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CENTRALAXONOMETRIC MAPPING IN COMPUTER-Graphics¹

Summary. This first part of this paper presents a syntetic method of genral central axonometric mapping in computer graphics. The input of the method is the projection of the origin, the unit-points and the infinite points of the Cartesian coordinate system on the image plane. In the second part I show a process to calculate the world-coordinates of the centre of the projection, when the central axonometric mapping is a central projection.

ZAGADNIEŃ AKSONOMETRII ŚRODKOWEJ W GRAFICE KOMPUTEROWEJ

Streszczenie. W pracy przedstawiono oparty na rozważaniach rzutowych sposób obliczenia współrzędnych punktu w aksonometrii środkowej, a następnie, w przypadku, kiedy taka aksonometria jest rzutem środkowym, wprowadzono wzory na obliczenie współrzędnych środka rzutów.

1. Introduction

The degenerated projective or central-axonometric mapping of the space to the plane can be given by the image of the origin, the unit-points and the infinite points of the axes of the Cartesian coordinate system O (E_x E_y E_z). We will denote these points by the symbol O^* (E_x^*, E_y^*, E_z^* , V_x^*, V_y^*, V_z^*).

In 1910 E. Krupa proved that the projective mapping of a figure is projective to the central-projection of the figure [1]. This affirmation can be refined, that is it can be proved that the projective mapping of a figure is affine to the central-projection of this figure, if the direction points V_x^*, V_y^*, V_z^*

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are not collinear. It is a well known fact that this predicate is not any more refinable, that is the projective mapping of a figure is not equivalent to the central-projection of the figure. First Krupa [2] gave a synthetic condition for when a central-axonometric mapping is a central projection. For the same affirmation J. Szabó gave several criteria which are easy to put into practice. [4], [5], [6].

In [3] J. Szabó gave an analytic method to calculate the user-coordinates of a point $P(x, y, z)$ of the space, if the system $O^*(E_x^*, E_y^*, E_z^*, V_x^*, V_y^*, V_z^*)$ is given.

In this paper first I will give a synthetic method to calculate the user-coordinates of a point P^* , which is the image of P .

Second, if the condition in [6] is satisfied, I show a process to calculate the world-coordinates of the centre of the projection.

In this way the description of a figure is more general, it is not necessary to use special mapping, e.g. perspective, special image planes, etc. If the user gives the system $O^*(E_x^*, E_y^*, E_z^*, V_x^*, V_y^*, V_z^*)$ on the image plane, he need not know the connections among the world and user coordinate-system and the centre of the projections.

The determination of the world coordinate of the centre is important in solving several problems, such that e.g. hidden line and hidden surfaces problems, contouring.

II Calculating the user-coordinates of a point $P(x, y, z)$

Let us suppose that the origin of the user-coordinate system is the point O^* , from the configuration $O^*(E_x^*, E_y^*, E_z^*, V_x^*, V_y^*, V_z^*)$ given by the user. Let us denote the user-coordinates of the other points of the system by:

$$E_x^*(t_1, t_2), E_y^*(t_3, t_4), E_z^*(t_5, t_6), V_x^*(v_1, v_2), V_y^*(v_3, v_4), V_z^*(v_5, v_6)$$

Let P_x, P_y, P_z be the vertices of the coordinate-cube of point P fitting to the axis. Let the central-axonometric images of these points be P_x^*, P_y^*, P_z^* . It is easy to calculate the user-coordinates of these points, using the invariance of the cross ratio. Let ξ_x, η_x the coordinates of P_x^* , then:

$$x = (P_x, E_x, O, V_x) = (P_x^*, E_x^*, O^*, V_x^*) - \frac{\xi_x}{t_1} : \frac{v_1 - \xi_x}{v_1 - t_1} - \frac{\eta_x}{t_2} : \frac{v_2 - \eta_x}{v_2 - t_2}, \text{ from which:}$$

$$\xi_x = \frac{t_1 x + v_1 x}{v_1 + x - t_1} \quad \text{and} \quad \eta_x = \frac{t_2 x + v_2 x}{v_2 + x - t_2}.$$

