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CONTROLLABILITY OF DYNAMICAL SYSTEMS WITH DAMPING TERM AND CONSTRAINED CONTROLS

Summary. In the paper presented the methodology of investigation of the controllability of an infinite dimensional second order dynamical systems with damping term. Following this aim spectral theory for linear unbounded operators is involved. In the first part of the paper the problem is stated and the methodology of transforming the second order equation to the set of the first order equations is reminded. Next the theorem on transforming considered infinite dimensional dynamical system to infinite series of finite dimensional systems is proved. Finally the theorem on necessary and sufficient conditions of constrained approximate controllability of considered system is formulated and proved.

STEROWALNOŚĆ UKŁADÓW DYNAMICZNYCH Z CZYNNIKIEM TŁUMIĄCYM I OGRANICZONYMI STEROWANIAMI

Streszczenie. W ramach pracy przedstawiono metodykę badania sterowalności nieskończenie wymiarowych układów dynamicznych rzędu drugiego z czynnikiem tłumiącym. Do tego celu wykorzystana została spektralna teoria liniowych operatorów nieograniczonych. W pierwszej części pracy został sformułowany problem i przypomniana została metodyka sprowadzenia rozpatrywanego układu drugiego rzędu do układu równań pierwszego rzędu. Następnie udowodniono twierdzenie o sprowadzeniu wyjściowego układu nieskończenie wymiarowego do nieskończonego ciągu układów skończenie wymiarowych. Na koniec zostało sformułowane i udowodnione twierdzenie podające warunki konieczne i wystarczające aproksymacyjnej sterowalności z ograniczeniami rozpatrywanego układu.

1. Basic concepts

Let us consider a dynamical system given by the following abstract differential equation:

$$\frac{d^2x(t)}{dt^2}+f(A)\frac{dx(t)}{dt}+Ax(t)=Bu(t),\quad t>0 \tag{1}$$

where the operator's A $f(A): X \supset D(A) \to X$ function is given by:

$$f(A)x = 2\alpha_0 Ax + \sum_{l=1}^{k} 2\alpha_l A^{\beta_l} x, \ x \in D(A) \tag{2}$$

where coefficients α_l, β_l fulfils the following inequalities:

$$\alpha_0 > 0, \ \alpha_l \geq 0, \frac{1}{2} \leq \beta_l < 1, \ l = 1,2,...,k \tag{3}$$

also are given initial conditions:

$$x(0) = x_0 \in D(A), \ \dot{x}(0) = x_1 \in X \tag{4}$$

The operator B is defined as follows:

$$Bu(t) = \sum_{i=1}^{p} b^i u_i(t), \ B \in L(R^p, X) \tag{5}$$

where:

$$B = [b^1, b^2, ..., b^p], \ u(t) = [u_1(t), u_2(t), ..., u_p(t)]^T, \ b^i \in X, \ u \in L^2([0,\infty), R^p) \tag{6}$$

Let us moreover assume that $A: X \supset D(A) \to X$ is a linear, generally unbounded, selfadjoint and positive-definite operator with domain D(A) dense in X and compact resolvent $R(\lambda,A)$ for all λ in the resolvent set ρ(A).

The physical interpretation of the equation encompasses a broad class of real systems in the form (1) and depends on a particular form on the A operator and of the coefficients and exponents of the damping term f(A) (2).

It is well known that the operator A has the following spectral properties [8]:

- Operator A has only purely discrete point spectrum consisting entirely of distinct real positive eigenvalues λ_i each with finite multiplicity m_i $(m_i < \infty)$:

$$0 < \lambda_1 < \lambda_2 < ... < \lambda_i < \lambda_{i+1} < ..., \ \lim_{i\to\infty} = \infty \tag{7}$$

- The eigenfunctions of operator A $\{\phi_{ij}, i = 1,2,3,..., j = 1,2,...,m_i\}$ form complete orthonormal system in Hilbert space X. Hence for every $x \in X$ the following unique expansion holds true

$$x = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} < x, \phi_{ij} >_X \phi_{ij} \tag{8}$$

- Operator A has the following spectral resolution:

$$\forall_{x\in D(A)} \ Ax = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} \lambda_i < x, \phi_{ij} >_X \phi_{ij} \tag{9}$$

$$D(A) = \left\{ x \in X : \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} \lambda_i^2 \left| < x, \phi_{ij} >_X \right|^2 < \infty \right\} \tag{10}$$

- The fractional power of operator A is defined as follows [7]:

$$\forall_{x\in D(A^{\beta}),\beta\in(0,1)} \quad A^{\beta}x=\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\lambda_i^{\beta}<x,\phi_{ij}>_X\phi_{ij} \tag{11}$$

$$D(A^{\beta})=\left\{x\in X:\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\lambda_i^{2\beta}\left|<x,\phi_{ij}>_X\right|^2<\infty\right\} \tag{12}$$

- Operator A^{β}, $0<\beta<1$ is also selfadjoint and positive-definite with domain $D(A^{\beta})$ dense in X.

2. Transformation of the given second order system (1) to the first order equation

The main aim of this paragraph is to present how to transform the given second order equation (1) to the first order one. As is shown in [8] given system (1) can be rewritten in equivalent form of the following system of two first order equations:

$$\frac{d}{dt}\begin{bmatrix}\xi(t)\\ \mu(t)\end{bmatrix}=\begin{bmatrix}A^{+} & 0\\ 0 & A^{-}\end{bmatrix}\begin{bmatrix}\xi(t)\\ \mu(t)\end{bmatrix}+\begin{bmatrix}g(A)^{-1}B\\ -g(A)^{-1}B\end{bmatrix}u(t) \tag{13}$$

where the operators A^{+} and A^{-} are defined by the following formulas:

$$A^{+}=-\alpha_0A-\sum_{l=1}^{k}\alpha_lA^{\beta_l}+g(A) \tag{14}$$

$$A^{-}=-\alpha_0A-\sum_{l=1}^{k}\alpha_lA^{\beta_l}-g(A) \tag{15}$$

$$A^{\pm}:X\supset D(A)\to X, D(A^{\pm})=D(A) \tag{16}$$

The operator $g(A)$ is defined as follows [2]:

$$\forall_{x\in D(g(A))} \quad g(A)x=\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}g(\lambda_i)<x,\phi_{ij}>_X\phi_{ij} \tag{17}$$

$$D(g(A))=\left\{x\in X:\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}g^2(\lambda_i)\left|<x,\phi_{ij}>_X\right|^2<\infty\right\} \tag{18}$$

with domain $D(g(A))=D(A)$ [6].

Appearing in the above formulas function $g(\lambda_i):R\to C$ is defined as follows:

$$g(\lambda_i)=\sqrt{\left(\alpha_0\lambda_i+\sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l}\right)^2-\lambda_i} \tag{19}$$

with additional assumption:

$$\left(\alpha_0\lambda_i + \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l}\right)^2 - \lambda_i \neq 0, \ i = 1,2,3,... \tag{20}$$

This condition (20) is necessary for the invertibility of the operator's function g(A). In [8] is showed that for all $i \in N$ the following inverse of g(A) linear operator can be defined:

$$\underset{x \in D(g^{-1}(A))}{\forall} \ g^{-1}(A)x = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} g(\lambda_i) < x, \phi_{ij} >_X \phi_{ij} \tag{21}$$

$$D(g^{-1}(A)) = \left\{ x \in X : \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} g^{-2}(\lambda_i) \left| < x, \phi_{ij} >_X \right|^2 < \infty \right\} \tag{22}$$

Operator of the system (13):

$$\Omega = \begin{bmatrix} A^+ & 0 \\ 0 & A^- \end{bmatrix} \tag{23}$$

has the following spectral properties [4]:

- Operator Ω has purely discrete point spectrum $\sigma(\Omega)$ of the following form:

$$\sigma(\Omega) = \{s_i^{\pm}, i \in N\} \cup \left\{\frac{1}{2\alpha_0}\right\} \tag{24}$$

where $s_i^{\pm}$ are distinct eigenvalues of Ω given by the formula:

$$s_i^{\pm} = -\alpha_0\lambda_i - \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l} \pm g(\lambda_i), \quad i = 1,2,3,... \tag{25}$$

- the set of eigenfunctions of the operator Ω:

$$\{[\phi_{ij}, 0]^T, [0, \phi_{ij}]^T, \ i = 1,2,3,... \ j = 1,2,..., m_i\} \tag{26}$$

is a complete orthonormal system in Hilbert space $X \times X$. Thus the following unique expansion holds true:

$$\underset{x \in X \times X}{\forall} \ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} \left\{ < x_1, \phi_{ij} >_X \begin{bmatrix} \phi_{ij} \\ 0 \end{bmatrix} + < x_2, \phi_{ij} >_X \begin{bmatrix} 0 \\ \phi_{ij} \end{bmatrix} \right\} \tag{27}$$

- operator Ω has the following unique spectral resolution:

$$\Omega \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} \left\{ s_i^+ < \xi, \phi_{ij} >_X \begin{bmatrix} \phi_{ij} \\ 0 \end{bmatrix} + s_i^- < \mu, \phi_{ij} >_X \begin{bmatrix} 0 \\ \phi_{ij} \end{bmatrix} \right\} \tag{28}$$

In the rest of this paper it will be assumed that all the previous assumptions are fulfilled.

