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THE SECOND BASIC 7R MECHANISM

Abstract. There are two kinds of homogenous, orthogonal, single-of-freedom linkages with seven binary links, connected by seven turning pairs, which rightly can be designated as "basic" spatial 7R mechanisms. The first, characterized by the system parameter structure: $a_i = 1$, $s_i = 0$, $\alpha_i = 90°$ ...... $i = 1(1)\ 7$, has been investigated in two previous papers [4, 5], and the second basic 7R mechanism, with the parameter structure: $a_i = 0$, $s_i = 1$, $\alpha_i = 90°$ .... $i = 1(1)\ 7$, is the subject of the present. For this second 7R mechanism - in some sense dual to the first - an algebraic displacement analysis is carried out, leading to input-output equations of degree 16 in the tan - half - angle of the input - and the output angular displacements. Extraneous roots in the algebraic input-output equations only could not be avoided in the case where the input angular displacements σ_i are related to the output angular displacements σ_{i+3} (or σ_{i+4}). This simply because in this case we were not able to expand the corresponding determinant (and then to split off the unwanted factor) with the devices at our diposition.

Therefore in this case the input-output equation has been given in the form of a 16 x 16 determinant equated to zero, leading to an algebraic equation of degree 24 (instead of 16).

Introduction

The input-output equation for the general single loop, single-degree-of-freedom, spatial 7R mechanism has been given by J. Duffy and C. Crane in 1980 [1]. It relates the input angular displacements σ_1 to the output angular displacement σ_7 and is presented in the form of a 16 x 16 determinant equated to zero, actually yielding an algebraic equation of degree 32 in tan $(\sigma_1/2)$ and in tan $(\sigma_7/2)$.

The present paper should be understood primarily as a contribution to the exploration of the limits of the long sought - for general solution of Duffy and Crane.

This exploration seems to be necessary because, though the solution is general, problems arise for special system parameters of the 7R mechanism. As the rank of the matrix, corresponding to the 16 x 16 determinant depends on the system parameters, it might happen that for special sets of parameters the determinant vanishes identically, making it impossible to determine the output angular displacements corresponding to a chosen input angular displacement by a numerical roots-finding procedure. Exactly this now occurs if one tries to analyse one of the two "basic" 7R mechanisms which can be characterized by their parameter structures: $a_i = 1$, $s_i = 0$, $\alpha_i = 90°$ and $a_i = 0$, $s_i = 1$, $\alpha_i = 90°$, respectively (a_i stands for the normal distance, s_i for the offset and α_i for the twistangle of the revolute axes on the link with the index i-1), on the basis of the general solution. Therefore in these cases an other set of initial equations has to be found, leading to a workable Euler-Sylvester-determinant. For the first basic 7R mechanism in [4] a solution (based on six initial equations instead of four) has been presented in the form of a 12 x 12 determinant equated to zero and leading to an algebraic equation of degree 16, and finally in [5] the algebraic input-output equation with the minimum degree (four) could be established for this mechanism. In 1979 J. Duffy and S. Derby lublished a paper [2] in which they treated the generalized lobster-arm, a 7R mechanism which has consecutive joint axes intersecting (with the exception of the first and the last). The second basic 7R mechanism which we are going to treat here is a special lobster-arm mechanism. The reason for taking up once again the same problem even in a specific form is twofold. First, we intend to find all possible input-output equations whatever angular displacement will be chosen as input - or as output angular displacement, and then we wish to show that a "modified fourth Duffy-equation" can be found and used to determine all possible input-output equations by equating corresponding 16 x 16 determinants to zero. The regular "fourth Duffy-equation" (equation 14 in [1]) yields $0 = 0$ for the special set of system parameters of the second basic 7R mechanism.

The starting position

The second basic 7R mechanism is a homogeneous, orthogonal space mechanism with consecutive joint axes intersecting. The parameter structure of this mechanism is given by:

$$a_i = 0, \quad s_i = 1, \quad \alpha_i = 90° \qquad i = 1(1)7 \tag{1}$$

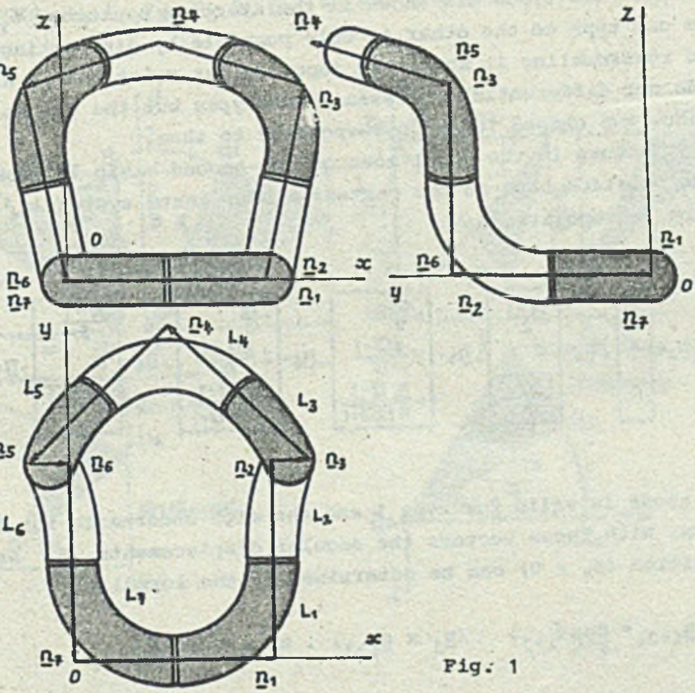


Fig. 1

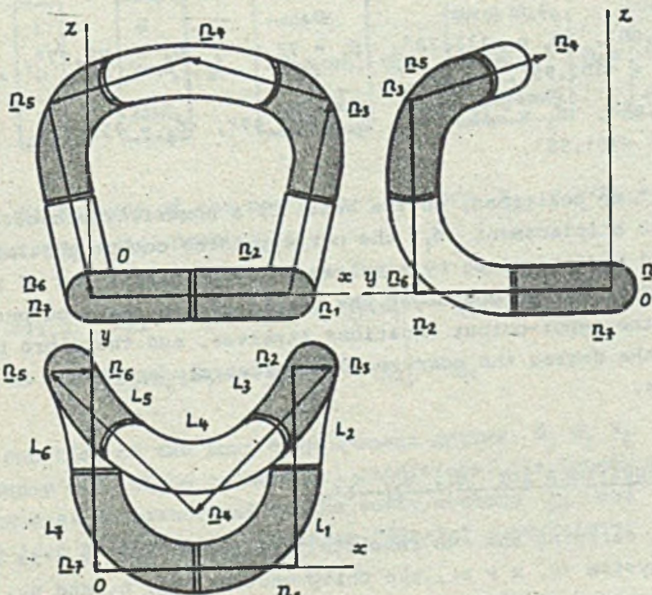


Fig. 2

We have to distinguish between two types of this mechanism. In Fig. 1 and Fig. 2 these two types are shown in the starting position $\sigma_1 = 0$. To change from one type to the other is only possible by dismounting the mechanism and reassembling it anew. The input-output relations we are going to derive do not differentiate between these types but the graphs of these relations show two closed loops corresponding to them.

