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POSITIVE REALIZATIONS WITH DELAYS OF PROPER TRANSFER MATRIX*

Summary. A method for finding positive realizations of a given proper transfer matrix of discrete-time systems with delays is proposed. Sufficient conditions for the existence of positive realizations of a given transfer matrix are established. A procedure for finding positive realizations of a given proper transfer matrix is presented and illustrated by a numerical example.

DODATNIA REALIZACJA Z OPÓŹNIENIAMI WŁAŚCIWEJ MACIERZY TRANSMITANCJI

Streszczenie. W pracy podano metodę wyznaczania dodatnich realizacji z opóźnieniami dla zadanej właściwej macierzy transmitancji układu dyskretnego. Podano warunki wystarczające istnienia realizacji dodatnich z opóźnieniami oraz procedurę wyznaczania takich realizacji. Procedurę zilustrowano przykładem liczbowym.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in standard delays systems is given in [4] and in positive systems theory is given in the monographs [3, 7]. Recent developments in positive theory and some new results are given in [8]. Realizations problem of positive linear systems without time-delays has been considered in many papers and books [1, 3, 7].

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Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [2, 14]. The realization problem for positive multivariable discrete-time systems and continuous-time systems with delays was formulated and solved in [6, 9, 10, 11, 12].

In this paper a new method for finding positive realizations of a given proper transfer matrix of discrete-time systems with delays will be proposed. The method can be considered as an extension of well-known classical Gilbert method [13, 5] for positive systems with delays in state vector and in inputs.

2. Preliminaries and problem formulation

Let $R_+^{n\times m}$ be the set of $n\times m$ real matrices with nonnegative entries and $R_+^m = R_+^{m\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n\times n$ identity matrix by I_n.

Consider the discrete-time system with one delay in state and one delay in input

$$\begin{aligned} x(i+1) &= A_1x(i) + A_0x(i-1) + B_1u(i) + B_0u(i-1) \\ y(t) &= Cx(i) + Du(i) \end{aligned} \tag{1}$$

where $x(i)\in R^n$, $u(i)\in R^m$, $y(i)\in R^p$ are the state, input and output vectors and $A_k \in R^{n\times n}$, $B_k \in R^{n\times m}$, $k = 0,1$, $C\in R^{p\times n}$, $D\in R^{p\times m}$.

Definition 1. The system (1) is called (internally) positive if for every initial condition $x(-k)\in R_+^n$, $k = 0,1$, $u(-1)\in R_+^m$, and all input sequences $u(i)\in R_+^m, i\in Z_+$ we have $x(i)\in R_+^n$ and $y(i)\in R_+^p$ for $i\in Z_+$.

Theorem 1. The system (1) is (internally) positive if and only if

$$A_k \in R_+^{n\times n},\ B_k \in R_+^{n\times m},\ k = 0,1,\ C\in R_+^{p\times n},\ D\in R_+^{p\times m} \tag{2}$$

The proof is given in [9, 10].

The transfer matrix of (1) has the form

$$T(z) = C[I_n z^2 - A_1 z - A_0]^{-1}(B_1 z + B_0) + D \tag{3}$$

Definition 2. The matrices (2) are called a positive realization of a given proper transfer matrix $T(z)$ if they satisfy the equality (3). The realization is called minimal if the dimension of matrices A_k, $k = 0,1$ is minimal among all realizations of $T(z)$.

The positive realization problem can be stated as follows. Given a proper rational matrix $T(z)\in R^{p\times m}$ (the set of $p\times m$ rational matrices in z), find a positive realization of $T(z)$.

In this paper sufficient conditions for the existence of a positive realization (2) will be established and a procedure for finding a positive (minimal) realization of T(z) will be proposed.

