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THE ARTIFICIAL HEAT SOURCE METHOD IN NUMERICAL MODELLING OF NON-LINEAR CONDUCTION PROBLEMS

Summary. In the paper a certain algorithm which can be called the artificial heat source method is presented. Proposed approach is useful in the case of non-linear and non-steady heat conduction problems.

METODA SZTUCZNEGO ŹRÓDŁA CIEPŁA W MODELOWANIU NUMERYCZNYM NIELINIOWYCH ZADAŃ PRZEWODNICTWA

Streszczenie. W pracy przedstawiono pewien algorytm, który nazwano metodą sztucznego źródła ciepła. Metoda może być wykorzystana do numerycznego modelowania nieliniowych i niestacjonarnych zagadnień przewodnictwa cieplnego.

МЕТОД ИСКУССТВЕННОГО ИСТОЧНИКА ТЕПЛА В ЧИСЛЕННОМ МОДЕЛИРОВАНИИ НЕЛИНЕЙНЫХ ПРОБЛЕМОВ ТЕПЛОПРОВОДНОСТИ

Резюме. Представлены основы алгоритма который назван методом искуственного источника тепла. Этот метод может быть применен к численному моделированию нелинейных задач теплопроводности.

1. GOVERNING EQUATIONS

A homogenous domain D limited by boundary Γ is considered. The heat conduction process in this area is described by the following energy equation

$$c(T)\rho(T)\frac{\partial T(X, t)}{\partial t} = \operatorname{div}[\lambda(T)\operatorname{grad}T(X, t)] \tag{1}$$

and boundary-initial conditions of the form (Figure 1)

$$
\begin{aligned}
X \in \Gamma_1 &: \quad T(X, t) = T_1(X, t) \\
X \in \Gamma_2 &: \quad -\lambda n \cdot \mathrm{grad}\, T(X, t) = q_n(X, t) \\
X \in \Gamma_3 &: \quad -\lambda n \cdot \mathrm{grad}\, T(X, t) = \alpha \left[T(X, t) - T_\infty \right] \\
t = 0 &: \quad T(X, 0) = T_0(X)
\end{aligned}
\tag{2}
$$

where c, ρ, λ are the thermophysical parameters (specific heat, mass density, thermal conductivity), T, X, t denote a temperature, spatial co-ordinates and time, $n \cdot \mathrm{grad} T$ is a normal derivative at boundary point X, q_n is a given heat flux, α, T_∞ are the heat transfer coefficient and ambient temperature, T_1, T_0 are the boundary and initial temperatures.

The basic mathematical model can be rebuilt by the introduction of so-called Kirchhoff's temperature, it means

$$
U(T) = \int_{T_r}^{T} \lambda(\mu)\, \mathrm{d}\mu \tag{3}
$$

where T_r is an arbitrary assumed reference level.

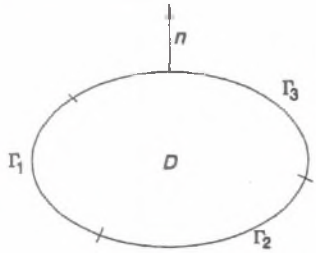

Fig 1. Considered domain D
Rys. 1. Rozważany obszar D

The Kirchhoff's transformation linearizes the right-hand side of energy equation (1), namely

$$
\mathrm{div}[\lambda(T)\, \mathrm{grad}\, T(X, t)] = \mathrm{div}[\mathrm{grad}\, U(X, t)] \tag{4}
$$

The left-hand side of equation (1) can be transformed by introducing to the considerations the physical enthalpy related to an unit of volume

$$
H(T) = \int_{T_r}^{T} c(\mu)\rho(\mu)\, \mathrm{d}\mu \tag{5}
$$

Because H and U are the functions of temperature and there are monotone ones so, it is possible to construct the function $H=\Phi(U)$ – comp. Figures 2, 3, 4. Additionally

$$
\frac{\partial H(X, t)}{\partial t} = \frac{\mathrm{d}H(U)}{\mathrm{d}U} \frac{\partial U(X, t)}{\partial t} = \Phi'(U) \frac{\partial U(X, t)}{\partial t} = \Psi(U) \frac{\partial U(X, t)}{\partial t} \tag{6}
$$

The final form of considered differential equation is the following

$$
\Psi(U) \frac{\partial U(X, t)}{\partial t} = \mathrm{div}[\mathrm{grad}\, U(X, t)] \tag{7}
$$

A course of derivative $\Psi(U)$ for considered material is shown in Figure 5.

