that $\nu_{k+1}\approx\dot{s}_{k+1}$. Substituting in (9) we have[3]

$$x \approx \frac{1}{\omega}\sum_{k=0}^{j-1}(\dot{s}_{k+1}-\dot{s}_k)\sin\omega(t-t_k). \tag{11}$$

In calculating the maximum displacement, (11) would be more convenient than (9) since $\dot{s}$ could be obtained by a single integration of $a(t)$.

A REMARK ON THE RECTIFICATION OF THE JOUKOWSKI PROFILE*

By CHARLES SALTZER (*Brown University*)

The Joukowski profile is usually defined as the image under the Joukowski transformation,

$$\zeta = z + c^2/z \tag{1}$$

of a circle passing through the point $(-c, 0)$ whose center lies in the first quadrant, and whose radius is $c(1+\epsilon)$ where c, and $\epsilon>0$. Although this representation gives the complex potential of the incompressible flow about a Joukowski profile very readily, the representation of this profile as the inverse of a parabola[1] has the advantage, as will be shown below, of introducing a parameter with direct geometrical meaning which permits the immediate rectification of the Joukowski profile in closed form.

In the z_1-plane consider the parabola

$$y_1 = \tfrac{1}{2}x_1^2 \tag{2}$$

---

[3] It is interesting to note that the sum in (11) is the so-called left Cauchy-Stieltjes sum corresponding to D.

* Received Aug. 17, 1945.

[1] In this way the profile later called "Joukowski profile" was introduced by Chaplygin. See Chaplygin's Collected Papers, Leningrad 1933, vol. 2, pp. 144–178, in particular §6.

and the point $-(a+bi)$ as a center of inversion. The coefficient of x_1^2 may be taken as 1/2 since it enters only as a scale factor. Only the case $a>0$, $b>-1/2$ will be treated here.[2] The transformation

$$\zeta_1 = (z_1 + a + ib)^{-1} \tag{3}$$

maps the exterior of the parabola in the z_1-plane on the exterior of a Joukowski profile in the ζ_1-plane. For the proper choice of parameters the profiles in the ζ and ζ_1-planes can be mapped on each other by a linear transformation and a reflection.

Letting $ds = |d\zeta_1|$ and $ds_1 = |dz_1|$ we have for the element of arc length on the Joukowski profile by (2) and (3),

$$ds = |z_1 + a + ib|^{-2}|dz_1| = \frac{4(1 + x_1^2)^{1/2}}{4(x_1 + a)^2 + (x_1^2 + 2b)^2}\,dx. \tag{4}$$

This expression can be simplified by separation into partial fractions. The roots of the denominator can be obtained by equating the latter to zero, transposing one term, extracting the square roots of both sides, and solving the two resulting quadratic equations. For the non-symmetrical case $(a>0)$ this enables us to write

$$s(x_1) = A\left[I\left(\frac{\alpha}{\beta} - \beta, -\beta, d, x_1\right) - I\left(\frac{\alpha}{\beta} + \beta, \beta, f, x_1\right)\right], \tag{5}$$

where

$$I(m, p, q, x_1) = \int_0^{x_1} \frac{(x + m)(1 + x^2)^{1/2}}{x^2 + px + q}\,dx \tag{6}$$

and

$$\alpha = \frac{\sqrt{2}}{g}[1 + (1 + 4a^2g^4)^{1/2}]^{1/2}, \qquad \beta = \frac{4a}{\alpha}, \qquad g = (1 + 2b)^{-1/2}, \tag{7}$$

$$A = -8\beta[\beta^4 + 4\beta^2 + \alpha^2(\beta^2 + 4)]^{-1},\ d = \tfrac{1}{4}[\beta^2 + (\alpha + 2)^2],\ f = \tfrac{1}{4}[\beta^2 + (\alpha - 2)^2].$$

