ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1994

Seria: MECHANIKA z. 116 Nr kol. 1231

Valeriy PILIPCHUK
Higher Mathematics Department
State Technology & Chemistry University of Ukraine
Dniepropietrowsk

O TEORII ZJAWISK OSCYLACYJNYCH I FALOWYCH OPARTEJ NA NAJPROSTSZYM UKŁADZIE WIBRACYJNO - UDAROWYM

Streszczenie. W pracy zastosowano ogólne własności symetrii czasowej procesów dynamicznych, wykorzystując niegładkie transformacje czasu. W efekcie zmienne pozycyjne stają się elementami algebry bez dzielenia. Możliwości metody zilustrowano szeregiem praktycznych przykładów dotyczących systemów o silnie nieliniowych charakterystykach.

ON OSCILLATORY AND WAVE PHENOMENA THEORY BASED ON THE SIMPLEST VIBROIMPACT SYSTEM

Summary. The general properties of dynamical processes time symmetry by means of the non-smooth transformations of time are used. As a consequence the positional variables become the elements of algebra without division. The corresponding special analytical method is developed. The possibilities of the method are illustrated by the series of practically useful systems having strongly nonlinear characteristics.

О ТЕОРИИ КОЛЕБАТЕЛЬНЫХ И ВОЛНОВЫХ ЯВЛЕНИИ НА БАЗЕ ПРОСТЕЙШЕЙ ВИБРОУДАРНОЙ СИСТЕМЫ

Резюме. Используются общие свойства временной симметрии динамических процессов посредством негладких преобразований времени. Вследствие этого позиционные переменные образуют элементы алгебры без деления. Возможности метода иллюстрируются на ряде практически полезных систем, имеющих сильно нелинейные характеристики.

1. INTRODUCTION

It seems reasonable to say that harmonic oscillator is one of the earliest and fundamental vibrating model. The pair of trigonometrical functions *sine, cosine,* generated by the corresponding differential equation, are the base ones of the most of the classical theories dealing with oscillatory and wave phenomena [1,2]. We keep in mind the Fourier transforms and a variety of trigonometrical series and expansions. Of course, the matter is that the above mentioned functions possess a number of convenient mathematical properties. Most probably the nature of last fact is associated with Euclidean space motion group or more specifically, with rotation group. In accordance with aforesaid it should be noted that translation group in combinations with reflection group provides other basic functions, which are no less convenient than *sine, cosine*. This functions are *saw-tooth sine* and *right - angled cosine*, namely piecewise-linear function $\tau(t)\equiv(2/\pi)\arcsin[\sin(\pi t/2)]$ and its generalized derivative $e(t)=\tau'(t)$. The mechanical model is the simplest vibroimpact system with two rigid barriers. Starting from the results of some previous paper [3-5] it will be shown that the functions $\{\tau(t),\ e(t)\}$ can be correctly used to analytically compute both the non-smooth and smooth oscillatory processes. The principal facts will be presented in the second and third subsections of the next section.

2. THEORY

2.1. The Objects

The principal manipulations will be illustrated by the mechanical systems of sufficiently general view.

Lagrang's system:

$$L = \frac{1}{2}\dot{x}^T\dot{x} - U(x)\ , \quad x\in R^n\ . \tag{1}$$

Here $U(x)$ is a potential energy.

System of the second-order equations:

$$\ddot{x}+f(x,\dot{x},t) = 0\ , \quad x\in R^n\ , \tag{2}$$

where the vector-function f is continuous and either periodically depended on t with period equal $4a$ or parameter of time t is absent.

Impulsive excitation. The subject of consideration will be a dynamical system subjected to impulsive excitation:

$$\dot{x} = \varepsilon f(x,\varphi) + pe'(\varphi) \; ; \quad x \in R^n \, , \quad p = p(\varepsilon t) \, , \quad \dot{\varphi} = \omega(\varepsilon t) \, , \tag{3}$$

where

$$e'(\varphi) = 2\sum_{k=-\infty}^{\infty} [\delta(\varphi + 1 - 4k) - \delta(\varphi - 1 - 4k)] \; ;$$

ε is a small parameter; the vector-function f is supposed to be continuous as a function of x and piece wise-continuous as a function of φ.

