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ZASTOSOWANIE STOCHASTYCZNEJ METODY ELEMENTÓW BRZEGOWYCH W MODELOWANIU LOSOWYCH UKŁADÓW MECHANICZNYCH

Streszczenie. W pracy przedstawiono podstawowe koncepcje stochastycznej metody elementów brzegowych. Niepewność w układach mechanicznych modelowana jest za pomocą pól losowych, których momenty opisują stochastyczne warunki brzegowe, losowe własnosci materiału i stochastyczny kształt brzegu.

APPLICATION OF STOCHASTIC BOUNDARY ELEMENT METHOD TO MODELLING OF UNCERTAIN MECHANICAL SYSTEMS

Summary. Fundamental concepts of the stochastic boundary element method are presented. Uncertainties in mechanical systems are modelled by means of random fields whose moments specify stochastic boundary conditions, random material properties and stochastic shape of a boundary.

ANWENDUNG DER STOCHASTISCHEN RANDELEMENTENMETHODE BEIM MODELLIEREN VON STOCHASTISCHEN MECHANISCHEN SYSTEMEN

Zusammenfassung. In der Arbeit wurden die Grundkonzeptionen der stochastischen Randelementenmethode dargestellt. Die Unsicherheit in mechanischen Systemen wird mittels der stochastischen Felder, die Momente von welchen stochastische Randbedingungen, stochastische Werkstoffeigenschaften und stochastische Randform beschreiben, modelliert.

1. INTRODUCTION

The functionality of many modern engineering systems depends to large extent on their ability to perform adequately and with a high level of reliability under not absolutely controllable conditions. In response to these problems, computer methods have been developed to deal with the statistical nature of loads, material properties and the shape of a structure. Stochastic boundary element method (SBEM), as an alternative numerical technique to the stochastic finite element method [cf. e.g. Ghanem and Spanos (1991), Kleiber and Hien (1992)] belongs to these methods.Many different interpretations are possible for terminology of the SBEM. This term is used here to refer to the boundary element method which accounts for uncertainties in: boundary conditions or material properties of a structure, as well as the shape of a boundary. Such uncertainties are usually distributed on a boundary (Γ) or within a domain (Ω) of the structure and should be modelled as spatial $\mathbf{r}(\mathbf{x},\gamma)$ or spatially-temporal $\mathbf{r}(\mathbf{x},t,\gamma)$ stochastic fields, where $\mathbf{x}=(x_k)$ denotes spatial position in Ω or Γ, $t\in T$ is time, γ is an elementary event ($\gamma\in\mathscr{E}$). Such stochastic fields are defined on the probability space ($\mathscr{E}$, $\mathscr{F}$, $\mathcal{P}$), where $\mathscr{E}$ is the space of elementary events, $\mathscr{F}$ is a s-algebra of subsets of $\mathscr{E}$ and $\mathcal{P}$ is a probability [Sobczyk (1984)].

Applications of the SBEM appear to have been initiated in the early 1980's. The earliest application used the boundary integral equation method to solving stochastic boundary value problems of elastostatics [Burczyński (1981)]. Next the SBEM was used to stochastic potential problems [Burczyński (1985a)], stochastic heat conduction problems [Drewniak (1985)], groundwater flow [Cheng and Lafe (1991)] and dynamical problems [Burczyński (1985b), (1988a), Burczyński and John (1985a), (1985b), (1991), Spanos and Ghanem (1991)]. SBEM was also extended to problems with random media [Burczyński (1986a)] and [Ettouney *et al.* (1989), (1993), Manolis (1993)] and to problems with uncertain boundaries [Nakagiri *et al.* (1983), Nakagiri and Suzuki (1989), Burczyński (1986b), (1988b), (1992)].

A concise presentation of the SBEM to problems with stochastic boundary conditions, stochastic material properties, stochastic shape sensitivity and identification problems was given in the form of chapters by Burczyński (1989), (1993a), (1993b), (1993c).

