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FINITE TIME BLOW-UP IN SEMILINEAR WAVE EQUATIONS

Streszczenie. W pewnych zjawiskach przebiegających w przyrodzie i technice opisanych procesami ciągłymi pojawia się niebezpieczny efekt polegający na tym, że proces narasta w sposób nieograniczony w skończonym czasie. Opisane zjawisko znane jest jako "blow-up". Rozpatrujemy problem początkowo-brzegowy opisany równaniem falowym. Niech $u: [0, T[\rightarrow X$, $X=L^2(G)$, będzie silnym rozwiązaniem tego problemu. Pokazano, że możemy znaleźć warunki początkowe u_0 i v_0 , $u_0 = u(0)$, $v_0 = u_t(0)$, takie, że rozwiązanie, o ile istnieje, może istnieć tylko na skończonym przedziale $[0, T[$, $T < \infty$ oraz $\lim_{t \rightarrow T^-} \|u(t)\|_2^2 = \infty$.

Резюме. В некоторых явлениях природы и техники описываемых при помощи непрерывных процессов, встречается опасный эффект, смысл которого в том, что процесс возрастает к бесконечности в конечном времени. Описываемое явление известно под названием "blow-up". Мы рассматриваем начально-береговую проблему, описанную волновым уравнением. Пусть $u: [0, T[\rightarrow X$, $X=L^2(G)$, будет сильным решением этой задачи. Можно найти такие начальные условия u_0 и v_0 , $u_0 = u(0)$, $v_0 = u_t(0)$, что решение, когда существует, может существовать только на конечном интервале $[0, T[$, $T < \infty$ и следовательно $\lim_{t \rightarrow T^-} \|u(t)\|_2^2 = \infty$.

Summary. In time-dependent continuous processes in nature or technics, such dangerous effects where the process breaks down after a finite time are called blowing-up effects. We consider the initial-boundary value problem. Let $u: [0, T[\rightarrow X$, $X=L^2(G)$, be a strong solution to this problem. We can prove that there exist initial data u_0 and v_0 , $u_0 = u(0)$, $v_0 = u_t(0)$, so that if solutions exist, it can exist on a finite interval $[0, T[$, $T < \infty$ only.

1. INTRODUCTION

In time-dependent continuous processes in nature or technics, such dangerous effects where the process breaks down after a finite time are

called blowing-up effects. The reason can be that the dynamical system leases too much of its initial energy through dissipation or that it is unstable. The engineer naturally is interested in avoiding such blowing-up effects. In the case of dynamical systems described by a wave equation

$$u_{tt} - \Delta u = f(t, x, u)$$

where f is linear with respect to u , the solutions exist for all points in time [7,8]. Thus, blowing-up effects can occur only in nonlinear systems.

2. PRELIMINARIES

We consider the initial-boundary value problem

$$u_{tt} - \Delta u + m^2 u = f(u) \quad \text{on } G \times]0, T[, \quad (1)$$

$$\begin{aligned} u(x, 0) &= u_0(x) && \text{on } G , \\ u_t(x, 0) &= v_0(x) && \text{on } G , \\ u(x, t) &= 0 && \text{on } \partial G \times]0, T[. \end{aligned}$$

Let $x = \{x_1, x_2, \dots, x_n\}$, $D_i = \partial/\partial x_i$, $u_t = \partial/\partial t$, $u = u(x, t)$, $\Delta u = \sum_{i=1}^n D_i^2 u$,

$\nabla u = \{D_1 u, D_2 u, \dots, D_n u\}$. Here G is a bounded region in \mathbb{R}^n , $n \geq 1$, with a boundary ∂G which is sufficiently smooth to admit of applications of the divergence theorem, $T \in \mathbb{R}^+$, m is a real parameter, $f: G \rightarrow \mathbb{R}^1$, $f(0)=0$. In the case where $f() = 0$, this is the so-called Klein-Gordon equation.

A number of authors [2-6, 9] have shown that solutions to the initial value problem or to initial - boundary value problems for classical nonlinear wave equations in one, two, three or more dimensions are not stable in time for arbitrary initial data and arbitrary nonlinearities. A special attention was given to a nonlinearity $f(u)=|u|^p$, for $p > 1$. It is well known that this problem does not admit a global solution for any such p when the initial values u_0 and v_0 are large in some sense.