Wave systems: the wave operators of type $\partial/\partial t \pm \partial/\partial x$ will be considered.

2.2. "Oscillating time"

The time transformations $(-\infty,\infty) \ni t \to I$, were I is half-limited or else limited domain, are used by number of Nonlinear Mechanics methods as a preliminary stage. Such transformations are useful for the equations of motion analysis by analytical iteration procedures (in detail see [6]). In present paper the Non-smooth transformations are discussed. For simplest cases it can be written as: $t \to |t|$. We consider it is periodic version

$$t \to \tau(t) \, . \tag{4}$$

It should be noted that the metric of time is preserved by this transformation because

$$e^2 = 1 \, , \quad e = e(t) \, . \tag{5}$$

(In present paper the equalities are understood in sense of distributions). Possibility of using of such time parameters is connected with the identity [3]

$$x = X(\tau) + Y(\tau)e \, , \tag{6}$$

where $x(t)$ is an arbitrary function, which is periodic with period, equal 4;

$$X = \frac{1}{2}[x(\tau) + x(2-\tau)] \, , \quad Y = \frac{1}{2}[x(\tau) - x(2-\tau)] \, . \tag{7}$$

2.3. The Algebraic structures

Single-phase case. The elements (6) are elements of hyperbolic numbers algebra on account of (5). Hence, for any function $f(x)$ we have

$$f(X+Ye) = R_f + I_f e \ ; \tag{8}$$

where

$$R_f = \frac{1}{2}[f(X+Y) + f(X-Y)] \ , \quad I_f = \frac{1}{2}[f(X+Y) - f(X-Y)] \ .$$

The first order generalized derivative of expression (6) is

$$\dot{x} = \frac{d}{dt}(X+Ye) = Y' + X'e + \underline{Y\dot{e}} \ , \tag{9}$$

where prime denotes differentiation with respect to τ. If the necessary conditions of continuity for function $x(t)$, i.e. equalities

$$Y|_{\tau=\pm1} = 0 \tag{10}$$

take place, then the underlined addend in (9) should be ignored. So *the result of differentiation remains in considered algebra.* In a similar manner one can consider the second derivative x, and so on.

Slow component. If the process has a slow component then the above introduced presentations should be "deformed" by the slow variables. As example, for oscillating process with one phase the deformed presentation is

$$x = X(\tau,t^0) + Y(\tau,t^0)e \ ; \quad \tau = \tau(\varphi) \ , \quad \dot{\varphi} = \omega(t^0) \ , \tag{11}$$

where $t^0 = \epsilon t$ is the "slow time".
For **wave processes** we can use the presentation of the form

$$u = U(\tau;x,t) + V(\tau;x,t)e \ ; \quad \tau = \tau(\phi) \ , \tag{12}$$

where $\phi = \phi(x,t)$ in comparison with x and t variables is a fast phase.

2.4. The Systems Transformations

The presentations (6),(11),(12) may be used for transformations of the motion equations.

The Lagrang's system. First of all consider *a periodic case* for illustration. Let the system position is defined by the periodic with period equal $4a$ n- dimensional vector-function $x(t)$. As a rule in a autonomous case the quarter of period a is a priori unknown value. Thanks to identity (6) the $X(\tau)$ - and $Y(\tau)$ - components of solution may be determined, where in given case the time parameter is $\tau = \tau(t/a)$. Consider the lagrangian (1). Using (6)-(9) for the set of periodic solutions we have

$$L = R_L + I_{L}e .$$

It may be easily shown that equations for functions X,Y follow from the variational principle:

$$\delta_X \int_{-1}^{1} R_L d\tau = 0 , \quad \delta_Y \int_{-1}^{1} R_L d\tau = 0 .$$

Finally, one obtains the boundary problem

$$X'' + a^2 R_f = 0 , \quad X' |_{r=\pm 1} = 0 , \qquad (13)$$

$$Y'' + a^2 I_f = 0 , \quad Y |_{r=\pm 1} = 0 ; \qquad (14)$$

$$f = \nabla_x U(x) ,$$

where the expressions R^f, I^f are defined by (8).