2. COMPUTATIONAL METHODOLOGY OF SBEM

Using SBEM to solving boundary value problems with random boundary conditions in the form of prescribed stochastic fields of displacements $\mathbf{u}(\mathbf{x},\gamma)=\mathbf{u}^0(\mathbf{x},\gamma)$, $\mathbf{x}\in\Gamma_1$, and tractions $p(x,\gamma)=p^0$, $x \in \Gamma_2$ respectively, where $\Gamma=\Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2=\emptyset$, as a result a vector stochastic boundary integral equation is obtained:

$$c(x)\,u(x,\gamma) = \int_{\Gamma} U(x,y)\,p(y,\gamma)\,d\Gamma(y) - \int_{\Gamma} P(x,y)\,u(y,\gamma)\,d\Gamma(y) \tag{1}$$

where $\mathbf{U}(x,y)$ and $\mathbf{P}(x,y)$ are deterministic fundamental solutions for elastostatics.
Practically not stochastic solutions of these equations are sought but their moments (i.e. expectations, covariances). In order to solve this problem it is possible to distinguish two general approaches, namely continuous and discrete approaches.
In the continuous approach deterministic boundary integral equations for moments are formulated. Boundary integral equations for expectations are identical to those for deterministic systems.
Boundary integral equations for covariances differ from the traditional ones in that they involve double, instead of single, boundary integrations [cf. Burczyński (1981), (1985a), Cheng and Lafe (1991), (1993)]. This approach can be useful to problems with the Dirichlet (essential) or Neumann (natural) type of stochastic boundary conditions.
In the case of mixed stochastic boundary conditions it is convenient to use the discrete approach in which stochastic boundary integral equations are discretized into a system of random algebraic equations:

$$[A]\{X(\gamma)\} = [B]\{Y(\gamma)\}\ , \tag{2}$$

where

- $[\mathbf{A}]$ and $[\mathbf{B}]$ are deterministic matrices dependent on boundary integrals of fundamental solutions $\mathbf{P}$ and $\mathbf{U}$,
- $\{\mathbf{X}(\gamma)\}$ is a column matrix of unknown random nodal values of boundary displacements and tractions,
- $\{\mathbf{Y}(\gamma)\}$ is a column matrix of random nodal values of boundary displacements and tractions prescribed by boundary conditions.

For the boundary element discretization it is necessary to approximate stochastic fields of boundary displacements and tractions into random vector representations $\{\mathbf{X}(\gamma)\}$ and $\{\mathbf{Y}(\gamma)\}$. Two methods of discretization can be proposed.

The midpoint method

The field value of $\mathbf{r}(x,\gamma)$, $\mathbf{r}=\mathbf{u},\mathbf{p}$, over a boundary element Γ^e is represented by its value at the midpoint x^e of the element Γ^e:

$$r^e(\gamma) = r(x^e,\gamma)\ , \quad x^e \in \Gamma^e \tag{3}$$

The boundary averaging method

The value for a boundary element Γ^e is represented by the spatial average of the random field over the element Γ^e:

$$r^e(\gamma) = \frac{1}{|\Gamma^e|}\int_{\Gamma^e} r(x,\gamma)\,d\Gamma(x)\ , \tag{4}$$

where $|\Gamma^e|$ is the area of Γ^e.

Discretization of random fields of displacements $\mathbf{u}(\mathbf{x},\gamma)$ and tractions $\mathbf{p}(\mathbf{x},\gamma)$ implies that the column matrices $\{X(\gamma)\}$ and $\{Y(\gamma)\}$ consist of variables whose covariance depends on the boundary element mesh.

Consequently, the stochastic character of the boundary element values may be expected to depend significantly on the chosen mesh geometry.

Thus, the important problem in using the discrete approach is the selection of the mesh size. Numerical calculations show [cf. Burczyński (1993a, 1993b, 1993c)] that the midpoint discretization method tends to over-represent the variability within the boundary element whereas the spatial averaging method tends to under-represent the same variability. The two methods coincide when the random field mesh is sufficiently fine in relation to the correlation length. The system of random algebraic equations (2) is the base for evaluating of moments:

- mean value:

$$\{m_X\} = [A]^{-1}[B]\{m_Y\}\ , \tag{5}$$

- covariance matrix:

$$[K_X] = [A]^{-1}[B][K_Y][B]^T[A]^{-1T}\ . \tag{6}$$

The cross-covariance matrices are evaluated as follows:

$$[K_{XY}] = [A]^{-1}[B][K_Y]\ , \qquad [K_{YX}] = [K_Y][B]^T[A]^{-1T}\ . \tag{7}$$

Having obtained probabilistic characteristics of unknown random values of displacements and tractions, together with the specified values of tractions and displacements, the interior values of displacements and stresses can be calculated [cf. Burczyński (1988a), (1993a), (1993b)].

