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PEWNE WYNIKI NUMERYCZNE DLA ZADANIA OPTYMALIZACJI PLYTY KIRCHHOFFA

Streszczenie. W pracy rozważa się zadanie optymalizacji płyty sprężystej opisywanej równaniem Kirchhoffa dla przypadku statycznego z losowym obciążeniem. Przedstawione są warunki konieczne optymalności pierwszego rzędu. Rozwiązania zadania optymalizacji w przypadku obciążenia deterministycznego i losowego są porównane na podstawie przeprowadzonych obliczeń numerycznych.

SOME NUMERICAL RESULTS FOR THE KIRCHHOFF PLATE OPTIMIZATION PROBLEM

Summary. The optimization problem of an elastic plate governed by the Kirchhoff equation for the static case with a random loading is considered. The first order necessary conditions of optimality are presented. Optimal solutions obtained for deterministic and random cases are compared on the basis of numerical examples.

НЕСКОЛЬКО ЧИСЛЕННЫХ РЕЗУЛЬТАТОВ ДЛЯ ПРОБЛЕМЫ ОПТИМИЗАЦИИ ПЛАСТИНКИ КИРХГОФФА

Резюме. В работе рассматривается задачу оптимизации упругой пластинки описанной статичным уравнением Кирхгоффа для случайной нагрузки. Представляется необходимые условия оптимальности первого порядка. Решения задачи оптимизации для детерминистической и случайной нагрузки сравнены на основе численных расчетов.

1. INTRODUCTION

Deterministic optimization problems for the Kirchhoff plate are considered e.g. in [7]. Some related results on the existence of an optimal solution and the necessary optimality conditions are given in [5] (for random loadings) and in [7] in deterministic case. The finite element method is applied to obtain the finite dimensional approximations of the problem under consideration, as it is described in [3],[9]. Numerical methods of optimization used here are given in [4].

2. CONTROL PROBLEMS WITH LOADING AS A RANDOM PARAMETER

Let $\mathcal{O} \subset \mathbf{R}^2$ be the domain occupied by the plate and $\partial\mathcal{O} = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let $W = \{v \in H^1(\mathcal{O}) | v = 0 \text{ on } \Gamma_1\}$ and $V = L^2(\Omega; H^2(\mathcal{O}) \cap W)$. Let (Ω, \mathcal{F}, P) be a probabilistic space, where Ω is a discrete set of $\omega_i, i \in \mathbb{N}$, \mathcal{F} is a σ -algebra spanned on Ω , $P: \mathcal{F} \rightarrow \{0, 1\}$ is a probabilistic measure on Ω , $P(\{\omega_i\}) = p_i$.

In the case of a transversal force $f(\omega; x_1, x_2), (x_1, x_2) \in \mathcal{O}$, for given $\omega \in \Omega$ acting on the plate, the state equation for the Kirchhoff plate model is the same as that in the deterministic case. It can be written for an orthogonal coordinate system for dimensionless variables as follows [7]:

$$\sum_{i,j,k,l=1}^2 \frac{\partial^2}{\partial x_j \partial x_i} \left(D_{ijkl}(x_1, x_2) \frac{\partial^2 w(\omega; x_1, x_2)}{\partial x_k \partial x_l} \right) = f(\omega; x_1, x_2) \quad (1)$$

for a.e. $(x_1, x_2) \in \mathcal{O}$, a.s. in Ω ,

where $D = (D_{ijkl}), i, j, k, l = 1, 2$ is a tensor characterizing plate stiffness, $w : \Omega \times \mathcal{O} \rightarrow \mathbf{R}$ is displacement, $w \in V$, $f(\omega; x_1, x_2)$ is given.

There exists tensor $b = (b_{ijkl}), i, j, k, l = 1, 2$ such that:

$$D_{ijkl}(x_1, x_2) = h^3(x_1, x_2) b_{ijkl}, \quad i, j, k, l = 1, 2 \quad (2)$$

where $h : \mathcal{O} \rightarrow \mathbf{R}$ is thickness, $h \in L^\infty(\mathcal{O})$. We will assume that the tensor b is symmetric, i.e.

