ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1995

Seria : MECHANIKA z. 122 Nr. kol. 1267

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DETERMINISTIC AND STOCHASTIC MODELLING OF CRACK GEOMETRY- A BOUNDARY ELEMENT APPROACH

Summary. Application of the boundary element method to numerical analysis of cracks with deterministic and stochastic geometry is presented. The dependence of selected mechanical characteristics with respect to shape and position of cracks is examined by means of the path-independent integrals. Numerical results for the deterministic and stochastic shape sensitivity of cracks are presented.

DETERMINISTYCZNE I STOCHASTYCZNE MODELE GEOMETRII PĘKNIĘCIA - UJĘCIE BRZEGOWO-ELEMENTOWE

Streszczenie. W pracy przedstawiono zastosowanie metody elementów brzegowych w analizie numerycznej pęknieć, których geometria opisana jest w sposób deterministyczny i stochastyczny. Zależność wybranych charakterystyk mechanicznych względem kształtu i położenia szczeliny badano za pomocą całek niezależnych od drogi całkowania. Wyniki obliczeń przedstawione zostały dla deterministycznej i stochastycznej analizy wrażliwości kształtu szczeliny.

DETERMINISTISCHE UND STOCHASTISCHE MODELLE DER SPALTGEOMETRIE -EINE RANDELEMENTENAUFFASSUNG

Zusammenfassung. Im Aufsatz hat man die Anwendung der Randelementenmethode zur numerischen Analysis der Spalten, deren Geometrie auf deterministischer und stochastischer Weise beschrieben ist, dargestellt. Die Abhängigkeit der gewelten mechanischen Charakteristiken bezogen auf die Form und Lage der Spalte wurde mittelst der Integrale, die von der Integrierungsweise unabhängig sind, untersucht. Die Ergebnise der zur deterministischen und stochastischen Analysis der Formempfindlichkeit der Spalte durch geführten Berechnungen wurden dargestellt.

1. INTRODUCTION

During the last years the boundary element method (BEM) turned out to be an appropriate numerical technique for solving fracture mechanics problems. This method is the most suitable for crack problems, since it describes the behaviour of a body in terms of displacements and tractions on the boundaries and is capable of accuracy modelling the high stress gradients near crack (e.g. [1.], [.7]).

There are a few numerical approaches based on BEM for crack analysis problems. The short description of these approaches is presented in Section 2. The application of BEM to examining stresses, strains, displacements and in the general case an arbitrary functional with respect to deterministic shape of a crack is presented in Sections 3. Crack geometry is often uncertain and vary unpredictably during the service life of the structure. Therefore special attention to the uncertainty in the crack geometry is given in Section 4. Numerical examples are described in Section 5.

2. CRACKED BODY MODELLING BY BOUNDARY ELEMENTS

Two problems of the crack modelling by boundary elements need special consideration. One problem concerns the geometrical modelling of the crack itself. The other problem refers to the numerical modelling of the elastic fields in the vicinity of the crack tip. Cracks are usually modelled as linear cuts (for 2-D problems) or planar cuts (for 3-D problems) with zero thickness. Such theoretical formulation is not possible in numerical analysis by means of boundary elements because it would lead to a singular matrix. The second problem involves the difficulties of adequately modelling singularities in the stress field that occur at crack tip.

Let an elastic body occupy a domain Ω bounded by an external boundary $\partial\Omega$ on which fields of displacements $\mathbf{u}^0(\mathbf{x}), \mathbf{x} \in \Gamma_u$, and tractions $\mathbf{p}^0(\mathbf{x}), \mathbf{x} \in \Gamma_p$, are prescribed. The body contains an internal crack which is a boundary Γ_o. For the a body which is not subjected to body forces, the displacement of a point $\mathbf{x}$ can be represented by the following boundary integral equation (cf. [4]):

$$\mathbf{c}(\mathbf{x})\mathbf{u}(\mathbf{x}) = \int_\Gamma \mathbf{U}(\mathbf{x},\mathbf{y})\mathbf{p}(\mathbf{y})d\Gamma(\mathbf{y}) - \int_\Gamma \mathbf{P}(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y})d\Gamma(\mathbf{y}) \tag{1}$$

where $\Gamma = \Gamma_u \cup \Gamma_p \cup \Gamma_o$, $\mathbf{U}$ and $\mathbf{P}$ are fundamental solutions of elastostatics; $\mathbf{u}$ and $\mathbf{p}$ are displacements and tractions, respectively, at the boundary; $\mathbf{c}(\mathbf{x})$ is a constant which depends on the position of the collocation point $\mathbf{x}$; $\mathbf{y}$ is the boundary point.

