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QUARTERLY OF APPLIED MATHEMATICS

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ON THE MOTION OF A SPINNING SHELL*

BY

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1. **Introduction.** Next after the problem of the motion of a particle in a resisting medium, the problem of the motion of a spinning shell is the major problem of exterior ballistics. Many crude treatments have been given, but the problem was first discussed exhaustively by Fowler, Gallop, Lock and Richmond.[1,2] Reference may also be made to treatments by Cranz[3] and Moulton.[4]

An exact treatment of the motion of a spinning shell as a hydrodynamical problem is obviously out of the question. The problem must be treated aerodynamically. This means that the forces exerted on the shell by the air must be regarded as dependent only on the instantaneous motion of the shell. The connection between the aerodynamic force system and the motion cannot be deduced logically. It must appear in the mathematical theory as a hypothesis, preferably supported by experimental observations.

But although mathematical theory cannot supply the aerodynamic forces, it does give us some information about them. Two basic ideas are important here.

First, the shell has an axis of symmetry. This fact has been used in all existing theories.

The second idea is a little more subtle. It concerns the connection between the position of the mass center (or center of gravity) of the shell and the aerodynamic force system. In one manner of speaking, there is no such connection. For two shells, moving with identical motions but with different mass-distributions, the aerodynamic forces are the same. But we cannot introduce the aerodynamic force system into the mathematical argument without expressing that force system mathematically as a force and a couple (or something equivalent). To do this, we must use a base-point,

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† On leave with U. S. Naval Ordnance Plant, Indianapolis, Ind.

[1] R. H. Fowler, E. G. Gallop, C. N. H. Lock, H. W. Richmond, *The aerodynamics of a spinning shell*, Phil. Trans. Roy. Soc. London (A) **221**, 295–387 (1920).

[2] R. H. Fowler, C. N. H. Lock, *The aerodynamics of a spinning shell*, Part II, Phil. Trans. Roy. Soc. London (A) **222**, 227–249 (1921).

[3] C. Cranz, *Lehrbuch der Ballistik*, J. Springer, Berlin, 1925, p. 358.

[4] F. R. Moulton, *New methods in exterior ballistics*, University of Chicago Press, Chicago, 1926, chap. 6.

and reduce the force system to a force at that base-point, together with a couple. It is well known that, for a given force system, the force is independent of the base-point, but the couple is not.

Also, to describe the motion of the shell mathematically, we must use a base-point. The motion is described by the velocity of that base-point and an angular velocity. The angular velocity is independent of the choice of base-point, but the velocity is not.

Now it is natural to use the mass center as base-point. If there are two shells, S_1 and S_2, with mass centers O_1 and O_2, we may use O_1 as base-point for S_1 and O_2 as base-point for S_2. Suppose that the two shells are of identical geometrical form (but O_1 and O_2 are not geometrically corresponding points) and that their motions at the instant are the same. (This means that geometrically corresponding points have equal velocities; the velocities of O_1 and O_2 are *not* the same.) Then the force systems on the two shells are the same. But the moments about O_1 and O_2 are *not* the same.

If we set out to formulate aerodynamic laws, using the mass center as base-point, we must exercise great care. We must ensure *invariance with respect to shift of mass center*. We must make sure, in the case described above, that when we apply our law, first to S_1 and then to S_2, we get equivalent force systems.

Unfortunately, Fowler et al.[1] paid no attention to this fact in formulating their aerodynamic laws (pp. 302–305), although they draw attention to the necessity for invariance (p. 305), and in fact make use of it. By considering a special case, it is easy to see the fallacy in their basic laws.

Consider the two shells described above. Let the velocity of O_1 be directed along the axis of the shell, and let the shell have an angular velocity represented by a vector perpendicular to the axis (plane motion). The yaw is zero, and the effect of the air is a drag along the axis. But now consider S_2. On account of the angular velocity, the velocity of O_2 is not along the axis; there is a yaw, and hence a cross wind force in addition to a drag. It is easy to see that the force systems on the two shells are not equivalent, as they ought to be since the motions are the same.

Thus the theory of Fowler et al. contains a logical contradiction. It is very difficult to discuss critically a theory containing a logical contradiction, for from inconsistent hypotheses we may arrive almost anywhere (at $1=0$, for example.) It may well be, however, that the logical contradiction does not invalidate the physical conclusions of their paper. In the example given above, the yaw of S_2 may well be very small indeed in cases of practical interest, and the logical inconsistency may be no more serious than that involved in writing $\pi=3.14$. Used in one way, this statement leads to $1=0$; used in another way, it leads to important practical results.

Nevertheless it is sound policy, in building up a theory in applied mathematics to make it logically consistent as far as possible. In the present paper we shall take care to state the aerodynamic laws in such a way as to avoid logical inconsistency.

Apart from the thorough treatment of the theory of the aerodynamic force system in sections 3 and 4, the following features of the present paper may be summarized here.

The exact equations of motion of the shell (independent of any aerodynamic hypothesis) are given a very compact form in (2.6). In section 5 it is shown how the aerodynamic functions may be found from high frequency photographs of a shell. Such observations should provide the ultimate test of the validity of the aerodynamic method. In view of the success of the cruder jump card method of Fowler et al., it seems probable that the aerodynamic hypothesis is valid, and, if so, the proposed method of observation should give us all information required concerning the aerodynamic functions.

There are three conditions for the stability of a spinning shell (section 7), but they are too complicated to interpret in the general case. If Magnus effects are absent (section 8), they become much simpler, and in fact there is then just one stability condition (8.19). In this condition the effect of the position of the mass center is shown explicitly. The condition is stronger than the usual condition (8.13b) based on the stability factor; a shell which is considered stable on the basis of the usual condition may in fact be unstable. We are very much indebted to Professor E. J. McShane for his critical comments on this paper in its original form. He has informed us that the existence of second stability condition, stronger than the usual one, has already been pointed out by R. H. Kent (Report No. 85, Ballistic Research Laboratory). This condition is implicit in the paper by Fowler et al. (1.332, equation 3.6234, and 4.12); this is discussed in section 10, where their method is brought into line with the more general method of the present paper.

Some well known facts are confirmed by theory in section 9. For a stable shell, after the oscillations have been damped out, the axis of the shell always points above the trajectory and to the right if the spin is right-handed. The phenomenon of trailing is explained; the axis of the shell turns downward at a rate approximately equal to the rate of turning of the tangent to the trajectory.

Drift also is discussed in section 9. A general condition (9.17) is obtained for standard drift, i.e., drift to the right for right-handed spin. When we specialize to subsonic velocity and flat trajectory, this condition simplifies to (9.20). When the numerical values of Fowler et al. are inserted, this inequality is liberally satisfied, so that the present theory is in agreement with the observed facts.

2. Exact equations of motion. We shall now develop the equations of motion of a shell in convenient form. No assumption is made here regarding the aerodynamic forces, and the only assumption regarding the shell is that it has an axis of dynamic symmetry (i.e., the momental ellipsoid at the mass center is a spheroid). Thus our equations would apply, for example, to a homogeneous projectile of square section or to a bomb with three or more fins, placed symmetrically.

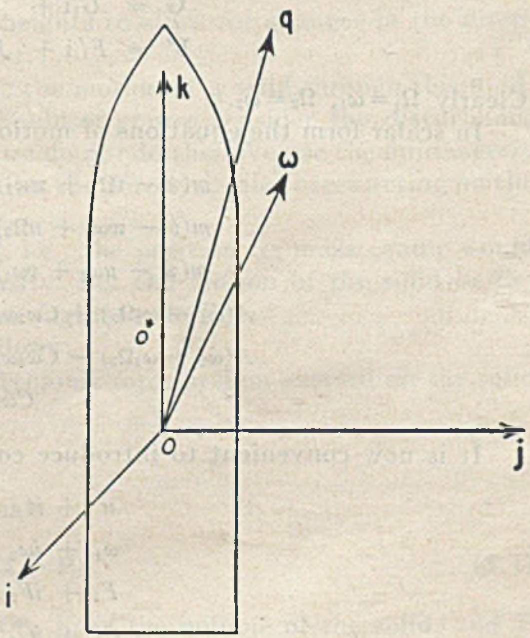

FIG. 1

We shall use the following notation, the motion being referred to a Newtonian reference system:

O = mass center of shell,
m = mass of shell,
A, C = transverse and axial moments of inertia at O,
$\mathbf{q}$ = velocity of O,
$\boldsymbol{\omega}$ = angular velocity of shell,
$\mathbf{h}$ = angular momentum of shell about O,
$\mathbf{F}$ = vector sum of aerodynamic forces acting on shell,
$\mathbf{G}$ = moment of aerodynamic forces about O,
$\mathbf{F}'$ = weight of shell.

Then the equations of motion are

$$m\dot{\mathbf{q}} = \mathbf{F} + \mathbf{F}', \qquad \dot{\mathbf{h}} = \mathbf{G}. \tag{2.1}$$

We introduce a right-handed unit orthogonal triad, $\mathbf{i}, \mathbf{j}, \mathbf{k}$, fixed neither in space nor in the shell (Fig. 1). We take $\mathbf{k}$ along the axis of the shell, and $\mathbf{i}, \mathbf{j}$ perpendicular to $\mathbf{k}$, but the final choice of $\mathbf{i}, \mathbf{j}$ is deferred for the present. Let $\boldsymbol{\Omega}$ be the angular velocity of the triad.

We may now resolve the vectors as follows:

$$\left.\begin{aligned}
\mathbf{q} &= u\mathbf{i} + v\mathbf{j} + w\mathbf{k},\\
\boldsymbol{\omega} &= \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k},\\
\boldsymbol{\Omega} &= \Omega_1\mathbf{i} + \Omega_2\mathbf{j} + \Omega_3\mathbf{k},\\
\mathbf{h} &= A\omega_1\mathbf{i} + A\omega_2\mathbf{j} + C\omega_3\mathbf{k},\\
\mathbf{F} &= F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k},\\
\mathbf{G} &= G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k},\\
\mathbf{F}' &= F_1'\mathbf{i} + F_2'\mathbf{j} + F_3'\mathbf{k}.
\end{aligned}\right\} \tag{2.2}$$

Clearly $\Omega_1 = \omega_1$, $\Omega_2 = \omega_2$.

In scalar form the equations of motion (2.1) then read

$$\left.\begin{aligned}
m(\dot{u} - v\Omega_3 + w\omega_2) &= F_1 + F_1',\\
m(\dot{v} - w\omega_1 + u\Omega_3) &= F_2 + F_2',\\
m(\dot{w} - u\omega_2 + v\omega_1) &= F_3 + F_3',
\end{aligned}\right\} \tag{2.3}$$

$$\left.\begin{aligned}
A(\dot{\omega}_1 - \omega_2\Omega_3) + C\omega_3\omega_2 &= G_1,\\
A(\dot{\omega}_2 + \omega_1\Omega_3) - C\omega_3\omega_1 &= G_2,\\
C\dot{\omega}_3 &= G_3.
\end{aligned}\right\} \tag{2.4}$$

It is now convenient to introduce complex variables. We write

$$\left.\begin{aligned}
u + iv &= \xi,\\
\omega_1 + i\omega_2 &= \eta,\\
F_1 + iF_2 &= F,\\
G_1 + iG_2 &= G,\\
F_1' + iF_2' &= F'.
\end{aligned}\right\} \tag{2.5}$$

We multiply the second equation of (2.3) by i and add it to the first, and deal similarly with the equations (2.4). Thus we reduce the equations of motion to the form:

$$\left.\begin{aligned} \dot{\xi} + i\xi\Omega_3 - iw\eta &= (F + F')/m, \\ \dot{\eta} + i\eta\Omega_3 - iC'\omega_3\eta &= G/A, \qquad (C' = C/A), \\ \dot{w} - u\omega_2 + v\omega_1 &= (F_3 + F_3')/m, \\ \dot{\omega}_3 &= G_3/C. \end{aligned}\right\} \tag{2.6}$$

These equations are exact; no approximations have been made.

3. **The general aerodynamic hypothesis.** What is here set down is probably a little more general and explicit than previous statements about aerodynamic force systems. There is no implication that the hypothesis is physically accurate in all cases. All we can hope is that deductions from these assumptions lead in suitable cases to results in fair agreement with observation. But it seems best to make the hypothesis mathematically clear.

First we consider a fluid, at rest or in motion. We are not particularly conerned with the properties of the fluid. The important thing is that it defines

(i) a scalar field of density ρ;

(ii) a scalar field of local sound velocity c;

(iii) a vector field of velocity $\mathbf{W}$.

This last field defines two other vector fields, vorticity ($\mathbf{V} = 1/2 \text{ rot } \mathbf{W}$) and acceleration ($\mathbf{a} = d\mathbf{W}/dt$).

Usually in ballistics we deal with the static case in which $\mathbf{W} = 0$ and ρ, c are functions of height only. A more accurate model is that in which $\mathbf{W}$ is horizontal, but in different directions at different heights to allow for changes in the direction of the wind with variation of height.

Now suppose we wish to investigate the motion of a solid through this fluid. To treat the problem adequately we should of course consider the disturbance produced in the fluid by the solid. But we do not do this. We use the fluid merely to compute from its undisturbed motion the aerodynamic forces acting on the solid.

Let O^* be the centroid of the solid, i.e., the position its mass center would occupy were the solid of uniform density. Let the motion of the solid be described by the velocity $\mathbf{q}^*$ of O^* and the angular velocity $\boldsymbol{\omega}^*$.

The basic hypothesis is then as follows:

Aerodynamic hypothesis: The aerodynamic force system exerted on the solid by the fluid consists of

(i) an aerostatic force;

(ii) an aerokinetic force system.

The aerostatic force acts at O^* and equals

$$\rho V_0(\mathbf{a} - \mathbf{P}) \tag{3.1}$$

where ρ is the density of the fluid at O^*, V_0 is the volume of the solid, and $\mathbf{P}$ is the body force per unit mass acting on the fluid at O^*. (Note that if $\mathbf{a} = 0$ and $\mathbf{P}$ is gravity, this is simply the Archimedean buoyancy.) *The aerokinetic force sys-*

tem is represented by a force $\mathbf{F}^*$ at O^* and a couple $\mathbf{G}^*$; these are functions of ρ and c at O^* and of the vectors

$$\mathbf{q}^* - \mathbf{W}, \qquad \omega^* - \mathbf{V}. \tag{3.2}$$

If $\mathbf{q}^* = \mathbf{W}$ and $\omega^* = \mathbf{V}$, then $\mathbf{F}^* = 0$ and $\mathbf{G}^* = 0$.

Henceforth we shall assume $\mathbf{W} = 0$, and so $\mathbf{F}^*$, $\mathbf{G}^*$ depend only on ρ, c, $\mathbf{q}^*$, ω^*, while the aerostatic force is $-\rho V_0 \mathbf{P}$. If we were discussing the aerodynamics of a dirigible, the aerostatic force would be very important. For a shell it is quite trivial and we shall omit it altogether.

Thus for our purposes the aerodynamic force system consists of the force $\mathbf{F}^*$ at O^* and the couple $\mathbf{G}^*$; they are functions of ρ, c, $\mathbf{q}^*$, and ω^*.

It will be observed that our base-point O^* has been chosen in a definite way with respect to the *geometry* of the solid, and not with respect to its *mass-distribution*. This frees our laws from the objection raised in the Introduction to the laws of Fowler et al.

It is to be noted that it is by no means essential to select the centroid as base point. But it is least confusing to choose, once and for all, a point simply related to the geometry of the solid, and the centroid seems the most natural point to take.

4. **The aerodynamic force system for a shell with an axis of symmetry.** We now consider a shell with an axis of aerodynamic symmetry. By this we mean that its exterior is a surface of revolution. We might proceed for the present without introducing the mass-distribution of the shell, but it seems simpler to proceed at once to the case of complete symmetry. We shall therefore suppose that the shell has a common axis of aerodynamic and dynamic symmetry. All that is stated in section 2 is then valid and we shall use the same notation.

The mass center of the shell is at O and its centroid at O^*. Let us write

$$\overrightarrow{OO^*} = r\mathbf{k}, \tag{4.1}$$

and

$$\left.\begin{aligned} \mathbf{q}^* &= \text{velocity of } O^*, \\ \omega^* &= \text{angular velocity of shell}, \\ \mathbf{F}^* &= \text{vector sum of aerodynamic forces}, \\ \mathbf{G}^* &= \text{moment of aerodynamic forces about } O^*. \end{aligned}\right\} \tag{4.2}$$

Then

$$\left.\begin{aligned} \mathbf{q}^* &= \mathbf{q} + \omega \times r\mathbf{k}, \qquad \omega^* = \omega, \\ \mathbf{F}^* &= \mathbf{F}, \qquad \mathbf{G}^* = \mathbf{G} + \mathbf{F} \times r\mathbf{k}. \end{aligned}\right\} \tag{4.3}$$

In the notation of (2.5) with asterisks attached to the symbols referring to O^*, we have in consequence

$$\left.\begin{aligned} \xi^* &= \xi - ir\eta, & w^* &= w, \\ \eta^* &= \eta, & \omega_3^* &= \omega_3, \\ F^* &= F, & F_3^* &= F_3, \\ G^* &= G - irF, & G_3^* &= G_3. \end{aligned}\right\} \tag{4.4}$$

Now $\mathbf{F}^*$ and $\mathbf{G}^*$ depend on $\mathbf{q}^*$ and $\boldsymbol{\omega}^*$. It follows from the aerodynamic symmetry that if the pair of vectors $\mathbf{q}^*$, $\boldsymbol{\omega}^*$ is given a rigid body rotation about the axis of symmetry, then the pair of vectors $\mathbf{F}^*$, $\mathbf{G}^*$ is also rotated rigidly about the axis through the same angle. Hence the following ten scalar quantities are unaltered by such a rotation:

$$F_3, \qquad G_3,$$

$$\left.\begin{array}{llll} u^*F_1^* + v^*F_2^*, & v^*F_1^* - u^*F_2^*, & u^*G_1^* + v^*G_2^*, & v^*G_1^* - u^*G_2^*, \\ \omega_1^*F_1^* + \omega_2^*F_2^*, & \omega_2^*F_1^* - \omega_1^*F_2^*, & \omega_1^*G_1^* + \omega_2^*G_2^*, & \omega_2^*G_1^* - \omega_1^*G_2^*. \end{array}\right\} \tag{4.5}$$

But, to within such a rotation, the vectors $\mathbf{q}^*$, $\boldsymbol{\omega}^*$ are determined by the quantities

$$w,\ \omega_3, \quad u^{*2} + v^{*2}, \quad \omega_1^{*2} + \omega_2^{*2}, \quad u^*\omega_1^* + v^*\omega_2^*, \quad u^*\omega_2^* - v^*\omega_1^*, \tag{4.6}$$

between which there exists the identity

$$(u^{*2} + v^{*2})(\omega_1^{*2} + \omega_2^{*2}) - (u^*\omega_1^* + v^*\omega_2^{*2})^2 = (u^*\omega_2^* - v^*\omega_1^*)^2. \tag{4.7}$$

Therefore the quantities (4.5) are functions of the quantities (4.6); in fact, for a shell of given size and shape, (4.5) are functions only of (4.6) and the air scalars ρ, c at O^*.

We now write

$$u^*F_1^* + v^*F_2^* = s_1, \qquad v^*F_1^* - u^*F_2^* = s_2. \tag{4.8}$$

Multiplying the second equation by i and subtracting it from the first, we get

$$\bar{\xi}^*F^* = s_1 - is_2, \tag{4.9}$$

the bar denoting the complex conjugate. Dealing similarly with the other quantities in (4.5), we see that

$$\left.\begin{array}{ll} \bar{\xi}^*F^*, & \bar{\xi}^*G^*, \\ \bar{\eta}^*F^*, & \bar{\eta}^*G^*, \end{array}\right\} \tag{4.10}$$

are complex functions of the real quantities in (4.6).

We cannot proceed further without an additional hypothesis. *We shall assume that*

$$F_1^*, \qquad F_2^*, \qquad G_1^*, \qquad G_2^*$$

are linear functions of

$$u^*, \qquad v^*, \qquad \omega_1^*, \qquad \omega_2^*.$$

This is certainly a reasonable assumption when the latter quantities are small.

We can then write

$$\left.\begin{aligned} F^* &= \alpha_1 u^* + \alpha_2 v^* + \beta_1\omega_1^* + \beta_2\omega_2^*, \\ G^* &= \gamma_1 u^* + \gamma_2 v^* + \delta_1\omega_1^* + \delta_2\omega_2^*, \end{aligned}\right\} \tag{4.11}$$

where the eight complex coefficients are functions of w, ω_3, ρ and c. When we form the quantities (4.10) and use the fact that these must be functions of the quantities (4.6), we find $\alpha_2 = i\alpha_1$, $\beta_2 = i\beta_1$, etc., and so

$$\left.\begin{aligned} F^* &= \xi^*P^* + \eta^*Q^*, \\ G^* &= \xi^*P'^* + \eta^*Q'^*, \end{aligned}\right\} \tag{4.12}$$

where P^*, Q^*, P'^*, Q'^* are complex functions of w, ω_3, ρ, c.

The components F_3, G_3 are functions of the quantities (4.6). *We shall assume that they are functions only of* w, ω_3, ρ, c. This also is a plausible assumption when u^*, v^*, ω_1^*, ω_2^* are small.

To sum up: *There are ten real aerodynamic functions of* w, ω_3, ρ, c, *contained in the set*

$$P^*, \quad Q^*, \quad P'^*, \quad Q'^*, \quad F_3^*, \quad G_3^*. \tag{4.13}$$

Let us see what these assumptions amount to in the case of a shell in a wind-tunnel. We think of the shell as moving and the air at rest. We put

$$v^* = 0, \qquad \omega_1^* = \omega_2^* = \omega_3^* = 0,$$

and (4.12) gives

$$F_1^* + iF_2^* = u^*P^*, \qquad G_1^* + iG_2^* = u^*P'^*.$$

In this simple case we must have, by symmetry since $\omega_3^* = 0$,

$$F_2^* = G_1^* = G_3^* = 0,$$

and so we have

$$F_1^* = u^*P^*, \qquad iG_2^* = u^*P'^*. \tag{4.14}$$

It is easy to see that these equations imply that (for small yaw), the cross wind force and the moment are proportional to the yaw. This is the usual assumption.

We now pass from the centroid O^* to the mass center O by the transformation (4.4). We get for the force system **F**, **G** on the shell

$$\left.\begin{aligned} F &= F_1 + iF_2 = \xi P + \eta Q, \qquad & F_3 &= F_3^*, \\ G &= G_1 + iG_2 = \xi P' + \eta Q', \qquad & G_3 &= G_3^*, \end{aligned}\right\} \tag{4.15}$$

where P, Q, P', Q' are complex functions of w, ω_3, ρ, c, given by

$$\left.\begin{aligned} P &= P^*, \qquad & Q &= Q^* - irP^*, \\ P' &= P'^* + irP^*, \qquad & Q' &= Q'^* - irP'^* + ir(Q^* - irP^*). \end{aligned}\right\} \tag{4.16}$$

This gives the transformation of the aerodynamic functions when we pass from the centroid O^* to the mass center O. Actually this is the transformation for passage from any base-point to any other, provided of course that both lie on the axis.

To show the real and imaginary parts of the aerodynamic functions, we shall write (with similar equations in asterisked form)

$$\left.\begin{aligned} P &= P_1 + iP_2, \qquad & Q &= Q_1 + iQ_2, \\ P' &= P_1' + iP_2', \qquad & Q' &= Q_1' + iQ_2'. \end{aligned}\right\} \tag{4.17}$$

The transformation (4.16) then gives

$$\left.\begin{aligned} P_1 &= P_1^*, \\ P_2 &= P_2^*, \\ Q_1 &= Q_1^* + rP_2^*, \\ Q_2 &= Q_2^* - rP_1^*, \\ P_1' &= P_1'^* - rP_2^*, \\ P_2' &= P_2'^* + rP_1^*, \\ Q_1' &= Q_1'^* + rP_2'^* + r(-Q_2^* + rP_1^*), \\ Q_2' &= Q_2'^* - rP_1'^* + r(Q_1^* + rP_2^*). \end{aligned}\right| \tag{4.18}$$

The method used above for the resolution of the aerodynamic force system is not the usual one. Three important vectors are involved: $\mathbf{k}$ the axis of the shell, $\mathbf{q}$ the velocity of the mass center, ω the angular velocity. In resolving vectors, it is necessary to pick out one of these three as a fundamental vector and build a basic triad on it. The traditional plan is to pick out $\mathbf{q}$ as fundamental and take $\mathbf{k}$ as a secondary vector, so that $\mathbf{q}$ and $\mathbf{k}$ together give one of the planes of the basic triad. Resolution of $\mathbf{F}$ along $\mathbf{q}$ and perpendicular to $\mathbf{q}$ in this plane gives the usual drag and lift. However convenient this may be for wind-tunnel work in which $\mathbf{q}$ is fixed while $\mathbf{k}$ is altered, it certainly appears less convenient than the method of the present paper for a simple mathematical formulation of the problem of the spinning shell. There is a further objection to the usual plan; the direction of $\mathbf{q}$ depends on the mass center.

The conventional terminology does not suit the present resolution. The following is suggested. The asterisk indicates that the centroid is used as base-point. The same notation without asterisks refers to the mass center.

$$\left.\begin{aligned} u^*\mathbf{i} + v^*\mathbf{j} &= \text{cross velocity},\\ w\mathbf{k} &= \text{axial velocity},\\ \omega_1\mathbf{i} + \omega_2\mathbf{j} &= \text{cross spin},\\ \omega_3\mathbf{k} &= \text{axial spin}.\end{aligned}\right\} \tag{4.19}$$

$$\left.\begin{aligned} P_1^*\,|\,\xi^*\,| &= \text{cross force due to cross velocity } (-),\\ P_2^*\,|\,\xi^*\,| &= \text{Magnus force due to cross velocity } (+),\\ Q_1^*\,|\,\eta^*\,| &= \text{Magnus force due to cross spin } (+),\\ Q_2^*\,|\,\eta^*\,| &= \text{cross force due to cross spin } (+),\\ F_3 &= \text{axial force } (-).\end{aligned}\right\} \tag{4.20}$$

$$\left.\begin{aligned} P_1'^*\,|\,\xi^*\,| &= \text{Magnus torque due to cross velocity } (-),\\ P_2'^*\,|\,\xi^*\,| &= \text{cross torque due to cross velocity } (-),\\ Q_1'^*\,|\,\eta^*\,| &= \text{cross torque due to cross spin } (-),\\ Q_2'^*\,|\,\eta^*\,| &= \text{Magnus torque due to cross spin } (+),\\ G_3 &= \text{Magnus axial torque } (-).\end{aligned}\right\} \tag{4.21}$$

It is a consequence of symmetry that where the word "Magnus" is included above, the quantity in question changes sign with ω_3; where the word "Magnus" does not occur, the quantity in question does not change sign with ω_3. For uniformity, we have called the axial (viscous) torque "Magnus"; there is justification for this in the fact that it is the viscous torque that sets up the circulation which is responsible for the other Magnus effects. The signs in parentheses indicate probable signs of the various quantities when ω_3 is positive, assuming a center of pressure in front of the centroid.

Since

$$|\,\xi^*\,| = q^* \sin(\mathbf{q}^*, \mathbf{k}), \qquad |\,\eta^*\,| = \omega \sin(\omega, \mathbf{k}), \tag{4.22}$$

it is clear that the usual sine law of variation is implicit in (4.20), (4.21). But since we suppose the angles in question to be small, the sine, tangent and circular measure are not distinguishable.

It is convenient to introduce positive dimensionless aerodynamic functions, as is done by Fowler et al. So we write, paying attention to dimensions and signs,

$$\left.\begin{aligned} P_1^* &= -\rho a^2 w f_1^*, & P_2^* &= \rho a^3 \omega_3 f_2^*, \\ Q_1^* &= \rho a^4 \omega_3 g_1^*, & Q_2^* &= \rho a^3 w g_2^*, \\ P_1'^* &= -\rho a^4 \omega_3 f_1'^*, & P_2'^* &= -\rho a^3 w f_2'^*, \\ Q_1'^* &= -\rho a^4 w g_1'^*, & Q_2'^* &= \rho a^5 \omega_3 g_2'^*. \end{aligned}\right\} \tag{4.23}$$

Here ρ is the air-density and a the radius of the cross section of the shell. The functions (f^*, g^*) depend certainly on w/c, and possibly also on $a\omega_3/c$ and the Reynolds number. The above equations may be regarded as definitions of the eight aerodynamic functions (f^*, g^*), which are analogous to the f_L, f_M, etc. of Fowler et al. To the above equations we may add

$$F_3 = -\rho a^2 w^2 f_3, \qquad G_3 = -\rho a^4 w \omega_3 g_3, \tag{4.24}$$

where f_3 and g_3 are dimensionless; f_3 is the usual drag except for the slight difference that we resolve along the axis of the shell and treat w as basic instead of q^*.

As the notation is necessarily somewhat complicated, let us summarize as follows:

Askerisked quantities refer to the centroid, unasterisked to the mass center.

The aerodynamic force system is denoted by

$$F^* = F_1^* + iF_2^*, \qquad G^* = G_1^* + iG_2^*, \qquad F_3, \qquad G_3.$$

There are ten real aerodynamic functions contained in the set

$$P^*, \qquad Q^*, \qquad P'^*, \qquad Q'^*, \qquad F_3, \qquad G_3,$$

and these may be expressed in terms of the ten positive dimensionless aerodynamic functions

$$f_1^*, \quad f_2^*, \quad g_1^*, \quad g_2^*, \quad f_1'^*, \quad f_2'^*, \quad g_1'^*, \quad g_2'^*, \quad f_3, \quad g_3.$$

The same notation may be used with reference to the mass center, but since the aerodynamic force system has nothing to do with the mass center as such, the asterisked quantities are the more fundamental. If we wish to pass from O^* to O, we must transform by (4.18) and (4.23). Thus $f_1^* = f_1$, $f_2^* = f_2$, $f_3^* = f_3$, $g_3^* = g_3$, but the other functions change.

One more notation will be introduced for convenience in (6.4).

It is clear from (4.20), (4.21), (4.23) that if the dimensionless aerodynamic functions (f^*, g^*) are constants, we have the following proportionalities, δ denoting the small yaw:

$$\left.\begin{aligned} \text{cross force due to cross velocity} &\propto w^2\delta, \\ \text{cross torque due to cross velocity} &\propto w^2\delta, \\ \text{axial force} &\propto w^2, \\ \text{axial torque} &\propto w\omega_3. \end{aligned}\right\} \tag{4.25}$$

The first three of these are in agreement with experiment for subsonic velocities—the effects vary as the square of the velocity. The last (axial torque) requires comment.

The form of G_3 in (4.24) agrees with Fowler et al., but one may ask why (apart from the theory of dimensions) the factor w should be present. The following is a possible explanation. The rotation of the shell generates a rotating wake. If this wake has, throughout, the same spin as the shell, it has angular momentum $\frac{1}{2}\pi\rho a^4\omega_3$ per unit length. In unit time a length w of wake is generated, and so, by the conservation of angular momentum, the rate of loss of angular momentum of the shell is

$$-G_3 = \tfrac{1}{2}\pi\rho a^4 w\omega_3.$$

This argument not only confirms the form G_3 of (4.24); it gives

$$g_3 = \tfrac{1}{2}\pi. \tag{4.26}$$

A crude argument of this sort must be accepted only provisionally in the absence of experimental check.

5. **Determination of the aerodynamic functions by observation.** Fowler et al. stressed the importance of avoiding the simple empirical assumptions previously em-

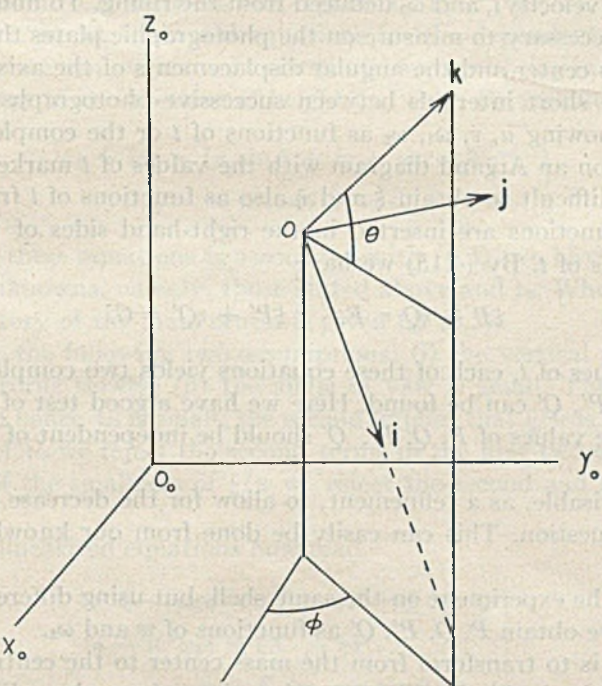

Fig. 2

ployed. As in the case of the drag function, it is necessary to determine the aerodynamic functions experimentally. What follows is a refinement and generalization of the jump card method of Fowler et al. Unless there are technical difficulties, or unless the basic aerodynamic hypothesis is wrong, the following method should yield *all* the aerodynamic functions quite simply, except perhaps g_3, and no doubt a method could be devised for it also.

Let a shell be fired horizontally and observations made of it not long after it leaves the muzzle. These observations consist of high-frequency photographs, one set of photographs being taken vertically and the other set horizontally from the side. These photographs show successive positions of the shell at short intervals of time.

We now turn to the exact equations of motion (2.6). There is some indeterminacy in these because we have not yet chosen the vector **i** definitely. Let us choose it in the vertical plane through the axis of the shell (**k**), pointing downward (Figure 2).

Then

$$F' = mg \cos \theta, \tag{5.1}$$

and the first two equations of (2.6) may be written

$$\left. \begin{aligned} F &= m(\dot{\xi} + i\xi\Omega_3 - iw\eta) - mg \cos \theta, \\ G &= A(\dot{\eta} + i\eta\Omega_3 - iC'\omega_3). \end{aligned} \right\} \tag{5.2}$$

These equations are exact. We may put $\cos \theta = 1$, since the axis of the shell is approximately horizontal. Then $\Omega_3 = 0$ by (6.2).

Now m, A, C' are known for the shell; w may be found from the observations or otherwise (muzzle velocity), and ω_3 deduced from the rifling. To find ξ, η as functions of t, it is merely necessary to measure on the photographic plates the linear displacements of the mass center and the angular displacements of the axis of the shell, corresponding to the short intervals between successive photographs. Smooth graphs might be made showing u, v, ω_1, ω_2 as functions of t or the complex quantities ξ, η might be plotted on an Argand diagram with the values of t marked in. In any case it should not be difficult to obtain $\dot{\xi}$ and $\dot{\eta}$ also as functions of t from these graphs.

When these functions are inserted in the right-hand sides of (5.2), we have F and G as functions of t. By (4.15) we have

$$\xi P + \eta Q = F, \qquad \xi P' + \eta Q' = G. \tag{5.3}$$

If we use two values of t, each of these equations yields two complex equations, and from them P, Q, P', Q' can be found. Here we have a good test of the aerodynamic hypothesis, for the values of P, Q, P', Q' should be independent of the particular instants chosen.

It may be advisable, as a refinement, to allow for the decrease in w between the two instants in question. This can easily be done from our knowledge of the drag function.

By repeating the experiment on the same shell, but using different muzzle velocities and riflings, we obtain P, Q, P', Q' as functions of w and ω_3.

The next step is to transform from the mass center to the centroid. This is done by (4.16), and we obtain P^*, Q^*, P'^*, Q'^* as functions of w and ω_3. Finally, the dimensionless aerodynamic functions (f^*, g^*) are found from (4.23).

It should be stressed that these last functions are characteristic of the *form* of the shell and completely independent of the mass distribution. Indeed, to a certain extent they will be independent of the size of the shell, but this must be accepted with caution.

6. **Plan of solution and partial linearization of the equations.** We now introduce fixed axes $O_0x_0y_0z_0$, O_0z_0 being directed vertically upward. Let θ be the inclination of **k** to the horizontal (Figure 2), and ϕ the inclination of the horizontal projection of **k** to O_0x_0. We have already made the vector **i** definite in section 5. We have then

$$\left. \begin{aligned} \mathbf{F}' &= mg \cos \theta \mathbf{i} - mg \sin \theta \mathbf{k}, \\ \mathbf{\Omega} &= -\dot{\phi} \cos \theta \mathbf{i} - \dot{\theta} \mathbf{j} + \dot{\phi} \sin \theta \mathbf{k}. \end{aligned} \right\} \tag{6.1}$$

Hence

$$\eta = -(\dot{\phi} \cos \theta + i\dot{\theta}), \qquad \Omega_3 = \dot{\phi} \sin \theta. \tag{6.2}$$

We substitute from (4.15) in (2.6), and the equations of motion become

$$\left.\begin{aligned} \dot\xi + i\xi\Omega_3 - iw\eta &= \xi X + \eta Y + g\cos\theta, \\ \dot\eta + i\eta\Omega_3 - iC'\omega_3\eta &= \xi X' + \eta Y', \\ \dot w - u\omega_2 + v\omega_1 &= F_3/m - g\sin\theta, \\ \dot\omega_3 &= G_3/C, \end{aligned}\right\} \tag{6.3}$$

where

$$X = P/m, \qquad Y = Q/m, \qquad X' = P'/A, \qquad Y' = Q'/A. \tag{6.4}$$

If we substitute from (6.2) for η, Ω_3 and regard X, Y, X', Y', F_3, G_3 as known functions of w, ω_3, ρ, c, we have six real equations for the dependent variables u, v, w, θ, ϕ, ω_3. But unless we assume ρ, c to be constants, we must bring in further equations. Let us assume them to be functions of height (z_0) only. By resolution of velocity we have

$$\left.\begin{aligned} \dot x_0 + i\dot y_0 &= (u\sin\theta + iv + w\cos\theta)e^{i\phi}, \\ \dot z_0 &= -u\cos\theta + w\sin\theta. \end{aligned}\right\} \tag{6.5}$$

When the last of these equations is associated with (6.3), we have seven real equations for seven unknowns, namely, those stated above and z_0. When they have been solved, the trajectory of the mass center is given by (6.5).

We now make the following two assumptions: (i) the vertical plane through the axis of the shell turns slowly; (ii) the angle of yaw is small. The first assumption implies that $\dot\phi$ and hence Ω_3 is small; the second implies that ξ/w is small. On account of the smallness of Ω_3 we reject the second terms in the first two equations of (6.3), and on account of the smallness of ξ/w we reject the second and third terms in the third equation.

Our partially linearized equations now read

$$\left.\begin{aligned} \dot\xi - iw\eta &= \xi X + \eta Y + g\cos\theta, \\ \dot\eta - iC'\omega_3\eta &= \xi X' + \eta Y', \\ \dot w &= F_3/m - g\sin\theta, \\ \dot\omega_3 &= G_3/C, \end{aligned}\right\} \tag{6.6}$$

where

$$\eta = -(\dot\phi\cos\theta + i\dot\theta). \tag{6.7}$$

7. **The stability of a spinning shell.** In discussing rapid oscillations of the shell, we treat w and ω_3 as constants in the first two equations of (6.6). Consequently X, Y, X', Y' are constants. In rapid oscillations differentiation with respect to t greatly increases the importance of a term. Hence we shall treat $\cos\theta$ as a constant in the first equation of (6.6); the term corresponding to a small change in θ will be negligible in comparison with the terms in η.