3. Theorem 1

The infinite dimensional dynamical system (13) can be rewritten in equivalent form of the following two infinite series of finite dimensional dynamical systems:

$$\begin{cases} \varsigma_i^{+}(t) = A_i^{'+}\varsigma_i^{+}(t) + B_i^{'+}u(t) \\ \varsigma_i^{-}(t) = A_i^{'-}\varsigma_i^{-}(t) + B_i^{'-}u(t) \end{cases} \quad i = 1,2,3,... \tag{29}$$

Where A'^{+}_i, A'^{-}_i and B'^{+}_i, B'^{-}_i are the following matrixes:

$$A_i^{'+} = diag[s_i^{+},...,s_i^{+}],\ \dim A_i^{'+} = m_i \times m_i \tag{30}$$

$$A_i^{'-} = diag[s_i^{-},...,s_i^{-}],\ \dim A_i^{'-} = m_i \times m_i \tag{31}$$

$$B_i^{'+} = g^{-1}(\lambda_i)\begin{bmatrix} b_{i1}^1 & \cdots & b_{i1}^k & \cdots & b_{i1}^p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{il}^1 & \cdots & b_{il}^k & \cdots & b_{il}^p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{im_i}^1 & \cdots & b_{im_i}^k & \cdots & b_{im_i}^p \end{bmatrix} \tag{32}$$

$$B_i^{'-} = -B_i^{'+} \tag{33}$$

Furthermore:

$$b_{ij}^k = \langle b^k, \phi_{ij} \rangle_X \quad i = 1,2,3,... \quad j = 1,2,...,m_i, \quad k = 1,2,...,p \tag{34}$$

The vectors $\varsigma_i^{+}(t), \varsigma_i^{-}(t)$ are given by:

$$\varsigma_i^{+}(t) = \left[\xi_{i1}(t), \xi_{i2}(t),...,\xi_{im_i}(t)\right]^T \tag{35}$$

$$\varsigma_i^{-}(t) = \left[\mu_{i1}(t), \mu_{i2}(t),...,\mu_{im_i}(t)\right]^T \tag{36}$$

where $\xi_{ij}(t)$, $\mu_{ij}(t)$ denotes the i[th] coefficient of the Fourier series of spectral representation for the element x in the state space X. The coefficients are explicit given by the inner product between element in the state space X and the appropriate eigenfunction ϕ_{ij} of the operator A:

$$\xi_{ij}(t) = \langle \xi(t), \phi_{ij} \rangle_X,\ \mu_{ij}(t) = \langle \mu(t), \phi_{ij} \rangle_X,\ i = 1,2,3,...\ j = 1,2,...,m_i \tag{37}$$

Proof

Let us remind the form of considered infinite dimensional system (13):

$$\frac{d}{dt}\begin{bmatrix} \xi(t) \\ \mu(t) \end{bmatrix} = \begin{bmatrix} A^{+} & 0 \\ 0 & A^{-} \end{bmatrix}\begin{bmatrix} \xi(t) \\ \mu(t) \end{bmatrix} + \begin{bmatrix} g(A)^{-1}B \\ -g(A)^{-1}B \end{bmatrix}u(t) \tag{38}$$

To proof the thesis of the theorem first of all let us take into account the fact that the operator Ω is complete and orthonormal and can be used use the spectral resolution of the operator Ω Using (5) and (27) we can rewrite (13) in form:

$$\begin{cases} \sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\dot{\xi}_{ij}(t)\phi_{ij} = A^{+}\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\xi_{ij}(t)\phi_{ij} + g^{-1}(A)\sum_{k=1}^{p}b^k u_k(t) \\ \sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\dot{\mu}_{ij}(t)\phi_{ij} = A^{-}\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\mu_{ij}(t)\phi_{ij} - g^{-1}(A)\sum_{k=1}^{p}b^k u_k(t) \end{cases} \tag{39}$$

Considering that the operator $g^{-1}(A)$ is linear and using formula (21) we can obtain the following equality:

$$g^{-1}(A)\sum_{k=1}^{p}b^k u_k(t) = \sum_{k=1}^{p}u_k(t)\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}g^{-1}(\lambda_i)b_{ij}^k\phi_{ij} \tag{40}$$

Next using the spectral resolution of the operator Ω (28) and the last formula the set of equations (39) receives form:

$$\begin{cases} \sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\dot{\xi}_{ij}(t)\phi_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i}s_i^{+}\xi_{ij}(t)\phi_{ij} + \sum_{k=1}^{p}u_k(t)\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}g^{-1}(\lambda_i)b_{ij}^k\phi_{ij} \\ \sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\dot{\mu}_{ij}(t)\phi_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i}s_i^{-}\mu_{ij}(t)\phi_{ij} - \sum_{k=1}^{p}u_k(t)\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}g^{-1}(\lambda_i)b_{ij}^k\phi_{ij} \end{cases} \tag{41}$$

Now let us calculate the inner product on both sides of last set of equations. Taking into account the fact that the eigenfunctions $\{\phi_{ij}, i=1,2,3,\ldots, j=1,2,\ldots,m_i\}$ form complete orthonormal system in Hilbert space X it gives:

$$\begin{cases} \dot{\xi}_{ij}(t) = s_i^{+}\xi_{ij}(t) + g^{-1}(\lambda_i)\sum_{k=1}^{p}u_k(t)b_{ij}^k \\ \dot{\mu}_{ij}(t) = s_i^{-}\mu_{ij}(t) - g^{-1}(\lambda_i)\sum_{k=1}^{p}u_k(t)b_{ij}^k \end{cases} \quad i=1,2,3,\ldots \quad j=1,2,\ldots,m_i \tag{42}$$

Now let us rewrite above set of equations (42) for fixed i in the following form:

$$\begin{cases} \frac{d}{dt}\begin{bmatrix}\xi_{i1}\\ \vdots \\ \xi_{im_i}\end{bmatrix} = \begin{bmatrix} s_i^{+} & & 0 \\ & \ddots & \\ 0 & & s_i^{+}\end{bmatrix}\begin{bmatrix}\xi_{i1}\\ \vdots \\ \xi_{im_i}\end{bmatrix} + \begin{bmatrix} g^{-1}(\lambda_i)b_{i1}^1 & \cdots & g^{-1}(\lambda_i)b_{i1}^k & \cdots & g^{-1}(\lambda_i)b_{i1}^p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ g^{-1}(\lambda_i)b_{il}^1 & \cdots & g^{-1}(\lambda_i)b_{il}^k & \cdots & g^{-1}(\lambda_i)b_{il}^p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ g^{-1}(\lambda_i)b_{im_i}^1 & \cdots & g^{-1}(\lambda_i)b_{im_i}^k & \cdots & g^{-1}(\lambda_i)b_{im_i}^p \end{bmatrix} u(t) \\ \frac{d}{dt}\begin{bmatrix}\mu_{i1}\\ \vdots \\ \mu_{im_i}\end{bmatrix} = \begin{bmatrix} s_i^{-} & & 0 \\ & \ddots & \\ 0 & & s_i^{-}\end{bmatrix}\begin{bmatrix}\mu_{i1}\\ \vdots \\ \mu_{im_i}\end{bmatrix} - \begin{bmatrix} g^{-1}(\lambda_i)b_{i1}^1 & \cdots & g^{-1}(\lambda_i)b_{i1}^k & \cdots & g^{-1}(\lambda_i)b_{i1}^p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ g^{-1}(\lambda_i)b_{il}^1 & \cdots & g^{-1}(\lambda_i)b_{il}^k & \cdots & g^{-1}(\lambda_i)b_{il}^p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ g^{-1}(\lambda_i)b_{im_i}^1 & \cdots & g^{-1}(\lambda_i)b_{im_i}^k & \cdots & g^{-1}(\lambda_i)b_{im_i}^p \end{bmatrix} u(t) \end{cases} \tag{43}$$

It can be easily seen that above set of equations after substitutions (30) to (36) has form (29).

