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## APPROXIMATE ANALYTICAL SOLUTION OF HELMHOLTZ'S EQUATION IN AN INHOMOGENEOUS CONDUCTING REGION USING HYBRID SYNTHESIS METHOD

**Summary.** The paper presents the analysis of the electromagnetic field in a basic model of an induction motor. The model contains an inhomogeneous conducting region. Field distributions are obtained using a hybrid synthesis method. In order to take into account field inhomogeneous problems the method employs special integral approach. It is based on analytical formulae describing the eddy current distribution in an equivalent homogenous region. Basic features of the method are discussed. Attention is focused on the error estimation. Comparison with the results obtained when using FEM is provided.

## PRZYBŁIŻONE ROZWIĄZANIE ANALITYCZNE RÓWNANIA HELMHOLTZA W NIEJEDNORODNYM OBSZARZE PRZEWODZĄCYM WYKORZYSTUJĄCE HYBRYDOWĄ METODĘ SYNTEZY POLA ELEKTROMAGNETYCZNEGO

**Streszczenie.** Artykuł przedstawia analizę pola elektromagnetycznego w uproszczonym modelu silnika indukcyjnego. Model ten zawiera niejednorodną warstwę przewodzącą. Rozkład pola wyznaczono wykorzystując hybrydową metodę syntezy. W metodzie tej wykorzystano specjalne podejście całkowe, aby uwzględnić niejednorodność analizowanego obszaru. Jest ono oparte na wyrażenях analitycznych opisujących rozkład prądów wirowych w zastępczym obszarze jednorodnym. Opisano podstawowe własności metody. Uwaga została skupiona na analizie błędu metody. Załączono porównanie uzyskanych wyników z rezultatami otrzymanymi przy wykorzystaniu metody elementów skończonych (MES).

## 1. INTRODUCTION

Purely analytical solutions describing the distribution of the electromagnetic field are restricted to the analysis of homogenous problems [1]-[3]. This also applies to the steady-state analysis of eddy currents described in the presented paper. This limitation can be easily overcome by the use of numerical methods [1]-[3],[5],[6]. However, internal errors encountered in numerical solutions of the field equations cause serious problems in certain applications [1], [2]. One of these applications is the evaluation of the electromagnetic force/torque [5], [6]. It led to the development of various hybrid field methods [4], [5]. They incorporate advantages of both analytical and numerical methods. Different approach is presented in [6] where the comparison between the results obtained with different methods is considered. Proposed hybrid synthesis method of solution of the Helmholtz's equation, in the opinion of the authors, is supplementary to the solution of the Poisson's equation presented by Rogowski ([1], pp.82). It applies both to the model structure and the employment of the Fourier's series analysis in the mathematical formulation of the method. The hybrid synthesis method was employed in the evaluation of parasitic torque in an induction motor [4]. The paper presents results of researches aimed at the estimation of the error introduced by the method.

## 2. DESCRIPTION OF THE MODEL

The inhomogeneous basic model of an induction squirrel cage motor analysed in the paper is shown in Fig.1a. It is a part of an infinitely long region of width  $g$  and length  $2\tau$  ( $\tau$ - pole pitch). This region is bounded by two parallel plane surfaces of infinite permeability ( $\mu=\infty$ ). These are the stator and rotor cores, respectively. In the lower of these, there lies a series of conducting rectangular bars ( $\gamma_c \neq 0$ ,  $\mu=\mu_0$ ) of height  $h$ . The current sheet of the stator (s) is placed on the surface of the stator core [1], [4].

The steady state distribution of the electromagnetic field in the model is described by the Helmholtz's equation. It is formulated in the reference frame fixed in the rotor. Resulting two-dimensional (2D) field equations are formulated in terms of the complex magnetic field vector potential  $\underline{A} = [0, 0, A]$ . For a the  $v$ -th field space harmonic they have the following general form:

$$\nabla^2 \underline{A}(x, y) = p\mu_0\gamma \underline{A}(x, y) \quad (1)$$

where  $A$  is complex magnitude of the vector potential,  $p=j\omega$  is the operator of time differentiation,  $\omega$  is relative angular frequency of the  $v$ -th magnetic field space harmonic in the rotor reference frame,  $\mu_0, \gamma$  are material parameters.