The System of second-order equations. Substituting the relation (6) into the left part of equation (2), and taking into account the results of subsection 2.3, one obtains an element of algebra. Furthermore, separation of the "real" and "imaginary" parts gives the equations (13),(14) where

$$R_f = \frac{1}{2}\left[f\left(X+Y, \frac{Y'+X'}{a}, ar \right) + f\left(X-Y, \frac{Y'-X'}{a}, 2a-ar \right) \right] ,$$

$$I_f = \frac{1}{2}\left[f\left(X+Y, \frac{Y'+X'}{a}, ar \right) - f\left(X-Y, \frac{Y'-X'}{a}, 2a-ar \right) \right] .$$

The Impulsive excitation. Substitution (11) into the equation (3) gives

$$\omega \frac{\partial Y}{\partial \tau} + \varepsilon \left(\frac{\partial X}{\partial t^0} - R_f \right) + \left[\omega \frac{\partial X}{\partial \tau} + \varepsilon \left(\frac{\partial Y}{\partial t^0} - I_f \right) \right] e + \underline{(Y\omega - p)e'} = 0 .$$

Eliminating the periodic singular term (one is underlined in equation) and equating separately the "real" and "imaginary" parts to zero, one obtains

$$\omega \frac{\partial Y}{\partial \tau} + \varepsilon \left(\frac{\partial X}{\partial t^0} - R_f \right) = 0 , \quad \omega \frac{\partial X}{\partial \tau} + \varepsilon \left(\frac{\partial Y}{\partial t^0} - I_f \right) = 0 ; \quad Y |_{r=\pm 1} = \frac{P}{\omega} . \qquad (15)$$

Now the system is not containing the singular terms and averaging techniques can be correctly used.

The Wave operators. Substituting (12) into the wave operators gives:

$$\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right) u = W_{\pm}(V,U) + W_{\pm}(U,V)e\,, \quad if \quad V|_{\tau=\pm 1} = 0\,.$$

Here

$$W_{\pm}(A,B) = (\phi_t \pm \phi_x)\frac{\partial A}{\partial \tau} + \left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right) B\,.$$

So the result is in hyperbolic numbers algebra.

2.5. Analysis of the transformed systems

The transformed equations as compared with origin ones are formally more complicated, however we have some essential advantages [3-5]. Now a priori information on the solutions qualitative properties is included in the systems by means of the "oscillating" variable $|\,\tau\,| \le 1$. One from advantages of the transformed equation of the type (13),(14) is the possibility to choose the systems $X''=0$, $Y''=0$, having the simple mechanical sense, as the generating ones. The corresponding procedures for problems of the Oscillation and Waves Theory give the expansions, where the pair of functions $\{\tau, e\}$ is the basis one. Some examples of such procedures can be found in [3,4].

2.6. Example of the procedure

τ-approximation. Let us consider the n- degree-of-freedom conservative system having the symmetric potential energy: $U(-x)=U(x)$, $x \in R^n$ Then, starting from (13), (14), one obtains:

$$\begin{gathered} X'' + a^2 f(X) = 0\,, \\ X'|_{\tau=1} = 0\,, \quad X(-\tau) = -X(\tau)\,; \quad Y \equiv 0\,. \end{gathered} \tag{16}$$

The solutions of the non-linear boundary problem can be found in the series form

$$\begin{gathered} X = X^0(\tau) + X^1(\tau) + X^2(\tau) + \ldots\,, \\ a^2 = h_0(1 + \gamma_1 + \gamma_2 + \ldots)\,, \end{gathered}$$

where $h_0\, \gamma_1, \gamma_2 \ldots$ are unkown constans. All terms of the expansions are found from sequence of the problems:

$$\begin{gathered} X^{0\,''} = 0\,; \\ X^{1\,''} = -h_0 f(X^0)\,, \quad (X^{0\,'} + X^{1\,'}|_{\tau=1} = 0\,; \\ X^{2\,''} = -h_0[\,\gamma_1 f(X^0) + f'_x(X^0)X^1\,]\,, \quad X^{2\,'}|_{\tau=1} = 0\,; \ldots\,. \end{gathered}$$

