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Valery PILIPCHUK
Higher Mathematics Department
State Technology & Chemistry University of Ukraine

ON CHOOSING A GENERATING MODEL FOR STRONGLY NON-LINEAR VIBRATING SYSTEMS COMPUTATION

Summary. The generating model for analyzing strongly non-linear vibrations of mechanical systems is discussed. The corresponding analytical technique, that has been proposed in previous papers, is illustrated on essentially non-linear elastic systems.

О ВЫБОРЕ ПОРОЖДАЮЩЕЙ МОДЕЛИ ДЛЯ РАСЧЕТА СИЛЬНО НЕЛИНЕЙНЫХ КОЛЕБАТЕЛЬНЫХ СИСТЕМ

Резюме. Обсуждается порождающая модель для анализа сильно нелинейных колебаний механических систем. Соответствующая аналитическая техника, предложенная в предыдущих работах, иллюстрируется на существенно нелинейных упругих системах.

O WYBORZE MODELU TWORZĄCEGO DO OBLICZEŃ SILNIE NIELINIOWYCH UKŁADÓW DRGAJĄCYCH

Streszczenie. Przedyskutowany jest model tworzacy do analizy silnie nieliniowych drgań układów mechanicznych. Związana z nim technika analityczna – zaproponowana w poprzednich pracach – jest zilustrowana na przykładzie nieliniowych układów sprężystych.

1. INTRODUCTION

There exist numerous quantitative techniques for computing nonlinear dynamic responses. The majority of these techniques are carried out under the assumption of weak nonlinearity. Assuming that the nonlinear system "neighbors" a linear one, a perturbation parameter is introduced to denote the small magnitudes of the nonlinear terms, and the nonlinear response is constructed "close" to a linear generating solution. Since the generating functions are harmonic, the weakly nonlinear responses are constructed using complete bases of trigonometric functions. An obvious disadvantage of such techniques is that they can not be applied to strongly nonlinear or non-linearizable oscillators. To circumvent this deficiency of weakly nonlinear techniques, an alternative class of strongly nonlinear ones was developed. These techniques relax the assumption of weak nonlinearity by utilizing nonlinear generating solutions, thereby assuming that the strongly nonlinear systems under consideration neighbor simplified, but otherwise, nonlinear systems. These strongly nonlinear techniques are highly specialized and can not be employed for the analysis of general classes of nonlinear problems. The main reason is that multi-dimensional nonlinear systems are generically non-integrable, and, hence, the nonlinear generating solutions are seldom available in closed-form [1].

From the above remarks it is concluded, that a strongly nonlinear analytical technique with wide range of applicability must employ nonlinear generating systems which,

(i) are sufficiently general so that can be used in a broad range of nonlinear applications

(ii) possess a sufficiently simple structure in order to enable the correction of efficient iterative perturbation schemes for computing the nonlinear response

(iii) possess an additional with reference to linear generating system properties.

It must be noted that, especially requirement (i) seems to contradict the well-known "individuality" of nonlinear systems, which generally prohibits the concievement of analytical methodologies which are applicable to general classes of strongly nonlinear systems.

The harmonic oscillator (the linear generating system) is probably the most fundamental model in vibrating analysis. The reason for the wide applicability of this simple mechanical model is that the generated trigonometric functions $\{sint, cost\}$ possesses a number of convenient mathematical properties associated with the group of motions in Euclidean space, such as, rotation-subgroup. In the same spirit, one could consider an additional pair of (non-smooth) functions, which have relative simple forms associated with translation- and reflection- subgroups in the group of Euclidean motions. These functions will be termed the saw-tooth sine $\tau(t)$ and right-angled cosine $e(t)$, respectively, and are defined as, $\tau(t) = (2/\pi)arcsin[sin(\pi t/2)]$ and $e(t) = \tau'(t)$, where prime denotes the generalized derivative. The mechanical model which generates these functions is the vibro-impact oscillator moving with constant velocity between two rigid barriers.

Interestingly enough, there is a remarkable relation between the harmonic oscillator and the vibroimpact one, since both can be viewed as limiting cases of the same nonlinear oscillator:

$$\ddot{x} + x^m = 0; \quad x \in R, \tag{1}$$

$$t = 0, \quad x = 0, \quad \dot{x} = 1, \tag{2}$$

where m is an arbitrary positive odd integer; over dot denotes differentiation with respect to time t.

