III INTERNATIONAL CONFERENCE ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 2003
TRANSPORT SYSTEMS TELEMATICS TST'03 TRANSPORT z.51, nr kol. 1608

traffic, optimal control, Bang-bang control

Youssef AHADAR [1]
Rachid BOUYEKHF
Abdellah EL MOUDNI

OPTIMAL CONTROL OF AN ISOLATED TWO-PHASE INTERSECTION

In this paper we consider an isolated two-phase intersection streets with controllable traffic lights on each corner. Isolated implies that we will develop signal control at the intersection without considering adjacent intersections. We construct a model that describes the evolution of the queue lengths in each lane as a function of cycle. We discuss how optimal control can minimize queues in the oversaturation conditions. An algorithm will be proposed to illustrate the results obtained.

OPTYMALNE STEROWANIE DWUFAZOWYMI IZOLOWANYMI SKRZYŻOWANIAMI

W artykule rozważa się wyizolowane dwufazowe skrzyżowania ulic wyposażone w sygnalizację świetlną. Wyizolowanie implikuje, iż będzie opracowane sterowanie sygnałami na skrzyżowaniu bez uwzględniania sąsiadujących skrzyżowań. Zbudowano model, który opisuje zmianę długości kolejki na każdym pasie jako funkcję cyklu. Rozważane jest optymalne sterowanie, które może minimalizować kolejkę w warunkach przesycenia. Zaproponowano algorytm do zilustrowania uzyskanych wyników.

1. INTRODUCTION

As the number of vehicles and the need for transportation grow, cities around the world face serious road traffic congestion problems. Costs include lost work and leisure time, increased fuel consumption, air pollution, health problems... In general there exist different methods to tackle the traffic congestion problem. The most effective measures in the battle against traffic congestion seem to be a selective construction of new roads and a better control of traffic through traffic management. Traffic light control can be used to augment the flow of traffic in urban environments by providing a smooth circulation of the traffic or to regulate the access to highways or main roads.

Traffic light controls have been the subject of several investigations. In particular, [1] has shown how an optimal traffic light switching scheme for an intersection of two streets can

[1] Laboratoire SeT-Université de Technologie Belfort-Montbéliard – UTBM 90010, Belfort cedex, France youssef.ahadar@edu.univ-fcomte.fr.

be determined. In general this leads to a minimization problem over the solution set of an extended linear complementarity problem (ELCP). However, it is well known that the ELCP is NP-hard problem. Hens, as claimed by [1] this approach is not feasible if the number of switching cycles is large. [2] constructed models for oversaturation control. But their models are all continuous types and do not address the problem of optimizing cycle length. The only work that merits our attention is [3]. In that paper the authors have developed a discrete dynamic optimization models in term of delay. The optimal cycle length and the optimal assigned green time are determined for the case of two-phase control.

In this paper, firstly, based on the work of [3]-[4]-[5] in which the authors calculate the delay during a period of oversaturated conditions in a crossroads, we adopt a model that describes the evolution of the queue lengths in discrete time according to the cycles. This model provides an improved level of control by using better modelling of traffic variables. Secondly, the problem of finding the set of admissible control to minimize the total length queue is considered in the entire oversaturated period. In order to apply the optimal control strategy with respect to real comportment of the system, an algorithm will be proposed

2. DEFINITION OF THE SYSTEM

We consider an isolated two-phase intersection as shown in Fig.1. There are two lanes and on each corner of the intersection there is a traffic light. Hence, there are two traffic flows of vehicles to be served in the intersection.

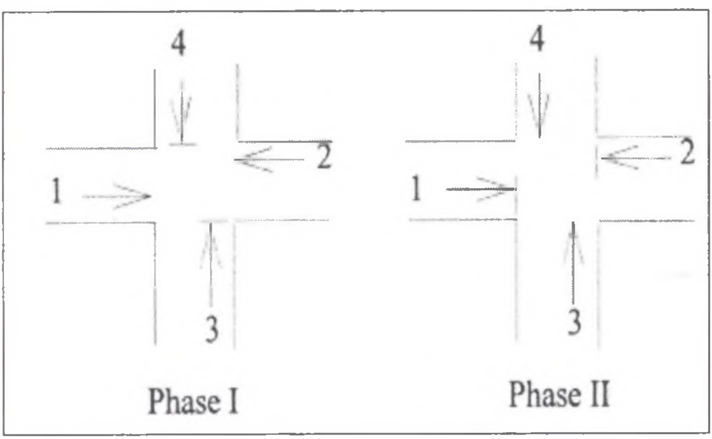

Fig.1. Four-leg intersection with two-phase signal control

To be able to present the model, a certain number of definitions are necessary:

Effective green time

As a vehicle approaches an intersection displaying a red signal the driver decelerates and stops either at the stop line or at the end of a queue Fig.2. When the signal turns green the driver accelerates until the vehicle reaches its desired or maximum possible speed. It is usually [6], [4] assumed that after startup lost time the saturation flow rate remains constant until the beginning of the yellow change interval. The effective green time is:

$$g_e = g + y - W = g + y - (W_1 + W_2) \tag{1}$$

where the lost time W is the sum of startup lost time W_1 and clearance lost time W_2.