The discrete version of (1) is obtained by discretizing the boundary by series of boundary elements and approximating boundary displacements and tractions by means of nodal values and interpolating functions, and finally a system of algebraic equations takes the form:

$$\mathbf{AX} = \mathbf{BY} \tag{2}$$

where **A** and **B** are coefficient matrices derived from integrals of **U** and **P** ; **X** contains unknown nodal displacements and tractions; **Y** contains prescribed boundary conditions.
If the crack is modelled as a cut and the discretization on the two surface is the same, then the application of equation (1) leads to pairs of identical columns in **A** matrix and **A** is singular.

In order to overcome these problems Snyder and Cruse [12] proposed a very elegant method based on the use of specific Green's functions which include the exact solution for a traction-free crack. Unfortunately this approach is restricted to two-dimensional problems.

A second possibility - the multiregion modelling approach proposed by Blandford *et al.* [2] is the decoupling of both crack surfaces by the help of the subregions technique, requiring higher discretization expense and bigger systems of linear equations.

There is a third method , which is refereed to as the displacement discontinuity method, was given by Crouch and Starfield [6]. In this formulation the crack is directly treated as a single surface across which the displacements are discontinuous. An integral equation written in terms of the applied crack surface tractions can be formulated and solved numerically.

Recently a fourth technique, so-called the dual boundary integral equation method, which describes two different boundary equations was suggested by Gray *et al.* [9], Portela *et al.* [10]. The two boundary equations applied are equation (1) and an additional hypersingular tractions integral equation

$$\tfrac{1}{2}\mathbf{p}(\mathbf{x}) = \mathbf{n}^{T}\left[\int_{\Gamma}\mathbf{D}(\mathbf{x},\mathbf{y})\mathbf{p}(\mathbf{y})d\Gamma(\mathbf{y}) - \int_{\Gamma}\mathbf{S}(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y})d\Gamma(\mathbf{y})\right] \tag{3}$$

where **n** is the outward normal at the collocation point **x**; and **D** and **S** are other fundamental solutions of elastostatics which contain higher-order singularities then in equation (1). The are available special techniques for evaluating such hypersingular integrals. The distinct set of the boundary integral equations is obtained by applying the displacement integral equation (1) on one side of the crack surface and hypersingular traction boundary integral equation (3) on the opposite side.

The stress fields at a crack tip are singular and the accurate modelling of these singularities requires modifications to standard boundary element method. The use of a quarter-point boundary element on the crack face at the tip is common. It models the $r^{1/2}$ dependence of the crack face displacements. Similarly a traction singular quarter-point boundary element ahead of the crack tip models the $r^{-1/2}$ dependence of the traction. Similar modelling can be achieved by the use of special shape functions. These techniques are fully described in [1], [7], together with the procedures used for the derivation of the stress intensity factors from the displacements or tractions near the crack tip.
Other important means of obtaining stress intensity factors is to calculate the path-independent J-integral, introduced by Rice [11] which contains work-like terms integrated round an arbitrary contour which encloses the crack tip.

All described above boundary element approaches enable to calculate stress intensity factors at crack tips for mixed mode conditions for given geometry of the crack. For each shape variation of a crack one should solve a new boundary value problem. Determination of the effect of shape change of a crack geometry is the problem of shape sensitivity of cracks [5].

3. SHAPE SENSITIVITY ANALYSIS OF CRACKS

It is obvious that state field as stresses, strains or displacements and in the general case an arbitrary functional F depend on the location and shape of the crack. The problem of the dependence of these quantities with respect to crack geometry can be solved in terms of shape sensitivity analysis. In fracture mechanics, shape sensitivity analysis can play an important role because a crack growth process can be treated as shape variation of the crack.