The unity vectors in the joint axes of the second basic 7R mechanism in the starting position have in the cartesian coordinate system (0, x y z) the following decompositions:

$$\underline{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{n}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underline{n}_3 = \begin{bmatrix} \frac{\sqrt{2}-1}{2} \\ 0 \\ \frac{\sqrt{2\sqrt{2}+1}}{2} \end{bmatrix}, \underline{n}_4 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \mp \frac{2\sqrt{2}-1}{\sqrt{7}} \\ -\frac{\sqrt{2}}{2}\frac{\sqrt{2}-1}{\sqrt{2\sqrt{2}+1}} \end{bmatrix}, \underline{n}_5 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \mp \frac{2\sqrt{2}-1}{\sqrt{7}} \\ -\frac{\sqrt{2}}{2}\frac{\sqrt{2}-1}{\sqrt{2\sqrt{2}+1}} \end{bmatrix}, \underline{n}_6 = \begin{bmatrix} \frac{\sqrt{2}-1}{2} \\ 0 \\ -\frac{\sqrt{2\sqrt{2}+1}}{2} \end{bmatrix}, \underline{n}_7 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

The sign above is valid for type 1 and the sign underneath for type 2 of the mechanism. With these vectors the angular displacements σ_i at the starting position ($\sigma_1 = 0$) can be determined by the formulas:

$$-\underline{n}_i \cdot \underline{n}_{i+2} = \cos\sigma_{i+1}, \quad (\underline{n}_i \times \underline{n}_{i+1}) \cdot n_{i+2} = \sin\sigma_{i+1} \tag{2}$$

i = 1(1)7 cyclic. Their evaluation leads (for type 1 and 2) to the following results:

$\sigma_1 = 0$, $\sigma_2 = 101{,}05°$, $\sigma_3 = -133{,}72°$, $\sigma_4 = 72{,}97°$, $\sigma_5 = -72{,}97°$,
$\sigma_6 = 133{,}72°$, $\sigma_7 = -101{,}95°$,

$\sigma_1 = 0$, $\sigma_2 = 101{,}95°$, $\sigma_3 = -46{,}28°$, $\sigma_4 = -72{,}97°$, $\sigma_5 = 72{,}97°$,
$\sigma_6 = 46{,}28°$, $\sigma_7 = -101{,}95°$.

With these starting positions, on the basis of a numerical procedure to every input angular displacement σ_1 the corresponding output angular displacements could be determined by simultaneously solving a set of non-linear equations. This way we would get the graphs of all input-output relations but not the input-output equations itself, and therefore no information about the degree the corresponding algebraic equations in the tan - half - angles.

Three "fit-in" equations for $\sigma_1, \sigma_2, \sigma_6, \sigma_7$

If the link L_7, carrying the two revolute joints R_7 and R_1, is fixed in the coordinate system (0, x y z), the unity vectors $\underline{n}_2$, $\underline{n}_3$ and $\underline{n}_5$, $\underline{n}_6$ can be given as functions of the angles σ_1, σ_2 and σ_6, σ_7 respectively.

From Fig. 3 and Fig. 4 we immediately read off the representations of the unity vectors $\underline{n}_1\ \underline{n}_2\ \underline{n}_3,\ \underline{n}_5\ \underline{n}_6\ \underline{n}_7$ in the coordinate system (0, x y z):

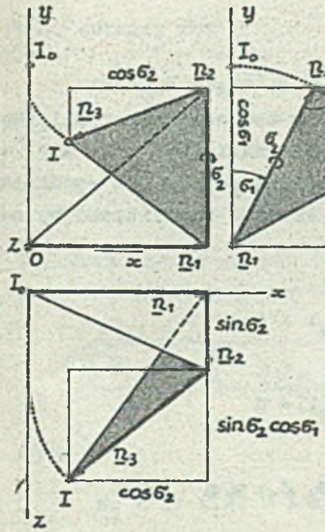


Fig. 3

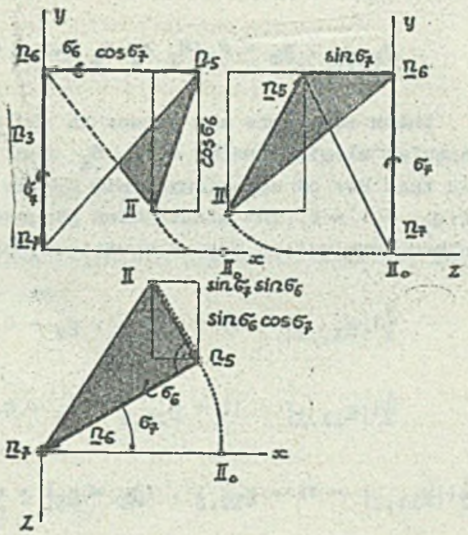


Fig. 4

$$\underline{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \underline{n}_2 = \begin{bmatrix} 0 \\ \cos\sigma_1 \\ \sin\sigma_1 \end{bmatrix}, \underline{n}_3 = \begin{bmatrix} -\cos\sigma_2 \\ -\sin\sigma_2 \sin\sigma_1 \\ \sin\sigma_2 \cos\sigma_1 \end{bmatrix}, \underline{n}_5 = \begin{bmatrix} \sin\sigma_6 \sin\sigma_7 \\ \cos\sigma_6 \\ \sin\sigma_6 \cos\sigma_7 \end{bmatrix} \quad \underline{n}_6 = \begin{bmatrix} -\cos\sigma_7 \\ 0 \\ \sin\sigma_7 \end{bmatrix}, \underline{n}_7 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

The difference of the position - vectors $x_{II} = -(\underline{n}_5 + \underline{n}_6 + \underline{n}_7)$ and $\underline{x}_I = \underline{n}_1 + \underline{n}_2 + \underline{n}_3$ namely:

$$\underline{x}_{II,I} = \underline{x}_{II} - x_I = -\big[\underline{n}_1 + \underline{n}_2(\sigma_1) + \underline{n}_3(\sigma_1,\sigma_2) + $$
$$+ \underline{n}_5(\sigma_6,\sigma_7) + \underline{n}_6(\sigma_7) + \underline{n}_7\big] = \underline{n}_4 \tag{3}$$

is a function of the four displacement angles $\sigma_1\ \sigma_2\ \sigma_6$ and σ_7 and has to comply with three geometrical conditions: Its length must be equal to 1 and it must be orthogonal to the unity vectors $\underline{n}_3$ and $\underline{n}_5$. These conditions lead to the following three "fit-in" - equations:

$$\frac{1}{2}(|\underline{x}_{II,I}| - 1) = F_1(\sigma_1\ \sigma_2\ \sigma_6\ \sigma_7) = 0 \tag{4}$$

$$\underline{x}_{II,I} \cdot \underline{n}_3 = F_2(\sigma_1\ \sigma_2\ \sigma_6\ \sigma_7) = 0 \tag{5}$$

$$\underline{x}_{II,I} \cdot \underline{n}_5 = F_3(\sigma_1\ \sigma_2\ \sigma_6\ \sigma_7) = 0 \tag{6}$$