$$b_{ijkl} = b_{jikl} = b_{klji}, \quad i, j, k, l = 1, 2 \quad (3)$$

The following boundary conditions, suitable for a simply supported plate on Γ_1 and free on Γ_2 (these conditions are satisfied simultaneously, with probability equal to 1) [7] are considered:

$$w(\omega) = 0 \quad \text{on } \Gamma_1 \quad \text{a.s. in } \Omega, \quad M_n(\omega) = 0 \quad \text{on } \Gamma_1 \quad \text{a.s. in } \Omega, \quad (4)$$

$$M_n(\omega) = 0 \quad \text{on } \Gamma_2 \quad \text{a.s. in } \Omega, \quad M'_n(\omega) = 0 \quad \text{on } \Gamma_2 \quad \text{a.s. in } \Omega, \quad (5)$$

where M_n is the so-called bending moment. M_n and M'_n can be expressed by formulae

$$M_n = h^3 \left[\nu \left(\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} \right) + (1 - \nu) \left(\frac{\partial^2 w}{\partial x_1^2} n_1^2 + 2 \frac{\partial^2 w}{\partial x_1 \partial x_2} n_1 n_2 + \frac{\partial^2 w}{\partial x_2^2} n_2^2 \right) \right], \quad (6)$$

$$M'_n = \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(h^3 \left(\frac{\partial^2 w}{\partial x_i^2} n_i + \nu \frac{\partial^2 w}{\partial x_j^2} n_i + (1 - \nu) \frac{\partial^2 w}{\partial x_i \partial x_j} n_j \right) \right), \quad (7)$$

where ν is Poisson ratio which characterizes plate material, $\nu \in (0, 0.5)$; $j \in \{1, 2\}$, $j \neq i$ and (n_1, n_2) is a normal vector on $\partial\mathcal{O}$.

Let $a(\omega; \cdot, \cdot) : V \times V \rightarrow \mathbf{R}$ be the bilinear form associated with the operator (1):

$$a(\omega; w, \phi) = \int_{\mathcal{O}} \sum_{i,j,k,l=1}^2 D_{ijkl} \frac{\partial^2 w}{\partial x_k \partial x_l} \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_1 dx_2 \quad (8)$$

and $F(\omega)(\cdot) : L^2(\Omega; (H^2(\mathcal{O}) \cap W)^{-1}) \rightarrow \mathbf{R}$ be the functional defined by f in standard way. Then the equation (1) can be rewritten in the variational form:

$$a(\omega; w, \phi) = F(\omega)(\phi) \quad \forall \phi \in V \quad \text{a.s. in } \Omega \quad (9)$$

If we take $\omega = \omega_i$, i is fixed, we get the deterministic equation of the Kirchhoff plate. With this assumption it is well known that: (a) the equation (1) with the boundary conditions (4)–(5) is equivalent to (9), (b) the solution $w(\omega) \in H^2(\mathcal{O}) \cap W$ is unique.

The results (a) and (b) can be obtained as well as for ω , a random variable, as in the case considered here.

Therefore we can formulate the following control problem:

$$\inf_{h \in \mathcal{U}_{ad}} J(h) \quad (10)$$

where J is defined by

$$J(h) = E[\int_{\mathcal{O}} w^2(\omega, h, x_1, x_2) dx_1 dx_2], \quad (11)$$

w is a solution to (1)–(5) for a given control h , $E[\cdot]$ denotes the mean value, \mathcal{U}_{ad} is the set of admissible controls,

$$\begin{aligned} \mathcal{U}_{ad} = & \{h \in L^\infty(\mathcal{O}) | h_{min} \leq h(x_1, x_2) \leq h_{max} \text{ a.e. on } \mathcal{O}, \\ & 0 < h_{min} < h_{max}, \int_{\mathcal{O}} h(x_1, x_2) dx_1 dx_2 = c\} \\ & \cap \{h \in H^s(\mathcal{O}) | \|h\|_{H^s(\mathcal{O})} \leq M\}, \end{aligned} \quad (12)$$

$h_{\min}, h_{\max}, c, M, s > 0$ are given constants. As a particular case, for Ω a finite set, we have:

$$J(h) = \sum_{i=1}^N (p_i \int_{\mathcal{O}} w^2(\omega_i, h, x_1, x_2) dx_1 dx_2). \quad (13)$$

3. THE EXISTENCE OF OPTIMAL SOLUTIONS AND THE NECESSARY OPTIMALITY CONDITIONS

Lemma 1

There exists a local solution $\bar{h} \in \mathcal{U}_{ad}$ to (10).

Since the proof is standard we omit it.

The necessary optimality conditions for the problem under consideration can be formulated as follows [2],

$$dJ(h_*, v - h_*) \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \quad (14)$$

for any local solution h_* to (10), where $dJ(h_*, v)$ denotes the directional derivative of J at h_* in a direction v .

It is easy to show, applying the implicit function theorem [6], that:

$$dJ(h_*, v) = -E \left[\sum_{i,j,k,l=1}^2 \int_{\mathcal{O}} (3h_*^2 vb)_{ijkl} \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 p}{\partial x_k \partial x_l} dx_1 dx_2 \right]. \quad (15)$$

where p is a solution to the adjoint state equation, $p \in V$.