3. Problem solution

From (3) we have

$$D = \lim_{s\to\infty} T(z) \tag{4}$$

since $\lim_{z\to\infty}[I_n z^2 - A_1 z - A_0]^{-1} = 0.$

The strictly proper part of $T(z)$ is given by

$$T_{sp}(z) = T(z) - D = \frac{N(z)}{d(z)} \quad (5)$$

where

$$N(z) = N_{2n-1}z^{2n-1} + \ldots + N_1 z + N_0 \quad (N_i \in R^{p\times m}, \quad i = 0,1,\ldots,2n-1) \quad (6)$$

$$d(z) = z^{2n} + a_{2n-1}z^{2n-1} + \ldots + a_1 z + a_0 \quad (7)$$

Assumption 1.
It is assumed that polynomial (7) can be decomposed as follows

$$d(z) = d_1(z)\, d_2(z) \ldots d_n(z) \quad (8a)$$

where

$$d_j(z) = z^2 - a_{j1}z - a_{j0}, \quad j = 1,\ldots,n \quad (8b)$$

and

$$a_{jk} \geq 0 \text{ for } j = 1,\ldots,n;\ k = 0,1 \quad (9)$$

For given $d_j(z)$, $j = 1,\ldots,n$ and $N(z)$ we may find the matrices T_{jk}, $j = 1,\ldots,n$; $k = 0,1$ satisfying the equality

$$\frac{T_{11}z + T_{10}}{d_1(z)} + \frac{T_{21}z + T_{20}}{d_2(z)} + \ldots + \frac{T_{n1}z + T_{n0}}{d_n(z)} = \frac{N(z)}{d(z)} \quad (10)$$

as follows.

From (10) we have

$$\begin{aligned}&(T_{11}z + T_{10})d_2(z)\ldots d_n(z) + (T_{21}z + T_{20})d_1(z)d_3(z)\ldots d_n(z) + \ldots \\ &+ (T_{n1}z + T_{n0})d_1(z)\ldots d_{n-1}(z) = N(z)\end{aligned} \quad (11)$$

Comparing the matrix coefficients at the same powers of z of (11) we obtain the matrix equation

$$HX = F \quad (12)$$

where $H \in R^{2np\times 2np}$, $F \in R^{2np\times m}$, $X \in R^{2np\times m}$.

The entries of H depend on the coefficients a_{jk}, $j = 1,\ldots,n$; $k = 0,1$ of (8b) and

$$X = \begin{bmatrix} T_{10} \\ T_{11} \\ T_{20} \\ \vdots \\ T_{n1} \end{bmatrix}, \quad F = \begin{bmatrix} N_{2n-1} \\ N_{2n-2} \\ \vdots \\ N_0 \end{bmatrix}$$

Assumption 2.
It is assumed that

$$\operatorname{rank} H = \operatorname{rank}[H, F] \tag{13}$$

and the equation (12) has a nonnegative solution $X \in R_+^{2np \times m}$ for given H and F.

Sufficient conditions for the existence of nonnegative solution of (12) are given in [11].

Let

$$\operatorname{rank}[T_{j0}, T_{j1}] = r_j \le \min(p, 2m) \quad j = 1,\dots,n \tag{14}$$

Then for given $[T_{j0}, T_{j1}] \in R_+^{p\times 2m}$ it is always possible to choose

$$C_j \in R_+^{p\times r_j} \text{ and } [B_{j0}, B_{j1}] \in R_+^{r_j \times 2m}, \ \operatorname{rank} C_j = \operatorname{rank}[B_{j0}, B_{j1}] = r_j, \ j = 1,\dots,n \tag{15}$$

such that

$$[T_{j0}, T_{j1}] = C_j\,[B_{j0}, B_{j1}], \ \ j = 1,\dots,n \tag{16}$$

Theorem 2. There exist a positive realization (2) of $T(z)$ if

i) $T(\infty) \in R_+^{p\times m}$

ii) Assumptions 1 and 2 are satisfied.

The desired positive realization (2) of $T(z)$ is given by

$$A_k = \text{block diag}[I_{r_1} a_{1k}, I_{r_2} a_{2k}, \dots, I_{r_n} a_{nk}] \in R_+^{M\times M}, \quad M = \sum_{j=1}^{n} r_j \tag{17}$$

$$B_k = \begin{bmatrix} B_{1k} \\ B_{2k} \\ \vdots \\ B_{nk} \end{bmatrix} \in R_+^{M\times m}, \ \ k = 0,1; \quad C = [C_1 \ \ C_2 \ \ \dots \ \ C_n] \in R_+^{p\times M}$$