These equations are linear in the sines and in the cosines of all the angular displacements $\sigma_1\ \sigma_2\ \sigma_6$ and σ_7. The elimination of σ_2 and σ_6 in this set of equations would result in the first input-output equation $f(\sigma_1, \sigma_7) = 0$. The elimination process however is much facilitated by using other, equivalent set of equations, namely:

$$\frac{1}{2}(|\underline{x}_{II,I}| - 1) + \underline{x}_{II,I} \cdot \underline{n}_3 = G_1(\sigma_1\ \sigma_2\ \sigma_7) = 0 \tag{7}$$

$$\frac{1}{2}(|\underline{x}_{II,I}| - 1) + \underline{x}_{II,I} \cdot \underline{n}_5 = G_2(\sigma_1\ \sigma_6\ \sigma_7) = 0 \tag{8}$$

$$\frac{1}{2}(|\underline{x}_{II,I}| - 1) + \underline{x}_{II,I} \cdot (\underline{n}_3 + \underline{n}_5) = G_3(\sigma_1\ \sigma_2\ \sigma_6\ \sigma_7) = 0 \tag{9}$$

In the equation (7) the angular displacement σ_6 does not enter, and in equation (8) on the other hand σ_2 is not present. The equations (7, 8, 9) written in full length read:

$$G_1 := (\tfrac{3}{2} - \cos\sigma_1 - \cos\sigma_7 + \sin\sigma_1\sin\sigma_7) +$$
$$+ \sin\sigma_2(\sin\sigma_1 + \cos\sigma_1\sin\sigma_7) + \cos\sigma_2(\cos\sigma_7 - 1) = 0,$$

$$G_2 := (\tfrac{3}{2} - \cos\sigma_7 - \cos\sigma_1 + \sin\sigma_7\sin\sigma_1) +$$
$$+ \sin\sigma_6(\sin\sigma_7 + \cos\sigma_7\sin\sigma_1) + \cos\sigma_6(\cos\sigma_1 - 1) = 0,$$

$$G_3 := (\tfrac{1}{2} - \cos\sigma_1 - \cos\sigma_7 + \sin\sigma_1\sin\sigma_7) +$$
$$+ \sin\sigma_2(\sin\sigma_1\cos\sigma_6 - \cos\sigma_1\cos\sigma_7\sin\sigma_6) + \cos\sigma_2\sin\sigma_6\sin\sigma_7 = 0.$$

The elimination of the angular displacements σ_2 and σ_6 from this set of equations gives not only the relation between the angular displacements σ_1 and σ_7 but, by cyclic exchangements also the equations relating any two neighbouring angular displacements (σ_i and σ_{i+1}; $i = 1(1)7$). This is a consequence of the fact that we are analysing a closed, homogeneous kinematical chain.

Introducing now new wariables x_α, connected with the angular displacements σ_α by

$$x_\alpha = \tan(\sigma_\alpha/2) \tag{10}$$

i.e. setting for

$$\sin\sigma_\alpha = 2x_\alpha/(1 + x_\alpha^2)$$

and for

$$\cos\sigma_\alpha = (1 - x_\alpha^2)/(1 + x_\alpha^2) \qquad \alpha = 1,2,5,6$$

converts the equations (7, 8, 9) into algebraic equations. This way we get then the following new set of equations:

$$\boxed{H_1 = a_2 x_2^2 + a_1 x_2 + a_o = 0} \tag{11}$$

with

$$a_2 = 11\,x_1^2 x_7^2 + 3\,x_1^2 + 8\,x_1 x_7 + 7\,x_7^2 - 1$$

$$a_1 = 8(-x_1^2 x_7 + x_7^2 x_1 + x_1 + x_7)$$

$$a_o = 3\,x_1^2 x_7^2 + 3\,x_1^2 + 8\,x_1 x_7 - x_7^2 - 1$$

and

$$\boxed{H_2 = b_2 x_6^2 + b_1 x_6 + b_o = 0} \tag{12}$$

wigh

$$b_2 = 11\,x_1^2 x_7^2 + 7\,x_1^2 + 8\,x_1 x_7 + 3\,x_7^2 - 1$$

$$b_1 = 8(x_1^2 x_7 - x_1 x_7^2 + x_1 + x_7)$$

$$b_o = 3\,x_1^2 x_7^2 - x_1^2 + 8\,x_1 x_7 + 3\,x_7^2 - 1$$

and finally

$$\boxed{H_3 = c_2 x_2^2 + c_1 x_2 + c_o = 0} \tag{13}$$

with

$$c_2 = c_{22}x_6^2 + c_{21}x_6 + c_{20}$$

where

$$c_{22} = 5\ x_1^2x_7^2 + x_1^2 + 8\ x_1x_7 + x_7^2 - 3$$

$$c_{21} = -8\ x_7(1 + x_1^2)$$

$$c_{20} = 5\ x_1^2x_7^2 + x_1^2 + 8\ x_1x_7 + x_7^2 - 3 = c_{22}$$

$$c_1 = c_{12}x_6^2 + c_{11}x_6 + c_{10}$$

$$c_{12} = 8\ x_1(1 + x_7^2)$$

$$c_{11} = 8(-x_1^2x_7^2 + x_1^2 + x_7^2 - 1)$$

$$c_{10} = 8\ x_1(1 + x_7^2) = c_{12}$$

$$c_o = c_{o2}x_6^2 + c_{o1}x_6 + c_{\infty}$$

$$c_{o2} = 5\ x_1^2x_7^2 + x_1^2 + 8\ x_1x_7 + x_7^2 - 3 = c_{22} = c_{20}$$

$$c_{o1} = 8\ x_7(1 + x_1^2) = -c_{21}$$

$$c_{oo} = 5\ x_1^2x_7^2 + x_1^2 + 8\ x_1x_7 + x_7^2 - 3 = c_{22} = c_{o2} = c_{20}.$$

The input-output equations

The mechanism we are investigating is a homogeneous one, i.e., identical links are connected by identical joint and thus forming a single loop. For such a mechanism there exist only three essentially different (algebraic) input-output equations, namely:

$$f_I(x_i,\ x_{i+1}) = 0$$

$$f_{II}(\ _i,\ x_{i+2}) = 0$$

$$f_{III}(x_i,\ x_{i+3}) = 0 \qquad i = 1(1)7 \text{ cyclic}$$

In addition, these equations are symmetric in their variables; the direction of the cyclic exchangement of the index i does not effect the result (Fig. 5):