So the zero-th order approximate solution is:

$$X^0 = A^0\tau\ , \quad R^n \ni A^0 = const.$$

This solution describes the simplest family of vibroimpact systems with two rigid barriers; in so doing the length of vector A^0 is equal to the barrier spacing. This vector will be defined in the next approximation.
The first approximation is

$$X^1 = -h_0\int_0^\tau(\tau - \xi)(A^0\xi)d\xi\ ,$$

and the vector A^0 direction is defined from the boundary condition by the nonlinear eigenvector problem:

$$\int_0^\tau(A^0\tau)d\tau = h_0^{-1}A^0\ , \quad where \quad h_0 = A^{0T}A^0 / \left[A^{0T}\int_0^1 f(A^0\tau)d\tau\right] .$$

On the next step in particular, one obtains:

$$\gamma_1 = -A^{0T}\int_0^1 f_x'(A^0\tau)X^1 d\tau \Big/ \left[A^{0T}\int_0^1 f(A^0\tau)d\tau\right] .$$

<u>e-approximation</u>. In addition the boundary problem (16), from (13),(14), the following one can be obtained:

$$Y'' + a^2 f(Y) = 0\ ,$$
$$Y'|_{\tau=1} = 0\ , \quad Y(-\tau) = Y(\tau)\ ; \quad X \equiv 0\ . \tag{17}$$

This equations are convenient for description of the almost separatrix periodic regimes. Let us consider the 1-degree-of-freedom oscillator having the stable point of equilibrium $x=0$, and two unstable ones: $x \pm K$. For this case the periodic solution can be derived in the form of series expansions

$$x = Y(\tau)e = [K + Y_1(\tau) \ = \ Y_2(\tau) \ + \ ...]e\ ,$$
$$a^2 = a_0^2 / (1 - \lambda_1 + - \lambda_2 + ...)\ .$$

The terms of these expansions can be found from the sequence of linear boundary problems:

$$Y_1^{//} - p^2 a_0^2 Y_1 = 0 , \qquad Y_1|_{r=1} = -K ;$$
$$Y_2^{//} - p^2 a_0^2 Y_2 = \lambda_1 Y_1^{//} - (a_0^2/2!) f^{/}(K) Y_1^2 , \quad Y_2|_{r=1} = 0 ; \ldots ,$$

where $p^2 = -f'(K) > 0$; $Y_i(-\tau) \equiv Y_i(\tau)$, $(i = 1, 2, \ldots)$.

The "resonance" terms $sinh(pa_0\tau)$, $cosh(pa_0\tau)$ appearing in the right parts of the equations are bound to be excluded by means of special choosing of values λ_1, λ_2· analogously Lindstedt's procedure.

It should be noted that the procedures for non-symmetric and non-conservative cases are more complicated. The present paper contains examples of solutions only.

3. EXAMPLES OF THE SOLUTIONS

Almost separatrix periodic regime of the mathematical pendulum:

$$\ddot{x} + \sin(x) = 0 .$$

Two steps of the iteration procedure for the high-energy periodic regime yield:

$$x \approx \pi\left[1 - \frac{\cosh(a_0\tau)}{\cosh(a_0)}\right]\tau^{/}; \quad \tau = \tau\left(\frac{t}{a}\right); \quad a^2 = a_0^2 \Big/ \left[1 - \frac{\pi^2}{8\cosh^2(a^0)}\right],$$

where a_0 is arbitrary parameter.

Self-excited nonlineariseable oscillator:

$$\ddot{x} + (bx^2 - 1)\dot{x} + x^m = 0 ; \quad (m = 1, 3, 5, \ldots) .$$

For the limited circle by means of two steps we have:

$$x \approx A\left(\tau - \frac{\tau^{m+2}}{m+2}\right) + \frac{A}{2\omega_0}(\tau^2 - \tau^4)\tau^{/}; \quad \tau = \tau\left(\frac{t}{a}\right),$$

$$bA^2 = 6 ; \quad \frac{1}{a} \approx \omega_0\left[1 - \frac{m}{4(m+2)}\right], \quad \omega_0^2 = \frac{A^{m-1}}{m+1} .$$