The boundary and initial conditions should be also transformed in adequate way [1]

$$
\begin{aligned}
X\in\Gamma_1 : &\quad U(X,t) = U_1(x,t)\\
X\in\Gamma_2 : &\quad -n\cdot \operatorname{grad} U(X,t) = q_n(X,t)\\
X\in\Gamma_3 : &\quad -n\cdot \operatorname{grad} U(X,t) = \alpha\left[T(x,t) - T_\infty\right]\\
t = 0 : &\quad U(X,0) = U_0(X)
\end{aligned}
\tag{8}
$$

Because

$$U - U_\infty = \int_{T_\infty}^{T} \lambda(\mu)\,d\mu = \lambda_m (T - T_\infty) \tag{9}$$

where λ_m is an integral mean of thermal conductivity for $[T_\infty, T]$, so the boundary condition for Γ_3 can be written in the form

$$-n\cdot \operatorname{grad} U(X,t) = \alpha_m\left[U(X,t) - U_\infty\right] \tag{10}$$

where $\alpha_m=\alpha/\lambda_m$. It should be pointed that this non-linearity of condition (10) does not cause the essential difficulties in numerical realization.

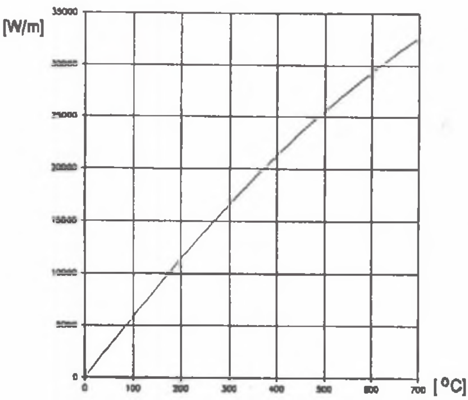

Fig. 2. Kirchhoff's temperature $U=U(T)$
Rys. 2. Temperatura Kirchhoffa $U=U(T)$

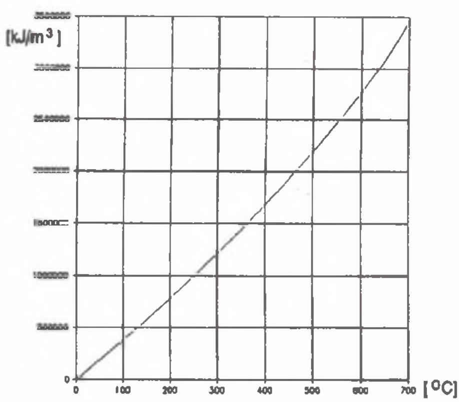

Fig. 3. Enthalpy function $H=H(T)$
Rys. 3. Funkcja entalpii $H=H(T)$

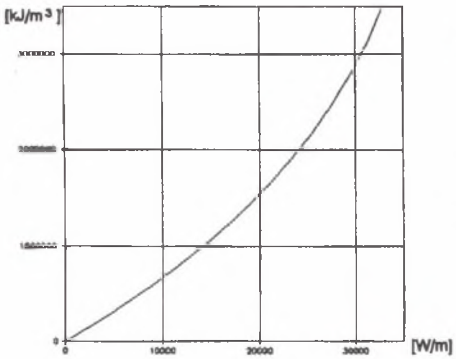

Fig. 4. Function $H=H(U)$
Rys. 4. Funkcja $H=H(U)$

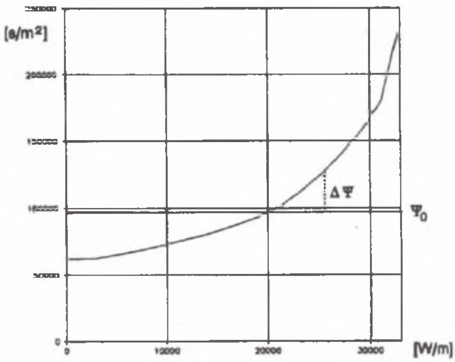

Fig. 5. Function $\Psi=\Phi'$
Rys. 5. Funkcja $\Psi=\Phi'$

2. THE ARTIFICIAL HEAT SOURCE METHOD (AHSM)

Consider now, a function $\Psi(U)$ which is conventionally expressed as a sum of two components, it means a constant part Ψ_0 and a certain increment $\Delta\Psi$

$$\Psi(U) = \Psi_0 + \Delta\Psi(U) \tag{11}$$

The energy equation (7) can be written in the form

$$\Psi_0 \frac{\partial U(X,\, t)}{\partial t} = \operatorname{div}[\operatorname{grad} U(X,\, t)] - \Delta\Psi \frac{\partial U(X,\, t)}{\partial t} \tag{12}$$

or

$$\Psi_0 \frac{\partial U(X,\, t)}{\partial t} = \operatorname{div}[\operatorname{grad} U(X,\, t)] + q_V(X,\, t) \tag{13}$$