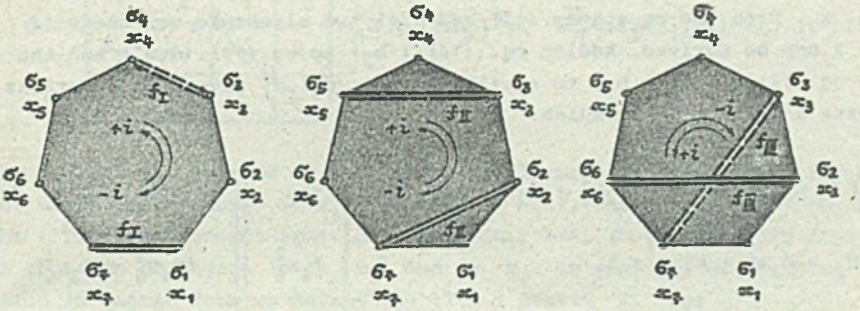


Fig. 5

$$f_I(x_i,\ x_{i+1}) = 0 = f_I(x_{i+1},\ x_i)$$

$$f_{II}(x_i,\ x_{i+1}) = 0 = f_{II}(x_{i+2},\ x_i)$$

$$f_{III}(x_i,\ x_{i+3}) = 0 = f_{III}(x_{i+3},\ x_i)$$

<u>The first input-output equation $f_I(x_i,\ x_{i+1}) = 0$</u>

To find the first input-output equation we eliminate (in two steps) from the equations (11, 12, 13) the variables σ_2 and σ_6 with the Euler-Sylvester resultant method [3].

Adding eq. (11).$(-c_2)$ to eq.(13).(a_2) and then adding eq.(11).$-(c_2x_2+c_1)$ to eq.(13).$(a_2x_2 + a_1)$ gives two equations, linear in x_2. The elimination of x_2 from these equations results in an algebraic equation of degree 4 in x_6. In formulas:

$$\begin{array}{ll|c|l}
H_1 = a_2x_2^2 + a_1x_2 + a_o = 0 & & -c_2 & -(c_2x_2 + c_1) \\
H_3 = c_2x_2^2 + c_1x_2 + c_o = 0 & & a_2 & (a_2x_2 + a_1)
\end{array}$$

$$\Longrightarrow \begin{vmatrix} a_2 a_1 \\ c_2 c_1 \end{vmatrix} x_2 + \begin{vmatrix} a_2 a_o \\ c_2 c_o \end{vmatrix} = 0 \ \vdots\ - \begin{vmatrix} a_2 a_o \\ c_2 c_o \end{vmatrix}$$

$$\begin{vmatrix} a_2 a_o \\ c_2 c_o \end{vmatrix} x_2 + \begin{vmatrix} a_1 a_o \\ c_1 c_o \end{vmatrix} = 0 \ \vdots\ \begin{vmatrix} a_2 a_1 \\ c_2 c_1 \end{vmatrix} \Longrightarrow$$

$$K = \begin{vmatrix} a_2 a_1 \\ c_2 c_1 \end{vmatrix} \begin{vmatrix} a_1 a_o \\ c_1 c_1 \end{vmatrix} - \begin{vmatrix} a_2 a_o \\ c_2 c_o \end{vmatrix}^2 = d_4 x_6^4 + d_3 x_6^3 + d_2 x_6^2 + d_1 x_6 + d_o = 0 \tag{14}$$

The coefficients d_i in this equation are polynoms of degree 8 in x_1 and in x_7. From the equations (12) and (14) two algebraic equations of degree 3 can be derived. Adding eq. (14).$(-b_2)$ to eq.(12).$(d_2 x_6^2)$ and the adding eq.(14).$-(b_2 x_6 + b_1)$ to eq.(12).$(d_4 x_6 + d_3).x_6^2$ gives two equations of degree 3 in x_6. In formulas:

$$\begin{array}{ll|c||c} K = d_4 x_6^4 + d_3 x_6^3 + d_2 x_6^2 + d_1 x_6 + d_o & = 0 & -b_2 & -(b_2 x_6 + b_1) \\ H_2 = b_2 x_6^2 + b_1 x_6 + b_o & = 0 & f_4 x_6^2 & (d_4 x_6^3 + d_3 x_6^2) \end{array}$$

$$\Longrightarrow \begin{vmatrix} d_4 d_3 \\ b_2 b_1 \end{vmatrix} x_6^3 + \begin{vmatrix} d_4 d_2 \\ b_2 b_o \end{vmatrix} x_6^2 + \begin{vmatrix} d_4 d_1 \\ b_2 \ominus \end{vmatrix} x_6 + \begin{vmatrix} d_4 d_o \\ b_2 \ominus \end{vmatrix} = 0$$

and

$$\begin{vmatrix} d_4 d_2 \\ b_2 b_o \end{vmatrix} x_6^3 + \left(\begin{vmatrix} d_4 d_1 \\ b_2 \ominus \end{vmatrix} + \begin{vmatrix} d_3 d_2 \\ b_1 b_o \end{vmatrix} \right) x_6^2 + \left(\begin{vmatrix} d_4 d_o \\ b_2 \ominus \end{vmatrix} + \begin{vmatrix} d_3 d_1 \\ b_1 \ominus \end{vmatrix} \right) x_6 - d_o b_1 = 0$$

or

$$e_3 x_6^3 + e_2 x_6^2 + e_1 x_6 + e_o = 0 \tag{15}$$

and

$$f_3 x_6^3 + f_2 x_6^2 + f_1 x_6 + f_o = 0. \tag{16}$$

The coefficients e_α and f_α $\alpha = 0\text{-}3$ are polynoms of degree 10 in x_1 and in x_7. These two equations, together with equation (12) multiplied by x_6 and the equation (12) itself can be written in the form of a homogeneous matrix equation:

$$\begin{bmatrix} e_3 & e_2 & e_1 & e_o \\ f_3 & f_2 & f_1 & f_o \\ b_2 & b_1 & b_o & 0 \\ 0 & b_2 & b_1 & b_o \end{bmatrix} \cdot \begin{bmatrix} x_6^3 \\ x_6^2 \\ x_6 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

As a trivial solution of this equation has to be excluded, the determinant of the coefficient matrix must vanish:

$$\det \begin{vmatrix} e_3 & e_2 & e_1 & e_o \\ f_3 & f_2 & f_1 & f_o \\ b_2 & b_1 & b_o & 0 \\ 0 & b_2 & b_1 & b_o \end{vmatrix} = 0 = (1+x_1^2)^4 \cdot (1+x_7^2)^4 \cdot f_I(x_1, x_7) \tag{18}$$

The determinant equated to zero gibes an algebraic equation (of degree 24) of which two factors, $(1+x_1^2)^4$ and $(1+x_7^2)^4$, can be splitted off. The sought - for input-output equation $f_I(x_1, x_7) = 0$ is therefore an algebraic equation of degree 16 in x_1 and in x_7. In oder to write this equation in matrix form we define the 17 x 1 matrix $\underline{X}_i$ by:

$$\underline{X}_i^T = \begin{bmatrix} x_i^{16} & x_i^{15} & x_i^{14} & x_i^{13} & x_i^{12} & x_i^{11} & x_i^{10} & x_i^{9} & x_i^{8} & x_i^{7} & x_i^{6} & x_i^{5} & x_i^{4} & x_i^{3} & x_i^{2} & x_i^{1} & 1 \end{bmatrix} \tag{19}$$

and the 17 x 17 matrix $\underline{A}$, which is symmetric, and has a chess-board structure i.e.:

$$A_{\alpha\beta} = A_{\beta\alpha} = \begin{cases} 0 & \text{if } \alpha+\beta \text{ is an odd number} \\ \neq 0 & \text{if } \alpha+\beta \text{ is an even number} \end{cases} \tag{20}$$