We have then linear equations with constant coefficients, which have solutions of the form

$$\left.\begin{aligned} \xi &= A_1e^{\alpha_1 t} + A_2e^{\alpha_2 t} + A_3, \\ \eta &= B_1e^{\alpha_1 t} + B_2e^{\alpha_2 t} + B_3, \end{aligned}\right\} \tag{7.1}$$

where α_1, α_2 are the roots of the equation

$$\alpha^2 - (iC'\omega_3 + X + Y')\alpha + i(C'\omega_3 X - wX') + XY' - X'Y = 0, \tag{7.2}$$

and

$$\left.\begin{aligned} A_3 &= -\frac{g}{E}\cos\theta(iC'\omega_3 + Y'), \\ B_3 &= \frac{g}{E}\cos\theta\cdot X', \\ E &= i(C'\omega_3 X - wX') + XY' - X'Y. \end{aligned}\right\} \tag{7.3}$$

The condition for stability is that both roots of (7.2) should have non-positive real parts.

If we write

$$\left.\begin{aligned} K_1 &= X_1 + Y_1', \\ K_2 &= C'\omega_3 + X_2 + Y_2', \\ K_3 &= -C'\omega_3 X_2 + wX_2' + X_1Y_1' - X_2Y_2' - X_1'Y_1 + X_2'Y_2, \\ K_4 &= C'\omega_3 X_1 - wX_1' + X_1Y_2' + X_2Y_1' - X_1'Y_2 - X_2'Y_1, \end{aligned}\right\} \tag{7.4}$$

then (7.2) becomes

$$\alpha^2 - (K_1 + iK_2)\alpha + (K_3 + iK_4) = 0. \tag{7.5}$$

The condition for stability may be written

$$K_1 + \zeta\cos\chi \leqq 0, \tag{7.6}$$

where ζ, χ are defined by

$$\left.\begin{aligned} \zeta^4 &= (K_1^2 - K_2^2 - 4K_3)^2 + 4(K_1K_2 - 2K_4)^2, \qquad \zeta \geqq 0, \\ \zeta^2\sin 2\chi &= 2(K_1K_2 - 2K_4), \\ \zeta^2\cos 2\chi &= K_1^2 - K_2^2 - 4K_3, \qquad -\tfrac{1}{2}\pi \leqq \chi \leqq \tfrac{1}{2}\pi. \end{aligned}\right\} \tag{7.7}$$

It is immediately evident that there is instability if $K_1 > 0$. If $K_1 \leqq 0$, then the condition (7.6) is equivalent to

$$K_1^2 \geqq \zeta^2\cos^2\chi, \tag{7.8}$$

or

$$2K_1^2 \geqq \zeta^2(1 + \cos 2\chi). \tag{7.9}$$

On substituting for $\zeta^2\cos 2\chi$ from (7.7), this becomes

$$K_1^2 + K_2^2 + 4K_3 \geqq \zeta^2. \tag{7.10}$$

Thus there is instability if $K_1 \leqq 0$, $K_1^2 + K_2^2 + 4K_3 < 0$. If $K_1 \leqq 0$, $K_1^2 + K_2^2 + 4K_3 \geqq 0$, the condition (7.10) is equivalent to

$$(K_1^2 + K_2^2 + 4K_3)^2 \geqq \zeta^4, \tag{7.11}$$

and, on substitution from (7.7), this becomes

$$K_1^2 K_3 + K_1 K_2 K_4 - K_4^2 \geqq 0. \tag{7.12}$$

To sum up, *the motion of the shell is stable if, and only if, the following three conditions are all satisfied*:

$$K_1 \leqq 0, \tag{7.13a}$$

$$K_1^2 + K_2^2 + 4K_3 \geqq 0, \tag{7.13b}$$

$$K_1^2 K_3 + K_1 K_2 K_4 - K_4^2 \geqq 0. \tag{7.13c}$$

The K's are given by (7.4).

These conditions are more general than any given previously.

If there is strong stability (i.e., if the real parts of α_1, α_2 are negative and large), then the first terms in (7.1) die away quickly. In fact, the rapid oscillations are damped out, and we are left with

$$\left.\begin{aligned} \xi &= -\frac{g}{E}\cos\theta\cdot(iC'\omega_3 + Y'), \\ \eta &= \frac{g}{E}\cos\theta\cdot X'. \end{aligned}\right\} \tag{7.14}$$

With these we associate the last two equations of (6.6), viz.

$$\left.\begin{aligned} \dot{w} &= F_3/m - g\sin\theta, \\ \dot{\omega}_3 &= G_3/C, \end{aligned}\right\} \tag{7.15}$$

and also $\eta = -(\phi\cos\theta + i\dot{\theta})$.

In (7.14), (7.15) and the last of (6.5) we have seven real equations for the seven quantities u, v, w, θ, ϕ, ω_3, z_0. E is a function of w and ω_3 as in (7.3); it also involves z_0, since the properties of the air depend on z_0 and aerodynamic functions X, Y, X', Y' depend on the properties of the air. The above equations determine the motion of the stable shell.

We note that the equations (7.14), (7.15) are simply (6.6) with the terms $\dot{\xi}$, $\dot{\eta}$ deleted. To test whether this treatment is valid, we should solve (7.14), (7.15) for ξ, η, calculate $\dot{\xi}$, $\dot{\eta}$ by differentiating these solutions, and compare these calculated values with the other terms in (6.6). They should, of course, turn out to be small.

8. **Stability in the absence of Magnus effects.** If we accept the linear law (4.11), the aerodynamic force system (4.13) is the most general possible. As we shall see in section 10, the force system of Fowler et al. is a special case. The system (4.13) contains ten real functions, and it appears impossible to make any deductions of physical interest without introducing some simplifications. We shall retain a force system a little more general than that of Fowler et al.; our system satisfies the fundamental condition of invariance with respect to shift of mass center, whereas theirs does not.

Let us refer to (4.20), (4.21), and assume that *all Magnus effects vanish, except* G_3; this means that

$$P_2^* = Q_1^* = P_1'^* = Q_2'^* = 0. \tag{8.1}$$

This leaves us with four real aerodynamic functions, in addition to F_3 and G_3

$$P_1^* < 0, \qquad Q_2^* > 0, \qquad P_2'^* < 0, \qquad Q_1'^* < 0. \tag{8.2}$$

There can be no doubt that these inequalities are physically valid.

We now transform to the mass center O by (4.18). We find

$$P_2 = Q_1 = P_1' = Q_2' = 0. \tag{8.3}$$

Thus the Magnus effects do not reappear with change of base-point; in fact, *the vanishing of Magnus effects is an invariant condition.* For base-point O there are again just four real aerodynamic functions in addition to F_3 and G_3:

$$\left.\begin{aligned} P_1 &= P_1^*, \\ Q_2 &= Q_2^* - rP_1^*, \\ P_2' &= P_2'^* + rP_1^*, \\ Q_1' &= Q_1'^* + rP_2'^* + r(-Q_2^* + rP_1^*). \end{aligned}\right\} \tag{8.4}$$

Then by (6.4), (7.4) and (8.3),

$$\left.\begin{aligned} K_1 &= X_1 + Y_1' = P_1/m + Q_1'/A, \\ K_2 &= C'\omega_3, \\ K_3 &= wX_2' + X_1Y_1' + X_2'Y_2 = \frac{wP_2'}{A} + \frac{1}{mA}(P_1Q_1' + P_2'Q_2), \\ K_4 &= C'\omega_3 X_1 = \frac{C'\omega_3 P_1}{m}. \end{aligned}\right\} \tag{8.5}$$

The stability conditions (7.13) read

$$X_1 + Y_1' \leqq 0, \tag{8.6a}$$

$$(C'\omega_3)^2 + 4wX_2' + (X_1 + Y_1')^2 + 4(X_1Y_1' + X_2'Y_2) \geqq 0, \tag{8.6b}$$

$$X_1Y_1'(C'\omega_3)^2 + (X_1 + Y_1')^2(wX_2' + X_1Y_1' + X_2'Y_2) \geqq 0. \tag{8.6c}$$

These are the stability conditions in the absence of Magnus effects. Now by (4.23), (6.4), (8.4), we have (since $A^* = mr^2 + A$)

$$\left.\begin{aligned} X_1 &= -\frac{\rho a^2 w}{m} f_1^*, \\ Y_2 &= \frac{\rho a^3 w}{m}\left(g_2^* + \frac{r}{a} f_1^*\right), \\ X_2' &= -\frac{\rho a^3 w}{A}\left(f_2'^* + \frac{r}{a} f_1^*\right) \\ Y_1' &= -\frac{\rho a^4 w}{A}\left[g_1'^* + \frac{r}{a}(g_2^* + f_2'^*) + \frac{r^2}{a^2} f_1^*\right], \\ X_1 + Y_1' &= -\frac{\rho a^4 w}{A}\left[g_1'^* + \frac{r}{a}(g_2^* + f_2'^*) + \frac{A^*}{ma^2} f_1^*\right], \\ X_1Y_1' + X_2'Y_2 &= \frac{\rho^2 a^6 w^2}{mA}(f_1^* g_1'^* - g_2^* f_2'^*). \end{aligned}\right\} \tag{8.7}$$

If we substitute these expressions in (8.6) we get stability conditions in terms of the functions (f^*, g^*). However, these conditions are somewhat complicated, and we shall make approximations.

The f's of Fowler et al. hardly exceed 10 in value. Our (f^*, g^*) functions are defined in a slightly different way, but it certainly seems legitimate to assert that the dimensionless quantities

$$\epsilon = \frac{\rho a^3}{m} f \tag{8.8}$$

are much less than unity, f standing for any one of the (f^*, g^*) functions. Then it is clear that

$$(X_1 + Y_1')^2, \qquad X_1 Y_1' + X_2' Y_2$$

are both small relative to wX_2'. Consequently our stability conditions (8.6) may be simplified to

$$X_1 + Y_1' \leqq 0, \tag{8.9a}$$

$$(C'\omega_3)^2 + 4wX_2' \geqq 0, \tag{8.9b}$$

$$X_1 Y_1' (C'\omega_3)^2 + (X_1 + Y_1')^2 wX_2' \geqq 0. \tag{8.9c}$$

It will be noticed that Y_2 has disappeared from the stability conditions in the last approximation. This aerodynamic function corresponds to cross force due to cross spin relative to the mass center [cf. (6.4) and (4.20)]. Thus it might be asserted that, for the discussion of stability in the absence of Magnus effects, cross force due to cross spin may be neglected. But this statement is not entirely correct, because this cross force contributes to the moment Y_1', and Y_1' remains in the stability conditions.

Let us examine the first stability condition (8.9a). On substitution from (8.7) it reads

$$\frac{r}{a}(g_2^* + f_2'^*) + g_1'^* + \frac{A^*}{ma^2} f_1^* \geqq 0. \tag{8.10a}$$

If r is positive (so that the mass center lies behind the centroid), this inequality is certainly satisfied; it is also satisfied for some negative range of r. But an interesting question arises: Can we make the shell unstable by pushing its mass center forward towards the nose? This is hardly to be expected on physical grounds, and it may well be that (8.10a) is satisfied for all permissible values of r, i.e., all values which place the mass center inside the shell.

It is tedious (and perhaps of little physical interest) to discuss the other stability conditions for sufficiently large negative values of r. We shall therefore assume either that r is positive, or, if it is negative, it is such that (8.10a) is satisfied and also

$$X_2' < 0, \qquad Y_1' < 0. \tag{8.11}$$

Let us write

$$s = \frac{(C'\omega_3)^2}{-4wX_2'}. \tag{8.12}$$

This is essentially the same as the usual stability factor.[5] Then the second stability condition (8.9b) takes the familiar form

$$s \geqq 1, \tag{8.13b}$$

while the third condition (8.9c) may be written

$$s \geqq \frac{(X_1 + Y_1')^2}{4X_1Y_1'}. \tag{8.13c}$$

Since the fraction on the right is never less than unity, this condition replaces (8.13b).

Let us substitute in (8.13c) from (8.7) and sum up as follows:

STABILITY CONDITION. The following assumptions are made:

(i) Magnus effects are negligible (except that G_3 may exist).

(ii) The quantities ϵ of (8.8) are very small.

(iii) The mass center is behind the centroid, or, if in front, its negative coordinate r is such that (8.10a) is satisfied and also

$$\left.\begin{aligned} &f_2'^* + \frac{r}{a} f_1^* > 0, \\ &g_1'^* + \frac{r}{a}(g_2^* + f_2'^*) + \frac{r^2}{a^2} f_1^* > 0. \end{aligned}\right| \tag{8.14}$$

Then the motion of the shell is stable if, and only if,

$$s \geqq \frac{ma^2}{4A} \frac{[g_1'^* + (r/a)(g_2^* + f_2'^*) + (A^*/ma^2)f_1^*]^2}{f_1^*[g_1'^* + (r/a)(g_2^* + f_2'^*) + (r^2/a^2)f_1^*]}, \tag{8.15}$$

where s is as in (8.12), or equivalently

$$s = \frac{C^2\omega_3^2}{4\rho a^3 A w^2[f_2'^* + (r/a)f_1^*]}, \qquad A = A^* - mr^2. \tag{8.16}$$

To show the dependence on r more explicitly, we introduce the dimensionless quantity

$$p = \frac{C^2\omega_3^2}{4\rho a^5 m w^2}, \tag{8.17}$$

so that

$$p = s\frac{A}{ma^2}\left(f_2'^* + \frac{r}{a} f_1^*\right). \tag{8.18}$$

Then *the sole condition for stability reads*

$$p \geqq \frac{(f_2'^* + (r/a)f_1^*)[g_1'^* + (r/a)(g_2^* + f_2'^*) + (A^*/ma^2)f_1^*]^2}{4f_1^*[g_1'^* + (r/a)(g_2^* + f_1'^*) + (r^2/a^2)f_1^*]}. \tag{8.19}$$

[5] T. J. Hayes, *Elements of ordnance*, J. Wiley and Sons, New York, 1938, p. 418.

Since A^* is the transverse moment of inertia at the centroid, the position of the mass center is involved in this formula only in the symbol r.

We see therefore that the usually accepted criterion for stability (8.13b) is not the true one; it must be replaced by one of the inequalities (8.13c), (8.15) or (8.19), which are of course equivalent to one another. As we remarked in the Introduction, the existence of a second condition for stability has been noticed by R. H. Kent. We shall refer to stability again in section 10.

9. **The trajectory of a stable shell in the absence of Magnus effects.** Let us assume, as in the preceding section, that Magnus effects are absent, except that G_3 may exist. Then, using (8.3) and (6.4) with (7.14), we get for the trajectory of a stable shell, after the disturbance has been damped out,

$$\left.\begin{aligned} \xi &= -\frac{g}{E}\cos\theta(iC'\omega_3 + Y_1'), \\ \eta &= i\frac{g}{E}\cos\theta\cdot X_2', \qquad \eta = -\dot{\phi}\cos\theta - i\dot{\theta}. \end{aligned}\right\} \tag{9.1}$$

Here E is as in (7.3); let us make the approximation indicated above (8.9), so that

$$E = wX_2' + iC'\omega_3 X_1. \tag{9.2}$$

Splitting (9.1) into real and imaginary parts we get

$$\left.\begin{aligned} u &= -\frac{g}{|E|^2}\cos\theta\left[X_1(C'\omega_3)^2 + wX_2'Y_1'\right] \\ v &= -\frac{g}{|E|^2}\cos\theta\cdot C'\omega_3 wX_2', \end{aligned}\right\} \tag{9.3}$$

(where we have dropped a term X_1Y_1' in comparison with wX_2') and

$$\left.\begin{aligned} \dot{\phi} &= -\frac{g}{|E|^2}C'\omega_3X_2'X_1, \\ \dot{\theta} &= -\frac{g}{|E|^2}\cos\theta\cdot w(X_2')^2. \end{aligned}\right\} \tag{9.4}$$

We shall assume, as in section 8, that X_1, X_2', Y_1' are all negative. Further, since the shell is stable, we have as in (8.9c)

$$X_1Y_1'(C'\omega_3)^2 + (X_1 + Y_1')^2wX_2' \geqq 0. \tag{9.5}$$

But

$$(X_1 + Y_1')^2 > Y_1'^2, \qquad X_2' < 0,$$

and therefore

$$X_1Y_1'(C'\omega_3)^2 + Y_1'^2wX_2' \geqq 0. \tag{9.6}$$

It follows at once from (9.3) that u is positive. This means that *the nose of the shell points above the trajectory.*

From (9.4) we see that $\dot{\phi}<0$ if $\omega_3>0$. Thus *for positive (right-handed) spin the vertical plane through, the axis of the shell turns to the right.*[6] For negative spin it turns to the left.

These two facts are well known to be true in practice.

There remain two outstanding physical facts to explain. These are (i) the trailing of the shell along the trajectory, (ii) the drift.

We see from (9.4) that $\dot{\theta}$ is negative, i.e., the inclination of the axis of the shell to the horizontal decreases steadily. But does it decrease at that rate required for trailing? We must be careful to avoid a circular argument. We have *assumed* that trailing takes place—otherwise the yaw is not small, and all our arguments are based on the smallness of the yaw. We must now *verify* that $\dot{\theta}$, as given by (9.4), is approximately equal to the rate of turning of the tangent to the trajectory of the mass center. The theory of the plane particle-trajectory gives, on resolution along the normal,

$$\dot{\theta}_0 = -\frac{g\cos\theta_0}{w}, \tag{9.7}$$

where θ_0 is the inclination of the tangent to the horizontal. To establish the required result, we must compare this with (9.4), and show that

$$\frac{|E|^2}{(wX_2')^2} = 1, \tag{9.8}$$

approximately. Now by (9.2), (8.12), (8.7), this fraction is

$$1 + X_1^2\left(\frac{C'\omega_3}{wX_2'}\right)^2 = 1 - \frac{4sX_1}{wX_2'}^2$$

$$= 1 + 4s\,\frac{\rho a^3}{m}\,\frac{A}{ma^2}\,\frac{f_1^{*2}}{f_2^{*\prime} + (r/a)f_1^*}\cdot \tag{9.9}$$

The last expression here is of the order of $s\epsilon$, where ϵ is as in (8.8). Hence, unless the stability factor s is very great, this expression is very small, and the condition of trailing is approximately fulfilled.

It is interesting that if s is very great the verification breaks down, for this is just what we would expect. If, by some mechanism, an enormous spin were imparted to a shell, the gyroscopic stability would be so great that the direction of the axis would remain fixed and the shell would not trail.

To discuss the drift, we write down (6.5) again:

$$\dot{x}_0 + i\dot{y}_0 = (u\sin\theta + iv + w\cos\theta)e^{i\phi}. \tag{9.10}$$

This is the horizontal velocity of the mass center in complex form. Consider the complex quantity

$$\alpha + i\beta = \frac{\ddot{x}_0 + i\ddot{y}_0}{\dot{x}_0 + i\dot{y}_0}\cdot \tag{9.11}$$

It is obvious that the vector $\dot{x}_0+i\dot{y}_0$ turns to the left if β is positive, and to the right if β is negative. It is our business to investigate the sign of β.

[6] Hayes, op. cit., 420.

We differentiate (9.10) logarithmically and simplify the result by the fact that u/w and v/w are small. This gives

$$\beta = \frac{d}{dt}\left(\frac{v}{w\cos\theta}\right) + \dot{\phi}. \tag{9.12}$$

With the approximation (9.8), we have from (9.3), (9.4)

$$\frac{v}{w\cos\theta} = -\frac{gC'\omega_3}{w^2X_2'}, \qquad \dot{\phi} = -\frac{gC'\omega_3}{w^2X_2'}X_1, \tag{9.13}$$

and so

$$\frac{\beta}{Z} = \frac{d}{dt}\log|Z| + X_1, \tag{9.14}$$

where

$$Z = -\frac{gC'\omega_3}{w^2X_2'}. \tag{9.15}$$

As a terminology, let us say that a shell has standard *drift* when it goes to the right ($\beta<0$) for right-handed spin ($\omega_3>0$), and vice versa. Now Z has the same sign as ω_3. Hence we get a standard drift if

$$\frac{\beta}{Z} \equiv \frac{d}{dt}\log|Z| + X_1 < 0. \tag{9.16}$$

Substituting from (8.7), we see that this *condition for standard drift* reads

$$f_1^* > \frac{m}{\rho a^2 w}\frac{d}{dt}\log\frac{\omega_3}{\rho w^3(f_2'^* + (r/a)f_1^*)}. \tag{9.17}$$

Let us look into the meaning of this inequality, assuming that the dimensionless aerodynamic functions are constants. This corresponds to a subsonic velocity [cf. (4.25)]. Further, let the trajectory be flat, so that ρ is constant and θ so small that it may be neglected.

Then by (6.6) and (4.24)

$$\dot{w} = -\frac{\rho a^2}{m}w^2f_3, \qquad \dot{\omega}_3 = -\frac{\rho a^4}{C}w\omega_3g_3. \tag{9.18}$$

Let s be the arc length of the trajectory (do not confuse with the stability factor). Then $w=ds/dt$, $\dot{w}=wdw/ds$, and so we have

$$\frac{1}{w}\frac{dw}{ds} = -\frac{\rho a^2}{m}f_3, \qquad \frac{1}{\omega_3}\frac{d\omega_3}{ds} = -\frac{\rho a^4}{C}g_3. \tag{9.19}$$

The right-hand side of (9.17) becomes

$$\frac{m}{\rho a^2}\frac{d}{ds}\log\frac{\omega_3}{w^3} = 3f_3 - \frac{ma^2}{C}g_3,$$

and so the condition for standard drift reads

$$f_1^* + \frac{ma^2}{C} g_3 > 3f_3. \tag{9.20}$$

We may write f_1 in place of f_1^*—they are equal. We note that ma^2/C will lie between 1 and 2.

We observe from (9.20) that cross wind force and axial torque tend to give standard drift, but axial force acts the other way. Let us use the numerical values of Fowler et al. in (9.20). We have ([1], pp. 306, 309)

$$\left.\begin{aligned} f_1 &= f_L + f_R = f_N = 3.34, \\ f_3 &= f_R = 0.34. \end{aligned}\right\} \tag{9.21}$$

We see that (9.20) is liberally satisfied, even if $g_3 = 0$. Thus the present theory appears adequate to explain drift without bringing in Magnus effects.

10. The aerodynamic force system of Fowler, Gallop, Lock, and Richmond.[1] In the preparation of this section we are very much indebted to Professor E. J. McShane, who read our paper in its original form and pointed out in detail the connections between our work and that of Fowler et al.

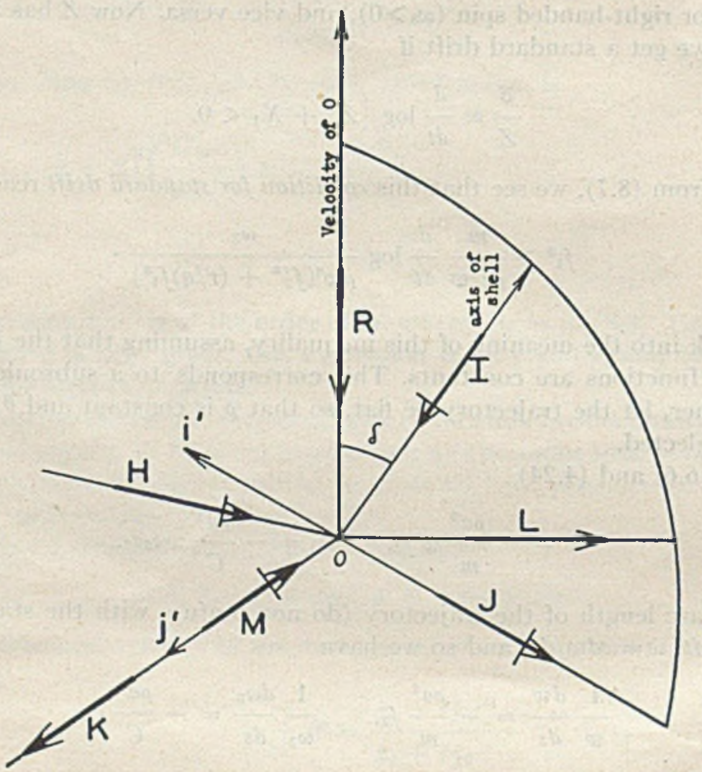

Fig. 3

The axis of the shell is indicated in Figure 3; O is the mass center and δ the yaw. The aerodynamic force system of Fowler et al. is represented by seven vectors—three forces (plain arrows) and four couples (arrows with crossbars). Their terminology is as follows:

$$\left.\begin{aligned} \mathbf{R} &= \text{drag}, \\ \mathbf{L} &= \text{cross wind force}, \\ \mathbf{K} &= \text{swerving force}, \\ \mathbf{M} &= \text{moment tending to increase yaw}, \\ \mathbf{H} &= \text{yawing moment due to yawing}, \\ \mathbf{I} &= \text{axial moment}, \\ \mathbf{J} &= \text{swerving moment}. \end{aligned}\right\} \qquad (10.1)$$

We shall use the notation of the present paper for velocity, angular velocity and the radius of cross section of the shell (a), and consider only the case of small yaw ($\delta = |\xi|/w$). Then the dimensionless aerodynamic functions of Fowler et al. are defined by

$$\left.\begin{aligned} R &= \rho a^2 w^2 f_R, \\ L &= \rho a^2 w^2 \delta f_L = \rho a^2 w |\xi| f_L, \\ K &= \rho a^3 w \omega_3 \delta f_K = \rho a^3 \omega_3 |\xi| f_K, \\ M &= \rho a^3 w^2 \delta f_M = \rho a^3 w |\xi| f_M, \\ H &= \rho a^4 w |\eta| f_H, \\ I &= \rho a^4 w \omega_3 f_I, \\ J &= \rho a^4 w \omega_3 \delta f_J = \rho a^4 \omega_3 |\xi| f_J. \end{aligned}\right\} \qquad (10.2)$$

Let $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}$ be an orthogonal triad of unit vectors, with $\mathbf{k}$ along the axis of the shell. The vector $\mathbf{i}'$ lies as shown in the plane containing $\mathbf{k}$ and the velocity of O. Then, to the first order in δ,

$$\left.\begin{aligned} \mathbf{R} &= -R\delta\mathbf{i}' - R\mathbf{k}, \\ \mathbf{L} &= -L\mathbf{i}', \\ \mathbf{K} &= K\mathbf{j}', \\ \mathbf{M} &= -M\mathbf{j}', \\ \mathbf{H} &= -H\frac{\omega_1'}{|\eta|}\mathbf{i}' - H\frac{\omega_2'}{|\eta|}\mathbf{j}', \\ \mathbf{I} &= -I\mathbf{k}, \\ \mathbf{J} &= -J\mathbf{i}', \end{aligned}\right\} \qquad (10.3)$$

where ω_1', ω_2' are the components of ω along $\mathbf{i}'$, $\mathbf{j}'$.

Let $\mathbf{i}$, $\mathbf{j}$ be any orthogonal unit vectors, perpendicular to $\mathbf{k}$, so that the triad $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is that considered in the present paper. It does not matter at present whether $\mathbf{i}$ lies in the vertical plane through $\mathbf{k}$. We have

$$\mathbf{i}' = \frac{u\mathbf{i} + v\mathbf{j}}{|\xi|}, \qquad \mathbf{j}' = \frac{-v\mathbf{i} + u\mathbf{j}}{|\xi|}, \qquad \omega_1'\mathbf{i}' + \omega_2'\mathbf{j}' = \omega_1\mathbf{i} + \omega_2\mathbf{j}. \qquad (10.4)$$

The total aerodynamic force is

$$\mathbf{F} = \mathbf{R} + \mathbf{L} + \mathbf{K} = \mathbf{i}\left(-R\delta\frac{u}{|\xi|} - L\frac{u}{|\xi|} - K\frac{v}{|\xi|}\right)$$
$$+ \mathbf{j}\left(-R\delta\frac{v}{|\xi|} - L\frac{v}{|\xi|} + K\frac{u}{|\xi|}\right)$$
$$- \mathbf{k}R, \tag{10.5}$$

and so, since $w\delta = |\xi|$,

$$\left.\begin{aligned} F = F_1 + iF_2 &= \xi\left(-\frac{R}{w} - \frac{L}{|\xi|} + \frac{iK}{|\xi|}\right), \\ F_3 &= -R. \end{aligned}\right\} \tag{10.6}$$

The total aerodynamic couple is

$$\mathbf{G} = \mathbf{M} + \mathbf{H} + \mathbf{I} + \mathbf{J} = \mathbf{i}\left(M\frac{v}{|\xi|} - H\frac{\omega_1}{|\eta|} - J\frac{u}{|\xi|}\right)$$
$$+ \mathbf{j}\left(-M\frac{u}{|\xi|} - H\frac{\omega_2}{|\eta|} - J\frac{v}{|\xi|}\right)$$
$$- \mathbf{k}I, \tag{10.7}$$

and so

$$\left.\begin{aligned} G = G_1 + iG_2 &= -iM\frac{\xi}{|\xi|} - H\frac{\eta}{|\eta|} - J\frac{\xi}{|\xi|}, \\ G_3 &= -I. \end{aligned}\right\} \tag{10.8}$$

Certain quantities are defined as follows:

$$\left.\begin{aligned} &\kappa = \frac{L}{m|\xi|}, \quad \lambda = \frac{K}{m\omega_3|\xi|}, \quad \mu = \frac{Mw}{|\xi|}, \\ &\nu = \kappa + \frac{R}{mw}, \quad h = \frac{H}{A|\eta|}, \quad \Gamma = \frac{I}{C\omega_3}, \quad \gamma = \frac{Jw}{C\omega_3|\xi|}. \end{aligned}\right\} \tag{10.9}$$

Then (10.6), (10.8) give

$$\left.\begin{aligned} &F = \xi(i\lambda m\omega_3 - m\nu), \qquad F_3 = -mw(\nu - \kappa), \\ &G = \xi\left(-i\frac{\mu}{w} - \frac{\gamma C\omega_3}{w}\right) - \eta A h, \qquad G_3 = -C\omega_3\Gamma. \end{aligned}\right\} \tag{10.10}$$

Comparing these with (4.15), we see that the force system of Fowler et al. is a particular case of our general system, with

$$\left.\begin{aligned} P &= -m\nu + im\lambda\omega_3, \\ Q &= 0, \\ P' &= -\frac{\gamma C\omega_3}{w} - i\frac{\mu}{w}, \\ Q' &= -Ah. \end{aligned}\right\} \tag{10.11}$$

The general system has eight real parts in these terms; the system of Fowler et al. has only five:

$$\left.\begin{aligned} P_1 &= -m\nu, & P_2 &= m\lambda\omega_3, \\ P_1' &= -\frac{\gamma C\omega_3}{w}, & P_2' &= -\frac{\mu}{w}, \\ Q_1' &= -Ah. \end{aligned}\right| \tag{10.12}$$

It is clear from the transformation (4.16) that $Q=0$ is not invariant with respect to shift of mass center. Thus Eqs. (10.11) describing the aerodynamic force system cannot be valid in general. It may happen of course that they are true for one particular mass center, but they cannot remain true when we shift the mass center.

Fowler et al. find little evidence for the existence of the Magnus effects J, K, or equivalently γ, λ. If we put them equal to zero, the survivors in (10.12) are

$$P_1 = -m\nu, \qquad P_2' = -\frac{\mu}{w}, \qquad Q_1' = -Ah. \tag{10.13}$$

These should be compared with (8.4), which are the general survivors in the absence of Magnus effects. We note that Q_2 is absent from (10.13), which means that the mass center is chosen so that $Q_2^* - rP_1^*$ is zero, or at least negligible.

By (6.4) we obtain from (10.13)

$$X_1 = -\nu, \qquad X_2' = -\frac{\mu}{Aw}, \qquad Y_1' = -h, \tag{10.14}$$

and so the stability condition (8.13c) reads

$$s \geqq \frac{(\nu + h)^2}{4\nu h}. \tag{10.15}$$

We have referred in the Introduction to a second stability condition implicit in the work of Fowler et al.; it is

$$s \geqq \frac{(\kappa + h)^2}{4\kappa h}. \tag{10.16}$$

The difference between (10.15) and (10.16) does not appear to be very great in practice. It is a question of replacing ν by κ, and by (10.9), (10.2)

$$\frac{\nu - \kappa}{\kappa} = \frac{R}{L}\,\frac{|\xi|}{w} = \frac{f_R}{f_L} = \frac{1}{10}, \tag{10.17}$$

roughly.

There are very simple relationships between the dimensionless aerodynamic functions in the two theories. We take the mass center O as base-point, and use (4.23) without asterisks, together with (10.12), (10.9), (10.2); we find

$$\left.\begin{aligned} f_1 &= f_R + f_L = f_N, & f_2 &= f_K, \\ f_1' &= f_J, \quad f_2' = f_M, & g_1' &= f_H. \end{aligned}\right\} \tag{10.18}$$

The functions g_1, g_2, g_2' are zero in the theory of Fowler et al.

As the paper of Fowler et al. is one of the basic papers of modern ballistics, it will be useful to summarize our criticisms as follows:

(i) Their aerodynamic force system is not the most general system consistent with

(a) the aerodynamic hypothesis,

(b) linear dependence on the cross components in the case of small yaw,

(c) the symmetry of the shell.

(ii) Their system does not satisfy the fundamental requirement of invariance with respect to shift of mass center.

(iii) If only shells with mass centers near their centroids are considered, it may be that the above theoretical objections are of small practical importance.

We believe that our exact dynamical equations (6.3) provide a clearer approach to the problem of the spinning shell than do the dynamical equations of Fowler et al. But it is frankly admitted that our simple treatment of the equations of motion in section 7 does not appear to be as satisfactory mathematically as their method. We have made the plausible but rather crude assumption that it is permissible to regard $\cos\theta$, w, ω_3 as constant during the oscillation. It would be interesting to apply their more refined methods to our differential equations, but this we must defer for the present.

TRANSFORMATION GROUPS OF THE THERMODYNAMIC VARIABLES*

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Abstract. A certain class of transformations on the thermodynamic variables E, H, F, G, S, T, P, and V which leave the fundamental equations invariant is investigated and found to form a group of order thirty-two. The quotient group with respect to a normal subgroup of order four gives the octic group obtained by other investigators, the normal subgroup containing trivial but non-excludable transformations. In contradistinction to previous investigators, it is not necessary to use absolute values or a rule of signs. Examples are given of the application of the transformations.

Certain transformations on the fundamental thermodynamic variables will change members of a large class of thermodynamic equations valid for reversible processes into other valid equations of similar form. These transformations have been investigated by Koenig[1] and Buckley[2] and found to form the group of order eight called the octic group. Koenig restricted his transformations to pure substitutions, or permutations, took care of a difficulty in sign by introducing absolute values and a rule of signs, and discussed a geometric method of exhibiting the transformation group. Buckley showed that Koenig's group could be derived in part by Lie's theory of contact transformations, and listed a number of families of thermodynamic equations to which Koenig's transformations apply. Although of course mathematically correct, the application of Lie's theory is not essential in this case.

In order to eliminate the inconvenient and somewhat disturbing use of absolute values and a rule of signs, the following exposition of the theory of these transformations is presented. The transformations considered are not limited to pure permutations but allow changes in sign, and the octic group is finally obtained out of a larger transformation group as a quotient group without the necessity of using absolute values or a rule of signs. The transformations are represented by matrices whose elements in any single row or column are all null except for one element which equals 1 or -1.

The thermodynamic quantities involved are[3]: the internal energy E, the enthalpy H, Helmholtz' function F, Gibbs' function G, the entropy S, the absolute temperature T (intensive), the absolute pressure P (intensive), and the volume V. All these quantities except T and P are extensive quantities. The quantities H, F, and G are defined relative to E by

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[1] F. O. Koenig, *Families of thermodynamic equations, I*, Journal of Chemical Physics, **3**, 29 (1935).

[2] F. Buckley, *Transformations of the fundamental equations of thermodynamics*, Journal of Research NBS, **33**, 213 (1944).

[3] This notation is perhaps the most common in scientific literature, with the symbol U often used in place of E. The functions F and G, invented by Helmholtz and Gibbs respectively, are generally known as the free energy and the thermodynamic potential, while the American Standards Association has used the term free enthalpy for the function G. The standard usage of American chemists is that of Lewis and Randall, where the functions F and G are denoted by A and F, respectively, and are termed the work function and the free energy.

$$H = E + PV, \tag{1a}$$

$$F = E - ST, \tag{1b}$$

$$G = E + PV - ST. \tag{1c}$$

These definitions, together with the fundamental thermodynamic equation for dE give the equations

$$dE = TdS - PdV, \tag{2a}$$

$$dH = TdS + VdP, \tag{2b}$$

$$dF = -\,SdT - PdV, \tag{2c}$$

$$dG = -\,SdT + VdP. \tag{2d}$$

The transformations considered are all transformations which leave Eqs. (1) and (2) invariant, such transformations preserving the validity of any equations derived from Eqs. (1) and (2). The class of equations to which the transformations apply is therefore the class of equations thus derived. If the symbol x is used to denote undetermined matrix elements, the transformations will be of the form

$$\begin{bmatrix} E' \\ H' \\ F' \\ G' \\ S' \\ T' \\ P' \\ V' \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \end{bmatrix} \begin{bmatrix} E \\ H \\ F \\ G \\ S \\ T \\ P \\ V \end{bmatrix}. \tag{3}$$

From the invariance of Eqs. (2), the following seven observations on the transformations are made:

I. The off-diagonal 4 by 4 submatrices are necessarily null. This fact allows the transformations to be put in the separated form

$$\begin{bmatrix} E' \\ H' \\ F' \\ G' \end{bmatrix} = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \begin{bmatrix} E \\ H \\ F \\ G \end{bmatrix}, \tag{4a}$$

$$\begin{bmatrix} S' \\ T' \\ P' \\ V' \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} x & x \\ x & x \end{bmatrix} & \begin{bmatrix} x & x \\ x & x \end{bmatrix} \\ \begin{bmatrix} x & x \\ x & x \end{bmatrix} & \begin{bmatrix} x & x \\ x & x \end{bmatrix} \end{bmatrix} \begin{bmatrix} S \\ T \\ P \\ V \end{bmatrix}. \tag{4b}$$

II. If one of the variables ($EHFG$) is changed in sign, all of them must be thus changed. Such transformations may at this point be excluded as trivial. This exclusion limits the transformation (4a) to pure permutations.

III. The invariance of the equation

$$E - H - F + G = 0 \tag{5}$$

derivable from (1) may be used to limit the transformations (4a) to eight in number, all of which are of the type considered.

IV. Since S is always associated with T in Eqs. (1) and (2), as is P with V, the form of the transformations (4b) must be as shown with two diagonally opposed 2 by 2 submatrices null.

V. The 2 by 2 submatrices of Eqs. (4b) are necessarily of one of five forms, which are abbreviated thus:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e, \tag{6a}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -e, \tag{6b}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i, \tag{6c}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -i, \tag{6d}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = o. \tag{6e}$$

VI. Transformations of the type in which, for example, both P and V are changed in sign are admittedly trivial but cannot be excluded because they are necessary for closure of the group of transformations (4b).

VII. A given transformation (4b) defines at most one transformation (4a). The converse is not true, however, and the correspondence is found to be four to one. Thus the number of transformations (4b) is thirty-two.

The eight transformations (4a), the corresponding thirty-two transformations (4b) expressed using the abbreviations in Eqs. (6), together with eight symbols representing group elements, are listed in Table I.

