ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1987

Seria: MECHANIKA z. 86 Nr kol. 1012

XI OGÓLNOPOLSKA KONFERENCJA TEORII MASZYN I MECHANIZMÓW

11th POLISH CONFERENCE ON THE THEORY OF MACHINES AND MECHANISMS

27—30. 04. 1987 ZAKOPANE

R. KRUSE

Institut für Informatik, TU Braunschweig,
Bültenweg 74/75, D-3300 Braunschweig, West Germany

K.D. MEYER

Institut für Mathematische Stochastik, TU Braunschweig,
Pockelsstr. 14, D-3300 Braunschweig, West Germany

FUZZY MARKOV CHAINS AND THEIR APPLICATIONS TO PROCESSOR POWER CONSIDERATIONS

Summary. Markov chains are used in queuing theory and in many other computer science applications. In this paper we propose fuzzy Markov chains as the first approach to fuzzy queuing theory. We prove a fuzzy ergodic theorem and estimate the limit probabilities. With help of this we calculate the effective processor power for a computer system consisting of N independent processors and M independent memory module, where the requests for memory moduls are not distributed equally likely.

Fuzzy numbers and fuzzy probabilities

Def. 1.1. A fuzzy number of the real line $\mathbb{R}$ is characterized by its membership function $\mu : \mathbb{R} \to [0,1]$. It is assumed that for all μ there is an $x \in \mathbb{R}$ such that $\mu(x) = 1$. The set of all fuzzy numbers will be denoted by $F(\mathbb{R})$; we will identify a fuzzy number with its membership function. An important tool for handling fuzzy numbers are the level sets and - more generally - the set representations.

Def. 1.2. Let $\mu \in F(\mathbb{R})$. $\{A_\alpha \mid \alpha \in (0,1)\}$ is called a set representation for μ, if

(i) $0 < \alpha < \beta < 1 \qquad A_\beta \subset A_\alpha \subseteq \mathbb{R}$,

(ii) $\forall x \in \mathbb{R}: \quad \mu(x) = \sup\{\alpha\, \mathbb{1}_{A_\alpha}(x) \mid \alpha \in (0,1)\}$

where $\mathbb{1}_{A_\alpha}$ denotes the indicator function of A_α. Let $F([0,1])$ denote the set of all fuzzy numbers of $[0,1]$.
The strong α-cut $\tilde{A}_\alpha$ of $\tilde{A}$ is the nonfuzzy set defined by

$$\tilde{A}_\alpha = \{x \in [0,1] \mid \mu_{\tilde{A}}(x) > \alpha\}, \quad 0 \leq \alpha < 1.$$

A fuzzy set may be decomposed into its level sets by

$$\mu_{\tilde{A}}(x) = \sup_{\alpha \in [0,1]} \min(\alpha, \mathbb{1}_{\tilde{A}_\alpha}(x)),$$ where $\mathbb{1}_B$ denotes the indicator function of B.
If $\tilde{A}$ and $\tilde{B}$ are fuzzy numbers, then

$$d(\tilde{A},\tilde{B}) := \sup_{\alpha > 0} d_H(\tilde{A}_\alpha, \tilde{B}_\alpha), \text{ where}$$

$$d_H(V,W) := \max(\sup_{v \in V} \inf_{w \in W} |v-w|, \sup_{w \in W} \inf_{v \in V} |w-v|),$$

is called the generalized Hausdorff distance between $\tilde{A}$ and $\tilde{B}$. d is a metric.
Let $\Omega = \{\omega_1, \ldots, \omega_n\}$, $n \in \mathbb{N}$.
A map $\tilde{P} : \Omega \to F([0,1])$ is called a fuzzy probability if and only if there exists $(u_1, \ldots, u_n) \in [0,1]^n$ such that $u_1 + \ldots + u_n = 1$ and $\tilde{P}(\omega_i)(u_i) = 1$ for $i = 1, \ldots, n$.
The fuzzy probability of a set $A \subset \Omega$ is the fuzzy number of $[0,1]$ defined by

$$\tilde{P}(A)(z) := \sup_{\substack{(u_1,\ldots,u_n) \in [0,1]^n: \\ \sum_{i=1}^n u_i = 1, \ \sum_{i:\omega_i \in A} u_i = z}} \ \min_{i=1,\ldots,n} \tilde{P}(\omega_i)(u_i)$$

if $A \neq \emptyset$, and $\tilde{P}(A)(z) := \mathbb{1}_{\{0\}}(z)$, if $A = \emptyset$.
The fuzzy expectation $\tilde{E}(V)$ of a function $V : \Omega \to \mathbb{R}$ is defined by

$$\tilde{E}V(z) = \sup_{\substack{(u_1,\ldots,u_n) \in [0,1]^n: \\ \sum_{i=1}^n u_i = 1, \ \sum_{i=1}^n u_i V(\omega_i) = z}} \ \min_{i=1,\ldots,n} \tilde{P}(\omega_i)(u_i).$$

$\tilde{E}(V)$ is an interactive sum ([4]).