Proof. First we shall show that (17) is a positive realization of $T(z)$. If A_k, B_k, $k = 0,1$ and C have the forms (17) then using (8)-(10) we obtain

$$C[I_M z^2 - A_1 z - A_0]^{-1}(B_1 z + B_0) =$$

$$= [C_1 \ \ C_2 \ \ \dots \ \ C_n] \begin{bmatrix} I_{r_1}(z^2 - a_{11}z - a_{10}) & 0 & \dots & 0 \\ 0 & I_{r_2}(z^2 - a_{21}z - a_{20}) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_{r_n}(z^2 - a_{n1}z - a_{n0}) \end{bmatrix}^{-1} \begin{bmatrix} B_{11}z + B_{10} \\ B_{21}z + B_{20} \\ \vdots \\ B_{n1}z + B_{n0} \end{bmatrix} =$$

$$= [C_1 \ \ C_2 \ \ \dots \ \ C_n] \begin{bmatrix} I_{r_1}(z^2 - a_{11}z - a_{10})^{-1} & 0 & \dots & 0 \\ 0 & I_{r_2}(z^2 - a_{21}z - a_{20})^{-1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_{r_n}(z^2 - a_{n1}z - a_{n0})^{-1} \end{bmatrix} \begin{bmatrix} B_{11}z + B_{10} \\ B_{21}z + B_{20} \\ \vdots \\ B_{n1}z + B_{n0} \end{bmatrix} =$$

$$= \sum_{j=1}^{n} \frac{C_j(B_{j1}z + B_{j0})}{z^2 - a_{j1}z - a_{j0}} = \sum_{j=1}^{n} \frac{T_{j1}z + T_{j0}}{d_j(z)} = \frac{N(z)}{d(z)} = T_{sp}(z)$$

From (4) it follows that the condition i) implies $D \in R_+^{p\times m}$. If the Assumption 1 is satisfied then $a_{jk} \geq 0$, $j = 1,...,n$; $k = 0,1$ and $A_k \in R_+^{M\times M}$ for $k = 0,1$. If the Assumption 2 is met then $X \in R_+^{2np\times m}$ and there exist matrices (15) satisfying the equalities (16). Therefore, the desired realization is a positive one. ∎

Remark 1. The dimension $M \times M$ of the matrices A_0 and A_1 given by (17) is minimal.

Remark 2. From (6) and (11) follows that if $N_{2n-1} = 0$ then $T_{j1} = 0$ for $j = 1,...,n$.

If the conditions of Theorem 2 are satisfied then a positive realization (2) can be found by the use of the following procedure.

PROCEDURE

Step 1. Using (4) and (5) find the matrix D and the strictly proper transfer matrix $T_{sp}(z)$.

Step 2. Knowing the denominator $d(z)$ of $T_{sp}(z)$ and decomposing it find the polynomials $d_j(z)$, $j = 1,...,n$ satisfying (8) and (9).

Step 3. Comparing the matrix coefficients at the same powers of z of the equality (11) find the matrix equation (12) and its nonnegative solution X and the matrices T_{j1}, T_{j0} for $j = 1,...,n$.

Step 4. For the known matrices $[T_{j0}, T_{j1}]$ find their rank r_j and matrices (15) satisfying (16).

Step 5. Using (4) and (17) find the desired positive realization (2) of $T(z)$.

4. Example

Find a positive realization (2) of the transfer matrix

$$T(z) = \begin{bmatrix} \dfrac{2z^3 - 2z^2 - 3z}{z^4 - 2z^3 - 2z^2 + 3z + 2}, & \dfrac{2z^3 - 4z - 3}{z^4 - 2z^3 - 2z^2 + 3z + 2}, & \dfrac{2z + 3}{z^2 - z - 1} \\ \dfrac{1}{z^2 - z - 2}, & \dfrac{z + 1}{z^2 - z - 2}, & \dfrac{2z + 1}{z^2 - z - 2} \end{bmatrix} \tag{18}$$

In this case $p = 2$ and $m = 3$. The transfer matrix (18) can be rewritten as

$$T(z) = \frac{N(z)}{d(z)} \tag{19}$$

where

$$N(z) = \begin{bmatrix} 2z^3 - 2z^2 - 3z, & 2z^3 - 4z - 3, & 2z^3 + z^2 - 7z - 6 \\ z^2 - z - 1, & z^3 - 2z - 1, & 2z^3 - z^2 - 3z - 1 \end{bmatrix} \tag{20}$$

$$d(z) = z^4 - 2z^3 - 2z^2 + 3z + 2 \tag{21}$$

From (7) and (21) it follows that $n = 2$.