The user-coordinates of points P_y^* , P_z^* can be calculated in the same way:

$$\xi_y = \frac{t_3 y + v_3 y}{v_3 + y - t_3} \quad \eta_y = \frac{t_4 y + v_4 y}{v_4 + y - t_4}$$

$$\xi_z = \frac{t_5 z + v_5 z}{v_5 + z - t_5} \quad \eta_z = \frac{t_6 z + v_6 z}{v_6 + z - t_6}$$

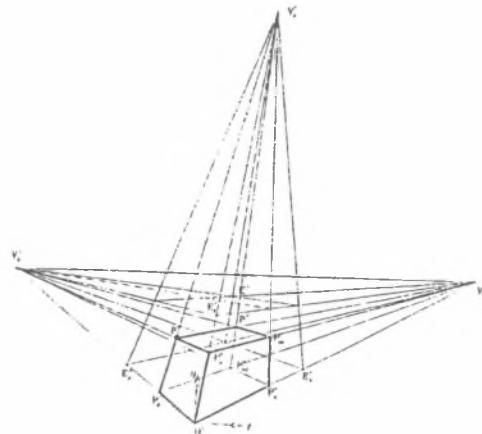


Fig. 1

Let P_{xy} , P_{yz} , P_{xz} be the vertices of the coordinate-cube of point P fitting to the coordinate-planes. Let the central-axonometric mapping of these points be P_y^* , P_z^* , P_x^* .

As Figure 1. shows, for example the point P_{xy}^* arises as the intersection of the lines P_y^* , V_x and P_x^* , V_y . Similary, the point P_{yz}^* is the intersection P_y^* , V_z and P_z^* , V_y . And finally, the user-coordinates of the point P^* can be calculated as the intersection of the lines P_{xy}^* , V_z and P_{yz}^* , V_x .

III Calculating the world-coordinates of the centre

The condition given in [6], using the notations of Figure 2., is the next:

$$\left(\frac{i}{j}\right)^2 : \left(\frac{k}{l}\right)^2 : \left(\frac{m}{n}\right)^2 = \operatorname{tg}\alpha : \operatorname{tg}\beta : \operatorname{tg}\gamma \quad [i]$$

A configuration $O^*(E_x^*, E_y^*, E_z^*, V_x^*, V_y^*, V_z^*)$ which defines a central-projection can be given by the configuration $O^*(E_x^*, V_x^*, V_y^*, V_z^*)$, and the points E_y^*, E_z^* must be determined based on the condition [i].

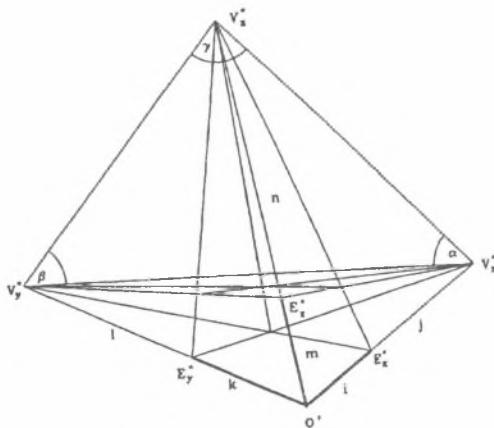


Fig. 2

For example the determining of point E_y^* is based on the relation $\left(\frac{k}{l}\right)^2 = \left(\frac{i}{j}\right)^2 \frac{\operatorname{tg}\beta}{\operatorname{tg}\alpha}$. So it is sure that the central-axonometric mapping is a central-projection and the viewpoint exists.

To find the world-coordinates of the viewpoint is a classic problem of photogrammetry. It is well known that the orthogonal projection of the viewpoint is the orthocentre of the triangle V_x^*, V_y^*, V_z^* ; let us denote it by C_1 . Let us rotate the point C around the line V_x^*, V_y^* to the image-plane and denote the rotated point by C° . As Figure 3. shows, C° is the intersection point of the through C_1 perpendicular to V_x^*, V_z^* and the circle with diameter V_x^*, V_z^* .