The integrand of (6) can be rationalized by the substitution

$$x = (1 - u^2)/2u \tag{8}$$

which gives

$$I(m, p, q, x_1) = -\frac{1}{2}\int_1^{r - x_1} \frac{(1 + 2mu - u^2)(1 + u^2)^2}{u^2[(u^2 - pu - 1)^2 + (4q - p^2)u^2]}\,du, \tag{9}$$

where $r = \sqrt{1 + x_1}$. The factors of the denominator of this integrand can be found in the same way as the factors of the denominator of (4) were found, and the integration can be carried out directly after expanding (9) in partial fractions with linear and quadratic denominators. It may be remarked that the I's taken individually may not converge over the entire range of u.

The case $a=0$ gives a symmetrical Joukowski profile for which the rectification can be carried out in a simpler way. Here equation (4) becomes

[2] This configuration represents the most frequently used Joukowski profiles to within a scale factor and a reflection. The case $a<0$, by reason of symmetry, can be regarded as a reflection of the case $a>0$, and the case $b \leqq -1/2$ can be treated in a way similar to the treatment of the case $b>-1/2$.

$$ds = 4(1 + x_1^2)^{1/2}[4x_1^2 + (x_1^2 + 2b)^2]^{-1} dx_1, \tag{10}$$

and equation (5) is replaced by

$$s(x_1) = \int_0^{x_1} g(1 + x^2)^{1/2}[\{x^2 + (1 - 1/g)^2\}^{-1} - \{x^2 + (1 + 1/g)^2\}^{-1}]dx. \tag{11}$$

Setting

$$x = t(1 - t^2)^{-1/2} \tag{12}$$

we get, after simplifying and expanding in fractions with quadratic denominators,

$$s(x_1) = g \int_0^{x_1} [(\sigma_1^2 - t^2)^{-1} - (\sigma_2^2 - t^2)^{-1}]dt, \tag{13}$$

where

$$\sigma_1^2 = 1 + g^2(1 + 2g)^{-1}, \qquad \sigma_2^2 = 1 + g^2(1 - 2g)^{-1}. \tag{14}$$

Therefore

$$s(x_1) = g[\sigma_1^{-1} \tanh^{-1} \{x_1\sigma_1^{-1}(1 + x_1^2)^{-1/2}\} - \sigma_2^{-1} \tanh^{-1} \{x_1\sigma_2^{-1}(1 + x_1^2)^{-1/2}\}]. \tag{15}$$

If we denote the slope of the parabola at the point $(x_1, x_1^2/2)$ by $\tan \gamma$ and note that $x_1 = \tan \gamma$, we can write (15) as

$$s(\gamma) = g[\sigma_1^{-1} \tanh^{-1} (\sigma_1^{-1} \sin \gamma) - \sigma_2 \tanh^{-1} (\sigma_2^{-1} \sin \gamma)], \tag{16}$$

where s is measured from the point furthest from the trailing edge.

In order to introduce the usual parameters ϵ and c for the symmetrical case of the Joukowski profile[3] consider the circle in the z-plane,

$$z(\phi) = c[\epsilon + (1 + \epsilon)e^{i\phi}] \tag{17}$$

which is the image of a symmetrical Joukowski profile in the ζ-plane. The distance between the leading and trailing edges of the Joukowski profile in the ζ-plane is (recalling Eq. (1))

$$\zeta[z(0)] - \zeta[z(\pi)] = \frac{4c(1 + \epsilon)^2}{1 + 2\epsilon}. \tag{18}$$

From (3) the corresponding length in the ζ_1-plane is seen to be $1/b$ (i.e. the length in the ζ_1-plane of the image of the upper half of the imaginary axis in the z_1-plane). If the profile in the ζ_1-plane is identified with the profile in the ζ-plane then

$$b = \tfrac{1}{4}(1 + 2\epsilon)(1 + \epsilon)^{-2}c^{-1}. \tag{19}$$

If the vertex of the parabola in the z_1-plane which corresponds to the given Joukowski profile in the ζ-plane is at the origin, and if the x_1-axis coincides with the tangent to the parabola at this point, then it is readily seen by comparing the positions of the profiles in the ζ-plane and the ζ_1-plane that

$$\zeta_1 = - i(\zeta + 2c). \tag{20}$$

Since $a = 0$, Eq. (3) can be written

$$z_1 = \frac{1}{\zeta_1} - ib. \tag{21}$$

[3] See, for instance, H. Glauert, *The elements of aerofoil and air screw theory*, The University Press, Cambridge, 1930, pp. 71–75.

