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ANALIZA WRAŻLIWOŚCI ZE WZGLĘDU NA KSZTALT DLA PEWNEJ KLASY FUNKCJONAŁÓW W OPTYMALIZACJI KONSTRUKCJI

Streszczenie. W pracy przedstawiono zagadnienie analizy wrażliwości funkcjonałów, które występują w zagadnieniach optymalnego projektowania ze względu na minimalną podatność przy jednoczesnej optymalizacji topologii i kształtu konstrukcji.

SHAPE SENSITIVITY ANALYSIS OF OPTIMAL COMPLIANCE FUNCTIONALS

Summary. This paper deals with the shape design sensitivity analysis of domain dependent functionals that arise in optimal compliance design for simultaneous optimization of material and structure.

FORMEMPFLINDLICH KEITSANALYSE FÜR EINE GEWISSE KLASSE DER FUNKTIONALE BEI DER KONSTRUKTIONSOPTIMIERUNG

Zusammenfaßung. In der Arbeit wurde das Problem der Formempfindlichkeitsanalyse fü r die Funktionale, welche in den eine optimale Projektierung unter Beachtung der minimalen Nachgiebigkeit bei gleichzeitiger Optimierung der Topologie und der Form der Konstruktion bezweckten Aufgaben auftreten, dargestellt.

1. INTRODUCTION

In this paper we study the shape design sensitivity analysis of a structural optimization problem which encompasses the design of structural material, the design of topology and the design of shape.

The strong interrelation between the fields of optimal design and materials science has been underlined in recent years and optimal design with advanced materials and optimal topology design using homogenization methods have been the subject of intensive research (Bendsøe and Mota Soares, 1993, Pedersen, 1993), and the well- known lack of existence to generalized shape design problems and the resulting regularization has provided a natural mathematical angle to the simultaneous design of material and overall structure.

2. THE INNER MATERIAL DESIGN PROBLEM FOR FIXED DOMAIN

In a recent paper (Bendsøe et.al, 1993) on the optimization of structures, the distribution of material as well as the material properties themselves have been considered as design variables. The goal of this study is to formulate a structural optimization problem in a form that encompasses the design of structural material in a broad sense, while encompassing the provision of predicting the structural topologies and shapes associated with the optimum distribution of the optimized materials. This goal is accomplished by representing as design variables the material properties in the most general form possible for a (locally) linear elastic continuum namely as the unrestricted set of elements of positive semi-definite constitutive tensors.

The problem we consider is an extension of the problem of minimizing the compliance of a structure made of a given material, to the situation where the material properties themselves appear in the role of design variables. We formulate the problem for a reference domain Ω.

$$\min_{E,u}\Big\{\ell(u) = \int_\Omega f \cdot u dx + \int_{\Gamma_1} P \cdot u d\Gamma(x)\Big\} \tag{1}$$

subject to :

$$E \geq 0$$

$$\int_\Omega \Phi(E) dx \leq M$$

$$\int_\Omega \epsilon(u): E: \epsilon(u) dx = \int_\Omega E_{ijkl}\epsilon_{ij}\epsilon_{kl} dx = \ell(u) \text{ for all } v \in \mathcal{U}$$

We take the minimization over all positive, semi-definite rigidity tensors E_{ijkl} and use the integral over the domain of some invariant $\Phi(E)$ of the rigidity tensor as the measure of cost. In (1) we have written the equilibrium equation in the weak form, using $\mathcal{U}$ to denote the space of kinematically admissible trial functions. Note that we treat a single loading conditions, represented by the specified body forces f and boundary traction P, and we use the compliance for this loading case as the objective for our optimal design problem.