Q.E.D.

4. Basic criteria of controllability of finite dimensional systems with constrained controls

It is given stationary finite dimensional system described by the following equations:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \ t \geq 0 \\ y(t) = Cx(t) + Du(t), \ t \geq 0 \end{cases} \tag{44}$$

where A,B,C,D are constants matrices with dimensions respectively n×n, n×m, p×n, p×m.

Definition 1

The dynamical system (44) is said to be U-controllable to zero from given initial state in the state space, if for any initial state $x(t_0)=x_0$, there exist an admissible control $u \in L^2_{loc}([0,\infty),U)$ such that the corresponding trajectory $x(t,x(t_0),u)$ of the dynamical system satisfies for some $t \in [t_0,\infty)$ the condition:

$$x(t_1, x(t_0), u) = 0 \tag{45}$$

Theorem 2 [3]

The dynamical system (44) is globally U-controllable to zero if and only if the following conditions are satisfied simultaneously:

(1) There exists a $w \in U$ such that $Bw=0$

(2) The convex hull CH(U) has a nonempty interior in the space R^p.

(3) $rank\left[B \mid AB \mid A^2 B \mid ... \mid A^{n-1} B\right] = n$

(4) There is no real eigenvector $v \in R^n$ of matrix A^T satisfying $v^T Bw \leq 0$ for all $w \in U$

(5) No eigenvalue of matrix A has a positive real part

Basing on Theorem 2 it can be formulated the criteria of controllability with constrained controls for the infinite dimensional system (1). It is the main outcome of this article. In this theorem we will also, similarily like in the theorem 1, assume that all the previous assumptions holds true.

5. Theorem 3

The infinite dimensional system (1) is globally approximately U-controllable to zero if and only if the following conditions are satisfied simultaneously:

(1) There exists a $w \in U$ such that $B_i^{+}w=0$ for every i=1,2,3,...

(2) The convex hull CH(U) has a nonempty interior in the space R^p.

(3) rank[B_i'']=m_i , for every i=1,2,3,...

(4) There is no real eigenvector $v_i \in R^{m_i}$ of matrices A'^{+}_i, A'_i satisfying $v_i^T B_i^{'+} w \le 0$ for all $w \in U$, for every i=1,2,3,...

where:

$$B_i^{*} = \begin{bmatrix} b_{i1}^1 & \cdots & b_{i1}^k & \cdots & b_{i1}^p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{il}^1 & \cdots & b_{il}^k & \cdots & b_{il}^p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{im_i}^1 & \cdots & b_{im_i}^k & \cdots & b_{im_i}^p \end{bmatrix} \quad (46)$$

and matrices A'^{+}_i, A'^{-}_i are introduced in theorem 1.

Proof

The proof bases on applying theorem 2 to system (1) in form of the two infinite series of finite dimensional systems (20).

- The condition 2 of the Theorem 2 can be rewritten in the same form, because the control space of each subsystem remains the same set by assumption

- The conditions 1, 4 in the Theorem 3 follows immediately from applying the Theorem 2 for every of finite dimensional subsystems in the form (29) and matrixes A'^{+}_i, A'^{-}_i are diagonal and so symmetric.

- So that testify the condition (3) of the Theorem 2 let us rewrite this condition after applying to the i[th] subsystem of the first series of (29):

$$rank[B_i^{'+} \mid A_i^{'+} B_i^{'+} \mid (A_i^{'+})^2 B_i^{'+} \mid \ldots \mid (A_i^{'+})^{(m_i-1)} B_i^{'+}] = m_i \quad i = 1,2,3,\ldots \quad (47)$$

As proved in [5] equation (47) with diagonal matrix $A_i^{'+}$ reduces to:

$$rank[B_i^{'+}] = m_i \quad i = 1,2,3,\ldots \quad (48)$$

By assumption (20) $g(\lambda_i) \neq 0$, so also $g^{-1}(\lambda_i) \neq 0$ and this does not affect on the rank of the matrix $B_i^{'+}$, because $B_i^{'+} = g^{-1}(\lambda_i) B_i^{*+}$ and condition (47) reduces to condition 3 of proved theorem.

Proof for the second series of (29) goes similarly and yields the same equation.

- condition 5 of the Theorem 2

The matrices A'^{+}_i, A'^{-}_i have diagonal form so their eigenvalues are equal to elements on the diagonals. At first let's check this condition for the matrix A'^{-}_i. Let us remind the formula (25) of the eigenvalues s_i^-:

$$s_i^- = -\alpha_0 \lambda_i - \sum_{l=1}^{k} \alpha_l \lambda_i^{\beta_l} - \sqrt{\left(\alpha_0 \lambda_i + \sum_{l=1}^{k} \alpha_l \lambda_i^{\beta_l}\right)^2 - \lambda_i} \quad (49)$$

considering conditions (3) it can be easy seen that for every i=1,2,3,... inequality $Re(s_i^-)<0$ is satisfied and condition 5 of theorem 2 is fulfilled for the first series of (29).

Now let's check this condition for the matrix A_i^+. To this purpose the following two cases will be distinguished:

Case A:

$$\left(\alpha_0\lambda_i + \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l}\right)^2 - \lambda_i \geq 0,\ i = 1,2,3,... \tag{50}$$

To prove this case let's take into account the fact that all the eigenvalues of the operator A are positive (7):

$$\lambda_i > 0 \tag{51}$$

Now let's perform on both sides of above inequality multiplication by (-1) and add an element $\left(\alpha_0\lambda_i + \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l}\right)^2$:

$$\left(\alpha_0\lambda_i + \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l}\right)^2 - \lambda_i < \left(\alpha_0\lambda_i + \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l}\right)^2 \tag{52}$$

The left side of the last inequality is non-negative in this case by assumption, therefore the right side is positive. The square root in the R_+ domain has different values for different arguments, so from the inequality (52) we can obtain:

$$\sqrt{\left(\alpha_0\lambda_i + \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l}\right)^2 - \lambda_i} < \alpha_0\lambda_i + \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l} \tag{53}$$

thus:

$$s_i^+ = -\alpha_0\lambda_i - \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l} + \sqrt{\left(\alpha_0\lambda_i + \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l}\right)^2 - \lambda_i} < 0 \tag{54}$$

Q.E.D.

Case B:

$$\left(\alpha_0\lambda_i + \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l}\right)^2 - \lambda_i < 0,\ i = 1,2,3,... \tag{55}$$

The square root of negative real number has only imaginary part, so we can write:

$$s_i^+ = -\alpha_0\lambda_i - \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l} + i\sqrt{\lambda_i - \left(\alpha_0\lambda_i + \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l}\right)^2} \tag{56}$$

Respect to inequalities (3) we can write:

$$-\alpha_0\lambda_i - \sum_{l=1}^{k}\alpha_l\lambda_i^{\beta_l} < 0 \tag{57}$$

Thus:

$$\operatorname{Re} s_i^+ < 0 \tag{58}$$

and condition 5 of the Theorem 2 is fulfilled.

Q.E.D.

6. Examples

6.1. Example 1: Two-dimensional system with single eigenvalues

Let us consider the following second order dynamical system with damping term:

$$\frac{d^2x(t)}{dt^2} + (4A + 6A^{\frac{3}{4}} + A^{\frac{1}{2}})\frac{dx(t)}{dt} + Ax(t) = Bu(t) \tag{59}$$

The state space X is R^2. Let the operator A be given by the following matrix:

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}, \quad A: R^2 \to R^2 \tag{60}$$

Operator B is as follows:

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \; B: R^2 \to R^2 \tag{61}$$

The control vector:

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{62}$$

The controls are constrained as follows:

$$u_1(t) \geq 0, \; u_2(t) \geq 0 \tag{63}$$

Problem Statement

The aim of this example is to verify the U-controllability of given dynamical system and rewriting him in the form of the first order systems' finite series.