To analyse the field, stator and rotor cores are represented by the appropriate boundary conditions applied on their bounds [1]. The remaining model region is divided into two subregions having different material properties:

- air-gap region (G), ( $\gamma=0, \mu=\mu_0$ ),
- electrically inhomogeneous region (C), ( $\mu=\mu_0$ ), where conductivity is described by the following expression :

$$\gamma(x) = (1 - z(x))\gamma_c \quad (2)$$

where  $z(x)$  is an auxiliary function (Fig.1b),  $\gamma_c$  is the conductivity of the bar.

In this inhomogeneous model analytical solution of the field equations could not be obtained. It is due to the spatial dependence of the conductivity in the region (C) [1]. Therefore, to obtain field distribution for the inhomogeneous model, we have to resort to a numerical approximation.

In order to obtain analytical solution the hybrid synthesis method is applied [4]. The method is based on the analytical solution of Helmholtz's equation in the homogenous conducting region [1], [3].

The proposed approach introduces two model changes, which are crucial to the problem analysis:

- geometry alteration, which affects the conductivity of the conducting region, changing the inhomogeneous region (C) into an equivalent homogeneous one (H),
- insertion of an additional compensating winding in the region (H), [4].

Resulting homogenous machine model is shown in Fig.1c.

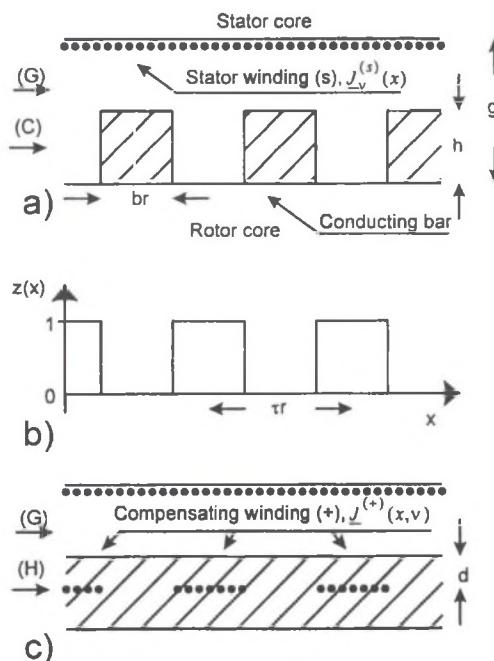


Fig. 1. a) Inhomogeneous machine model, b) auxiliary function  $z(x)$ , c) equivalent homogeneous machine model  
Rys. 1.a) Niejednorodny model maszyny, b) funkcja pomocnicza  $z(x)$ , c) zastępczy jednorodny model maszyny

The homogenous model is excited by two sources:

- $\underline{J}_v^{(s)}(x) e^{j\omega t}$  - the stator winding current sheet of the  $v$ -th space harmonic, being the driving one in the model,

$$\underline{J}_v^{(s)}(x) = \underline{a}_v \cos(v\pi x / \tau) + \underline{b}_v \sin(v\pi x / \tau) \quad (3)$$

where  $\underline{a}_v$ ,  $\underline{b}_v$  are known current coefficients of the two-phase stator winding,

- $\underline{J}^{(+)}(x, v) e^{j\omega t}$  - the compensating winding current sheet, placed on an arbitrary depth  $d$  in region (H),

$$\underline{J}^{(+)}(x, v) = \sum_{\lambda \in \Lambda} \underline{J}_{\lambda}^{(+)}(x, v) \quad (4)$$

where  $\Lambda = \{\lambda_1, \dots, \lambda, \dots, \lambda_N\}$  is a set of  $N$  space harmonics of the compensating winding related to the  $v$ -th stator space harmonic,

$$\underline{J}_{\lambda}^{(+)}(x, v) = \underline{a}_{\lambda}(v) \cos(\lambda\pi x / \tau) + \underline{b}_{\lambda}(v) \sin(\lambda\pi x / \tau) \quad (5)$$

where  $\underline{a}_{\lambda}(v)$ ,  $\underline{b}_{\lambda}(v)$  are **unknown** current coefficients for each  $\lambda$ -th space harmonic of the two phase compensating winding.