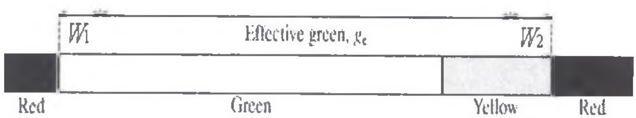

Fig.2. Effective green time

Rate of saturation

Rate of saturation s is defined as being the maximum number of vehicles being able to use the corridor without interruption during the effective time of the green light g_e.

Oversaturated conditions

If arrival rate exceeds capacity, the intersection is oversaturated. A phase is saturated when a vehicle at least is constrained to await more than one cycle to cross the crossroads. The crossroads is saturated when at least one of its phases is saturated. Formally, the number of arrivals $A(k)$ during cycle $k \in \mathrm{N}$ is defined by the equation:

$$A(k) = A[kc] - A[(k-1)c] \tag{2}$$

$A[kc]$ is defined as the number of the arrivals at the end of the cycle. The departure rate is s during the effective time of the green light g_e, so that:

$$D(t + c) - D(t) = s \cdot g_e \tag{3}$$

where $D(t)$ is the number of departures at the instant t. Fig.3 displays a case where demand flow rate instantaneously increases above the capacity at the beginning of a cycle. The capacity curve $C(t)$ is not the saw-toothed departure curve $D(t)$, so that the area between $A(t)$ and $C(t)$ curves is the overflow delay [7].

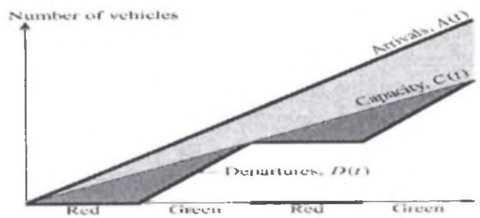

Fig.3. Model for overflow delay

Let $D_k(c)$ the number of the departures during the cycle k, then the condition of oversaturation is defined by:

$$A(k) > D_k(c) \tag{4}$$

3. MODELING

Continuous type models are limited in that the switch-over point does not necessarily occur at the end of a cycle, neither does the termination of the oversaturated period occur only at the end of the final cycle. On the other hand, the switch-over points determined by a discrete model occur exactly at the termination of a cycle. Discrete operation provides a smooth, regular, and ordered transfer of control. Calculating queue is more reliable. Fig.4 illustrates the situation of queue during oversaturation and shows the duality between the two phases. To keep this duality, it is necessary that the queue represented when the effective green time terminates at a certain cycle state k.

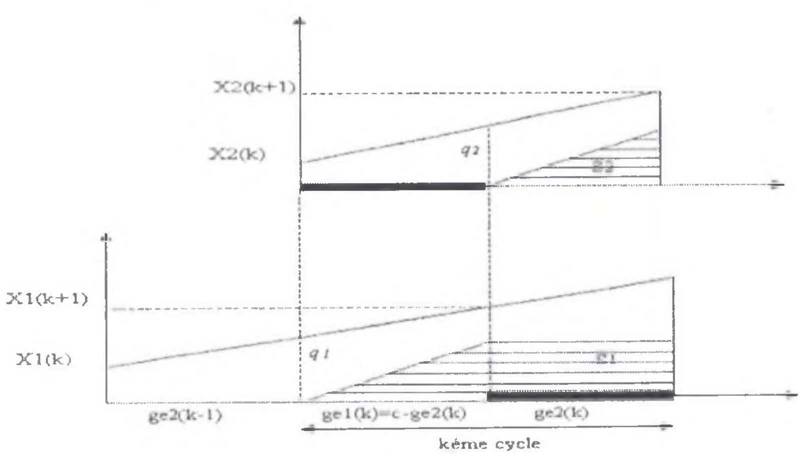

Fig.4. Queue of a four-leg intersection with two-phase control

Let $X_2(k)$ be a queue length of approach 2 when the effective green time $g_{e2}(k)$ terminates at a certain cycle state k. According to Fig 4, the relation of the queue lengths between state k and $k+1$ can be represented by the following equations:

$$X_2(k+1) = X_2(k) + A_2(k) - D_2(k) \quad (5)$$

where $A_2(k) = q_2 \cdot c$, represents the number of arrivals during cycle k and $D_2(k) = S_2 \cdot g_{e2}(k)$ the number of departures during cycle k. The equation (5) becomes:

$$X_2(k+1) = X_2(k) + Q_2 \cdot c - S_2 \cdot g_{e2}(k) \quad (6)$$

Similarly, let $X_1(k)$ be a queue length of approach 1 when the effective green time $g_{e1} = c - g_{e2}(k)$ terminates at a certain cycle state k. Then

$$X_1(k+1) = X_1(k) + A_1(k) - D_1(k) \quad (7)$$

where :