Consider the problem of evaluating first order sensitivity of an arbitrary functional of the form

$$J = \int_{\Omega(\mathbf{a})} \Psi(\sigma,\varepsilon,\mathbf{u})d\Omega + \int_{\Gamma(\mathbf{a})} \phi(\mathbf{u},\mathbf{p})d\Gamma \tag{4}$$

where $\mathbf{a}$ is a vector which contains geometrical shape parameters of the crack.
The shape variation of the crack is introduced by a special kind of shape transformation in the form of:

- translation (T_{x_k}), by prescribing variations δb_k, $k=1,2,3$, where b_k are translation parameters (Fig.1);

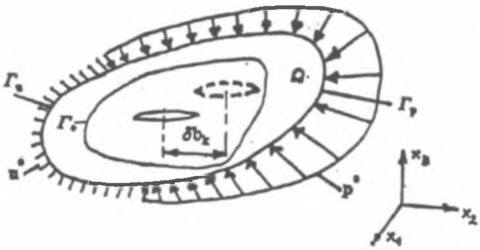

Fig.1. Translation of crack
Rys.1. Translacja pęknięcia

- rotation (R_{x_p}), by prescribing $\delta\omega_p$, $p=1,2,3$, where ω_p are rotation parameters (Fig.2);

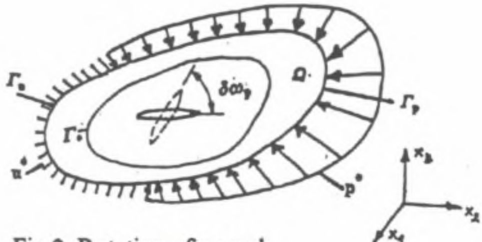

Fig.2. Rotation of a crack
Rys.2. Obrót pęknięcia

- scale change (expansion or contraction) (E) of the crack by prescribing $\delta\eta$, where η is a scale change parameter (Fig.3);

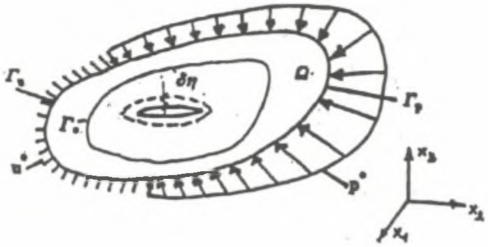

Fig.3. Expansion of crack
Rys.3. Ekspansja pęknięcia

The first variation of F is expressed by:

$$\delta F = \mathbf{S}^T \delta \mathbf{a}, \tag{5}$$

where

$$\delta \mathbf{a} = \mathrm{col}\{\delta a_q\}, \quad q = 1,2,\ldots,Q, \tag{6}$$

is the vector of variation of shape parameters, and

$$\mathbf{S} = \mathrm{col}\left\{\frac{DF}{Da_q}\right\}, \quad q = 1,2,\ldots,Q, \tag{7}$$

is the sensitivity vector.
For 3-D problems there are Q=7 shape parameters:

$$\mathbf{a} = \mathrm{col}\{a_q\} \equiv [b_1, b_2, b_3, \omega_1, \omega_2, \omega_3, \eta]^T, \tag{8}$$

and for 2-D problems there are Q=4 shape parameters:

$$\mathbf{a} = \mathrm{col}\{a_q\} \equiv [b_1, b_2, \omega_3, \eta]^T. \tag{9}$$

Elements of the sensitivity vector $S_q \equiv DF/Da_q$ are expressed by path-independent integrals along an arbitrary closed surface (for 3-D) or contour (for 2-D problems) $\Gamma_\bullet$ enclosing the crack (cf. [8], [5]).

$$\frac{DF}{Da_q} = \int_{\Gamma_\bullet} Z_L^q(\sigma, \varepsilon, \mathbf{u}, \sigma^a, \varepsilon^a, \mathbf{u}^a)\, d\Gamma_\bullet, \; L=T,R,E; \quad q=1,2,\ldots,Q \tag{10}$$

Integrands Z_L^q depend on state fields of primary and adjoint solutions:
- for translation (k=1,2,3):

$$Z_T^k = \left(\Psi\delta_{kj} + \sigma_{ij} u_{i,k}^a + \sigma_{ij}^a u_{i,k} - \sigma_{il} \varepsilon_{il}^a \delta_{kj}\right) n_j, \tag{11}$$