The transformations (4a) form a group of order eight which is designated as M. The transformations (4b) form a group of order thirty-two which is designated as G. The group G is four to one homomorphic to M, the normal subgroup

$$N = \left(\begin{bmatrix} e & a \\ o & e \end{bmatrix} \begin{bmatrix} c & o \\ o & -e \end{bmatrix} \begin{bmatrix} -e & o \\ o & e \end{bmatrix} \begin{bmatrix} -e & o \\ o & -e \end{bmatrix} \right) \tag{7}$$

of G corresponding to the indentity element of M. From this correspondence is established the isomorphism

$$M \simeq G/N \tag{8a}$$

or the congruence

$$M \equiv G \bmod N. \tag{8b}$$

This group is the octic group, whose multiplication table in terms of the group elements shown in Table I is given in Table II. This multiplication table is consistent with matrix multiplication of the matrices representing either transformations (4a)

TABLE I: The transformations of the thermodynamic variables.

Group Element	Transformations of $\begin{bmatrix} E \\ H \\ F \\ G \end{bmatrix}$	Transformations of $\begin{bmatrix} S \\ T \\ P \\ V \end{bmatrix}$
m_1	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} e & o \\ o & e \end{bmatrix}, \begin{bmatrix} -e & o \\ o & e \end{bmatrix}, \begin{bmatrix} e & o \\ o & -e \end{bmatrix} \begin{bmatrix} -e & o \\ o & -e \end{bmatrix}.$
m_2	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} i & o \\ o & i \end{bmatrix}, \begin{bmatrix} -i & o \\ o & i \end{bmatrix}, \begin{bmatrix} i & o \\ o & -i \end{bmatrix}, \begin{bmatrix} -i & o \\ o & -i \end{bmatrix}.$
m_3	$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} o & e \\ e & o \end{bmatrix}, \begin{bmatrix} o & e \\ -e & o \end{bmatrix}, \begin{bmatrix} o & -e \\ e & o \end{bmatrix}, \begin{bmatrix} o & -e \\ -e & o \end{bmatrix}.$
m_4	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} o & i \\ i & o \end{bmatrix}, \begin{bmatrix} o & i \\ -i & o \end{bmatrix}, \begin{bmatrix} o & -i \\ i & o \end{bmatrix}, \begin{bmatrix} o & -i \\ -i & o \end{bmatrix}.$
m_5	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} e & o \\ o & i \end{bmatrix}, \begin{bmatrix} -e & o \\ o & i \end{bmatrix} \begin{bmatrix} e & o \\ o & -i \end{bmatrix}, \begin{bmatrix} -e & o \\ o & -i \end{bmatrix}.$
m_6	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} i & o \\ o & e \end{bmatrix}, \begin{bmatrix} -i & o \\ o & e \end{bmatrix}, \begin{bmatrix} i & o \\ o & -e \end{bmatrix}, \begin{bmatrix} -i & o \\ o & -e \end{bmatrix}.$
m_7	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} o & e \\ i & o \end{bmatrix}, \begin{bmatrix} o & e \\ -i & o \end{bmatrix}, \begin{bmatrix} o & -e \\ i & o \end{bmatrix}, \begin{bmatrix} o & -e \\ -i & o \end{bmatrix}.$
m_8	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} o & i \\ e & o \end{bmatrix}, \begin{bmatrix} o & i \\ -e & o \end{bmatrix}, \begin{bmatrix} o & -i \\ e & o \end{bmatrix}, \begin{bmatrix} o & -i \\ -e & o \end{bmatrix}.$

or (4b). The non-identical transformations of the subgroup N are those of the trivial type mentioned in observation VI, and their elimination in the process giving Eqs. (8) is tantamount to disregarding a change in sign of both S and T or of both P and V.

To illustrate the general application of these transformations the equations obtained by transforming two given thermodynamic equations are shown in Table III. The two given equations are those shown in the table opposite the symbol m_1, which

TABLE II: Multiplication table for the octic group.

	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
m_1	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
m_2	m_2	m_1	m_4	m_3	m_6	m_5	m_8	m_7
m_3	m_3	m_4	m_1	m_2	m_8	m_7	m_6	m_5
m_4	m_4	m_3	m_2	m_1	m_7	m_8	m_5	m_6
m_5	m_5	m_6	m_7	m_8	m_1	m_2	m_3	m_4
m_6	m_6	m_5	m_8	m_7	m_2	m_1	m_4	m_3
m_7	m_7	m_8	m_5	m_6	m_4	m_3	m_2	m_1
m_8	m_8	m_7	m_6	m_5	m_3	m_4	m_1	m_2

TABLE III: Examples of equations obtained from the transformations.

Transformation	First Example	Second Example
m_1	$\left(\frac{\partial S}{\partial T}\right)_V = \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_V$	$\left(\frac{\partial P}{\partial T}\right)_V = \frac{P}{T} + \frac{1}{T}\left(\frac{\partial E}{\partial V}\right)_T$
m_2	$\left(\frac{\partial T}{\partial S}\right)_P = -\frac{1}{S}\left(\frac{\partial G}{\partial S}\right)_P$	$\left(\frac{\partial V}{\partial S}\right)_P = \frac{V}{S} - \frac{1}{S}\left(\frac{\partial G}{\partial P}\right)_S$
m_3	$\left(\frac{\partial P}{\partial V}\right)_T = \frac{1}{V}\left(\frac{\partial G}{\partial V}\right)_T$	$\left(\frac{\partial S}{\partial V}\right)_T = \frac{S}{V} + \frac{1}{V}\left(\frac{\partial G}{\partial T}\right)_V$
m_4	$\left(\frac{\partial V}{\partial P}\right)_S = -\frac{1}{P}\left(\frac{\partial E}{\partial P}\right)_S$	$\left(\frac{\partial T}{\partial P}\right)_S = \frac{T}{P} - \frac{1}{P}\left(\frac{\partial E}{\partial S}\right)_P$
m_5	$\left(\frac{\partial S}{\partial T}\right)_P = \frac{1}{T}\left(\frac{\partial H}{\partial T}\right)_P$	$\left(\frac{\partial V}{\partial T}\right)_P = \frac{V}{T} - \frac{1}{T}\left(\frac{\partial H}{\partial P}\right)_P$
m_6	$\left(\frac{\partial T}{\partial S}\right)_V = -\frac{1}{S}\left(\frac{\partial F}{\partial S}\right)_V$	$\left(\frac{\partial P}{\partial S}\right)_V = \frac{P}{S} + \frac{1}{S}\left(\frac{\partial F}{\partial V}\right)_S$
m_7	$\left(\frac{\partial P}{\partial V}\right)_S = \frac{1}{V}\left(\frac{\partial H}{\partial V}\right)_S$	$\left(\frac{\partial T}{\partial V}\right)_S = \frac{T}{V} - \frac{1}{V}\left(\frac{\partial H}{\partial S}\right)_V$
m_8	$\left(\frac{\partial V}{\partial P}\right)_T = -\frac{1}{P}\left(\frac{\partial F}{\partial P}\right)_T$	$\left(\frac{\partial S}{\partial P}\right)_T = \frac{S}{P} + \frac{1}{P}\left(\frac{\partial F}{\partial T}\right)_P$

represents the identity transformation. As an example of the carrying out of one of these transformations, the m_7 transformation of the second equation of Table III is here given in detail. The transformed value of E is shown by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} E \\ H \\ F \\ G \end{bmatrix} = \begin{bmatrix} H \\ G \\ E \\ F \end{bmatrix} \tag{9a}$$

to be H. For the $(STPV)$ transformation any one of the four matrices given may be used, as

$$\begin{bmatrix} o & e \\ -i & o \end{bmatrix} \begin{bmatrix} S \\ T \\ P \\ V \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S \\ T \\ P \\ V \end{bmatrix} = \begin{bmatrix} P \\ V \\ -T \\ S \end{bmatrix}. \tag{9b}$$

Hence the equation

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{P}{T} + \frac{1}{T}\left(\frac{\partial E}{\partial V}\right)_T \tag{10a}$$

is transformed into

$$\left(\frac{\partial T}{\partial V}\right)_S = \frac{T}{V} - \frac{1}{V}\left(\frac{\partial H}{\partial S}\right)_V. \tag{10b}$$

Since $m_7 = m_5 m_3$ from Table II, Eq. (10b) can also be obtained by applying m_3 to the equation obtained by the m_5 transformation of Eq. (10a).

CONSTRUCTION OF A COMPLETE SET OF SOLUTIONS OF A LINEAR PARTIAL DIFFERENTIAL EQUATION IN TWO VARIABLES, BY USE OF PUNCH CARD MACHINES*

BY

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1. Introduction. Many problems in Engineering and Physics lead to the determination of a function U satisfying the partial differential equation

$$L(U) \equiv \frac{1}{4}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{1}{2}\alpha(x, y)\frac{\partial U}{\partial x} + \frac{1}{2}\beta(x, y)\frac{\partial U}{\partial y} + \gamma(x, y)U = 0 \quad (1.1)$$

in a domain D, which function assumes certain prescribed values on the boundary of D. Here α, β, γ are polynomials in x, y (or can be approximated by such polynomials).

Recently a procedure, "the method of particular solutions," has been developed for the solution of problems of this kind [2].† The idea of this method is to determine at first a "complete" set of particular solutions of (1.1), i.e., *a set of functions each of which satisfies* (1.1); *this set* [denoted by $p_\nu(x, y)$, $(\nu = 1, 2, \cdots)$], *is chosen in such manner as to possess the property that every solution U can be approximated in any simply connected domain by a conveniently chosen finite combination*

$$\sum_{\nu=1}^{N} a_\nu p_\nu(x, y) \quad (1.2)$$

of the above particular solutions. In the case of the Laplace equation $(\partial^2 U/\partial x^2) + (\partial^2 U/\partial y^2) = 0$, such a set can be obtained by taking the real and imaginary parts of the powers $(x+iy)^{\nu-1}$, $(\nu = 1, 2, \cdots)$; i.e., $p_1 = 1$, $p_2 = x$, $p_3 = y$, $p_4 = x^2 - y^2$, $p_5 = 2xy$, etc. In the following, a procedure will be described for finding analogous solutions for any equation of the form (1.1). The second step of the method consists in indicating a rule for determining the a_ν so that (1.2) assumes on the boundary of D values which approximate the prescribed values of U. In order to apply the method of particular solutions to an actual problem and obtain numerical results of interest to an engineer or physicist, it is frequently necessary to carry out lengthy computations. These computations can as a rule be performed most efficiently by the use of special computing devices, such as punch card machines. Before the computations can be carried out on such machines, however, it is necessary to organize the computations so that they can be given to the operators of the machines. This organization is often a problem in itself, as in the case of the example given below.

In the present paper there is described a working procedure for computing the above set of particular solutions, and for carrying out the associated numerical computations by the use of punch card machines.

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† Numbers in brackets refer to references at the end of this paper.

2. Notation. In the following, it is convenient to use the complex variables z and $\bar{z}$ instead of x and y. We have $z=x+iy$, $\bar{z}=x-iy$, where x and y are cartesian coordinates in the plane. We then have

$$U_z = \frac{\partial U}{\partial z} = \frac{1}{2}\left(\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}\right), \qquad U_{\bar{z}} = \frac{\partial U}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right),$$

$$U_{z\bar{z}} = \frac{1}{4}\Delta U = \frac{1}{4}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right).$$

In terms of z and $\bar{z}$ Eq. (1.1) then assumes the form

$$U_{z\bar{z}} + AU_z + BU_{\bar{z}} + CU = 0, \tag{2.1}$$

where $2A=\alpha+i\beta$, $2B=\alpha-i\beta$, $C=\gamma$.

REMARK. In the above case, $B=\bar{A}$. However, in the following we derive formulas without making this assumption and we note that in general (2.1) is equivalent to a system of two equations, one corresponding to the real part and one to the imaginary part.

3. Some previous results on particular solutions. The method to be described in the following is based on the following theorem, proved in [2] p. 542.

THEOREM. *For every partial differential equation* (2.1) *there exists a function*

$$E(z, \bar{z}, t) = \exp\left[-\int_0^{\bar{z}} A\,d\bar{z}\right]\left[1 + \sum_{p=1}^{\infty} t^{2p} z^p Q^{(p)}(z, \bar{z})\right] \tag{3.1}$$

*such that the functions**

$$\left.\begin{aligned} p_{2\nu+1}(z, \bar{z}) &= \mathrm{Re}\left[z^\nu \int_{-1}^{1} E(z, \bar{z}, t)(1-t^2)^{\nu-1/2} dt\right], \\ p_{2\nu+2}(z, \bar{z}) &= \mathrm{Im}\left[z^\nu \int_{-1}^{1} E(z, \bar{z}, t)(1-t^2)^{\nu-1/2} dt\right], \end{aligned}\right\} \quad (\nu = 0, 1, 2, \cdots), \tag{3.2}$$

*form the required complete set of particular solutions,** i.e., a set possessing the properties mentioned in* §1. *The* $Q^{(p)}$ *are*† *given by the recurrence formula*

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* As has been proved in [1], the series (3.1) converges for $|x| < \infty$, $|y| < \infty$, $|t| \leq 1$ and therefore the functions p_ν are entire functions. We note that they are independent of the domain for which the boundary value problem is considered.

In the N.A.C.A. Technical Notes Nos. 972, 973, 1018 and 1096 the functions $Q^{(p)}(z, \bar{z})$ have been computed for the compressibility equations.

** If $E(z, \bar{z}, t)$ is real, then $p_2(z, \bar{z})$ vanishes identically and we have to change the numeration of the $p_\nu(z, \bar{z})$ accordingly. The proof that (7.1) where f is an arbitrary function of a complex variable is a solution of (1.1) is given in [1] §1. Substituting $Q^{(p)}$ instead of $\phi^{(2p)}/z^p$ into (1.16) of [1] p. 1174 and integrating with respect to $\bar{z}$ we obtain (3.3). (Note that there are some misprints in [1] which are indicated in the Trans. of the Amer. Math. Soc. vol. 57, p. 311, Footnote 15.)

† Unless otherwise indicated the arguments of the functions $Q^{(p)}$, F, A, etc., are $(z, \bar{z})$.

$$\left.\begin{aligned} Q^{(1)} &= -2\int_0^{\bar z} F d\bar z, \\ Q^{(p)} &= -\frac{2}{2p-1}\left[Q_z^{(p-1)} - Q_z^{(p-1)}(z,0) + \int_0^{\bar z}(DQ_z^{(p-1)} + FQ^{(p-1)})d\bar z\right], \\ &\qquad (p = 2, 3, 4, \cdots). \end{aligned}\right\} \tag{3.3}$$

$$Q^{(p)}(z, 0) = 0, \qquad (p = 1, 2, 3, \cdots),$$

where

$$D = -\int_0^{\bar z} A_z d\bar z + B, \qquad F = -AB - A_z + C. \tag{3.4}$$

If the coefficients A, B, C of (2.1) are polynomials

$$A = \sum_{m=0}^{N}\sum_{n=0}^{N} a_{m,n}z^m\bar z^n, \qquad B = \sum_{m=0}^{N}\sum_{n=0}^{N} b_{m,n}z^m\bar z^n, \qquad C = \sum_{m=0}^{N}\sum_{n=0}^{N} c_{m,n}z^m\bar z^n, \tag{3.5}$$

then for the coefficients $d_{m,n}$ and $f_{m,n}$ of the developments

$$D = \sum_{m=0}^{M}\sum_{n=0}^{M} d_{m,n}z^m\bar z^n, \qquad F = \sum_{m=0}^{M}\sum_{n=0}^{M} f_{m,n}z^m\bar z^n, \tag{3.6}$$

we obtain the relations

$$d_{0,n} = b_{0,n}, \qquad d_{m,n} = -\frac{(m+1)a_{m+1,n-1}}{n} + b_{m,n}, \tag{3.7}$$

$$f_{m,n} = -\left[(m+1)a_{m+1,n} + \sum_{i=0}^{i=m}\sum_{j=0}^{j=n} a_{m-i,n-j}b_{i,j}\right] + c_{m,n}. \tag{3.8}$$

4. Determination of the $Q^{(p)}$. Since the A, B, C (see (2.1)) are assumed to be polynomials, D and F are also polynomials in z, $\bar z$, as is indicated in (3.6). (Both F and D are assumed to be of the same degree; this is always permissible, since some f's or d's can be assumed to equal zero.) We now write*

$$Q^{(p-1)} = \sum_i\sum_j q_{i,j}^{(p-1)} z^i\bar z^j, \qquad q_{i,0}^{(p-1)} = 0, \tag{4.1}$$

and proceed to the representation of $q_{m,n}^{(p)}$ in terms of the $q_{i,j}^{(p-1)}$. Formal computations yield

$$Q_z^{(p-1)} = \sum_i\sum_j jq_{i,j}^{(p-1)} z^i\bar z^{j-1} = \sum_i\sum_j (j+1)q_{i,j+1}^{(p-1)} z^i\bar z^j,$$

---

* The relations $q_{i,0}^{(p-1)} = 0$ follow from the equation $Q^{(p-1)}(z, 0) = 0$. See (3.3). Note that if one of the subscripts m, n of $d_{m,n}$ becomes negative or larger than M, it is necessary to substitute 0 for the corresponding $d_{m,n}$. The same holds for $f_{m,n}$.

$$FQ^{(p-1)} = \left[\sum_\nu z^\nu (f_{\nu,0} + f_{\nu,1}\bar{z} + f_{\nu,2}\bar{z}^2 + \cdots)\right]\left[\sum z^i (q_{i,1}\bar{z} + q_{i,2}\bar{z}^2 + \cdots)\right]$$

$$= \sum_m \sum_n \left(\sum_{i=1}^{m}\sum_{j=1}^{n} f_{m-i,n-j} q_{i,j}^{(p-1)}\right) z^m \bar{z}^n, \tag{4.2}$$

$$DQ_{\bar{z}}^{(p-1)} = \left[\sum_\nu z^\nu (d_{\nu,0} + d_{\nu,1}\bar{z} + d_{\nu,2}\bar{z}^2 + \cdots)\right]\left[\sum_i z^i (q_{i,1} + 2q_{i,2}\bar{z} + 3q_{i,3}\bar{z}^2 + \cdots)\right]$$

$$= \sum_m \sum_n \left(\sum_{i=0}^{m}\sum_{j=0}^{n} (j+1) d_{m-i,n-j} q_{i,j+1}^{(p-1)}\right) z^m \bar{z}^n$$

$$= \sum_m \sum_n \left(\sum_{i=0}^{m}\sum_{j=1}^{n+1} j d_{m-1,n-j+1} q_{i,j}^{(p-1)}\right) z^m \bar{z}^n,$$

$$FQ^{(p-1)} + DQ_{\bar{z}}^{(p-1)} = \sum_m z^m \sum_{i=0}^{m} \{d_{m-1,0}q_{i,1} + [2d_{m-i,0}q_{i,2} + (d_{m-1,1} + f_{m-i,0})q_{i1}\bar{z}]$$

$$+ [3d_{m-i,0}q_{i,3} + (2d_{m-i,1} + f_{m-i,0})q_{i,2} + (d_{m-i,2} + f_{m-i,1})q_{i,1}]\bar{z}^2 + \cdots\}$$

$$= \sum_m \sum_n \left[\sum_{i=0}^{m}\sum_{j=1}^{n} (jd_{m-i,n-j+1} + f_{m-i,n-j}) q_{i,j}^{(p-1)} + (n+1)\sum_{i=0}^{m} d_{m-i,0} q_{i,n+1}^{(p-1)}\right] z^m \bar{z}^n. \tag{4.3}$$

Therefore

$$q_{m,k}^{(1)} = -\frac{2}{k} f_{m,k-1}, \tag{4.4}$$

$$\left.\begin{aligned} q_{m,1}^{(p)} &= -\frac{2}{2p-1}\left[\sum_{i=0}^{m} d_{m-i,0} q_{i,1}^{(p-1)} + (m+1) q_{m+1,1}^{(p-1)}\right], \quad (p = 2, 3, \cdots), \\ q_{m,n}^{(p)} &= -\frac{2}{2p-1}\left[\frac{1}{n}\sum_{i=0}^{m}\sum_{k=1}^{n-1} (kd_{m-i,n-k} + f_{m-i,n-k-1}) q_{i,k}^{(p-1)} \right. \\ &\quad \left. + \sum_{i=0}^{m} (d_{m-i,0} q_{i,n}^{(p-1)} + (m+1) q_{m+1,n}^{(p-1)})\right], \\ &(m = 0, 1, 2, \cdots, n = 2, 3, 4, \cdots, p = 2, 3, \cdots). \end{aligned}\right| \tag{4.5}$$

5. Evaluation of the $q_{m,n}^{(p)}$. If we assume that the $d_{m,n}$ and $f_{m,n}$ are determined, the evaluation of the $q_{m,n}^{(p)}$ represents a considerable computational task. Merely writing down the various pairs which have to be multiplied together and adding the resulting products is quite laborious. Therefore, it is convenient to organize the determination of the $q_{m,n}^{(p)}$ in such manner that one can see automatically what quantities have to be multiplied and added. After such a procedure has been developed it can be conveniently employed for performing the computation on punch card machines.

We shall describe such a procedure, assuming for the sake of simplicity that M in (4.1) is a given number, namely $M=3$. The changes that have to be made when M assumes other values are in many instances clear. If however this is not the case, then we shall write these formulae out explicitly for the general case, i.e., retaining M. The expressions $q_{m,n}^{(p)}$ $(p>1)$ to be evaluated are given in (4.5) in the form of sums of products of two sets of numbers; the first set are the coefficients of $q_{i,k}^{(p-1)}$, and the second set are $q_{i,k}^{(p-1)}$. This suggests the construction of a set of cards and stencils. The coefficients of $q_{i,k}^{(p-1)}$ are entered on the cards. The stencils are cards with holes cut in them, and the quantities $q_{i,k}^{(p-1)}$ are entered on the stencils in such a way that when a stencil is placed over a card properly, quantities on the cards appear in the holes in the stencil beside the $q_{i,k}^{(p-1)}$ by which they are to be multiplied.

A card is constructed for each value of n; these cards are called the [S, n] cards. A stencil is constructed for each value of p; these stencils are called the [Q, p] stencils. In order to compute $q_{m,n}^{(p)}$ and hence construct the [Q, p] stencil, we require the [Q, $p-1$] stencil and the [S, n] cards corresponding to* $n=1, 2, \cdots, p(M+1)$.

Each [S, n] card contains a matrix $\{s_{r,k}^{(n)}\}$ of $M+2$ rows and $M+1$ columns; r indicates the row and k the column. Also when $M=3$,

$$s_{r,k}^{(n)} = \frac{1}{n}\left[(n + r - 5)d_{4-k,5-r} + f_{4-k,4-r}\right].$$

$s_{1,1}^{(n)}= f_{3,3}/n$	$s_{1,2}^{(n)}= f_{2,3}/n$	$s_{1,3}^{(n)} f_{1,3}/n$	$s_{1,4}^{(n)}= f_{0,3}/n$
$s_{2,1}^{(n)}=[(n-3)d_{3,3}+f_{3,2}]/n$	$s_{2,2}^{(n)}=[(n-3)d_{2,3}+f_{2,2}]/n$	$s_{2,3}^{(n)}=[(n-3)d_{1,3}+f_{1,2}]/n$	$s_{2,4}^{(n)}=[(n-3)d_{0,3}+f_{0,2}]/n$
$s_{3,1}^{(n)}=[(n-2)d_{3,2}+f_{3,1}]/n$	$s_{3,2}^{(n)}=[(n-2)d_{2,2}+f_{2,1}]/n$	$s_{3,3}^{(n)}=[(n-2)d_{1,2}+f_{1,1}]/n$	$s_{3,4}^{(n)}=[(n-2)d_{0,2}+f_{0,1}]/n$
$s_{4,1}^{(n)}=[(n-1)d_{3,1}+f_{3,0}]/n$	$s_{4,2}^{(n)}=[(n-1)d_{2,1}+f_{2,0}]/n$	$s_{4,3}^{(n)}=[(n-1)d_{1,1}+f_{1,0}]/n$	$s_{4,4}^{(n)}=[(n-1)d_{0,1}+f_{0,0}]/n$
$s_{5,1}^{(n)}= d'_{3,0}$	$s_{4,1}^{(n)}= d'_{2,0}$	$s_{5,3}^{(n)}= d'_{1,0}$	$s_{5,4}^{(n)}= d'_{0,0}$

FIG. 1. A [S, n] card** in the case $M=3$.

For $n \leqq M+1$ (i.e., in our case for $n \leqq 4$), the elements of the first $M+2-n$ lines are to be set equal to zero, i.e., in our case $s_{kr}^{(n)}=0$ for $k \leqq 5-n$.

A [Q, p] stencil consists of $(p+1)(M+1)-p$ columns and $(p+1)(M+1)$ rows. The $q_{m,n}^{(p)}$ in the last M columns and $M+1$ last lines are equal to zero. In the case $M=3$, typical stencils are as shown in Figs. 2 and 3 (toe interiors of the rectangles should be cut out).

The number of remaining columns in any stencil will be three greater and the number of rows four greater than in the stencil for the preceding value of p. The [Q, 1] stencil has 7 columns, 7 "deleted" columns, and 8 rows; hence the [Q, 2] stencil will have 10 columns, 10 "deleted" columns, and 12 rows; etc. (The [Q, 1] stencil has 7 columns (+7 "deleted" columns) and 8 rows and the S cards have 4 columns and 5 rows.) See Fig. 2.

Each member $q_{m,n}^{(p)}$ on the [Q, p] stencils is specified by three numbers, one superscript and two subscripts. The superscript remains constant for each Q stencil, i.e., all numbers computed for the [Q, 2] stencil will have the form $q_{m,n}^{(2)}$, for the [Q, 3] stencil $q_{m,n}^{(3)}$, etc. The subscripts give the position of the number on the paper, indicating column and row respectively; $q_{4,3}^{(1)}$ means that the number is on the [Q, 1] stencil in the fourth column and third row; $q_{5,2}^{(2)}$ that the number is on the [Q, 2] stencil in the fifth column and second row; etc.

* We note that in increasing p to $p+1$ the number of [S, n] cards increases to $(p+1)(M+1)$; however $p(M+1)$ of these cards are exactly the same cards used at the pth stage, so that it is necessary to prepare only $M+1$ new [S, n] cards.

** The numbers on the S stencil are the coefficients of $q_{i,k}^{(p-1)}$, $q_{i,k}^{(p-1)}$ in (4.5).

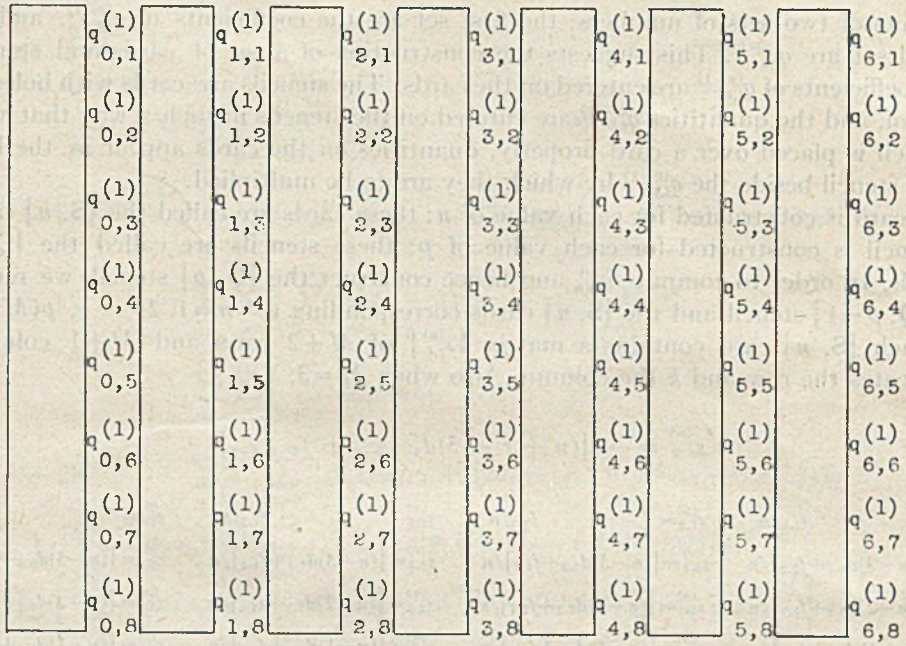

FIG. 2. Stencil [Q, 1].

Using (4.4), we determine $q^{(1)}_{m,p}$, $(m=0, 1, 2, 3, p=1, 2, 3, 4)$; $q^{(1)}_{m,p}$ $(m=4, 5, 6$ or $p=5, 6, 7, 8)$ we set equal to zero. Thus the stencil [Q, 1] has the appearance indicated in Fig. 2. The stencil [Q, 2] consists of 10 columns and 12 rows (besides the deleted columns, i.e. the holes in the stencils). See Fig. 3. We now proceed to the computation of the [Q, 2] stencil. To compute $q^{(2)}_{m,n}$ we proceed as follows: the second subscript (m, n) indicates the $q^{(1)}_{m,n}$ on which we must fix our attention. Since there are four more rows and three more columns on the [Q, 2] stencil than on the [Q, 1] stencil, there will be no corresponding $q^{(1)}_{m,n}$ for these last columns and rows—we fill in these last 3 columns and last four rows on the [Q, 2] stencil with zeros. To construct the remainder of the [Q, 2] stencil, we place the [Q, 1] stencil on top of the [S, n] card so that the number $s^{(n)}_{M+2,M+1}$ on the [S, n] card occupies the space to the left of $q^{(1)}_{m,n}$.* There are then 20 numbers on the S card adjacent (on the left) to 20 numbers on the [Q, 1] stencil, which indicates the products (20 in all) which have to be obtained; once the products are computed they are to be summed.

In this manner we may tentatively fill in the whole [Q, 2] stencil. Now we compute the product of $(m+1)q^{(1)}_{m+1,n}$ and add this to each of the "tentative $q^{(2)}_{m,n}$." If this number is multiplied by $-2/3$ [in the case of an arbitrary p by $-2/(2p-1)$], we obtain the "final $q^{(2)}_{m,n}$" of the stencil [Q, 2].

Having completed the [Q, 2] stencil we may repeat the above operations but with [Q, 2] and $q^{(2)}_{m,n}$ replacing [Q, 1] and $q^{(1)}_{m,n}$, respectively, to compute the [Q, 3] stencil. Similarly, we can compute as many Q stencils as desired.

---

* See Fig. 4, where the [Q, 1] stencil covers the [S, 4] card, and the arrangement for the computation of $q^{(2)}_{5,4}$ is indicated.

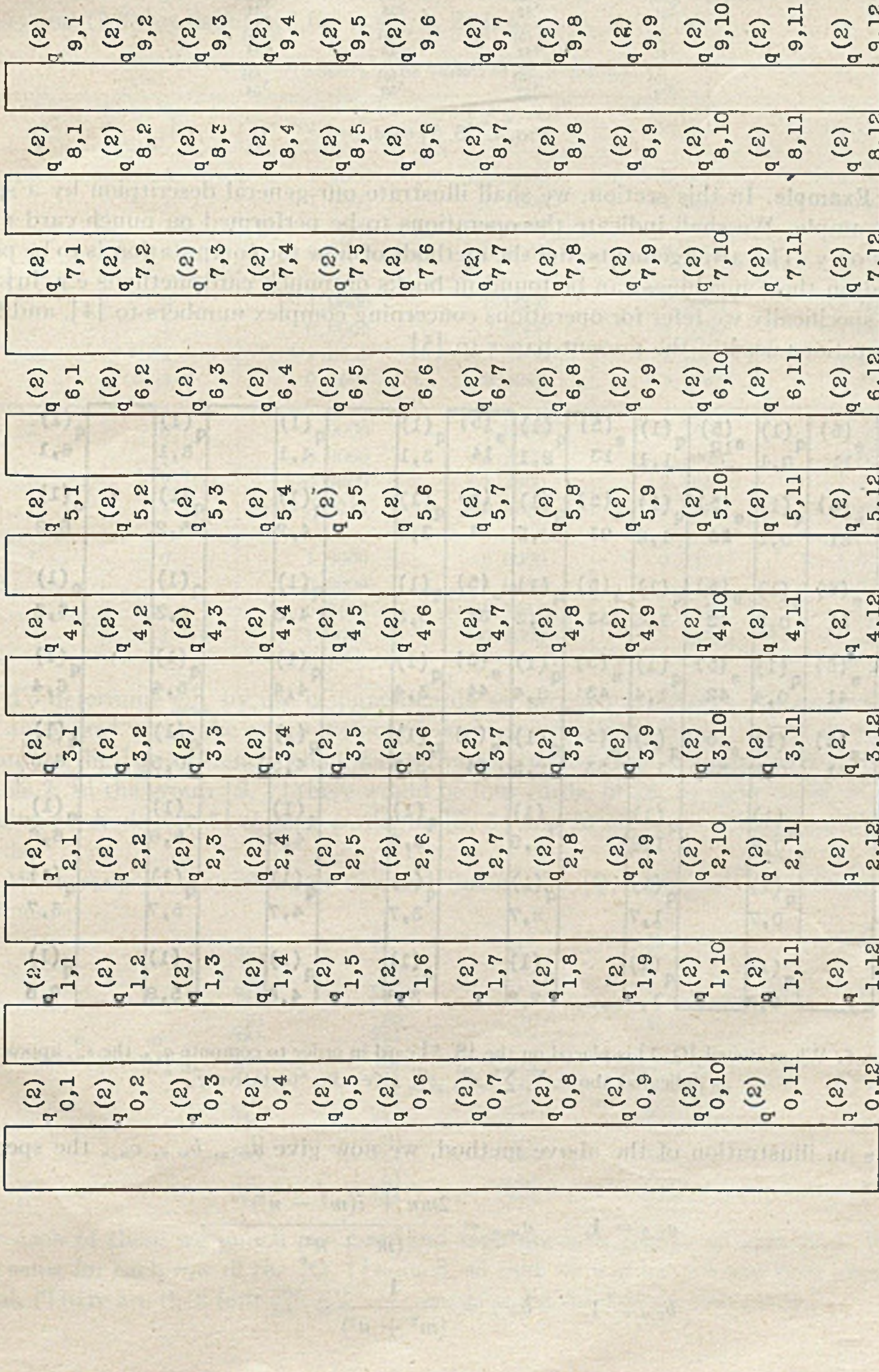

FIG. 3. Stencil [Q, 2].

$s^{(5)}_{11}$	$s^{(5)}_{12}$	$s^{(5)}_{13}$	$s^{(5)}_{14}$
$s^{(5)}_{21}$	$s^{(5)}_{22}$	$s^{(5)}_{23}$	$s^{(5)}_{24}$
$s^{(5)}_{31}$	$s^{(5)}_{32}$	$s^{(5)}_{33}$	$s^{(5)}_{34}$
$s^{(5)}_{41}$	$s^{(5)}_{42}$	$s^{(5)}_{43}$	$s^{(5)}_{44}$
$s^{(5)}_{51}$	$s^{(5)}_{52}$	$s^{(5)}_{53}$	$s^{(5)}_{54}$

FIG. 4. [S, 5] card.

6. Example. In this section, we shall illustrate our general descritpion by a specific example. We shall indicate the operations to be performed on punch card machines only. The arrangements and the methods of how the computation is to be performed on these machines can be found in books on punch card methods e.g. in [3]. More specifically we refer for operations concerning complex numbers to [4], and for computations used in the present paper to [5].

$s^{(5)}_{11}$	$q^{(1)}_{0,1}$	$s^{(5)}_{12}$	$q^{(1)}_{1,1}$	$s^{(5)}_{13}$	$q^{(1)}_{2,1}$	$s^{(5)}_{14}$	$q^{(1)}_{3,1}$		$q^{(1)}_{4,1}$		$q^{(1)}_{5,1}$		$q^{(1)}_{6,1}$
$s^{(5)}_{21}$	$q^{(1)}_{0,2}$	$s^{(5)}_{22}$	$q^{(1)}_{1,2}$	$s^{(5)}_{23}$	$q^{(1)}_{2,2}$	$s^{(5)}_{24}$	$q^{(1)}_{3,2}$		$q^{(1)}_{4,2}$		$q^{(1)}_{5,2}$		$q^{(1)}_{6,2}$
$s^{(5)}_{31}$	$q^{(1)}_{0,3}$	$s^{(5)}_{32}$	$q^{(1)}_{1,3}$	$s^{(5)}_{33}$	$q^{(1)}_{2,3}$	$s^{(5)}_{34}$	$q^{(1)}_{3,3}$		$q^{(1)}_{4,3}$		$q^{(1)}_{5,3}$		$q^{(1)}_{6,3}$
$s^{(5)}_{41}$	$q^{(1)}_{0,4}$	$s^{(5)}_{42}$	$q^{(1)}_{1,4}$	$s^{(5)}_{43}$	$q^{(1)}_{2,4}$	$s^{(5)}_{44}$	$q^{(1)}_{3,4}$		$q^{(1)}_{4,4}$		$q^{(1)}_{5,4}$		$q^{(1)}_{6,4}$
$s^{(5)}_{51}$	$q^{(1)}_{0,5}$	$s^{(5)}_{52}$	$q^{(1)}_{1,5}$	$s^{(5)}_{53}$	$q^{(1)}_{2,5}$	$s^{(5)}_{54}$	$q^{(1)}_{3,5}$		$q^{(1)}_{4,5}$		$q^{(1)}_{5,5}$		$q^{(1)}_{6,5}$
	$q^{(1)}_{0,6}$		$q^{(1)}_{1,6}$		$q^{(1)}_{2,6}$		$q^{(1)}_{3,6}$		$q^{(1)}_{4,6}$		$q^{(1)}_{5,6}$		$q^{(1)}_{6,6}$
	$q^{(1)}_{0,7}$		$q^{(1)}_{1,7}$		$q^{(1)}_{2,7}$		$q^{(1)}_{3,7}$		$q^{(1)}_{4,7}$		$q^{(1)}_{5,7}$		$q^{(1)}_{6,7}$
	$q^{(1)}_{0,8}$		$q^{(1)}_{1,8}$		$q^{(1)}_{2,8}$		$q^{(1)}_{3,8}$		$q^{(1)}_{4,8}$		$q^{(1)}_{5,8}$		$q^{(1)}_{6,8}$

FIG. 5. When stencil [Q, 1] is placed on the [S, 5] card in order to compute $q^{(2)}_{3,5}$, the $s^{(5)}_{m,r}$ appear as indicated above. $\sum_n \sum_m s^{(5)}_{m,n+1} q^{(1)}_{n,m}$ gives the "tentative $q^{(2)}_{3,5}$."

As an illustration of the above method, we now give $a_{m,n}$, $b_{m,n}$, $c_{m,n}$ the specific values,

$$a_{0,0} = 1, \qquad a_{m,n} = \frac{2mn + i(m^2 - n^2)}{(m^2 + n^2)},$$

$$b_{0,0} = 1, \qquad b_{m,n} = \frac{1}{(m^2 + n^2)},$$

$$c_{0,0} = 1, \qquad c_{m,n} = \frac{(m+n) + i(m-n)}{(m^2+n^2)} \cdot$$

In Table 1 the values of $d_{m,n}$ and $f_{m,n}$ which have been determined according to (3.7) and (3.8) are tabulated for $m=0, 1, 2, 3, 4, n=0, 1, 2, 3, 4$.

