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DISPLACEMENT AND FORCE ANALYSIS OF THE SPHERICAL PLATFORM MECHANISM 3(SPS)-S WITH COMPLIANCIES OF LINEAR ACTUATORS TAKEN INTO ACCOUNT

Summary. The displacement equations of the 3(SPS)-S spherical platform mechanism (with three linear actuators and three degrees of freedom) are derived by using the solutions of the vector tetrahedron equations in the form of three unit vectors formula [6]. The compliancies of the actuator system are taken into consideration, leading to an algorithm for determining the mechanism compliant characteristics. Some numerical examples are solved.

1. Introduction

Solar panels, telescopes, radar and satellite antennas, mirrors for laser beams and manipulator end-effectors are some of the most outstanding application of the spherical platform mechanism driven by three linear actuators with multi-loop kinematic chain of the 3(SPS)-S structure (S-spherical or universal joint, P-prismatic joint). When a precise orientation control, high stiffness and favourable load capacity to mechanism weight ratios are the major requirement with respect to manoeuverability, platform-type mechanisms with parallel structure of actuator system can provide higher performances than serial ones.

In spite of their very attractive performances, few spherical platform mechanism for an orientation of rigid body have been proposed. In particular, a fully-parallel three degree-of-freedom spherical wrist with coaxial actuators has been considered in [2,3] from the viewpoint of the optimization of some kinematic performance.

An important problem for trajectory planning of parallel manipulators is represented by direct position analysis (DPA), one of the subjects of this paper. The DPA consists of finding the orientation of the platform when the set of actuator displacements is given. The DPA of spherical platform can

be reduced to the successive solution of two second order trigonometric equations, hence obtaining four or eight configurations at most [5,8].

When the actuators of the parallel manipulator are all locked, then it reduces to a statically determined structure, and the solution of the DPA is equivalent to finding all possible closures of the structure itself.

The compliant characteristics of a robot manipulator to a large extent determine the limits of its performance. Effects of serial manipulator compliance are the subject of many investigations. There are, however, relatively limited previous investigations into the compliance of parallel manipulator and its matrix representation [4].

In this investigation a payload/manipulator system is modelled as kinematically constrained rigid body, supported by an elastic system and is represented by a symmetric 3 x 3 compliance matrices.

2. Direct position analysis by using vector method

The platform (see Fig.1) is connected to the base through one spherical joint centered at point Q, and three legs between three base points A_i and platform points B_i (i = 1, 2, 3), where spherical joints are centered. The rotational freedom of each leg about the line through its two terminal joints does not affect the platform orientation; however, it can be eliminated by substituting a universal joint at the base.

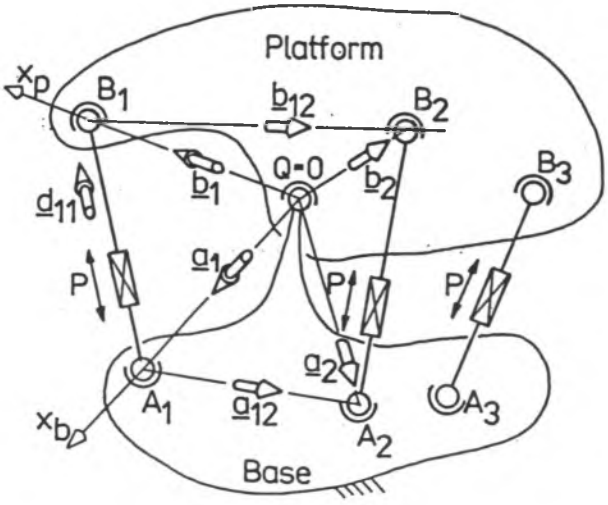

Fig.1. The 3(SPS$_u$)-S spherical platform mechanism with three linear actuators. Notation: S-spherical joints centered at points Q and B_i, S_u-universal joints at A_i(i=1,2,3), P-prismatic joints on each leg A_iB_i

The geometry of the spherical platform mechanism is given. In particular, the position of points A_i are given in an arbitrary reference system fixed to the base; the position of points B_i are given in an arbitrary reference system fixed to the platform; the lengths d_i of legs A_iB_i are known. Without loss of generality, reference systems are chosen with origins coincident with point Q, x_b-axis directed from point Q to point A_1, x_p-axis to point B_1, y_b-axis is in the plane OA_1A_2, and y_p-axis is in the plane OB_1B_2.