Solve

The matrix's operators are obviously linear. Additionally given operator A (60) is symmetrical and thus self-adjoined. Moreover, their major minors are positive so is also positive-defined. It's easy to see that equation (59) has form of the dynamical system (1) after introduction of the following coefficients:

$$\alpha_0 = 2,\ \ \alpha_1 = 3,\ \ \alpha_2 = \frac{1}{2},\ \ \beta_1 = \frac{3}{4},\ \ \beta_2 = \frac{1}{2} \tag{64}$$

All these coefficients fits into their proper ranges given by the inequalities (3), so all the assumptions of the theorem 3 are fulfilled. Now let's check in sequence conditions this theorem is consisted of.

- Condition 1

Lets substitute $w = [0\ 0]^T$. Then $B_1^{"}w = 0 \wedge B_2^{"}w = 0$ and condition is fulfilled.

- Condition 2

Since $u_1(t) \geq 0$, $u_2(t) \geq 0$ the convex hull has a nonempty interior in R^2 and condition holds true.

- Condition 3

The operator A has two single eigenvalues:

$$\lambda_1 = 3 - 2\sqrt{2},\ \lambda_2 = 3 + 2\sqrt{2} \tag{65}$$

And it's eigenvectors has form:

$$\phi_1 = \begin{bmatrix} \dfrac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \\ \dfrac{1}{\sqrt{4+2\sqrt{2}}} \end{bmatrix} \quad \phi_2 = \begin{bmatrix} \dfrac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \dfrac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix} \tag{66}$$

In finite n-dimensional state space X the scalar product b_{ij}^k is given by the following sum:

$$b_{ij}^k = < b^k, \phi_{ij} >_X = \sum_{p=1}^{n} b_{pk}\phi_{ij}^{(p)} \tag{67}$$

In this example all the eigenvalues are single (m_i=1), so according to (67) the matrices B_1'' and B_2'' have form:

$$B_1^{"} = \left[b_1^1\ b_1^2\right] = \left[< b^1, \phi_1 >_X\ \ < b^2, \phi_1 >_X\right] = \left[b_{11}\phi_1^{(1)} + b_{21}\phi_1^{(2)}\ \ \ b_{12}\phi_1^{(1)} + b_{22}\phi_1^{(2)}\right] = \\ = \begin{bmatrix} \dfrac{(2+\sqrt{2})b_{11} + \sqrt{2}b_{21}}{2\sqrt{2+\sqrt{2}}} & \dfrac{(2+\sqrt{2})b_{12} + \sqrt{2}b_{22}}{2\sqrt{2+\sqrt{2}}} \end{bmatrix} \tag{68}$$

$$B_2^{"} = \left[b_2^1\ b_2^2\right] = \left[< b^1, \phi_2 >_X\ \ < b^2, \phi_2 >_X\right] = \left[b_{11}\phi_2^{(1)} + b_{21}\phi_2^{(2)}\ \ \ b_{12}\phi_2^{(1)} + b_{22}\phi_2^{(2)}\right] = \\ = \begin{bmatrix} \dfrac{(1-\sqrt{2})b_{11} + b_{21}}{\sqrt{4-2\sqrt{2}}} & \dfrac{(1-\sqrt{2})b_{12} + b_{22}}{\sqrt{4-2\sqrt{2}}} \end{bmatrix} \tag{69}$$

In this case the condition 3 has form:

$$rank[B_1^-] = 1 \wedge rank[B_2^-] = 1 \tag{70}$$

Using the equations (68), (69) the equation (70) receives form:

$$\left[b_{21} \neq -(1+\sqrt{2})b_{11} \vee b_{12} \neq -\frac{\sqrt{2}b_{22}}{2+\sqrt{2}}\right] \wedge \left[b_{21} \neq (-1+\sqrt{2})b_{11} \vee b_{12} \neq \frac{b_{22}}{\sqrt{2}-1}\right] \tag{71}$$

- Condition 4

The eigenvalues are single, so the matrices A'^{+}_{i}, A'^{-}_{i} are degenerated to scalar and such a problem has been investigated in the paper [5] and it yielded the following condition, after adaptation to current symbols:

$$b_1^1 b_1^2 < 0 \wedge b_2^1 b_2^2 < 0 \tag{72}$$

which is equivalent to:

$$\left[\left(1+\sqrt{2}\right)b_{11} + b_{21}\right]\left[\left(1+\sqrt{2}b_{12}\right) + b_{22}\right] < 0 \wedge \left[\left(1-\sqrt{2}\right)b_{11} + b_{21}\right]\left[\left(1-\sqrt{2}\right)b_{12} + b_{22}\right] < 0 \tag{73}$$

Combining conditions (71), (73) we can obtain:

$$\begin{cases} \left[b_{21} \neq -(1+\sqrt{2})b_{11} \vee b_{12} \neq -\frac{\sqrt{2}b_{22}}{2+\sqrt{2}}\right] \wedge \left[b_{21} \neq (-1+\sqrt{2})b_{11} \vee b_{12} \neq \frac{b_{22}}{\sqrt{2}-1}\right] \\ \left[\left(1+\sqrt{2}\right)b_{11} + b_{21}\right]\left[\left(1+\sqrt{2}b_{12}\right) + b_{22}\right] < 0 \wedge \left[\left(1-\sqrt{2}\right)b_{11} + b_{21}\right]\left[\left(1-\sqrt{2}\right)b_{12} + b_{22}\right] < 0 \end{cases} \tag{74}$$

Now let's see how the second order dynamical system (59) looks like in the form of the series of first order dynamical system. To do that at first let us obtain the form of the function $g(\lambda_i)$. The function $g(\lambda_i)$ receives form:

$$g(\lambda_i) = \sqrt{\left(2\lambda_i + 3\lambda_i^{\frac{3}{4}} + \frac{1}{2}\lambda_i^{\frac{1}{2}}\right)^2 - \lambda_i} = \sqrt{-\frac{3}{4}\lambda_i + 3\lambda_i^{\frac{5}{4}} + 11\lambda_i^{\frac{3}{2}} + 12\lambda_i^{\frac{7}{4}} + 4\lambda_i^2} \tag{75}$$

Substituting into equation (75) eigenvalues of our operator A we can achieve:

$$g(\lambda_1) = \sqrt{\frac{263}{4} - \frac{93}{\sqrt{2}} - 22\sqrt{6-4\sqrt{2}} + \left(9-6\sqrt{2}\right)\left(3-2\sqrt{2}\right)^{\frac{1}{4}} + 33\sqrt{3-2\sqrt{2}} + \left(36-24\sqrt{2}\right)\left(3-2\sqrt{2}\right)^{\frac{3}{4}}} \tag{76}$$

$$g(\lambda_2) = \sqrt{\frac{263}{4} + \frac{93}{\sqrt{2}} + 33\sqrt{3+2\sqrt{2}} + 3\left(3+2\sqrt{2}\right)^{\frac{5}{4}} + 12\left(3+2\sqrt{2}\right)^{\frac{7}{4}} + 22\sqrt{6+4\sqrt{2}}} \tag{77}$$

Now let us calculate the matrices A'^{+}_{i}, A'^{-}_{i}. In our example the dynamical system (59) is a second order system, so, i=1,2 and considering that has single eigenvalues they are degenerated to scalar:

$$A'^{\pm}_1 = a_1^{\pm} = -\frac{1}{2}\sqrt{3-2\sqrt{2}} - 3\left(3-2\sqrt{2}\right)^{\frac{3}{4}} - 2\left(3-2\sqrt{2}\right) \pm g(\lambda_1) \tag{78}$$

$$A_2^{'\pm} = a_2^{\pm} = -\frac{1}{2}\sqrt{3+2\sqrt{2}-3\left(3+2\sqrt{2}\right)^{\frac{3}{4}}-2\left(3+2\sqrt{2}\right)} \pm g(\lambda_2) \tag{79}$$

Existing in the above formulas (78), (79) the terms $g(\lambda_1)$, $g(\lambda_2)$ are given explicitly by the equalities (76), (77). The terms appearing in the form (29) matrix $B_i'^{+}$ and $B_i'^{-}$ are equal to matrix B_i'' with accuracy to the $\pm g^{-1}(\lambda_i)$ term. These term $g^{-1}(\lambda_1)$ and $g^{-1}(\lambda_2)$ we can obtain from the equation:

$$g(\lambda_i) = \sqrt{-\frac{3}{4}g^{-1}(\lambda_i) + 3g^{-\frac{5}{4}}(\lambda_i) + 11g^{-\frac{3}{2}}(\lambda_i) + 12g^{-\frac{7}{4}}(\lambda_i) + 4g^{-2}(\lambda_i)} \quad i = 1,2 \tag{80}$$

It is feasible to give the solutions of both the above equations (80) explicitly, but they have very sophisticated form and will not be presented because it does not concerns the essence of presented example.