TABLE 1. The values of $d_{m,n}$ and $f_{m,n}$.

m	n	$\mathrm{Re}(d_{m,n})$	$\mathrm{Im}(d_{m,n})$	$\mathrm{Re}(f_{m,n})$	$\mathrm{Im}(f_{m,n})$
0	0	1.0000	0.0000	0.0000	−1.0000
	1	1.0000	0.0000	−1.0000	0.0000
	2	0.2500	0.0000	−0.5500	2.1000
	3	0.1111	0.0000	−0.3778	2.7167
1	0	1.0000	0.0000	0.0000	−2.0000
	1	0.0000	−2.0000	−2.1000	−1.2000
	2	−0.8000	−0.6000	−3.4000	1.6500
	3	−0.6667	0.0000	−3.1962	3.5581
2	0	1.0000	0.0000	0.2500	−4.5000
	1	0.0000	−3.2000	−3.2000	−4.0550
	2	−0.9000	−1.2000	−5.4942	−1.1538
	3	−0.9231	−0.3846	−6.0154	1.6216
3	0	1.0000	0.0000	0.2222	−5.9157
	1	0.0000	−4.0000	−3.2324	−6.3183
	2	−0.9412	−1.7647	−6.2154	−4.0216
	3	−1.0667	−0.8000	−7.6534	−1.1200

To determine $q^{(2)}_{m,n}$ by use of punch cards we proceed as follows: for each number on an S card, we make one punch-card, and separate into groups. Thus, in order to compute the [Q, 2] stencil which has 8 rows we would need 8 groups of S cards. See Table 2. In the group [S, 1] there would be four cards, in [S, 2] eight cards, in [S, 3] twelve cards, in [S, 4] sixteen cards, in [S, 5] and in all succeeding groups twenty cards.

We arrange the numbers in the first line of the [Q, 1] stencil (see Table 3) as follows:

$$\begin{array}{cccc}
q^{(1)}_{0,1} & 0 & 0 & 0 \\
q^{(1)}_{1,1} & q^{(1)}_{0,1} & 0 & 0 \\
q^{(1)}_{2,1} & q^{(1)}_{1,1} & q^{(1)}_{0,1} & 0 \\
q^{(1)}_{3,1} & q^{(1)}_{2,1} & q^{(1)}_{1,1} & q^{(1)}_{0,1} \\
q^{(1)}_{4,1} & q^{(1)}_{3,1} & q^{(1)}_{2,1} & q^{(1)}_{1,1} \\
q^{(1)}_{5,1} & q^{(1)}_{4,1} & q^{(1)}_{3,1} & q^{(1)}_{2,1} \\
q^{(1)}_{6,1} & q^{(1)}_{5,1} & q^{(1)}_{4,1} & q^{(1)}_{3,1}
\end{array}$$

For each of these we punch one card and separate into groups as indicated. We do the same for each row of the [Q, 1] stencil, so that we will have seven such groups in total. (There are then four $q^{(1)}_{0,1}$, $q^{(1)}_{1,1}$, $\cdots$ cards and a total of six zero cards.)

TABLE 2. The values of $s_{i,j}^{(n)}$, for $n=1, 2, 3, 4, 5$.*

n	i	j	$\mathrm{Re}(s_{i,j}^{(n)})$	$\mathrm{Im}(s_{i,j}^{(n)})$
1	5	4	1.0000	0.0000
	5	3	1.0000	0.0000
	5	2	1.0000	0.0000
	5	1	1.0000	0.0000
2	5	4	1.0000	0.0000
	4	4	0.5000	−0.5000
	5	3	1.0000	0.0000
	4	3	0.0000	−2.0000
	5	2	1.0000	0.0000
	4	2	0.1250	−3.7500
	5	1	1.0000	0.0000
	4	1	0.1111	−4.9583
3	5	4	1.0000	0.0000
	4	4	0.6667	−0.3333
	3	4	−0.2500	0.0000
	5	3	1.0000	0.0000
	4	3	0.0000	−2.0000
	3	3	−0.9667	−0.6000
	5	2	1.0000	0.0000
	4	2	0.0833	−3.5000
	3	2	−1.3667	−1.7500
	5	1	1.0000	0.0000
	4	1	0.0741	−4.6389
	3	1	−1.3912	−2.6943
4	5	4	1.0000	0.0000
	4	4	0.7500	−0.2500
	3	4	−0.1250	0.0000
	2	4	−0.1097	0.5250
4	5	3	1.0000	0.0000
	4	3	0.0000	−2.0000
	3	3	−0.9250	−0.6000
	2	3	−1.0167	0.4125
	5	2	1.0000	0.0000
	4	2	0.0625	−3.3750
	3	2	−1.2500	−1.6125
	2	2	−1.6043	−0.3846
	5	1	1.0000	0.0000
	4	1	0.0555	−4.4792
	3	1	−1.2787	−2.4619
	2	1	−1.8205	−1.2054
5	5	4	1.0000	0.0000
	4	4	0.8000	−0.2000
	3	4	−0.0500	0.0000
	2	4	−0.0655	0.4200
	1	4	−0.0755	0.5433
	5	3	1.0000	0.0000
	4	3	0.0000	−2.0000
	3	3	−0.9000	−0.6000
	2	3	−0.9467	0.3300
	1	3	−0.6392	0.7116
	5	2	1.0000	0.0000
	4	2	0.5000	−3.3000
	3	2	−1.1800	−1.5300
	2	2	−1.4681	−0.3846
	1	2	−1.2031	0.3243
	5	1	1.0000	0.0000
	4	1	0.0444	−4.3833
	3	1	−1.2112	−2.3225
	2	1	−1.6697	−1.1243
	1	1	−1.5307	−0.2240

TABLE 3. The values of $q_{m,k}^{(1)}$.

m	k	$\mathrm{Re}(q_{m,k}^{(1)})$	$\mathrm{Im}(q_{m,k}^{(1)})$
0	1	0.0000	2.0000
	2	1.0000	0.0000
	3	0.3666	−1.4000
	4	0.1889	−1.3583
1	1	0.0000	4.0000
	2	2.1000	1.2000
	3	2.2667	−1.1000
	4	1.5981	−1.7791
2	1	−0.5000	9.0000
	2	3.2000	4.0500
	3	3.6628	0.7692
	4	3.0077	−0.8108
3	1	−0.4444	11.8333
	2	3.2324	6.3183
	3	4.1436	2.6810
	4	3.8267	0.5600

* The $s_{i,j}^{(n)}$'s for $n=6, 7, 8$ have to be computed in a similar manner.

Starting with the group of cards for line one, we take the "dot" product for each sub-group with the [S, 1] cards.* Let the numbers in the first row of the [S, 1] card be denoted by $s^{(1)}_{51}$, $s^{(1)}_{52}$, $s^{(1)}_{53}$, $s^{(1)}_{54}$; we then have to compute the following products:

$$
\begin{array}{l}
q^{(1)}_{0,1}\cdot s^{(1)}_{5,4} + 0\cdot s^{(1)}_{5,3} + 0\cdot s^{(1)}_{5,2} + 0\cdot s^{(1)}_{5,1}\\
q^{(1)}_{1,1}\cdot s^{(1)}_{5,4} + q^{(1)}_{0,1}\cdot s^{(1)}_{5,3} + 0\cdot s^{(1)}_{5,2} + 0\cdot s^{(1)}_{5,1}\\
q^{(1)}_{2,1}\cdot s^{(1)}_{5,4} + q^{(1)}_{1,1}\cdot s^{(1)}_{5,3} + q^{(1)}_{0,1}\cdot s^{(1)}_{5,2} + 0\cdot s^{(1)}_{5,1}\\
q^{(1)}_{3,1}\cdot s^{(1)}_{5,4} + q^{(1)}_{2,1}\cdot s^{(1)}_{5,3} + q^{(1)}_{1,1}\cdot s^{(1)}_{5,2} + q^{(1)}_{0,1}\cdot s^{(1)}_{5,1}\\
q^{(1)}_{4,1}\cdot s^{(1)}_{5,4} + q^{(1)}_{3,1}\cdot s^{(1)}_{5,3} + q^{(1)}_{2,1}\cdot s^{(1)}_{5,2} + q^{(1)}_{1,1}\cdot s^{(1)}_{5,1}
\end{array}
$$

The actual procedure is to form the products $q^{(1)}_{0,1}\cdot s^{(1)}_{5,4}$, $q^{(1)}_{1,1}\cdot s^{(1)}_{5,4}$, $q^{(1)}_{2,1}\cdot s^{(1)}_{5,4}, \cdots$; $0\cdot s^{(1)}_{5,3}$, $q^{(1)}_{0,1}\cdot s^{(1)}_{5,3}$, $q^{(1)}_{1,1}\cdot s^{(1)}_{5,3}$ and enter these together with appropriate signs etc. on the corresponding cards. These cards are then assorted in sub-groups as indicated and the products summed to yield the desired "dot-product."

To compute the numbers which are to appear in the second line of the [Q, 2] stencil, the numbers in the second line of the [Q, 1] stencil are arranged as follows:

$$
\begin{array}{cccc}
q^{(1)}_{0,2} & 0 & 0 & 0\\
q^{(1)}_{1,2} & q^{(1)}_{0,2} & 0 & 0\\
q^{(1)}_{2,2} & q^{(1)}_{1,2} & q^{(1)}_{0,2} & 0\\
q^{(1)}_{3,2} & q^{(1)}_{2,2} & q^{(1)}_{1,2} & q^{(1)}_{0,2}\\
\cdot & \cdot & \cdot & \cdot\\
q^{(1)}_{6,2} & q^{(1)}_{5,2} & q^{(1)}_{4,2} & q^{(1)}_{3,2}
\end{array}
$$

For each of these numbers one card is punched and separated into groups of eight cards as indicated below, using the cards previously punched for the first line of the [Q, 1] stencil:

$$
\begin{array}{cccccccc}
q^{(1)}_{0,2} & q^{(1)}_{0,1} & 0 & 0 & 0 & 0 & 0 & 0\\
q^{(1)}_{1,2} & q^{(1)}_{1,1} & q^{(1)}_{0,2} & q^{(1)}_{0,1} & 0 & 0 & 0 & 0\\
q^{(1)}_{2,2} & q^{(1)}_{2,1} & q^{(1)}_{1,2} & q^{(1)}_{1,1} & q^{(1)}_{0,2} & q^{(1)}_{0,1} & 0 & 0\\
q^{(1)}_{3,2} & q^{(1)}_{3,1} & q^{(1)}_{2,2} & q^{(1)}_{2,1} & q^{(1)}_{1,2} & q^{(1)}_{1,1} & q^{(1)}_{0,2} & q^{(1)}_{0,1}\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

The dot product of each of these groups in the [S, 2] cards are then taken. Thus:

$$
\begin{array}{l}
q^{(1)}_{0,2}\cdot s^{(2)}_{5,4} + q^{(1)}_{0,1}\cdot s^{(2)}_{4,4} + 0\cdot s^{(2)}_{5,3} + 0\cdot s^{(2)}_{4,3} + 0\cdot s^{(2)}_{5,2} + 0\cdot s^{(2)}_{4,2} + 0\cdot s^{(2)}_{5,1} + 0\cdot s^{(2)}_{4,1}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
q^{(1)}_{3,2}\cdot s^{(2)}_{5,4} + q^{(1)}_{3,1}\cdot s^{(2)}_{4,4} + q^{(1)}_{2,2}\cdot s^{(2)}_{5,3} + q^{(1)}_{2,1}\cdot s^{(2)}_{4,3} + q^{(1)}_{1,2}\cdot s^{(2)}_{5,2} + q^{(1)}_{1,1}\cdot s^{(2)}_{4,2} + q^{(1)}_{0,2}\cdot s^{(2)}_{5,1} + q^{(1)}_{0,1}\cdot s^{(2)}_{4,1}
\end{array}
$$

The actual products are again taken as indicated before, and then reassorted and summed.

This process is repeated for each line of the [Q, 1] stencil until each group has twenty cards in it, after which the first line is discarded when a new line is added so that there are never more than twenty cards. In this way the "tentative [Q, 2] sten-

* Note that, in general, both $q^{(p-1)}_{m,n}$ and $s^{(n)}_{r,k}$ are complex, and in order to compute $q^{(p-1)}_{m,n}\cdot s^{(n)}_{r,k}$ on punch card machines we have to use rules for the evaluation of complex numbers.

cil" is computed and it is necessary only to add $(m+1)q^{(1)}_{m+1,n}$ and multiply the result by $-2/3$ to obtain the actual values of the elements of the stencil [Q, 2]. This procedure may then be repeated to obtain the values of the elements of the stencils [Q, 3], (i.e. the quantities $q^{(3)}_{m,p}$) [Q, 4] etc.

In Tables 2 and 3 the values of the elements of the cards [S, n] and those of the stencil [Q, 1] are given.

7. Conclusion. Application of the differential analyzer to the construction of particular solutions. As soon as a sufficient number of [Q, p] stencils (i.e. $q^{(p)}_{m,n}$) have been prepared we can, by (3.1) and (3.2) and standard methods of evaluation of polynomials on punch card machines, determine the values of $\mathfrak{p}_{2\nu-1}$ and $\mathfrak{p}_{2\nu}$ for a sufficiently dense lattice of points.

The second step, the determination of the coefficients a_ν in the expression (1.2) in order to obtain a solution of (1.1) which assumes the prescribed boundary values, will be discussed in a subsequent paper. The basic idea of the procedure to be employed has already been indicated in [2]; we shall however discuss this in more detail from the point of view of numerical analysis.

Sometimes we need solutions of (1.1) in connection with questions other than the boundary value problem, and it is convenient to apply the method of particular solutions in the following slightly modified form.

As was proved in [1], §1,

$$U(z, \bar{z}) = \int_{-1}^{1} \mathrm{E}(z, \bar{z}, t) f[(z/2)(1 - t^2)]dt/(1 - t^2)^{1/2}, \tag{7.1}$$

where f is an arbitrary analytic function of a complex variable, which is regular at the origin, is a solution of (1.1).

There are instances in which a large number of solutions of the same equation are required, and the corresponding functions f are known. (See (7.1).) (This situation occurs, for example, if an "atlas" of solutions has been prepared.) In these cases it is then very convenient to employ graphical methods. In the following we shall indicate two graphical methods for the evaluation of (7.1). Both can be performed conviently by use of a differential analyzer.

I. One prepares once and for all for a given equation (1.1) diagrams in which the curves

$$\begin{aligned} Y &= E_1(z_\nu, \bar{z}_\nu, t) = \mathrm{Re}\,[\mathrm{E}(z_\nu, \bar{z}_\nu, t)], \quad -1 \leqq t \leqq 1, \quad (z_\nu, \bar{z}_\nu) \text{ fixed}, \\ Y &= E_2(z_\nu, \bar{z}_\nu, t) = \mathrm{Im}\,[\mathrm{E}(z_\nu, \bar{z}_\nu, t)], \quad -1 \leqq t \leqq 1, \quad (z_\nu, \bar{z}_\nu) \text{ fixed}, \end{aligned} \tag{7.2}$$

for a number of points $(x, y) = (x_\nu, y_\nu)$, $\nu = 1, 2, 3, \cdots$ are drawn. Further one has to prepare tables for the values $\frac{1}{2}z_\nu(1-t_\mu^2)$, for $t_\mu = -1, -1+\alpha, -1+2\alpha, \cdots, 1$, where α is a sufficiently small positive constant. $z_\nu = x_\nu + iy_\nu$ denote the coordinates of points mentioned above. If now the function $f(z) = u(z, \bar{z}) + iv(z, \bar{z})$ is given, say in the form of two diagrams for curves $u(z, \bar{z}) =$ const. and $v(z, \bar{z}) =$ const., we draw (using the tables mentioned above) the curves

$$Y = u[z_\nu(1 - t^2), \bar{z}_\nu(1 - t^2)], \qquad Y = v[z_\nu(1 - t^2), \bar{z}_\nu(1 - t^2)], \qquad -1 \leqq t \leqq 1. \tag{7.3}$$

Using these diagrams and those mentioned above and employing a differential analyzer (or simply an integrator) we compute the real part of (7.1),

$$\int_{t=-1}^{t=1} \left\{ E_1(z_\nu, \bar{z}_\nu, t) u\left[\tfrac{1}{2}z_\nu(1 - t^2), \tfrac{1}{2}\bar{z}_\nu(1 - t^2)\right] \right.$$
$$\left. - E_2(z_\nu, \bar{z}_\nu, t) v\left[\tfrac{1}{2}z_\nu(1 - t^2), \tfrac{1}{2}\bar{z}_\nu(1 - t^2)\right] \right\} dt, \tag{7.4}$$

and analogously its imaginary part.

II. Sometimes it is not sufficient to determine the values of (7.1) at a set of points (x_ν, y_ν) which are prescribed in advance. Then one can apply the following procedure which was suggested to the author by Mr. Hans Kraft.

One prepares (once and for all) diagrams

$$E_1(z, \bar{z}, t_\nu) = \text{const.}, \; E_2(z, \bar{z}, t_\nu) = \text{const.}, \; t_\nu \text{ const.} \tag{7.5}$$

for a set of values $t_\nu = -1, \; -1+\alpha, \; -1+2\alpha, \; \cdots, \; 1$.

Using these diagrams and the tables (described in method I, for *every* required value of z we can easily determine the curve (7.3) and evaluate the real and imaginary part of (7.1).

Remark. The procedure I can be performed by the use of punch card machines. In this case instead of diagrams (7.2) it is necessary to prepare master cards.

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THE LIFT OF A DELTA WING AT SUPERSONIC SPEEDS*

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1. Introduction. The use of the two dimensional linearized theory of supersonic flows in the solution of airfoil problems as introduced by Ackeret[1] has been extremely successful in solving these problems and the results have generally been completely satisfactory for engineering purposes. The generalization of these results to the three dimensional finite span problems has, however, progressed rather slowly due to mathematical complications. The flow near the tip of a rectangular wing was given (incorrectly) by Schlichting.[2] The drag of a "delta" wing (a wing having an isosceles triangle for its planform with the symmetric vertex pointing into the oncoming flow as in Fig. 1) has been determined by Puckett.[3] These two flow patterns and many other technically interesting finite span flow problems are particular cases of conical flows. A conical flow is one for which the fluid properties (pressure, velocity, etc.) are constant along each radial line emanating from the given origin. The concept of a conical flow was given by Busemann[4] who developed certain general techniques for treating these flows and who applied the method to several problems including Schlichting's problem.

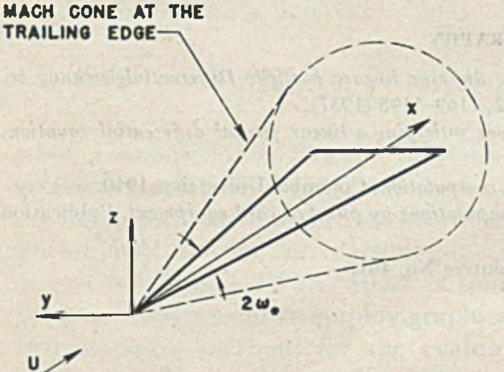

FIG. 1. Delta wing in a supersonic flow.

The methods of analysis used by Busemann have, however, proved to be rather obscure, and it has been found difficult to follow these methods in the solution of additional conical flow problems, in particular the currently very interesting problem of the lift of a delta wing. A new method of treating these conical flow airfoil problems which uses the well known theory of conformal transformation has been devised. It is the purpose of the present paper to discuss this method and to apply this method to the problem of the lift of a delta wing. In this application it is only necessary to consider the case for which the leading edges of the delta wing are within the Mach cone from the vertex. The other case for which the leading edges are outside the Mach cone has already been solved by Puckett.

In the present method no essential mathematical difference is found in the solution of the two cases.

* Received May 21, 1946.

[1] J. Ackeret, Z.F.M., **16**, 72 (1925).

[2] H. Schlichting, Luftfahrtforschung, **13**, 320 (1936).

[3] A. J. Puckett, Aero. Sci. (To be published shortly).

[4] A. Busemann, Schriften der Deutschen Akademie für Luftfahrtforschung, **7B**, 105 (1943). Also Luftfahrtforschung, **12**, 210 (1935).

2. General theory of conical flows. It is well known that the linearized theory of steady supersonic flows is based on the Prandtl-Glauert equation,

$$(1 - M^2)\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = 0, \tag{1}$$

where the undisturbed flow of Mach number M is taken to be parallel to the x axis. Here, P may denote a velocity or acceleration potential, or one of the velocity components u, v, w in rectangular Cartesian coordinates x, y, z, or a property of the state of this fluid such as pressure or enthalpy. It can be seen that the coordinate transformation

$$\left.\begin{aligned} r &= \left[\frac{x^2}{M^2 - 1} - (y^2 + z^2)\right]^{1/2} = \frac{R \cos \omega}{\mu\sqrt{M^2 - 1}}, \\ \mu &= \left[1 - (M^2 - 1)\frac{y^2 + z^2}{x^2}\right]^{-1/2} = [1 - (M^2 - 1)\tan^2 \omega]^{-1/2}, \\ \theta &= \tan^{-1}(y/z), \end{aligned}\right\} \tag{2}$$

where $R = (x^2 + y^2 + z^2)^{1/2}$ and $\omega = \tan^{-1}[(y^2 + z^2)^{1/2}/x]$, transforms the Prandtl-Glauert equation, Eq. (1), into

$$r^2\frac{\partial^2 P}{\partial r^2} + 2r\frac{\partial P}{\partial r} + \frac{\partial}{\partial \mu}\left[(1 - \mu^2)\frac{\partial P}{\partial \mu}\right] + \frac{1}{1 - \mu^2}\frac{\partial^2 P}{\partial \theta^2} = 0. \tag{3}$$

The surfaces on which θ is constant are the meridional planes through the x axis; the surfaces on which μ is constant are cones about the x axis; and the surfaces on which r is constant are hyperboloids. It may also be noted that $r = 0$ and $\mu = \infty$ on the Mach cone through the origin. Both μ and r are real within the Mach cone and complex outside it. The harmonic solutions of Eq. (3) may be written in the form

$$P = \sum_{m,n} A_{mn} \begin{Bmatrix} r^n \\ r^{-n-1} \end{Bmatrix} \begin{Bmatrix} P_n^m(\mu) \\ Q_n^m(\mu) \end{Bmatrix} \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix}, \tag{4}$$

by the well known theory of the Laplace equation. Here, P_n^m and Q_n^m denote Legendre functions of the first and second kind, respectively. By introducing the normal spherical coordinates as given in Eq. (2), Eq. (4) is seen to give the harmonic solutions of the Prandtl-Glauert equation in spherical coordinates.

Busemann's conical flows are included in the general solution of Eq. (4) as a special case. For example, if P is a velocity potential, then $n = 1$. On the other hand, if P is one of the Cartesian velocity components (u, v, w), a property of the state of the fluid such as the pressure or enthalpy, or the acceleration potential, then $n = 0$. It is the latter case which is of particular interest here, for P is then independent of r, and Eq. (3) becomes

$$(\mu^2 - 1)\frac{\partial}{\partial \mu}\left[(\mu^2 - 1)\frac{\partial P}{\partial \mu}\right] + \frac{\partial^2 P}{\partial \theta^2} = 0. \tag{5}$$

It is apparent that this may be reduced to the Laplace equation in two dimensions;* in fact, if

* This result was first communicated to the author by Mr. W. D. Hayes.

$$s = \sqrt{\frac{\mu - 1}{\mu + 1}},$$

Eq. (5) becomes

$$s \frac{\partial}{\partial s}\left(s \frac{\partial P}{\partial s}\right) + \frac{\partial^2 P}{\partial \theta^2} = 0. \tag{6}$$

This is the normal form of the Laplace equation in two dimensional polar coordinates. It is seen that s is a function only of μ and is thus constant on any one of the cones for which ω is constant. The relations between s and ω are as follows:

$$s = \frac{\sqrt{M^2 - 1} \tan \omega}{1 + \sqrt{1 - (M^2 - 1) \tan^2 \omega}}, \tag{7}$$

$$\sqrt{M^2 - 1} \tan \omega = \frac{2s}{1 + s^2} \cdot \tag{8}$$

It may further be noted that $s = 1$ on the Mach cone through the origin.

Since the reduction to Eq. (6) is possible, any of the quantities which P may represent can be written as the real (or imaginary) part of an analytic function of the complex variable ζ where

$$\zeta = se^{i\theta}. \tag{9}$$

Furthermore, all the methods of treatment of such functions, in particular the method of conformal transformation, may be used in the analysis of these quantities. If $\bar{P}$ is the harmonic conjugate of P and

$$P + i\bar{P} = \mathrm{P}(\zeta), \tag{10}$$

the Cauchy-Riemann equations for these conjugate functions may be written

$$\begin{aligned} s \frac{\partial P}{\partial s} &= \frac{\partial \bar{P}}{\partial \theta} = (\mu^2 - 1) \frac{\partial \bar{P}}{\partial \mu}, \\ -\frac{\partial P}{\partial \theta} &= s \frac{\partial \bar{P}}{\partial s} = (\mu^2 - 1) \frac{\partial \bar{P}}{\partial \mu} \cdot \end{aligned} \tag{11}$$

In the direct airfoil problem, the airfoil geometry is given, and if the z axis is taken normal to the airfoil plane, the boundary conditions for determining the flow are thus given in terms of the disturbance velocity component w. It is desired in this case to compute the pressure distribution which may be easily expressed in terms of the axial disturbance velocity component u. In the inverse airfoil problem, a pressure distribution is defined, and it is desired that the airfoil shape be computed. In either case the boundary condition is given in terms of one velocity component and another velocity component gives the desired result. For a conical flow there are simple relations between the complex functions representing the various Cartesian velocity components. The use of these relations is the essence of the present method of treatment of conical flows. These relations between the complex functions corresponding to the Cartesian disturbance velocity components u, v, w which will be written

$$u + i\bar{u} = U(\zeta), \qquad v + i\bar{v} = V(\zeta), \qquad w + i\bar{w} = W(\zeta), \tag{12}$$

are essentially the vorticity relations.

The fundamental linearized relations governing the steady flow of a fluid at supersonic speeds are the vorticity relations

$$\frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \quad (13) \qquad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad (14) \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad (15)$$

and the linearized equation of continuity

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = (M^2 - 1)\frac{\partial u}{\partial x}. \tag{16}$$

If these are solved simultaneously, it is easily seen that each of the velocity components obeys the Prandtl-Glauert equation, Eq. (1). For any conical flow each of the velocity components must be a function only of the coordinates μ and θ. By means of this fact, Eqs. (13) to (16), respectively, may be written as follows:

$$\mu(\mu^2 - 1)\left(\cos\theta\frac{\partial v}{\partial \mu} - \sin\theta\frac{\partial w}{\partial \mu}\right) = \left(\cos\theta\frac{\partial w}{\partial \theta} + \sin\theta\frac{\partial v}{\partial \theta}\right), \tag{17}$$

$$-\frac{(\mu^2 - 1)^{3/2}}{\sqrt{M^2 - 1}}\frac{\partial w}{\partial \mu} = \mu(\mu^2 - 1)\cos\theta\frac{\partial u}{\partial \mu} - \sin\theta\frac{\partial u}{\partial \theta}, \tag{18}$$

$$-\frac{(\mu^2 - 1)^{3/2}}{\sqrt{M^2 - 1}}\frac{\partial v}{\partial \mu} = \mu(\mu^2 - 1)\sin\theta\frac{\partial u}{\partial \mu} + \cos\theta\frac{\partial u}{\partial \theta}, \tag{19}$$

$$-\sqrt{M^2 - 1}\,(\mu^2 - 1)^{3/2}\frac{\partial u}{\partial \mu} = \mu(\mu^2 - 1)\left(\sin\theta\frac{\partial v}{\partial \mu} + \cos\theta\frac{\partial w}{\partial \mu}\right) + \left(\cos\theta\frac{\partial v}{\partial \theta} - \sin\theta\frac{\partial w}{\partial \theta}\right). \tag{20}$$

If Eqs. (18) and (19) are combined, it is seen that

$$\frac{\partial u}{\partial \theta} = -\frac{(\mu^2 - 1)^{3/2}}{\sqrt{M^2 - 1}}\left(\cos\theta\frac{\partial v}{\partial \mu} - \sin\theta\frac{\partial w}{\partial \mu}\right), \tag{21}$$

$$\frac{\partial u}{\partial \mu} = -\frac{(\mu^2 - 1)^{1/2}}{\sqrt{M^2 - 1}}\left(\sin\theta\frac{\partial v}{\partial \mu} + \cos\theta\frac{\partial w}{\partial \mu}\right). \tag{22}$$

Furthermore, Eqs. (20) and (22) show that

$$\frac{\mu^2 - 1}{\mu}\left(\sin\theta\frac{\partial v}{\partial \mu} + \cos\theta\frac{\partial w}{\partial \mu}\right) = -\left(\cos\theta\frac{\partial v}{\partial \theta} - \sin\theta\frac{\partial w}{\partial \theta}\right). \tag{23}$$

If the derivatives with respect to μ are eliminated from Eqs. (17) and (23) by means of the Cauchy-Riemann equations [cf. Eq. (11)] for v, w, $\bar{v}$ and $\bar{w}$, these equations may be written as follows:

$$\frac{1}{\mu}\left(\sin\theta\,\frac{\partial v}{\partial\theta}+\cos\theta\,\frac{\partial w}{\partial\theta}\right)-\left(\cos\theta\,\frac{\partial\bar{v}}{\partial\theta}-\sin\theta\,\frac{\partial\bar{w}}{\partial\theta}\right)=0, \tag{24}$$

$$\frac{1}{\mu}\left(\sin\theta\,\frac{\partial\bar{v}}{\partial\theta}+\cos\theta\,\frac{\partial\bar{w}}{\partial\theta}\right)+\left(\cos\theta\,\frac{\partial v}{\partial\theta}-\sin\theta\,\frac{\partial w}{\partial\theta}\right)=0. \tag{25}$$

If Eq. (25) is multiplied by i and added to Eq. (24), it is seen that

$$\frac{1}{\mu}\left(\sin\theta\,\frac{\partial \mathrm{V}}{\partial\theta}+\cos\theta\,\frac{\partial \mathrm{W}}{\partial\theta}\right)+i\left(\cos\theta\,\frac{\partial \mathrm{V}}{\partial\theta}-\sin\theta\,\frac{\partial \mathrm{W}}{\partial\theta}\right)=0. \tag{26}$$

Since V and W are functions of the complex variable ζ, this may further be written

$$\frac{d\mathrm{V}}{d\zeta}=\frac{d\mathrm{W}}{d\zeta}\,\frac{i\mu\sin\theta-\cos\theta}{\sin\theta+i\mu\cos\theta} \tag{27}$$

and, by the definition of s and Eq. (9),

$$\frac{d\mathrm{V}}{d\zeta}=i\,\frac{1-\zeta^2}{1+\zeta^2}\,\frac{d\mathrm{W}}{d\zeta}\,. \tag{28}$$

A similar treatment of Eqs. (21) and (22), V being eliminated by Eqs. (17) and (27), shows that

$$\frac{d\mathrm{U}}{d\zeta}=-\,\frac{2\zeta}{(1+\zeta^2)\sqrt{M^2-1}}\,\frac{d\mathrm{W}}{d\zeta}\,. \tag{29}$$

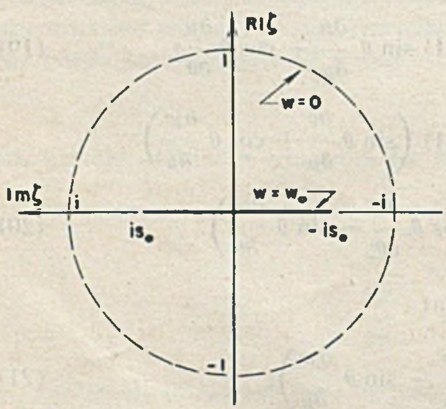

FIG. 2. Boundary conditions in the ζ plane.

These two relations, Eqs. (28) and (29), are the fundamental relations for the present treatment of conical supersonic flow problems.

3. Example. Lift of a delta wing. The general techniques developed in the previous section will now be used to compute the lift of a delta wing at a small angle of attack for the case in which the leading edges are inside of the Mach cone (see Fig. 1). The z axis is taken normal to the airfoil. The conditions in the ζ plane are shown in Fig. 2. Note that the airfoil cuts the ζ plane on the imaginary axis. The boundary conditions for determining the vertical velocity w are then

$$\begin{aligned} w &= 0 \quad \text{on} \quad s = 1, \\ w &= w_0 = -\,U\alpha \text{ on the airfoil,} \end{aligned} \tag{30}$$

where U is the velocity of the mean flow and α is the angle of attack of the airfoil.

This boundary value problem can be solved by conformal transformation. First, apply the transformation

$$\zeta_1=-\,\frac{i}{2}\left(\zeta-\frac{1}{\zeta}\right). \tag{31}$$

This maps the interior of the unit circle in the ζ plane into the entire ζ_1 plane with the region $Re\zeta>0$ corresponding to the region $Im\zeta_1>0$. This transformed plane is

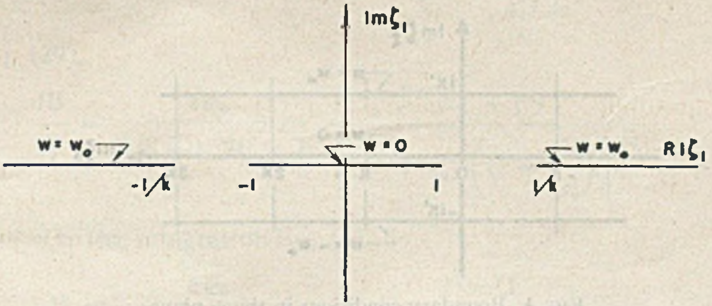

FIG. 3. Boundary conditions in the ζ_1 plane.

shown in Fig. 3. The points at the wing tips, $\zeta=\pm is_0$, are transformed into the points $\zeta_1=\pm 1/k$ where

$$k=\frac{2s_0}{1+s_0^2}=\sqrt{M^2-1}\tan\omega_0. \tag{32}$$

Second, apply the transformation

$$\zeta_2=\int_0^{\zeta_1}\frac{d\zeta_1}{\sqrt{(1-\zeta_1^2)(1-k^2\zeta_1^2)}}, \tag{33}$$

i.e.,

$$\zeta_1=sn(\zeta_2), \tag{34}$$

where the elliptic function has the modulus k. Then the region $Im\zeta_1>0$ is mapped into the rectangle having its corners at $\zeta_2=\pm K$, $iK'\pm K$ where K and K' are the complete elliptic integrals of the first kind having a modulus of k and k' where

$$k'=\sqrt{1-k^2}=\frac{1-s_0^2}{1+s_0^2}. \tag{35}$$

By integrating around the slit from -1 to 1 in the ζ_1 plane, it is seen that the region $Im\zeta_1<0$ maps into the rectangle having corners at $\zeta_2=2K\pm K$, $2K\pm K+iK'$.

Now, the transformation given by Eq. (31) is double valued, i.e., two points in the ζ plane correspond to each point in the ζ_1 plane. The ζ_1 plane must thus be considered as a two sheeted Riemann surface with one sheet corresponding to the interior of the unit circle in the ζ plane and the other sheet corresponding to the exterior of the unit circle in the ζ plane. Furthermore, the value of the downwash velocity w must be equal and opposite at inverse points in the ζ plane. This permits the analytic continuation of w throughout the entire ζ plane; in particular it is seen that $w=-w_0$ on the exterior points corresponding to the airfoil. The two sheets in the ζ_1 plane are connected through the slit from -1 to $+1$. A contour cutting this line passes from the upper to the lower sheet or vice-versa. The sheet which corresponds to the exterior region of the ζ plane is thus seen to be mapped into the rectangle having corners at $\zeta_2=K\pm 2K$, $K\pm 2K-iK'$. The entire plane is mapped into a basic rectangle in the ζ_2 plane as shown in Fig. 4. As ζ_1 has periods of $4K$, $2iK'$ [see Eq. (34)] in ζ_2, this pattern is repeated throughout the ζ_2 plane.

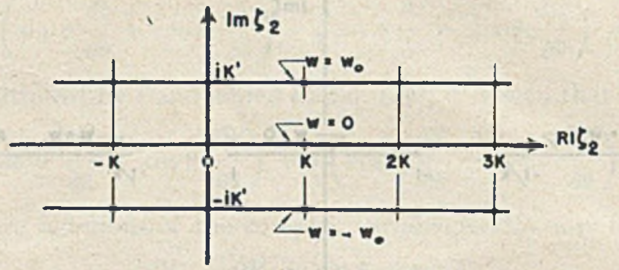

FIG. 4. Boundary conditions in the ζ_2 plane.

The function $d\mathbf{W}/d\zeta_2$ (but not $\mathbf{W}$ itself) must be doubly periodic in the ζ_2 plane with periods $4K$ and $2iK'$, the first corresponding to a loop around the points $\zeta_1 = \pm 1$ and the second corresponding to a loop around the points $\zeta_1 = 1$, $1/k$ or -1, $-1/k$. The only singularities of $\mathbf{W}$ or $d\mathbf{W}/d\zeta_2$ must be at the points corresponding to the airfoil leading edges, i.e., at the points conjugate to $iK' \pm K$. Finally $d\mathbf{W}/d\zeta_2$ must be pure imaginary on the lines $Im\zeta_2 = nK'$ and $Re\zeta_2 = K + 2nK$ (n being any integer). All of these conditions are satisfied by the Jacobian elliptic function

$$\frac{d\mathbf{W}}{d\zeta_2} = iDcd^{2n}(\zeta_2), \tag{36}$$

where n is any positive integer and D is a real constant. If this is integrated it is seen that for $n > 0$, $\mathbf{W}$ has a pole of order $2n-1$ at the wing tips. The cases for $n > 1$ can then be discarded as the singularity at the wing tips is seen to correspond to a source-sink complex which has an infinite total lift. Furthermore, the case for $n = 0$ may be discarded as [see Eq. (29)] it requires that $\mathbf{U}(\zeta)$ have a logarithmic singularity on the Mach cone. The appropriate solution is thus

$$\frac{d\mathbf{W}}{d\zeta_2} = iDcd^2(\zeta_2). \tag{37}$$

The constant D may be evaluated from the fact that

$$w_0 = Rl\left\{\int_0^{iK'} \frac{d\mathbf{W}}{d\zeta_2} d\zeta_2\right\}. \tag{38}$$

If this integration is carried out, it is seen that

$$D = -\frac{k^2 w_0}{E(k')}, \tag{39}$$

where $E(k')$ is the complete elliptic integral of the second kind having a modulus k' as given by Eq. (35).