The main feature of the compensating winding can be described in the following way: *reduction of the influence of the current flowing in region (H) outside the bar subregion (i.e. for  $z(x)=l$ ) on the overall field distribution.*

The above current sheets constitute set of boundary conditions. The resulting field problems described by (1) are solved separately for:

- the stator  $v$ -th space harmonic,
- each  $\lambda$ -th space harmonic of the compensating winding.

The general solutions can be written in the following form:

- for the air-gap region (G)

$$\underline{A}_n^{(i)} = \sum_n \left( \underline{G}_{1n}^{(i)} \cos(nx) + \underline{G}_{2n}^{(i)} \sin(nx) \right) \left( \underline{G}_{3n}^{(i)} e^{ny} + \underline{G}_{4n}^{(i)} e^{-ny} \right), \quad (6)$$

- for the homogeneous conducting region (H)

$$\underline{A}_n^{(i)} = \sum_n \left( \underline{H}_{1n}^{(i)} \cos(nx) + \underline{H}_{2n}^{(i)} \sin(nx) \right) \left( \underline{H}_{3n}^{(i)} e^{\sigma_n y} + \underline{H}_{4n}^{(i)} e^{-\sigma_n y} \right) \quad (7)$$

where  $\underline{G}_{1n}^{(i)}, \underline{H}_{1n}^{(i)}$ ,  $\dots$  are constants adjusted to fit the particular boundary conditions resulting from the continuity of

- the normal component of the flux density  $\bar{B}$ ,
- the tangential component of the field strength  $\bar{H}$ ,

for stator ( $n=v\pi/\tau$ ,  $i=s$ ) or compensating winding ( $n=\lambda\pi/\tau$ ,  $i=+$ ) respectively,  $\sigma_n = \sqrt{\mu_0 \gamma + n^2}$ .

In order to evaluate  $\underline{J}^{(+)}(x, v)$ , the following constraint equation is formulated [4]:

$$\underline{J}^{(+)}(x, v) + z(x) \int_g^g j(x, y) dy = 0. \quad (8)$$

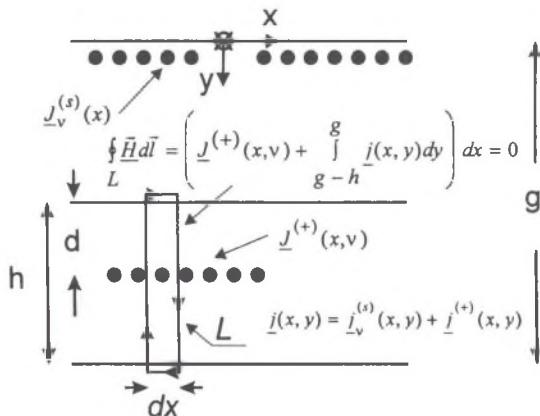


Fig.2. Simplified graphical interpretation of the constraint equation (8)

Rys.2. Uproszczona graficzna interpretacja równania więzów (8)

It results from the implementation of the Ampere's law to eddy currents  $j(x,y)$  induced in the homogenous conducting region  $H$  by both windings (Fig.2):

$$\underline{j}(x, y) = \underline{j}_v^{(s)}(x, y) + \underline{j}_+^{(+)}(x, y), \quad (9)$$

$$\underline{j}_v^{(s)}(x, y) = -p\gamma \frac{\underline{A}_v^{(s)}}{c}(x, y), \quad (10)$$

$$\underline{j}_+^{(+)}(x, y) = -p\gamma \sum_{\lambda} \underline{A}_{\lambda}^{(+)}(x, y) = \sum_{\lambda} \underline{j}_{\lambda}^{(+)}(x, y). \quad (11)$$

Application of the Fourier's analysis to (8) leads to the algebraic system of equations [4]. The unknowns are  $2N$  current coefficients  $a_{\lambda}(v)$ ,  $b_{\lambda}(v)$  of the compensating winding. The system matrix is full which is typical for the integral field methods (e.g. BEM). Then the final field distribution is obtained by superposition of solutions (6), (7) for both windings.

### 3. EVALUATION OF THE ERROR

The constraint (6) is satisfied only in an average (integral) sense. It results in error in the obtained field distribution.

The following three approaches are feasible to improve the accuracy of the proposed method:

- increase of the number of harmonics  $N$  in the analysis [4],
- changing the position of the compensating winding (analysed in the paper),
- increase of the number of the winding current sheets.