In order to formulate (1) as an operator equation of the form

$$u'' + Au = f(u) \quad \text{on }]0, T[, \quad (2)$$

$$u(0) = u_0 ,$$

$$u'(0) = v_0 ,$$

we set $X = L_2(G)$, $D(A_0) = C_0^\infty(G)$, and

$$A_0 u = (-\Delta + m^2)u.$$

For all $u \in D(A_0)$,

$$(A_0 u | u) = (-\Delta u | u) + m^2(u | u) \geq m^2(u | u) \geq 0 \quad (3)$$

Let $A : D(A) \subseteq X \rightarrow X$ denote the Friedrichs extension of A_0 . Then the energetic spaces of A_0 and $-A$ are the same up to an equivalent norm, i.e. the energetic space of A is equal to $X_E = W_2^1(G)$. We say that the function $u : [0, T] \rightarrow X$ is a solution to (2) if, for each $t, u(t)$ and u_t belong to X (u_t being the strong derivative of u in the norm $\|\cdot\|_X$), u_{tt} exists and is strongly continuous in the sense of the norm on X .

It is assumed in the future that for each $u_0 \in X_E$ and $v_0 \in X$ there exists a local strong solution of (2).

The equation (2) possesses a real-valued energy form

$$E(t) = \frac{1}{2} \int_G \left[u_t^2 + m^2 u^2 + \sum_{i=1}^n (D_i u)^2 - 2P(u) \right] dx \quad (4)$$

where $P(u) = \int_0^u f(y) dy$. Since u is a sufficiently smooth solution of (2),

then integration by parts yields

$$E'(t) = \int_G u_t \left[u_{tt} + m^2 u^2 - \Delta u - f(u) \right] dx = 0. \quad (5)$$

i.e., $E(t) = \text{const}$. This describes the conservation of energy.

Thus we do not expect global existence for any $f(\cdot)$. We will show that if u_0 and v_0 are chosen correctly, then

$$F(t) = \int_G u(x, t)^2 dx$$

goes to infinity in finite time.

Let $\|\cdot\|_2$ and $(\cdot | \cdot)$ denote the norm and the scalar product in $L^2(G)$ respectively, defined in usual way.

3. NONEXISTENCE OF GLOBAL SOLUTIONS

THEOREM. Consider the initial-boundary value problem (1-2) in the above formulation. Let $u: [0, T] \rightarrow X$ be a solution to this problem in the prescribed sense. Suppose that we can find an $\alpha > 0$ and initial data u_0 and v_0 so that the following statements hold:

$$(A1) \quad (u_0 | v_0) \geq 0 ,$$

$$(A2) \quad \int_G P(u_0(x)) dx \geq 2 \left[\|v_0\|_2^2 + \|u_0\|_2^2 + \|\nabla u_0\|_2^2 \right] .$$

$$(A3) \quad (u_0 | f(u_0)) \geq 2(2\alpha + 1) \int_G P(u_0(x)) dx .$$

then the solution can only exist on a finite interval $[0, T]$, $T < \infty$ and

$$\lim_{t \rightarrow T^-} \|u(t)\|_2^2 = \infty .$$

Proof. Let $R(t) = F(t)^{-\alpha}$ and the following hypotheses are satisfied

$$(H1) \quad R'(t) < 0 \quad \text{at } t = 0$$

$$(H2) \quad R''(t) \leq 0 \quad \text{for all } t \geq 0 .$$

Then $R(t)$ will go to zero in finite time. We have $F'(t) = 2(u(t)|u_t(t))$,

$$R'(t) = -\alpha F(t)^{-\alpha-1} F'(t)$$

$$R'(0) = -2\alpha F(0)^{-\alpha-1} (u_0 | v_0) \quad (6)$$

Thus the hypothesis (H1) is satisfied following the assumption (A1). It remains to arrange for (H2) to hold. Let

$$R''(t) = \alpha(\alpha+1)R(t)^{-2}F'(t)^2 - \alpha R(t)^{-1}F''(t) \quad (7)$$

We define

$$Q(t) = (-\alpha)^{-1} F(t)^{\alpha+2} R''(t)$$

and following (7) we obtain

$$Q(t) = F(t)F''(t) - (\alpha+1)F'(t)^2$$

Since $F(t) \geq 0$, to examine (H2) this is the same to show that $Q(t) \geq 0$. We have

$$F'(t) = 2(u|u_t)$$

$$\begin{aligned} F''(t) &= 2(u|u_{tt}) + \|u_t\|_2^2 = \\ &= 4(\alpha+1)\|u\|_2^2 + 2[(u|u_{..}) - (2\alpha+1)\|u\|_2^2] \end{aligned}$$

and hence

$$\begin{aligned} Q(t) &= 4(\alpha+1) \left[\|u\|_2^2 \|u_t\|_2^2 - (u|u_t)^2 \right] \\ &\quad + 2F(t) \left[(u|u_{tt}) - (2\alpha+1) \|u_t\|_2^2 \right] \end{aligned} \quad (8)$$