$\longrightarrow \beta$

$\alpha \downarrow$

261121	1786568	1316380	64120	-237626	12152	96028	35784	3969
	2076928	14724992	9806848	-1431936	-2804480	-402304	181760	40320
	19594912	56435616	26401632	-10528336	-8771232	-1027424	146592	11592
		121910400	156354432	34570368	-22310272	-5713536	1209984	260480
		338635920	256285024	1456808	-24473760	2281616	1352096	-24548
			391229184	60241536	-95220224	-12288896	3895040	21888
			58708640	-256572464	-71246688	14775904	2340192	-52744
				343221120	-150997120	52297088	11496320	-1022592
				-185991132	116588496	27612840	-5274192	61894
					149808384	40382848	14530580	308352
					44699296	-25118624	764768	18680
						-29831040	1209728	90240
						1370256	199840	1820
							258816	6272
							8864	72
								128
								1

this means that only 81 of the 289 elements of $\underline{A}$ do not vanish. These elements can be written down in the following condensed form:

$$A_{\alpha\beta} \; : \quad \alpha = 1(2)17 \wedge \beta = \alpha\,(2) \begin{cases} 17 & \text{if } \alpha \text{ is odd} \\ 16 & \text{if } \alpha \text{ is even} \end{cases} \tag{21}$$

With these two matrices the first input-output equation can be written in the form:

$$f_I(x_i,\, x_{i+1}) = \underline{X}_i^T \underline{A}\, \underline{X}_{i+1} = 0 \tag{22}$$

The second input-output equation $f_{II}(x_i,\, x_{i+2}) = 0$

The elimination of the variables x_2 and x_7 from the set of equations (11, 12, 13) leads to a relation between x_1 and x_6, and there with all the other input-output equations $f_{II}(x_i,\, x_{i+2}) = 0$ are determined. Of course, if we are only interested in numerical results, with the known relation between x_1 and x_7 we could determine the variable x_6 corresponding to any pair of the variablex x_1 and x_7. As the coefficient determinant (equation 18) vanishes we could solve eq.(17) for x_6:

$$\tan(\sigma_6/2) = x_6(x_1, x_7(x_1)) = \det \begin{vmatrix} e_3 & e_2 & -e_o \\ f_3 & f_2 & -f_o \\ b_2 & b_1 & 0 \end{vmatrix} : \det \begin{vmatrix} e_3 & e_2 & e_1 \\ f_3 & f_2 & f_1 \\ b_2 & b_1 & b_o \end{vmatrix} \tag{23}$$

But now to find the immediate relation between x_6 and x_1, we shall start again with the fundamental "fit-in" equations (11, 12, 13). These equations can be rewritten as algebraic equations of degree 2 in x_7:

$$\boxed{H_1 = A_2 x_7^2 + A_1 x_7 + A_o = 0} \tag{24}$$

with

$$A_2 = 11\, x_1^2 x_2^2 + 3\, x_1^2 + 8\, x_1 x_2 + 7\, x_2^2 - 1$$

$$A_1 = 8(-x_1^2 x_2 + x_2^2 x_1 + x_1 + x_2)$$

$$A_o = 3\, x_1^2 x_2^2 + 3\, x_1^2 + 8\, x_1 x_2 - x_2^2 - 1$$

and

$$\boxed{H_2 = B_2 x_7^2 + B_1 x_7 + B_o = 0} \tag{25}$$

with

$$B_2 = 11 x_1^2 x_6^2 + 3 x_6^2 - 8 x_1 x_6 + 3 x_1^2 + 3$$

$$B_1 = 8(x_1 x_6^2 + x_1^2 x_6 + x_1 + x_6)$$

$$B_o = 7 x_1^2 x_6^2 - x_6^2 + 8 x_1 x_6 - x_1^2 - 1$$

and finally

$$\boxed{H_3 = C_2 x_7^2 + C_1 x_7 + C_o = 0} \tag{26}$$

with

$$C_2 = C_{22} x_2^2 + C_{21} x_2 + C_{20}$$

where

$$C_{22} = (5 x_1^2 + 1)(x_6^2 + 1)$$

$$C_{21} = 8\left[-x_1 x_6^2 + (1 - x_1^2) x_6 + x_1\right]$$

$$C_{20} = (5 x_1^2 + 1)(x_6^2 + 1) = C_{22}$$

$$C_1 = C_{12} x_2^2 + C_{11} x_2 + C_{10}$$

$$C_{12} = 8\left[x_1 x_6^2 - (1 + x_1^2) x_6 + x_1\right]$$

$$C_{11} = 0$$

$$C_{10} = 8\left[x_1 x_6^2 + (1 + x_1^2) x_6 + x_1\right]$$

$$C_o = C_{o2}x_2^2 + C_{o1}x_2 + C_{oo}$$

$$C_{o2} = (x_1^2 - 3)(x_6^2 + 1)$$

$$C_{o1} = 8\left[-x_1x_6^2 - (1 - x_1^2)x_6 + x_1\right] = C_{21}$$

$$C_{oo} = (x_1^2 - 3)(x_6^2 + 1) = C_{o2}$$

The elimination of the variable x_7 first from the equations (24) and (25) and then from the equations (25) and (26) gives two algebraic equations of degree 4 in x_2. The elimination process is the same as that leading to equation (14). These two equations read

$$L = D_4x_2^4 + D_3x_2^3 + D_2x_2^2 + D_1x_2 + D_o = 0 \tag{27}$$

$$M = E_4x_2^4 + E_3x_2^3 + E_2x_2^2 + E_1x_2 + E_o = 0 \tag{28}$$

where in the coefficients D_i are polynoms in the variables x_1 and x_6 of degree 8 and 4, respectively, and the coefficients E_i are polynoms in x_1 and x_6 of degree 8. From them we can derive four algebraic equations of degree 3 in x_2:

$$\begin{array}{l|l|l|l|l}
L = 0 & -E_4 & -(E_4x_2+E_3) & -(E_4x_2^2+E_3x_2+E_2) & -(E_4x_2^3+E_3x_2^2+E_2x_2+E_1) \\
M = 0 & D_4 & (D_4x_2+D_3) & (D_4x_2^2+D_3x_2+D_2) & (D_4x_2^3+D_3x_2^2+D_2x_2+D_1)
\end{array}$$

$$M.D_4 - L.E_4 = K_{11}x_2^2 + K_{12}x_2^2 + K_{13}x_2 + K_{14} = 0$$

$$M.(D_4x_2+D_3) - L.(E_4x_2+E_3) = K_{21}x_2^3 + K_{22}x_2^2 + K_{23}x_2 + K_{24} = 0$$

$$M.(D_4x_2^2+D_3x_2+D_2) - L.(E_4x_2^2+E_3x_2+E_2) = K_{31}x_2^3 + K_{32}x_2^2 + K_{33}x_2 + K_{34} = 0$$

$$M.(D_4x_2^3+D_3x_2^2+D_2x_2+D_1) - L.(E_4x_2^3+E_3x_2^2+E_2x_2+E_1) = K_{41}x_2^3 + K_{42}x_2^2 + K_{43}x_2 + K_{44} = 0$$

or written in matrix form:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{47} & K_{42} & K_{34} & K_{44} \end{bmatrix} \cdot \begin{bmatrix} x_2^3 \\ x_2^2 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{29}$$