The first one is related to the basic properties of the Fourier series describing auxiliary function  $z(x)$  (8). The third of the above methods is proposed as an extension to the basic idea described in the paper. It will be based on the division of the region ( $C$ ) into subregions along its height and application of the proposed method to each of these.

In the paper only the second method is described. Two cases are distinguished, based on the value of the skin depth  $\delta = \sqrt{2/(\omega\gamma\mu)}$  [2,3]. The optimum position of the compensating winding is:

- for the resistance-limited case ( $\delta > h$ )  $\Rightarrow d=0.5h$  – in that case the eddy current density is almost constant along the height of the region (H). Therefore to compensate the magnetic field generated by the current flowing outside bar subregion (i.e. for  $z(x)=l$ ), the compensating winding should be placed near the middle of the height of the region (H),
- for the inductance-limited case ( $\delta < h$ )  $\Rightarrow d=0$  – in that case the eddy current is concentrated near the surface of the region (H). To compensate the magnetic field generated by the current flowing outside bar subregion the compensating winding should be placed on the boundary of the region (H) and the air gap (G).

Above cases are based on the properties of the magnetic field generated by the currents distributed in the homogenous space. However, as there are two infinitely permeability surfaces on both sides of the analysed region, infinite number of images influences the distribution of the field [1]. Therefore in practice the solutions for  $d=0$  and  $d=0.5h$  should be thought as a measure of the error limit for the actual field distribution.

#### 4. RESULTS

The calculations were performed for the following data:  $\gamma_c=58e6$  [S/m],  $\tau=0.1$  [m],  $r=\pi/4$ ,  $br/\tau=0.5$ ,  $v=1$ ,  $N=100$ ,  $\omega=2\pi50$  [rad/s],  $g=5.5e-3$  [m],  $h=0.9g$ . Resulting model of half of the bar region shown in Fig.3.

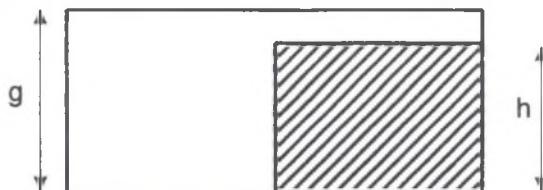


Fig.3. Half of the bar region of the analysed machine model  
Rys.3. Połowa obszaru pręta w analizowanym modelu maszyny

The comparisons were drawn between the results for:

- the homogenous machine model (Fig.2c), obtained using the method proposed in this paper. Two values of  $d$  were chosen:  $d=0$ ,  $d=0.5h$ ,
- the inhomogeneous machine model (Fig.2a) in which the electromagnetic field is evaluated with the help of the finite element method (FEM, ANSYS®). A small part of the finite element mesh is shown in Fig.4.

The finite element model was formulated using the following assumptions (Fig.4):

- it covers one pole pitch of the machine,

- additional layers of elements, representing rotor and stator cores, with relative permeability  $\mu_r=10000$ , are placed along the active region (Fig.1a),
- appropriate boundary conditions are applied on the borders of the model to account for the symmetry of the field ( $A(\Gamma_0)=-A(\Gamma_\tau)$ ) and high permeability of the machine cores (Neumann's  $\partial A / \partial y = 0 ; \Gamma = \Gamma_s \cup \Gamma_r$ ) [1],[3],
- current is injected into the nodes on the stator inner surface according to the formula:

$$\underline{I}_m = \cos(\nu\pi x_m / \tau) + j \sin(\nu\pi x_m / \tau) \quad (12)$$

where  $m=1,2,3,\dots,M+1$  is the numbering of nodes,  $M=800$  is the number of elements along the pole,  $x_m$  - position of the m-th node.

The stator winding current coefficients  $a_v$ ,  $b_v$  in the analytical field model were adjusted to ensure the same magnetomotive force MMF as in the case of the finite element model. Results obtained along half of a pole pitch ( $\tau/2$ ) are shown in Fig.5.