where $q_V(X, t)$ is a source function (a capacity of internal heat sources). The essential feature of equation (13) consists in a fact, that leaving out the last term one obtains the linear form of energy equation. Taking into account the possibilities of boundary element method application in the range of non-steady problems modelling, it is the very convenient form of basic differential equation (a non-linearity appears only in the component determining the internal heat sources, and the function describing so-called fundamental solution for considered problem is well known). The calculation of a source function requires, of course, the introduction of a certain iterative procedure (the details connected with numerical aspects of proposed algorithm will be presented in the further part of the paper), but it should be pointed that if $|\Delta\Psi| < \Psi_0$ then the adequate iterative algorithm is convergent. In this paper the AHSM will supplement a variant of boundary element method called the BEM using discretisation in time [2, 3, 4, 5].

3. BOUNDARY ELEMENT METHOD USING DISCRETISATION IN TIME

The 1D problem will be considered, it means the energy equation in the form

$$\Psi_0 \frac{\partial U(x, t)}{\partial t} = \frac{\partial^2 U(x, t)}{\partial x^2} + q_V(x, t) \tag{14}$$

with assumed boundary conditions for $x=x_1$, $x=x_2$ and initial condition for $t=0$. It should be pointed that a generalization of presented algorithm on 2D or 3D problem is very simple [5].

At first the Green's function of the form

$$U^*(\xi, x) = -\frac{\sqrt{a\Delta t}}{2} \exp\left(-\frac{|x-\xi|}{\sqrt{a\Delta t}}\right), \quad a = \frac{1}{\Psi_0} \tag{15}$$

is introduced (x is a spatial co-ordinate whereas ξ is a point where a concentrated heat source is applied).

The function (15) fulfills the equation

$$\frac{\partial^2 U^*(\xi, x)}{\partial x^2} - \frac{1}{a\Delta t} U^*(\xi, x) = \Delta(\xi, x) \tag{16}$$

where $\Delta(\xi, x)$ is a Dirac's function.