Next, we reformulate problem (1) into a convenient form that emphasizes the role of the material design:

$$\max_{\substack{rigidity E \geq 0 \\ \int_\Omega \Phi(E)dx \leq M}} \min_{u \in \mathcal{U}} \Pi(E, u) \tag{2}$$

with

$$\Pi(E, u) = \frac{1}{2} \int_\Omega E_{ijkl} \epsilon_{ij} \epsilon_{kl} dx - \ell(u) \tag{3}$$

In (2), the equilibrium requirement is represented via minimization of the potential energy (3) with respect to deformation. Also, the minimum compliance problem in (2) is stated as a maximization of potential energy with respect to design, since the measure of compliance equals the negative of twice the value of the potential energy at equilibrium.

For physical reasons, the possible rigidity tensors in the above design formulation are restricted to the set of positive semi-definite, symmetric tensors and suitable cost functions must have the property of frame indifference, for example in terms of invariants of the constitutive tensor itself:

$$\begin{aligned} \text{Case A:} \quad \Phi_A(E) &= \rho_A = E_{ijij} \,, \\ \text{Case B:} \quad \Phi_B(E) &= \rho_B = [E_{ijkl} E_{ijkl}]^{\frac{1}{2}} \end{aligned} \tag{4}$$

i.e., respectively, the trace and the Frobenius norm of the tensor E. For the sake of simplifying the derivation, we have introduced the resource density functions, ρ_A and ρ_B and we choose to restate the design problem in the form:

$$\text{Case A:} \quad \max_{\substack{density \rho_A \\ 0 < \rho_{min} \leq \rho_A \leq \rho_{max} < \infty \\ \int_\Omega \rho_A dx \leq M}} \quad \max_{\substack{rigidity E \geq 0 \\ \Phi_A(E) \leq \rho_A}} \quad \min_{u \in \mathcal{U}} \Pi(\rho_A, E, u) \tag{5}$$

$$\text{Case B:} \quad \max_{\substack{density \rho_B \\ 0 < \rho_{min} \leq \rho_B \leq \rho_{max} < \infty \\ \int_\Omega \rho_B dx \leq M}} \quad \max_{\substack{rigidity E \geq 0 \\ \Phi_B(E) \leq \rho_B}} \quad \min_{u \in \mathcal{U}} \Pi(\rho_A, E, u) \tag{6}$$

This separation of the design variables provides a separation between the properties of the tensor E that can be optimized locally (at each point in the structure) and those that must be treated as a distributed parameter problem over the full domain. In (5) and (6) we have introduced upper and lower bounds on the resource densities in order to ensure that the problem is well posed.

We can perform a simplification of problems (5) and (6) by exchanging the order of the min and max operations in the two inner problem and solve for E (see Bendsøe et. al., 1993). The resulting reduced problem is for both cost measures of the form:

$$\max_{\rho \in \mathcal{G}} \{ \min_u \left[\frac{1}{2} \int_\Omega \rho \epsilon(u) : \epsilon(u) dx - \ell(u) \right] \} \tag{7}$$

with the density being restricted to the closed, convex and weak-$*$ compact constraint set $\mathcal{G}$ in $L^\infty(\Omega)$:

$$\mathcal{G} = \{\rho \in L^\infty(\Omega) | \int_\Omega \rho dx \le M,\ 0 < \rho_{\min} \le \rho \le \rho_{\max} < \infty\}$$

The design variables can be removed entirely from the problem, and the resulting problem becomes a non–linear and non-smooth, convex, analysis–only problem, for which existence of solution is assured, but for which solutions may not be unique.

We can also give problem (7) in its equivalent complementary energy form as

$$\min_{\rho\in\mathcal{G}}\Big\{ \min_{\substack{\sigma \\ div\sigma = f \\ \sigma\cdot n = P}} \Big[\int_\Omega \frac{1}{\rho}\sigma : \sigma dx\Big]\Big\} \tag{8}$$

and we note that the derivation of (7) and (8) both extend readily to the case of a contact problem, for which the contact condition is stated as a design independent, convex constraint on the displacements. It is the problem (7) that we in the following will analyse with respect to shape variations of the reference domain.