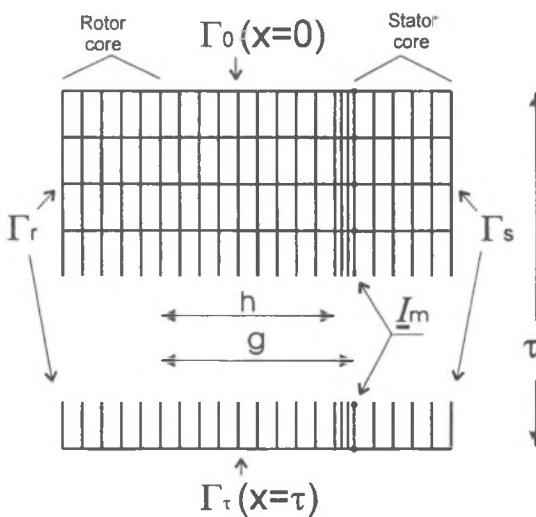


Fig.4. Finite element model of the analysed machine with the description of the boundaries and model dimensions

Rys.4. Model analizowanej maszyny wykorzystujący metodę elementów skończonych z opisem brzegów i wymiarów modelu

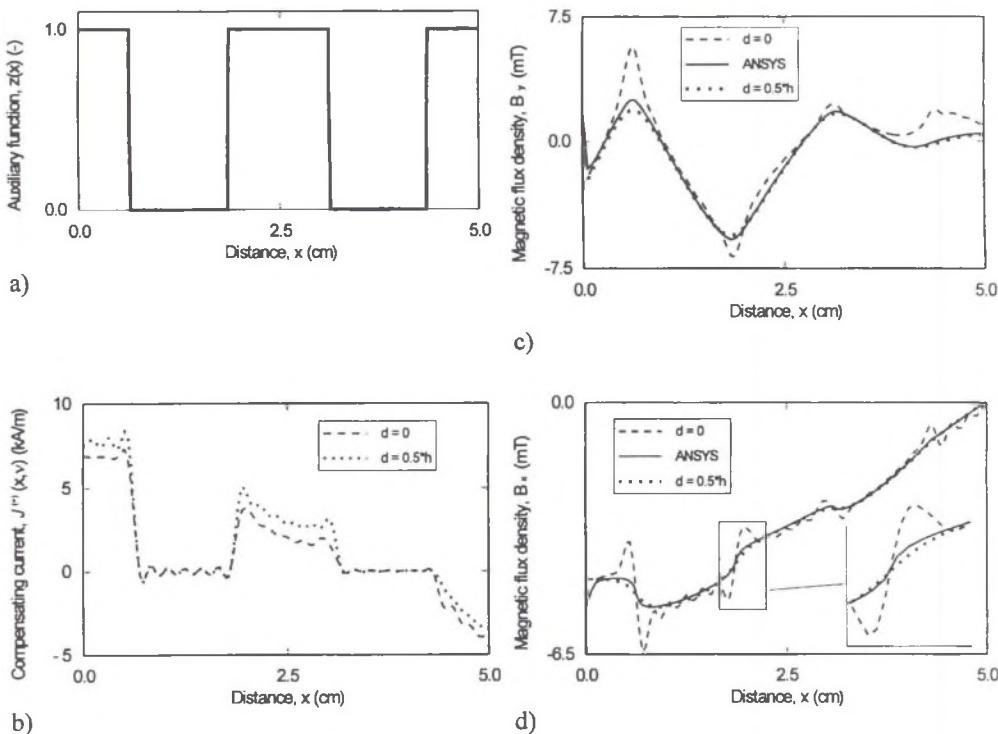


Fig.5. a) Auxiliary function  $z(x)$ . Spatial distribution of the real component of: b) the compensating winding current, c) the normal component of the magnetic flux density on the surface of the stator ( $y=0$ ), d) the tangential component of the magnetic flux density on the surface of the homogenous conducting region ( $y=(g-h)$ )

Rys.5. a) Funkcja pomocnicza  $z(x)$ . Rozkład przestrzenny części rzeczywistej: b) prądu uzwojenia kompensacyjnego, c) składowej normalnej wektora indukcji magnetycznej na powierzchni stojana ( $y=0$ ), d) składowej stycznej wektora indukcji magnetycznej na powierzchni jednorodnej warstwy przewodzącej ( $y=(g-h)$ )

## 5. CONCLUSIONS

Comparison has been made between the results obtained using proposed hybrid synthesis method and the FE method. As the value of the skin depth for the analysed model is  $\delta \approx 0.9e-3$  [m], and therefore  $\delta \approx h/5$ , neither of the two conditions mentioned before (for resistance-limited and inductance-limited case, respectively) is fulfilled.