4. NUMERICAL EXAMPLES

Let J_{Δ} denotes numerical approximation of J where Δ is a parameter of approximation. A pointwise transversal force acting at $n(\Delta)$ points on rectangular plate is considered and it is assumed that

$$\Gamma_1 = \{(x_1, d) | 0 < x_1 \leq b\} \cup \{(b, x_2) | 0 \leq x_2 < d\}, \quad (16)$$

where b, d are given. Computations are performed with the following values of parameters:

$$\nu = 0.3, h_{\min} = 0.8, h_{\max} = 1.2, c = 0.25, b = d = 0.5.$$

It is assumed that the plate is divided into 16 rectangles by partition of every side into 4 equal parts. Moreover starting values $h_1 = \dots = h_{n(\Delta)} = 1$ are taken.

Computations are performed for the following cases:

- 1) for the deterministic loading $f_r = 1$, $r = 1, \dots, n(\Delta)$;
- 2) for the deterministic loading $f_r = 1$ for all indexes r such that (z_1, z_2) is a node point, except the point $(0.125, 0.375)$ where $f_r = 50$;
- 3) for the deterministic loading $f_r = 1$ for all indexes r such that (z_1, z_2) is a node point, except the points $(0.125, 0.375)$ and $(0.375, 0.125)$ where $f_r = 50$;
- 4) for the random loading:
 - ω_1 takes place with the probability $p_1 = 0.75$ and $f_r(\omega_1) = 1$ for every node different from $(0.125, 0.375)$, where $f_r(\omega_1) = 50$;
 - ω_2 takes place with the probability $p_2 = 0.25$ and $f_r(\omega_2) = 1$ for every node different from $(0.375, 0.125)$, where $f_r(\omega_2) = 50$.
- 5) for the random loading:
 - ω_1 takes place with the probability $p_1 = 0.75$ and $f_r(\omega_1) = 1$ for every node different from $(0.125, 0.125)$, where $f_r(\omega_1) = 50$;
 - ω_2 takes place with the probability $p_2 = 0.25$ and $f_r(\omega_2) = 1$ for every node different from $(0.375, 0.375)$, where $f_r(\omega_2) = 50$.
- 6) for the random loading:
 - ω_1 takes place with the probability $p_1 = 0.75$ and $f_r(\omega_1) = 1$ for every node different from $(0.375, 0.375)$, where $f_r(\omega_1) = 50$;
 - ω_2 takes place with the probability $p_2 = 0.25$ and $f_r(\omega_2) = 1$ for every node different from $(0.125, 0.125)$, where $f_r(\omega_2) = 50$.

The results of computations are presented in tables 1-6.

In Case 1,3,5,6 the optimal plate thickness is symmetric because of the symmetric loading. In each Case great concentration of material along simply supported part of boundary (Γ_1) is observed.

In Case 2 the growth of concentration of plate material takes place around the node with coordinates $(0.125, 0.375)$ because of greater loading value in this point.

In Case 3 additional regions of concentration of material (except Γ_1) are situated around points $(0.125, 0.375)$ and $(0.375, 0.125)$.

The optimal plate thickness in Case 4 (random loading) differs from the results of earlier cases. A greater concentration of material around point $(0.125, 0.375)$ appears in comparison with Case 1. The optimal plate thickness isn't symmetric as in Case 3 although loading values at points $(0.125, 0.375)$ and $(0.375, 0.125)$ are the same. It is caused by different probabilities for these loading values. Neighbourhood of the point $(0.125, 0.375)$, has a greater concentration of material.

Table 1

Plate thickness after optimization: Case 1. Value of J_Δ at the starting point: $J_\Delta(h_0) = 1.825 \cdot 10^{-2}$. Value of J_Δ after optimization: $J_\Delta(h_*) = 1.619 \cdot 10^{-2}$.

x_1	0.000	0.125	0.250	0.375	0.500
x_2					
0.500	1.01	1.01	1.06	1.11	1.07
0.375	1.00	1.00	1.09	1.19	1.11
0.250	0.95	0.93	1.01	1.09	1.06
0.125	0.88	0.82	0.93	1.00	1.01
0.000	0.92	0.88	0.95	1.00	1.01

Table 2

Plate thickness after optimization: Case 2. Value of J_Δ at the starting point: $J_\Delta(h_0) = 2.153 \cdot 10^{-1}$. Value of J_Δ after optimization: $J_\Delta(h_*) = 1.940 \cdot 10^{-1}$.

x_1	0.000	0.125	0.250	0.375	0.500
x_2					
0.500	1.11	1.10	1.13	1.13	1.16
0.375	1.10	1.00	1.01	1.02	1.12
0.250	1.07	0.95	0.96	0.81	1.09
0.125	0.94	0.90	0.92	0.94	1.07
0.000	0.93	0.92	1.00	1.06	1.04

Table 3

Plate thickness after optimization: Case 3. Value of J_Δ at the starting point: $J_\Delta(h_0) = 6.271 \cdot 10^{-1}$. Value of J_Δ after optimization: $J_\Delta(h_*) = 5.070 \cdot 10^{-1}$.

