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P-VERSION FINITE ELEMENT FOR TORSION OF THIN-WALLED BEAM

Summary. In this paper p-version torsional finite element stiffness matrix and load vector are derived for thin-walled beam of open and closed cross-sections.

ИССЛЕДОВАНИЕ КРУЧЕНИЯ ТОНКОСТЕННЫХ СТЕРЖНЕЙ С КОНЕЧНЫМИ ЭЛЕМЕНТАМИ р-ВЕРСИИ.

Резюме. В роботе рассматривается построение матриц жёсткостей и узлобых нагрузок для стержней тонкостенных открытых и закрытых профилей при кручении. Строятся конечные элементы р-версии

ZASTOSOWANIE METODY ELEMENTÓW SKOŃCZONYCH DO BADANIA SKRĘCANIA PRĘTÓW CIENKOŚCIENNYCH

Streszczenie. W artykule wyprowadzona zostaje macierz sztywności skończonego elementu skrętnego wersji p i wektor obciążenia dla belki cienkościennej o przekroju otwartym i zamkniętym.

1. INTRODUCTION

Thin-walled structures are used in wide-range for civil and mechanical engineering purposes. Many papers can be found concerning the theoretical [1-3] and computational [4-6] aspects of the mechanical models. Applying the theory of Vlasov and Umansky and supposing their terms to be known we give the formulae which can be coded easily for the finite element matrices.

2. OPEN CROSS-SECTION

Let us consider a prismatic beam denoted its ax by z, cross-section by A, center of gravity by S, principal axes by x and y, center point of shear or torsion by C. The displacement due to torsion:

$$u = -\varphi\, y \;, \qquad v = \varphi\, x \;, \qquad w = -\varphi'\,\phi \tag{1}$$

where $(\ )' \equiv \frac{\partial}{\partial z}$, $\varphi=\varphi(z)$ is torsional function, $\phi = \phi(x,y)$ is the warping function satisfying the conditions of $\int \phi dA = \int x\phi dA = \int y\phi dA = 0$. The strain energy due to the displacement filed $\mathbf{u} = u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k}$ of (1) is

$$U = \frac{1}{2} \int_L \{ G\, I_T (\varphi')^2 + E\, I_\phi\, (\phi'')^2 \}\, dz \tag{2}$$

where I_T is the torsional rigidity, I_ϕ is the second order distortion moment. Loading of the beam consist of distributed moments m along its length, couple M_i and bimoment B_i at the two ends , the work of it can be written

$$W = \sum_i (M_i \varphi_i - B_i \varphi'_i) + \int_L m\, \varphi\, dz \tag{3}$$

The $\delta\Pi=0$ first variation of the potential energy $\Pi=U-W$ gives the well known equations

$$M = GI_T\varphi' - EI_\phi\varphi''' , \qquad B = -EI_\phi\varphi'' , \tag{4}$$

and the differential equation of the torsion warping taking into considera- tion

$$EI_\phi\varphi^{iv} - GI_T\varphi'' = m \tag{5}$$

As in the equation $\delta\Pi=0$ the first variations of φ and φ' can be found the approximation must be C^1 continuous. Hence the torsional field approximated in the following form

$$\varphi(z) = [\, 1-3\bar z^2+2\bar z^3,\ L(\bar z-2\bar z^2+\bar z^3),\ (3\bar z^2-2\bar z^3),\ L(\bar z^3-\bar z^2)\,] \begin{bmatrix} \varphi_o \\ \varphi'_o \\ \varphi_L \\ \varphi'_L \end{bmatrix}$$

$$+ [\ \dots\ \{(\bar z^{j+3} + \bar z^2 j - \bar z^3 (j+1)\}\ \dots\ ]_{j=1,\dots,NPF}\ \hat{a} \tag{6a}$$

$$\varphi(z) = N(z)\, q + \hat{N}(z)\, \hat{a} \qquad (\ \bar z = z/L\ ) , \tag{6b}$$