The idea of discussed variant of the BEM consists in a substitution of a time derivative by its first order approximation and then the equation (14) is transformed to the form

$$\frac{\partial^2 U(x, t+\Delta t)}{\partial x^2} - \frac{1}{a\Delta t} U(x, t+\Delta t) = -q_V(x, t) - \frac{1}{a\Delta t} U(x, t) \tag{17}$$

Multiplying both sides of equation (17) by $U^*(\xi, x)$ and integrating over x_1, x_2 yields

$$\begin{gathered}\int_{x_1}^{x_2} \left[\frac{\partial^2 U(x, t+\Delta t)}{\partial x^2} - \frac{1}{a\Delta t} U(x, t+\Delta t)\right] U^*(\xi, x)\,\mathrm{d}x = \\ = \int_{x_1}^{x_2} \left[-q_V(x, t) - \frac{1}{a\Delta t} U(x, t)\right] U^*(\xi, x)\,\mathrm{d}x\end{gathered} \tag{18}$$

The last integral equation corresponds to so-called Weighted Residual Method criterion. Integration by parts the left-hand side of equation (18) and using equation (15) leads to

$$U(\xi,t+\Delta t)=\left[\frac{\operatorname{sgn}(x-\xi)}{2}\exp\left(-\frac{|x-\xi|}{\sqrt{a\Delta t}}\right)U(x,t+\Delta t)+\frac{\sqrt{a\Delta t}}{2}\exp\left(-\frac{|x-\xi|}{\sqrt{a\Delta t}}\right)q_n(x,t+\Delta t)\right]_{x_1}^{x_2}+$$
$$+\int_{x_1}^{x_2}\left[\frac{\sqrt{a\Delta t}}{2}q_V(x,\,t)+\frac{1}{2\sqrt{a\Delta t}}U(x,\,t)\right]\exp\left(-\frac{|x-\xi|}{\sqrt{a\Delta t}}\right)\mathrm{d}x \tag{19}$$

For $\xi \to x_1$ and $\xi \to x_2$ one obtains two following boundary equations

$$\begin{bmatrix} -\dfrac{\sqrt{a\Delta t}}{2} & \dfrac{\sqrt{a\Delta t}}{2}\exp\left(\dfrac{|x_2-x_1|}{\sqrt{a\Delta t}}\right) \\ -\dfrac{\sqrt{a\Delta t}}{2}\exp\left(-\dfrac{|x_1-x_2|}{\sqrt{a\Delta t}}\right) & \dfrac{\sqrt{a\Delta t}}{2} \end{bmatrix}\begin{bmatrix} q_n(x_1,\,t+\Delta t) \\ q_n(x_2,\,t+\Delta t) \end{bmatrix}=$$
$$=\begin{bmatrix} -\dfrac{1}{2} & \dfrac{1}{2}\exp\left(\dfrac{|x_2-x_1|}{\sqrt{a\Delta t}}\right) \\ -\dfrac{1}{2}\exp\left(-\dfrac{|x_1-x_2|}{\sqrt{a\Delta t}}\right) & -\dfrac{1}{2} \end{bmatrix}\begin{bmatrix} U(x_1,\,t+\Delta t) \\ U(x_2,\,t+\Delta t) \end{bmatrix}+ \tag{20}$$
$$+\begin{bmatrix} \displaystyle\int_{x_1}^{x_2}\frac{\sqrt{a\Delta t}}{2}q_V(x,\,t)+\frac{1}{2\sqrt{a\Delta t}}U(x,\,t)\exp\left(-\frac{|x-x_1|}{\sqrt{a\Delta t}}\right)\mathrm{d}x \\ \displaystyle\int_{x_1}^{x_2}\frac{\sqrt{a\Delta t}}{2}q_V(x,\,t)+\frac{1}{2\sqrt{a\Delta t}}U(x,\,t)\exp\left(-\frac{|x-x_2|}{\sqrt{a\Delta t}}\right)\mathrm{d}x \end{bmatrix}$$

or

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} q_n(x_1,\,t+\Delta t) \\ q_n(x_2,\,t+\Delta t \end{bmatrix}=\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}\begin{bmatrix} U(x_1,\,t+\Delta t) \\ U(x_2,\,t+\Delta t) \end{bmatrix}+\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tag{21}$$

Above system of equations allows to find the 'missing' boundary Kirchhoff's temperatures or heat fluxes, and next applying the equation (19) one can determine a searched function $U(x, t+\Delta t)$ at the internal points of domain D. The heat flux continuity condition (10) can be also taken into account.