If the variable ζ_2 is eliminated from Eq. (37) by means of Eq. (31) and (33), it is seen that

$$\frac{dW}{d\zeta} = -\frac{2w_0}{kE(k')}\frac{(1+\zeta^2)^2}{\left[(\zeta^2+s_0^2)\left(\zeta_2+\dfrac{1}{s_0^2}\right)\right]^{3/2}}. \tag{40}$$

Thus, from Eq. (29),

$$\frac{dU}{d\zeta} = \frac{4w_0}{kE(k')\sqrt{M^2-1}}\frac{\zeta(1+\zeta^2)}{\left[(\zeta^2+s_0^2)\left(\zeta^2+\dfrac{1}{s_0^2}\right)\right]^{3/2}}. \tag{41}$$

Since $\mathbf{U}=0$ at $\zeta=1$, the integral of Eq. (41) is

$$\mathbf{U} = \frac{kw_0}{E(k')\sqrt{M^2-1}}\frac{\zeta^2-1}{\left[(\zeta^2+s_0^2)\left(\zeta^2+\dfrac{1}{s_0^2}\right)\right]^{1/2}}. \tag{42}$$

On the top side of the airfoil $\zeta=i\eta$ where $-s_0<\eta<s_0$, so

$$u = -\frac{kw_0}{E(k')\sqrt{M^2-1}}\frac{1+\eta^2}{\left[(s_0^2-\eta^2)\left(\dfrac{1}{s_0^2}-\eta^2\right)\right]^{1/2}}. \tag{43}$$

This result may be considerably simplified if we introduce [from Eq. (8) and (30)]

$$\begin{aligned} k &= \sqrt{M^2-1}\tan\omega_0 \\ w_0 &= -U\alpha \end{aligned} \tag{44}$$

and

$$t = \frac{\tan\omega}{\tan\omega_0}.$$

Equation (43) then becomes

$$\frac{4}{\alpha}\left(\frac{u}{U}\right) = \frac{4\tan\omega_0}{E(k')\sqrt{1-t^2}}. \tag{45}$$

The slope of the lift curve $dC_L/d\alpha$ is given by the mean value of $4/\alpha(u/U)$ over the surface of the wing; thus

$$\frac{dC_L}{d\alpha} = \frac{4}{E(k')}\int_0^{\omega_0}\frac{\sec^2\omega d\omega}{\sqrt{1-t^2}} \tag{46}$$

and, by Eq. (44),

$$\frac{dC_L}{d\alpha} = \frac{2\pi\tan\omega_0}{E(k')}. \tag{47}$$

In the limit for which ω_0 or $s_0 \to 0$, $k' \to 1$; so $E(k') \to 1$. For this case which was given by Jones[5]

[5] R. T. Jones, N.A.C.A., Technical Note 1032 (1945).

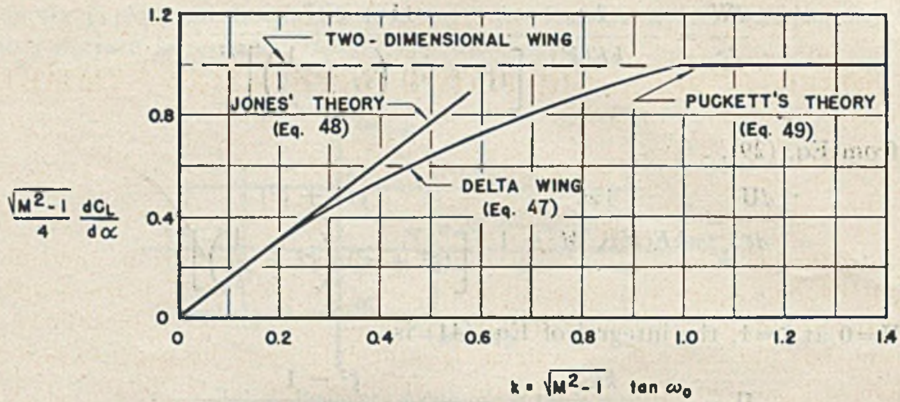

FIG. 5. Lift of a delta wing.

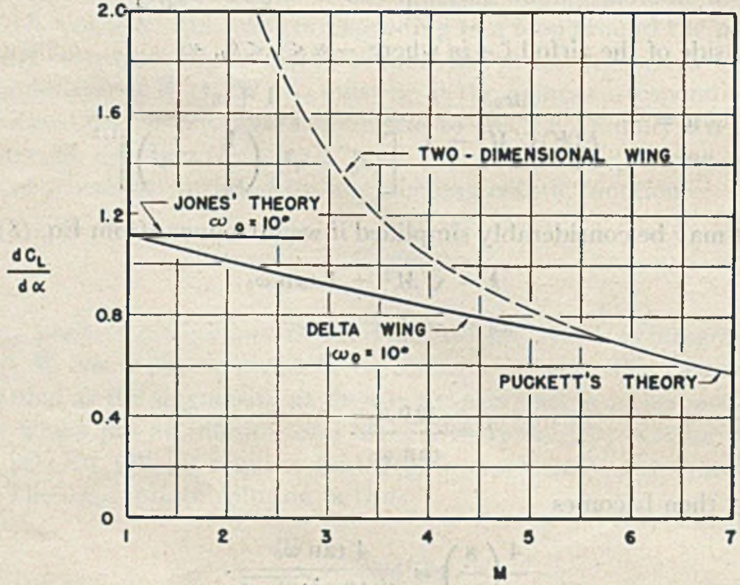

FIG. 6. $dC_L/d\alpha$ vs. M for a delta wing with $\omega = 10°$.

$$\frac{dC_L}{d\alpha} = 2\pi \tan \omega_0. \tag{48}$$

On the other limit for which $s_0 \to 1$, $k' \to 0$; so $E(k') \to \pi/2$. For this case

$$\frac{dC_L}{d\alpha} = 4 \tan \omega_0 = \frac{4}{\sqrt{M^2-1}}. \tag{49}$$

This limit, the same as the two dimensional solution, had previously been obtained by Puckett.

It may further be noted that the quantity $\frac{1}{4}\sqrt{M^2-1}\ dC_L/d\alpha$ is a function only of the parameter $k = \sqrt{M^2-1}\ \tan \omega_0$. This result is shown graphically in Fig. 5, and the slope of the lift curve for a particular case, $\omega_0 = 10°$, is shown as a function of Mach number in Fig. 6.

LINEARIZED SUPERSONIC FLOWS WITH AXIAL SYMMETRY*

BY

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1. Introduction. The study of spatial linearized supersonic flow may be aided by the study of some simple fundamental flows with axial symmetry. Through the principle of superposition, these flows may be combined to give more general flows about various objects and about lifting systems. It is the purpose of this paper to express the equations of linearized supersonic flow in a system of conical coordinates, to develop a theory for fundamental flows with axial symmetry, and to describe examples of such flows and of their combination by superposition.

Various examples of the fundamental equations and solutions here described will be given in later papers, together with the development of some concepts useful in this field.

2. The velocity potential. Steady-state compressible irrotational flow can be described by a velocity potential ϕ whose gradient is the velocity vector. Under the assumption that the velocity deviations from a uniform supersonic flow of the Mach number M are small, the differential equation for this potential takes the linear form[1,2]

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} - (M^2 - 1)\phi_{zz} = 0 \tag{1}$$

in cylindrical coordinates.

The fundamental uniform flow is given by the potential $\phi_0 = Vz$ where V is the velocity corresponding to the Mach number M. Equation (1) will be considered as yielding velocity deviations which must be added to the velocity of the fundamental flow to describe the net flow.

A new coordinate is introduced to replace the coordinate r:

$$t = (r/z)\sqrt{M^2 - 1}. \tag{2}$$

This quantity is the ratio of the tangents of the polar angle and of the Mach angle. Equation (1) with r eliminated and t introduced becomes

$$(1 - t^2)\phi_{tt} + \frac{1}{t}(1 - 2t^2)\phi_t + \frac{1}{t^2}\phi_{\theta\theta} + 2tz\phi_{tz} - z^2\phi_{zz} = 0. \tag{3}$$

By separation of variables a solution of the form

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* Received March 25, 1946.

** This paper was prepared while the writer was employed by the Lockheed Aircraft Corporation.

[1] R. Sauer, *Theoretische Einführung in die Gasdynamik*, Springer, Berlin, 1943. Reprinted by Edwards Bros., Ann Arbor, 1945.

[2] G. I. Taylor, and J. W. Maccoll, *The mechanics of compressible fluids*, in Durand, *Aerodynamic theory*, vol. 3, Berlin, 1935.

$$\phi = z^n \Phi(t, \theta) \tag{4a}$$

or

$$\phi = z^n \sin (m\theta + \beta) T(t) \tag{4b}$$

is found. The function Φ satisfies the equation

$$(1 - t^2)\Phi_{tt} + \frac{1}{t}(1 + 2(n-1)t^2)\Phi_t - n(n-1)\Phi + \frac{1}{t^2}\Phi_{\theta\theta} = 0 \tag{5}$$

and may be called the velocity potential for generalized conical flow. If $n = 1$, the function Φ describes conical flow. The function T satisfies the equation

$$(1 - t^2)T_{tt} + \frac{1}{t}(1 + 2(n-1)t^2)T_t - \frac{1}{t^2}(m^2 + n(n-1)t^2)T = 0. \tag{6}$$

Superposition of solutions of the type of (4a), (4b) will give a general solution.

The velocity components are

$$u = \frac{\sqrt{M^2 - 1}}{z}\phi_t \tag{7a}$$

in the radial direction,

$$v = \frac{\sqrt{M^2 - 1}}{tz}\phi_\theta \tag{7b}$$

in the azimuthal direction, and

$$w = \phi_z - \frac{t}{z}\phi_t \tag{7c}$$

in the axial direction. The pressure in linearized supersonic flow is given in terms of the velocity components by

$$p = -\rho\left(Vw + \frac{u^2 + v^2}{2}\right), \tag{8a}$$

and the pressure coefficient by

$$C_p = -2\left(\frac{w}{V} + \frac{u^2 + v^2}{2V^2}\right). \tag{8b}$$

The part of Eqs. (8a), (8b) in u and v is not necessary if w is of the same magnitude as u and v. In many important cases, however, $u^2 + v^2$ is of the same magnitude as Vw and Eqs. (8a), (8b) must be used in its complete form. In these cases the validity of the solution should be checked.

The singularity of (5) or (6) at $t = \pm 1$ corresponds to the two Mach cones extending from the origin in the three dimensional flow. Various ranges of t correspond to various regions of flow, as shown in the following table.

Range of t	Region of Flow
$0 \leqq t < 1$	inside downstream cone
$-1 < t \leqq 0$	inside upstream cone
$1 < t < \infty$ $-\infty < t < -1$	outside both cones

3. Solutions of the differential equation (6). The parameter n is restricted to integers and the parameter m to non-negative integers. The differential equation for T, Eq. (6), has regular singularities of exponents $(+m, -m)$ at $t=0$, $(0, n+\frac{1}{2})$ at $t=\pm 1$, and $(-n, 1-n)$ at $t=\infty$. The solutions about the origin are

$$T = t^m F\left(\frac{-n+m}{2}, \frac{-n+m+1}{2}; 1+m; t^2\right) \tag{9a}$$

$$= t^m(1-t^2)^{n+\frac{1}{2}} F\left(\frac{n+m+1}{2}, \frac{n+m+2}{2}; 1+m; t^2\right), \tag{9b}$$

$$T = t^{-m} F\left(\frac{-n-m}{2}, \frac{-n-m+1}{2}; 1-m; t^2\right) \tag{10a}$$

$$= t^{-m}(1-t^2)^{n+\frac{1}{2}} F\left(\frac{n-m+1}{2}, \frac{n-m+2}{2}; 1-m; t^2\right), \tag{10b}$$

where F denotes the hypergeometric function. The solution of negative exponent, Eqs. (10a), (10b) is not well defined.

It is of considerably more value to express the solutions about $t^2=1$, since then both solutions are well defined and two distinct types of solution may be distinguished. One type of solution, designated as type I, is the solution of zero exponent at $t^2=1$ and is real throughout the range of t. The resulting solution for ϕ has no singularity on the Mach cones. The other type of solution, designated as type II, is the solution of exponent $n+\frac{1}{2}$ at $t^2=1$ and is real only for $t^2<1$ or only for $t^2>1$. The resulting solution for ϕ is defined only within the Mach cones or only outside the Mach cones. These solutions may be expressed as follows:

I) $$T = t^m F\left(\frac{-n+m}{2}, \frac{-n+m+1}{2}; -n+\tfrac{1}{2}; 1-t^2\right) \tag{11a}$$

$$= t^{-m} F\left(\frac{-n-m}{2}, \frac{-n-m+1}{2}; -n+\tfrac{1}{2}; 1-t^2\right), \tag{11b}$$

II) $$T = t^m(1-t^2)^{n+\frac{1}{2}} F\left(\frac{n+m+1}{2}, \frac{n+m+2}{2}; n+\tfrac{3}{2}; 1-t^2\right) \tag{12a}$$

$$= t^{-m}(1-t^2)^{n+\frac{1}{2}} F\left(\frac{n-m+1}{2}, \frac{n-m+2}{2}; n+\tfrac{3}{2}; 1-t^2\right). \tag{12b}$$

Three special cases are distinguished according to the relative values of n and m:

case A: $-\infty < n \leqq -m-1$,
case B: $-m \leqq n \leqq m-1$,
case C: $m \leqq n < \infty$

The distribution of these cases for small values of m and n is shown in the table:

m \ n	-3	-2	-1	0	1	2
0	A	A	A	C	C	C
1	A	A	B	B	C	C
2	A	B	B	B	B	C
3	B	B	B	B	B	B

From a consideration of Eqs. (9) to (12) the forms of the two types of solutions in the various cases may be found. For all solutions except solutions I-A (i.e., solutions of type I in case A) and solutions II-C, the form is explicit in terms of a polynomial in t^2 or in $(1-t^2)$. Solutions I-A and II-C have logarithmic singularities at $t=0$ and are discussed later. The polynomial forms are expressed as follows:

Solutions	*Form*	*Order of $P(t^2)$* whichever is integral of	*Equation for Calculation*
I-A	logarithmic	———	———
II-A	$t^m(1-t^2)^{n+\frac{1}{2}}P(t^2)$	$\frac{1}{2}(-n-m-1)$ or $\frac{1}{2}(-n-m-2)$	(9b) or (12a)
I-B	$t^{-m}P(t^2)$	$\frac{1}{2}(n+m)$ or $\frac{1}{2}(n+m-1)$	(10a) or (11b)
II-B	$t^{-m}(1-t^2)^{n+\frac{1}{2}}P(t^2)$	$\frac{1}{2}(-n+m-1)$ or $\frac{1}{2}(-n+m-2)$	(10b) or (12b)
I-C	$t^mP(t^2)$	$\frac{1}{2}(n-m)$ or $\frac{1}{2}(n-m-1)$	(9a) or (11a)
II-C	logarithmic	———	———

There is a connection between the solutions of Eq. (6) and the Legendre functions with the same values of n and m, except that $-n-1$ is used when n is negative. However, since Legendre functions are customarily defined only for $m \leqq n$ or $m \leqq n-1$, respectively, they are of assistance here only in cases A and C. The polynomial solutions are

$$\text{I-C)} \qquad T = (1-t^2)^{n/2}P_n^m\left[(1-t^2)^{-\frac{1}{2}}\right], \tag{13a}$$

$$\text{II-A)} \qquad T = (1-t^2)^{n/2}P_{-n-1}^m\left[(1-t^2)^{-\frac{1}{2}}\right]. \tag{13b}$$

These solutions are most easily obtained by transforming Eq. (1) into Laplace's equation by introducing the variable $iz/\sqrt{M^2-1}$.

The polynomial solutions may be obtained in another form from an expression given by A. R. Forsyth,[3] and the law for differentiating the hypergeometric functions. When n is not positive, these solutions are

[3] A. R. Forsyth, *A treatise on differential equations*, 6th ed., Macmillan, London, 1933, p. 235.

I) $$T = t^{-m} \frac{d^{-n}}{d(t^2)^{-n}} \left[(1 + \sqrt{1 - t^2})^{-n+m} + (1 - \sqrt{1 - t^2})^{-n+m}\right], \tag{14}$$

II) $$T = t^{\pm m} \frac{d^{-n}}{d(t^2)^{-n}} \left[(1 + \sqrt{1 - t^2})^{-n\mp m} - (1 - \sqrt{1 - t^2})^{-n\mp m}\right], \tag{15}$$

and when $n+1$ is not negative,

I) $$T = t^{\pm m}(1 - t^2)^{n+\frac{1}{2}} \frac{d^{n+1}}{d(t^2)^{n+1}} \left[(1 + \sqrt{1 - t^2})^{n+1\mp m} - (1 - \sqrt{1 - t^2})^{n+1\mp m}\right], \tag{16}$$

II) $$T = t^{-m}(1 - t^2)^{n+\frac{1}{2}} \frac{d^{n+1}}{d(t^2)^{n+1}} \left[(1 + \sqrt{1 - t^2})^{n+1+m} + (1 - \sqrt{1 - t^2})^{n+1+m}\right]. \tag{17}$$

4. Logarithmic solutions. The logarithmic solutions I-A and II-C are most easily expressed in terms of the Legendre functions, as in Eqs. (13a), (13b). They are

I-A) $$T = (1 - t^2)^{n/2} Q^m_{-n-1}\left[(1 - t^2)^{-\frac{1}{2}}\right], \tag{18a}$$

II-C) $$T = (1 - t^2)^{n/2} Q^m_n \left[(1 - t^2)^{-\frac{1}{2}}\right]. \tag{18b}$$

Since the validity of Eqs. (14) to (17) does not depend upon m being an integer, and since an appropriate solution of the form of these equations vanishes as a logarithmic solution is approached, the logarithmic solutions may also be obtained by differentiating such solutions with respect to m. The logarithmic solutions in this form are

I-A) $$T = t^{\pm m} \frac{d^{-n}}{d(t^2)^{-n}} \Big[(1 + \sqrt{1 - t^2})^{-n\mp m} \log (1 + \sqrt{1 - t^2}) + (1 - \sqrt{1 - t^2})^{-n\mp m} \log (1 - \sqrt{1 - t^2})\Big], \tag{19}$$

II-C) $$T = t^{\pm m}(1 - t^2)^{n+\frac{1}{2}} \frac{d^{n+1}}{d(t^2)^{n+1}} \Big[(1 + \sqrt{1 - t^2})^{n+1\mp m} \log (1 + \sqrt{1 - t^2}) + (1 - \sqrt{1 - t^2})^{n+1\mp m} \log (1 - \sqrt{1 - t^2})\Big]. \tag{20}$$

5. Generating equations. The fundamental equation, Eq. (1), expressed in Cartesian coordinates is invariant under differentiation with respect to any of these coordinates. Solutions of the type of Eq. (4b) expressed in Cartesian coordinates and differentiated with respect to these coordinates are still solutions of Eqs. (1) and (3). This fact permits a given solution of parameters n and m to yield solutions of parameters $n-1$ and m, $m+1$, or $m-1$:

$$T(n - 1, m) = nT - tT_t = - t^{n+1} \frac{d}{dt} (t^{-n}T), \tag{21a}$$

$$T(n - 1, m + 1) = \frac{m}{t} T - T_t = - t^{+m} \frac{d}{dt} (t^{-m}T), \tag{21b}$$

$$T(n - 1, m - 1) = \frac{m}{t} T + T_t = + t^{-m} \frac{d}{dt} (t^{+m}T). \tag{21c}$$

The procedures yielding these new solutions can be considered procedures of superposition; for example, one solution superposed on its negative an infinitesimal distance downstream yields the new solution given by Eq. (21a)

These equations are not to be considered recurrence relations, as no system has been established for specifying solutions with respect to the multiplicative arbitrary constant.

In a similar manner solutions with the parameter n increased by 1 may be obtained by reversing Eqs. (21a), (21b), and (21c) with suitable integrations.

6. Integral relation. An integral relation connecting two solutions whose parameters differ in value may be obtained either from the corresponding relation for the Legendre functions or directly from Eq. (6). If T_1 denotes a solution corresponding to n_1 and m_1 and T_2 a solution corresponding to n_2 and m_2, the relation is

$$
\begin{aligned}
(m_1^2 - m_2^2) \int_a^b & t^{-1}(1 - t^2)^{-\frac{1}{2}(n_1+n_2+1)} T_1 T_2 dt \\
& + (n_1 - n_2)(n_1 + n_2 + 1) \int_a^b t(1 - t^2)^{-\frac{1}{2}(n_1+n_2+3)} T_1 T_2 dt \\
= \Bigg[& t(1 - t^2)^{-\frac{1}{2}(n_1+n_2-1)} \left(T_2 \frac{dT_1}{dt} - T_1 \frac{dT_2}{dt} \right) \\
& + (n_1 - n_2) t^2 (1 - t^2)^{-\frac{1}{2}(n_1+n_2+1)} T_1 T_2 \Bigg] \Bigg|_a^b . \qquad (22)
\end{aligned}
$$

Setting $n_1 = n_2$ or $m_1 = m_2$, we obtain simpler equations as special cases which may be used to obtain orthogonality relations between solutions.

7. Two-dimensional cross-flow. The solutions of type I for which $m = |n|$ are given by $T = t^n$. The corresponding solutions for ϕ in cylindrical coordinates are

$$\phi = r^n \sin(|n| \theta + \beta).$$

These solutions give two-dimensional cross-flow because of the fact that z does not appear. This cross-flow, as a result of the linearizing assumptions, appears as an incompressible flow.

8. Conical flow. The solutions of either type for which $n = 1$ give solutions of conical flow, of which only those of type II are here treated. Since the exponent of these solutions at $t^2 = 1$ is 3/2, both the potential and the velocity vanish on the Mach cone. The first few solutions are:

$m = 0$) $\quad \sqrt{1-t^2} - \tanh^{-1}\sqrt{1-t^2}$,

$m = 1$) $\quad t^{-1}\sqrt{1-t^2} - t \tanh^{-1}\sqrt{1-t^2}$,

$m = 2$) $\quad t^{-2}(1-t^2)^{3/2}$,

$m = 3$) $\quad t^{-3}(1-t^2)^{3/2}$,

The flow about an infinitesimal circular cone at zero incidence is given by the first solution (II, 1, 0), the solution of type II with $n = 1$ and $m = 0$. The flow about the same cone at a small angle of attack is obtained by superposing solutions (II, 1, 1) and (I, 1, 1) with appropriate constants on the (II, 1, 0) solution. A standard treatment of this case will be found on pp. 46 to 49 of Sauer's book.[1]

9. Infinitesimal horseshoe vortices. An infinitesimal horseshoe vortex can be represented by a semi-infinite line dipole in the same manner as a planar vortex system can be represented by a planar dipole system. Thus a lifting element and other lifting systems can be represented by solutions of type II with $m=1$, as shown in the following table.

Solution	Semi-Infinite Line Dipole of Strength	Designation in Terms of Lifting Properties
(II, -1, 1): $t^{-1}(1-t^2)^{-\frac{1}{2}}$	Constant	"Lifting element"
(II, 0, 1): $t^{-1}(1-t^2)^{+\frac{1}{2}}$	Proportional to z	"Lifting line"
(II, 1, 1): $t^{-1}(1-t^2)^{+\frac{1}{2}}-t\tanh^{-1}(1-t^2)^{+\frac{1}{2}}$	Proportional to z^2	"Lifting infinitesimal triangle"

The "lifting element" solution, since the potential has exponent $-\frac{1}{2}$ and the velocities $-\frac{3}{2}$ at $t^2=1$, has a troublesome singularity on the Mach cones. A simpler singularity has the "lifting line" solution, whose potential vanishes and whose velocities have exponent $-\frac{1}{2}$ at $t^2=1$. When these solutions are superposed to give lifting systems of finite dimension, the singularity in the velocities usually disappears. The third solution is the same as the one which gives the lift on an inclined circular cone. Examples of the superposition of such solutions to form a lifting system will be given in a later paper.

10. The acceleration potential. Since the axial velocity component w is a derivative of the velocity potential in Cartesian coordinates, it satisfies the same equations, Eqs. (1) to (6), as does the velocity potential. The acceleration potential, whose fundamental theory will be found in a paper by L. Prandtl,[4] equals $-p/\rho$ for linearized flow, and also equals $Vw+\frac{1}{2}(u^2+v^2)$ from Eq. (8a). Hence the approximate acceleration potential defined by $\psi=Vw$ satisfies Eqs. (1) to (6). The relation of this quantity to the velocity potential for a given fundamental flows is of the type which leads to Eq. (21a), and hence the corresponding acceleration potential is given by a solution with n decreased by 1. It is important to note that this does not give the true linearized acceleration potential where u^2+v^2 is not of smaller magnitude than Vw. Thus the two-dimensional cross-flow described above has no approximate acceleration potential, and the acceleration potential is given incorrectly in the vicinity of the axis for other flows. However, the true acceleration potential may not itself be superposed, and often the difference between the approximate and true linearized acceleration potential disappears under superposition.

The "lifting element," "lifting line," and " lifting infinitesimal triangle" have approximate acceleration potential solutions (II, -2, 1), (II, -1, 1), and (II, 0, 1), respectively. With conical flow, the approximate acceleration potential is a function only of t and θ and can be shown to satisfy Laplace's equation in two dimensions.

[4] L. Prandtl, *Theorie des Flugzeugtragflügels in zusammendrückbarem Medium*, Luftfahrtforschung, 13, 313 (1936).

THE OPENING OF A GRIFFITH CRACK UNDER INTERNAL PRESSURE*

BY

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1. The determination of the distribution of stress in the neighbourhood of a crack in an elastic body is of importance in the discussion of certain properties of the solid state. The theory of cracks in a two-dimensional elastic medium was first developed by Griffith[1] who succeeded in solving the equations of elastic equilibrium in two dimensions for a space bounded by two concentric coaxial ellipses; by considering the inner ellipse to be of zero eccentricity and by assuming that the major axis of the outer ellipse was very large Griffith then derived the solution corresponding to a very thin crack in the interor of an infinite elastic solid. Because of the nature of the coordinate system employed by Griffith the expressions he derives for the components of stress in the vicinity of the crack do not lend themselves easily to computation. An alternative method of determining the distribution of stress in the neighbourhood of a Griffith crack was given recently by one of us[2] making use of a complex stress-function stated by Westergaard.[3] This method suffers from the disadvantage that the Westergaard stress-function refers only to the case in which the Griffith crack is opened under the action of a *uniform* internal pressure; the stress-function corresponding to a variable internal pressure does not appear to be known.

In the present note we discuss the distribution of stress in the neighbourhood of a Griffith crack which is subject to an internal pressure, which may vary along the length of the crack, by considering the corresponding boundary value problem for a semi-infinite two-dimensional medium. The analysis is the exact analogue of that for the three-dimensional "circular" cracks developed in the previous paper[2] except that now we employ a Fourier cosine transform method in place of the Hankel transform method used there. A method is given for determining the shape of the crack resulting from the application of a variable internal pressure to a very thin crevice in the interior of an elastic solid, and for determining the distribution of stress throughout the solid. The converse problem of determining the distribution of pressure necessary to open a crevice to a crack of prescribed shape is also considered. As an example of the use of the method the expressions for the components of stress, due to the opening of a crack under a uniform pressure, are derived and are found to be in agreement with those found in the earlier paper.[2]

2. We consider the distribution of stress in the interior of an infinite two-dimensional elastic medium when a very thin internal crack $-c \leqq y \leqq c$, $x=0$ is opened under the action of a pressure which may be considered to vary in magnitude along the length of the crack. For simplicity we shall consider the symmetrical case in which the applied pressure is a function of $|y|$ but the analysis may easily be extended to the

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[1] A. A. Griffith, Phil. Trans. (A) **221**, 163 (1921).

[2] I. N. Sneddon, Proc. Roy. Soc. (A) (in the press).

[3] H. M. Westergaard, J. Appl. Mech. **6**, A49 (1939).

more general case in which there is no such symmetry. The stress in such a medium may be described by three components of stress σ_x, σ_y and τ_{xy}; the corresponding components of the displacement vector will be denoted by u_x and u_y. The differential equations determining the stress-components are[4]

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad (1) \qquad\qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \qquad (2)$$

The boundary conditions to be satisfied are that all the components of stress and of the displacement vector must tend to zero as x^2+y^2 tends to infinity, and that

$$\tau_{xy} = 0, \qquad \sigma_x = -p(y), \tag{3}$$

when $x=0$ and $-c \leqq y \leqq c$.

It is obvious from the symmetry about the axis $x=0$ that the problem of determining the distribution of stress in the neighbourhood of the crevice is equivalent to that of determining the stress in the semi-infinite elastic medium $x \geqq 0$ when the boundary $x=0$ is subjected to the following conditions:

(i) $\tau_{xy}=0$, for all values of y,

(ii) $\sigma_x = -p(y)$, $|y| \leqq c$,

$u_x = 0$ $\qquad |y| \geqq c$.

From the symmetry about the second axis $y=0$ we may take as solutions of the elastic equations (1) and (2) the expressions:[5]

$$\sigma_x = \frac{2}{\pi}\int_0^\infty \phi(p)(1+px)e^{-px}\cos py\,dp, \tag{4}$$

$$\sigma_y = \frac{2}{\pi}\int_0^\infty \phi(p)(1-px)e^{-px}\cos py\,dp, \tag{5}$$

$$\tau_{xy} = \frac{2x}{\pi}\int_0^\infty p\phi(p)e^{-px}\sin py\,dp. \tag{6}$$

These expressions satisfy the equations of equilibrium and the boundary condition (i) above; the function $\phi(p)$ is determined from the set of conditions (ii). The components of the displacement vector are similarly found to be

$$u_x = -\frac{2(1+\sigma)}{\pi E}\int_0^\infty \phi(p)e^{-px}\{2(1-\sigma)+px\}\frac{\cos py}{p}\,dp, \tag{7}$$

$$u_y = \frac{2(1+\sigma)}{\pi E}\int_0^\infty \phi(p)e^{-px}\{(1-2\sigma)-px\}\frac{\sin py}{p}\,dp. \tag{8}$$

When $x=0$, equations (4) and (7) reduce to

$$\sigma_x = \frac{2}{\pi}\int_0^\infty \phi(p)\cos py\,dp, \tag{9}$$

$$u_x = -\frac{4(1-\sigma^2)}{\pi E}\int_0^\infty \phi(p)\frac{\cos py}{p}\,dp. \tag{10}$$

[4] A. E. H. Love, *The mathematical theory of elasticity*, 4th ed., Cambridge, 1934, p. 208.

[5] I. N. Sneddon, Proc. Cambridge Phil. Soc. 40, 229 (1944).

If we insert the boundary conditions (ii) into Eqs. (9) and (10) and make the substitutions

$$\rho = \xi/c, \qquad y = \eta c, \qquad g(\eta) = -c\left(\frac{\pi}{2\eta}\right)^{1/2} p(\eta c), \qquad \Phi\left(\frac{\xi}{c}\right) = \xi^{1/2}F(\xi), \tag{11}$$

we obtain a pair of "dual" integral equations

$$\left.\begin{aligned} &\int_0^\infty \xi F(\xi) J_{-1/2}(\xi\eta)\,d\xi = g(\eta), \qquad 0 < \eta < 1 \\ &\int_0^\infty F(\xi) J_{-1/2}(\xi\eta)\,d\xi = 0, \qquad \eta > 1 \end{aligned}\right\} \tag{12}$$

for the determination of the function $F(\xi)$. Once $F(\xi)$ has been found, $\Phi(\rho)$ can be written down and the components of stress calculated by means of Eqs. (4), (5) and (6).

3. The dual integral equations (12) are a special case of a pair of equations considered by Busbridge;[6] the solution may be obtained by substituting $\alpha=1$, $\nu=-1/2$ in the general solution given in the paper.[6] In this we obtain

$$F(\xi) = \sqrt{\frac{2}{\pi}}\,\xi^{1/2}\left[J_0(\xi)\int_0^1 y^{1/2}(1-y^2)^{1/2}g(y)\,dy \right.$$
$$\left. + \xi\int_0^1 u^{1/2}(1-u^2)^{1/2}du \int_0^1 g(yu)\,y^{5/2}J_1(\xi y)\,dy\right]. \tag{13}$$

Thus if the pressure $p(y)$ is given by a Taylor series of the form

$$p(y) = p_0 \sum_{n=0}^\infty a_n \left(\frac{y}{c}\right)^n, \tag{14}$$

convergent for $-c \leqq y \leqq c$, then the corresponding expression for $\Phi(\rho)$ is readily found to be

$$\Phi(\rho) = -\tfrac{1}{2}p_0 c^2 \pi^{1/2}\rho \sum_{n=0}^\infty \frac{\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{1}{2}n+2)}\,a_n \left\{J_0(c\rho) + c\rho\int_0^1 y^{n+2}J_1(c\rho y)\,dy\right\}. \tag{15}$$

Substituting for $\Phi(\rho)$ from Eq. (15) into Eq. (10) and making use of the results[7]

$$\int_0^\infty J_0(c\rho)\cos\rho y\,d\rho = \frac{1}{\sqrt{c^2-y^2}}, \qquad 0 < y < c$$

$$\int_0^\infty \rho J_1(c\rho)\cos\rho y\,d\rho = \frac{c}{(c^2-y^2)^{3/2}}, \qquad 0 < y < c$$

we find that the normal component of the displacement along the crack is given by w, where

$$w = \frac{2(1-\sigma^2)p_0 c}{\sqrt{\pi}\cdot E}\sum_{n=0}^\infty \frac{\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{1}{2}n+2)}\,a_n\left\{\frac{c}{\sqrt{c^2-y^2}} + \left(\frac{y}{c}\right)^{n+1}\int_1^{c/y}\frac{u^{n+3}du}{(u^2-1)^{3/2}}\right\}. \tag{16}$$

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[6] I. W. Busbridge, Proc. London Math. Soc. (2) **44**, 115 (1938).

[7] G. N. Watson, *The theory of Bessel functions*, 2nd ed., Cambridge, 1944, p. 405.

For the case of a uniform pressure p_0 we take $a_0=1$, $a_n=0$, $n\geqq 1$ and find

$$w=\frac{2(1-\sigma^2)p_0}{E}\sqrt{c^2-y^2}. \tag{17}$$

If we write

$$b=2(1-\sigma^2)p_0c/E,$$

Eq. (17) reduces to the form

$$\frac{y^2}{c^2}+\frac{w^2}{b^2}=1$$

which shows that the effect of the uniform pressure is to widen the crevice into an elliptic crack.

4. It is also of interest to determine what distribution of pressure will produce a crack of prescribed shape. In this case we assume that the value of the normal displacement u_x is known *all* along the y-axis; we have

$$u_x=\begin{cases} w(y), & y\leqq |c|, \quad x=0,\\ 0, & y\geqq |c|, \quad x=0.\end{cases}$$

Inverting Eq. (10) by the Fourier cosine rule and substituting this value for u_x we have

$$\bar{\phi}(\rho)=-\frac{E}{2(1-\sigma^2)}\rho\int_0^c w(y)\cos\rho y\,dy. \tag{18}$$

With this value of $\bar{\phi}(\rho)$ in Eqs. (4), (5) and (6) we obtain expressions for the components of stress in the interior of the elastic solid.

For example if we take

$$w(y)=\epsilon\left(1-\frac{y^2}{c^2}\right),$$

then, from Eq. (18)

$$\bar{\phi}(\rho)=-\frac{E\epsilon}{(1-\sigma^2)c\rho}\left(\frac{\sin c\rho}{c\rho}-\cos c\rho\right). \tag{19}$$

Substituting from (19) into Eq. (9) we obtain for the normal component of the stress along $x=0$,

$$\sigma_x=-\frac{2E\epsilon}{\pi(1-\sigma^2)c}\left[1-\frac{y}{c}\int_0^\infty\frac{\sin u\sin\dfrac{yu}{c}}{u}du\right]. \tag{20}$$

Now

$$\int_0^\infty\frac{\cos qx-\cos px}{x}dx=\tfrac{1}{2}\log\frac{p^2}{q^2},$$

so that Eq. (20) reduces to

$$\sigma_x=-\frac{2E\epsilon}{\pi(1-\sigma^2)c}\left[1-\frac{y}{2c}\log\frac{c+y}{c-y}\right],\qquad 0<y<c, \tag{21}$$

giving the normal component of stress along the crack. This stress is negative when $y=0$ but becomes positive for a value of y between 0 and c, so that if a crack of this shape is to be maintained the applied stress must be tensile (and very large) near the edges $y=\pm c$ of the crack.

5. Expressions for the potential functions $\omega(z)$, $\Omega(z)$ of Stevenson corresponding to this problem can easily be deduced from the analysis of Section 3. It was shown by Stevenson,[8] that if we write

$$\Theta = \sigma_x + \sigma_y; \qquad \Phi = \sigma_x - \sigma_y + 2i\tau_{xy}, \qquad D = u_x + iu_y$$

then the components of the stress and the displacement can be expressed in terms of two "potential" functions $\omega(z)$, $\Omega(z)$ by means of the equations

$$\left.\begin{aligned} D &= \frac{1+\sigma}{4}E\{(3-4\sigma)\Omega(z) - z\bar{\Omega}'(\bar{z}) - \bar{\omega}'(\bar{z})\} \\ 2\Theta &= \Omega'(z) + \bar{\Omega}'(\bar{z}) \\ -2\Phi &= z\bar{\Omega}''(\bar{z}) + \bar{\omega}''(\bar{z}) \end{aligned}\right\} \tag{22}$$

in the absence of body forces.

It follows from Eqs. (4) to (8) that the stresses and the components of the displacement vector may be derived from the potential functions

$$\Omega(z) = -\frac{4}{\pi}\int_0^\infty \frac{\phi(\rho)}{\rho}e^{-\rho z}d\rho, \qquad \omega'(z) = \frac{4}{\pi}\int_0 \frac{\phi(\rho)}{\rho}(1+\rho z)e^{-\rho z}d\rho, \tag{23}$$

where $\phi(\rho)$ is given by Eq. (15) in the case where the applied internal pressure is given by Eq. (14).

6. We now consider the distribution of stress in the solid when the crevice $-c\leqq y\leqq c$, $x=0$ is opened up by the action of a uniform pressure p_0. Taking $a_0=1$, $a_n=0$, $n>0$, in Eq. (15) we obtain for $\phi(\rho)$ the expression

$$\phi(\rho) = -\tfrac{1}{4}\pi p_0 c^2\rho\left\{J_0(c\rho) + \frac{1}{c^2\rho^2}\int_0^{c\rho} z^2 J_1(z)dz\right\}.$$

Now,

$$\int_0^{c\rho} z^2 J_1(z)dz = c^2\rho^2 J_2(c\rho)$$

and, by a well-known recurrence relation,

$$J_0(c\rho) + J_2(c\rho) = \frac{2}{c\rho}J_1(c\rho)$$

so that

$$\phi(\rho) = -\tfrac{1}{2}\pi p_0 c J_1(c\rho). \tag{24}$$

Substituting from Eq. (24) into (4), (5) and (6) we obtain the equations

$$\tfrac{1}{2}(\sigma_x + \sigma_y) = -p_0 c\int_0^\infty e^{-\rho x}\cos\rho y J_1(c\rho)d\rho, \tag{25}$$

[8] A. C. Stevenson, Phil. Mag., (7) **34**, 766 (1943).

$$\tfrac{1}{2}(\sigma_y - \sigma_x) = p_0 c x \int_0^\infty \rho e^{-\rho x} \cos \rho y J_1(c\rho) d\rho, \tag{26}$$

$$\tau_{xy} = - p_0 c x \int_0^\infty \rho e^{-\rho x} \sin \rho y J_1(c\rho) d\rho \tag{27}$$

for the determination of the components of stress.