The first term in (8) is positive by the Schwartz inequality, so we'll discuss the second part, denoted by $H(t)$, only

$$\begin{aligned} H(t) &= (u|u_{tt}) - (2\alpha+1) \|u_t\|_2^2 \\ &= \left(u \left| f(u) \right. \right) - m^2 \|u\|_2^2 + (u|\Delta u) - (2\alpha+1) \|u_t\|_2^2 = \\ &= \left(u \left| f(u) \right. \right) - m^2 \|u\|_2^2 - (\nabla u|\nabla u) - (2\alpha+1) \|u_t\|_2^2 \end{aligned} \quad (9)$$

the conserved energy for the equation (1-2) is given by (4) and is independent of t . Thus if we choose α as in assumptions (A2) and (A3) we obtain

$$\begin{aligned} H(t) &= -2(2\alpha+1)E(t) + 2\alpha m^2 \|u\|_2^2 + 2\alpha(\nabla u|\nabla u) \\ &\quad + \left(u \left| f(u) \right. \right) - 2(2\alpha+1) \int_G P(u) dx \\ &= -2(2\alpha+1)E(0) + 2\alpha m^2 \|u\|_2^2 + 2\alpha(\nabla u|\nabla u) \\ &\quad + \left(u_0 \left| f(u_0) \right. \right) - 2(2\alpha+1) \int_G P(u_0) dx \end{aligned} \quad (10)$$

Thus, if $E(0)<0$, it is the consequence of (A2), $H(t)$ is always strictly positive since the second and third components of (10) are always nonnegative and the last term is nonnegative following the assumption (A3).

For any such initial data and a parameter α $F(t)$ goes to infinity in a finite time. ■

4. EXAMPLES

Example I. Let $m=0$ and $f(u)=u^p, p>1$. Now choosing $u_0 \geq 0, v_0 \geq 0$ so that (A1) is satisfied, we can choose α so that $2(2\alpha+1)=p+1$ and u_0 in such a matter, to satisfy $E(0)<0$ (assumption (A2)) and (A3). For any such initial data $F(t)$ goes to infinity in finite time. ■

Example II. Let $m=0$ and $f(u)=-u^p, p > 1$. If p is even then by choosing $u_0 \leq 0, v_0 \leq 0$ thus satisfying (A1), we can choose α as in example I, and u_0 sufficiently large to obtain (A2) (i.e. $E(0)<0$), and thus for any such initial data the solution blows up in finite time.

If on the other hand, p is odd, then $E(t) \geq 0$ so the assumption (A2) is not satisfied and the above arguments doesn't work. ■

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O NIEOGRAŃCZONOŚCI ROZWIĄZAŃ NIELINIOWYCH RÓWNAŃ FALOWYCH

W pewnych zjawiskach przebiegających w przyrodzie i technice opisanych procesami ciągłymi pojawia się niebezpieczny efekt polegający na tym, że proces narasta w sposób nieograniczony w skończonym czasie. Opisane zjawisko znane jest jako "blow-up". Rozpatrujemy problem początkowo-brzegowy opisany równaniem falowym

$$u_t - \Delta u + m^2 u = f(u) \quad \text{on } G \times [0, T],$$

Niech $u: [0, T] \rightarrow X$, $X = L^2(G)$, będzie silnym rozwiązaniem tego problemu, przyjmijmy, że możemy znaleźć $\alpha > 0$ i warunki początkowe u_0 i v_0 , $u_0 = u(0)$, $v_0 = u_t(0)$ takie, że spełnione są następujące założenia:

$$\begin{aligned} \left(u_0 | v_0 \right) &\geq 0, \quad \int_G P(u_0(x)) dx \geq 2 \left[\|v_0\|_2^2 + m^2 \|u_0\|_2^2 + \|\nabla u_0\|_2^2 \right], \\ \left(u_0 | f(u_0) \right) &\geq 2(2\alpha + 1) \int_G P(u_0(x)) dx, \quad P(u) = \int_0^u f(y) dy, \end{aligned}$$

wówczas rozwiązanie, o ile istnieje, może istnieć tylko na skończonym przedziale $[0, T]$, $T < \infty$ oraz

$$\lim_{t \rightarrow T^-} \|u(t)\|_2^2 = \infty.$$