The iterative process of source function determination is the following.

1. Transition from $t^0=0$ to $t^1=t^0+\Delta t$:
   - it is assumed that $q_V(x_i, t^1)=0$, at the same time x_i denotes a central point of internal cells distinguished in considered domain,
   - for this assumption the Kirchhoff's temperature field for whole domain is calculated,
   - the local cooling rates $[U(x_i, t^1)-U(x_i, t^0)]/\Delta t$ are estimated,
   - a local values of source function $q_V(x_i, t^1)$ are corrected,
   - the iterative process is stopped if required accuracy is obtained.
2. Transition from t^{f-1} to t^f, $f=2, 3, ..., F$:
   - it is assumed that $q_V(x_i, t^f)$ is equal to the last value of q_V found during the previous iterative process (at considered point),
   - for this assumption the Kirchhoff's temperature field for whole domain is calculated,
   - the local cooling rates $[U(x_i, t^f)-U(x_i, t^{f-1})]/\Delta t$ are estimated,
   - a local values of source function $q_V(x_i, t^f)$ are corrected,
   - the iterative process is stopped if required accuracy is obtained.

The test computations show (and it can be probably proved in analytic way) that the iterative process is convergent if $|\Delta\Psi| < \Psi_0$.

4. EXAMPLE OF NUMERICAL SIMULATION

The steel plate with thickness $L=0.1$[m] has been considered. Thermophysical parameters of the material (C=0.08, Si=0.08, Mn=0.31, S=0.05, P=0.029, Cr=0.045, Ni=0.07, Mo=0.02) have been assumed on the basis of experimental data quoted in [6]. The problem is strongly non-linear because, for example, specific heat (related to an unit of volume) changes from $3.5\cdot10^6$ to $8.7\cdot10^6$[J/m^{3}K], whereas the thermal conductivity from 30 to 50[W/mK]. Using the numerical integration methods the functions $H(T)$ and $U(T)$ have been found – Figures 2 and 3, next function $H=H(U)$ has been constructed (Figure 4) and finally its derivative (Figure 5). The following boundary-initial conditions have been assumed: $x_1=0$: $q_n(0, t)=0$, $x_2=L$: $-\partial U/\partial x=\alpha_m(U-U_\infty)$, at the same time $\alpha=300$, $U_\infty=0$ (comp. equation (11)), for $t=0$: $U_0=27510$ (this value corresponds to $T=553$°C).

In Figure 6 the cooling curves at selected points of interior D are shown, in particular: 1: $x=0.095$, 2: $x=0.065$, 3: $x=0.035$, 4: $x=0.05$[m]. The full lines illustrate the numerical solution obtained on the basis of repeatedly verified FDM algorithm for non-linear equations, whereas the symbols show the numerical solution found by means of artificial heat source method. The maximum difference between presented numerical solutions does not exceed 0.5%. It seems that proposed method can be very useful in the case of the BEM application for numerical computations of non-steady and non-linear heat conduction problems.

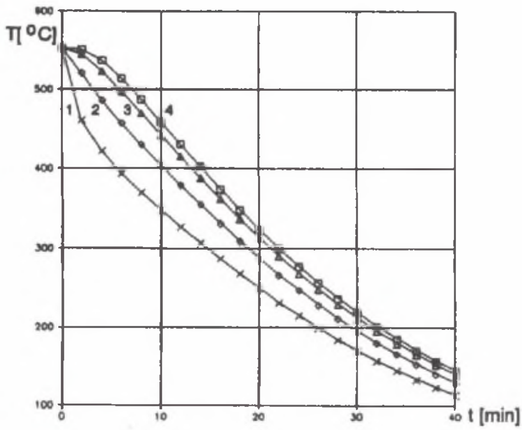

Fig. 6. Cooling curves at selected points of domain *D*
Rys. 6. Krzywe stygnięcia w wybranych punktach obszaru *D*

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