3. APPLICATION OF SBEM TO DYNAMIC SYSTEMS

Two different boundary element approaches to dynamic analysis can be distinguished, namely: the time domain approach and the integral transform (Laplace or Fourier) domain approach. Both can be used to stochastic dynamic analysis but it seems that the application of the Fourier integral transform domain approach offers a special convenience consisting in the possibility of employing spectral densities in description of spatially-temporal stochastic boundary fields of displacements **u**$(\mathbf{x},t,\gamma)$ and tractions **p**$(\mathbf{x},t,\gamma)$. This spectral approach was used to boundary element analysis of stochastic vibration of elastic and visco-elastic systems by Burczyński (1985b, 1988a), Burczyński and John (1985a, 1985b, 1991). An alternative time domain approach to stochastic vibration of elastic systems was also proposed by Burczyński and John (1985b).
The stochastic boundary integral equation for dynamic problems in the Fourier transform domain has the form:

$$c(x)\,\bar{u}(x,\omega,\gamma) = \int_{\Gamma} \bar{U}(x,y,\omega)\,\bar{p}(y,\omega,\gamma)\,d\Gamma(y) - \int_{\Gamma} \bar{P}(x,y,\omega)\,\bar{u}(y,\omega,\gamma)\,d\Gamma(y)\ , \qquad (8)$$

where $\bar{\mathbf{u}}\ (\mathbf{x},\omega,\gamma)$ and $\bar{\mathbf{p}}\ (\mathbf{x},\omega,\gamma)$ are the Fourier transform of stochastic spatially-temporal fields of displacements and tractions, respectively:

$$\bar{r}(x,\omega,\gamma) = \int_{-\infty}^{\infty} r(x,t,\gamma)\exp(-i\omega t)\,dt\ , \qquad r = u,p \qquad (9)$$

and $\bar{\mathbf{U}}\ (\mathbf{x},\mathbf{y},\omega)$ and $\bar{\mathbf{P}}\ (\mathbf{x},\mathbf{y},\omega)$ are the Fourier transform of deterministic fundamental solutions $\mathbf{U}(\mathbf{x},\mathbf{y},t)$ and $\mathbf{P}(\mathbf{x},\mathbf{y},t)$, respectively.
Discrete version of equation (8) for $\omega=(\omega^l)$, $l=1,2,..L$, takes the form:

$$[\bar{A}(\omega)]\,\{\bar{X}(\omega,\gamma)\} = [\bar{B}(\omega)]\,\{\bar{Y}(\omega,\gamma)\}\ , \qquad (10)$$

where

$[\mathbf{A}(\omega)]$ and $[\mathbf{B}(\omega)]$ are deterministic complex square matrices dependent on boundary integrals of Fourier transforms of fundamental solutions **P** and **U**,

$\{\mathbf{X}(\omega,\gamma)\}$ is a column matrix of unknown Fourier transform of random nodal values of boundary displacements and tractions,

$\{\mathbf{Y}(\omega,\gamma)\}$ is a column matrix of Fourier transform of random boundary displacements and tractions prescribed by boundary conditions.

General spectral density of solution of equation (10) has the form:

$$[S_X(\omega_1,\omega_2)] = [\bar{A}(\omega_1)]^{-1}[\bar{B}(\omega_1)][S_Y(\omega_1,\omega_2)][\bar{B}(\omega_2)]^H[\bar{A}(\omega_2)]^{-1H}$$
$$\omega_k = (\omega_k^l)\ , \quad k = 1,2\ , \quad l = 1,2,...L\ . \tag{11}$$

Cross-spectral densities are expressed as follows:

$$[S_{XY}(\omega_1,\omega_2)] = [\bar{A}(\omega_1)]^{-1}[\bar{B}(\omega_1)][S_Y(\omega_1,\omega_2)]\ , \tag{12}$$

$$[S_{XY}(\omega_1,\omega_2)] = [S_Y(\omega_1,\omega_2)][\bar{B}(\omega_2)]^H[\bar{A}(\omega_2)]^{-1H}\ , \tag{13}$$

for $\omega=(\omega_k^l)$, k=1,2, l=1,2,..L. The $[.]^H$ denotes conjugation and transposition of the matrix [.].