Now,

$$\int_0^\infty \rho e^{-\rho z} J_1(c\rho) d\rho = c(c^2 + z^2)^{-3/2},$$

so that writing

$$z = x + iy = re^{i\theta}, \qquad z - ic = r_1 e^{i\theta_1}, \qquad z + ic = r_2 e^{i\theta_2} \tag{28}$$

we obtain the formula

$$\int_0^\infty J_1(c\rho)\rho e^{-\rho x}(\cos \rho y - i \sin \rho y) d\rho = \frac{c}{(r_1 r_2)^{3/2}} e^{-i3/2(\theta_1+\theta_2)}. \tag{29}$$

In a similar way we can establish that

$$\int_0^\infty J_1(c\rho) e^{-\rho x}(\cos \rho y - i \sin \rho y) d\rho = \frac{1}{c}\left\{1 - \frac{r}{(r_1 r_2)^{1/2}} e^{i(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2)}\right\}. \tag{30}$$

Equating real and imaginary parts in Eqs. (29) and (30) and substituting into (25), (26), and (27) we obtain the expressions

$$\left.\begin{aligned} \tfrac{1}{2}(\sigma_x + \sigma_y) &= p_0\left\{\frac{r}{(r_1 r_2)^{1/2}} \cos\left(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2\right) - 1\right\}, \\ \tfrac{1}{2}(\sigma_y - \sigma_x) &= p_0 \frac{r \cos \theta}{c}\left(\frac{c^2}{r_1 r_2}\right)^{3/2} \cos \frac{3}{2}(\theta_1 + \theta_2), \\ \tau_{xy} &= - p_0 \frac{r \cos \theta}{c}\left(\frac{c^2}{r_1 r_2}\right)^{3/2} \sin \frac{3}{2}(\theta_1 + \theta_2) \end{aligned}\right| \tag{31}$$

for the components of stress. Equations (31) are agreement with those derived in the previous paper;[2] in making the comparison it should be noted that the angles θ, θ_1, θ_2 of this note are the complements of the angles denoted by these symbols in the paper quoted.

It follows from Eqs. (23) that these equations are a consequence of the Stevenson equations (22) if we write

$$\Omega(z) = 2p_0\left[\sqrt{c^2 + z^2} - z\right], \qquad \omega'(z) = - 2p_0 c^2 (c^2 + z^2)^{-\frac{1}{2}}.$$

ANALYSIS OF SHEAR LAG IN BOX BEAMS BY THE PRINCIPLE OF MINIMUM POTENTIAL ENERGY*

BY

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1. Introduction. Let us consider a thin-walled box beam of web height $2h$ and cover sheet width $2w$ which is bent in such a way that one of the cover sheets is in tension while the opposite cover sheet is in compression (Fig. 1). In elementary beam theory the assumption is made that the normal stress in the cover sheets does not

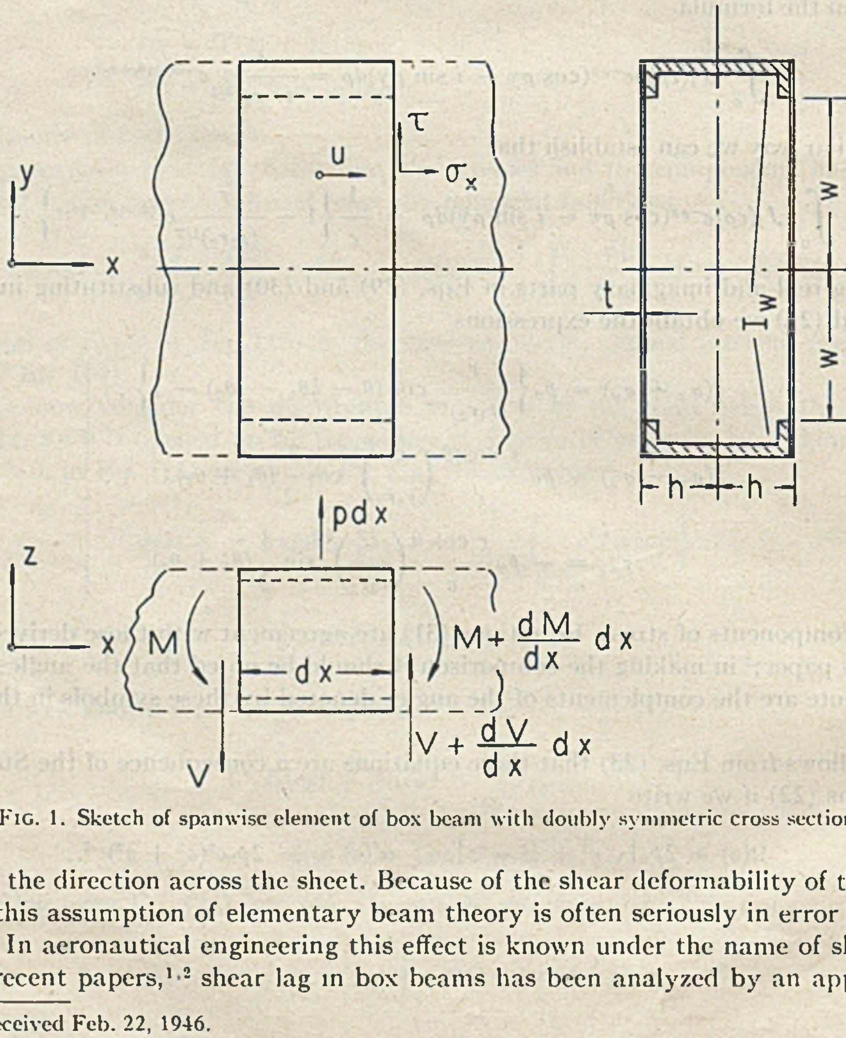

FIG. 1. Sketch of spanwise element of box beam with doubly symmetric cross section.

vary in the direction across the sheet. Because of the shear deformability of the cover sheets this assumption of elementary beam theory is often seriously in error for wide beams. In aeronautical engineering this effect is known under the name of shear lag.

In recent papers,[1,2] shear lag in box beams has been analyzed by an application

* Received Feb. 22, 1946.

[1] E. Reissner, *Least work solutions of shear lag problems*, Journal of the Aeronautical Sciences, 8, 284–291 (1941).

of the theorem of least work which is the basic minimum principle for the *stresses*. The present paper contains an application to the problem of shear lag of the theorem of minimum potential energy, which is the basic minimum principle for the *strains*.[3] It is shown that application of the theorem of minimum potential energy to the present problem leads to simpler and more general results than the application of the theorem of least work. While the least-work method furnishes the stresses in box beams with no cut-outs, application of the minimum-potential-energy method furnishes, in a simpler manner, the stresses in beams without or with cut-outs. It also furnishes beam deflections, and is equally convenient for beams supported in statically determinate or in statically indeterminate manner.

Application, in the manner described below, of the minimum-potential-energy principle to the problem of bending of thin-walled box beams leads to a differential equation for the beam deflection which is a generalization of the relation $z'' = -M/EI$; this differential equation contains an additional term proportional to the fourth derivative of z which takes into account the shear deformability of the cover sheets. As the order of the differential equation in this theory is higher than the order of the differential equation of elementary beam theory, boundary conditions appear in addition to those of elementary beam theory. These additional boundary conditions are different for beams with cut outs and for beams without cut outs.

The manner of application of the results obtained in the present paper is shown by solving explicitly the following four examples.

1. Simply supported beam. Load distributed according to a cosine law.
2. Cantilever beam with uniform load distribution. Cover sheets fixed at the support.
3. Cantilever beam with uniform load distribution. Cover sheets not fixed at the support.
4. Beam with both ends built in. Uniform load distribution.

For the sake of simplicity, it is assumed in what follows that the cross sections of the beams are rectangular and doubly symmetrical. It also is assumed that there is no continuous variation of cross-sectional properties.

The author believes that the way in which the principle of minimum potential energy is here applied to the problem of shear lag will prove useful in other problems of structural mechanics. As an example of such future application, the theory for combined torsion and bending of beams with open or closed cross sections is mentioned.

2. Formulation and solution of problem. In the following, we analyze a box beam of doubly symmetrical rectangular cross section, composed of cover sheets, sidewebs and flanges. A given distribution of loads is applied to the sidewebs, acting normal to the plane of the cover sheets (Fig. 1). To this load distribution there corresponds a distribution of bending moments $M(x)$. The spanwise coordinate being x, let y be the coordinate in the plane of the cover sheets perpendicular to the x direction, and $z(x)$ the deflection of the neutral axis of the beam.

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[2] F. B. Hildebrand and E. Reissner, *Least work analysis of the problem of shear lag in box beams*, N.A.C.A. Technical Note No. 893 (1943).

[3] For a formulation of these theorems see for instance I. S. Sokolnikoff and R. D. Specht, *Mathematical theory of elasticity*, McGraw-Hill Book Co., Inc., New York, 1946, pp. 275–287.

The potential energy of the bent beam may be considered as composed of three parts. The first part is the potential energy of the load system. This may be written in the form

$$\Pi_l = \int M(x) \frac{d^2z}{dx^2} dx, \tag{1}$$

the integral being extended over the entire length of the beam.[4] The second part is the strain energy of sidewebs and flanges. This may be written in the form

$$\Pi_w = \frac{1}{2} \int EI_w \left(\frac{d^2z}{dx^2}\right)^2 dx, \tag{2}$$

the quantity I_w denoting the principal moment of inertia of the two sidewebs and flanges.

The third part is the strain energy of the two cover sheets. If it is assumed that the normal strains in the chordwise direction in the sheets may be neglected, as discussed in the reference given in Footnote 1, then the strain energy of the two sheets is given by the integral

$$\Pi_s = \frac{1}{2} \iint 2t[E\epsilon_x^2 + G\gamma^2] dx dy, \tag{3}$$

where the quantity t denotes the cover sheet thickness, and where E and G are the effective moduli of elasticity and rigidity. Spanwise normal strain ϵ_x and shear strain γ are then expressed in terms of the spanwise sheet displacement u as follows

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \gamma = \frac{\partial u}{\partial y}. \tag{4}$$

The theorem of minimum potential energy states that the total potential energy

$$\Pi = \Pi_s + \Pi_w + \Pi_l \tag{5}$$

becomes a minimum for the correct displacement functions u and z, if only such displacement functions are compared which satisfy all conditions of support and continuity imposed on the displacements.

Direct application of this condition by means of the calculus of variations leads to a partial differential equation for u and to a complete system of boundary conditions. In what follows, an ordinary differential equation for the beam deflection z and boundary conditions for it are obtained instead. This is done by making a suitable approximation for the sheet displacements u and by applying the rules of the calcula- of variations to the resultant approximate expression for the potential energy function.

A reasonable assumption for the spanwise sheet displacements is

$$u(x, y) = \pm h \left[\frac{dz}{dx} + \left(1 - \frac{y^2}{w^2}\right) U(x)\right]. \tag{6}$$

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[4] Eq. (1) implies that the beam is supported in such a manner that the end forces and moments can do no work. This restriction shortens the developments slightly.

The second term on the right of Eq. (6) represents the correction due to shear lag. Instead of the vanishing chordwise variation of the sheet displacements of elementary beam theory, we now assume a parabolic variation. The relative magnitude of the function U is a measure for the magnitude of the shear lag effect. The form of the correction is such that continuity of the displacements along the flanges, that is along $y=\pm w$, is preserved.

Denoting differentiation with respect to x by primes, we obtain the following expressions for the strains in the sheets from Eqs. (6) and (4):

$$\epsilon_x = \pm h\left[z'' + \left(1 - \frac{y^2}{w^2}\right)U'\right], \tag{7}$$

$$\gamma = \mp \frac{2h}{w}\frac{y}{w} U. \tag{8}$$

On the basis of Eqs. (7) and (8) the following expression for the strain energy of the sheets is obtained:

$$\Pi_s = \iint th^2\left\{E\left[z'' + \left(1 - \frac{y^2}{w^2}\right)U'\right]^2 + G\left[\frac{2}{w}\frac{y}{w}U\right]^2\right\}dydx. \tag{9}$$

In Eq. (9) the integration with respect to y is carried out. Setting

$$I_s = 4wth^2, \qquad I = I_s + I_w, \tag{10}$$

we have

$$\Pi_s = \frac{1}{2}\int EI_s\left\{(z'')^2 + \frac{8}{15}(U')^2 + \frac{4}{3}z''U' + \frac{G}{E}\frac{4}{3w^2}U^2\right\}dx. \tag{11}$$

Substituting Eqs. (11), (12) and (1) into Eq. (5), we obtain the following expression for the potential energy of the system

$$\Pi = \int\left\{\frac{1}{2}EI(z'')^2 + Mz''\right\}dx$$
$$+ \int\frac{1}{2}EI_s\left\{\frac{8}{15}(U')^2 + \frac{4}{3}z''U' + \frac{G}{E}\frac{4}{3w^2}U^2\right\}dx. \tag{12}$$

Differential equations and boundary conditions for z and U are obtained by making

$$\delta\Pi = 0. \tag{13}$$

Thus, with x_1 and x_2 denoting the ends of the interval of integration,

$$\delta\Pi = \int\left\{\left[EIz'' + M + \frac{2}{3}EI_sU'\right]\delta z''\right.$$
$$\left. + EI_s\left[-\frac{8}{15}U'' - \frac{2}{3}z''' + \frac{G}{E}\frac{4}{3w^2}U\right]\delta U\right\}dx$$
$$+ \left\{EI_s\left[\frac{8}{15}U' + \frac{2}{3}z''\right]\delta U\right\}_{x_1}^{x_2} = 0. \tag{14}$$

As $\delta z''$ and δU are arbitrary in the interior of the interval (x_1, x_2) the terms multiplying them must vanish. This gives the following two differential equations

$$z'' + \frac{2}{3}\frac{I_s}{I}U' + \frac{M}{EI} = 0, \tag{15}$$

$$EI_s\left[U'' - \frac{5}{2}\frac{G}{E}\frac{U}{w^2} + \frac{5}{4}z'''\right] = 0. \tag{16}$$

The integrated portion of Eq. (14) defines the boundary and transition conditions for the function U. At a section where the sheet is fixed, $\delta U = 0$ and

$$U = 0. \tag{17}$$

At a section where the sheet is not fixed and consequently δU is arbitrary,

$$EI_s[U' + \tfrac{5}{4}z''] = 0. \tag{18}$$

Transitions conditions for adjacent bays with different stiffness are:

$$U \text{ and } EI_s[U' + \tfrac{5}{4}z''] \text{ continuous.} \tag{19}$$

The above boundary and transition conditions are in addition to those imposed on z and M in elementary beam theory, as may be verified by repeated integration by parts of the term containing $\delta z''$ in the integral of Eq. (14).

3. The modified beam equation and its boundary conditions. By eliminating the quantity U from Eqs. (15) to (19), we obtain a system of relations containing the beam deflection z only.

The differential equation for z is derived by differentiating Eq. (16) and substituting U' from Eq. (15). There follows

$$z'' + \frac{M}{EI} - w^2\frac{E}{G}\left[\frac{2}{5}\left(z'' + \frac{M}{EI}\right)'' - \frac{I_s}{3I}z^{\mathrm{IV}}\right] = 0. \tag{20}$$

When the shear deformability of the sheets is neglected, that is when it is assumed that $G = \infty$, Eq. (20) reduces to the well known result of elementary beam theory.

Equation (20) may be written in the alternate form

$$z'' - \frac{2}{5}\frac{E}{G}\left(1 - \frac{5}{6}\frac{I_s}{I}\right)\frac{z^{\mathrm{IV}}}{w^2} = -\frac{M}{EI} + \frac{2}{5}\frac{E}{G}\frac{M''}{w^2EI}. \tag{21}$$

With the help of Eqs. (15) and (16), the boundary condition (17), which holds when the sheet is attached to the support, is transformed into

$$\left(1 - \frac{5}{6}\frac{I_s}{I}\right)z''' + \frac{M'}{EI} = 0. \tag{22}$$

Similarly, the boundary condition (18), which holds when the sheet is not attached to the support, becomes

$$\left(1 - \frac{5}{6}\frac{I_s}{I}\right)z'' + \frac{M}{EI} = 0. \tag{23}$$

The continuity conditions (19) may be transformed in an analogous manner.

The values of the sheet stresses may be obtained from Eqs. (9) and (10). From Eq. (9) it follows that the flange stress is given by

$$\sigma_f = \pm Ehz''. \tag{24}$$

For the application of the results it may be noted that the differential equation (21) can first be solved for the value of z'' which, according to (24), gives directly the approximate value of the flange stress σ_f. The magnitude of the deflection z can then be found from the value of z'' as in elementary beam theory.

For the evaluation of the solution we define the following two parameters

$$n = \frac{1}{1 - 5I_s/6I}, \tag{25}$$

$$k = \frac{1}{w}\sqrt{\frac{5n}{2}\frac{G}{E}}\cdot \tag{26}$$

With (25) and (26) the differential equation (21) becomes

$$z'' - \frac{1}{k^2}z^{IV} = -\frac{M}{EI} + \frac{n}{k^2}\frac{M''}{EI}; \tag{27}$$

the boundary condition at an end section where the sheet is attached to the support becomes

$$z''' = -n\frac{M'}{EI}; \tag{28}$$

and the boundary condition at an end section where the sheet is not attached to the support becomes

$$z'' = -n\frac{M}{EI}\cdot \tag{29}$$

4. Examples of applications (Fig. 2). 1. *Simply supported beam. Load distributed according to a cosine law.* Designating the span length of the beam by l and assuming the origin of the coordinate system at the center of the beam, we consider the moment distribution

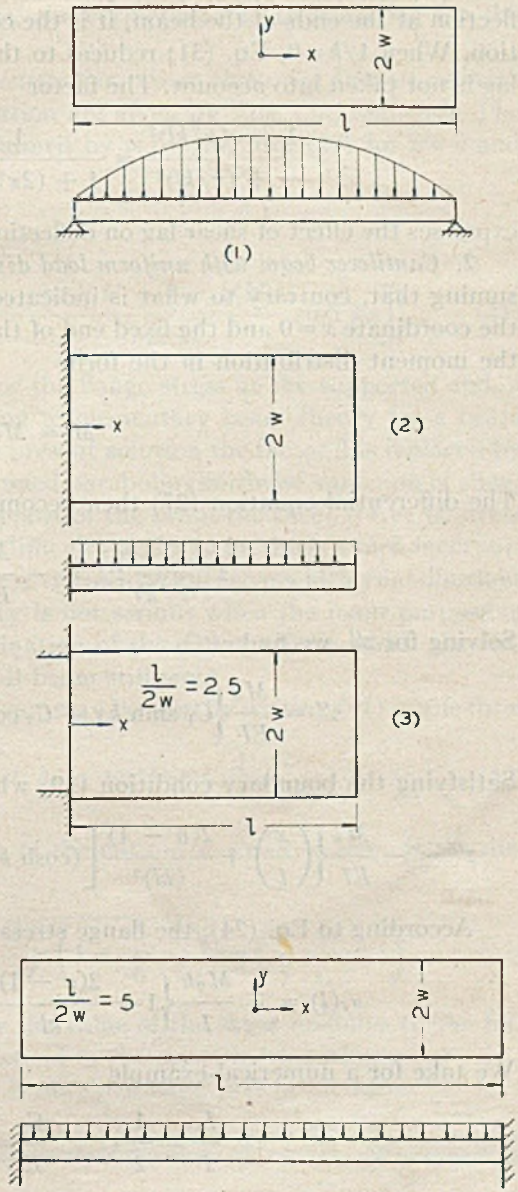

FIG. 2. Diagrammatic sketches of beams analyzed as examples of application of the theory.

$$M = M_0 \cos \pi \frac{x}{l} \cdot \tag{30}$$

A particular solution of Eq. (27) is

$$z = \left(\frac{l}{\pi}\right)^2 \frac{M_0}{EI} \frac{1 + n(\pi/kl)^2}{1 + (\pi/kl)^2} \cos \pi \frac{x}{l} \cdot \tag{31}$$

As Eq. (31) satisfies the boundary condition (29) and the condition of vanishing deflection at the ends of the beam, it is the complete expression for the deflection function. When $1/k = 0$, Eq. (31) reduces to the expression for z in the case where shear lag is not taken into account. The factor

$$\frac{1 + n(\pi/kl)^2}{1 + (\pi/kl)^2} = \frac{1 + (2\pi^2 E/5G)(w/l)^2}{1 + (2\pi^2 E/5G)(w/l)^2(1 - 5I_s/6I)} \cdot \tag{32}$$

expresses the effect of shear lag on deflection and flange stresses.

2. *Cantilever beam with uniform load distribution. Cover sheets fixed at support.* Assuming that, contrary to what is indicated in Fig. 2, the free end of the beam has the coordinate $x = 0$ and the fixed end of the beam the coordinate $x = l$, we may write the moment distribution in the form

$$M = M_0 \left(\frac{x}{l}\right)^2. \tag{33}$$

The differential equation (27) then becomes

$$z'' - \frac{1}{k^2} z^{\mathrm{IV}} = -\frac{M_0}{EI}\left[\left(\frac{x}{l}\right)^2 - \frac{2n}{(kl)^2}\right]. \tag{34}$$

Solving for z'', we find

$$z'' = \frac{M_0}{EI}\left\{C_1 \sinh kx + C_2 \cosh kx - \left(\frac{x}{l}\right)^2 + \frac{2(n-1)}{(kl)^2}\right\} \cdot \tag{35}$$

Satisfying the boundary condition (29) when $x = 0$ and (28) when $x = l$, we obtain

$$z'' = -\frac{M_0}{EI}\left\{\left(\frac{x}{l}\right)^2 + \frac{2(n-1)}{(kl)^2}\left[(\cosh kx - 1) - \frac{\sinh kl - kl}{\cosh kl} \sinh kx\right]\right\} \cdot \tag{36}$$

According to Eq. (24), the flange stress at the fixed end of the beam becomes

$$\sigma_f(l) = \mp \frac{M_0 h}{I}\left\{1 + \frac{2(n-1)}{kl}\left[\tanh kl - \frac{1}{kl} + \frac{1}{kl \cosh kl}\right]\right\} \cdot \tag{37}$$

We take for a numerical example

$$\frac{I_s}{I} = \frac{1}{2}, \qquad \frac{G}{E} = \frac{3}{8}, \qquad \frac{l}{2w} = \frac{5}{2}, \tag{38}$$

so that according to Eqs. (25) and (26)

$$n = 1.714, \qquad kl = 6.34, \tag{39}$$

and we find

$$\sigma_f(l) = \mp \frac{M_0 h}{I} \{1 + .190\}. \tag{40}$$

By application of the least work method[1,2] a factor 1.186 is obtained instead of the factor 1.190 in Eq. (40).

The deflection of the beam is obtained from Eq. (36) by integrating twice and making $z(l) = z'(l) = 0$. In the present case, the correction due to shear lag for the maximum deflection is about ten percent.

3. *Cantilever beam with uniform load distribution. Cover sheets not fixed at support.* Moment distribution and differential equation are given by Eqs. (33) and (34). The constants of integration in (36) are determined by satisfying Eq. (29) for $x=0$ and for $x=l$. There follows

$$z'' = -\frac{M_0}{EI}\left\{\left(\frac{x}{l}\right)^2 + \frac{2(n-1)}{(kl)^2}\left[(\cosh kx - 1) - \frac{\cosh kl - 1 - \frac{1}{2}(kl)^2}{\sinh kl} \sinh kx\right]\right\}. \tag{41}$$

Taking again $I_s/I = .5$, we should have, for the flange stress at the supported end, a value twice as large as the stress according to elementary beam theory for a beam with sheet attached to the support. In the present solution the factor 2 is replaced by $n = 1.714$. This indicates that with the assumed parabolic chordwise variation of sheet displacement the condition that at the support of the beam the sheet is free of stress is only approximately satisfied. The same difficulty arises in methods which incorporate the ability of the sheet to carry normal stresses as effective width contributions to the strength of stiffners.[5] This difficulty is not serious when the main purpose of such "cut-out" calculations is the determination of the distance over which the cut-out is effective and its effect on the over all beam stiffness.[6]

The localization of the effect of the cut-out may be seen by writing (41) in the form

$$z'' \approx -\frac{M_0}{EI}\left\{\left(\frac{x}{l}\right)^2 + (n-1)e^{-kl(1-x/l)}\right\}. \tag{42}$$

This equation indicates that the influence of the cut-out is small as soon as the distance $l-x$ satisfies the inequality

$$l - x \gg \frac{-\log(n-1)}{k} = w\sqrt{\frac{2E}{5G}\left(1 - \frac{5}{6}\frac{I_s}{I}\right)} \log\left(\frac{6}{5}\frac{I}{I_s} - 1\right). \tag{43}$$

Thus, the wider the sheet and the smaller the value of the shear modulus G, the farther away does the effect of the cut-out extend in the spanwise direction.

The magnitude of the beam deflection is obtained from (41) in the form

[5] P. Kuhn and P. Chiarito, *Shear lag in box beams—methods of analysis and experimental investigations*, N.A.C.A. Technical Report No. 739 (1943).

[6] Exact solutions of problems of this kind have been obtained by F. B. Hildebrand, *The exact solution of shear-lag problems in flat panels and box beams assumed rigid in the transverse direction*, N.A.C.A., Technical Note No. 894 (1943).

$$z(x) = \int_l^x \int_l^{x_1} z''(x_2)dx_2dx_1 \tag{44}$$

which determines the constants of integration such that $z(l) = z'(l) = 0$. For the deflection at the free end of the beam, we have

$$z(0) = \frac{M_0 l^2}{EI}\left\{\frac{1}{4} + \frac{2(n-1)}{kl}\left[\left(\frac{1}{2} + \frac{1}{(kl)^2}\right)\coth kl - \frac{1}{kl} - \frac{1}{(kl)^2 \sinh kl}\right]\right\}. \tag{45}$$

For a beam with dimensions as in (38) and (39), Eq. (45) becomes

$$z(0) = \frac{M_0 l^2}{EI}(.25 + .083). \tag{46}$$

This indicates that for a beam with dimensions as given shear lag due to lack of sheet restraint at the supported end of the beam is responsible for a thirty percent increase of the maximum beam deflection as compared with the result of elementary beam theory for a beam fully restrained at the supported end. This increase of deflection of thirty percent compares with one of hundred per cent which is obtained if the contribution of the cover sheets is neglected.

4. *Beam with both ends built-in. Uniform load distribution.* The distribution of bending moments may be written as

$$M = M_0\left(\frac{x}{l}\right)^2 + M_1. \tag{47}$$

The value of M_0 is determined by the load intensity, the value of M_1 in this statically indeterminate problem has to be determined from the displacement boundary conditions. The boundary conditions are

$$z\left(\pm\frac{l}{2}\right) = 0, \quad (48) \qquad z'\left(\pm\frac{l}{2}\right) = 0, \tag{49}$$

$$z'''\left(\pm\frac{l}{2}\right) = -n\frac{M'(\pm l/2)}{EI}. \tag{50}$$

For these boundary conditions the moment distribution is not affected by shear lag, provided the moment distribution is symmetrical about the mid-span section of the beam. Indeed, the differential equation (27) may be integrated to give

$$z' - \frac{z'''}{k^2} = -\int_0^x \frac{M}{EI}dx + \frac{n}{k^2}\frac{M'}{EI}, \tag{51}$$

the limits of integration being so chosen that Eq. (51) satisfies the conditions of zero slope and zero vertical shear at the mid-span section. In view of (49) and (50), Eq. (51) implies

$$\int_0^{l/2} \frac{M}{EI}dx = 0, \tag{52}$$

regardless of whether or not shear lag is taken into account. A considerably less simple proof of the same fact by means of the least work method has been given in the reference quoted in Footnote 2. For the moment distribution of Eq. (47) there follows, from (52),

$$\frac{M_0}{24} + \frac{M_1}{2} = 0, \tag{53}$$

and hence

$$M = M_0\left[\left(\frac{x}{l}\right)^2 - \frac{1}{12}\right]. \tag{54}$$

With this value of M and the requirement that z'' be an even function of x, Eq. (27) is solved in the form

$$z'' = -\frac{M_0}{EI}\left\{\left(\frac{x}{l}\right)^2 - \frac{1}{12} - \frac{2(n-1)}{(kl)^2} + C_2 \cosh kx\right\}. \tag{55}$$

The constant C_2 is determined from Eq. (50). There follows,

$$z'' = -\frac{M_0}{EI}\left\{\left(\frac{x}{l}\right)^2 - \frac{1}{12} + \frac{n-1}{kl}\left[\frac{\cosh kx}{\sinh kl/2} - \frac{1}{kl/2}\right]\right\}. \tag{56}$$

Taking a beam five times as long as wide, that is $l/2w=5$, and assuming the remaining parameters as in (38) and (39), we obtain the following expressions for the flange stresses at the built-in section and at the center section of the beam

$$\sigma_f\left(\frac{l}{2}\right) = \pm \frac{M_0 h}{I}\frac{1}{6}(1 + .283), \tag{57}$$

$$\sigma_f(0) = \mp \frac{M_0 h}{I}\frac{1}{12}(1 + .106). \tag{58}$$

These results agree to within a fraction of a percent with the corresponding results obtained by the least work method.[2] It is worthy of note that, for this beam with both ends built-in, shear lag is considerably larger than for a cantilever beam with the same load, same width and half the span of the beam with both ends built-in. If both beams had the same span, the discrepancy would be even larger.

The deflection z of the beam is obtained from (56) and (48) in the form

$$z = -\frac{M_0 l^2}{EI}\left\{\frac{1}{12}\left(\frac{x}{l}\right)^4 - \frac{1}{24}\left(\frac{x}{l}\right)^2 + \frac{1}{192} + \frac{n-1}{(kl)^2}\left[\frac{1}{4} - \left(\frac{x}{l}\right)^2 - \frac{\cosh (kl/2) \cosh kx}{kl \sinh kl/2}\right]\right\}. \tag{59}$$

Corresponding to the stresses of Eqs. (57) and (58) we find for the deflection at midspan

$$z(0) = -\frac{1}{192}\frac{M_0 l^2}{EI}(1 + .145). \tag{60}$$

Shear lag in this beam is thus responsible for an almost fifteen percent increase in deflection. This percentage increase of deflection, while appreciable, is considerably smaller than the percentage increase of maximum flange stress.

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THE ANALOGY BETWEEN MULTIPLY-CONNECTED SLICES AND SLABS*

BY

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1. Introduction. The analogy between the two-dimensional field of stress and the transverse flexure of a thin plate was first applied by K. Wieghardt[1] to the solution of a problem involving boundary loading of a simply-connected body. As is well known, the analogy establishes the proportionality of the curvatures of the surface of the plate to the components of stress in the two-dimensional field of stress. H. M. Westergaard[2] introduced the useful terminology of slab and slice, free slice and constrained slice, and gave the boundary conditions for the slab when the slice is multiply-connected and is stressed by boundary loads having no resultant force on an internal boundary. Westergaard also proposed the use of the analogy in the investigation of the stresses in the Boulder Canyon Dam,[3] a problem involving gravity and boundary loading of a simply connected body. An improvement in experimental technique was contributed by H. Cranz[4] in introducing an optical spherometer[5] for measuring the components of surface curvature. Cranz's application was to boundary load problems in simply connected bodies.

It is the purpose of this paper to give the general boundary conditions for the slab when the slice is multiply-connected and is stressed by any combination of boundary loading, body forces, dislocations and thermal dilatations. The analogy has, in fact, its most useful applications in the last three cases as they are either difficult to reproduce, or the resulting stresses are difficult to measure, in an experimental model of the slice itself, while the analogous conditions for the slab, developed below, are easy to handle.

In order to proceed, it is necessary, first, to set down the general boundary value problem for the slice. It is convenient to do this along the lines established by Michell,[6] with the additional consideration of dislocations and thermal dilatations.

2. Airy's stress function and its differential equations. In a state of plane strain defined by setting

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* Received April 9, 1946.

[1] K. Wieghardt, *Über ein neues Verfahren, verwickelte Spannungsverteilungen in elastischen Körpern auf experimentellem Wege zu finden*, Mitteilungen über Forschungsarbeiten a. d. Gebiete d. Ingenieurwesens, 49, 15–30 (1908).

[2] H. M. Westergaard, *Graphostatics of stress functions*, Transactions, Amer. Soc. Mech. Eng., 56, 141–150 (1934).

[3] United States Department of the Interior, Bureau of Reclamation, Boulder Canyon Project, Final Reports (1938), Part V, Technical Investigations: Bulletin 2, *Slab analogy experiments*; Bulletin 4, *Stress studies for Boulder Dam*.

[4] H. Cranz, *Die experimentelle Bestimmung der Airyschen Spannungsfunktion mit Hilfe des Plattengleichnisses*, Ingenieur-Archiv, 10, 159–166 (1939).

[5] E. Einsporn, *Ebenheit*, Zeitschrift für Instrumentenkunde, 57, 265–285 (1937).

[6] J. H. Michell, *On the direct determination of stress in an elastic solid, with application to the theory of plates*, Proc. London Math. Soc., 31, 100–124 (1899).

$$\gamma_{yz} = \gamma_{zx} = \epsilon_z = 0$$

and restricting the displacements u and v to be functions of x and y only, the relations between strain, displacement, stress and temperature in an isotropic elastic body are

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E_1}\left[(1-\nu_1^2)\sigma_x - \nu_1(1+\nu_1)\sigma_y\right] + (1+\nu_1)\alpha_1 T, \tag{2.1a}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E_1}\left[(1-\nu_1^2)\sigma_y - \nu_1(1+\nu_1)\sigma_x\right] + (1+\nu_1)\alpha_1 T, \tag{2.1b}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(1+\nu_1)}{E_1}\tau_{xy}. \tag{2.1c}$$

These are the relations for a constrained slice. The notations for stress, strain and displacement are the usual ones and E_1, ν_1 are Young's Modulus and Poisson's ratio for the material of the slice, α_1 is the coefficient of linear thermal expansion, and T is the temperature in excess of a uniform initial temperature.

When the stresses are expressed in terms of Airy's stress function (ϕ) and a body force potential (V) by

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} + V, \qquad \sigma_y = \frac{\partial^2\phi}{\partial x^2} + V, \qquad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}, \tag{2.2}$$

the equations of equilibrium are satisfied and the strain relation

$$\frac{\partial^2\epsilon_x}{\partial y^2} + \frac{\partial^2\epsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y} \tag{2.3}$$

yields the differential equation governing ϕ:

$$\nabla^4\phi = -\frac{1-2\nu_1}{1-\nu_1}\nabla^2 V - \frac{1+\nu_1}{1-\nu_1}\alpha_1\nabla^2 T. \tag{2.4}$$

In a state of plane stress, defined by

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0,$$

the strain-displacement-stress-temperature relations become

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E_1}(\sigma_x - \nu_1\sigma_y) + \alpha_1 T, \tag{2.5a}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E_1}(\sigma_y - \nu_1\sigma_x) + \alpha_1 T, \tag{2.5b}$$

$$\epsilon_z = \frac{\partial w}{\partial z} = -\frac{\nu_1}{E}(\sigma_x + \sigma_y) + \alpha_1 T, \tag{2.5c}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(1+\nu_1)}{E_1}\tau_{xy}. \tag{2.5d}$$

These are the relations for a free slice. If the components of stress are again expressed in terms of an Airy function and a body force potential by (2.2), the equilibrium equations are identically satisfied and the strain relations reduce to

$$\nabla^4\phi = -(1-\nu_1)\nabla^2 V - (1+\nu_1)\alpha\nabla^2 T \tag{2.6}$$

if terms associated with the coordinate z are neglected.

In what follows, the case of plane strain (constrained slice) will be treated, but the results are directly applicable to plane stress (neglecting z-dependent terms) if Young's modulus E_1, Poisson's ratio ν_1 and the linear thermal expansion coefficient α_1 are replaced by E_1', ν_1' and α_1' where

$$E_1' = \frac{E_1(1+2\nu_1)}{(1+\nu_1)^2}, \qquad \nu_1' = \frac{\nu_1}{1+\nu_1}, \qquad \alpha_1' = \frac{\alpha_1(1+\nu_1)}{1+2\nu_1}. \tag{2.7}$$

3. Conditions on ϕ at a point on a boundary of the slice. Michell[6] gave the conditions to be satisfied, at each point of each boundary, by ϕ and its derivative normal to the boundary:

$$\phi = \int_0^s (Bl - Am)ds + \alpha x + \beta y + \gamma, \tag{3.1}$$

$$\frac{d\phi}{dn} = Al + Bm + \alpha l + \beta m, \tag{3.2}$$

where α, β, γ are constants, in general different for each boundary, ds is an element of arc of a boundary, dn an element of normal to that boundary, and

$$l = \frac{dy}{ds}, \qquad m = \frac{-dx}{ds}, \tag{3.3}$$

$$A = -\int_0^s \overline{Y}ds + \int_0^s Vmds, \qquad B = \int_0^s \overline{X}ds - \int_0^s Vlds, \tag{3.4}$$

$$\overline{X} = \sigma_x l + \tau_{xy}m, \qquad \overline{Y} = \tau_{xy}l + \sigma_y m. \tag{3.5}$$

In a simply connected body, α, β, γ may be assigned arbitrary (including zero) values as the addition of a linear function of x and y to ϕ does not affect the stresses. In a multiply-connected body, three additional conditions on ϕ are required for determining α, β, γ, on each additional boundary. Equations (3.1) to (3.5) are not altered by introducing thermal dilatations and dislocations of the type considered here.

4. Conditions on ϕ for each boundary of the slice. The additional conditions on ϕ are obtained by assuming the strains (and hence the stresses) to be continuous and requiring the rotations and displacements (a) to be single-valued or (b) to have prescribed discontinuities (dislocations). Michell[6] gave the conditions for case (a). The conditions for case (b), including, also, thermal dilatations, are derived by following Michell's procedure with modifications along the lines indicated by Volterra.[7]

(*i*) *Rotation condition.* Considering the rotation

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{4.1}$$

[7] Love, *Theory of elasticity*, 4th ed., Cambridge Univ. Press, Cambridge, 1927, pp. 221–228.

we require that the line integral of its differential have a value, say c, after one complete circuit around (and along) a boundary. Thus,

$$c = \oint d\omega_z. \tag{4.2}$$

Now,

$$\oint d\omega_z = \oint \frac{\partial \omega_z}{\partial x} dx + \frac{\partial \omega_z}{\partial y} dy$$

$$= \oint \left(\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}\right) dx + \left(\frac{\partial \epsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y}\right) dy. \tag{4.3}$$

Replacing the strain components by their expressions in terms of ϕ, V and T, we find

$$\frac{E_1 c}{1+\nu_1} = (1-\nu_1)\oint \left(\frac{\partial}{\partial x}(\nabla^2\phi)dy - \frac{\partial}{\partial y}(\nabla^2\phi)dx\right) + (1-2\nu_1)\oint \left(\frac{\partial V}{\partial x}dy - \frac{\partial V}{\partial y}dx\right)$$

$$+ E_1\alpha_1 \oint \left(\frac{\partial T}{\partial x}dy - \frac{\partial T}{\partial y}dx\right).$$

Then

$$\oint \frac{d(\nabla^2\phi)}{dn} ds = \frac{E_1 c}{1-\nu_1^2} - \frac{1-2\nu_1}{1-\nu_1}\oint \frac{dV}{dn} ds - \frac{E_1\alpha_1}{1-\nu_1}\oint \frac{dT}{dn} ds. \tag{4.4}$$

This is the first of Michell's three conditions on ϕ for each boundary of the slice. It may be observed that, if the circuit of the line integral in (4.3) were reducible, the integral would vanish because, by Green's theorem,

$$\oint \left(\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}\right) dx + \left(\frac{\partial \epsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y}\right) dy$$

$$= \iint \left(\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}\right) dxdy; \tag{4.5}$$

and the surface integral vanishes by virtue of (2.3).