- for rotation (p=1,2,3):

$$Z_R^{p+3} = e_{kpl}\left(\Psi x_l \delta_{kj} - \sigma_{iq}\varepsilon_{iq}^{a} x_l \delta_{kj} + \sigma_{ij} u_{i,k}^{a} x_l + \sigma_{lj} u_k^{a} + \sigma_{lj}^{a} u_k + \sigma_{ij}^{a} u_{i,k} x_l\right) n_j,) \tag{12}$$

- for expansion or contraction:

$$Z_S^7 = \left[\frac{2}{\alpha}\sigma_{ij}^{a} u_i - \left(\frac{2}{\alpha} - \beta\right)\sigma_{ij} u_i^{a} + x_k \sigma_{ij}^{a} u_{i,k} - x_k \sigma_{ij} u_{i,j}^{a} \delta_{jk} + x_k \sigma_{ij} u_{i,k}^{a}\right] n_j, \tag{13}$$

where α is the order of $\phi=\phi(\mathbf{u})$, $\beta=1$ for 3-D and $\beta=0$ for 2-D and $\Upsilon=0$ in the case of equation (13).

It is seen from equations (11) to (13) that in order to calculate sensitivity information one should solve primary and adjoint problems. State fields in the form of stresses σ^a, strains ε^a and displacements $\mathbf{u}^a$ are obtained for an adjoint elastic body with prescribed boundary conditions in the form of

$$\mathbf{u}^{a0} = -\frac{\partial\phi(\mathbf{u},\mathbf{p})}{\partial \mathbf{p}} \text{ on } \Gamma_u, \qquad \mathbf{p}^{a0} = \frac{\partial\phi(\mathbf{u},\mathbf{p})}{\partial \mathbf{u}} \text{ on } \Gamma_p, \tag{14}$$

and with initial strain ε^{ai}, and stress σ^{ai} fields and body forces $\mathbf{b}^a$ within the domain Ω

$$\varepsilon^{ai} = \frac{\partial\Psi(\sigma,\varepsilon,\mathbf{u})}{\partial\sigma}, \quad \sigma^{ai} = \frac{\partial\Psi(\sigma,\varepsilon,\mathbf{u})}{\partial\varepsilon}, \quad \mathbf{b}^{ai} = \frac{\partial\Psi(\sigma,\varepsilon,\mathbf{u})}{\partial\mathbf{u}} \; . \tag{15}$$

The boundary integral equations for the adjoint body have the same form as for the primary cracked body.

If functional F expresses potential energy then first-order sensitivity of F with respect to translation of a crack determines the J-integral (cf.[8]).

4. STOCHASTIC GEOMETRY OF CRACKS

Boundaries of real cracks can be very complicated as far as their geometrical shape is concerned. Usually they are uneven and the irregularities do not easily lead to a unique deterministic description. Therefore boundaries of such cracks can be defined stochastically.

Stochastic geometry of the crack can be specified by random shape parameters $\mathbf{a}(\gamma)$ which can be expressed as follows:

$$\mathbf{a}(\gamma) = \mathbf{a}_0 + \delta\mathbf{a}(\gamma), \quad \mathbf{E}\delta\mathbf{a}(\gamma) = 0, \tag{16}$$

where variation $\delta\mathbf{a}(\gamma)$ represents fluctuation of random parameters; γ is an element of a sample space (cf. [13]); $\mathbf{E}$ indicates expectation; $\mathbf{a}_0 = \mathbf{E}\mathbf{a}(\gamma)$ is the mean value of shape parameters $\mathbf{a}(\gamma)$. If $\mathbf{a}(\gamma)$ has the Gaussian distribution then it is completely described by the mean value $\mathbf{a}_0$ and the covariance matrix $[\mathbf{K}]$ which is given as follows:

$$[\mathbf{K}] = \mathbf{E}\left[\delta\mathbf{a}\delta\mathbf{a}^T\right] \tag{17}$$