4. APPLICATION OF SBEM TO RANDOM CONTINUOUS SYSTEMS

Stochastic nature of the material results mainly from the nonhomogeneity and indeterminacy of the media structure. The complexity and irregularity of the properties of real media leads to a stochastic description of these media [Sobczyk (1984)].
It is difficult to obtain fundamental solutions for stochastic media because of the complexity of their mathematical formulations. To solve this problem perturbation techniques can be used (cf. Burczyński (1986a), Ettouney *et al.* (1989), (1993), Manolis (1993)]. However, this approach requires the assumption that random fluctuations of stochastic properties of a medium are small.
It is possible to apply the other approach [cf. Burczyński (1993a)] which does not require such assumption. This approach consists in using the isotropic deterministic fundamental solutions corresponding to a reference elastic model $\mathbf{C}^0$, whose properties may be found by averaging the stochastic medium:

$$C(x,\gamma) = C^o + \bar{C}(x,\gamma)\ , \qquad x\in\Omega\ , \tag{14}$$

where $\mathbf{C}^0=\mathbf{E}[\mathbf{C}(\mathbf{x},\gamma)]$ is the mean value of the elastic moduli tensor, and $\mathbf{C}(\mathbf{x},\gamma)$ is a random field characterizing the medium fluctuations with mean value $\mathbf{E}[\mathbf{C}(\mathbf{x},\gamma)]=0$ and the correlation moment $\mathbf{K}_C(x_1,x_2)=[K_{kjmnprst}(x_1,x_2)]$. In the general case for the three-dimensional problems $\mathbf{C}$ is a 6x6 matrix and the number of different elastic constants in $\mathbf{C}(\mathbf{x},\gamma)$ is 21.

As a result the following vector integral equation is obtained:

$$c(x)u(x,\gamma) = \int_{\Gamma} U(x,y)p(y)d\Gamma - \int_{\Gamma} P(x,y)u(y)d\Gamma - \int_{\Omega} R(x,y)\bar{\sigma}(y,\gamma)d\Omega(y) \quad (15)$$

where **U**, **P** and **R** are deterministic fundamental solutions for the medium with material constants $\mathbf{C}^0$, and

$$\bar{\sigma}(y,\gamma) = \bar{C}(y,\gamma)\varepsilon(y) , \qquad y \in \Omega . \quad (16)$$

Equation (15) is similar to deterministic problems with only the addition of a new stochastic domain term which depends on the elastic moduli tensor $C(x,\gamma)=C_{kjmn}(x,\gamma)$ characterizing the random fluctuations of medium. This term can be integrated over the domain using internal cells.

The problem is solved iteratively by finding the first solution with $\bar{\sigma}(y,\gamma)=0$ and then computing their values and resolving the system as many times as required.

5. APPLICATION OF SBEM TO SHAPE DESIGN SENSIVITY ANALYSIS

One knows that boundaries bounding real bodies are very complicated as far as their geometrical shape is concerned. Usually they are uneven and the irregularities do not easily lead to a unique deterministic description. Therefore boundaries of such bodies can be defined stochastically. SBEM is a very useful and natural technique for modelling such problems. In order to solve these problems it is possible to use an approach based on the idea of *stochastic shape design sensitivity analysis* [cf. Burczyński (1986b, 1988b, 1992, 1993a, 1993b, 1993c)]. The application of this approach to examine stresses, strains, displacements, natural frequencies and in the general case an arbitrary functional with respect to stochastic shape of the boundary is presented. For the simplicity of further considerations boundary conditions have been assumed deterministic.

One assumes that stochastic shape of the boundary Γ^* may be defined by prescribing a stochastic vector field $\delta \mathbf{g}(\mathbf{x},\gamma)=(\delta g_k(\mathbf{x},\gamma))$, so that:

$$x^*(\gamma) = x + \delta g(x,\gamma) \quad , \qquad E\,\delta g = 0 , \quad (17)$$

where the deterministic variable $\mathbf{x}$ is related to the baseline of the boundary Γ.

The stochastic transformation field $\mathbf{g}(\mathbf{x},\mathbf{a}(\gamma))$ modifies the external boundary Γ, where $a(\gamma)=(a_r(\gamma))$, $r=1,2,..R$, is a set of random shape parameters, which specifies the actual stochastic tolerance range of the structure.

The variation of the transformation field $\delta \mathbf{g}$ is expressed as

$$\delta g_k = v_k^r \delta a_r(\gamma) \ , \tag{18}$$

where $v_k^r = \partial g_k / \partial a_r$ can be considered as a velocity transformation field which is associated with a shape design parameter $a_r(\gamma)$. One assumes that random shape parameters can be expressed as follows:

$$\boldsymbol{a}(\gamma) = \boldsymbol{a}_o + \delta \boldsymbol{a}(\gamma) \ , \qquad E\,\delta \boldsymbol{a}(\gamma) = 0 , \tag{19}$$

where variation $\delta \mathbf{a}(\gamma)$ represents fluctuation of random parameters and $\mathbf{a}_0 = \mathbf{E}\mathbf{a}(\gamma)$ is the mean value of shape parameters $\mathbf{a}(\gamma)$.