(*ii*) *Displacement conditions*. We admit a translational dislocation a parallel to x and set

$$a = \oint du = \oint \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \oint \left(\epsilon_x dx + \frac{1}{2}\gamma_{xy} dy\right) - \oint \omega_z dy.$$

Now

$$\oint \omega_z dy = y_0 \oint d\omega_z - \oint y d\omega_z = y_0 c - \oint y\, d\omega_z,$$

where y_0 is the y-coordinate of the starting point of integration. Also

$$\oint y d\omega_z = \oint y\left(\frac{\partial \omega_z}{\partial x} dx + \frac{\partial \omega_z}{\partial y} dy\right)$$

$$= \oint y\left(\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}\right) dx + \oint y\left(\frac{\partial \epsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y}\right) dy.$$

Hence

$$a + y_0 c = \oint \left[\epsilon_x + y\left(\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}\right)\right] dx + \oint \left[\frac{1}{2}\gamma_{xy} + y\left(\frac{\partial \epsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y}\right)\right] dy. \tag{4.6}$$

We now note that

$$\oint \epsilon_x dx + \frac{1}{2}\gamma_{xy} dy = [x\epsilon_x]_0^0 + \frac{1}{2}[y\gamma_{xy}]_0^0 - \oint \left(x\frac{\partial \epsilon_x}{\partial x} dx + \frac{1}{2} y \frac{\partial \gamma_{xy}}{\partial y} dy\right)$$
$$= -\oint \left(x\frac{\partial \epsilon_x}{\partial x} dx + \frac{1}{2} y \frac{\partial \gamma_{xy}}{\partial y} dy\right),$$

the terms outside the integrals vanishing because of the assumption of continuous strains. Equation (4.6) then becomes

$$a + y_0 c = \oint \left[y\left(\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}\right) - x\frac{\partial \epsilon_x}{\partial x}\right] dx + \oint y\left(\frac{\partial \epsilon_y}{\partial x} - \frac{\partial \gamma_{xy}}{\partial y}\right) dy. \tag{4.7}$$

When the strain components in (4.7) are replaced by their expressions in terms of ϕ, V and T, we find

$$\frac{E_1(a + y_0 c)}{1 + \nu_1} = (1 - \nu_1) \oint y\left[\frac{\partial}{\partial x}(\nabla^2\phi) dy - \frac{\partial}{\partial y}(\nabla^2\phi) dx\right]$$
$$+ (1 - 2\nu_1) \oint y\left(\frac{\partial V}{\partial x} dy - \frac{\partial V}{\partial y} dx\right)$$
$$+ E_1\alpha_1 \oint y\left(\frac{\partial T}{\partial x} dy - \frac{\partial T}{\partial y} dx\right)$$
$$- \oint x\left[(1 - \nu_1)\frac{\partial}{\partial x}(\nabla^2\phi) + (1 - 2\nu_1)\frac{\partial V}{\partial x} + E_1\alpha_1 \frac{\partial T}{\partial x}\right] dx$$
$$+ \oint \left(x\frac{\partial^3 \phi}{\partial x^3} dx + y\frac{\partial^3 \phi}{\partial x \partial y^2} dy\right). \tag{4.8}$$

Now,

$$\oint \left(x\frac{\partial^3 \phi}{\partial x^3} dx + y\frac{\partial^3 \phi}{\partial x \partial y^2} dy\right) = \left[x\frac{\partial^2 \phi}{\partial x^2} + y\frac{\partial^2 \phi}{\partial x \partial y}\right]_0^0 - \oint \left(\frac{\partial^2 \phi}{\partial x^2} dx + \frac{\partial^2 \phi}{\partial x \partial y} dy\right)$$
$$= -\oint \frac{d}{ds}\left(\frac{\partial \phi}{\partial x}\right) ds,$$

the term outside the integral vanishing because the stresses are continuous. But, from (2.2), (3.3) and (3.5),

$$\frac{d}{ds}\left(\frac{\partial \phi}{\partial x}\right) = Vm - \overline{Y}.$$

Hence (4.8) may be written:

$$\oint \left(y \frac{d(\nabla^2\phi)}{dn} - x \frac{d(\nabla^2\phi)}{ds} \right) ds = \frac{E_1(a + y_0 c)}{1 - \nu_1^2} - \frac{1 - 2\nu_1}{1 - \nu_1} \oint \left(y \frac{dV}{dn} - x \frac{dV}{ds} \right) ds$$
$$- \frac{E_1\alpha_1}{1 - \nu_1} \oint \left(y \frac{dT}{dn} - x \frac{dT}{ds} \right) ds$$
$$- \frac{1}{1 - \nu_1} \oint (\overline{Y} - Vm) ds. \tag{4.9}$$

This is Michell's second condition on ϕ for each boundary of the slice.

Similarly, admitting a translational dislocation b in the y-component of displacement, we set

$$b = \oint dv$$

and we find

$$\oint \left(y \frac{d(\nabla^2\phi)}{ds} + x \frac{d(\nabla^2\phi)}{dn} \right) ds = - \frac{E_1(b - x_0 c)}{1 - \nu_1^2} - \frac{1 - 2\nu_1}{1 - \nu_1} \oint \left(y \frac{dV}{ds} + x \frac{dV}{dn} \right) ds$$
$$- \frac{E_1\alpha_1}{1 - \nu_1} \oint \left(y \frac{dT}{ds} + x \frac{dT}{dn} \right) ds$$
$$- \frac{1}{1 - \nu_1} \oint (\overline{X} - Vl) ds, \tag{4.10}$$

which is the last of Michell's three conditions.

Corresponding to (4.5), a similar application of Green's theorem to (4.6) reveals that the right hand side of the latter vanishes for reducible circuits and the same result is found for the corresponding step in the development of Michell's third condition.

The differential equation (2.6), the boundary conditions (3.1) and (3.2), and the three conditions (4.4), (4.9) and (4.10) constitute a statement of the boundary value problem of plane elasticity for stresses induced by boundary loading, body forces, dislocations, and thermal dilatations. The general formulation of the problem reveals the analogies, discovered by M. A. Biot,[8] between gravity loading and boundary pressures, and between thermal loading and boundary pressures and dislocations.

5. The slab equations. In the approximate theory of the bending of thin plates[9] (slabs), the deflection (w) is governed by the differential equation

$$D\nabla^4 w = Z, \tag{5.1}$$

where D is the flexural rigidity of the plate and Z is the surface load, normal to the middle plane.

The components of curvature in the y, z and x, z planes are given by

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[8] M. A. Biot, *Distributed gravity and temperature loading in two-dimensional elasticity replaced by boundary pressures and dislocations*, J. Appl. Mech., **2**, A 41–A 95 (1935).

[9] Love, loc. cit., p. 487.

$$\kappa_x = \frac{\partial^2 w}{\partial y^2}, \qquad \kappa_y = \frac{\partial^2 w}{\partial x^2}. \tag{5.2}$$

On a boundary of the slab, the shearing force (N) normal to the middle plane, the flexural couple (G), and the torsional couple (H) (all per unit of arc length s) are

$$N = -D \frac{\partial}{\partial n}(\nabla^2 w), \tag{5.3a}$$

$$G = -D\left[\frac{\partial^2 w}{\partial n^2} + \nu_2\left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{\rho'}\frac{\partial w}{\partial n}\right)\right], \tag{5.3b}$$

$$H = (1 - \nu_2) D \frac{\partial}{\partial n}\left(\frac{\partial w}{\partial s}\right), \tag{5.3c}$$

where ρ' is the radius of curvature of the boundary of the unflexed slab and ν_2 is Poisson's ratio for the slab material.

The resultant force and the components, parallel to the x and y axes, of the resultant couple on a complete boundary are[10]

$$F_z = \oint \left(N - \frac{\partial H}{\partial s}\right) ds, \tag{5.4a}$$

$$M_x = \oint \left[y\left(N - \frac{\partial H}{\partial s}\right) + G\frac{dx}{ds}\right] ds, \tag{5.4b}$$

$$M_y = \oint \left[G\frac{dy}{ds} - x\left(N - \frac{\partial H}{\partial s}\right)\right] ds. \tag{5.4c}$$

Substituting (5.3) in (5.4) we find

$$F_z = -D \oint \left[\frac{\partial}{\partial n}(\nabla^2 w) + (1 - \nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial s}\right)\right] ds, \tag{5.5a}$$

$$\begin{aligned} M_x = -D \oint \Bigg\{ & y\left[\frac{\partial}{\partial n}(\nabla^2 w) + (1 - \nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial s}\right)\right] \\ & + \frac{dx}{ds}\left[\frac{\partial^2 w}{\partial n^2} + \nu_2\left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{\rho'}\frac{\partial w}{\partial n}\right)\right]\Bigg\} ds, \end{aligned} \tag{5.5b}$$

$$\begin{aligned} M_y = -D \oint \Bigg\{ & \frac{dy}{ds}\left[\frac{\partial^2 w}{\partial n^2} + \nu_2\left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{\rho'}\frac{\partial w}{\partial n}\right)\right] \\ & - x\left[\frac{\partial}{\partial n}(\nabla^2 w) + (1 - \nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial s}\right)\right]\Bigg\} ds. \end{aligned} \tag{5.5c}$$

6. The analogy for singly-connected bodies. Noting the similarity between the differential equations (2.6) and (5.1) for ϕ and w, we set

$$w = K\phi, \tag{6.1}$$

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[10] Love, loc. cit., p. 460.

where K is a conversion constant having the dimensions of length/force.

Then, from (6.1) and (2.6),

$$\nabla^4 w = -\frac{1-2\nu_1}{1-\nu_1} K\nabla^2 V - \frac{1+\nu_1}{1-\nu_1} K\alpha_1 \nabla^2 T \tag{6.2}$$

becomes the differential equation for the deflection of the analogous slab. Hence

$$Z = -\frac{1-2\nu_1}{1-\nu_1} KD\nabla^2 V - \frac{1+\nu_1}{1-\nu_1} KD\alpha_1 \nabla^2 T \tag{6.3}$$

is the normal surface loading to be applied to the face of the slab. In the case of a steady state temperature distribution,

$$\nabla^2 T = 0. \tag{6.4}$$

If, in addition, the body force potential is harmonic, the slab is subjected to edge loading only. If either V or T is not harmonic, transverse loading is required on the surface of the slab, and the load may vary slowly with time.

The edge conditions (i.e., the elevation and slope at each point of a boundary) of the slab are specified by substituting $w=K\phi$ in (3.1) and (3.2). Thus

$$\frac{w}{K} = \int_0^s (Bl - Am)ds + \alpha x + \beta y + \gamma, \tag{6.5}$$

$$\frac{1}{K}\frac{dw}{dn} = Al + Bm + \alpha l + \beta m. \tag{6.6}$$

The normal components of stress in the slice are obtained by combining (2.2), (5.2) and (6.1), with the result

$$\sigma_x = \frac{\kappa_x}{K} + V, \qquad \sigma_y = \frac{\kappa_y}{K} + V. \tag{6.7}$$

The principal stresses and their directions may be calculated from two sets of curvature measurements at each point.[4] If the boundary of the slab is a scale model of the boundary of the slice, e.g., if the ratio of a linear dimension of the slab to the corresponding linear dimension of the slice is k, the stress components in the slice are given by

$$\sigma_x = \frac{k^2\kappa_x}{K} + V, \qquad \sigma_y = \frac{k^2\kappa_y}{K} + V. \tag{6.8}$$

For a singly-connected body, (6.1) to (6.8) completely specify the analogy, since the unknown constants α, β, γ may be given arbitrary values.

7. Additional conditions on the slab for multiply-connected bodies. For a multiply-connected body, α, β, γ must be prescribed for each boundary. Now, it will be observed, from (6.5) and (6.6), that α, β, γ specify a rigid body translation and rotation of each complete boundary of the slab. Such rigid body movements may be effected by applying, on each boundary, a resultant force, normal to the middle plane of the slab, and a couple about an axis properly oriented in the plane of the slab. The magnitudes of the force and the x and y components of the couple on each boundary are determined by expressing F_z, M_x, M_y (see (5.5)) in terms of the specified boundary loadings, body forces, dislocations, and temperature distribution of the slice.

i. Resultant force on a boundary of the slab. Replacing w by $K\phi$ in (5.5a), we have

$$F_z = -KD\oint\left[\frac{\partial}{\partial n}(\nabla^2\phi) + (1-\nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]ds. \tag{7.1}$$

Now

$$\oint\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)ds = 0$$

because of the assumption of continuity of the components of stress in the slice. Hence

$$F_z = -KD\oint\frac{\partial}{\partial n}(\nabla^2\phi)ds. \tag{7.2}$$

Therefore, from (4.4),

$$\frac{(1-\nu_1)F_z}{KD} = -\frac{E_1c}{1+\nu_1} + (1-2\nu_1)\oint\frac{dV}{dn}ds + E_1\alpha_1\oint\frac{dT}{dn}ds, \tag{7.3}$$

whereby F_z is expressed in terms of known quantities.

ii. x-component of couple on a boundary of the slab. Substituting $K\phi$ for w in (5.5b):

$$M_x = -KD\oint\left\{y\left[\frac{\partial}{\partial n}(\nabla^2\phi) + (1-\nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]\right.$$
$$\left.+\frac{dx}{ds}\left[\frac{\partial^2\phi}{\partial n^2} + \nu_2\left(\frac{\partial^2\phi}{\partial s^2} + \frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right]\right\}ds. \tag{7.4}$$

Eliminating

$$\oint y\frac{\partial}{\partial n}(\nabla^2\phi)ds$$

between (7.4) and (4.9), we find

$$\frac{M_x}{KD} = -\oint\left\{x\frac{d(\nabla^2\phi)}{ds} + (1-\nu_2)y\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right.$$
$$\left.+\frac{dx}{ds}\left[\frac{\partial^2\phi}{\partial n^2} + \nu_2\left(\frac{\partial^2\phi}{\partial s^2} + \frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right]\right\}ds$$
$$-\frac{E_1(a+y_0c)}{1-\nu_1^2} + \frac{1-2\nu_1}{1-\nu_1}\oint\left(y\frac{dV}{dn} - x\frac{dV}{ds}\right)ds$$
$$+\frac{E_1\alpha_1}{1-\nu_1}\oint\left(y\frac{dT}{dn} - x\frac{dT}{ds}\right)ds + \frac{1}{1-\nu_1}\oint(\bar{Y} - Vm)ds. \tag{7.5}$$

Now,

$$\oint\left[x\frac{\partial(\nabla^2\phi)}{\partial s}+(1-\nu_2)y\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]ds=\left[x\nabla^2\phi\right]_0^0+(1-\nu_2)\left[y\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]_0^0$$
$$-\oint\left[\frac{dx}{ds}\nabla^2\phi+(1-\nu_2)\frac{dy}{ds}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]ds. \quad (7.6)$$

The terms outside the integral vanish on account of the assumption of continuity of the stress components. Therefore the first integral on the right hand side of (7.5) becomes

$$\oint\left\{(1-\nu_2)\frac{dy}{ds}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)+\frac{dx}{ds}\left[\nabla^2\phi-\frac{\partial^2\phi}{\partial n^2}-\nu_2\left(\frac{\partial^2\phi}{\partial s^2}+\frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right]\right\}ds. \quad (7.7)$$

On a boundary

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial n^2}+\frac{1}{\rho'}\frac{\partial\phi}{\partial n}+\frac{\partial^2\phi}{\partial s^2}, \quad (7.8)$$

so that (7.7) becomes

$$(1-\nu_2)\oint\left[\frac{dy}{ds}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)+\frac{dx}{ds}\left(\frac{\partial^2\phi}{\partial s^2}+\frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right]ds. \quad (7.9)$$

However, along a boundary,

$$\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)=-\tau_{ns}, \quad (7.10) \qquad \frac{\partial^2\phi}{\partial s^2}+\frac{1}{\rho'}\frac{\partial\phi}{\partial n}=\sigma_n-V, \quad (7.11)$$

$$\frac{dy}{ds}\tau_{ns}-\frac{dx}{ds}\sigma_n=\overline{Y}. \quad (7.12)$$

Hence, (7.9) becomes

$$-(1-\nu_2)\oint\left(\overline{Y}+V\frac{dx}{ds}\right)ds.$$

Substituting back in (7.5), we have, finally,

$$\frac{(1-\nu_1)M_x}{KD}=-\frac{E_1(a+y_0c)}{1+\nu_1}+(1-2\nu_1)\oint\left(y\frac{dV}{dn}-x\frac{dV}{ds}\right)ds$$
$$+E_1\alpha_1\oint\left(y\frac{dT}{dn}-x\frac{dT}{ds}\right)ds$$
$$-\left[(1-\nu_2)(1-\nu_1)+1\right]\oint(\overline{Y}-Vm)ds. \quad (7.13)$$

This gives M_x in terms of known quantities.

iii. y-component of couple on a boundary of the slab. Substituting $K\phi$ for w in (5.5c),

$$M_y = -KD \oint \left\{ \frac{dy}{ds}\left[\frac{\partial^2\phi}{\partial n^2} + \nu_2\left(\frac{\partial^2\phi}{\partial s^2} + \frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right] - x\left[\frac{\partial(\nabla^2\phi)}{\partial n} + (1-\nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]\right\} ds. \tag{7.14}$$

Eliminating

$$\oint x\frac{\partial}{\partial n}(\nabla^2\phi)ds$$

between (7.14) and (4.10), we have

$$\begin{aligned}\frac{M_y}{KD} = &-\oint \left\{ y\frac{d}{ds}(\nabla^2\phi) - (1-\nu_2)x\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right) + \frac{dy}{ds}\left[\frac{\partial^2\phi}{\partial n^2} + \nu_2\left(\frac{\partial^2\phi}{\partial s^2} + \frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right]\right\} ds \\ &- \frac{E_1(b - x_0c)}{1-\nu_1^2} - \frac{1-2\nu_1}{1-\nu_1}\oint\left(y\frac{dV}{ds} + x\frac{dV}{dn}\right)ds \\ &- \frac{E_1\alpha_1}{1-\nu_1}\oint\left(y\frac{dT}{ds} + x\frac{dT}{dn}\right)ds - \frac{1}{1-\nu_1}\oint(X - Vl)ds.\end{aligned} \tag{7.15}$$

Now,

$$\begin{aligned}\oint\left[y\frac{d(\nabla^2\phi)}{ds} - (1-\nu_2)x\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]ds = {}&[y\nabla^2\phi]_0^0 - (1-\nu_2)\left[x\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]_0^0 \\ &- \oint\left[\frac{dy}{ds}\nabla^2\phi - (1-\nu_2)\frac{dx}{ds}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]ds.\end{aligned} \tag{7.16}$$

The terms outside the integrals in (7.16) vanish on account of the assumption of continuity of stresses. Therefore the first integral on the right hand side of (7.15) becomes

$$\oint\left\{\frac{dy}{ds}\left[\nabla^2\phi - \frac{\partial^2\phi}{\partial n^2} - \nu_2\left(\frac{\partial^2\phi}{\partial s^2} + \frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right] - (1-\nu_2)\frac{dx}{ds}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right\}ds. \tag{7.17}$$

Then, using (7.8), (7.10) and (7.11) and noting that

$$\tau_{ns}\frac{dx}{ds} + \sigma_n\frac{dy}{ds} = \overline{X}, \tag{7.18}$$

(7.17) may be written in the form

$$(1-\nu_2)\oint(\overline{X} - Vl)ds. \tag{7.19}$$

Substituting back in (7.15), we have

$$\frac{(1-\nu_1)M_y}{KD} = -\frac{E_1(b-x_0c)}{1+\nu_1} - (1-2\nu_1)\oint\left(y\frac{dV}{ds}+x\frac{dV}{dn}\right)ds$$
$$-E_1\alpha_1\oint\left(y\frac{dT}{ds}+x\frac{dT}{dn}\right)ds$$
$$+[(1-\nu_2)(1-\nu_1)-1]\oint(\overline{X}-Vl)ds. \qquad (7.20)$$

8. Recapitulation. The stresses, in a multiply-connected slice, resulting from boundary loadings $\overline{X}$, $\overline{Y}$, a body force potential V, dislocations a, b, c and temperatures T, are related to the curvatures of a slab according to

$$\sigma_x = \frac{\kappa_x}{K} + V, \qquad \sigma_y = \frac{\kappa_y}{K} + V$$

if the following conditions are satisfied on the slab:
(i) The surface loading on the slab is

$$Z = -\frac{1-2\nu_1}{1-\nu_1}KD\nabla^2V - \frac{1+\nu_1}{1-\nu_1}KD\alpha_1\nabla^2T; \qquad (6.3)$$

(ii) The boundaries of the slab are geometrically identical with those of the slice, with elevations and normal slopes given by

$$\frac{w}{K} = \int_0^s (Bl-Am)ds + \alpha x + \beta y + \gamma, \qquad \frac{1}{K}\frac{dw}{dn} = Al + Bm + \alpha l + \beta m, \qquad (6.5)$$

at each point of each boundary;
(iii) There are a resultant force (F_z) and resultant couples (M_x) and (M_y), on each boundary, with magnitudes given by

$$\frac{(1-\nu_1)F_z}{KD} = -\frac{E_1c}{1+\nu_1} + (1-2\nu_1)\oint\frac{dV}{dn}ds + E_1\alpha_1\oint\frac{dT}{dn}ds, \qquad (7.2)$$

$$\frac{(1-\nu_1)M_x}{KD} = -\frac{E_1(a+y_0c)}{1+\nu_1} + (1-2\nu_1)\oint\left(y\frac{dV}{dn}-x\frac{dV}{ds}\right)ds$$
$$+E_1\alpha_1\oint\left(y\frac{dT}{dn}-x\frac{dT}{ds}\right)ds$$
$$-[(1-\nu_1)(1-\nu_2)-1]\oint(\overline{Y}-Vm)ds, \qquad (7.13)$$

$$\frac{(1-\nu_1)M_y}{KD} = -\frac{E_1(b-x_0c)}{1+\nu_1} - (1-2\nu_1)\oint\left(y\frac{dV}{ds}+x\frac{dV}{dn}\right)ds$$
$$-E_1\alpha_1\oint\left(y\frac{dT}{ds}+x\frac{dT}{dn}\right)ds$$
$$+[(1-\nu_1)(1-\nu_2)-1]\oint(\overline{X}-Vl)ds. \qquad (7.20)$$

ON AN EXTENSION OF THE VON KÁRMÁN-TSIEN METHOD TO TWO-DIMENSIONAL SUBSONIC FLOWS WITH CIRCULATION AROUND CLOSED PROFILES*

BY

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1. Introduction. The method for treating compressible flows, as developed by Chaplygin,[1] von Kármán[2] and Tsien[3] leads to a successful solution of the flow pattern past solid bodies when the flow has no circulation. When the flow has a finite circulation, as in the case of airfoils, the profile shapes furnished by this theory are not closed. It is doubtless desirable to develop the theory so as to remove this difficulty.

Recently, Bers[4] succeeded in obtaining flows with circulation around closed profiles. As is usual in the case of a first success, the new method has a few disadvantages. In the first place, the mapping between the actual compressible flow and the associated incompressible flow is not regular at the stagnation points. Thus, if the profile in the associated incompressible flow is regular everywhere, angular points would appear in the profile in the compressible case; and vice versa. The application of the method is further complicated by the fact that the angle thus generated depends on the free-stream Mach number. For the engineer, the treatment has the additional inconvenience of involving the concepts of Riemannian geometry (which are avoided in the present treatment).

In the present article we shall describe a method which is free from the disadvantages mentioned above. The derivation is *very* simple, and no reference is made to Riemannian geometry. Yet the result includes all the previous ones as special cases. Indeed, the treatment seems to be now in the most natural and the most general form which is obtainable from the line of study of Chaplygin, von Kármán and Tsien. It also has great flexibility. Given one incompressible flow, there is still an analytic function at our disposal for constructing compressible flows. This freedom of choice enables us to avoid much unnecessary numerical labor in constructing flows of certain general types. A large number of compressible flows can be derived from a given incompressible flow by the present method without numerical integration.

Apart from giving a useful method for constructing compressible flow patterns, the present development has the following significance. First, the freedom of disposing of one analytic function leads to the solution of the direct problem,—namely to

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[1] S. A. Chaplygin, *On gas jets.* Scientific Memoirs, Moscow Univ., Math. Phys. Sec. 21, 1–121 (1902). (English translation published by Brown University, 1944. Also NACA TM No. 1063, 1944.)

[2] Th. von Kármán, *Compressibility effects in aerodynamics.* Jour. Aero. Sci. 8, 337–356 (1941).

[3] Hsue-Shen Tsien, *Two-dimensional subsonic flow of compressible fluids.* Jour. Aero. Sci. 6, 339–407 (1939).

[4] L. Bers, *On a method of constructing two-dimensional subsonic compressible flows around closed profiles,* NACA TN No. 969 (1945); *On the circulatory subsonic flow of a compressible fluid past a circular cylinder,* NACA TN No. 970 (1945). See also S. Bergman, *On two-dimensional flows of compressible fluids,* NACA TN No. 972 (1945).

calculate the compressible flow past a *given* profile. Indeed, *the solution of problems of compressible flow, either direct or inverse,*—namely, the construction of flows around profiles either given beforehand or not—*is now an a parallel footing with the incompressible case.* In either case, the direct problem requires a method of successive approximations.[5] Secondly, the application of the pressure coefficient formula of von Kármán and Tsien to flows with circulation is justified on the same basis as in the circulation-free case. Experimentally, the formula has been found to be successful even when there is circulation, although the theory has so far been incomplete. The original development of von Kármán and Tsien leaves the body not closed, while the profiles given by the method of Bers do not have the same analytic nature in the incompressible and the compressible cases. The present method removes these difficulties.

Further investigations of the significance of the present method are being carried out. The present article contains only the essential result and its proof. It is hoped that a complete discussion of further developments may be published very soon.

2. Method of constructing two-dimensional subsonic flows with circulation around profiles. Let p, ρ, u, v be the pressure, density and components of velocity of a steady two-dimensional irrotational flow in the x, y plane. Let p be a function of the density ρ only (given either by the isentropic relation or any other approximate relation). Then there exist the velocity potential ϕ and the stream function ψ defined by the following differential relations:

$$d\phi = udx + vdy, \tag{2.1}$$

$$d\psi = -\rho vdx + \rho udy. \tag{2.2}$$

The velocity components u, v and the density ρ are further connected by Bernoulli's equation

$$\frac{q^2}{2} + \int \frac{dp}{\rho} = \text{const.}, \qquad (q^2 = u^2 + v^2). \tag{2.3}$$

It is convenient to refer the density of the gas to that at the stagnation point and to refer all the velocities to the velocity of sound at stagnation. The coordinates x, y may be regarded as referred to the size of the body, and the pressure as referred to the product of stagnation density and the square of the velocity of sound at stagnation. Throughout this article, this process shall be implied, and all the quantities under discussion are dimensionless.

As is well-known, the problem simplifies greatly if the pressure-density relation is approximated by

$$p = A - \frac{B}{\rho}\cdot \tag{2.4}$$

This is the basis of the method of Chaplygin, von Kármán and Tsien. A discussion of its physical interpretation may be found in the papers of these authors. Equation (2.3) leads to

[5] For the incompressible case, see T. Theodorsen, *Theory of wing sections of arbitrary shapes*, NACA Rep. No. 411 (1931).

T. Theodorsen and I. E. Garrick, *General potential theory of arbitrary wing sections*, NACA Rep. No. 452 (1933).

$$c = \frac{1}{\rho} = \sqrt{1 + q^2}, \tag{2.5}$$

where c is the local velocity of sound. Indeed, B must be equal to unity if the reference stagnation quantities are calculated from (2.3) by the use of (2.4)

Under the approximation (2.4), the following method may be used for constructing two-dimensional subsonic flows with circulation around closed profiles.

Given an incompressible flow past a profile P_0 in the ζ-plane ($\zeta=\xi+i\eta$) described by the complex stream function $F(\zeta)$ and the complex velocity $w_0(\zeta)$, choose a function $k(\zeta)$, regular in the region exterior to P_0 and including the point at infinity, having no root in R_0, and such that R_0

$$\left| \tfrac{1}{2} w_0(\zeta) \right| < \left| k(\zeta) \right| < \infty \quad \text{on} \quad P_0, \tag{2.6}$$

and that

$$\oint k(\zeta) d\zeta - \frac{1}{4} \oint \overline{\frac{w_0^2(\zeta)}{k(\zeta)} d\zeta} = 0, \tag{2.7}$$

where the integration is performed along any contour enclosing P_0. Then

$$x + iy = \int k(\zeta) d\zeta - \frac{1}{4} \int \overline{\frac{w_0^2(\zeta)}{k(\zeta)} d\zeta}, \tag{2.8}$$

$$\frac{2q}{1 + \sqrt{1 + q^2}} e^{-i\theta} = \frac{w_0(\zeta)}{k(\zeta)}, \tag{2.9}$$

$$\phi + i\psi = F(\zeta) \tag{2.10}$$

gives the parametric representation of a compressible flow past a profile P in the x, y plane with ζ as parameter, where P has the same general analytic nature (e.g. same number of corners, etc.) as the original profile P_0. In these formulae, ϕ, ψ are the velocity potential and stream function defined by (2.1) *and q, θ denote the magnitude and direction of the velocity of the compressible flow.*

3. Proof. The proof consists of two parts. First, it is necessary to show that, after the auxiliary variable ζ is eliminated from (2.8)–(2.10), we obtain proper functional relations between ϕ, ψ, q, θ and x, y. Secondly, we must show that the profile P is a closed curve and is mapped into P_0 by a regular mapping such that the regions R, exterior to P, and R_0, exterior to P_0, are mapped into each other in a one-to-one manner.

(a) The first part of the proof is simple. It is well known[6] that under the approximation (2.4), the relations

$$dz = \frac{dF}{w^*} - \frac{1}{4} \overline{w^* dF}, \qquad F = F(w^*) \tag{3.1}$$

with z, F, w^* defined by

$$\left.\begin{aligned} z &= x + iy, \\ F &= \phi + i\psi, \\ w^* &= \frac{2q}{1 + \sqrt{1 + q^2}} e^{-i\theta} \end{aligned}\right\} \tag{3.2}$$

[6] Cf., for example, Eqs. (23), (26) of Tsien's paper quoted in Footnote 3.

would give a solution of the *differential* equations of compressible flow. Instead of trying to establish a relation between F and w^* directly, we introduce an *auxiliary variable* ζ such that $F(\zeta)$ and $w^*(\zeta)$ are analytic functoins. It is well known that great simplification is obtained by taking $F(\zeta)$ to be the complex potential of an incompressible flow similar to the compressible flow we desire. However, the extent of arbitariness in the choice of $w^*(\zeta)$ has not been carefully examined. It is clear that any choice of $w^*(\zeta)$ will be sufficient so far as satisfying the *differential* equations is concerned.

In the present case we dispose of the arbitrary function by writing

$$w^*(\zeta) = w_0(\zeta)/k(\zeta). \tag{3.3}$$

The only requirements on $k(\zeta)$ are the general conditions of regularity and the relations (2.6) and (2.7), which will be discussed immediately.

(b) The second part of the proof is also very simple. In the first place, the profile P is closed by virtue of (2.7).[6] The regularity of the mapping is established if the Jacobian of the transformation maintains the same sign and does not vanish or become infinite in the region R_0, including the boundary P_0 and the point at infinity. It can be easily verified that the Jacobian is

$$J = \frac{\partial(x, y)}{\partial(\xi, \eta)} = Im\{\bar{z}_\xi z_\eta\} = |k|^2\left\{1 - \frac{1}{16}\left|\frac{w_0}{k}\right|^4\right\}. \tag{3.4}$$

From this expression, it is clear that the requirement is satisfied when k satisfies the restrictions specified in the last section.*

4. Discussion. (a) *The function* $k(\zeta)$. To make use of the freedom in choosing the function $k(\zeta)$ is the essential improvement made in the present paper. Von Kármán and Tsien gave an interpretation of $w^*(\zeta)$ by identifying it with the complex velocity $w_0(\zeta)$ in the associated incompressible flow. This means that they put

$$k(\zeta) \equiv 1. \tag{4.1}$$

They were therefore unable to meet the requirement (2.7) for closing the profile, for flows with circulation. Bers overcame this difficulty by virtually taking

$$k(\zeta) = \text{const.}\ \{w_0(\zeta)\}^{1-\sqrt{1-M_\infty^2}}, \tag{4.2}$$

where M_∞ is the free-stream Mach number. However, at the stagnation points of the incompressible flow

$$|k(\zeta)| = 0, \tag{4.3}$$

and the mapping between the profiles P and P_0 is not regular there. In applying his method to the calculation of compressible flow past a circle, Bers had to start with the rather complicated problem of finding the incompressible flow past two intersecting circular arcs with the stagnation points at the points of intersection and with the

* *Note added in proof*: Professor K. O. Friedricks pointed out to the author that, in the compressible as well as the incompressible case, a mapping with non-vanishing Jacobian does not always lead to a useful result: the region obtained may be simply-connected but multiply covered. This difficulty has not been experienced so far in some numerical examples which have been worked out.

angles properly adjusted in relation to the free-stream Mach number. Indeed, it seems that after reaching the relation

$$dz = \frac{dF}{w^*} - \frac{1}{4}\overline{w^* dF},$$

the most natural development of the Kármán-Tsien method is to leave $w^*(\zeta)$ quite free, as we have done here, instead of connecting it definitely with w_0, as was done by previous authors. The present method of leaving $k(\zeta)$ free seems to be the most general scheme.

If we deliberately want to introduce some singular points in P by starting from a profile P_0 without a singular point, $|k(\zeta)|$ should be allowed to take the limiting values specified in (2.6); e.g., $k(\zeta)=0$ where $w_0(\zeta)=0$.

Although $k(\zeta)$ cannot be taken to be unity when the flow has a circulation, it should not depart from unity very much if the profiles P and P_0 are not to differ much from each other. This is easily seen from a comparison of terms in (cf. (2.8))

$$dz = kd\zeta - \frac{1}{4}\overline{\frac{w_0^2}{k}d\zeta}. \tag{4.4}$$

They have the ratio (cf. (2.9) and (2.5))

$$\lambda = \left(\frac{q}{1+\sqrt{1+q^2}}\right)^2 = \frac{M^2}{(1+\sqrt{1-M^2})^2}, \tag{4.5}$$

where M is the local Mach number q/c. This value is much smaller than unity, except for values of M close to unity.[7] Hence, (4.4) is approximately the identity transformation if k is very close to unity. This approximately preserves the shape of the profile during the transformation.

(b) *Conformal mapping of compressible flows.* If we make a conformal transformation of the ζ-plane into the $\tilde{\zeta}$-plane by the analytic relation

$$\zeta = h(\tilde{\zeta}), \tag{4.6}$$

we are merely making a change of the auxiliary variable in (2.8)–(2.10). Indeed, we have

$$x + iy = \int \tilde{k}d\tilde{\zeta} - \frac{1}{4}\int \overline{\frac{\tilde{w}_0^2(\tilde{\zeta})}{\tilde{k}(\tilde{\zeta})}d\tilde{\zeta}}, \tag{4.7}$$

$$\frac{2q}{1+\sqrt{1+q^2}}e^{-i\theta} = \frac{\tilde{w}_0(\tilde{\zeta})}{\tilde{k}(\tilde{\zeta})}, \tag{4.8}$$

$$\phi + i\psi = \tilde{F}(\tilde{\zeta}); \tag{4.9}$$

where

$$\tilde{F}(\tilde{\zeta}) = F(\zeta), \tag{4.10}$$

$$\tilde{w}_0(\tilde{\zeta}) = \tilde{F}'(\tilde{\zeta}) = F'(\zeta)\frac{d\zeta}{d\tilde{\zeta}}, \tag{4.11}$$

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[7] Cf., Fig. 3 of Tsien's paper quoted in Footnote 3.

and

$$\tilde{k}(\tilde{\zeta}) = k(\zeta)\frac{d\zeta}{d\tilde{\zeta}}\cdot \tag{4.12}$$

We note that the equations (4.7)–(4.9) are of the same nature as (2.8)–(2.10). However, the profile $\tilde{P}_0$, into which the profiles P and P_0 are mapped, may bear no resemblance at all to the original profiles. Indeed, there is no loss in generality in taking P_0 to be a circle. *The relations* (4.7)–(4.9) *thus serve to transform the incompressible flow past a circle into a compressible flow past a profile of a quite arbitrary shape.* Referring to (4.6) and (4.12) and to the fact that $k(\zeta)$ should be taken not far from unity, we see that $\tilde{k}(\tilde{\zeta})$ should be chosen so that it is not very much different from the derivative of the function mapping P into a circle.

(c) *Formulation of the direct problem.* If we disregard the intermediate step of the ζ-plane and drop the tilde, we have a mapping of the nature described in Section 2, but with $k(\zeta)$ so chosen that a circle will be mapped into some profile P. The function $k(\zeta)$ must satisfy the requirements established there, but it should not be very much different from the function $k_0(\zeta)$ which maps the circle into P_0 by the relation

$$z = \int k_0(\zeta)d\zeta.$$

Thus for each profile P, the determination of the compressible flow past it is equivalent to the determination of a proper $k(\zeta)$ mapping it into a circle by (2.8), where $w_0(\zeta)$ is the flow past the circle. There is no question about the existence of such a mapping function. It is clear that to each purely subsonic flow, where the approximation (2.4) is accurate enough and therefore $F=F(w^*)$, we may find a certain $k_1(\zeta)$ mapping the compressible flow into *some* incompressible flow. By considering successive conformal transformations, we can therefore *always* map the flow into a circle. The actual method of finding $k(\zeta)$ is all that remains to be done in the direct problem.

The theory of subsonic compressible flows (so long as the approximation (2.4) is valid) is now put on an equal footing with the incompressible flows. The inverse problem is complete, the direct problem of finding a mapping function for a given profile can only be solved (practically) by successive approximations, *even in the incompressible case.*[8] To develop a method of successive approximations for the direct problem in the compressible case seems to be a natural next step.*

5. Application of von Mises' method of generating airfoils. In the incompressible case, von Mises transforms a circle in the ζ-plane into an airfoil of a quite general shape by the transformation

$$\frac{dz}{d\zeta} = \left(1 - \frac{\lambda_0}{\zeta}\right)\left(1 - \frac{\lambda_1}{\zeta}\right)\cdots\left(1 - \frac{\lambda_n}{\zeta}\right) \tag{5.1}$$

[8] Cf., e.g. the papers quoted in Footnote 5.

* *Note added in proof*: The essential difference of the two cases lies in the existence problem. While the existence of the incompressible flow follows from well-known results concerning the Laplace equation, very little seems to be known about the existence problem for compressible flows.

where $\lambda_1, \cdots, \lambda_r$ are points *inside* the circle, and λ_0 is a point on the boundary (which transforms into the trailing edge of the airfoil). A similar method can be used here. In (2.8)–(2.10), we put

$$k(\zeta) = \left(1 - \frac{\lambda_0}{\zeta}\right)\left(1 - \frac{\lambda_1}{\zeta}\right) \cdots \left(1 - \frac{\lambda_n}{\zeta}\right) \tag{5.2}$$

with the same general restrictions on the points λ. The condition that the points λ are inside the circle is exactly the condition required of $k(\zeta)$ as stated in Section 2.

The condition (2.7) for the closure of the body has also its equivalence in the case considered by von Mises. There it is[9]

$$\oint k(\zeta)d\zeta = 0.$$

Here, it differs by the presence of another term. The condition that

$$| k(\zeta) | > | \tfrac{1}{2}w_0(\zeta) |$$

on the circle is the only additional restriction in the present case. As it is a mere inequality, there is no great difficulty in ensuring it to be satisfied.