The successive substitution into Eq. (21) of Eqs. (20), (1), (17), and (19) gives after simplification

$$z_1 = x_1 + iy_1 = \frac{\epsilon}{2c(1+\epsilon)^2} \tan \tfrac{1}{2}\phi + i \frac{1}{4c(1+\epsilon)^2} \tan^2 \tfrac{1}{2}\phi. \tag{22}$$

This is the parabola

$$y_1 = c\epsilon^{-2}(1+\epsilon)^2 x_1^2. \tag{23}$$

Since a change in the value of c, effects only a change of scale in the ζ-plane, c may be taken without loss of generality as

$$c = \tfrac{1}{2}\epsilon^2(1+\epsilon)^{-2}, \tag{24}$$

and this parabola becomes the one considered in Eq. (2). Setting this value of c in (19) yields

$$b = \tfrac{1}{2}(1+2\epsilon)\epsilon^{-2}. \tag{25}$$

Hence, by Eqs. (7) and (14),

$$\sigma_1^2 = (1+2\epsilon)^2/(1+\epsilon)(1+3\epsilon), \qquad \sigma_2^2 = 1/(1-\epsilon^2), \qquad g = \epsilon/(1+\epsilon). \tag{26}$$

Formulae (15) and (16) are valid for $\sigma_1 > 0$, $\sigma_2 > 0$, i.e., for $\epsilon < 1$ and thus include those profiles whose thicknesses is less than about 4/5 of their lengths.

It may also be noted that in terms of the variable γ, the slope, $\theta(\gamma)$, and the curvature, $d\theta(\gamma)/ds$, for the symmetrical profile may be written as

$$\theta(\gamma) = \gamma - \arctan \left\{ \frac{4 \tan \gamma(\tan^2 \gamma + (1+2\epsilon)/\epsilon^2)}{4 \tan^2 \gamma - [\tan^2 \gamma + (1+2\epsilon)/\epsilon^2]^2} \right\}, \tag{27}$$

$$\frac{d\theta(\gamma)}{ds} = \frac{\sec \gamma}{8c} [(1+3\epsilon)(1-\epsilon)\epsilon^{-2} \cos^4 \gamma + 6 \cos^2 \gamma - 3\epsilon^2/(1+\epsilon)^2]. \tag{28}$$

CORRECTION AND SUPPLEMENT TO OUR PAPER

THE CYLINDRICAL ANTENNA: CURRENT AND IMPEDANCE*

QUARTERLY OF APPLIED MATHEMATICS 3, 302–335 (1946)

By RONOLD KING AND DAVID MIDDLETON (*Harvard University*)

Equation (58) should be written as follows:

$$\psi \equiv \overline{\Psi} = \begin{cases} |\Psi_{K1}(0)| = |\psi_1(0)|/\sin \beta h; & \beta h \leqq \pi/2 \\ |\Psi_{K1}(h-\lambda/4)| = |\psi_1(h-\lambda/4)|; & \beta h \geqq \pi/2. \end{cases} \tag{58}$$

Two lines before this equation $|\psi_1(0)|/\sin \beta h$ should be written instead of $|\psi_1(0)|$.