If $\mathbf{a}(\gamma)$ has the Gaussian distribution then it is completely described by the mean value $\mathbf{a}_0$ and the covariance matrix $[\mathbf{K}] = \mathbf{E}[\delta \mathbf{a} \delta \mathbf{a}^T]$.

Due to small random variations of the boundary, resulting stochastic fields of stresses $\boldsymbol{\sigma}(\mathbf{x},\gamma)$, strains $\boldsymbol{\varepsilon}(\mathbf{x},\gamma)$, displacements $\mathbf{u}(\mathbf{x},\gamma)$ and natural frequency $\omega(\gamma)$ can be expressed as follows:

$$\sigma(x,\gamma) = \sigma_0(x) + \delta\sigma(x,\gamma) \ , \qquad E\,\delta\sigma(x,\gamma) = 0 \ , \qquad x \in \Omega \ , \tag{20}$$

$$\varepsilon(x,\gamma) = \varepsilon_0(x) + \delta\varepsilon(x,\gamma) \ , \qquad E\,\delta\varepsilon(x,\gamma) = 0 \ , \qquad x \in \Omega \ , \tag{21}$$

$$u(x,\gamma) = u_0(x) + \delta u(x,\gamma) \ , \qquad E\,\delta u(x,\gamma) = 0 \ , \qquad x \in \Omega \ \ or \ \ x \in \Gamma \ , \tag{22}$$

$$\omega(\gamma) = \omega_0 + \delta\omega(\gamma) \ , \qquad E\,\delta\omega(\gamma) = 0 \ , \tag{23}$$

where $\mathbf{q}_0 = \mathbf{E}\mathbf{q}(\mathbf{x},\gamma)$, $\mathbf{q} = \boldsymbol{\sigma},\boldsymbol{\varepsilon},\mathbf{u}$ and $\omega_0 = \mathrm{E}\omega(\gamma)$ are identified with the mean value of state fields calculated for the untransformed boundary $\boldsymbol{\Gamma}$ with the deterministic base shape parameters $\mathbf{a}_0 = (a_{0r})$, $r = 1,2,..R$.

In the general case it is possible to consider an arbitrary functional

$$J = \int_{\Omega(a)} \Psi(\sigma,\varepsilon,u)\, d\Omega + \int_{\Gamma(a)} \varphi(u,p)\, d\Gamma \ , \tag{24}$$

where $\boldsymbol{\Psi} = (\boldsymbol{\sigma},\boldsymbol{\varepsilon},\mathbf{u})$ is an arbitrary function of stresses $\boldsymbol{\sigma}$, strains $\boldsymbol{\varepsilon}$ and displacements $\mathbf{u}$ within the domain $\boldsymbol{\Omega}^* = \boldsymbol{\Omega}(\mathbf{a})$, and $\varphi(\mathbf{u},\mathbf{p})$ is an arbitrary function of displacements $\mathbf{u}$ and tractions $\mathbf{p}$ on the boundary $\boldsymbol{\Gamma}^* = \boldsymbol{\Gamma}(\mathbf{a})$.

The functional $\mathbf{J}$ can express global mechanical characteristics (e.g. potential or complementary energy) as well as local stresses, strains or displacements [cf. Burczyński (1992)].

Due to stochastic shape variation the functional **J** can be expressed as follows:

$$J(a_0 + \delta a(\gamma)) = J(a_0) + \delta J(\gamma) \ , \tag{25}$$

where the first variation of the functional $\delta J(\gamma)$ can be expressed analytically utilizing the adjoint approach.
Independently of the primary system, a concept of an *adjoint system* with the adjoint solution $\mathbf{u}^a$, ε^a and σ^a is introduced.
The adjoint system is an elastic body with identical configuration and physical properties as the primary system but with other boundary conditions and body forces. On the boundary Γ there are prescribed boundary conditions in the form:

$$u^{ao} = -\frac{\partial \phi(u,p)}{\partial p} \quad on\ \Gamma_1 \ , \quad p^{ao} = \frac{\partial \phi(u,p)}{\partial u} \ , \quad on\ \Gamma_2 \ , \tag{26}$$

and within the domain Ω initial strain ε^{ai}, and stress σ^{ai} fields and body forces $\mathbf{b}^a$ are specified:

$$\varepsilon^{ai} = \frac{\partial \Psi(\sigma,\varepsilon,u)}{\partial \sigma} \ , \quad \sigma^{ai} = \frac{\partial \Psi(\sigma,\varepsilon,u)}{\partial \varepsilon} \ , \quad b^a = \frac{\partial \Psi(\sigma,\varepsilon,u)}{\partial u} \tag{27}$$