Nonlineariseable damped oscillator:

$$\ddot{x} + 2\mu\dot{x} + x^m = 0 ; \quad (m = 3,5,7\ldots, \quad 0 < \mu << 1) .$$

One step yields:

$$x \sim A\left(\tau - \frac{\tau^{m+2}}{m+2}\right); \quad A = A_0 e^{\frac{-4\mu t}{m+3}}, \quad \tau = \tau(\varphi), \quad \varphi \sim \varphi_\infty\left(1 - e^{-2\mu\frac{m-1}{m+3}t}\right);$$

$$\varphi_\infty = \frac{1}{2\mu}\frac{m+3}{m-1}\frac{A_0^{-(m-1)/2}}{\sqrt[2]{2(m+1)}}, \quad A_0 = const.$$

Linear dynamical system subjected to periodic excitation:

$$\dot{x} = Mx + pe^{l}(\varphi), \quad x \in R^n, \quad \varphi = \omega t,$$

where M is constant non-singular matrix; p is constant vector. The exact periodic solution will be derived in the united analytical form. The equations (15) in this case are presented as:

$$\omega\frac{\partial Y}{\partial \tau} = MX, \quad \omega\frac{\partial X}{\partial \tau} = MY, \quad Y|_{\tau=\pm 1} = \frac{p}{\omega}.$$

Excluding the vector $X=\omega M^{-1}(\bullet Y/\bullet\tau)$ from the second equation we'll have the equation

$$\omega^2\frac{\partial^2 X}{\partial \tau^2} - M^2 Y.$$

Denoting with ζ_j, λ_j^2 $(j=1,...,n)$ the orthonormal set of eigenvectors and eigenvalues of the matrix M^2 we'll find

$$Y = \frac{1}{\omega}\sum_{j=1}^{l} p^j \xi_j \frac{\cosh(\lambda_j\tau/\omega)}{\cosh(\lambda_j/\omega}.$$

Analysis of the solution shows that the resonance takes place in the case, if the value of ω satisfies the equation $cosh(\lambda_j/\omega)=0$ for any $j=1, ..., n$. The corresponding value λ_j must be imaginary.

REFERENCES

[1] Nayfeh A.H. and Mook D.T.: Nonlinear Oscillations. John Wiley & Sons,Inc., 1979, 444 p.

[2] Whitham G.B.: The Linear and Nonlinear Waves. Mir, Moscow: 1977, 622 p., (in Rus.).

[3] Pilipchuk V.N.: The vibrating systems transformation by means of the pair of non-smooth periodic functions. "Ukrainian Acad. of Sci. Reports, Ser.A", No.4,~1988, p.37-40. (in Rus.).

[4] Manevich L.I., Mikhlin Ju.V. and Pilipchuk V.N.: Method of Normal Vibrations for Essentially Nonlinear Systems. Nauka, Moscow: 1989, 216p. (in Rus.).

[5] Pilipchuk V.N.: On calculation of periodic processes in mechanical systems with impulsive excitation. "ZN Pol. Śl., s. Mechanika", z.107, 1992, s.335-343 (in Rus.).

[6] Panchenkov A.I.: The Principles of the Limit Correction Theory. Nauka, Moscow: 1976, 240p. (in Rus.).

Recenzent: Prof. dr hab. inż. Eugeniusz Świtoński

Wpłynęło do Redakcji w grudniu 1993 r.

Streszczenie

Rozpatruje się ogólne własności symetrii czasowej procesów dynamicznych za pomocą niegładkich przekształceń czasu. W rezultacie zmienne pozycyjne stają się elementami algebry bez dzielenia. Opracowana została odpowiednia specjalna metoda analityczna, a zwłaszcza sformalizowano koncepcję wyboru najprostszych systemów z barierami sztywnymi jako modeli podstawowych dla teorii oscylacji i fal. Możliwości metody ilustruje szereg praktycznie przydatnych systemów, które charakteryzują się silnie nieliniowymi własnościami. Utworzone algorytmy mogą być łatwo realizowane w systemie automatycznym obliczeń symbolicznych.