Because the crack boundary undergoes small random variations then resulting stochastic fields of stresses $\sigma(\mathbf{x},\gamma)=[\sigma_{qr}(\mathbf{x},\gamma)]$, strains $\varepsilon(\mathbf{x},\gamma)=[\varepsilon_{qr}(\mathbf{x},\gamma)]$ and displacements $u(\mathbf{x},\gamma)=[u_q(\mathbf{x},\gamma)]$ be expressed as follows:

$$\sigma(\mathbf{x},\gamma)=\sigma_0(x)+\delta\sigma(\mathbf{x},\gamma),\quad \mathbf{E}\delta\sigma(\mathbf{x},\gamma)=0, \tag{18}$$
$$\varepsilon(\mathbf{x},\gamma)=\varepsilon_0(x)+\delta\varepsilon(\mathbf{x},\gamma),\quad \mathbf{E}\delta\varepsilon(\mathbf{x},\gamma)=0, \tag{19}$$
$$\mathbf{u}(\mathbf{x},\gamma)=\mathbf{u}_0(x)+\delta\mathbf{u}(\mathbf{x},\gamma),\quad \mathbf{E}\delta\mathbf{u}(\mathbf{x},\gamma)=0, \tag{20}$$

where $\mathbf{q}_0=\mathbf{E}\mathbf{q}(\mathbf{x},\gamma)$, $\mathbf{q}=\sigma,\varepsilon,\mathbf{u}$, are identified with the mean value of state fields calculated for the untransformed shape of crack boundary Γ with the deterministic base shape parameters $\mathbf{a}_0=[a_{0q}]$, $q=1,2,\ldots,Q$.
Due to stochastic shape variation the functional F can be expressed as follows:

$$F(\mathbf{a}_0+\delta\mathbf{a}(\gamma))=F(\mathbf{a}_0)+\delta F(\gamma), \tag{21}$$

where the first variation of the functional $\delta F(\gamma)$ can be expressed analytically utilizing the adjoint approach:

$$\delta F(\gamma)=\mathbf{S}^T\delta\mathbf{a}(\gamma) \tag{22}$$

where elements of sensitivity matrix are expressed by (10).
Covariances of the stochastic functional F and displacements and stresses for the two fixed points $\mathbf{x}_1$ and $\mathbf{x}_2$ are calculated from:

$$\mathbf{K}_F=\mathbf{S}^T[\mathbf{K}]\mathbf{S}, \tag{23}$$

$$K^u_{qs}(\mathbf{x}_1,\mathbf{x}_2)=\mathbf{E}\left[\delta u_q(\mathbf{x}_1,\gamma)\delta u_s(\mathbf{x}_2,\gamma)\right]=\left\{\mathbf{S}^u_q(\mathbf{x}_1)\right\}^T[\mathbf{K}]\left\{\mathbf{S}^u_s(\mathbf{x}_2)\right\}. \tag{24}$$

and

$$K^\sigma_{qrsh}(\mathbf{x}_1,\mathbf{x}_2)=\mathbf{E}\left[\delta\sigma_{qr}(\mathbf{x}_1,\gamma)\delta\sigma_{sh}(\mathbf{x}_2,\gamma)\right]=\left\{\mathbf{S}^\sigma_{qr}(\mathbf{x}_1)\right\}^T[\mathbf{K}]\left\{\mathbf{S}^\sigma_{sh}(\mathbf{x}_2)\right\}. \tag{25}$$

In order to calculate the sensitivity matrices $\{\mathbf{S}^u\}$ and $\{\mathbf{S}^\sigma\}$ one should introduce special form of integrands ϕ and Ψ.

5. NUMERICAL EXAMPLES

The path-independent integrals for sensitivity of boundary displacements were tested numerically for several problems of plane elasticity with internal and edge cracks using the boundary element method [5]. The vector boundary integral equation for primary and adjoint problems has the form:

$$\mathbf{c}(\mathbf{x})\mathbf{u}^w(\mathbf{x})=\int_\Gamma\left[\mathbf{U}(\mathbf{x},y;e)\mathbf{p}^w(y)-\mathbf{P}(\mathbf{x},y;e)\mathbf{u}^w(y)\right]d\Gamma(y) \tag{26}$$

w=primary problem
w=adjoint problem

where $U(x,y;e)$ and $P(x,y;e)$ are fundamental solutions which include the exact solution for a traction-free crack [12]. The surface of crack $L(\bar{x}_1) = 2e$ is flat and extends from $\bar{x}_1 = -e$ to $\bar{x}_1 = e$, where $\bar{x}_1$ is a local axis (Fig.4)

