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CONTROLLABILITY OF FRACTIONAL DISCRETE-TIME SYSTEMS WITH DELAY

Summary. In the paper unconstrained controlability problem of finite-dimensional fractional-discrete time linear systems with delay in control is addressed. Using formula of solution necessary and sufficient condition for controllablity in a given number of steps is formulated and proved. Simple illustrative example is also presented.

STEROWALNOŚĆ UŁAMKOWYCH UKŁADÓW DYSKRETNYCH Z OPÓŹNIENIEM

Streszczenie. W pracy sformułowano i rozwiązano problem sterowalności bez ograniczeń dla liniowego ułamkowego skończenie-wymiarowego układu dyskretnego o stałych współczynnikach oraz z opóźnieniem w sterowaniu. Wykorzystując postać rozwiązania, sformułowano i udowodniono warunek konieczny i wystarczający sterowalności w zadanej liczbie kroków. Podano prosty przykład liczbowy.

1. Introduction

Controllability is one of the fundamental concepts in modern mathematical control theory. This is qualitative property of control systems and is of particular importance in control theory. Systematic study of controllability was started at the beginning of sixties in XX century and theory of controllability is based on the mathematical description of the dynamical system.

Many dynamical systems are such that the control does not affect the complete state of the dynamical system but only a part of it. On the other hand, very often in real industrial processes it is possible to observe only a certain part of the complete state of the dynamical system. Therefore, it is very important to determine whether or not control of the complete state of the dynamical system is possible. Roughly speaking, controllability generally means, that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls.

Controllability plays an essential role in the development of the modern mathematical control theory [4-10]. There are important relationships between controllability, stability and stabilizability of linear control systems. Controllability is also strongly connected with the theory of minimal realization of linear time-invariant control systems [3]. Moreover, it should be pointed out that there exists a formal duality between the concepts of controllability and observability. Moreover, controllability is strongly connected with so-called minimum energy control problem [10]. It should be pointed out that in the literature there are many results concerning controllability and minimum energy control, which depend on the type of dynamical control system (see [1], [2], and [10] for more details).

During last few years many results concerning theory of fractional control systems have been published [1-5] in the literature. Howvere, controllability problems studied in these papers concern fractional discrete-time control systems without delays. In the present paper unconstrained controlability problem of finite-dimensional fractional-discrete time linear systems with delay in control is addressed. Using formula of solution necessary and sufficient condition for controllablity in a given number of steps is formulated and proved.

The paper is organized as follows. In section 2 the solution of the difference state equation finite-dimensional fractional systems with delay is recalled. Necessary and sufficient condition for controllability of the fractional discrete-time control system with constant parameters and single delay in control is established in section 3. Section 4 contains simple numerical example, which illustrates theoretical considerations. Finally, concluding remarks are given in section 5.

2. Fractional systems

The set of nonnegative integers will be denoted by Z_+. Let $x_k \in R^n$, $u_k \in R^m$, $k \in Z_+$. In this paper extended definition of the fractional difference of the form [1-5]

$$\Delta^\alpha x_k = \sum_{j=0}^{j=k} (-1)^j \binom{\alpha}{j} x_{k-j} \qquad \text{for} \quad n\text{-}1<\alpha<n\in N=\{1,2,\ldots\},\ k\in Z_+ \tag{1}$$

will be used, where $\alpha \in R$ is the order of the fractional difference and

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j=0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j=1,2,\ldots \end{cases} \tag{2}$$

Let us consider the fractional discrete linear system, described by the diference state-space equations with delayed control

$$\Delta^\alpha x_{k+1} = Ax_k + B_0 u_k + B_1 u_{k-1} \tag{3}$$

where $x_k \in R^n$, $u_k \in R^m$ are the state and input and A and B are $n\times n$ and $n\times m$ constant matrices, respectively.

Using definition (1) we may write the equations (3) in the form

$$x_{k+1} + \sum_{j=1}^{j=k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + B_0 u_k + B_1 u_{k-1} \tag{4}$$

Lemma 1. [5] The solution of difference equation (4) with initial condition $x_0 \in R^n$ is given by

$$\begin{aligned} x_k &= \Phi_k x_0 + \sum_{i=0}^{i=k-1} (\Phi_{k-i-1}(B_0 u_i + B_1 u_{i-1})) = \\ &= \Phi_k x_0 + \Phi_{k-1} B_1 u_{-1} + \sum_{i=0}^{i=k-1} (\Phi_{k-i-1} B_0 + \Phi_{k-i-2} B_1) u_i \end{aligned} \tag{5}$$

where $n \times n$ dimensional matrices Φ_k, k=0,1,2,... are determined by the equation

$$\Phi_{k+1} = (A + I_n \alpha)\Phi_k + \sum_{i=2}^{i=k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1} \tag{6}$$

with $\Phi_0 = I$, where I is $n \times n$ dimensional identity matrix and $\Phi_k = 0$ for $k<0$.