Summary of the Example 1

- The dynamical system (59) with constrained controls (63) is *U*-controllable if and only if

$$\left(b_{21} \neq -(1+\sqrt{2})b_{11} \vee b_{12} \neq -\frac{\sqrt{2}b_{22}}{2+\sqrt{2}}\right) \wedge \left(b_{12} \neq \frac{b_{22}}{\sqrt{2}-1}\right) \wedge \left(b_{11}b_{21} < 0\right) \wedge \left(b_{21}b_{22} < 0\right) \tag{81}$$

- The dynamical system (59) can be represented in equivalent form of four first order ordinary differential equations:

$$\begin{cases} \dfrac{d\varsigma_1^+(t)}{dt} = a_1^+\varsigma_1^+(t) + b_1^1u_1(t) + b_1^2u_2(t) \\ \dfrac{d\varsigma_1^-(t)}{dt} = a_1^-\varsigma_1^-(t) - b_1^1u_1(t) - b_1^2u_2(t) \\ \dfrac{d\varsigma_2^+(t)}{dt} = a_2^+\varsigma_2^+(t) + b_2^1u_1(t) + b_2^2u_2(t) \\ \dfrac{d\varsigma_2^-(t)}{dt} = a_2^-\varsigma_2^-(t) - b_2^1u_1(t) - b_2^2u_2(t) \end{cases} \tag{82}$$

where:

$$\left[b_1^1\ b_1^2\right] = g^{-1}(\lambda_1)\left[\frac{(2+\sqrt{2})b_{11} + \sqrt{2}b_{21}}{2\sqrt{2+\sqrt{2}}} \quad \frac{(2+\sqrt{2})b_{12} + \sqrt{2}b_{22}}{2\sqrt{2+\sqrt{2}}}\right] \tag{83}$$

$$\left[b_2^1\ b_2^2\right] = g^{-1}(\lambda_2)\left[\frac{(1-\sqrt{2})b_{11} + b_{21}}{\sqrt{4-2\sqrt{2}}} \quad \frac{(1-\sqrt{2})b_{12} + b_{22}}{\sqrt{4-2\sqrt{2}}}\right] \tag{84}$$

Where a_i^+ and a_i^- (i=1,2) are given by the equations (78)-(80).

6.2. Example 2: Three-dimensional system with multiple eigenvalues

Now let us consider another dynamical system with damping term:

$$\frac{d^2x(t)}{dt^2}+(2A+8A^{\frac{4}{5}}+6A^{\frac{3}{4}})\frac{dx(t)}{dt}+Ax(t)=Bu(t) \tag{85}$$

Now the state space X is R^3. In this case let us consider the following matrix as operator A:

$$A=\begin{bmatrix} 5 & -\frac{5}{2}i & -\frac{23}{6\sqrt{5}} \\ -\frac{5}{2}i & 10 & -\frac{61}{3\sqrt{5}}i \\ -\frac{23}{6\sqrt{5}} & -\frac{61}{3\sqrt{5}}i & -8 \end{bmatrix},\quad A:R^3\to R^3 \tag{86}$$

Operator B is as follows:

$$B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},\quad B:R^3\to R^3 \tag{87}$$

The control vector:

$$u(t)=[u_1(t)\quad u_2(t)\quad u_3(t)]^T \tag{88}$$

The controls are also non-negative:

$$u_1(t)\ge 0,\ u_2(t)\ge 0,\ u_3(t)\ge 0 \tag{89}$$

Problem Statement

The problem statement remains the same like in the example 1, that is the verification of the U-controllability of given dynamical system (85) and transforming him into the form of the first order systems' finite series.

Solve

It's easy to check that the operator A is linear, self-adjoined and positive-definite. Moreover, after defining the following coefficients:

$$\alpha_0=1,\ \alpha_1=4,\ \alpha_2=3,\ \beta_1=\frac{4}{5},\ \beta_2=\frac{3}{4} \tag{90}$$

the equation (85) receives form of the dynamical system (1), and are fulfilled all the assumptions of the theorem 3, like in the example 1. Now let's check in sequence conditions of the theorem 3.

- The conditions 1 and 2 remains similar to these in example 1 and obviously are also fulfilled.
- Condition 3

The operator A has three eigenvalues:

$$\lambda_1 = \lambda_{11} = \lambda_{12} = 2, \quad \lambda_2 = 3 \tag{91}$$

And now it's eigenvectors has form:

$$\phi_{11} = \begin{bmatrix} -\frac{\sqrt{2}}{7} \\ \frac{\sqrt{2}}{2} i \\ \frac{3}{7}\sqrt{\frac{5}{2}} \end{bmatrix} \quad \phi_{12} = \begin{bmatrix} \frac{\sqrt{2}(\sqrt{34740931} - 2925)}{41503} \\ \frac{\sqrt{34740931} - 90}{5929\sqrt{2}} i \\ \frac{3(358\sqrt{10} + \sqrt{347409310}}{83006} \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -16\sqrt{\frac{2}{4579}} \\ \frac{67}{\sqrt{9158}} i \\ 27\sqrt{\frac{5}{9158}} \end{bmatrix} \tag{92}$$

The martrices B_1'' and B_2'' has form:

$$B_1^{"} = \begin{bmatrix} b_{11}^1 & b_{11}^2 & b_{11}^3 \\ b_{12}^1 & b_{12}^2 & b_{12}^3 \end{bmatrix} = \begin{bmatrix} <b^1, \phi_{11}>_X & <b^2, \phi_{11}>_X & <b^3, \phi_{11}>_X \\ <b^1, \phi_{12}>_X & <b^2, \phi_{12}>_X & <b^3, \phi_{12}>_X \end{bmatrix} =$$

$$= \begin{bmatrix} b_{11}\phi_{11}^{(1)} + b_{21}\phi_{11}^{(2)} + b_{31}\phi_{11}^{(3)} & b_{12}\phi_{11}^{(1)} + b_{22}\phi_{11}^{(2)} + b_{32}\phi_{11}^{(3)} & b_{13}\phi_{11}^{(1)} + b_{23}\phi_{11}^{(2)} + b_{33}\phi_{11}^{(3)} \\ b_{11}\phi_{12}^{(1)} + b_{21}\phi_{12}^{(2)} + b_{31}\phi_{12}^{(3)} & b_{12}\phi_{12}^{(1)} + b_{22}\phi_{12}^{(2)} + b_{32}\phi_{12}^{(3)} & b_{13}\phi_{12}^{(1)} + b_{23}\phi_{12}^{(2)} + b_{33}\phi_{12}^{(3)} \end{bmatrix} =$$

$$= \begin{bmatrix} q_1(b_{11}, b_{21}, b_{31}) & q_1(b_{12}, b_{22}, b_{32}) & q_1(b_{13}, b_{23}, b_{33}) \\ q_2(b_{11}, b_{21}, b_{31}) & q_2(b_{12}, b_{22}, b_{32}) & q_2(b_{13}, b_{23}, b_{33}) \end{bmatrix} \tag{93}$$

where the functions $q_1(x,y,z)$ and $q_2(x,y,z)$ are defined as follows:

$$q_1(x,y,z) = \frac{-2x + 3\sqrt{5}z}{7\sqrt{2}} + \frac{y}{\sqrt{2}} i \tag{94}$$

$$q_2(x,y,z) = \frac{2(\sqrt{34740931} - 2925)x - 3\sqrt{5}(358 + \sqrt{34740931})z}{41503\sqrt{2}} - \frac{7(\sqrt{34740931} - 90)y}{41503\sqrt{2}} i \tag{95}$$

$$B_2^{"} = \begin{bmatrix} b_2^1 & b_2^2 & b_2^3 \end{bmatrix} = \begin{bmatrix} <b^1, \phi_2>_X & <b^2, \phi_2>_X & <b^3, \phi_2>_X \end{bmatrix} =$$

$$= \begin{bmatrix} b_{11}\phi_2^{(1)} + b_{21}\phi_2^{(2)} + b_{31}\phi_2^{(3)} & b_{12}\phi_2^{(1)} + b_{22}\phi_2^{(2)} + b_{32}\phi_2^{(3)} & b_{13}\phi_2^{(1)} + b_{23}\phi_2^{(2)} + b_{33}\phi_2^{(3)} \end{bmatrix} =$$

$$= \begin{bmatrix} q_3(b_{11}, b_{21}, b_{31}) & q_3(b_{12}, b_{22}, b_{32}) & q_3(b_{13}, b_{23}, b_{33}) \end{bmatrix} \tag{96}$$

where:

$$q_3(x,y,z) = \frac{-32x + 27\sqrt{5}z}{\sqrt{9158}} + \frac{67y}{\sqrt{9158}} i \tag{97}$$

In this case the condition 3 has form:

$$rank[B_1^{"}] = 2 \wedge rank[B_2^{"}] = 1 \tag{98}$$

Fulfilling this condition depends on particular values of the elements of the operator B.