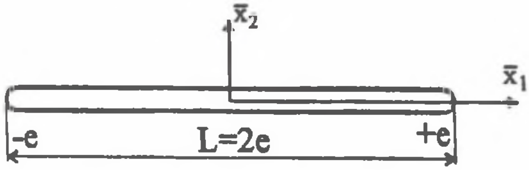

Fig.4. Crack geometry
Rys.4. Geometria pęknięcia

The discrete version of equation (22) is obtained by approximating the external boundary Γ by series of boundary linear elements. Finally, a system of algebraic linear equations is obtained:

$$AX^{w} = BY^{w}, \tag{27}$$

w=primary problem
w=adjoint problem

where all unknown boundary variables are written in the vector X^{w} and known from boundary conditions in Y^{w}. Square matrices A and B depending on boundary integrals of fundamental solutions are the same for primary as well as adjoint problems and therefore are calculated only once.
Having boundary variables X^{w} the displacement and stress fields are calculated along a path-independent contour Γ_{*}.

5.1. Deterministic crack geometry

A square plate (2x2) under tension with an internal crack of the length L=1 was considered (Fig. 5).

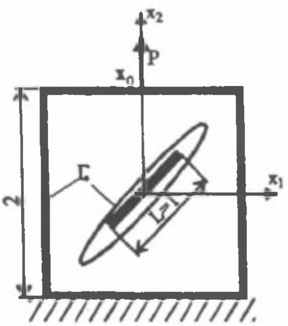

Fig.5. The square plate with internal crack
Rys.5. Tarcza kwadratowa z wewnętrznym pęknięciem

The boundary element model had 52 linear elements. Derivatives of a vertical displacement u_2, at the point x_0, where a load p was applied, with respect to translations T_{x_1} and T_{x_2}, rotation R_{x_3} and expansion E of the crack were calculated for many different internal contours such as an ellipse, a circle, a square and a rectangular (cf. [5]). In the Table 1 numerical results for an ellipse and a square are presented.

Table 1

No	Path description	$\frac{Du_2}{Db_1}\times 10^{-1}$ T_{x_1}	$\frac{Du_2}{Db_2}$ T_{x_2}	$\frac{Du_2}{D\omega_3}\times 10^{-1}$ R_{x_3}	$\frac{Du_2}{D\eta}$ E
1	ellipse (0.60;0.10) rot 45°	0.407756	0.102535	-0.854890	0.138631
2	square 1.98×1.98	0.408336	0.101790	-0.854059	0.137807

Obtained numerical calculations show a very good agreement for all these types of contours.

5.2. Stochastic crack geometry

A stochastic counterpart of previous example was also considered. The problem of an uncertain geometry of a crack of the mean length L=1 in a 2-D square plate (2×2) under tension is examined.

Four random shape parameters were chosen as design variables

$$\{\delta \mathbf{a}(\gamma)\} = [\delta a_1(\gamma), \delta a_2(\gamma), \delta a_3(\gamma), \delta a_4(\gamma)]^T = [\delta b_1(\gamma), \delta b_2(\gamma), \delta\omega_3(\gamma), \delta\eta(\gamma)]^T$$

which described translations T_{x_1} and T_{x_2}, rotation R_{x_3} and expansion E, respectively.

Prescribed variances of random shape parameters and calculated coefficients of displacement sensitivity analysis S_r^u are given in Table 2.

Table 2.

r	$\delta a_r(\gamma)$	shape transformation	$Var(a_r)\times 10^{-4}$	$S_r^u \times 10^{-4}$
1	δb_1	T_{x_1}	1.0	0.0408
2	δb_2	T_{x_2}	1.0	0.1020
3	$\delta\omega_3$	R_{x_3}	2.89	-0.0854
4	$\delta\eta$	E	1.0	0.1380

The mean value and the variance of a vertical displacement u_2 at the point where a load p has been applied are $(u_2)_0 = E[u_2(\gamma)] = 36.965\cdot 10^{-5}$ and $Var(u_2) = 5.220\cdot 10^{-12}$, respectively.

ACKNOWLEDGEMENTS

This work has been carried out within the support of a KBN grant PB0778/S6/94/06.

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Recenzent: prof. dr hab. inż. W. Gutkowski

Wpłynęło do Redakcji w grudniu 1994 r.