From Fig.5c,d it is apparent that the FE solution in the air gap for an inhomogeneous model is within the limits established by the field distributions obtained using the proposed method for ( $d=0$ ,  $d=0.5h$ ). Discrepancy on the edges of the model, near  $x=0$  and  $x=\tau$  (not included) may result from the FEM internal errors [1]. Presented results prove correctness of the proposed approach.

The most outstanding advantage of the proposed method is the computational time required to obtain solution. The analytical evaluation using MATLAB environment takes only about 1/300 of the computational time required to obtain the reasonable solution using FEM (ANSYS®) for the model considered. It is especially important when numerous field solutions are necessary, as it is the case in the evaluation of the parasitic torques in induction squirrel cage machines [4], [6]. We are working on the implementation of the Rogowski's method as the error estimator [1].

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## Streszczenie

Rozwiązania analityczne opisujące rozkład pola elektromagnetycznego ograniczają się do analizy problemów w obszarach jednorodnych [1]. Dotyczy to również przedstawionej w artykule analizy stanów ustalonych przy zasilaniu sinusoidalnym [2,3]. Ograniczenia powyższe można pokonać wykorzystując dyskretnie algorytmy numeryczne [1,2,3,5,6]. Ich zastosowanie w analizie sił oraz momentów elektromagnetycznych wiąże się jednak z licznymi problemami natury obliczeniowej [5,6]. Doprowadziło to do rozwoju wielu

rodzajów tzw. metod hybrydowych [4,5,6]. Wykorzystują one zalety obydwóch wspomnianych uprzednio podejść: analitycznego oraz numerycznego.

Analizowany w artykule niejednorodny model maszyny elektrycznej przedstawiono na rys.1a. W modelu tym nie jest możliwe wyznaczenie analitycznego rozwiązania układu równań pola [1]. Problem ten rozwiązano wykorzystując hybrydową metodę syntezy pola [4].

Otrzymany na jej podstawie zastępczy model jednorodny przedstawiono na rys.1c. W stosunku do uprzednio przedstawionej wersji metody [4] wprowadzono następujące zmiany:

- grubość warstwy przewodzącej  $h$  jest dowolna,
- uzwojenie kompensacyjne (+) może być usytuowane na dowolnej głębokości  $d$  (rys.1c).

Istnieją trzy sposoby poprawy dokładności metody:

- zwiększenie liczby harmonicznych pola  $N$  [4],
- zmiana położenia uzwojenia kompensacyjnego  $d$ ,
- zmiana liczby warstw prądowych tworzących uzwojenie kompensacyjne.

Pierwsza z powyższych metod wynika z własności szeregu Fouriera opisującego funkcję pomocniczą  $z(x)$  wykorzystaną w opisanym algorytmie. Z kolei trzecia proponowana metoda poprawy dokładności jest rozwinięciem proponowanego algorytmu.

W artykule uwzględniono jedynie drugą z przedstawionych metod. Opierając się na definicji głębokości wnikania  $\delta$  wyróżniono dwa skrajne przypadki:

- słabego efektu naskórkowości ( $\delta > h$ )  $\Rightarrow$  dla którego przyjęto  $d=0.5 h$ ,
- silnego efektu naskórkowości ( $\delta < h$ )  $\Rightarrow$  dla którego przyjęto  $d=0$ .

Rozwiązania dla powyższych dwóch przypadków stanowią miarę błędu ograniczając zakres rzeczywistego rozwiązania. Poprawność przedstawionej analizy uproszczonej zweryfikowano wykorzystując metodę elementów skończonych (MES) na przykładzie modelu przedstawionego na rys.3. Otrzymane wyniki przedstawiono na rys.5.

Podstawową zaletą przedstawionej metody w stosunku do MES stanowi czas obliczeń. W przypadku przedstawionego modelu stosunek czasów obliczeń wynosił 1/300. Jest to szczególnie istotne w przypadku modeli wymagających wielokrotnych obliczeń. Przykładem jest wyznaczanie momentów paozytywniczych w maszynach indukcyjnych klatkowych [4].

Aktualnie prowadzone są prace nad wykorzystaniem metody Rogowskiego w roli estymatora błędu [1].