- Condition 4

Verification of this condition to the 2[nd] eigenvalue, equal to 3, remains the same like in the example 1 and yields the following condition:

$$\underset{\substack{q,r\in\{1,2,3\}\\ q\neq r}}{\exists} b_2^q b_2^r < 0 \tag{99}$$

where the numbers b_2^q, $q=1,2,3$ are the proper elements of the matrix B_2'' defined in the equation (96). Substituting them into equation (99) we can receive:

$$\underset{\substack{q,r\in\{1,2,3\}\\ q\neq r}}{\exists} \left(27\sqrt{5}b_{3q} - 31b_{1q}\right)\left(27\sqrt{5}b_{3r} - 31b_{1r}\right) < 0 \tag{100}$$

Considered condition 4 is more sophisticated to the double eigenvalue of the operator A $\lambda_1 = 2$. In this case the eigenvector v_2 has two elements and as v_2 could be taken the following non-zero and linearly independent two vectors:

$$v_{11} = \begin{bmatrix} v_{11}^{(1)} \\ v_{11}^{(2)} \end{bmatrix} \quad v_{12} = \begin{bmatrix} v_{12}^{(1)} \\ v_{12}^{(2)} \end{bmatrix} \tag{101}$$

They are said to be non-zero and linearly independent, so their elements have to fulfil the following inequalities:

$$\left[v_{11}^{(1)}\right]^2 + \left[v_{11}^{(2)}\right]^2 \neq 0 \wedge \left[v_{12}^{(1)}\right]^2 + \left[v_{12}^{(2)}\right]^2 \neq 0 \wedge v_{11}^{(1)}v_{12}^{(2)} - v_{11}^{(2)}v_{12}^{(1)} \neq 0 \tag{102}$$

Let us calculate appearing in this condition 4 the matrix $B_1'^{+}w$:

$$B_1'^{+}w = g^{-1}(\lambda_1)\begin{bmatrix} b_{11}^1 u_1 + b_{11}^2 u_2 + b_{11}^3 u_3 \\ b_{12}^1 u_1 + b_{12}^2 u_2 + b_{12}^3 u_3 \end{bmatrix} \tag{103}$$

where the numbers b_{11}^i, b_{12}^i $i=1,2,3$ are the proper elements of the matrix B_1'' defined in the equation (93). Already we can calculate the terms $v_{11}^T B_1'^{+}w$ and $v_{12}^T B_1'^{+}w$:

$$v_{11}^T B_1'^{+}w = g^{-1}(\lambda_1)\left(v_{11}^{(1)}b_{11}^1u_1 + v_{11}^{(1)}b_{11}^2u_2 + v_{11}^{(1)}b_{11}^3u_3 + v_{11}^{(2)}b_{12}^1u_1 + v_{11}^{(2)}b_{12}^2u_2 + v_{11}^{(2)}b_{12}^3u_3\right) \tag{104}$$

$$v_{12}^T B_1'^{+}w = g^{-1}(\lambda_1)\left(v_{12}^{(1)}b_{11}^1u_1 + v_{12}^{(1)}b_{11}^2u_2 + v_{12}^{(1)}b_{11}^3u_3 + v_{12}^{(2)}b_{12}^1u_1 + v_{12}^{(2)}b_{12}^2u_2 + v_{12}^{(2)}b_{12}^3u_3\right) \tag{105}$$

Now let us perform the analysis under what circumstances is fulfilled the condition 4 of the theorem 2. From the conditions (102) yields that every element of both the vectors v_{11}, v_{12} can be chosen by arbitrary sign. This condition 4 is equivalent to the requirement so that for any admitable eigenvectors v_{11}, v_{12} both the the equations (104), (105) receives values of the opposite sign in the constrained control space U. Taking into account the fact that in considering example the controls are constrained to non-negative values, the change of the sign of the terms $v_{11}^T B_1'^{+}w$ and $v_{12}^T B_1'^{+}w$ will occur if and only if in every of the formulas (104), (105) exist two coefficients of the opposite sign in the terms in the controls' linear combinations in that formulas:

$$\underset{\substack{v_{11},v_{12}\\ v_{11}\neq\bar{0},\, v_{12}\neq\bar{0},\, v_{11}\neq\alpha v_{12},\,\alpha\in r}}{\forall}\ \underset{q=1,2}{\forall}\ \underset{r_1,r_2\in\{1,2\}}{\exists}\ \underset{p_1,p_2\in\{1,2,3\}}{\exists}(r_1,p_1)\neq(r_2,p_2)\wedge\left(v_{1q}^{(r_1)}\operatorname{Re}[b_{1r_1}^{p_1}]\right)\left(v_{1q}^{(r_2)}\operatorname{Re}[b_{1r_2}^{p_2}]\right)<0 \quad (106)$$

Considering the particular form of the controls' linear combinations in the formulas (104), (105) and mentioned earlier fact that every element of both the vectors v_{11}, v_{12} can be chosen by arbitrary sign, the condition (106) receives form:

$$\underset{r\in\{1,2\}}{\exists}\ \underset{\substack{p_1,p_2\in\{1,2,3\}\\ p_1\neq p_2}}{\exists}\operatorname{Re}[b_{1r}^{p_1}]\operatorname{Re}[b_{1r}^{p_2}]<0 \quad (107)$$

And considering the form of the matrix $B_1{}'^{+}$ the condition (107) can be expressed directly by the function of the elements of considered dynamical system's (87) operator B as follows:

$$\underset{\substack{q,r\in\{1,2,3\}\\ q\neq r}}{\exists}\left(3\sqrt{5}b_{3q}-2b_{1q}\right)\left(3\sqrt{5}b_{3r}-2b_{1r}\right)<0\vee\left(d_1b_{3q}-d_2b_{1q}\right)\left(d_1b_{3r}-d_2b_{1r}\right)<0 \quad (108)$$

where d_1 and d_2 are constant coefficients defined as follows:

$$d_1=2(\sqrt{34740931}-2925)\quad d_2=3\sqrt{5}(358+\sqrt{34740931}) \quad (109)$$

At the end let's transform considered in this example second order dynamical system (85) to the form form of the series of first order dynamical systems. In this example the function $g(\lambda_i)$ receives form:

$$g(\lambda_i)=\sqrt{\left(\lambda_i+4\lambda_i^{\frac{4}{5}}+3\lambda_i^{\frac{3}{4}}\right)^2-\lambda_i}=$$

$$=\sqrt{-\lambda_i+9\lambda_i^{\frac{3}{2}}+24\lambda_i^{\frac{31}{20}}+16\lambda_i^{\frac{8}{5}}+6\lambda_i^{\frac{7}{4}}+8\lambda_i^{\frac{9}{5}}+\lambda_i^2} \quad (110)$$

This function taken in the points equal to eigenvalues proper to the operator A considering in this example receives the following values:

$$g(\lambda_{11})=g(\lambda_{12})=\sqrt{2+18\sqrt{2}+48\cdot2^{\frac{11}{20}}+32\cdot2^{\frac{3}{5}}+12\cdot2^{\frac{3}{4}}+16\cdot2^{\frac{4}{5}}} \quad (111)$$

$$g(\lambda_2)=\sqrt{6+27\sqrt{3}+72\cdot3^{\frac{11}{20}}+48\cdot3^{\frac{3}{5}}+18\cdot3^{\frac{3}{4}}+24\cdot3^{\frac{4}{5}}} \quad (112)$$