where the first term is concerned the generalized torsion that is the normal approximation while the second one is the higher approximation with additional constants. Substituting (6) into (2) and (3) we get

$$\Pi = \frac{1}{2} [\, q^T \quad \hat{a}^T \,] \left(\begin{bmatrix} K_{qq} & K_{qa} \\ K_{aq} & K_{aa} \end{bmatrix} \begin{bmatrix} q \\ \hat{a} \end{bmatrix} - 2 \begin{bmatrix} f_q \\ f_a \end{bmatrix} \right) \tag{7}$$

where

$$K_{qq}(1,1) = K_{qq}(3,3) = -K_{qq}(1,3) = -K_{qq}(3,1) = \frac{12\, E\, I_\phi}{L^3} + \frac{6\, G\, I_T}{5\, L}$$

$$K_{qq}(1,2) = K_{qq}(2,1) = K_{qq}(1,4) = K_{qq}(4,1) = -K_{qq}(2,3) = -K_{qq}(3,2)$$

$$= -K_{qq}(3,4) = -K_{qq}(4,3) = \frac{6\, E\, I_\phi}{L^2} + \frac{G\, I_T}{10} \tag{8}$$

$$K_{qq}(2,2) = K_{qq}(4,4) = \frac{4\, E\, I_\phi}{L} + \frac{2\, L\, G\, I_T}{15}$$

$$K_{qq}(2,4) = K_{qq}(4,2) = \frac{2\, E\, I_\phi}{L} - \frac{L\, G\, I_T}{30}$$

further

$$K_{qa}(1,j) = -K_{qa}(3,j) = \frac{G\, I_T}{L} \left[6\,(j+3) \left(\frac{1}{j+5} - \frac{1}{j+4} \right) - \frac{j}{10} + \frac{9}{10} \right]$$

$$K_{qa}(2,j) = \frac{G\, I_T}{L} \left[-(j+3) \left(\frac{4}{j+4} - \frac{3}{j+5} \right) + \frac{j}{30} + \frac{12}{10} \right] \tag{9}$$

$$K_{qa}(4,j) = \frac{G\, I_T}{L} \left[(j+3) \left(\frac{3}{j+5} - \frac{2}{j+4} \right) - \frac{4j}{30} - \frac{3}{10} \right], \quad j=1,\dots,NPF$$

$$K_{aa}(i,j) = \frac{E\, I_\phi}{L^3} \left[\frac{(i+3)(i+2)(j+3)(j+2)}{i+j+3} - 6\,(i+j) - 4\,(ij+3) \right]$$

$$+ \frac{G\, I_T}{L} \left[\frac{(i+3)(j+3)}{i+j+5} - \frac{3}{i+5}(i+3)(j+1) - \frac{3}{j+5}(j+3)(i+1) \right. \tag{10}$$

$$\left. + \frac{2}{i+4}\, j\,(i+3) + \frac{2}{j+4}\, i\,(j+3) + \frac{4}{30}\, ij + \frac{3}{10}(i+j) + \frac{9}{5} \right]$$

$$i,j=1,\dots,NPF$$

$$f_q^T = \left[L\left(\frac{m_o}{2} + \frac{3}{20}(m_L - m_o) \right),\ L^2\left(\frac{m_o}{12} + \frac{1}{30}(m_L - m_o) \right), \right.$$

$$\left. L\left(\frac{m_o}{2} + \frac{7}{20}(m_L - m_o) \right),\ -L^2\left(\frac{m_o}{12} + \frac{1}{20}(m_L - m_o) \right) \right] \tag{11}$$

here it is supposed that $m = m_o + \frac{m_o - m_L}{L} z$, and m_o, m_L are the values of the moment intensity at the ends of the element .

$$f_a^T(j) = L\,[\,m_o(\frac{1}{j+4} + \frac{j}{12} - \frac{1}{4}) + (m_L - m_o)(\frac{1}{j+5} + \frac{j}{20} - \frac{1}{5})] \quad j=1,\dots,NPF \tag{12}$$

Finally from (7) by condensation technique $\hat{a}$ can be eliminated at finite element level.