We assume that a fuzzy probability $\tilde{P}$ is a fuzzy perception of a usual probability σ, which is called the original of $\tilde{P}$. The original is unknown. By fuzzy logic we are able to evaluate the acceptability that σ is an original of $\tilde{P}$. The statement "σ is an original of $\tilde{P}$" or equivalently "For all $A \subseteq \Omega$ the fuzzy-probability $\tilde{P}(A)$ assumes the value $\sigma(A)$" has the truth value

$$\min_{A \subset \Omega} \left\{ \tilde{P}(A)(\sigma(A)) \right\} .$$

Of course we have

$$\min_{A \subset \Omega} \left\{ \tilde{P}(A)(\sigma(A)) \right\} \leq \min_{i = 1,\ldots,m} \left\{ \tilde{P}(\omega_i)(\sigma(\omega_i)) \right.$$

On the other hand we have for every $A \subset \Omega$, $A \neq \emptyset$

$$P(A)(\sigma(A)) \geq \min_{i = 1,\ldots,n} \tilde{P}(\omega_i)(\sigma(\omega_i)) .$$

So the truth value of the statement "σ is an original of $\tilde{P}$" is

$$\min_{i = 1,\ldots,n} \left\{ \tilde{P}(\omega_i)(\sigma(\omega_i)) \right\} .$$

2. Fuzzy Markov chains

Consider a sequence of experiments, where a finite number of results $\omega_1,\ldots,\omega_n$ is possible. We call $\omega_1,\ldots,\omega_n$ the states of the experiment. We use the symbol $\omega_j^{(k)}$ to express that the state ω_j was realized in the k-th experiment.

Let furthermore denote $\tilde{P}_{ij}^{(k)}$ the fuzzy probability that if the system is in the state ω_i in the (k-1)th experiment, then it is in the state ω_j in the k-th experiment. We assume that the probability $\tilde{P}_{ij}^{(k)}$ do not depend on k. We say that the sequence of experiments is a homogenous fuzzy Markov chain.

A homogenous fuzzy Markov chain is characterized by the square matrix

$$\begin{bmatrix} \tilde{P}_{11} & \cdots\cdots & \tilde{P}_{1n} \\ \vdots & & \vdots \\ \tilde{P}_{n1} & & \tilde{P}_{nn} \end{bmatrix}$$

where the $\tilde{P}_{ij}$ are the one-step transition probabilities. We assume that the map

$$\tilde{P}_i \;:\Omega \longrightarrow F([0,1]), \quad \omega_j \longrightarrow \tilde{P}_{ij},$$

is a fuzzy probability for all $i = 1,\ldots,n$.

A fuzzy Markov chain is a perception of a usual Markov chain, which is characterized by a stochastic matrix

$$(\sigma_{ij}) \quad \begin{matrix} i = 1,\dots,n \\ j = 1,\dots,n \end{matrix}$$

This unknown Markov chain is called original of the fuzzy Markov chain. Its m-step transition probabilities can be computed using the Chapman Kolmogorov equations, we have

$$\sigma_{ij}(2) = \sum_{k=1}^{n} \sigma_{ik}\sigma_{kj}$$

$$\sigma_{ij}(m) = \sum_{k=1}^{n} \sigma_{ik}(t)\sigma_{kj}(m-t), \quad \text{where} \quad 1 \leq t < m.$$