Now similarly like in the example 1 let us calculate the matrices A'^{+}_i, A'^{-}_i. Following this aim the eigenvalues $s_i^{\pm}$ i=1,2 (25) will be necessary:

$$s_1^{\pm}=-2-3\cdot2^{\frac{3}{4}}-4\cdot2^{\frac{4}{5}}\pm g(\lambda_1)=$$

$$=-2-3\cdot2^{\frac{3}{4}}-4\cdot2^{\frac{4}{5}}\pm\sqrt{2+18\sqrt{2}+48\cdot2^{\frac{11}{20}}+32\cdot2^{\frac{3}{5}}+12\cdot2^{\frac{3}{4}}+16\cdot2^{\frac{4}{5}}} \quad (113)$$

$$s_2^{\pm}=-3-3\cdot3^{\frac{3}{4}}-4\cdot3^{\frac{4}{5}}\pm g(\lambda_2)=$$

$$=-3-3\cdot 3^{\frac{3}{4}}-4\cdot 3^{\frac{4}{5}}\pm\sqrt{6+27\sqrt{3}+72\cdot 3^{\frac{11}{20}}+48\cdot 3^{\frac{3}{5}}+18\cdot 3^{\frac{3}{4}}+24\cdot 3^{\frac{4}{5}}} \tag{114}$$

Involving above two equalities (113), (114) the matrices A'^{+}_{i} and A'^{-}_{i} can be expressed as follows:

$$A'^{\pm}_{1}=diag[s_1^{\pm}\quad s_1^{\pm}] \tag{115}$$

$$A'^{\pm}_{2}=\lfloor s_2^{\pm}\rfloor \tag{116}$$

The values $g^{-1}(\lambda_1)$ and $g^{-1}(\lambda_2)$ we can obtain from the following equation:

$$g(\lambda_i)=\sqrt{-g^{-1}(\lambda_i)+9g^{-\frac{3}{2}}(\lambda_i)+24g^{-\frac{31}{20}}(\lambda_i)+16g^{-\frac{8}{5}}(\lambda_i)+6g^{-\frac{7}{4}}(\lambda_i)+8g^{-\frac{9}{5}}(\lambda_i)+g^{-2}(\lambda_i)}$$

$$i=1,2 \tag{117}$$

Summary of the Example 2

- The dynamical system (85) with constrained controls (89) is *U*-controllable if and only if the following conditions are fulfilled simultaneously:

$$rank[B_1^{''}]=2\wedge rank[B_2^{''}]=1 \tag{118}$$

$$\underset{\substack{q,r\in\{1,2,3\}\\ q\neq r}}{\exists}\left(27\sqrt{5}b_{3q}-31b_{1q}\right)\left(27\sqrt{5}b_{3r}-31b_{1r}\right)<0 \tag{119}$$

$$\underset{\substack{q,r\in\{1,2,3\}\\ q\neq r}}{\exists}\left(3\sqrt{5}b_{3q}-2b_{1q}\right)\left(3\sqrt{5}b_{3r}-2b_{1r}\right)<0\vee\left(d_1b_{3q}-d_2b_{1q}\right)\left(d_1b_{3r}-d_2b_{1r}\right)<0 \tag{120}$$

$$d_1=2(\sqrt{34740931}-2925)\quad d_2=3\sqrt{5}(358+\sqrt{34740931}) \tag{121}$$

where matrices B_1'' and B_2'' are given by the formulas (93), (96), b_{ij} are the elements of the operator B (87).

- The dynamical system (85) can be represented in equivalent form of four first order linear ordinary differential equations:

$$\begin{cases}\dfrac{d\varsigma_1^{+}(t)}{dt}=A'^{+}_{1}\varsigma_1^{+}(t)+B'^{+}_{1}u(t)\\ \dfrac{d\varsigma_1^{-}(t)}{dt}=A'^{-}_{1}\varsigma_1^{-}(t)+B'^{-}_{1}u(t)\\ \dfrac{d\varsigma_2^{+}(t)}{dt}=A'^{+}_{2}\varsigma_2^{+}(t)+B'^{+}_{2}u(t)\\ \dfrac{d\varsigma_2^{-}(t)}{dt}=A'^{-}_{2}\varsigma_2^{-}(t)+B'^{-}_{2}u(t)\end{cases} \tag{122}$$

where:

-the matrices A'^{+}_{i} and A'^{-}_{i} are given by the formulas (115), (116)

-the matrices B'^{+}_{i} and B'^{-}_{i} are as follows:

$$\begin{cases} B_i^{'+} = g^{-1}(\lambda_i)B_i^{*} \\ B_i^{'-} = -B_i^{'+} \end{cases} \quad i = 1,2 \tag{123}$$

6.3. Example 3: System with distributed parameters

In this point will be verified the *U*-controllability with non-negative controls of a dynamical system with distributed parameters given by the following linear partial differential state equation:

$$\frac{\partial^2 x(z,t)}{\partial t^2} - 4\frac{\partial^3 x(z,t)}{\partial z^2 \partial t} - \frac{\partial^2 x(z,t)}{\partial z^2} = \sum_{k=1}^{p} b^k(z)h_k(t) \tag{124}$$

where:

$$z \in (0,1), \ t > 0 \tag{125}$$

the number of the control forces is greater than 1:

$$p \geq 2 \tag{126}$$

with boundary conditions:

$$x(0,t) = 0, \ t > 0 \qquad \frac{\partial x(1,t)}{\partial z} = 0, \ t > 0 \tag{127}$$

Solve

In the analysis of the controllability of given dynamical system (124) will be necessary its representation by the form of the abstract differential equation. Following this aim at first let us define linear unbounded differential operator "*A*" $A: D(A) \subset H \to H$ in the following way:

$$Ax(z) = -\frac{\partial^2 x(z)}{\partial z^2}, \ x \in D(A) \tag{128}$$

Domain of the operator A :

$$D(A) = \left\{ x(z) \in H^2(0,1) : \int_0^1 x^2(z)dz < \infty, \ x(0) = 0, \ \frac{dx}{dz}(1) = 0 \right\} \tag{129}$$

where $H = L^2(0,1)$ is a Hilbert space of functions integrated with square. Also it can be shown that respectively the eigenvalues λ_i and eigenfunctions $\phi_i(z)$ of the operator A have form:

$$\lambda_i = \left(\frac{\pi}{2} + i\pi\right)^2 \qquad \phi_i(z) = C\cos\left[\left(\frac{\pi}{2} + i\pi\right)z\right] \qquad i = 1,2,3,... \tag{130}$$

Presented properties of the operator A are sufficient for representation of the partial differential equation (124) in form of the linear abstract ordinary differential equation in the Hilbert space H:

$$\frac{d^2x(t)}{dt^2}+f(A)\frac{dx(t)}{dt}+Ax(t)=Bh(t),\ t>0 \tag{131}$$

where:

$$\frac{d^2x(t)}{dt^2},\frac{dx(t)}{dt},x(t)\in H \tag{132}$$

$$f(A)=\alpha_0 A,\quad \alpha_0=2 \tag{133}$$

the function $g(\lambda_i):R\to C$ (19) necessary in the used transformation of the equation (124) to abstract ordinary differential equation (1) in this example has form:

$$g(\lambda_i)=\sqrt{4\lambda_i^2-\lambda_i}\quad i=1,2,3,... \tag{134}$$

and the operator B is defined as follows:

$$B=\left[b^1\ \ b^2\ \ ...\ \ b^k\ \ ...\ \ b^p\right],\ b_j\in H,\ j=1,2,...,p \tag{135}$$

and controls:

$$h(t)=\left[h_1(t),\ h_2(t),...,h_p(t)\right]^T\in R^p \tag{136}$$

where $h_k(t)$ $k=1,2,...,p$ denote scalar controls.

The operator A is linear and has only real positive eigenvalues, so is self-adjoined and positive-defined. The dumping term f(A) fulfils the assumptions (3) so the equation (124) has form of the dynamical system (1) and in the investigating of the U-controllability can be used the theorem 3. Now let us verify the conditions of the theorem 3. The controls are nonnegative so the conditions 1 and 2 are fulfilled like in previous examples.