- $A_1(k) = q_1 \cdot g_{e2}(k-1) + q_1 \cdot (c - g_{e2}(k))$: represents the number of the arrivals at the end of $c - g_{e2}(k)$.
- $D_1(k) = S_1 \cdot (c - g_{e2}(k))$: represents the number of the departures at the end of $c - g_{e2}(k)$.

The equation (7) becomes:

$$X_1(k+1) = X_1(k) + q_1 \cdot g_{e2}(k-1) + q_1 \cdot (c - g_{e2}(k)) - S_1 \cdot (c - g_{e2}(k)) \quad (8)$$

Let $X(k)$ be the vector defined as $X(k) = (X_1(k) \;\; X_2(k))^T$, and $U(k)$ be the vector of control of the system defines by $U(k) = (g_{e2}(k) \;\; g_{e2}(k-1))^T$. Then we can gather the two equations (6) and (8) in the following matrix form:

$$X(k+1) = A \cdot X(k) + B \cdot U(k) + C \quad (9)$$

Where

$$X(k) = \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix}, A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} s_1 - Q & Q \\ -s_2 & 0 \end{bmatrix}, U(k) = \begin{bmatrix} g_{e2}(k) \\ g_{e2}(k-1) \end{bmatrix}, C = \begin{bmatrix} (Q - s_1) \cdot c \\ Q_2 \cdot c \end{bmatrix}$$

Notice that equation (9) is the linear form with respect to the state variable and control. Hens, it is easily amenable to mathematical analysis. Therefore, we can efficiently compute optimal traffic light control, which is the mean concern in the following section.

4. OPTIMAL LIGHTS CONTROL

Consider the problem of finding a control $U(k)$ which minimising on of the following possible criterions (10) defined as:

- $J_1 = \sum_{k=0}^{N} X(k)$: the sum of the queues during oversaturated period.

- $J_2 = \sum_{k=0}^{N} 1/2X^2(k)$: quadratic form of the queues length.
- $J_3 = \max_i (\sum_{k=0}^{N} X_i(k))$: the queue length over the worst queue.

subject to the dynamic equation:

$$X(k+1) = A \cdot X(k) + B \cdot U(k) + C \quad (11)$$

and the inequality constraints of the form:

$$\Omega = \{(U(k), X(k)) \in \Re^2 \times \Re^2 / U_{min} \le U(k) \le U_{max},\ X(k) \ge 0\} \quad (12)$$

where N is the terminative cycle of the oversaturated period and Ω is a given set of constraints.

Minimizing (10) subjected to (9), based on the optimal control theory [8], involves equivalently to minimizing the Hamiltonian formula (13). The Hamiltonian formula is defined as:

$$H(k) = \phi(k) + \lambda^T(k+1) \cdot [X(k) + B \cdot U(k) + C] \quad (13)$$

where $\lambda(k)$ is the costate corresponding to $X(k)$. Now, the task is to find an admissible sequence $U(k) \in \Omega$ such that (13) is minimal subject to (12). According to the optimal control theory, if an extremum of H exists, it must satisfy the following conditions:

$$\lambda(k) = \frac{\partial H}{\partial X(k)} = \frac{\partial \phi}{\partial X(k)} + \lambda(k+1) \quad (14)$$

$$\frac{\partial H}{\partial \lambda(k+1)} = X(k+1) = X(k) + BU(k) + C \quad (15)$$

$$\frac{\partial H}{\partial U(k)} = 0 \quad (16)$$

where $\phi(k)$ is the function which appears in the criterions forms. Equation (13) clearly reveals that the relation between H and the control variable $U(k)$ is linear. This leads to:

$$\frac{\partial H}{\partial U(k)} = \lambda^T(k+1)B \ne 0 \quad (17)$$

Hens, equation (17) obviously indicates that only the single control variable $U(k)$ can minimize H. In signal control, the control variable is related to the effective green time which should be taken as the value between predetermined upper and lower limits. This proposition is clearly a bang-bang control with a successive two-stage operation U_{min} and U_{max}:

$$\begin{cases} U(k) = U_{min} & if\ \lambda^T(k+1)B > 0 \\ U(k) = U_{max} & if\ \lambda^T(k+1)B < 0 \end{cases} \quad (18)$$

At first glance, (18) indicates that the implementation of the control strategy is note problematic. However, the difficulty arises in the determination of the initial value of $\lambda(k)$ which obeys the recurrent equation (14). This difficulty can be avoided by taking a set of certain initials values of $\lambda(k)$ that respects the maximal admissible queue length X_{max} which is defined by the oversaturation conditions. With this in mind, the following algorithm is proposed in order to take this consideration.