3. CLOSED CROSS-SECTION

Applying the theory of Umansky the coordinates of the displacement of the torsion :

$$u = -\varphi\, y\,, \quad v = \varphi\, x\,, \quad w = -\beta'\,\phi \tag{13}$$

where $(\ )' = \frac{\partial}{\partial z}$, and $\beta' = \beta'(z)$ is an independent field from φ, $\phi = \phi(x,y)$ generalized warping function. In this case the strain energy

$$U = \frac{1}{2}\int_L \{\, GI_p\,[(\varphi' - \kappa\beta')^2 + \kappa(1-\kappa)(\beta')^2] + EI_\phi(\beta'')^2 \,\}\, dz \tag{14}$$

while the work of the loading

$$W = \Sigma\, M_i\varphi_i - \Sigma\, B_i\beta'_i + \int_L m\,\varphi\, dz \tag{15}$$

where I_p is the torsional warping rigidity, $\kappa = \frac{I_p - I_T}{I_p}$.

From the first variation of the potential energy $\delta\Pi = 0$ can beshown that the fields of φ and β' must be C^ocontinuous, further

$$M = I_p\, G\,(\varphi' - \kappa\beta')\,, \qquad B = -I_\phi\, E\,\beta'' \tag{16}$$

and the differential equation of the problem :

$$E\, I_p\, \varphi^{iv} - \kappa\, I_T\, G\, \varphi'' = \kappa\, m - \frac{E\, I_\phi}{G\, I_p}\, m'' \tag{17}$$

We approximate the fields of φ and β'

$$\begin{bmatrix} \varphi(\bar{z}) \\ \beta'(\bar{z}) \end{bmatrix} = \begin{bmatrix} (1-\bar{z}) & 0 & \bar{z} & 0 \\ 0 & (1-\bar{z}) & 0 & \bar{z} \end{bmatrix} \begin{bmatrix} \varphi_o \\ \beta'_o \\ \varphi_L \\ \beta'_L \end{bmatrix} +$$

$$+\begin{bmatrix} \dots(\bar{z}^{j+1}-\bar{z})\dots & \dots & 0 & \dots \\ \dots & 0 & \dots & \dots(\bar{z}^{j+1}-\bar{z})\dots \end{bmatrix}\begin{bmatrix} \vdots \\ \hat{a}^{\varphi}_j \\ \vdots \\ \hat{a}^{\beta}_j \\ \vdots \end{bmatrix} \tag{18}$$

In the followings the vectors of the generalized displacements and the additional constants are denoted respectively by

$$q^T = [\ \varphi_o\ \ \beta'_o\ \ \varphi_L\ \ \beta'_L\ ] \quad \text{and} \quad \hat{a}^T = [\ (a^{\varphi})^T\ (a^{\beta})^T] \tag{19}$$

Then substituting of (17) into (13) and (14) we get the potential energy of (7) where

$$K_{qq}(1,1) = K_{qq}(3,3) = -K_{qq}(1,3) = -K_{qq}(3,1) = \frac{I_p G}{L}$$

$$K_{qq}(1,2) = K_{qq}(2,1) = K_{qq}(1,4) = K_{qq}(4,1) = -K_{qq}(2,3) = -K_{qq}(3,2)$$

$$= -K_{qq}(3,4) = -K_{qq}(4,3) = \kappa\,\frac{I_p G}{2}$$

$$K_{qq}(2,2) = K_{qq}(4,4) = \frac{E\ I_{\varphi}}{L} + I_p\,G\,\kappa\,\frac{L}{3}$$