The constitutive law for the adjoint system has the form:

$$\sigma^a = C(\varepsilon^a - \varepsilon^{ai}) - \sigma^{ai} \ , \tag{28}$$

where $C\varepsilon = c_{ijkl}\varepsilon^a_{kl}$, $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, λ and μ are the Lame constants.
The first variation of the functional **J** can be expressed as:

$$\delta J(\gamma) = \{S\}^T \{\delta a(\gamma)\} \ , \tag{29}$$

where

$$\{\delta a(\gamma)\} = [\delta a_1(\gamma), \delta a_2(\gamma), ..., \delta a_r(\gamma), .., \delta a_R(\gamma)]^T \ , \tag{30}$$

is the matrix of variation of random shape parameters, and

$$\{S\} = [S_1, S_2, ..., S_r, ..., S_R]^T \ , \tag{31}$$

is the sensitivity matrix whose elements S_r, r=1,2,..R, are expressed by total material derivative of the functional **J** with respect shape parameters, i.e. S =DJ/Da_r.

The total material derivatives of J has the form [cf. Mróz (1986), Burczyński (1992, 1993b, 1993c)]:

$$S_r = \int_{\Gamma}\left[\Psi - \sigma\cdot\varepsilon^a + b\cdot u^a + (\phi + p\cdot u^a)_{,n} - 2(\phi + p\cdot u^a)\Re\right]n_k v_k d\Gamma$$
$$+ \int_{\Gamma_1}\left(\frac{\partial\phi}{\partial u} - p^a\right)\left(\frac{Du^0}{Da_r} - u^0_{,k}v^r_k\right)d\Gamma_1 + \int_{\Gamma_2}\left(\frac{\partial\phi}{\partial p} + u^a\right)\left(\frac{Dp^0}{Da_r} - p^0_{,k}v^r_k\right)d\Gamma_2 \tag{32}$$
$$+ \int_L [\![\phi + p\cdot u^a]\!] v^r_\upsilon dL\ ,$$

where integrand $[\![\phi+p\cdot u^a]\!]=(\phi+p\cdot u^a)^+-(\phi+p\cdot u^a)^-$ represents the discontinuity of $(\phi+p\cdot u^a)$ along the curve L, which separates two parts of the boundary Γ_1 and Γ_2, $n=(n_k)$ is the unit normal vector, $\Re$ is the mean curvature of the boundary.

It is seen from (32) that sensitivities of J depend only on boundary state variables of the primary system and the adjoint system. This fact gives significant advantages in numerical calculations by means of the boundary element method.

Consider now a special kind of shape transformation in the form of translation, rotation and scale change (expansion or contractions). This case is especially interesting when a body contains inhomogeneity in the form of internal defects such as a crack, a cavity or an inclusion. Then the dependence of the functional J on stochastic shape and location of such internal defects can be examined.

One assumes that the stochastic shape transformation field $g(x,a(\gamma))$ modifies of a given initial shape of the internal defect in the form:

- stochastic translation (T), by prescribing $\delta g_k(x,\gamma)=\delta b_k(\gamma)$, k=1,2,3 where $b_k(\gamma)$ are random translation parameters,
- stochastic rotation (R), by prescribing $\delta g_k(x,\gamma)=e_{kpl}x_l\delta\omega_p(\gamma)$, p=1,2,3 where $\omega_p(\gamma)$ are random rotation parameters, e_{kpl} denotes the permutation tensor,
- stochastic scale change (expansion or contraction) (S) of the crack by prescribing $\delta g_k(x,\gamma)=x_k\delta\eta(\gamma)$, where $\eta(\gamma)$ is a random scale change parameter.