These changes involve no alternations in the figures. However, the function $|\psi(0)|$ plotted in Fig. 11 to the left of $\beta h = \pi/2$ is not the parameter of expansion ψ defined by (58) as modified above and as indicated in the caption. The parameter of expansion ψ as defined in (58) is plotted in Fig. 11a where the part to the right of $\beta h = \pi/2$ is the same as in Fig. 11, the part to the left of $\beta h = \pi/2$ is obtained from the curves in Fig. 11 by dividing by $\sin \beta h$.

* Received Jan. 25, 1946.

For small values of βh a convenient approximate formula is

$$\Psi_{K1}(0) = \Omega - 2 - j\beta h; \qquad \beta h < 0.5$$

so that

$$|\Psi_{K1}(0)| = \sqrt{(\Omega-2)^2 + \beta^2 h^2} \doteq \Omega - 2 + \tfrac{1}{2}\beta^2 h^2/(\Omega - 2).$$

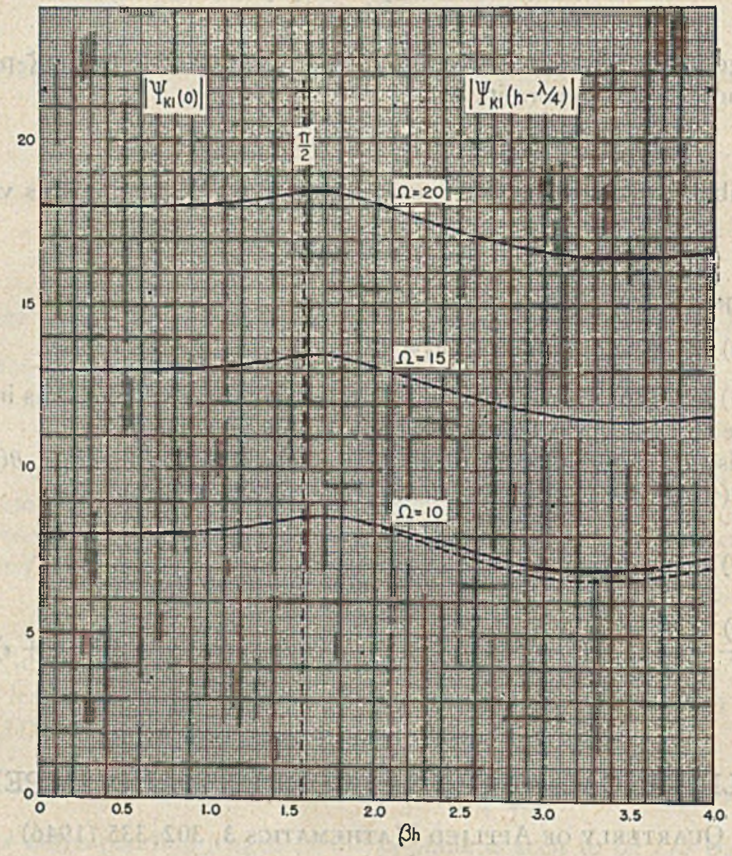

FIG. 11a. The expansion parameter ψ as defined in the corrected equation (58).

The following minor errors and misprints have been called to our attention:

page 312, Eq. (43) change Ψ to ψ,

page 319, Eqs. (59) and 62), change $\Psi_{K1}(z)$ to ψ; line following Eq. (61), delete the following: $\gamma(z)=0$ and

page 320, Eqs. (69) and (70), change b to Ω; Eq. (76), insert $1/(n-1)!$ after the first equality sign,

page 323, Eq. (77b), change 4 to ψ,

page 324, Eq. (79), insert ψ after R_c,

page 329, Eq. (19), third line, change $(R_{2h}+u_2)$ to $(R_{2h}-u_2)$,

page 330, Eqs. (23) and (27), page 335 Eqs. (45) and (46), and in the integral preceding Eq. (45), change R_{2h} to u_2, R_{1h} to u_1, throughout,

page 330, Eq. (24) replace by: $u_2=(h+z)$; $u_1=(h-z)$,

page 332, Eq. (43), add superscript bar over first three symbols Ci.

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