The integration required in establishing the relation between z and ζ can be readily performed, as it involves only rational functions. This ease of calculation, together with the flexibility of the choice of the points λ in controlling the shape of the airfoil, are the advantages of the method of von Mises which are still preserved in the present application.

The author is greatly indebted to Dr. J. B. Diaz for very helpful discussions in the course of the investigation and to Professors W. Prager and K. O. Friedrichs for their interest and discussions.*

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[9] Cf. W. F. Durand, *Aerodynamic theory*, vol. 2, J. Springer, Berlin, 1933, p. 78, Eq. (20.4).

* *Note added in proof*: After the paper was completed and presented in a colloquium at Brown University, Professor E. Reissner informed the author that Professor A. Gelbart had recently presented a somewhat similar approach in a lecture at M.I.T.

—NOTES—

ON THE ELASTIC DISTORTION OF A CYLINDRICAL HOLE BY A LOCALISED HYDROSTATIC PRESSURE*

By C. J. TRANTER (*Military College of Science, Shrivenham, England*)

When a hydrostatic pressure is applied over only a small part of the length of a cylindrical hole extending through an infinite elastic solid, the stresses and displacements differ considerably from those caused by the application of this pressure over the entire length of the hole. This problem has been discussed by H. M. Westergaard[1] using an approximate method but it is not easy to assess the accuracy of his numerical results. It is the purpose of the present note to give an exact solution and to compare numerical results with those given by Westergaard.

The analysis used here is a simple adaptation of that given by A. W. Rankin[2] for the similar problem of a band of uniform pressure applied to a long cylindrical shaft. The numerical calculations are not so formidable as would appear at first sight and a method given by L. N. G. Filon[3] for evaluating trigonometric integrals has proved very valuable in this connection. The results for the maximum radial displacement show that the approximation used by Westergaard is rather crude.

1. **The analytical solution.** We use cylindrical coordinates and consider the pressure loading as being given by $\sigma_r = -p$, $|z| < c$, $\sigma_r = 0$, $|z| > c$ on the surface of the cylindrical hole $r = a$. With the usual notation[4] we therefore require to find a stress function ϕ satisfying

$$\nabla^4\phi = 0, \quad r > a, \quad -\infty < z < \infty, \tag{1}$$

where ∇^2 denotes $\partial^2/\partial r^2 + (1/r)(\partial/\partial r) + \partial^2/\partial z^2$ and the boundary conditions

$$\left.\begin{aligned} \sigma_r = \frac{\partial}{\partial z}\left\{\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right\}\phi &= -p, \quad |z| < c, \\ &= 0, \quad |z| > c, \end{aligned}\right\} \quad r = a, \tag{2}$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\phi = 0, \quad -\infty < z < \infty, \quad r = a, \tag{3}$$

ν being Poisson's ratio for the material of the elastic solid.

Once ϕ has been found, the stresses σ_r, τ_{rz} are given by the expressions shown in (2) and (3) and the remaining stresses and the radial displacement are given by

$$\sigma_\theta = \frac{\partial}{\partial z}\left\{\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right\}\phi, \quad \sigma_z = \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\phi, \quad u = -\frac{1+\nu}{E}\cdot\frac{\partial^2\phi}{\partial r\partial z}, \tag{4}$$

E being the modulus of elasticity.

* Received May 8, 1946.

[1] H. M. Westergaard, Kármán Anniversary Volume, 1941, p. 154.

[2] A. W. Rankin, *Shrink-fit stresses and deformations*, Journ. Appl. Mech. 11, A77 (1944).

[3] L. N. G. Filon, *On a quadrature formula for trigonometric integrals*, Proc. Roy. Soc. Edin. 49, 38 (1928–29).

[4] S. Timoshenko, *Theory of elasticity*, McGraw-Hill Book Co., New York, 1934, p. 309.

Following Rankin, we take

$$\phi = \int_0^\infty Rb \sin c\lambda \sin z\lambda d\lambda \tag{5}$$

where R is a function of r only and b is a function of λ. To satisfy (1) we must have

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \lambda^2\right)^2 R = 0,$$

and the solution, finite as $r \to \infty$, of this equation is

$$R = AK_0(\rho) + B\rho K_1(\rho), \tag{6}$$

where $K_0(\rho)$, $K_1(\rho)$ are Bessel functions of imaginary argument, A and B are constants to be found and

$$\rho = \lambda r. \tag{7}$$

Using the well known relations

$$\left.\begin{aligned} K_0'(\rho) &= -K_1(\rho), \\ \rho K_1'(\rho) + K_1(\rho) &= -\rho K_0(\rho), \end{aligned}\right\} \tag{8}$$

we find

$$\nabla^2\phi = -2\int_0^\infty BK_0(\rho) b\lambda^2 \sin c\lambda \sin z\lambda d\lambda. \tag{9}$$

With

$$\alpha = \lambda a, \tag{10}$$

substitution from (9) into (3) yields

$$(\tau_{rz})_{r=a} = \int_0^\infty [-B\alpha K_0(\alpha) + \{2B(1-\nu) - A\} K_1(\alpha)] b\lambda^3 \sin c\lambda \sin z\lambda d\lambda,$$

so that, to satisfy (3)

$$A/B = 2(1-\nu) - \alpha K_0(\alpha)/K_1(\alpha). \tag{11}$$

We also find

$$(\sigma_r)_{r=a} = -\int_0^\infty \left[\{A + (2\nu - 1)B\} K_0(\alpha) + \left(\frac{A}{\alpha} + B\alpha\right) K_1(\alpha)\right] b\lambda^3 \sin c\lambda \cos z\lambda d\lambda,$$

and since the boundary condition (2) can be represented by

$$(\sigma_r)_{r=a} = -\frac{2p}{\pi}\int_0^\infty \frac{\sin c\lambda \cos z\lambda}{\lambda} d\lambda,$$

we have

$$b = \frac{2p}{\pi\lambda^4}\left[\{A + (2\nu - 1)B\} K_0(\alpha) + \left(\frac{A}{\alpha} + B\alpha\right) K_1(\alpha)\right]^{-1}. \tag{12}$$

Equations (11) and (12) yield

$$\left.\begin{aligned} \pi\lambda^4 D(\alpha) bA &= 2p[2(1-\nu)\alpha K_1(\alpha) - \alpha^2 K_0(\alpha)], \\ \pi\lambda^4 D(\alpha) bB &= 2p\alpha K_1(\alpha), \end{aligned}\right\} \tag{13}$$

where

$$D(\alpha) = \{\alpha^2 + 2(1-\nu)\} K_1^2(\alpha) - \alpha^2 K_0^2(\alpha). \tag{14}$$

bA, bB having now been found, the expressions for the stresses and radial displacement are found to be given by

$$\left.\begin{aligned}
\sigma_r &= \frac{2pa}{\pi r}\int_0^\infty \Big[\alpha\rho K_0(\alpha)K_0(\rho) + \alpha K_0(\alpha)K_1(\rho) - \rho K_0(\rho)K_1(\alpha) \\
&\qquad - \{\rho^2 + 2(1-\nu)\}K_1(\alpha)K_1(\rho)\Big] \frac{\sin\frac{c}{a}\alpha \cos\frac{z}{a}\alpha}{\alpha D(\alpha)}\, d\alpha \\
\tau_{rz} &= \frac{2p}{\pi}\int_0^\infty [\alpha K_0(\alpha)K_1(\rho) - \rho K_0(\rho)K_1(\alpha)] \frac{\sin\frac{c}{a}\alpha \sin\frac{z}{a}\alpha}{D(\alpha)}\, d\alpha \\
\sigma_\theta &= -\frac{2pa}{\pi r}\int_0^\infty \Big[\alpha K_0(\alpha)K_1(\rho) + (2\nu - 1)\rho K_0(\rho)K_1(\alpha) \\
&\qquad - 2(1-\nu)K_1(\alpha)K_1(\rho)\Big] \frac{\sin\frac{c}{a}\alpha \cos\frac{z}{a}\alpha}{\alpha D(\alpha)}\, d\alpha \\
\sigma_z &= -\frac{2p}{\pi}\int_0^\infty \Big[\alpha K_0(\alpha)K_0(\rho) + 2K_0(\rho)K_1(\alpha) \\
&\qquad - \rho K_1(\alpha)K_1(\rho)\Big] \frac{\sin\frac{c}{a}\alpha \cos\frac{z}{a}\alpha}{D(\alpha)}\, d\alpha \\
\frac{Eu}{1+\nu} &= -\frac{2pa}{\pi}\int_0^\infty \Big[\alpha K_0(\alpha)K_1(\rho) - \rho K_0(\rho)K_1(\alpha) \\
&\qquad - 2(1-\nu)K_1(\alpha)K_1(\rho)\Big] \frac{\sin\frac{c}{a}\alpha \cos\frac{z}{a}\alpha}{\alpha D(\alpha)}\, d\alpha
\end{aligned}\right\} \tag{15}$$

2. **Numerical results for the maximum radial displacement at $r=a$.** When $r=a$, $\rho=\alpha$ and the greatest displacement occurs when $z=0$, so that we have

$$\frac{E(u_{\max})_{r=a}}{1+\nu} = \frac{4pa(1-\nu)}{\pi}\int_0^\infty \frac{K_1^2(\alpha)}{\alpha D(\alpha)} \sin\frac{c}{a}\alpha d\alpha.$$

If the pressure p acts over the entire length of the hole, the displacement $(u')_{r=a}$ is given by

$$\frac{E(u')_{r=a}}{1+\nu} = pa,$$

so that

$$\frac{(u_{\max})_{r=a}}{(u')_{r=a}} = \frac{4(1-\nu)}{\pi}\int_0^\infty \frac{K_1^2(\alpha)}{\alpha D(\alpha)} \sin\frac{c}{a}\alpha d\alpha. \tag{16}$$

The numerical work was performed with $\nu=0.3$ and, above $\alpha=12$, it was found that the first three terms of the asymptotic expansion of $K_1^2(\alpha)/\alpha D(\alpha)$, viz.,

$$\frac{K_1^2(\alpha)}{\alpha D(\alpha)} \sim \frac{1}{\alpha^2} - \frac{0.4}{\alpha^3} - \frac{0.965}{\alpha^4}$$

gave an adequate representation. Integration by parts then leads to

$$\int_{12}^\infty \frac{K_1^2(\alpha)}{\alpha D(\alpha)} \sin\frac{c}{a}\alpha d\alpha = \left[.08176 + .01340\frac{c^2}{a^2}\right]\sin\frac{12c}{a} - .01778\frac{c}{a}\cos\frac{12c}{a}$$
$$+ .2\frac{c^2}{a^2}\left[\frac{\pi}{2} - \mathrm{Si}\left(\frac{12c}{a}\right)\right] - \left[1 + .16083\frac{c^2}{a^2}\right]\frac{c}{a}\,\mathrm{Ci}\left(\frac{12c}{a}\right),$$

where

$$\mathrm{Si}\,(x) = \int_0^x \frac{\sin x}{x}dx, \qquad \mathrm{Ci}(x) = -\int_x^\infty \frac{\cos x}{x}dx.$$

The evaluation of the integral in equation (16) from $\alpha=0$ to $\alpha=12$ was performed as follows. The function $K_1^2(\alpha)/\alpha D(\alpha)$ was computed at intervals of $\alpha=0.2$ from $\alpha=0$ to $\alpha=2$ and at intervals of $\alpha=0.5$ from $\alpha=2$ to $\alpha=12$. The integral was then evaluated by a method due to Filon[5] in which Simpson's rule is replaced by the formula

$$\int_A^B F(x)\sin kx dx = h[\alpha\{F(A)\cos kA - F(B)\cos kB\} + \beta S_{2s} + \gamma S_{2s-1}],$$

where the range of integration is divided into intervals of length h, S_{2s} is the sum of all the even ordinates of the curve $y=F(x)\sin kx$ between A and B inclusive less half the first and last ordinates, S_{2s-1} is the sum of all the odd ordinates, and α, β, γ are given in terms of $\psi=hk$ by

$$\alpha = \frac{1}{\psi} + \frac{\sin\psi\cos\psi}{\psi^2} - \frac{2\sin^2\psi}{\psi^3}, \qquad \beta = 2\left[\frac{1+\cos^2\psi}{\psi^2} - \frac{2\sin\psi\cos\psi}{\psi^3}\right],$$

$$\gamma = 4\left[\frac{\sin\psi}{\psi^3} - \frac{\cos\psi}{\psi^2}\right].$$

[5] Filon, loc. cit.

This formula holds even when k is large, provided that the function $F(x)$ can be fitted with reasonable accuracy over the range $2h$ by parabolic arcs.

To avoid an infinity at the origin, the integral actually evaluated was

$$I = \int_0^{12} \left[\frac{1}{1.4\alpha} - \frac{K_1^2(\alpha)}{\alpha D(\alpha)}\right] \sin \frac{c}{a} \alpha d\alpha,$$

and when this had been found, the required integral was given by

$$\frac{1}{1.4} \operatorname{Si}\left(\frac{12c}{a}\right) - I.$$

As a check that the substitution of the asymptotic series did not lead to unacceptable errors, the range of integration was also divided into 0 to 10, 10 to infinity and the infinite integral was similarly computed on this basis. Little extra work was involved and excellent agreement was obtained.

The results are shown below, together with those given by the approximate analysis by Westergaard. It is seen that even his second approximation is quite crude.

$\frac{c}{a}$	Values of $(u_{\max})_{r=c}/(u')_{r=a}$		
	Westergaard		Present Method
	First Approximation	Second Approximation	
0.25	0.557	0.537	0.450
0.50	0.806	0.770	0.633

ON THE REPEATED INTEGRALS OF BESSEL FUNCTIONS*

By J. C. JAEGER (*University of Tasmania*)

It is well known that

$$L\left\{\frac{nJ_n(t)}{t}\right\} = [(p^2+1)^{1/2} - p]^n, \qquad n > 0, \tag{1}$$

and

$$L\{J_n(t)\} = \frac{[(p^2+1)^{1/2} - p]^n}{(p^2+1)^{1/2}}, \qquad n \geqq 0, \tag{2}$$

* Received Jan. 25, 1946.

where $L\{f(t)\}$ is written for the Laplace transform of $f(t)$, that is,

$$L\{f(t)\} = \int_0^\infty e^{-pt}f(t)dt. \tag{3}$$

These results have important applications in the theory of the semi-infinite dissipationless artificial transmission line with simple terminations, and thus in the expression of the solutions of corresponding problems on finite lines in terms of multiply reflected waves.

In an important class of similar problems in which the line is terminated by a matching resistance, the Laplace transforms of the solutions contain powers of $[1+(p^2+1)^{1/2}]$ or $[p+1+(p^2+1)^{1/2}]$ in the denominator, and the functions which have such Laplace transforms do not seem to have been given. The object of this note is to show that they can be expressed in terms of repeated integrals of Bessel functions and that numerical values of these can readily be obtained.

We use the notation

$$\left.\begin{aligned} Ji_n^{(r)}(t) &= \int_0^t dt \cdots \int_0^t \frac{J_n(t)dt}{t}, && n > 0, \\ Ji_{n,r}(t) &= \int_0^t dt \cdots \int_0^t J_n(t)dt, && n \geqq 0, \end{aligned}\right\} \tag{4}$$

for the r-ple integrals of $J_n(t)/t$ and $J_n(t)$ respectively.

It is convenient to use both these types of integral though there are many relations between them, the simplest being

$$Ji_{n-1,r}(t) + Ji_{n+1,r}(t) = 2nJi_n^{(r)}(t) \tag{5}$$

and

$$Ji_{n-1,r}(t) - Ji_{n+1,r}(t) = 2Ji_{n,r-1}(t), \tag{6}$$

which follow immediately from the recurrence formulae for $J_n(t)$. $Ji_{0,1}(t)$ is tabulated[1] and $Ji_n^{(1)}(t) = (1/n) + Ji_n(t)$ where $Ji_n(t)$ is the ordinary Bessel integral function. For all values of n and r repeated application of the result

$$\int_0^t J_n(t)dt = 2\sum_{m=0}^\infty J_{n+2m+1}(t)$$

gives the formulae

$$Ji_{n,r}(t) = 2^r \sum_{m=0}^\infty \binom{m+r-1}{m} J_{n+2m+r}(t) \tag{7}$$

$$nJi_n^{(r)}(t) = 2^{r-1} \sum_{m=0}^\infty \frac{(2m+r-1)}{(m+r-1)} \binom{m+r-1}{m} J_{n+2m+r-1}(t). \tag{8}$$

For integral values of t, which are in fact close enough for many practical purposes, (7) and (8) may be evaluated rapidly from the Tables in Gray and Mathews.[2]

[1] Lowan and Abramowitz, J. Math. and Phys., **22**, 2 (1943).

[2] Gray and Mathews, *Treatise on Bessel functions*, 2nd ed., 1922, Table II.

The Laplace transforms referred to above may now be written down. Firstly we have immediately from (1) and (2)

$$L\{nJi_n^{(r)}(t)\} = p^{-r}[(p^2+1)^{1/2} - p]^n \tag{9}$$

$$L\{Ji_{n,r}(t)\} = \frac{[(p^2+1)^{1/2} - p]^n}{p^r(p^2+1)^{1/2}}. \tag{10}$$

Then, since

$$\frac{1}{1+(p^2+1)^{1/2}} = \left(1+\frac{1}{p^2}\right)\frac{1}{(p^2+1)^{1/2}} - \frac{1}{p^2},$$

it follows that

$$L\{J_0(t) + Ji_{0,2}(t) - t\} = \frac{1}{1+(p^2+1)^{1/2}}. \tag{11}$$

In the same way if $n>0$

$$L\{J_n(t) + Ji_{n,2}(t) - nJi_n^{(2)}(t)\} = \frac{[(p^2+1)^{1/2} - p]^n}{1+(p^2+1)^{1/2}}. \tag{12}$$

Similarly

$$L\{J_n(t) + 3Ji_{n,2}(t) + 2Ji_{n,4}(t) - 2nJi_n^{(2)}(t) - 2nJi_n^{(4)}(t)\}$$
$$= \frac{(p^2+1)^{1/2}[(p^2+1)^{1/2} - p]^n}{[1+(p^2+1)^{1/2}]^2}, \tag{13}$$

if $n>0$, and if $n=0$ the term $nJi_n^{(r)}(t)$ is to be replaced by $t^{r-1}/(r-1)!$. Again with this convention we have

$$L\{\tfrac{1}{2}nJi_n^{(1)}(t) - \tfrac{1}{2}(n+1)Ji_{n+1}^{(1)}(t)\} = \frac{[(p^2+1)^{1/2} - p]^n}{p+1+(p^2+1)^{1/2}}, \tag{14}$$

$$L\{J_{n+2}(t) - 2J_{n+1}(t) + J_n(t) + Ji_{n+2,2}(t) - 2Ji_{n+1,2}(t) + Ji_{n,2}(t)\}$$
$$= \frac{4(p^2+1)^{1/2}[(p^2+1)^{1/2} - p]^n}{[p+1+(p^2+1)^{1/2}]^2}. \tag{15}$$

These expressions may be transformed in many ways using (5) and (6) and general results for higher powers in the denominators[3] may be obtained in the same way.

As an example of the way in which the above functions arise, we consider a semi-infinite artificial transmission line with mid-series termination, in which the series elements are inductances L and the shunt elements are condensers of capacity C. Suppose that all condensers are charged to unit potential, and that at time $t=0$ the line is discharged through the matching resistance $\sqrt{(L/C)}$. Then if I_0 is the current in the resistance, I_n that in the nth inductance L, and Cv_n is the charge on the nth condenser, applying the Laplace transformation method in the usual way we find that

$$L\{I_r\} = \frac{aC[(1+p^2/a^2)^{1/2} - p/a]^{2r}}{2p[1+(1+p^2/a^2)^{1/2}]}, \qquad r = 0, 1, \cdots \tag{16}$$

[3] The extension of (15) is trivial; for that of (13) the results needed are given in Chrystal, *Textbook of algebra*, 2nd ed., 1906, vol. 2, pp. 204–205.

$$L\{v_r\} = \frac{1}{p} - \frac{[(1+p^2/a^2)^{1/2} - p/a]^{2r-1}}{p[1+(1+p^2/a^2)^{1/2}]}, \qquad r = 1, 2, \cdots, \tag{17}$$

where $a = 2(LC)^{-1/2}$.

It follows from (12) that

$$v_r = 1 - Ji_{2r-1,1}(at) - Ji_{2r-1,3}(at) + (2r-1)J\overset{(3)}{i}_{2r-1}(at) \tag{18}$$

$$I_r = (C/L)^{1/2}\{Ji_{2r,1}(at) + Ji_{2r,3}(at) - 2rJ\overset{(3)}{i}_{2r}(at)\} \tag{19}$$

$$I_0 = (C/L)^{1/2}\{Ji_{0,1}(at) + Ji_{0,3}(at) - \tfrac{1}{2}a^2t^2\} \tag{20}$$

$$= \tfrac{1}{2}(C/L)^{1/2}\{(1+a^2t^2)Ji_{0,1}(at) - a^2t^2(1+J_1(at)) + atJ_0(at)\}, \tag{21}$$

where (21) follows from (20) by integration by parts.

If the line is discharged into inductance $\frac{1}{2}L$ and resistance $\sqrt{(L/C)}$ in series, the solution follows from (14) in place of (12).

ON CERTAIN INTEGRALS IN THE THEORY OF HEAT CONDUCTION*

By STEWART PATERSON (*I.C.I. (Explosives) Limited, Stevenston, Scotland*)

In a recent note[1] W. Horenstein evaluates the integrals

$$\phi \equiv \int_0^t x^{-3/2} \exp\left(-\frac{a^2}{x} - b^2x\right)dx, \tag{1}$$

$$\psi \equiv \int_0^t x^{-1/2} \exp\left(-\frac{a^2}{x} - b^2x\right)dx, \tag{2}$$

in terms of the tabulated exponential and error functions. The evaluation of the more general integral, viz.

$$\int_k^t \exp(-s^2 - n^2/s^2)ds$$

from which ϕ and ψ are easily derived, was given by Riemann.[2]

Integrals of the above type arise in the solution by classical methods of various heat conduction problems. It is the purpose of this note to point out that treatment of many such problems by the Heaviside "operational" or equivalent Laplace transform method leads directly and naturally to the required solution in tabulated functions.

Thus, to take a simple case, the classical solution of

$$\frac{\partial\theta}{\partial t} = \frac{1}{4}\frac{\partial^2\theta}{\partial a^2} - b^2\theta; \qquad \theta \to 0,\ t \to 0,\ \theta \to 1,\ a \to 0+, \tag{3}$$

* Received Nov. 24, 1945.

[1] W. Horenstein, Quart. Appl. Math. 3, 183–184 (1945).

[2] B. Riemann, *Partielle Differentialgleichungen*, 2nd ed., 1876, p. 173.

(where θ is a function of a and t) will be

$$\theta = a\pi^{-1/2}\phi.$$

If, however,

$$\bar{\theta}(a, p) \equiv \int_0^\infty e^{-pt}\theta(a, t)dt,$$

equations (3) transform into

$$\frac{\partial^2\bar{\theta}}{\partial a^2} = 4(p + b^2)\bar{\theta}; \qquad \bar{\theta} \to \frac{1}{p}, \quad a \to 0 + \tag{4}$$

which lead at once to

$$\bar{\theta} = p^{-1} \exp\left[-2a\sqrt{(p + b^2)}\right]. \tag{5}$$

The inversion theorem for the Laplace transform then gives

$$\theta = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \lambda^{-1} \exp\left[\lambda t - 2a\sqrt{(\lambda + b^2)}\right]d\lambda, \tag{6}$$

along the usual contour.

By a series of obvious and natural steps,[3] it is easy to show that this is equal to

$$\frac{e^{2ab}}{4\pi i}\int_{\gamma' - i\infty}^{\gamma' + i\infty} \lambda^{-1} \exp\left[\lambda t - 2(a + bt)\sqrt{\lambda}\right]d\lambda + \frac{e^{-2ab}}{4\pi i}\int_{\gamma'' - i\infty}^{\gamma'' + i\infty} \lambda^{-1} \exp\left[\lambda t - 2(a - bt)\sqrt{\lambda}\right]d\lambda,$$

$$= \frac{e^{2ab}}{2}\left[1 - \operatorname{erf}\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)\right] + \frac{e^{-2ab}}{2}\left[1 - \operatorname{erf}\left(\frac{a}{\sqrt{t}} - b\sqrt{t}\right)\right],$$

and it can be verified that this satisfies (3).

NOTE ON A FORMULA FOR THE SOLUTION OF AN ARBITRARY ANALYTIC EQUATION*

By HERBERT E. SALZER (*Mathematical Tables Project, New York City*)

In a recent note D. R. Blaskett and H. Schwerdtfeger[1] give a fairly well known expansion for a root α of the equation $f(z) = 0$, as a power series in $f(z_0)$, where z_0 is near α, namely,

$$\alpha = \sum_{\nu=0}^\infty \frac{(-1)^\nu [f(z_0)]^\nu}{\nu!}\left[\frac{d^\nu f^{-1}(w)}{dw^\nu}\right]_{w=f(z_0)}, \tag{1}$$

where w denotes $f(z)$.

Of use in connection with (1) is a paper by Van Orstrand, "Reversion of Power Series," Phil. Mag., (6) 19, 366–376 (1910). Van Orstrand's article deals with the re-

[3] H. Jeffreys, *Operational methods in mathematical physics*, Cambridge, 1931, p. 70.

* Received January 26, 1946.

[1] This Quarterly 3, 266–268 (1945).

version of the special type of power series $y=a_0x+a_1x^2+a_2x^3+\cdots$ to obtain x as an integral power series in $z\equiv y/a_0$, whose coefficients are given as polynomials in $b_i\equiv -a_i/a_0$, as far as the term involving z^{13}. Now the explicit expansion for (1) in terms of the derivatives of $f(z)$ at z_0 can be written down immediately, as far as $\nu=13$, from Van Orstrand's expansion on pp. 369-370, merely by

(A) replacing b_i in his formula by $-f^{(i+1)}(z_0)/(i+1)!f'(z_0)$,

(B) replacing *his* z by $-f(z_0)/f'(z_0)$, and

(C) adding the constant term z_0.

The truth of the last statement is obvious from the fact that when (1) is applied at the origin it yields Van Orstrand's expansion and from the uniqueness of Van Orstrand's expansion.

A NOTE ON THE CORRECTION OF GEIGER MÜLLER COUNTER DATA*

By H. B. Mann (*Ohio State University*)

The correction of Geiger Müller Counter data has been considered in a previous paper by J. D. Kurbatov and the author.[1] According to the model described there the following result was proved: If the density of radiation is a constant a and if τ denotes the resolving time, $B(T)$ the number of discharges during the time T; then

$$B(T) = \frac{aT}{1+a\tau} + \eta, \tag{1}$$

where η is given by

$$\eta = -a\int_0^T \epsilon(t)dt \tag{2}$$

and $\epsilon(t)$ satisfies the conditions

$$\left.\begin{aligned} \epsilon(t) &= -a\int_{t-\tau}^{t} \epsilon(x)dx \qquad \text{for} \quad t \geqq \tau, \\ \epsilon(t) &= 1 - e^{-at} - \frac{a\tau}{1+a\tau} \qquad \text{for} \quad 0 \leqq t \leqq \tau. \end{aligned}\right\} \tag{3}$$

It was further shown that for $a\tau<1$.

$$|\eta| \leqq \frac{(a\tau)^2}{1-(a\tau)^2}\,[1-(a\tau)^{s+1}],$$

where s is the largest integer not larger than T/τ. In this paper an upper bound for $|\eta|$ will be derived without the restriction $a\tau<1$. We shall prove the following inequality:

* Received May 29, 1946.

[1] J. D. Kurbatov and H. B. Mann, *A correction for Geiger Müller counter data*, Phys. Rev. 68, 40–43 (1945).

Let $[x]$ denote the largest integer not larger than x; then

$$|\eta(T)| \leqq \frac{a\tau}{1+a\tau}(2e^{a\tau}-1) + \frac{a^2\tau}{1+a\tau}\left(T - \left[\frac{T}{\tau}\right]\tau\right)(1-e^{-a\tau})^{[T/2\tau]}. \tag{4}$$

Proof of the inequality (4). From (3) we see that $\epsilon(t)$ is a continuous function of t. Applying the mean value theorem to (3) we have

$$\epsilon(t) = -a\tau\epsilon(t^*), \qquad t-\tau \leqq t^* \leqq t.$$

Hence $\epsilon(t)$ changes its sign at least once in every open interval of length τ and will therefore be 0 at least once in every such interval. Hence we have

PROPOSITION 1. *In every open interval of length τ there is at least one point for which $\epsilon(t)=0$.*

Differentiating (3) with respect to t we obtain

$$\epsilon'(t) = a\epsilon(t-\tau) - a\epsilon(t). \tag{5}$$

In the interval $\bar{t} \leqq t \leqq \bar{t}+\tau$ Eq. (5) may be considered as a differential equation for $\epsilon(t)$ with the initial condition that its solution be equal to $\epsilon(\bar{t})$ at the point $\bar{t}$. Solving (5) with this initial condition we have, for $\bar{t} \leqq t \leqq \bar{t}+\tau$,

$$\epsilon(t) = \epsilon(\bar{t})e^{-a(t-\bar{t})} + ae^{-at}\int_{\bar{t}}^{t} e^{ax}\epsilon(x-\tau)dx. \tag{6}$$

Let $M(\bar{t})$ be the maximum of the absolute value of $\epsilon(t)$ in the interval $[\bar{t}-\tau, \bar{t}]$, then

$$|\epsilon(t)| \leqq M(\bar{t})e^{-a(t-\bar{t})} + e^{-at}M(\bar{t})(e^{at}-e^{a\bar{t}}) = M(\bar{t}) \quad \text{for} \quad \bar{t} \leqq t \leqq \bar{t}+\tau. \tag{7}$$

From (7) it follows that $|\epsilon(t)| \leqq M$ for $t \geqq \bar{t}$. Hence we have

PROPOSITION 2. *If $|\epsilon(t)| \leqq M$ for $\bar{t}-\tau \leqq t \leqq \bar{t}$, then $|\epsilon(t)| \leqq M$ for $t \geqq \bar{t}-\tau$.*

If $\epsilon(\bar{t})=0$ then we obtain from (6)

$$|\epsilon(t)| \leqq M(\bar{t})(1-e^{-a(t-\bar{t})}) \leqq M(\bar{t})(1-e^{-a\tau}) \quad \text{for} \quad \bar{t} \leqq t \leqq \bar{t}+\tau. \tag{8}$$

From (8) and Proposition 2 follows

PROPOSITION 3. *If $\epsilon(\bar{t})=0$ and $|\epsilon(t)| \leqq M$ for $\bar{t}-\tau \leqq t \leqq \bar{t}$, then $|\epsilon(t)| \leqq M(1-e^{-a\tau})$ for $t \geqq \bar{t}$.*

According to proposition (1) we have in the interval $\alpha\tau \leqq t \leqq (\alpha+1)\tau$ at least one point t_α for which $\epsilon(t_\alpha)=0$.

Consider the points $t_1, t_3, \cdots, t_{2n+1}$. If M is the maximum of $|\epsilon(t)|$ in $0 \leqq t \leqq \tau$ we must have, according to Propositions 1, 2, and 3,

$$|\epsilon(t)| \leqq M \quad \text{for} \quad 0 \leqq t \leqq t_1,$$
$$|\epsilon(t)| \leqq M(1-e^{-a\tau}) \quad \text{for} \quad t_1 \leqq t \leqq t_3,$$
$$\vdots$$
$$|\epsilon(t)| \leqq M(1-e^{-a\tau})^k \quad \text{for} \quad t_{2k-1} \leqq t \leqq t_{2k+1}.$$

Hence

$$|\eta| = \left| -a\int_0^T \epsilon(x)dx \right| = \left| \sum_{\alpha=1}^{\alpha=[T/\tau]} \epsilon(\alpha\tau) - a\int_{[T/\tau]\tau}^T \epsilon(x)dx \right|$$

$$\leqq M + 2M(1-e^{-a\tau}) + 2M(1-e^{-a\tau})^2 + \cdots + aM\left(T - \left[\frac{T}{\tau}\right]\tau\right)(1-e^{-a\tau})^{[T/2\tau]}$$

$$\leqq M(2e^{a\tau}-1) + aM\left(T - \left[\frac{T}{\tau}\right]\tau\right)(1-e^{-a\tau})^{[T/2\tau]}.$$

From (3) it can be seen that $M = a\tau/(1+a\tau)$ and (4) follows.

The inequality (4) is very satisfactory and shows that even for large values of $a\tau$ the quantity η will be very small compared to $aT/(1+a\tau)$ even if T is only a few minutes.

CORRECTIONS TO OUR PAPER

STABILITY OF COLUMNS AND STRINGS UNDER PERIODICALLY VARYING FORCES*

QUARTERLY OF APPLIED MATHEMATICS, 3, 215–236 (1943)

By S. LUBKIN AND J. J. STOKER (*New York University*)

The following errors were found in the tables printed on pp. 232–235.

β \ $\alpha(C_0)$	for	read
1.6	−0.77898	−0.77897
1.8	−0.92281	−0.92282
7.6	−5.71537	−5.71538
9.2	−7.11974	−7.11975

β \ $\alpha(C_1)$	for	read
3.6	−2.32402	−2.32401

β \ $\alpha(S_1)$	for	read
0.8	0.55906	0.55406
1.4	0.63015	0.63016
4.4	−0.29781	−0.29780
7.6	−2.08644	−2.08648
11.0	−4.29436	−4.29437

β \ $\alpha(S_2)$	for	read
3.8	−0.00468	−0.00464
6.8	−1.60383	−1.60379
8.4	−2.58478	−2.58477

β \ $\alpha(C_2)$	for	read
0.6	1.12806	1.12810
3.4	2.01478	2.01477
20.0	−5.05198	−5.05199

β \ $\alpha(C_3)$	for	read
0.6	2.26622	2.26621
1.0	2.28515	2.28516
2.2	2.31495	2.31493
2.8	2.29660	2.29661
5.6	1.85589	1.85591

* Received Aug. 16, 1946.

β \ $\alpha(S_3)$	for	read
1.6	2.51308	2.51309
2.0	2.66777	2.66776
3.8	3.46578	3.46579
4.4	3.69216	3.69215
5.6	4.01149	4.01150

β \ $\alpha(S_4)$	for	read
1.6	4.07660	4.07659
1.8	4.09433	4.09432
2.4	4.15212	4.15211
2.6	4.17199	4.17200
12.0	3.38817	3.38820

β \ $\alpha(C_4)$	for	read
1.6	4.09776	4.09777
2.4	4.24889	4.24891
2.8	4.35867	4.35865
4.8	5.18127	5.18128
6.0	5.74803	5.74303

β \ $\alpha(C_5)$	for	read
16.0	6.52721	6.52709

Column headings for $\alpha(S_5)$ and $\alpha(S_6)$ are interchanged on p. 234.

β \ $\alpha(S_5)$	for	read
0.4	6.25333	6.25334
2.2	6.35487	6.35488
3.2	6.48591	6.48590
4.8	6.86185	6.86180
5.2	6.99394	6.99396
5.6	7.14093	7.14116
20.0	10.33749	10.33744

β \ $\alpha(S_6)$	for	read
3.8	9.20714	9.20713
5.2	9.38281	9.38279
16.0	10.59848	10.59849
20.0	10.35813	10.35825

BOOK REVIEWS

Theory of Structures. By S. Timoshenko and D. H. Young. McGraw-Hill Book Company, Inc., New York and London, 1945. xiv+488 pp. $5.00.

This valuable addition to text-book literature is based on the senior author's earlier volume, published in Russia in 1926 (S. Timoshenko, Theory of Structures, Leningrad). The book is intended for engineering students with some background in mechanics. The keynote of this book is that familiarity with the general principles of mechanics is indispensable to a thorough understanding of the analysis of stresses in trusses and frames. For this reason two of the nine chapters are devoted to a comprehensive recapitulation of the rudiments of plane statics and of such general theorems on elastic systems as the Principle of Least Work, Castigliano's Theorem, Maxwell's Reciprocal Theorem, etc.

As one would expect from the authors, the book is very clearly written. It abounds in carefully constructed figures and diagrams, and contains a wealth of well-graded problems.

The chapter headings are as follows: Elements of Plane Statics, Statically Determinate Plane Trusses, Influence Lines, Statically Determinate Space Structures, General Theorems Relating to Elastic Systems, Deflection of Pin-jointed Trusses, Statically Indeterminate Pin-jointed Trusses, Beams and Frames, Arches.

This book will be of considerable interest to structural engineers and will be welcomed by the teachers of mechanics and theory of structures.

I. S. Sokolnikoff

Table of arc sin x. Prepared by the Mathematical Tables Project conducted under the sponsorship of the National Bureau of Standards. Official Sponsor: Lyman J. Briggs. Project Director: Arnold N. Lowan. Columbia University Press, New York, 1945. xix+121 pp. $3.50.

The main tables give the values of arc sin x to twelve decimal places, the intervals of the argument being .0001 in the range between 0 and .9890, and .00001 in the range between .98900 and unity. To facilitate interpolation the second (and, wherever necessary, the fourth) differences are tabulated, and auxiliary tables are given for the coefficients in the interpolation formulas of Newton-Gregory and Everett. For values of x exceeding 0.99950, interpolation by means of differences becomes unsatisfactory. For such values of x the use of the formula $\arcsin(1-v)=\pi/2-f(v)\sqrt{2v}$ is recommended, and $f(v)=1+v/12+3v^2/160+5v^3/896+\cdots$ is tabulated (with first and second differences) to thirteen decimal places at intervals of .00001 in the range from 0 to 0.00050.

W. PRAGER

Network Analysis and Feedback Amplifier Design. By Hendrick W. Bode. D. Van Nostrand Company, Inc., New York, 1945, xii+551 pp. $7.50.

This book is concerned with a complete exposition of electrical circuit theory, the properties and design of feedback amplifiers, non-feedback amplifiers, and the discussion of certain problems of wide band transmission. A great deal of the material presented in this book has not appeared before in text book form.

The book is divided into nineteen chapters. The first two chapters are devoted to the presentation of the fundamental principles of linear, passive, electrical circuits and to a formulation of the fundamental equations of these circuits from the mesh and nodal standpoints. The response of linear circuits to driving functions of the exponential type is considered and the very useful concept of the complex frequency plane in the study of the properties of linear circuits is introduced.

In the next four chapters, the basic principles and theorems of feedback are considered in detail. A thorough discussion of stability, physical realizability, contour integration, Nyquist's criterion for stability, and the physical representation of driving point impedance functions, occupies a central position in the book. The remaining chapters are devoted to the design of impedance functions, equalizers, interstage networks, single loop amplifiers, single loop feedback amplifiers, and a discussion of the relations between the real and imaginary components of network functions.

From a mathematical standpoint, the material presented in this book is a beautiful example of the power and utility of the application of the fundamental theorems of the complex variable to a most important physical problem. Since the subject of network analysis and synthesis is of such great importance not only in itself but also because it serves as a model for the analysis of mechanical and acoustical systems, the excellent original analysis of the problem presented by Dr. Bode in this book is a great contribution to the field of applied mathematics.

LOUIS A. PIPES

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