Now the total material derivatives DJ/Da_r, where $a(\gamma)=(a_r(\gamma))\equiv(b_1,b_2,b_3,\omega_1,\omega_2,\omega_3,\eta)$ take the form of the path-independent integrals:

$$S_r = \frac{DJ}{Da_r} = \int_{\Gamma_*} Z^r_L(\sigma,\varepsilon,u,\sigma^a,\varepsilon^a,u^a)d\Gamma_*\ , \qquad r=1,2,3,4,5,6,7;\quad L=T,R,S \tag{33}$$

where

$$Z^k_T = (\Psi\,\delta_{kj} + \sigma_{ij}u^a_{i,k} + \sigma^a_{ij}u_{i,k} - \sigma_{il}\varepsilon^a_{il}\delta_{kj})n_j\ , \tag{34}$$

for translation (k=1,2,3),

$$Z_R^{p+3} = e_{kpl}(\Psi x_l \delta_{kj} - \sigma_{iq}\varepsilon_{iq}^{a} x_l \delta_{kj} + \sigma_{ij} u_{i,k}^{a} x_l + \sigma_{lj} u_k^{a} + \sigma_{lj}^{a} u_k + \sigma_{ij}^{a} u_{i,k} x_l) n_j \;, \tag{35}$$

for rotation (p=1,2,3), and

$$Z_s^7 = \left[\frac{2}{\alpha}\sigma_{ij}^{a} u_i - \left(\frac{2}{\alpha} - \beta\right)\sigma_{ij} u_i^{a} + x_k \sigma_{ij}^{a} u_{i,k} - x_k \sigma_{ij} u_{i,j}^{a} \delta_{jk} + x_k \sigma_{ij} u_{i,k}^{a}\right] n_j \;, \tag{36}$$

for expansion or contraction.

In the last case, the form of Z_s^7 is valid $\Psi=0$ and $\varphi=\varphi(\mathbf{u})$ is a homogeneous function of order α and $\beta=1$ for 3-D and $\beta=0$ for 2-D. Γ_* is an arbitrary closed surface (or contour for 2-D) enclosing the defect. Primary $(u_i, \varepsilon_{ij}, \sigma_{ij})$ and adjoint $(u_i^a, \varepsilon_{ij}^a, \sigma_{ij}^a)$ solutions along Γ_* can be obtained using boundary element procedures. Enclosing the defect by a surface Γ_*, derivatives of **J** can be determined by calculating the respective path-independent integrals along any surface Γ_*. In particular case the surface Γ_* can be identified with the external boundary surface Γ.

Internal defects can introduce local gradients of displacements or stresses which reach great or infinite values as compared to respective values within the structure domain. Calculations of derivatives of **J** by means of path-independent integrals along fixed contours far from such singularities (e.g. cracks) ensure good accuracy of these derivatives [Burczyński and Polch (1993)].

The variation of natural frequency can be expressed by:

$$\delta\omega(\gamma) = \{S^{\omega}\}^T \{\delta a(\gamma)\} \;, \tag{37}$$

where elements of sensitivity matrix $\{S^{\omega}\}$ are evaluated by total material derivative of natural frequencies with respect to shape parameters [cf.Burczyński (1986b, 1988b, 1992, 1993a, 1993b, 1993c)]:

$$S_r^{\omega} = \frac{D\omega}{Da_r} = \frac{1}{2\omega_0}\int_{\Gamma}[\sigma(u)\cdot\varepsilon(u) - \omega_0^2 \rho\, u\cdot u]\, n_k v_k^r d\Gamma \;, \tag{38}$$

where ρ is the mass density and $\mathbf{u}(\mathbf{x})$ is the displacement amplitude.

It is interesting to notice that $\delta\omega(\gamma)$ is expressed by the boundary integral and depends on the natural frequency ω_0, the mode $\mathbf{u}(\mathbf{x})$ and the stochastic fluctuation of shape parameters $\delta a(\gamma)$. This is very important in numerical calculations using boundary elements.

The mean value ω_0 can be evaluated using the dual reciprocity approach for the free vibration problem. After discretization of the boundary Γ by means of boundary elements the following equation is obtained:

$$[H]\{u\} = \omega_0^2[M]\{u\} , \tag{39}$$

where $\{u\}$ is the column matrix of nodal values of displacement amplitude, $[M]$ is the mass matrix.
The covariance of stochastic stresses, strains and displacements for the two fixed points x_1 and x_2 is calculated from:

$$\begin{gathered} K_q(x_1,x_2) = E\{[q(x_1,\gamma) - q_0(x_1)][q(x_2,\gamma) - q_0(x_2)]\}y = \\ E[\delta q(x_1,\gamma)\delta q(x_2,\gamma)] = \{S(x_1)\}[K]\{S(x_2)\}^T \\ q = \sigma_{qs}, \varepsilon_{qs}, u_q \end{gathered} \tag{40}$$

The variance of the random natural frequency can be calculated using equation:

$$Var(\omega) = E[\omega(\gamma) - \omega_0]^2 = E[\delta\omega(\gamma)]^2 = \{S^\omega\}[K]\{S^\omega\}^T . \tag{41}$$

In order to calculate the sensitivity matrices $\{S\}$ or $\{S^\omega\}$ one should determine the transformation velocity field v_k^r, which is associated with the shape parameter $a_r(\gamma)$. The selection of shape design parameters is the key element in the shape sensitivity analysis and optimal design.