Let be given: $\mathbf{a}_i$ $(i=1,2,3)$ - the position vector of point A_i described in the base reference system; $\mathbf{b}_i$ $(i=1,2,3)$ - the position vector of point B_i in the platform reference system; $d_i = A_iB_i$ - the distance between two respective points of the base and the platform (the actuator leg length). The dot products of the corresponding unit vectors can be described as follows:

$$\begin{aligned} \mathbf{a}_i^o \cdot \mathbf{a}_{i+1}^o &= (a_i^2 + a_{i+1}^2 - a_{i,i+1}^2)/2a_ia_{i+1} \\ \mathbf{a}_i^o \cdot \mathbf{b}_i^o &= (a_i^2 + b_i^2 - d_i^2)/2a_ib_i \\ \mathbf{b}_i^o \cdot \mathbf{b}_{i+1}^o &= (b_i^2 + b_{i+1}^2 - b_{i,i+1}^2)/2b_ib_{i+1} \end{aligned} \tag{1}$$

where:

$$\mathbf{a}_{i,i+1} = \mathbf{a}_{i+1} - \mathbf{a}_i \quad , \quad \mathbf{b}_{i,i+1} = \mathbf{b}_{i+1} - \mathbf{b}_i \quad ,$$

$$\mathbf{a}_i^o = \mathbf{a}_i/a_i \quad , \quad \mathbf{b}_i^o = \mathbf{b}_i/b_i \quad , \quad i = 1,\ 2,\ 3$$

If the leg A_3B_3 is momentarily removed from the platform mechanism, then the point B_3 can be treated as the coupler point of a spherical 4-bar mechanism $(A_1B_1B_2A_2)$. The position of this mechanism can be described by an additional independent variable:

$$v = \mathbf{a}_2^o \cdot \mathbf{b}_1^o \tag{2}$$

It can be used to describe the unit vectors $\mathbf{b}_i^o$ of points B_i $(i=1,2,3)$ in the base reference system according to the general formula for three unit vectors [6]:

$$\mathbf{b}_1^o = \left\{\left[\left(\mathbf{a}_1^o \cdot \mathbf{b}_1^o\right) - v\left(\mathbf{a}_1^o \cdot \mathbf{a}_2^o\right)\right]\mathbf{a}_1^o + \left[v - \left(\mathbf{a}_1^o \cdot \mathbf{a}_2^o\right)\left(\mathbf{a}_1^o \cdot \mathbf{b}_1^o\right)\right]\mathbf{a}_2^o + \pm\sqrt{D_1}\left(\mathbf{a}_1^o \times \mathbf{a}_2^o\right)\right\}\Big/\left[1 - \left(\mathbf{a}_1^o \cdot \mathbf{a}_2^o\right)^2\right] \tag{3}$$

where $D_1 = 1 - (\mathbf{a}_1^o \cdot \mathbf{b}_1^o)^2 - (\mathbf{a}_1^o \cdot \mathbf{a}_2^o)^2 - v^2 + 2(\mathbf{a}_1^o \cdot \mathbf{b}_1^o)(\mathbf{a}_1^o \cdot \mathbf{a}_2^o)v$

$$\mathbf{b}_2^o = \left\{\left[\left(\mathbf{a}_2^o \cdot \mathbf{b}_2^o\right) - v\left(\mathbf{b}_1^o \cdot \mathbf{b}_2^o\right)\right]\mathbf{a}_2^o + \left[\left(\mathbf{b}_1^o \cdot \mathbf{b}_2^o\right) - v\left(\mathbf{a}_2^o \cdot \mathbf{b}_2^o\right)\right]\mathbf{b}_1^o + \pm\sqrt{D_2}\left(\mathbf{a}_2^o \times \mathbf{b}_1^o\right)\right\}\Big/\left(1 - v^2\right) \tag{4}$$

where $D_2 = 1 - (\mathbf{a}_2^o \cdot \mathbf{b}_2^o)^2 - (\mathbf{b}_1^o \cdot \mathbf{b}_2^o)^2 - v^2 + 2(\mathbf{a}_2^o \cdot \mathbf{b}_2^o)(\mathbf{b}_1^o \cdot \mathbf{b}_2^o)v$