We assume that the original of the fuzzy Markov chain is unknown but that the original is located in a set $\mathfrak{M}$ of Markov chains. The statement "(σ_{ij}) is an original of $(\tilde{P}_{ij})$" is fuzzy, by fuzzy logic we know that it has the truth value $\min\limits_{i,j}\{\tilde{P}_{ij}(\sigma_{ij})\}$. L. Zadeh's extension principle gives a proper definition for the m-step transition probabilities in the fuzzy case. We have

$$\tilde{P}_{ij}(m)(z) = \sup_{\substack{(\sigma_{ij}) \in \mathfrak{M} \\ \sum_{j=1}^{n} \sigma_{ij} = 1, \ i = 1,\dots,n, \\ \sigma_{ij}(m) = z}} \ \min_{\substack{i = 1,\dots,n \\ j = 1,\dots,n}} \{P_{ij}(\sigma_{ij})\}$$

for all $z \in [0,1]$ and all $m = 1,2,3,\dots$

We can formulate the following fuzzy ergodic theorem:

<u>Theorem</u> Let $(\tilde{P}_{ij})_{i,j=1,\dots,n}$ be the transition probabilities of a homogeneous fuzzy Markov chain with the property (*):

(*) There exists a real number $\sigma > 0$ and an integer $t > 0$ such that: If $R = (\sigma_{ij}) \in \mathfrak{M}$ is a stochastic matrix and if $\inf\limits_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \{\tilde{P}_{ij}(\sigma_{ij})\} > 0$, then in at least one column of the matrix R^t all numbers are bigger than σ.

Then $P_j := \lim\limits_{m\to\infty} \tilde{P}_{ij}(m)$ exists for $j = 1,\dots,n$ independent of i, and $\tilde{P} : \{\omega_1,\dots,\omega_n\} \longrightarrow F([0,1])$, $\omega_j \longrightarrow \tilde{P}_j$, is a fuzzy probability. The convergence is within the generalized Hausdorff-metric.

<u>Proof</u> For the proof of the theorem we refer to [5].

3. On the calculation of the effective processor power

Consider a computer system consisting of N independent processors (CPU's) and M independent memory modules, which are synchronized. A similary example would be a Local Area Network (LAN) consisting of Personal Computers and File Servers or a Distributed Database-Machine-System ([2], [9]). We assume (compare [1], [3], [8]) that each processor has always a memory request ready for a memory module to accept as soon as possible, and it is assumed that a processor can access each memory module. During one cycle every memory module can satisfy only one request, so at the beginning of each memory cycle each of the processors whose request from the previous cycle was satisfied makes a new request. A processor whose memory request from the last cycle was not honored must wait at least one more cycle before it is allowed to make another memory request. In our systems it may be that several requests are made to the same memory module during one memory cycle. While each memory module will service one request per cycle, the remaining ones are queued for future memory cycles.
Moreover we make the simplifying assumption that the N processors simultaneously make their memory requests and all those which are successful on the last cycle receive their data at the same time. The logical design of this computer model is shown in Fig. 1.

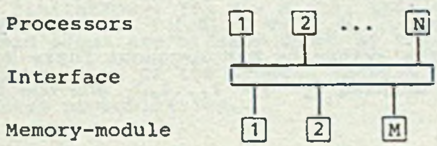

Fig. 1

The state of the system can be represented as an M-tuple $K = (k_1,...,k_M)$, where k_i is the number of access requests queued for memory module i. We have $k_1 + ... + k_M = N$, and there are $\binom{M+N-1}{M-1}$ different states.
At the end of a memory cycle the state of the system is represented by $H = (h_1,...,h_M)$, where $h_i := k_i-1$, if $k_i > 0$, and $h_i := 0$, otherwise. A state $G = (g_1,...,g_M)$ is reachable in one step from $K = (k_1,...,k_M)$ if and only if $g_i \geq k_i$ for all i.
In contrast to Allen's [1] model we consider the case when each memory module is not equally likely chosen. We suppose that the module i is chosen with probability σ_i, so we have $\sigma_1 + ... + \sigma_M = 1$. If a state $G = (g_1,...,g_M)$ is reachable from $K = (k_1,...,k_M)$ in one step, then the probability of transition from state K to G is given (with resp. to $\sigma = (\sigma_1,...,\sigma_M)$) by

$$P_\sigma(K,G) = \frac{x!}{\prod_{i=1}^{M}(d_i!)} \prod_{i=1}^{M} (\sigma_i^{(d_i)}),$$

where $d_i = g_i - h_i$, and $x = \sum_{i=1}^{M} d_i$.

The formula for $P_\sigma(K,G)$ can be simply pointed out, since we have a mod of a multinomial experience.

Clearly the system can reach any state from any other state in a finite number of steps with a probability larger than zero.