6. NUMERICAL EXAMPLES

Example 1
The chain link (Fig.1a) made up of the visco-elastic material is loaded by the slow-changeable spatially-temporal stationary stochastic field of tractions $p(x,t,\gamma)$ whose spectral density is given by

$$S_p(x_1,x_2,\omega) = 2\delta(x_1 - x_2)\sigma_p^2\alpha(\beta^2 + \alpha^2)/[\pi(\omega^2 - \beta^2 - \alpha^2)^2 + 4\alpha^2\omega^2]$$

where $\sigma_p^2 = 289[N^2]$, $\alpha=0.15[s^{-1}]$, $\beta=7[s^{-1}]$.

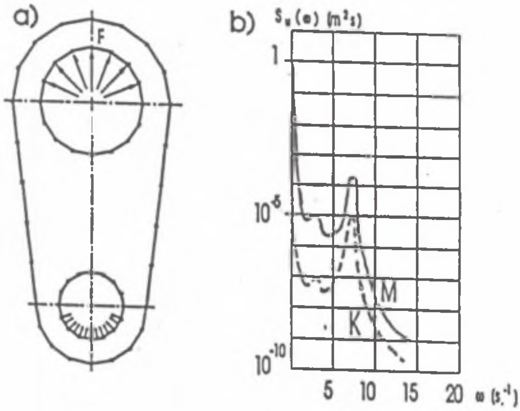


Fig. 1.

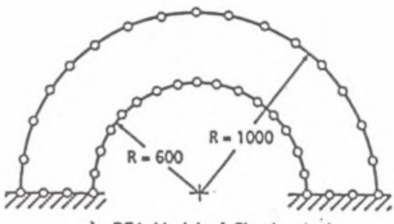
Numerical results of spectral densities of vertical displacements of the point F for the Maxwell (M) and Kelvin (K) models are presented in Fig.1b.

Example 2

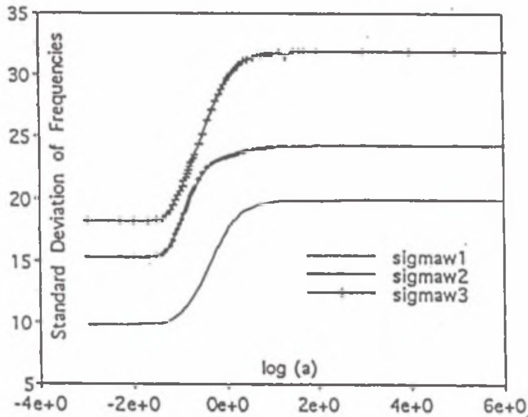
The problem of stochastic shape sensitivity analysis for a vibrating steel arch (Fig.2a) is considered. The boundary element model consists of 38 linear elements. It is assumed that the boundary undergoes small random variation measured along the radius with the covariance $K(x,y)=\sigma^2\exp[(-|x_1-x_2|-|y_1-y_2|)/a]$, where $\sigma^2=15 \cdot 10^{-6}$ [m^2].

The mean values of the first, second and third natural frequencies are $\omega_{01}=3034[s^{-1}]$, $\omega_{02}=4788[s^{-1}]$ and $\omega_{03}=7743[s^{-1}]$, respectively.

Fig.2b shows the dependence of standard deviations of natural frequencies as a function of the radius of correlation a [m].



a) BEA Model of Circular Arch



b) Frequency Standard Deviations Versus Correlation Length

Fig. 2.

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Streszczenie

W pracy przedstawiono podstawowe koncepcje stochastycznej metody elementów brzegowych w rozwiązywaniu zadań brzegowych teorii sprężystości z losowymi warunkami brzegowymi. Opisano dwa sposoby dyskretyzacji losowych pól przemieszczeń brzegowych i sił powierzchniowych prowadzące do układu losowych równań algebraicznych, które są podstawą do określania momentów.

Omówiono zastosowanie stochastycznej metody elementów brzegowych do zagadnień dynamicznych oraz do układów o losowych własnościach materiałowych. Przedstawiono także zastosowanie metody do zagadnień analizy wrażliwości, gdy brzeg układu sprężystego opisany jest z losową tolerancją. Podano przykłady numeryczne ilustrujące metodę.