As no trivial solution of this linear homogeneous matrix equation exists $(1 \neq 0!)$ the determinant of the matrix $\underline{K} = \| K_{ij} \|$ must vanish:

$$D(x_1, x_6) = \det \| K_{ij} \| = 0 = \text{factor}(x_1, x_6)(1+x_1^2)^4 (1+x_6^2)^4 \cdot f_{II}(x_1 x_6) \tag{30}$$

The elements of the matrix $\underline{K}$ are polynoms of degree 14 in x_1 and in x_6. The determinant of $\underline{K}$ equated to zero therefore will be an algebraic equation of degree 56 in the variables x_1 and x_6. This means that there is in eq. (30) a big factor containing all unwanted roots! By trying to factorize eq.(30) in a straightforward way, even using the newest version of the symbolic computation software REDUCE, we were not successful. The input-output equation $f_{II}(x_1, x_6) = 0$ finally could only be found by factorizing the determinant D for a number of specified values of x_6

$$D(x_1, x_6 = 0;\ 1;\ 2;\ 3;\ 1/2;\ 1/3) = 0,$$

and for $x_6 = \pm x_1$

$$D(x_1, x_6 = x_1) = 0; \quad D(x_1, x_6 = -x_1) = 0$$

This way a sufficient number of linear equations for the determination of the 81 coefficients B_{ij} entering $f_{II}(x_1, x_6) = 0$ could be found and solved. Again, with $f_{II}(x_1, x_6) = 0$ every input-output equation of the form $f_{II}(x_i, x_{i+2}) = 0$ is known.

With the 17 x 1 matrix $\underline{X}_i$ of eq.(19) and the 17 x 17 matrix $\underline{B}$ (eqs. 32, 33) the input-output equation $f_{II}(x_i, x_{i+2}) = 0$ can be written in the form of a matrix equation:

$$f_{II}(x_i, x_{i+2}) = \underline{X}_i^T \, \underline{B} \, \underline{X}_{i+2} = 0 \tag{31}$$

The matrix $\underline{B}$ in this equation is symmetric and has a chess-board structure, i.e.:

$$B_{\alpha\beta} = B_{\beta\alpha} = \begin{cases} 0 & \text{if } \alpha+\beta \text{ is an odd number} \\ \neg 0 & \text{if } \alpha+\beta \text{ is an even number} \end{cases} \tag{32}$$

Of the 17 17 = 289 elements of $\underline{B}$ only 81 do not vanish. These elements are given by:

$$B_{\alpha\beta} \quad \alpha = 1(2)17 \wedge \beta = \alpha\ (2) \begin{cases} 17 & \text{if} \quad \text{is odd} \\ 16 & \text{if} \quad \text{is even} \end{cases} \tag{33}$$

β

α

3969		18648		12804		3320		646		-88		4		8		1
	1008		14016		2368		-13712		-3440		-320		64		16	
		60294		-36600		-33134		-15600		-1480		1048		198		8
			125480		52408		-57840		5136		4904		1144		160	
				-180479		52044		-53106		-7572		5985		872		-12
					-128664		-149904		49264		24312		4328		480	
						216518		-72824		150		9196		-30		-232
							-83504		38512		19152		3248		336	
								91720		52520		-3506		-4240		-538
									-7088		-7472		-2896		-336	
										10726		-6724		-2206		-232
											-11288		-4232		-480	
												-1983		-152		-12
													-1496		-160	
														70		8
															-16	
																1

<u>The third input-output equation $f_{III}(x_i, x_{i+3}) = 0$ (unsolved)</u>

To find $f_{III}(x_i, x_{i+3}) = 0$ we would have to eliminate from the equations (24, 25, 26) the variables x_1 and x_7. The smalest matrix whose determinant equated to zero would give the sought - for equation has a size of 8 x 8 and the element of this matrix are polynoms of degree 12 i x_2 and in x_6. The elimination of the variable from the pairs of equations ($H_1 = 0$, $H_2 = 0$), ($H_1 = 0$, $H_3 = 0$= and ($H_2 = 0$, $H_3 = 0$) gives three equations of degree 8 (the first and the second is identical with eq. 27 and eq. 28, respectively, rewritten as polynoms in x_1):

$$L = \sum_0^8 L_\alpha x_1^\alpha = 0; \quad M = \sum_0^8 M\, x_1^\alpha = 0; \quad N = \sum_0^8 N_\alpha x_1^\alpha = 0 \tag{34}$$

The coefficients L_α, M_α and N_α are polynoms of degree (4,4), (8,4) and (4,8) in the variables (x_2, x_6), respectively. The elimination of the first terms from any pair of these equations can be carried out in seven different ways, therefore we can easely find eight equations of degree 7 in x_1, necessary to eliminate x_1 by the Euler-Sylvester method. This method leads to:

$$F_{III}(x,x_6) = \det\|P_{ij}\|_{8x8} = 0 = \text{factor}(x_2,x_6).f_{III}(x_2,x_6) \quad (35)$$

It was to expect that the evaluation of the determinant of the 8 x 8 matrix $\underline{P}$ (whose elements are polynoms of degree 12 in x_2 and in x_6) were not feasible. Even for x_2 = constant it was impossible to get a result. So the determination of $f_{III}(x_i,x_{i+3}) = 0$ with the minimum degree (16) remains an unsolved problem, though of course, equation (35) can be used to determine numerically the variablex x_6 corresponding to any chosen input variable x_2.

The one-step elimination procedure

Thus far we have only used the three fit - in equations in the elimination procedures. Since J. Duffy and C. Crane have published their paper [2] it is well known that there exist a fourth equation of the same type as the fit - in equations. With the aid of this fourth equation Duffy and Crane were able to find the input-output equation for the general 7R mechanism. As has been stated in the introduction there are some limits for this general solution. For the parameter structure of our second basic 7R mechanism the fourth Duffy-Crane equation seems not to exist. Equation 14 in [2] turns out to be an empty equation, i.e., it becomes 0 = 0 as the three constants K_1, K_2 and K_3 all vanish. The general solution of Duffy and Crane therefore must fail.