Using Procedure we obtain.

Step 1. The transfer matrix (18) is strictly proper ($D=0$) and $T_{sp}(z)=T(z)$.

Step 2. The polynomial (21) can be decomposed as follows $d(z)=d_1(z)d_2(z)$ and the polynomials

$$d_1(z)=z^2-z-2,\ d_2(z)=z^2-z-1 \tag{22}$$

satisfy the conditions (9).

Step 3. Comparison of the matrix coefficients at the same powers of z of the equality

$$(T_{11}z+T_{10})(z^2-z-1)+(T_{21}z+T_{20})(z^2-z-2)=$$

$$=\begin{bmatrix} 2z^3-2z^2-3z & , & 2z^3-4z-3 & , & 2z^3+z^2-7z-6 \\ z^2-z-1 & , & z^3-2z-1 & , & 2z^3-z^2-3z-1 \end{bmatrix}$$

yields the matrix equation

$$\begin{bmatrix} 0 & I_2 & 0 & I_2 \\ I_2 & -I_2 & I_2 & -I_2 \\ -I_2 & -I_2 & -I_2 & -2I_2 \\ -I_2 & 0 & -2I_2 & 0 \end{bmatrix}\begin{bmatrix} T_{10} \\ T_{11} \\ T_{20} \\ T_{21} \end{bmatrix}=\begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \\ 1 & 0 & -1 \\ -3 & -4 & -7 \\ -1 & -2 & -3 \\ 0 & -3 & -6 \\ -1 & -1 & -1 \end{bmatrix}$$

and its nonnegative solution is

$$T_{10}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ T_{11}=\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix},\ T_{20}=\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix},\ T_{21}=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \tag{23}$$

Step 4. Taking into account that for (23)

$$r_1=\operatorname{rank}[T_{10},T_{11}]=2 \text{ and } r_2=\operatorname{rank}[T_{20},T_{21}]=1$$

we choice

$$C_1=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B_{10}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ B_{11}=\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$C_2=\begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_{20}=\begin{bmatrix} 0 & 1 & 3 \end{bmatrix},\ B_{21}=\begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \tag{24}$$

Step 5. Using (17) and (24) we obtain

$$A_0=\begin{bmatrix} I_{r_1}a_{10} & 0 \\ 0 & I_{r_2}a_{20} \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ A_1=\begin{bmatrix} I_{r_1}a_{11} & 0 \\ 0 & I_{r_2}a_{21} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

$$B_0 = \begin{bmatrix} B_{10} \\ B_{20} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix},\ B_1 = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix},\ C = [C_1 \quad C_2] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The desired positive realization of (18) is given by (25) and $D = 0$.

5. Concluding remarks

A method for finding positive realizations of a given proper transfer matrix of discrete-time systems with delays has been proposed. Sufficient conditions for the existence of a positive realizations of a given transfer matrix have been established (Theorem 2). A procedure for finding of positive realizations of a transfer matrix has been proposed. The method has been presented for discrete-time systems with one delay in state vector and one delay in input but it can be easily extended for systems with many delays. Note that the presented method enable us to finding a positive realization with delays of a given transfer matrix when does not exist a positive realization without delays. An open problem is an extension of the proposed method for 2D systems with delays.

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Omówienie

W pracy podano metodę wyznaczania dodatnich realizacji z opóźnieniami dla zadanej właściwej macierzy transmitancji układu dyskretnego. Metoda ta stanowi uogólnienie znanej dla układów standardowych (niedodatnich) metody Gilberta na przypadek układów dodatnich dyskretnych z opóźnieniami. Podano warunki wystarczające istnienia realizacji dodatnich oraz procedurę wyznaczania takich realizacji dla zadanej właściwej macierzy transmitancji. Procedurę tę zilustrowano przykładem liczbowym. Należy podkreślić, że może istnieć i proponowana metoda pozwala wyznaczyć realizację dodatnią z opóźnieniami dla zadanej macierzy transmitancji, w przypadku gdy nie istnieje realizacja dodatnia bez opóźnień. Problemem otwartym jest uogólnienie proponowanej metody na układy dwuwymiarowe.