- Condition 3

The operator A has only single eigenvalues, so the condition 3 of the theorem 3, considering the number of the control forces, receives in the example the following form:

$$rank\left[b_i^1 ... b_i^k ... b_i^p\right]=1\quad i=1,2,3,... \tag{137}$$

As we can see the controllability matrix in this example is reduced to vector so the controllability condition can be rewritten in more compact form:

$$\underset{i=1,2,3,...}{\forall}\ \underset{1\le k\le p}{\exists}\ b_i^k\neq 0 \tag{138}$$

Now let us calculate the controllability matrix's element b_i^k, considering the proper form of the scalar product in the Hilbert space H:

$$b_i^k=\langle b^k,\phi_i\rangle_X=\int_0^1 b^k(z)\phi_i(z)dz=C\int_0^1 b^k(z)\cos\left[\left(\frac{\pi}{2}+i\pi\right)z\right]dz\quad i=1,2,3,... \tag{139}$$

- Condition 4

In this case at first we will calculate the $B_i^{*+}w$ factor as well:

$$B_i^{*+}w = g^{-1}(\lambda_i)\left[b_i^1 \dots b_i^k \dots b_i^p\right]\left[h_1(t),\ h_2(t),\dots,h_p(t)\right]^T \quad i=1,2,3,\dots \tag{140}$$

Considering that $g^{-1}(\lambda_i) \neq 0$ and h_i is non-negative the condition 4 of the theorem 3 reduces to requirement so that in the vector $\left[b_i^1 \dots b_i^k \dots b_i^p\right]$ $i=1,2,3,\dots$ exist elements of both signs:

$$\underset{i=1,2,3,\dots}{\forall}\ \underset{q,r\in(1,2,\dots,p),q\neq r}{\exists}\ b_i^q b_i^r < 0 \tag{141}$$

The condition (141) is valid for any control operator B. Now let us check how do they look like for given forces. Let us assume the control operator B of the following form:

$$b^k(z) = C_k e^{kz} \quad k=1,2,\dots,p \quad C_k \in R \tag{142}$$

After integrating the element b_i^k (139) receives in this case form:

$$b_i^k = 2C_k \frac{e^k\pi(-1)^i(1+2i)-2k}{4k^2+(\pi+2i\pi)^2} \quad i=1,2,3,\dots \tag{143}$$

For odd i coefficients the numerator of the above formula (143), with the accuracy to constant factor, is equal:

$$-\left(e^k\pi(1+2i)+2k\right) \quad i=1,2,3,\dots \tag{144}$$

and is obviously not equal 0. For even i coefficients the same term is equal:

$$e^k\pi(1+2i)-2k \quad i=1,2,3,\dots \tag{145}$$

Let us construct the following estimation:

$$\underset{i=1,2,3,\dots}{\forall}\ e^k\pi(1+2i) > 3e^k \quad i=1,2,3,\dots \tag{146}$$

Next let us expand the exponential function into the Taylor's series:

$$3e^k = 3\sum_{n=0}^{\infty}\frac{k^n}{n!} > 3k > 2k \tag{147}$$

Combining the inequalities (146), (147) we can state that:

$$\underset{i=1,2,3,\dots}{\forall}\ e^k\pi(1+2i)-2k > 0(\neq 0) \tag{148}$$

And so:

$$\underset{i=1,2,3,\dots}{\forall}\ b_i^k \neq 0 \Leftrightarrow C_k \neq 0 \tag{149}$$

So the statement (149) formulates necessary and sufficient condition of fulfilling condition 3 of the theorem 3 for investigated dynamical system (124). Now let us verify the condition 4 of this theorem in case of the control forces given explicitly by the equality (142). Following this aim let us calculate the $b_i^q b_i^r$ term:

$$b_i^q b_i^r = 4C_q C_r \frac{e^q\pi(-1)^i(1+2i)-2q}{4q^2+(\pi+2i\pi)^2}\cdot\frac{e^r\pi(-1)^i(1+2i)-2r}{4r^2+(\pi+2i\pi)^2} \quad i=1,2,3,... \tag{150}$$

From the inequality (149) follows that the term:

$$4\frac{e^q\pi(-1)^i(1+2i)-2q}{4q^2+(\pi+2i\pi)^2}\cdot\frac{e^r\pi(-1)^i(1+2i)-2r}{4r^2+(\pi+2i\pi)^2} \quad i=1,2,3,... \tag{151}$$

holds fixed sign for every i=1,2,3,... and every $q,r\in\{1,2,...,p\}$. So the condition (4) receives form:

$$\exists_{q,r\in\{1,2,...,p\},q\neq r} \quad C_q C_r < 0 \tag{152}$$

Received last condition (152) is stronger than (149) and so becomes the necessary and sufficient condition of the *U*-controllability of investigated dynamical system (124) with non-negative controls.

Now let's transform considered dynamical system with distributed parameters (120) to the form of the series of first order equations. In this example the state space is infinite dimensional so the series will be infinite as well. Necessary in this representation the state matrices A'^{+}_i and A'^{-}_i requires the values $g(\lambda_i)$:

$$g(\lambda_i)=\frac{1}{2}\pi(1+2i)\sqrt{\pi^2(1+2i)^2-1} \quad i=1,2,3,... \tag{153}$$

Calculating further:

$$s_i^{\pm}=-2\left(\frac{\pi}{2}+i\pi\right)^2 \pm\frac{1}{2}\pi(1+2i)\sqrt{\pi^2(1+2i)^2-1}=$$

$$=-\frac{1}{2}\pi(1+2i)\left[\pi(1+2i)\mp\sqrt{\pi^2(1+2i)^2-1}\right] \quad i=1,2,3,.. \tag{154}$$

Summary of the Example 3

-The dynamical system (124) with non-negative controls is *U*-controllable with control forces (142) if and only if there exist two forces of the opposite sign, what is equivalent to:

$$\exists_{q,r\in\{1,2,...,p\},q\neq r} \quad C_q C_r < 0 \tag{155}$$

- The dynamical system (124) can be represented in equivalent form of the infinite series of first order linear ordinary differential equations:

$$\begin{cases}\dfrac{d\varsigma_i^+(t)}{dt}=s_i^+\varsigma_i^+(t)+B_i'^+h(t)\\ \dfrac{d\varsigma_i^-(t)}{dt}=s_i^-\varsigma_i^-(t)-B_i'^+h(t)\end{cases} \quad i=1,2,3,... \tag{156}$$

where the matrix B'^{+}_i is defined by formula:

$$B_i^{*+} = g^{-1}(\lambda_i)\left[b_i^1 \dots b_i^k \dots b_i^p\right] \quad i = 1,2,3,\dots \tag{157}$$

Summary

Presented in this paper methodology of the verification of the *U*-controllability of second order dynamical dynamical systems with damping term can be applied for a broad class of the physical systems that can be expressed in the form (1). For instance state equations of such a form have mechanical systems containing of elastic beams with internal friction.

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Streszczenie

W artykule przedstawiono metodykę badania sterowalności nieskończenie wymiarowych układów dynamicznych rzędu drugiego z czynnikiem tłumiącym. Do tego celu wykorzystana została spektralna teoria liniowych operatorów nieograniczonych.

W pierwszej części pracy sformułowano problem i przypomniano niezbędne własności występujących w problemie operatorów. Następnie przedstawiono znaną metodykę sprowadzenia rozpatrywanego układu drugiego rzędu do układu równań pierwszego rzędu, a także niezbędne w dalszej części pracy własności użytych do tego celu operatorów. W kolejnym punkcie sformułowano i udowodniono twierdzenie o sprowadzeniu wyjściowego układu nieskończenie wymiarowego do dwóch nieskończonych ciągów układów skończenie wymiarowych. Dodatkowo przypomniano twierdzenie dotyczące warunków sterowalności z ograniczeniami układów skończenie wymiarowych. Na jego podstawie (i z wykorzystaniem udowodnionego wcześniej twierdzenia o sprowadzeniu wyjściowego układu nieśkończenie wymiarowego do dwóch nieskończonych ciągów układów skończenie wymiarowych) sformułowano i udowodniono twierdzenie podające warunki konieczne i wystarczające aproksymacyjnej sterowalności z ograniczeniami rozpatrywanego układu drugiego rzędu z czynnikiem tłumiącym. Uzyskany rezultat, ze względu na własności użytych w pracy operatorów, redukuje się do nieskończonego ciągu, którego każdy wyraz składa się z czterech warunków.