The exact solution of the initial problem (1),(2) can be expressed in closed form using the special Ljapunov's functions [2] or cam-functions [3], but these expressions are too mathematically complicated to provide aforementioned requirement (ii). Considering the range $1 \leq m \leq \infty$, one obtains the following limiting cases for the solutions:

$$\{x, \dot{x}\} = \{sint, cost\}, \quad if \;\; m = 1 \tag{3}$$

and

$$\{x, \dot{x}\} \to \{\tau(t), e(t)\}, \quad if \;\; m \to \infty. \tag{4}$$

The last case needs an extended concept of the solution, at the same time this case can be interpreted by means of the first integral of motion

$$\frac{\dot{x}^2}{2} + \frac{x^{m+1}}{m+1} = \frac{1}{2},$$

which is satisfied by the functions (4) almost everywhere as m tends to infinity. From viewpoint of physics the given case corresponds to the classical particle in the square-well potential or aforementioned vibroimpact system with two rigid barriers. The localized singularities of (4) occur at time instants $\{t : \tau(t) = \pm 1\}$ (i.e., at the instances of contact of the vibroimpact oscillator with its rigid boundaries), and are the cause of convergence problems of conventional analyzes based on trigonometric expansions when they are applied to strongly nonlinear problems.

So it has been shown that the oscillator (1) gives two simple limiting systems generating two simple pairs of periodic functions.

It should be noted that the oscillator (1) with general and concrete powers m was considered by number of authors from various viewpoints [2,4-10].

As a second example demonstrating the physical significance of the pair of functions $\{\tau(t), e(t)\}$ consider the Duffing oscillator with negative non-linear stiffness

$$\ddot{x} + x - x^3 = 0.$$

Denote by $T = T(E)$ the period of oscillation, where E is parameter of the total energy. When the energy is in the interval, $0 < E < 1/4$, the system performs periodic oscillations with amplitude A in the neighborhood of the stable fixed point $(x, \dot{x}) = (0, 0)$. For this type of motions, the exact solution can be expressed in terms of Jacobian elliptic functions, and it can be proven to satisfy the following asymptotic relations:

$$T \to 2\pi, \quad \frac{x}{A} \to cos\frac{\pi}{2}(\bar{t} + \alpha), \quad if \;\; E \to +0;$$

$$T \to \infty, \quad x \to e(\bar{t} + \alpha), \quad if \;\; E \to \frac{1}{4} - 0, \tag{5}$$

where $\bar{t} = 4t/T$ is nondimensioned time, and $\alpha = const$ is an arbitrary phase. Solution (5) is written in terms of right-angled cosine, and corresponds to motion of the system on a heteroclinic orbit in phase space. In scale of nondimensioned (own) time the system performs momentary "jumps" between the two unstable equilibrium points $(x, \dot{x}) = (\pm 1, 0)$. Increasing the energy above the critical value $E = 1/4$, leads to strongly nonlinear, non-periodic motions outside the heteroclinic loop. For values of the energy in the range $0 < 1 - 4E << 1$, standard perturbation methods based on trigonometric expansions encounter convergence problems and do not lead to accurate results. It will be shown that, by using as generating solutions the non-smooth functions $\{\tau(t), e(t)\}$, one can analytically study such essentially nonlinear solutions without encountering convergence problems. The corresponding technique is based on the saw-tooth time transformation (STTT) [11-14].

Note that the method of non-smooth transformations *for coordinates* (not for time) of impact systems has been developed in number of works [15,16]. It should be emphasized that the corresponding technique is technique of non-smooth *spatial* transformations of variables, that can be directly applied to *impact* or *vibro – impact* systems only. The STTT-technique contains a saw-tooth (non-smooth) *time* transformation, and the corresponding procedures are applied to systems with analytical restoring force characteristics.

2. THE SAW-TOOTH TIME TRANSFORMATIONS TECHNIQUE

The STTT-technique is based on the following propositions [12,14]:

1. **The general periodic function** $x = x(t)$ **with period** $T = 4$ **can be expressed as:**

$$x = X(\tau) + Y(\tau)e, \tag{6}$$

where $X(\tau) = [x(\tau) + x(2-\tau)]/2, \quad Y(\tau) = [x(\tau) - x(2-\tau)]/2;$

$$e^2 = 1, \quad e = \tau'(t). \tag{7}$$

2. **The elements (6) are elements of hyperbolic numbers algebra on account of (7),** and, thus, for any function $f(x)$ we have

$$f(X + Ye) = R_f + I_f e, \tag{8}$$

where $R_f = [f(X+Y) + f(X-Y)]/2, \quad I_f = [f(X+Y) - f(X-Y)]/2.$

3. **The result of differentiation remains in the algebra:** $\dot{x} = Y' + X'e$ under the conditions $Y|_{\tau=\pm1} = 0$. For continuous functions the conditions are satisfied automatically.