$$\mathbf{b}_3^o = \left\{\left[\left(\mathbf{b}_1^o \cdot \mathbf{b}_3^o\right) - \left(\mathbf{b}_1^o \cdot \mathbf{b}_2^o\right)\left(\mathbf{b}_2^o \cdot \mathbf{b}_3^o\right)\right]\mathbf{b}_1^o + \left[\left(\mathbf{b}_2^o \cdot \mathbf{b}_3^o\right) - \right.\right.$$
$$\left.\left. - \left(\mathbf{b}_1^o \cdot \mathbf{b}_2^o\right)\left(\mathbf{b}_1^o \cdot \mathbf{b}_3^o\right)\right]\mathbf{b}_2^o \pm \sqrt{D_3}\left(\mathbf{b}_1^o \times \mathbf{b}_2^o\right)\right\} \Big/ \left[1 - \left(\mathbf{b}_1^o \cdot \mathbf{b}_2^o\right)^2\right] \tag{5}$$

where $D_3 = 1 - (\mathbf{b}_1^o \cdot \mathbf{b}_2^o)^2 - (\mathbf{b}_2^o \cdot \mathbf{b}_3^o)^2 - (\mathbf{b}_1^o \cdot \mathbf{b}_3^o)^2 +$
$+ 2(\mathbf{b}_1^o \cdot \mathbf{b}_2^o)(\mathbf{b}_2^o \cdot \mathbf{b}_3^o)(\mathbf{b}_1^o \cdot \mathbf{b}_3^o)$

The closure equation for the loop with leg A_3B_3 can be written in the form

$$F(v) = \mathbf{a}_3^o \cdot \mathbf{b}_3^o - (a_3^2 + b_3^2 - d_3^2)/2a_3b_3 = 0 \tag{6}$$

where $\mathbf{b}_3^o$ is described as a nonlinear function of variable v by using formulas (3)-(5). The roots of equation (6) can be found in the range [-1;+1], by using a numerical method, for example Newton's method.

3. The Jacobian of the 3(SPS)-S manipulator

The input variables $d_i(t)$, $(i=1,2,3)$, and the output variables, for example α, β, γ - Euler angles, are related by the following equations:

$$g_i = a_i^2 + b_i^2 - 2a_ib_i(\mathbf{a}_i^o \cdot \mathbf{b}_i^o) - d_i^2 = 0, \quad (i=1,2,3), \tag{7}$$

where $\mathbf{b}_i^o$ - the unit vectors of points B_i $(i=1,2,3)$ described in the base reference system by using formulas (3)-(5) or by using the rotation matrix as follows

$$\mathbf{b}_i^o = \mathbb{R}\,\mathbf{b}_{ip}^o \tag{8}$$

where $\mathbf{b}_{ip}^o$ - the unit vectors of points B_i $(i=1,2,3)$ described in the platform reference system, $\mathbb{R}$ - the rotation matrix expressed by Euler angles. One of the 24 Euler angle set conventions (denoted in [1] by $\mathbb{R}_{X'Y'X'}$) is chosen (as given below):

$$\mathbb{R} = \begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma \\ -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix} \tag{9}$$

Differentiation of equation (7) with respect to time yields

$$\dot{\mathbf{g}} = \mathbb{U}_\omega \cdot \underline{\omega} - \mathbf{W} \cdot \dot{\mathbf{d}} = 0 \tag{10}$$

where: $\mathbf{g} = [g_1\ g_2\ g_3]^T = 0$, $\quad \dot{\mathbf{g}} = \dfrac{d\mathbf{g}}{dt}$

$\underline{\omega} = [\dot{\alpha}\ \dot{\beta}\ \dot{\gamma}]^T$ - output variables $\qquad \dot{\mathbf{d}} = [\dot{d}_1\ \dot{d}_2\ \dot{d}_3]^T$ - input variables

$$\mathbb{U}_\omega = \left[\frac{\partial g_i}{\partial \alpha}\ \frac{\partial g_i}{\partial \beta}\ \frac{\partial g_i}{\partial \gamma}\right], \qquad (i = 1,2,3) \tag{11a}$$

$$\mathbb{W} = \left[\frac{\partial g_i}{\partial d_1} \quad \frac{\partial g_i}{\partial d_2} \quad \frac{\partial g_i}{\partial d_3} \right], \qquad (i = 1,2,3) \tag{11b}$$

The diagonal matrix (11) can be written in the form

$$\mathbb{W} = -2 \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \tag{12}$$