Step 1: Let $k \in \angle$, initiate $\lambda(0)$.

Step 2: When $\lambda^T(k+1)B>0$ Do $U(k)=U_{min}$.

When $\lambda^T(k+1)B<0$ Do $U(k)=U_{max}$.

Step 3: Give $(X_{01}, X_{02})^T$, calculate $X(k)$.

Step 4: If $X_i(k)=0$ or $X_i(k)>X_{i\ max}$, back to Step1. And employing (14) to define a new value of $\lambda(k)$ to keep $X_i(k)>0$ and $X_i(k)<X_{i\ max}$.

Step 5: Calculate $X(k)$ until $k=N$.

End.

According to figure 4, step 4 indicates that the cumulative output curves do not intersect the cumulative input curves for any of the approaches. This fact implies that no queue becomes negative or zero before the end of the oversaturated period. If a queue becomes negative while the signal is green, the designed green time becomes invalid due to the waste of control time. In Fig.5 we have plotted the evolution of queue lengths according to the algorithm and the constraints. Clearly we observe that $X_1(k)$ decreases rapidly while $X_2(k)$ increases slowly until 90th cycle. Once $X_1(k)$ becomes zero the control is changed according to step 4. After 90th cycle $X_1(k)$ increases until $X_{1\max}=24$ which is the maximal value that we have permitted, of course $X_2(k)$ decreases in this case. All these observations indicate that the control strategy forces the queue lengths to change according to constraints and makes a flexibility to get out the oversaturation situation.

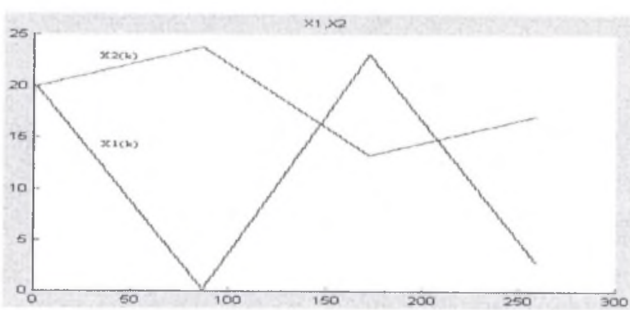

Fig.5. Evolution of the queues

5. CONCLUSION

When the traffic flow exceeds intersection capacity, this situation cause queuing of vehicles that cannot be eliminated in one signal cycle. To minimise this congestion, firstly, we explained a model that describes the evolution of the queue lengths at an intersection of two two-way streets with controllable traffic lights on each corner. Secondly, we have proceeded at the stage of the optimisation by defining the appropriate control strategy. This results in bang-bang like control, which is quite appropriate for oversaturation control. Finally, an algorithm has been proposed in order to take into account the constraints.

BIBLIOGRAPHY

[1] SCHUTTER, B.D., MOOR, B.D: Optimal traffic light for a single intersection.

[2] MICHALOPOULOS, P.G. and STEPHANOPOULOS, G., Oversaturated signal systems with queue length constraints - I: Single intersection. Transportation Research 11, 413-421. (1977a).

[3] Tang-Hsien Chang : Optimal signal timing for an oversaturated intersection, Department of Transportation Science, Tamkang University (1998).

[4] WEBSTER, F.V., 1958. Traffic signal settings. road research technical paper, No.39, Great Britain Road Research Laboratory, London.

[5] MAY, A.D., Jr., 1965. Traffic flow theory -the traffic engineer's challenge. In: Proceedings of the Institute Traffic Engineering. pp. 290±303.

[6] CLAYTON, A.J.H., 1941. Road traffic calculations. Journal of the Institution of Civil Engineers, vol. 16, no. 7, p. 247–284. With discussion and correspondence in no. 8, pp. 588–594.

[7] GAZIS, D.C., POTTS, R.B., 1965. The oversaturated intersection. In: Proceedings of the Second International Symposium on the Theory of Road Tra†c Flow. Organization for Economic Cooperation and Development, Paris, pp. 221±237.

[8] SAGE, Andrew P. and WHITE, Chelsea C.,III. Optimum System Control, Second edition (1977).

Reviewer: Prof. Romuald Szopa