The search for an other fourth equation which can be substituted for the lacking Duffy-Crane equation has lead to:

$$F_4(\vartheta_1\vartheta_2\vartheta_6\vartheta_7) = F_2.F_3 - (\underline{n}_3 \circ \underline{n}_5)F_1 = 0 \Longrightarrow H_4(x_1x_2x_6x_7) = 0 \quad (36)$$

where for F_1, F_2 and F_2 is to insert according to the equations (4, 5, 6). The equation (36) is linear in the sines and cosines of the angular displacements $\vartheta_1\ \vartheta_2\ \vartheta_6$ and ϑ_7. This equation together with the three fir-in equations enable us to eliminate two of the four variables in one step.

The input-output equation $f^{*}_{III}(x_i, x_{i+3}) = 0$

As the algebraic equations $f_I(x_i, x_{i+1}) = 0$ and $f_{II}(x_i, x_{i+2}) = 0$ have already been found with the minimum degree in their variables, we now focus our attention on the relation between the variables x_i and x_{i+3} only. The simultaneous elimination of the variables x_1 and x_7 from the four equations $H = 0 \ldots \alpha = 1(1)4$ leads to a 16 x 16 determinant equated to zero resulting in an algebraic equation of degree 24 (instead of 16) in the variables x_2 and x_6.

Let us define first a 4 x 9 matrix $\underline{R}$ with the four row-vectors $\underline{R}_1$, $\underline{R}_2$, $\underline{R}_3$ and $\underline{R}_4$ given by:

$$\underline{R}_1 = \begin{bmatrix} 3+11x_2^2 & 8x_2 & -1+7x_2^2 & -8x_2 & 8(1+x_2^2) & 8x_2 & 3(1+x_2^2) & 8x_2 & -(1+x_2^2) \end{bmatrix}$$

$$\underline{R}_2 = \begin{bmatrix} 3+11x_6^2 & -8x_6 & 3(1+x_6^2) & 8x_6 & 8(1+x_6^2) & 8x_6 & -1+7x_6^2 & 8x_6 & -(1+x_6^2) \end{bmatrix}$$

$$\underline{R}_3^T = \begin{bmatrix} 5-8x_2x_6+5x_2^2x_6^2+5x_2^2+5x_6^2 \\ 8x_2(1-x_6^2) \\ 1+8x_2x_6+x_2^2x_6^2+x_2^2+x_6^2 \\ 8x_6(1-x_2^2) \\ 8(1+x_2^2x_6^2+x_2^2+x_6^2) \\ 8(1-x_2^2)x_6 \\ 1+8x_2x_6+x_2^2x_6^2+x_2^2+x_6^2 \\ 8(1-x_6^2)x_2 \\ -3-8x_2x_6-3x_2^2x_6^2-3x_2^2-3x_6^2 \end{bmatrix} \qquad \underline{R}_4^T = \begin{bmatrix} 1-10x_2x_6+9x_2^2x_6^2-3x_2^2-3x_6^2 \\ 4x_6+6x_2-12x_2^2x_6+2x_2x_6^2 \\ -1+2x_2x_6+3x_2^2x_6^2+3x_2^2-x_6^2 \\ 4x_2+6x_6-12x_2x_6^2+2x_2^2x_6 \\ 16x_2x_6 \\ 4x_2-2x_6+4x_2x_6^2+10x_2^2x_6 \\ -1+2x_2x_6+3x_2^2x_6^2-x_2^2+3x_6^2 \\ -2x_2+4x_6+10x_2x_6^2+4x_2^2x_6 \\ 1+6x_2x_6+x_2^2x_6^2+x_6^2+x_2^2 \end{bmatrix} \qquad (37)$$

and then the vector $\underline{S}$:

$$\underline{S}^T = \begin{bmatrix} x_7^2x_1^2 & x_7^2x_1 & x_7^2 & x_7x_1^2 & x_7x_1 & x_7 & x_1^2 & x_1 & 1 \end{bmatrix} \qquad (38)$$

With these matrices the four equations (24, 25, 26 and 36) can be written in the form of matrix equations:

$$H_1 = 0 = \underline{R}_1\underline{S}, \quad H_2 = 0 = \underline{R}_2\underline{S}, \quad H_3 = 0 = \underline{R}_3\underline{S}, \quad H_4 = \underline{R}_4\underline{S} = 0 \qquad \underline{R}\ \underline{S} = \underline{0} \Longrightarrow (39)$$

An equivalent linear homogeneous matrix equation with a quadratic matrix can be derived in the following way. Multiplication of the equations $H = 0 \ldots = 1(1)4$ by x_1, x_7 and $x_1 \cdot x_7$ gives 12 additional equations:

$$H_\alpha = 0; \quad H_\alpha x_1 = 0; \quad H_\alpha x_7 = 0, \quad H_\alpha x_1 x_7 = 0 \tag{40}$$

With the aid of the matrices $\underline{M}$ and $\underline{Z}$ defined by:

$$\underline{Z}^T = \begin{bmatrix} x_7^3x_1^3 & x_7^3x_1^2 & x_7^3x_1 & x_7^3 & x_7^2x_1^3 & x_7^2x_1^2 & x_7^2x_1 & x_7^2 & x_7x_1^3 & x_7x_1^2 & x_7x_1 & x_7 & x_1^3 & x_1^2 & x_1 & 1 \end{bmatrix}$$

and ....(41)

$$\underline{M} = \begin{bmatrix}
&&&&&R_{11}&R_{12}&R_{13}&&R_{14}&R_{15}&R_{16}&&R_{17}&R_{18}&R_{19} \\
&&&&R_{11}&R_{12}&R_{13}&&R_{14}&R_{15}&R_{16}&&R_{17}&R_{18}&R_{19}& \\
&R_{11}&R_{12}&R_{13}&&R_{14}&R_{15}&R_{16}&&R_{17}&R_{18}&R_{19}&&&& \\
R_{11}&R_{12}&R_{13}&&R_{14}&R_{15}&R_{16}&&R_{17}&R_{18}&R_{19}&&&&& \\
&&&&&R_{21}&R_{22}&R_{23}&&R_{24}&R_{25}&R_{26}&&R_{27}&R_{28}&R_{29} \\
&&&&R_{21}&R_{22}&R_{23}&&R_{24}&R_{25}&R_{26}&&R_{27}&R_{28}&R_{29}& \\
&R_{21}&R_{22}&R_{23}&&R_{24}&R_{25}&R_{26}&&R_{27}&R_{28}&R_{29}&&&& \\
R_{21}&R_{22}&R_{23}&&R_{24}&R_{25}&R_{26}&&R_{27}&R_{28}&R_{29}&&&&& \\
&&&&&R_{31}&R_{32}&R_{33}&&R_{34}&R_{35}&R_{36}&&R_{37}&R_{38}&R_{39} \\
&&&&R_{31}&R_{32}&R_{33}&&R_{34}&R_{35}&R_{36}&&R_{37}&R_{38}&R_{39}& \\
&R_{31}&R_{32}&R_{33}&&R_{34}&R_{35}&R_{36}&&R_{37}&R_{38}&R_{39}&&&& \\
R_{31}&R_{32}&R_{33}&&R_{34}&R_{35}&R_{36}&&R_{37}&R_{38}&R_{39}&&&&& \\
&&&&&R_{41}&R_{42}&R_{43}&&R_{44}&R_{45}&R_{46}&&R_{47}&R_{48}&R_{49} \\
&&&&R_{41}&R_{42}&R_{43}&&R_{44}&R_{45}&R_{46}&&R_{47}&R_{48}&R_{49}& \\
&R_{41}&R_{42}&R_{43}&&R_{44}&R_{45}&R_{46}&&R_{47}&R_{48}&R_{49}&&&& \\
R_{41}&R_{42}&R_{43}&&R_{44}&R_{45}&R_{46}&&R_{47}&R_{48}&R_{49}&&&&&
\end{bmatrix} \tag{42}$$