Differentiation of (8) with respect to time gives

$$\dot{\mathbf{b}}_i^0 = \dot{\mathbb{R}}\, \mathbf{b}_{ip}^0 \tag{13}$$

substituting (8) to (13) one can obtain

$$\dot{\mathbf{b}}_i^0 = \dot{\mathbb{R}}\, \mathbb{R}^T\, \mathbf{b}_i^0 \tag{14}$$

It is known (for example from [1]) that

$$\dot{\mathbb{R}}\, \mathbb{R}^T = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \quad \text{where: } \underline{\omega} = \begin{bmatrix} \dot{\gamma} c\beta + \dot{\alpha} \\ \dot{\beta} c\alpha + \dot{\gamma} s\beta s\alpha \\ \dot{\beta} s\alpha - \dot{\gamma} s\beta c\alpha \end{bmatrix} \tag{15}$$

Introducing the diagonal matrix $\mathbb{W}$ (12) and the matrix $\mathbb{U}_\omega$(11a) into equation (9) we can write

$$[\dot{\alpha}\ \dot{\beta}\ \dot{\gamma}\ ]^T = \mathbb{J}_\omega^{-1}\ [\dot{d}_1\ \dot{d}_2\ \dot{d}_3]^T \tag{16}$$

where the Jacobian matrix of the platform angular velocity is given by:

$$\mathbb{J}_\omega^{-1} = \mathbb{U}_\omega^{-1}\mathbb{W} \tag{17}$$

For any position of the mechanism the Jacobian matrix $\mathbb{J}_\omega$(17) can be calculated. With equations (15) and (16) the angular velocity $\underline{\omega} = [\omega_x\ \omega_y\ \omega_z]^T$ can be found for any given set of relative velocities of the linear actuators $\dot{\mathbf{d}} = [\dot{d}_1\ \dot{d}_2\ \dot{d}_3]^T$. This result can be used for the determination of actuator forces if the platform is loaded only by a torque.

The position vector of point P given in the platform reference system

$$\mathbf{p}_p = [p_{xp}\ p_{yp}\ p_{zp}]^T$$

can be transformed into the base reference system using the formula

$$\mathbf{p}_b = \mathbb{R}\ \mathbf{p}_p \tag{18}$$

where $\mathbf{p}_b = [p_{xb}\ p_{yb}\ p_{zb}]^T$.

The Jacobian matrix of the linear velocity of point P can be obtained by differentiation of (18) with respect to time. According to formula (14) one can obtain

$$\dot{\mathbf{p}}_b = [\dot{p}_{xb}\ \dot{p}_{yb}\ \dot{p}_{zb}]^T = \frac{d\mathbf{p}}{dt} = \dot{\mathbb{R}}\ \mathbb{R}^T\mathbf{p}_b \tag{19a}$$

and $$[\dot{p}_{xb}\ \dot{p}_{yb}\ \dot{p}_{zb}]^T = J_p^{-1}\ [\dot{d}_1\ \dot{d}_2\ \dot{d}_3]^T \tag{19b}$$

The Jacobian matrix J_p can be immediately used to determine the actuator forces acting on the platform $\mathbf{f} = [f_1\ f_2\ f_3]^T$ if the external load can be reduced to $\mathbf{F}$ - the vector of force acting on the platform at the point P. The forces $\mathbf{F}$ and $\mathbf{f}$ are related by the equation

$$\mathbf{F} = J_p\ \mathbf{f} \tag{19c}$$

The equations (16) and (19) follow immediately from the principle of virtual displacement.

4. Global stiffness matrix of the 3(SPS)-S manipulator

Stiffness matrix transforms a differential displacement of the platform into an incremental change in force. The word "global" is introduced to denote that the mapping of stiffness changes and that it is a function of the manipulator configuration [4].

An external force $\mathbf{F}$ is applied to the platform at point P. An applied force $\mathbf{F}$ produces small changes in the platform orientation. The external force is in static equilibrium with the actuator forces and the manipulator remains in static equilibrium. Consider now that each actuator has a stiffness k_i (i = 1,2,3) and form a diagonal matrix $[k_i]$ and that $\delta f_i = k_i \delta d_i$, where δd_i is a small change in the leg length. Repeating for all legs and substituting into (18) yields:

$$\mathbf{F} = J_p^{-1}\ [k_i]\delta\mathbf{d} \tag{20}$$

where $\delta\mathbf{d} = [\delta d_1\ \delta d_2\ \delta d_3]^T$ and $[k_i] = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}$