4. **The result of integration remains in the algebra,** that is the equality $\int (X + Ye)dt = Q + Pe$ takes place under the following condition $\int_{-1}^{1} X(\tau)d\tau = 0$, where $Q = \int_{0}^{\tau} Y d\tau + C$; $P = \int_{-1}^{\tau} X d\tau$; C is an arbitrary constant.

5. Let the dynamical system is described with the set of second-order equations, which is written as:

$$\ddot{x} + f(x, \dot{x}, t) = 0, \quad x \in R^n, \tag{9}$$

where vector-function f is assumed to be sufficiently smooth, and to either depend periodically on time t with period equal to $T = 4a$, or to have no time dependence.

Taking into account the propositions 1-3, for the $X-$, and $Y-$components of periodic solution(s), one obtains the boundary value problem of the following form

$$X'' + a^2 R_f = 0, \quad X'|_{\tau=\pm1} = 0; \tag{10}$$

$$Y'' + a^2 I_f = 0, \quad Y\,|_{\tau=\pm1} = 0; \tag{11}$$

$$R_f = \frac{1}{2}\left[f\left(X+Y,\ \frac{Y'+X'}{a},\ a\tau\right) + f\left(X-Y,\ \frac{Y'-X'}{a},\ 2a-a\tau\right)\right],$$

$$I_f = \frac{1}{2}\left[f\left(X+Y,\ \frac{Y'+X'}{a},\ a\tau\right) - f\left(X-Y,\ \frac{Y'-X'}{a},\ 2a-a\tau\right)\right],$$

where prime denotes the differentiation with respect to new time variable $\tau = \tau(t/a)$.

Remark. Although the transformed equations are formally more complicated as compared with original equations, they possess certain significant advantages. Indeed, since the solutions qualitative properties "is included" in the system due to the "oscillating" variable τ, the solutions of the following simplified equations

$$X'' = 0, \quad Y'' = 0, \tag{12}$$

as generating solutions, can be employed for the perturbation method of successive approximations. This leads to simple perturbation solutions for strongly non-linear cases.

3. VIBROIMPACT APPROXIMATION OF A CONSERVATIVE SYSTEM

Let us consider the n-DOF unforced system

$$\ddot{x} + f(x) = 0, \quad x \in R^n, \tag{13}$$

where $f(x)$ is odd analytical vector-function: $f(-x) = -f(x)$. The one-parametric (except time translation) family of periodic solutions will be constructed. Searching even with respect to quarter of period solutions, one writes

$$x = X(\tau), \quad Y \equiv 0, \quad \tau = \tau(t/a). \tag{14}$$

Then, starting from (10),(11) one obtains the following simplified expressions:

$$X'' + a^2 f(X) = 0, \tag{15}$$

$$X'|_{\tau=1} = 0, \quad X(-\tau) = -X(\tau).$$

The solution of the nonlinear boundary value problem (15) can be found in series of successive approximations [17]:

$$X = X^0(\tau) + X^1(\tau) + X^2(\tau) + ...,$$

$$a^2 = h_0(1 + \gamma_1 + \gamma_2 + ...). \tag{16}$$

The generating solution is:

$$X^0 = A^0\tau, \quad R^n \ni A^0 = const.$$

This solution describes an $n-$DOF vibro-impact oscillator with two rigid barriers; in so doing the length of arbitrary vector A^0 is equal to the barrier spacing. Direction of the vector is defined by the following non-linear eigenvector problem relating the vector A^0:

$$\int_0^1 f(A^0\tau)d\tau = h_0^{-1}A^0, \quad h_0 = \frac{A^{0T}A^0}{A^{0T}\int_0^1 f(A^0\tau)d\tau}, \tag{17}$$

where $(\ )^T$ denotes the transpose of a vector.