Now it is necessary to calculate the equation of the image-plane in the world-system. For the first we determine the $N_x N_z$ traceline which is the intersection of the image-plane and the xy -coordinate plane, in the following way: In central-projection it is known that there is a central collineation between the central projection of a plane and the rotation of this plane to the projection plane. The rotation axis is of course the $N_x N_z$ traceline, and this is the axis of the central collineation, too. The centre of this collineation is C° .

Now we have the centre C° of that collineation, and we want to know where the axis is. This is a simple sizing problem of central collineation. The solving of this problem is based on the following: if

the unit square lying on the xz -coordinate plane is rotated on the image-plane, its edges can be seen in their real length.

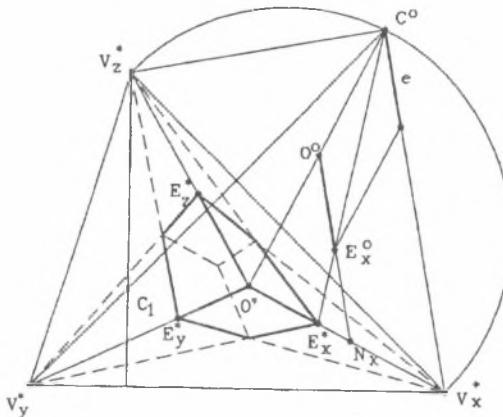


Fig. 3

The construction can be seen in Figure 3. In this way the user coordinates of the points N_x, N_y, N_z can be easily calculated. Let the user coordinates of point N_x be denoted by n_x^{ξ}, n_x^{η} and its world x -coordinate by n_x . Then n_x can be simply calculated based on the invariance of the cross ratio:

$$n_x = (N_x \text{ } E_x \text{ } O \text{ } V_x) = (N_x^* \text{ } E_x^* \text{ } O^* \text{ } V_x^*) = \frac{n_x^{\xi}}{t_1} : \frac{v_1 - n_x^{\xi}}{v_1 - t_1}$$

In the case of the other points:

$$n_y = (N_y \text{ } E_y \text{ } O \text{ } V_y) = (N_y^* \text{ } E_y^* \text{ } O^* \text{ } V_y^*) = \frac{n_y^{\xi}}{t_3} : \frac{v_3 - n_y^{\xi}}{v_3 - t_3}$$

$$n_z = (N_z \text{ } E_z \text{ } O \text{ } V_z) = (N_z^* \text{ } E_z^* \text{ } O^* \text{ } V_z^*) = \frac{n_z^{\xi}}{t_5} : \frac{v_5 - n_z^{\xi}}{v_5 - t_5}$$

The equation of the image plane in the world coordinate system is as follows:

$$\frac{1}{n_x} x + \frac{1}{n_y} y + \frac{1}{n_z} z = 1$$

Now the world coordinates of O^* o_x, o_y, o_z are determined.

Let d_x , d_y , d_z denote the distance between O^* and the points N_x , N_y , N_z . As the user coordinates of the points N_x , N_y , N_z are known, the distances d_x , d_y , d_z can be calculated.

The following equations can be gained:

$$\frac{o_x}{n_x} + \frac{o_y}{n_y} + \frac{o_z}{n_z} = 1, \quad (1)$$

$$(o_x - n_x)^2 + o_y^2 + o_z^2 = d_x^2, \quad (2)$$

$$o_x^2 + (o_y - n_y)^2 + o_z^2 = d_y^2, \quad (3)$$

$$o_x^2 + o_y^2 + (o_z - n_z)^2 = d_z^2. \quad (4)$$

Taking the differences (2) – (3) and (2) – (4), we will get a linear equation system consisting of three equations, whose solution is the following:

$$o_x = \frac{1}{n_x \left(\frac{1}{n_x^2} + \frac{1}{n_y^2} + \frac{1}{n_z^2} \right)} \left(2 - \frac{1}{2n_y^2} (n_x^2 + d_y^2 - d_x^2) - \frac{1}{2n_z^2} (n_x^2 + d_z^2 - d_x^2) \right),$$