$$K_{qq}(2,4) = K_{qq}(4,2) = -\frac{E\ I_{\phi}}{L} + I_p\,G\,\kappa\,\frac{L}{6} \tag{20}$$

$$K_{qa} = \left[\begin{array}{ccc|ccc} \dots & 0 & \dots & \dots & [-\kappa I_p G\frac{j}{2(j+2)}] & \dots \\ \dots & [\kappa I_p G\frac{j}{2(j+2)}] & \dots & \dots & [\kappa I_p G(\frac{1}{j+2}-\frac{1}{j+3}-\frac{1}{6})L] & \dots \\ \dots & 0 & \dots & \dots & [\ \kappa I_p G\frac{j}{2(j+2)}] & \dots \\ \dots & [-\kappa I_p G\frac{j}{2(j+2)}] & \dots & \dots & [-\kappa I_p G L\frac{j}{3(j+3)}] & \dots \end{array}\right] \tag{21}$$

$$j=1,\dots,NPF \qquad\qquad j=1,\dots,NPB$$

$$K_{aa}(i,j) = \frac{I_p G}{L}\,\frac{ij}{i+j+1} \qquad i=1,\dots,NPF\ ,\quad j=1,\dots,NPF$$

$$K_{aa}(i,j+NPF) = K_{aa}(j,i+NPF) = \kappa\ I_p G\ (\frac{i+1}{i+2} - \frac{i+1}{i+j+2} - \frac{j}{2(j+2)}\ )$$

$$i=1,\dots,NPF\ ,\quad j=1,\dots,NPB$$

$$K_{aa}(i+NPF, j+NPF) = \frac{E\ I_\phi}{L} \frac{ij}{i+j+1} + \kappa I_p GL \left(\frac{j}{3(j+3)} - \frac{1}{i+3} + \frac{1}{i+j+3} \right)$$

$$i=1,\dots,NPB \ , \quad j=1,\dots,NPB \qquad (22)$$

$$f_q^T = [\ L(\tfrac{1}{3} m_o + \tfrac{1}{6} m_L) \ , \quad 0 \quad , L(\tfrac{1}{6} m_o + \tfrac{1}{3} m_L) \ , \quad 0 \ ] \qquad (23)$$

$$f_a(k) = L \ [\ m_o(\frac{1}{k+2} - \frac{1}{2}) + (m_L - m_o)(\frac{1}{k+3} - \frac{1}{3}) \ ] \qquad k=1,\dots,NPF$$

$$f_a(k+NPF) = 0 \qquad k=1,\dots,NPB \qquad (24)$$

4. NUMERICAL EXPERIENCES

The above elements perform the better accuracy for the dinamical boundary and fitting conditions by the grater number of additional constants (NPF,NPB). Solving a lot of examples we experienced that the results are accurate enough if the number of additional constants were taken about five that is NPF≈5 , NPB≈5 . Practically exact solution can be gained by NPF=NPB=5 if the length L of the element is not longer than $L^*/2$, where

$$L^* = \pi \sqrt[4]{\frac{E\ I_\phi}{\kappa G I_p}}$$

5. SUMMARY

The above elements can be completed easily by truss and bending effects . Then the nodal point degree of freedom is equal to 7. The additional constants $\hat{a}$ can be calculated at element level. By the developed prismatic beam element [7] exact solution can be get for truss and bending effect in the case of linear distributed loads , for torsion the greater numbers of NPF and NPB the better solution can be gained .

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Streszczenie

Konstrukcje cienkościenne stosowane są w szerokim zakresie w inżynierii lądowej i wodnej oraz w technologii budowy maszyn. Istnieje wiele artykułów poruszających teoretyczne [1-3] i obliczeniowe [4-6] aspekty modeli mechanicznych. Stosując teorię Vlasowa i Umanskiego i zakładając, że ich terminy (elementy) są znane, podajemy wzory, które z łatwością mogą być kcdowane dla macierzy elementów skończonych.