x_1	0.000	0.125	0.250	0.375	0.500
x_2					
0.500	1.05	1.11	1.06	1.09	1.17
0.375	1.08	1.00	1.07	1.14	1.09
0.250	1.00	0.90	0.94	1.07	1.06
0.125	0.87	0.83	0.90	1.00	1.11
0.000	0.90	0.87	1.00	1.08	1.05

Table 4

Plate thickness after optimization: Case 4. Value of J_Δ at the starting point: $J_\Delta(h_0) = 2.153 \cdot 10^{-1}$. Value of J_Δ after optimization: $J_\Delta(h_*) = 1.869 \cdot 10^{-1}$.

x_1	0.000	0.125	0.250	0.375	0.500
x_2					
0.500	1.10	1.08	1.14	1.15	1.19
0.375	1.10	1.00	1.03	0.95	1.16
0.250	1.06	0.94	0.94	1.01	1.11
0.125	0.94	0.90	0.95	0.81	1.09
0.000	0.93	0.93	1.04	1.08	1.06

Table 5

Plate thickness after optimization: Case 5. Value of J_Δ at the starting point: $J_\Delta(h_0) = 8.360 \cdot 10^{-1}$. Value of J_Δ after optimization: $J_\Delta(h_*) = 8.070 \cdot 10^{-1}$.

x_1	0.000	0.125	0.250	0.375	0.500
x_2					
0.500	1.13	1.07	1.09	1.10	1.15
0.375	1.07	0.94	0.96	0.98	1.10
0.250	1.06	0.92	0.94	0.96	1.09
0.125	1.04	0.90	0.92	0.94	1.07
0.000	0.96	1.04	1.06	1.07	1.13

Table 6

Plate thickness after optimization: Case 6. Value of J_Δ at the starting point: $J_\Delta(h_0) = 3.209 \cdot 10^{-1}$. Value of J_Δ after optimization: $J_\Delta(h_*) = 2.693 \cdot 10^{-1}$.

x_1	0.000	0.125	0.250	0.375	0.500
x_2					
0.500	1.14	1.07	1.12	1.19	1.15
0.375	1.05	0.93	1.03	1.13	1.19
0.250	1.03	0.88	0.98	1.03	1.12
0.125	0.95	0.82	0.88	0.93	1.07
0.000	0.89	0.95	1.03	1.05	1.14

Case 5 and Case 6 differ because of different probabilities for loading values at points (0.125,0.125) and (0.375,0.375).

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Streszczenie

Praca przedstawia wyniki liczbowe dla problemu optymalnego minimalizowania normy ugięcia (wychylenia) cienkiej płyty sprężonej.

Rozdział 1 zawiera krótkie wprowadzenie, w którym dokonano przeglądu literatury.

W rozdziale 2 sformułowano problem optymalizacji. Pod uwagę brany jest model płyty opisany równaniem Kirchhoffa (1). Rozważa się przypadek statyczny. Przyjmuje się, że na płytę działa siła poprzeczna $f(\omega; x_1, x_2), \omega \in \Omega, (x_1, x_2) \in \mathcal{O}$, gdzie Ω jest zbiorem dyskretnym zdarzeń losowych, a \mathcal{O} stanowi dziedzinę płyty. Tensor sztywności D zależy od grubości płyty h w sposób podany przez (2). Zakłada się, że tensor D jest symetryczny (3). Warunki brzegowe dla (1) podane przez (4)–(7) odpowiadają przypadkowi płyty podpartej swobodnie na części T_1 obszaru ograniczonego i swobodnej na pozostałej części. Równania (8)–(9) stanowią słabą formę (1)–(7). Grubość płyty h jest wielkością regułowaną problemu optymalizacji (10). Funkcjonał kosztów określony jest przez (11) i jeśli Ω jest zbiorem skończonym, to ma on postać (13). Dopuszczalny zbiór regulacji podany jest przez (12).

W rozdziale 3 przedstawiono temat istnienia rozwiązania miejscowego dla (10). Konieczne warunki optymalności podane są w postaci ogólnej przez (14), a w postaci szczegółowej odpowiadającej rozważanemu problemowi optymalizacji podaje je (15).

Rozdział 4 przedstawia wyniki liczbowe. Przedmiotem rozważań jest płyta prostokątna z obciążeniem w jednym punkcie. T_1 jest opisane przez (16).

Przeprowadzono obliczenia dla trzech przypadków obciążenia zdeterminowanego i dla trzech przypadków obciążenia losowego. Otrzymane wyniki wykazują znaczną różnicę pomiędzy przypadkami wyznaczonymi (optymalny układ grubości płyty w tabelach 1–3) a przypadkami losowymi (tabele 4–6).