$$o_y = \frac{1}{n_y \left(\frac{1}{n_x^2} + \frac{1}{n_y^2} + \frac{1}{n_z^2} \right)} \left(2 - \frac{1}{2n_x^2} (n_y^2 + d_x^2 - d_y^2) - \frac{1}{2n_z^2} (n_y^2 + d_z^2 - d_y^2) \right),$$

$$o_z = \frac{1}{n_z \left(\frac{1}{n_x^2} + \frac{1}{n_y^2} + \frac{1}{n_z^2} \right)} \left(2 - \frac{1}{2n_y^2} (n_z^2 + d_y^2 - d_z^2) - \frac{1}{2n_x^2} (n_z^2 + d_x^2 - d_z^2) \right).$$

The coordinates c_x , c_y , c_z of the centre C are easy to calculate on the basis of the relation

$$(COO^*) = (V_x^* N_x O^*) - \frac{v_1}{n_x^2} \quad (\text{Figure 4.})$$

$$c_x = o_x \frac{v_1}{n_x}, \quad c_y = o_y \frac{v_3}{n_y}, \quad c_z = o_z \frac{v_5}{n_z}.$$

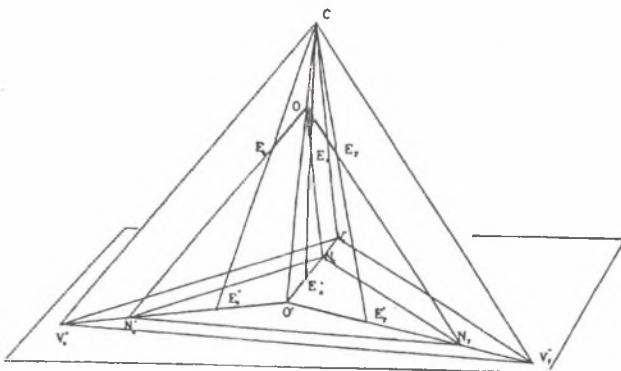


Fig. 4

IV Transformation from world coordinate system into user coordinate system

In order that the matrix of the transformation will be given, the world coordinates of the unit vectors $\bar{e}_1, \bar{e}_2, \bar{e}_3$ of the orthonormal basis are needed. We will get the world coordinates of the unit point of the axis ξ in the user coordinate system by rotating the normalised of vector $O^* N_x$ around the rotation axis, which is perpendicular to the image plane and parallel to the vector $\left(\frac{1}{n_x}, \frac{1}{n_y}, \frac{1}{n_z}\right)$. The

rotation angle $\arctg \frac{n_x}{n_\xi}$ is the angle of the ξ coordinate axis and of the vector $O^* N_x$. Let the vector

\bar{e}_3 be the normalised of the vector $\left(\frac{1}{n_x}, \frac{1}{n_y}, \frac{1}{n_z}\right)$, which is perpendicular to the image plane. The

third unit vector is given by $\bar{e}_2 = \bar{e}_1 \times \bar{e}_3$. The transformation matrix is of course given by the coordinates of the $\bar{e}_1, \bar{e}_2, \bar{e}_3$ column vectors.

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Streszczenie

W pracy omówiono sposób wyznaczenia ogólnej aksonometrii środkowej realizowanej metodami grafiki komputerowej. W założeniach wyjściowych niezbędne są informacje o rzutach na płaszczyznę obrazu: początku układu, punktów o współrzędnych jednostkowych oraz punktów niewłaściwych osi układu kartezjańskiego. Podano wzory na obliczenie współrzędnych punktu, które wyprowadzono opierając się na rozważaniach rzutowych. Podniesiono problem warunków, w których aksonometria środkowa jest jednocześnie rzutem środkowym. Przypominając prace E. Krupy i J. Szabo, precyzujące takie warunki, Autor korzysta z analitycznych wyników J. Szabo i dla przypadku, w którym zachodzi identyczność aksonometrii środkowej z rzutem środkowym, wyprowadza wzory na obliczenie współrzędnych środka rzutów.