The values of k_i are determined experimentally. The work done by all given leg forces is equal to the work done by the external force acting on the platform. Equating this two expressions for work, dividing both sides by f_i and repeating for all legs yields the matrix expression

$$\delta\mathbf{d} = J_p\ \delta\mathbf{p} \tag{21}$$

where $\delta\mathbf{p} = [\delta p_x\ \delta p_y\ \delta p_z]^T$ - the small displacement of the point P where the external force is applied. Equation (21) is the reverse kinematic solution. Substituting (21) into (20) gives the correlation form representation of the mapping of stiffness for the manipulator

$$\mathbf{F} = J_p^{-1}\ [k_i]\ J_p\ \delta\mathbf{p} \tag{22}$$

The stiffness matrix of the manipulator

$$\mathbf{S} = J_p^{-1}\ [k_i]\ J_p \tag{23}$$

contains nine independent parameters.

The joint reaction forces are dependent on external forces acting on the platform and the mechanism position which is dependent on the external load. Thus the equilibrium position has been determined using the method of successive approximation, making use of the results of the displacement analysis.

Having measured the compliance characteristics of the linear actuators (or legs), it is possible to determine the equilibrium position of the platform under static load, taking into account small actuator displacements caused by the compliances of their hydraulic drive system.

5. Numerical example

The numerical data for this example was determined on the basis of measurement results of the road building machine (Baukema SHM4-120A). The geometrical parameters are given below (in metres): $\mathbf{a}_1 = [\ 2.470 \quad 0 \quad 0\ ]^T$;

$$\mathbf{a}_2 = [2.250 \quad 1.020 \quad 0]^T; \qquad \mathbf{a}_3 = [-2.720 \quad 0.130 \quad -0.460]^T$$

The distances between the corresponding platform points are also given:

$$b_1 = 2.400 \qquad b_2 = 2.400 \qquad b_3 = 2.635$$

$$b_{12} = 1.030 \qquad b_{23} = 0.240 \qquad b_{13} = 1.070$$

The input variables as the actuator lengths are taken from the permissible ranges: $d_1 \in (0.530;\ 1.780)$, $d_2 \in (0.530;\ 1.780)$, $d_3 \in (0.810;\ 2.060)$
As a starting point the following values are taken:

$$d_1 = 0.780,\ d_2 = 0.570,\ d_3 = 1.320$$

Taking the values of v from the range [-1,1] and using a numerical method all real roots of equation (6) are found. For the starting point four real roots exist, determined as follows:

$v_1 = 0.98004$, $v_2 = 0.83904$, $v_3 = 0.91463$, $v_4 = 0.96707$

The results of calculations obtained for v_3 according to the formulas (2 ÷ 5) correspond to the actual configuration of the machine manipulator. The respective position vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ are calculated. The numerical results are presented below.

$$\mathbf{b}_1 = [2.278 \quad 0.290 \quad -0.698]^T; \quad \mathbf{b}_2 = [2.000 \quad 1.250 \quad -0.454]^T$$

$$\mathbf{b}_3 = [\ 2.200 \quad 1.345 \quad -0.541]^T$$

The following external force $\mathbf{F} = [-500 \quad -500 \quad -5000]$ N was applied to the platform at point P described by the vector $\mathbf{p}_p = [1.8 \quad 1.9 \quad -0.5]$ m.
The stiffness coefficients for the actuators are following:

$$k_1 = -7.7292 \cdot 10^6; \quad k_2 = -2.30083 \cdot 10^8\ ; \quad k_3 = -1.0165 \cdot 10^7 \quad \text{N/m}$$

By using the presented procedure the actuator forces corresponding to the platform equilibrium position are calculated: $\mathbf{f} = [5378 \quad -11584 \quad 4170]$ N
The stiffness matrix calculated in the given position of the mechanism is as follows:

$$\mathbf{S} = \begin{bmatrix} -8.614\cdot10^{6} & 1.861\cdot10^{7} & -6.401\cdot10^{6} \\ 1.861\cdot10^{7} & -4.041\cdot10^{7} & 1.263\cdot10^{7} \\ -6.401\cdot10^{7} & 1.263\cdot10^{7} & -1.137\cdot10^{7} \end{bmatrix} \quad \text{N/m}$$

6. Conclusion

The vector method of displacement and force analysis presented in the paper is very efficient and it can be implemented on IBM PC. The resultant equations are expressed in recursive notation, convenient for programming.

7. References

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