The first successive approximation to the solution is

$$X^1 = -h_0\int_0^\tau(\tau-\xi)f(A^0\xi)d\xi, \quad \gamma_1 = -\frac{A^{0T}\int_0^1 f'_x(A^0\tau)X^1 d\tau}{A^{0T}\int_0^1 f(A^0\tau)d\tau}. \tag{18}$$

4. EXAMPLES OF STRONGLY NON-LINEAR SYSTEMS

Single - mass mechanical system. Consider the transverse free vibrations of a particle, having mass m, which is clamped to a foundation by means of two horizontal springs, having the length l and the stiffness c. The corresponding equation for the vertical displacement of the mass, w, can be written as

$$\frac{d^2\overline{w}}{d\overline{t}^2} + \overline{w} - \frac{\overline{w}}{(1+\overline{w}^2)^{1/2}} = 0; \quad \overline{t} = (2c/m)^{1/2}t/l; \quad \overline{w} = w/l. \tag{19}$$

For the approximate solution, $\overline{w} = X(\tau), \quad \tau = \tau(\overline{t}/a)$, afore presented procedure gives:

$$X_0 = A\tau; \quad h_0 = \left[1/2 - (\sqrt{1+A^2}-1)/A^2\right]^{-1};$$
$$X_1 = -\frac{h_0}{2A^2}\left[X_0^3/3 + 2X_0 - X_0\sqrt{1+X_0^2} - ln(X_0 + \sqrt{1+X_0^2})\right]; \tag{20}$$
$$\gamma_1 = -\frac{h_0}{A^2}f(A)X_1|_{X_0=A} - \frac{1}{2},$$

where the first approximation is expressed in terms of the zero-th one.

A peculiarity is that the system becomes linear for large amplitude but not for small one. In fact for $A \to \infty$, we obtain: $X_1/A \to -\tau^3/3, \quad h_0 \to 2, \quad \gamma_1 \to 1/6$. The limiting case corresponds to the harmonic oscillator.

For the case $A \to 0$ we have the asymptotic: $h_0 \sim 8/A^2, \quad X_1/A \to -\tau^5/5, \quad \gamma_1 \to 3/10$. The corresponding limiting oscillator is: $d^2\overline{w}/d\overline{t}^2 + \overline{w}^3/2 = 0$.

It should be noted that the expansion is appropriate both for large and for small amplitudes of vibrations.

Two-mass chain with non-linear elastic springs. Consider the two-mass chain that is shown in the Fig. 1. Let the elastic energy of springs is defined by the following relationships:

$$\Pi = F(x_1) + G(x_1 - x_2) + F(x_2);$$

$$F(s) \equiv s^2/2 + s^4/4 + s^6/6; \quad G(s) \equiv c_1 s^2/2 + c_3 s^4/4 + c_5 s^6/6,$$

where c_i $(i = 1, 3, 5)$ are positive parameters.

The vector-function of the restoring force is: $f(x) = (\partial\Pi/\partial x_1, \partial\Pi/\partial x_2)^T$. It can be shown that the non-linear eigen-vector problem (17) describes the qualitative dissimilar situations in the following cases:

a) $c_3 > 1/4, \quad c_5 > 1/8;$

b) $c_3 < 1/4, \quad c_5 > 1/8; \quad 3(4c_3 - 1)^2 - 32c_1(8c_5 - 1) > 0;$

c) $c_3 < 1/4, \quad c_5 < 1/8.$

Fig.2a,b,c exhibit the solutions at the following quantities: a) $c_1 = 1/10$; $c_3 = 1/2$; $c_5 = 1/4$; b) $c_1 = 1/10$; $c_3 = 1/8$; $c_5 = 1/7$; c) $c_1 = 1/10$; $c_3 = 1/8$; $c_5 = 1/15$.

The solutions corresponding to symmetric mode are not shown in the figures. It should be noted that numerical procedure used for the eigen-vector problem is unstable about the symmetric mode. Fig.2b,c presents such cases that the nonlinear localized modes [13,18,19] can be realized. In the case (b) the oscillators are weakly coupled in the fourth degrees of potential energy. In the case (c) the oscillators are weakly coupled else in the fifth degrees.

The infinite chain of nonlinear oscillators. Consider an infinite chain of nonlinear oscillators, coupled by means of linear springs. The equation of motion of this system are given as:

$$\ddot{u}_n - u_{n-1} + 2u_n - u_{n+1} + f(u_n) = 0, \quad n = 0, \pm 1, \pm 2, \dots . \tag{21}$$

Function $f(u_n)$ denotes the nonlinear restoring grounding force acting on the n-th oscillator. This function is assumed to be analytic and odd, and to possesses a single zero at the equilibrium position $u_n = 0$. Traveling and stationary wave solutions of (21) will be computed by imposing continuoum approximations, and reducing this infinite set of equations to a single approximate nonlinear partial differential equation. In the continuoum limit, the displacements u_n become continuous function of the spatial and temporal variables, $u_n \to u(t, n)$.