3. THE DISPLACEMENTS BASED FORMULATION

The following domain functional is considered:

$$\mathcal{J}(\Omega) = \min_{u,\lambda\ge 0}\Big[\int_\Omega \max\{(\frac{1}{2}\epsilon(u):\epsilon(u) - \lambda)\rho_{max}, (\frac{1}{2}\epsilon(u):\epsilon(u) - \lambda)\rho_{min}\}dx$$

$$- \int_\Omega f\cdot u dx - \int_{\Gamma_1} P\cdot u d\Gamma(x) + \lambda M\Big] \tag{9}$$

where u belongs to the convex subset of the Sobolev space $H^1(\Omega)^3$ defined by the following relations (including the possibility of mechanical contact):

$$u = 0 \text{ on } \Gamma_0$$
$$u\cdot n \le 0 \text{ on } \Gamma_2$$

We are interested in the shape derivative, whenever it exists, of the functional $\mathcal{J}(\Omega)$, see e.g. Sokolowski and Zolesio, 1992, for a detailed definition and a description of the material derivative method .

To this end, a one parameter family of domains $\{\Omega_t\}$, $t \in [0,\delta)$, is defined as follows.

$$\Omega_t = T_t(\Omega) \text{ for } t \in [0,\delta)$$

$$\partial\Omega_t = T_t(\partial\Omega),$$

$$\Gamma_i^t = T_t(\Gamma_i) \text{ for } i = 0,1,2\ .$$

here $T_t = T_t(V)\ : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth transformation given by a sufficiently smooth vector field V with $V(t,x) = \frac{\delta T_t}{\partial t}(x)$

Then, the domain functional (10) defined in $\{\Omega_t\}$, takes the form

$$\begin{aligned}\mathcal{J}(\Omega_t) &= \min_{u,\lambda\geq 0}\Big[\int_{\Omega_t} \max\{(\frac{1}{2}\epsilon(u):\epsilon(u)-\lambda)\rho_{max},(\frac{1}{2}\epsilon(u):\epsilon(u)-\lambda)\rho_{min}\}dx \\ &- \int_{\Omega_t} f\cdot u dx - \int_{\Gamma_1^t} P\cdot u d\Gamma(x) + \lambda M\Big] \qquad (10)\end{aligned}$$

where u belongs to the convex subset of the Sobolev space $H^1(\Omega_t)^3$ defined by the following relations:

$$u = 0 \text{ on } \Gamma_0^t$$
$$u\cdot n \leq 0 \text{ on } \Gamma_2^t$$

For any $t \geq 0$ a minimizer is denoted by (u_t, λ_t), so

$$\begin{aligned}\mathcal{J}(\Omega_t) &= \Big[\int_{\Omega_t} \max\{(\frac{1}{2}\epsilon(u_t):\epsilon(u_t)-\lambda_t)\rho_{max},(\frac{1}{2}\epsilon(u_t):\epsilon(u_t)-\lambda_t)\rho_{min}\}dx \\ &- \int_{\Omega_t} f\cdot u_t dx - \int_{\Gamma_1^t} P\cdot u_t d\Gamma(x) + \lambda_t M\Big] \qquad (11)\end{aligned}$$

In order to evaluate the shape derivative $d\mathcal{J}(\Omega;V)$, the domain functional $\mathcal{J}(\Omega_t)$ is transported to the fixed domain Ω. The following notation is introduced, we refer the reader to Sokolowski and Zolesio, 1992, for details, in order to make the functional as well as the boundary conditions dependent of the domain Ω only:

$$f^t = {}^*DT_t\cdot f\circ T_t, \quad P^t = {}^*DT_t\cdot P\circ T_t$$

$$u^t = u_t\circ T_t$$

$$z^t = DT_t^{-1}\cdot u^t\epsilon^t(\phi) = \tfrac{1}{2}\{D(DT_t\cdot\phi)\cdot DT_t^{-1} + {}^*DT_t^{-1}\cdot{}^*(D(DT_t\cdot\phi))\}$$