Now we assume that the values of the probability that the module j is chosen are not numerical, as usually is the case but linguistic. This is reasonable since experts are often not able to express their meanings in terms of numbers but in terms of linguistic expressions. For example $\tilde{p}_j$ may represent the value "often", "rarely", "with high probability" etc. $\tilde{p}_j$ is the linguistic value of a linguistic variable [6, 7] $S = (T, [0,1]$, $G, M)$, where T is a term set, G is a set of syntactical rules that generate T from a set of primitive terms, and M is a set of semantic rules that assign to each value x of S $(x \in T)$ its meaning $M(x)$ whi is a fuzzy number (in the sense of D. Dubois [4]).

We assume that $\tilde{P} : \{\omega_1,\dots,\omega_M\} \longrightarrow F([0,1])$ is a fuzzy probability. Then we can describe the system by a homogenous fuzzy Markov chain with $J := \binom{M+N-1}{M-1}$ states, denoted by $G = 1,\dots,J$ and the fuzzy transition probabilities

$$\tilde{P}(K,G)(z) = \sup_{\substack{\sigma=(\sigma_1,\dots,\sigma_M) \in [0,1]^M: \\ \sum_{j=1}^{M} \sigma_j = 1, \; P_\sigma(K,G) = z}} \; \min_{j=1,\dots,M} \tilde{P}_j(\sigma_j)$$

where the set $\mathfrak{M}$ of all possible originals of $P(K,G)$ is

$$\mathfrak{M} \left\{ (P_\sigma(K,G))_{\substack{K=1,\dots,J \\ G=1,\dots,J}} \;\middle|\; \sigma = (\sigma_1,\dots,\sigma_M) \in [0,1]^M, \; \sigma_1 + \dots + \sigma_M = 1 \right\}$$

If $P_j(0) = P(\omega_j)(0) = 0$, then the assumptions of the Ergodic Theorem ar satisfied, therefore the limiting probabilities

$$\tilde{\pi}_G = \lim_{m \to \infty} \tilde{P}_{K,G}(m)$$

exist, they are independent of the original starting-state K. We can use the stationary fuzzy probabilities $\tilde{\pi}_G$ for calculating the effective processor power $\widetilde{EP}(M,N)$. The formula is

$$\widetilde{EP}(M,N)(z) = \sup_{\substack{\sigma=(\sigma_1,\ldots,\sigma_J)\in[0,1]^J:\\ \sum_{i=1}^{J}\sigma_i = 1,\ \prod_{i=1}^{J}\sigma_i\ \mathrm{proc}(i) = z}} \ \min_{i=1,\ldots,J} \{\tilde{\pi}_i(\sigma_i)\},$$

where proc(i) is the number of processors that are in operation just after a memory cycle in which the system was in state i. Thus proc(i) is the number of memory requests serviced during a memory cycle in which the system was in state i.

We illustrate our results by two examples. These examples were carried out computer-aided by using a program-system written in PASCAL on an IBM 4341-machine under CMS.

In our example we consider a little computer system consisting of two processors (N=2) and two memory modules (M=2).

This is the standard example proposed by Baskett and Smith [3] and Mills [8], see also Allen [1]. A more complex example is not considered in this paper for space limitations: If M = N = 4, then we have 35 different states. But the basic ideas are already shown by this example.

Note that the assumptions of the Ergodic Theorem in this case are satisfied for arbitrary fuzzy probabilities.

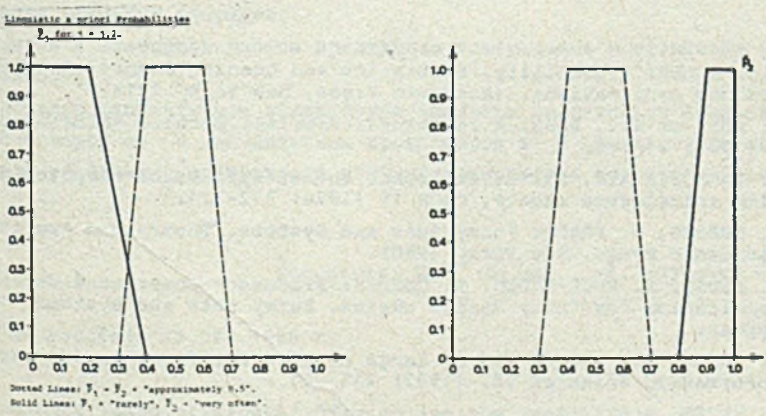

Fig. 2

In Fig. 2 example A is characterized by the solid lines, example B is described by the dotted lines.