So for long-wave motions one has the following Klein-Gordon nonlinear partial differential equation [20]:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial n^2} + f(u) = 0, \tag{22}$$

The travelling wave solutions of (22) are now sought in the form:

$$u = u(\phi), \quad \phi = kn - \omega t.$$

Considering ϕ as a new independent variable, one has the following ordinary differential equation:

$$(\omega^2 - k^2)\frac{d^2u}{d\phi^2} + f(u) = 0. \tag{23}$$

Let $f(u) \equiv \gamma u^m$; $\gamma = const$; $m = 2n - 1$. Putting $a^2 = 1/(\omega^2 - k^2)$ and using the STTT-technique, one obtains the following dispersion relation for the traveling wave:

$$\omega^2 = k^2 + \gamma\frac{A^{m-1}(m+4)}{2(m+1)(m+2)},$$

where A is related to the maximum amplitude of the wave by the relation, $A = \frac{m+2}{m+1}U_{max}$.

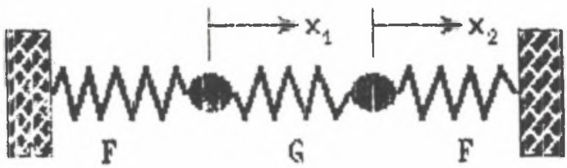

Fig.1. Two-mass chain with non-linear springs
Rys.1. Układ dwóch mas z nieliniowymi sprężynami

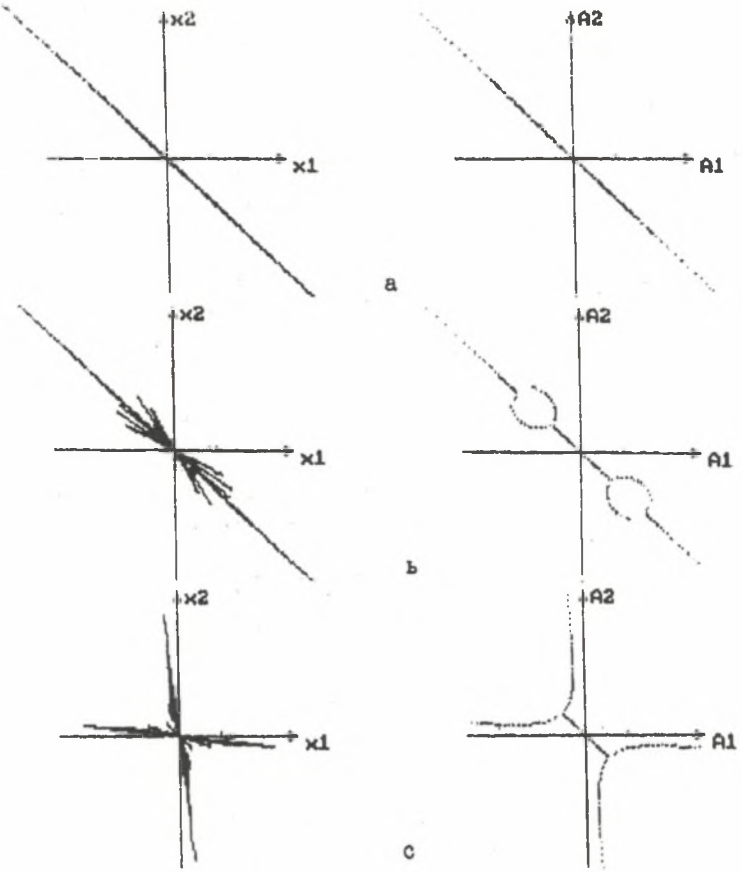

Fig.2. Numerical analysis of the non-linear eigen-vector problem (at the right of the figure) and corresponding trajectories in the configuration plane (at the left of the figure): a) strongly coupled oscillators; b) the coupling stiffness of the 3-rd degree is weak; c) coupling sniffinesses of the 3-rd and the 5-th degrees are weak

Rys.2. Analiza numeryczna nieliniowego zagadnienia własnego (po prawej) z odpowiadającymi trajektoriami na płaszczyźnie konfiguracyjnej (po lewej): a) silnie sprzężone oscylacje; b) sprzęźająca sztywność trzeciego stopnia jest słaba; c) sprzęźające sztywności trzeciego i piątego stopnia są słabe

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Recenzent: prof. dr hab. inż. E Świtoński

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