$$\gamma(t) = \det(DT_t)$$
$$\omega(t) = \|M(T_t)\cdot n\|^3_{\mathbb{R}},$$

$$M(T_t) = \det(DT_t)\,{}^*DT_t^{-1}$$

furthermore, the nonpenetration condition on Γ_2^t takes an equivalent form

$$z^t\cdot n \leq 0 \text{ on } \Gamma_2$$

The domain functional is thus transported to the fixed domain

$$\begin{aligned}\mathcal{J}_t(\Omega)\equiv\mathcal{J}(\Omega_t) &= \Big[\int_{\Omega} \max\{(\frac{1}{2}\epsilon(u_t)\circ T_t:\epsilon(u_t)\circ T_t - \lambda_t\circ T_t)\rho_{max}, \qquad (12)\\ & \qquad (13)\\ &(\quad \frac{1}{2}\epsilon(u_t)\circ T_t:\epsilon(u_t)\circ T_t - \lambda_t\circ T_t)\rho_{min}\}\gamma(t)dx \\ &- \int_{\Omega} f^t\cdot u^t\gamma(t)dx - \int_{\Gamma_1} P^t\cdot u^t\omega(t)d\Gamma(x) + \lambda_t\circ T_t M\Big]\end{aligned}$$

The transported domain functional can be evaluated for any value of $t \in [0, \delta)$ by a minimization procedure, i.e.

minimize with respect to (v, λ) the following functional

$$\begin{aligned} \mathcal{I}_t(v,\lambda) &= \Big[\int_\Omega \max\{(\frac{1}{2}\epsilon^t(v):\epsilon^t(v)-\lambda)\rho_{max},(\frac{1}{2}\epsilon^t(v):\epsilon^t(v)-\lambda\rho_{min})\}\gamma(t)dx \\ &- \int_\Omega f^t\cdot v\gamma(t)dx - \int_{\Gamma_1} P^t\cdot v\omega(t)d\Gamma(x)+\lambda M\Big] \end{aligned} \tag{14}$$

subject to the constraints

$$\begin{gathered} v \in H^1(\Omega)^3 \\ v = 0 \text{ on } \Gamma_0 \\ v\cdot n \le 0 \text{ on } \Gamma_2 \end{gathered}$$

In the following we assume, for simplicity of presentation, that the minimizers of problem (13) are unique, $t \in [0, \delta)$. If this is not the case, we have to invoke the procedure of section 3 in Bendsøe and Sokolowski, 1992, with each derivative being given below. Now, with this assumption, let (u, λ) denotes the unique minimizer for the functional (9). The following notation is introduced

$$\Omega_1 \equiv \Omega_1(V) = \{x\in\Omega \,|\, \tfrac{1}{2}\epsilon(u):\epsilon(u)-\lambda = 0 \text{ and}$$

$$\epsilon'(u):\epsilon(u) > 0 \text{ or } \tfrac{1}{2}\epsilon(u):\epsilon(u)-\lambda > 0\}$$

$$\begin{aligned} &\Omega_2 \equiv \Omega_2(V) = \Omega\setminus\Omega_1(V) \\ &\chi_i = \text{ characteristic function of } \Omega_i \\ &\chi_\Omega = \text{ characteristic function of } \Omega, \text{ i.e.} \\ &\chi_\Omega(x) = \begin{matrix} 1 & x\in\Omega \\ 0 & otherwise \end{matrix} \\ &\text{therefore } \chi_1(x)+\chi_2(x) = \chi_\Omega(x) \text{ a.e.} \end{aligned}$$

Here ϵ' denotes the derivative of ϵ^t and we have divided the domain in two parts, depending on the size of the specific strain energy relative to the multiplier λ. With this notation we can now state the result on the shape sensitivity:

Theorem 1.