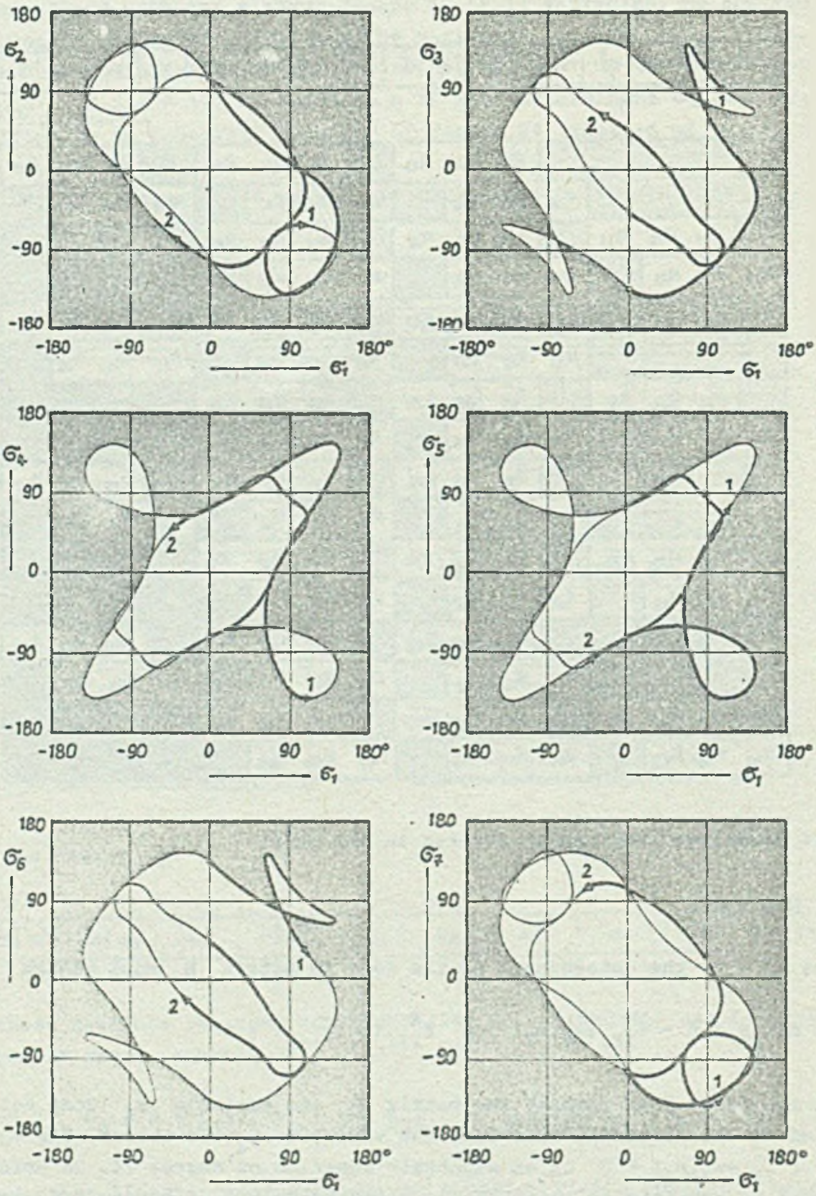
the 16 equations (40) can be written in the form:

$$\underline{M}\,\underline{Z} = \underline{\Theta}, \tag{43}$$

and as $\underline{Z} \neq \underline{0}$ the determinant of the 16 x 16 matrix $\underline{M}$ must vanish

$$\det \underline{M} = 0 = f^{*}_{III}(x_6, x_2) \implies f^{*}_{III}(x_i, x_{i+3}) = 0 \tag{44}$$

In the first four rows of the matrix $\underline{M}$ the variable x_6 does not enter and in the following four rows the variable x_2 is absent. The equation $f^{*}_{III}(x_2, x_6) = 0$ is an algebraic equation of degree 24. In principle it would be possible to split off an unwanted factor of degree 8 to get the minimum equation $f_{III}(x_2, x_6) = 0$.


180
σ_2
90
0
-90
-180
-180
-90
0
90
180°
σ_1
1
2
180
σ_3
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0
-90
-180
-180°
-90
0
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σ_1
1
2
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σ_4
90
0
-90
-180
-180
-90
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2
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σ_6
90
0
-90
-180
-180
-90
0
90
180°
σ_1
1
2
180
σ_7
90
0
-90
-180
-180
-90
0
90
180°
σ_1
2
1

Fig. 6

Numerical results

In Fig. 6 numerical results are given for the second basic 7R mechanism. The equations (22), (31) and (44) or (35) do not differentiate between the types of that mechanism, but every graph consists of two closed loops corresponding to the two types. In the algebraic form all the input-output equations are of degree 16 but there are for a given input angular displacement σ_1 at most six real corresponding output angular displacements. All graphs are not only symmetric in their variables σ_1 and σ_α but also skewsymmetric, i.e., the relations $f(\sigma_1,\sigma_\alpha)\ f(-\sigma_\alpha,-\sigma_1) = f(-\sigma_1,-\sigma_\alpha) = 0$ hold. The skew symmetry is a consequence of the fact that the exchanges $x_\alpha \leftrightarrow -x_\alpha \ldots \alpha = 1,2,6,7$ do not affect the equations $H_\alpha = 0 \ldots \alpha = 1(1)4$.

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DRUGI PODSTAWOWY MECHANIZM 7R

S t r e s z c z e n i e

Są 2 rodzaje "podstawowego" przestrzennego mechanizmu 7R. Pierwszy z nich omówiony był w poprzednich 2 artykułach, a dla drugiego przeprowadzona jest analiza przemieszczenia algebraicznego.

ВТОРОЙ ОСНОВНОЙ 7R МЕХАНИЗМ

Р е з ю м е

Существуют два вида "основного" пространственного механизма 7R. Первый был обсуждён с предыдущих двух статьях, а для второго проводится анализ алгебраического перемещения.

Recenzent: Prof. zw. dr inż. Adam Morecki

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