3. Controllability

First of all, in order to define controllability concept let us introduce the notion of reachable set or in other words attainable set in q steps for finite-dimensional discrete-time fractional control system (4).

Definition 1. For fractional system (4) controllable set in q steps from $x_0=0$ is defined as follows

$$K_q = \{x \in X: x \text{ is a solution of (4) for } k=q \text{ and for controls } u_0, u_1, \ldots u_k, \ldots, u_{q-1}\} \tag{7}$$

Definition 2. The fractional discrete-time control system (4) is controllable in q-steps if

$$K_q = R^n \tag{8}$$

Let us introduce the $n \times nm$ dimensional controllability matrix

$$R_q = [B_0, (\Phi_1 B_0 + B_1), (\Phi_2 B_0 + \Phi_1 B_1), \ldots, (\Phi_{k-i-1} B_0 + \Phi_{k-i-2} B_1), \ldots, (\Phi_{q-1} B_0 + \Phi_{q-2} B_1)] \tag{9}$$

Theorem 1. The fractional discrete-time system with delay in control (4) is controllable in q steps if and only if

$$rank\, R_q = n \tag{10}$$

Proof. Using formula (5) for $k = q$, $x_0 = 0$, $u_{-1} = 0$ we obtain

$$x_f = x_q = \sum_{i=0}^{i=k-1} (\Phi_{k-i-1} B_0 + \Phi_{k-i-2} B_1) u_i = R_q \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_i \\ \vdots \\ u_0 \end{bmatrix} \tag{11}$$

From Definition 2 and (11) it follows that for every final state $x_f \in R^n$ there exists a input sequence $u_i \in R^m$, $i = 0,1,\dots,q-1$ if and only if the image of controllability matrix ImR_q is the whole space R^n, so controllability matrix R_q has full row rank n■

Corollary 1. The fractional control system (4) is controllable in q steps if and only nxn dimensional constant matrix $R_q R_q^T$ is invertible, i.e. there exists inverse matrix $(R_q R_q^T)^{-1}$.

4. Example

Let us consider the fractional discrete-time system with delayed control of the form (4) for $0 \le \alpha \le 1$ with the following matrices

$$A = \begin{bmatrix} -\alpha & 0 \\ 2 & 1 \end{bmatrix} \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{12}$$

Hence we have

$$A + I\alpha = \begin{bmatrix} 0 & 0 \\ 2 & 1+\alpha \end{bmatrix}$$

Using formula (6) for k = 0 we obtain

$$\Phi_1 = (A + I\alpha)\Phi_0 = \begin{bmatrix} 0 & 0 \\ 2 & 1+\alpha \end{bmatrix}$$

Controllability matrix (10) for $q = 2$ has the form

$$R_2 = [B_0, (\Phi_1 B_0 + B_1)] = \begin{bmatrix} 0 & 1 \\ 1 & 1+\alpha \end{bmatrix}$$

Therefore, since *rank* $R_2=2=n$ then taking into account Theorem 1 the fractional discrete-time system with constant coefficients and delayed control (12) is controllable in two steps.

5. Concluding remarks

In the present paper unconstrained controlability problem of finite-dimensional fractional-discrete time linear systems with delay in control is addressed. Using solution formula necessary and sufficient condition for unconstrained controllability in q steps of the discrete time fractional control system has been established as rank condition of suitably defined controllability matrix. Simple illustrative example is also presented.

Moreover, it should be mentioned, that controllability considerations presented in the paper can be extended for fractional discrete-time linear systems with multiple delays both in the controls and in the state variables and for infinite-dimensional fractional discrete-time linear systems with constant parameters.

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Omówienie

Sterowalność, podobnie jak obserwowalność oraz stabilność, należy do podstawowych pojęć teorii sterowania. Ogólnie sterowalność oznacza, że w rozpatrywanym układzie dynamicznym możliwe jest osiągnięcie z danego stanu początkowego zadanego stanu końcowego przy użyciu odpowiednio dobranego sterowania dopuszczalnego należącego do zbioru sterowań dopuszczalnych. Zatem sterowalność zależy od postaci równania stanu układu dynamicznego oraz od zbioru sterowań dopuszczalnych.

W pracy przedstawiono problematykę badania sterowalności liniowych ułamkowych dyskretnych skończenie wymiarowych układów dynamicznych o stałych parametrach oraz z opóźnieniem w sterowaniu. Wykorzystując postać rozwiązania, sformułowano i udowodniono warunek konieczny i wystarczający sterowalności w zadanej liczbie kroków bez ograniczeń na sterowanie oraz podano prosty przykład liczbowy ilustrujący rozważania teoretyczne. Zaproponowano również kierunki dalszych badań w zakresie ułamkowych układów dyskretnych z opóźnieniami zarówno w sterowaniu, jak i we współrzędnych stanu.