If the minimizers (u_t, λ_t) of functional (13) are continuous in $H^1(\Omega)^3 \times \mathbb{R}$ with respect to the parameter t at $t = 0^+$, then the Eulerian derivative of the shape functional (3.1) takes the following form

$$
\begin{aligned}
d\mathcal{J}(\Omega;V) &= lim_{t\to 0}\frac{1}{t}(\mathcal{J}(\Omega_t)-\mathcal{J}(\Omega)) && (15)\\
& && (16)\\
&= \Big[\int_\Omega [\rho_{\max}\chi_1+\rho_{\min}\chi_2]\epsilon'(u):\epsilon(u)dx + && \\
& && (17)\\
&+ \int_\Omega \max\{[\frac{1}{2}\epsilon(u):\epsilon(u)-\lambda]\rho_{max},[\frac{1}{2}\epsilon(u):\epsilon(u)-\lambda]\rho_{min}\}\gamma' dx + && \\
&- \int_\Omega [(\nabla f\cdot V)\cdot u + f\cdot DV\cdot u + f\cdot u\gamma']dx + && \\
&- \int_{\Gamma_1} [(\nabla P\cdot V)\cdot u + P\cdot DV\cdot u + P\cdot u\omega']d\Gamma(x)\Big] &&
\end{aligned}
$$

where we have that

$$\epsilon'(\phi) = \frac{1}{2}\{D(DV\cdot\phi) + {}^*(D(DV\cdot\phi)) - D\phi\cdot DV - {}^*DV\cdot {}^*D\phi\},$$

$$\gamma' = \mathrm{div}V,$$

$$\omega' = \mathrm{div}_\Gamma V\,{}^*D\phi$$

is the transpose of the Jacobian matrix function

$$D\phi\ ;\ \phi\ \ \mathbb{R}^3 \to \mathbb{R}^3$$
$$V(0) = \frac{\partial T_t}{\partial t}|_{t=0}$$

We refer the reader to Bendsøe and Sokolowski (to appear) for a proof of Theorem 1.

We remark that under our assumptions, the Eulerian derivative of the shape functional (9) can be evaluated in the same way as for the multiple eigenvalues in Myslinski and Sokolowski, 1985, Sokolowski and Zolesio, 1992. If the measure of the set

$$\Omega_0 = \{x\in\Omega | \frac{1}{2}\epsilon(u):\epsilon(u)-\lambda = 0\}$$

is not zero then the mapping $V \to d\mathcal{J}(\Omega;V)$ fails to be linear, and only one sided directional derivatives exist.

4. CONCLUSIONS

We have presented here, on the basis of Bendsøe and Sokołowski (to appear) a general framework for an integrated design analysis of shape and topology variations of structures. The topology variations are treated as an inner problem, with shape variations as an outer level problem. The topology and material design problem employed used a completely free parametrization of rigidity, but the analysis extends readily to homogenization based topology problems with reduced design independent problems, as described in e.g. Allaire and Kohn, 1993, and Jog, Haber and Bendsøe, 1993. We note that the integrated shape and topology and material design formulation allows for the treatment of problems with shape dependent loadings, such as pressure loads; these problems cannot be handled by standard topology and material design methodology based on the use of a fixed reference domain.

The results on derivatives involves domain integrals and a computational implementation is straightforward. In order to preserve consistency with the computational procedure for the solution of the analysis problem, it is recommended that the sensitivity analysis is carried out for the discretized problem statement before implementation. Computational experience and details on the derivation for a discretized version of the problem are the subjects of a forthcoming study.

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Streszczenie

W pracy podano nowe wyniki z zakresu analizy wrażliwości uzyskane dla optymalnych funkcjonałów podatności ze względu na kształt obszaru geometrycznego. Wyniki te pozwalają na wykorzystanie dla rozwiazywania zadań optymalizacji konstrukcji połączonych metod obliczeniowych optymalizacji topologii i optymalizacji kształtu. W szczególności umożliwia to rozwiązywanie metodami optymalizacji topologii tych zadań optymalnego projektowania, w których obciążenie zależy od geometrii obszaru – np. ciśnienie – co nie było do tej pory robione ze względu na brak uzasadnienia teoretycznego. W pracy podano wyniki dla przykładowego zadania optymalizacji kształtu dla zagadnienia typu kontaktu ciała sprężystego ze sztywną przeszkodą.