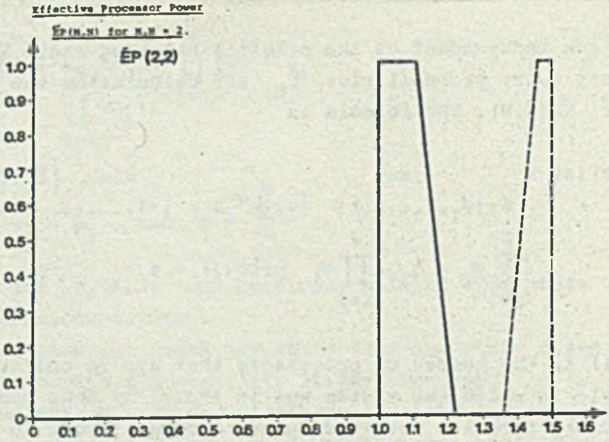

Fig. 3

In example A we assume that the memory module 1 is chosen "very often" and that module 2 is chosen "rarely", while in example B we assume that both modules are chosen with probability "approximately 0.5". It's suprising (Fig. 3) that in example B the resulting fuzzy number $\tilde{E}P(2,2)$ is rather sharp although the starting values P_j are "fuzzy". On the other hand we obviously have a maximum at 1.5, because this is the resulting value in the stationary stochastic case for EP(2,2) ([1]).

REFERENCES

[1] A.O. ALLEN: Probability, Statistics and Queuing Theory with Computer Science Applications. (Academic Press, New York, 1978).

[2] H. AUER et al., RDBM: A relational data base machine Information Systems 6 (1981) 91-100.

[3] F. BASKETT, A.J. SMITH: Inference in multiprocessor computer systems with interleaved memory, CACM 19 (1976) 372-334.

[4] D. DUBOIS, H. PRADE: Fuzzy Sets and Systems. Theory and Applications. (Academic Press, New York, 1980).

[5] R. KRUSE, R. BUCK-EMDEN, R. CORDES: Processor-power considerations - an application for fuzzy Markov chains, Fuzzy sets and systems, 1987, to appear.

[6] R. KRUSE: The strong law of large numbers for fuzzy random variables, Information Sciences 28, (1982) 233-241.

[7] R. KRUSE: Statistical estimation with linguistic data, Information Sciences 33 (1984).

[8] P.M. MILLS: A simple model for cost considerations in a batch multiprocessor environment, Performance Evaluation 5 (1976) 19-27.

[9] M. MISSIKOFF, M. TERRANOVA: The architecture of a relational database computer known as DBMAC, in: D.K. Hsiao, (Ed.) Advanced database machine architecture (Prentice Hall, New York, 1982) 87-107.

ROZMYTE ŁAŃCUCHY MARKOVA I ICH ZASTOSOWANIE DO OKREŚLANIA MOCY PROCESORÓW

S t r e s z c z e n i e

Łańcuchy Markova używane są w teorii porządkowania zadań i w różnych zastosowaniach komputerów.

W tej pracy proponujemy "rozmyte" łańcuchy Markova jako propozycję opisu rozmytych kolejek.

Dowodzimy rozmyte twierdzenie ergodyczne i oceniamy prawdopodobieństwa graniczne. Przy ich pomocy obliczamy skuteczną moc procesorów układu komputerowego składającego się z N niezależnych procesorów i M niezależnych modułów pamięci, gdzie żądania dostępu do pamięci nie są rozłożone równomiernie.

РАЗМЫТЫЕ ЦЕПИ МАРКОВА И ИХ ПРИМЕНЕНИЕ
ДЛЯ ОПРЕДЕЛЕНИЯ МОЩНОСТИ ПРОЦЕССОРОВ

Р е з ю м е

Цепи Маркова употребляются в теории систематизации задач и в разнообразных применениях компьютеров.

В этой работе предлагаются размытые (фаззи) цепи Маркова как предложение описания размытых очередей.

Доказываем в настоящей работе эргодичное утверждение и оцениваем крайние возможности.

С их помощью рассчитываем эффективную мощность процессоров компьютерной системы состоящей из Н независимых процессоров и М независимых модулей памяти, где требования доступности к памяти равномерно не раскладываются.

Recenzent: Doc. dr hab. inż. Wojciech Cholewa

Wpłynęło do redakcji 18.XII.1986 r.