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A CURVE ARC WITH TWO OPEN PROBLEMS[1]

Summary. The aim of the present paper is to determine an arc of curve, with a given initial point, endpoint, as well as a tangent straight line and an osculating circle in both these points.

DWA OTWARTE PROBLEMY W KONSTRUKCJI ŁUKU KRZYWEJ PŁASKIEJ

Streszczenie. Przedmiotem pracy jest określenie łuku krzywej Beziera, dla której dane są: punkt początkowy oraz końcowy łuku, proste styczne do krzywej w tych punktach oraz krzywizny łuku w obydwu punktach.

A great many methods of curve-approximation and -interpolation are employed in computer-graphics. Whatever starting point we may choose, the curve must ultimately have an analytical form. Since the drawing is made in the plane, the curves are always plane curves, and given in the form $y = f(x)$ or $\mathbf{r} = \mathbf{r}(t)$. Of the two kinds of curve, the latter one is the one of more general use.

We wish to obtain the parametric equation of the curve in the form

$$x(t) = a_{11}t^3 + a_{12}t^2 + a_{13}t + a_{14}$$
$$y(t) = a_{21}t^3 + a_{22}t^2 + a_{23}t + a_{24},$$

where $-1 \leq t \leq 1$.

The curve is determined by the coefficients of the equations. Thus the above equation can yield a Ferguson-arc, an Akimova-arc, a Bézier-curve of third degree, or a B-spline.

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The aim of the present paper is to determine an arc of curve, with a given initial point, endpoint, as well as a tangent straight line and an osculating circle in both these points.

If we build up a curve of arcs then in the joining points the osculating circles are not only identical but have a given radius.

Assume that the tangents of a certain arc at its initial and endpoint $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are $e_1(u_1, \upsilon_1)$ and $e_2(u_2, \upsilon_2)$, respectively. Assume also that the points P_1 and P_2 correspond to parametr values t = - 1 and t = + 1.

In this case our conditions are

$$x_1 = -a_{11} + a_{12} - a_{13} + a_{14} \quad (1)$$

$$x_2 = a_{11} + a_{12} + a_{13} + a_{14} \quad (2)$$

$$u_1 = 3a_{11} - 2a_{12} + a_{13} \quad (3)$$

$$u_2 = 3a_{11} + 2a_{12} + a_{13} \quad (4)$$

$$y_1 = -a_{21} + a_{22} - a_{23} + a_{24} \quad (5)$$

$$y_2 = a_{21} + a_{22} + a_{23} + a_{24} \quad (6)$$

$$\upsilon_1 = 3a_{21} - 2a_{22} + a_{23} \quad (7)$$

$$\upsilon_2 = 3a_{21} + 2a_{22} + a_{23}. \quad (8)$$

After calculations we get:

$$a_{12} = (u_2 - u_1)/4 \quad (9)$$

$$a_{14} = (x_1 + x_2)/2 - a_{12} \quad (10)$$

$$a_{11} = (u_1 + u_2 - x_2 + x_1)/4 \quad (11)$$

$$a_{13} = (x_2 - x_1)/2 - a_{11} \quad (12)$$

$$a_{22} = (\upsilon_2 - \upsilon_1)/4 \quad (13)$$

$$a_{24} = (y_1 + y_2)/2 - a_{22} \quad (14)$$

$$a_{21} = (\upsilon_1 + \upsilon_2 - y_2 + y_1)/4 \quad (15)$$

$$a_{23} = (y_2 - y_1)/2 - a_{21}. \quad (16)$$

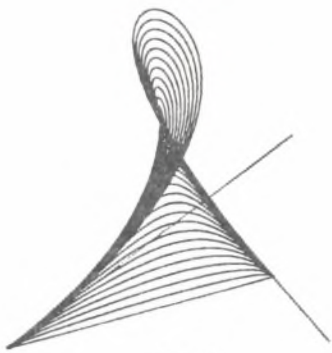

Fig. 1

The arcs have the same initial and endpoints and tangential lines at these points on Fig. 1, the lengths of the tangent vectors, however, vary between 1 and 30. The signed curvatures are computed too.

At the first arc the values of the curvatures are -0,24 and -0,087.

When the length of the tangents is 6 the curvature at the initial point changes its sign. This change occurs at the endpoint at value 13. Finally, when the length of the tangents are 30 the values of the radii are 26.15 and 55.43.

Let us denote the radii of the osculating circles by r_1 and r_2 ; let the corresponding curvatures be κ_1 and κ_2.

It is known that:

$$k(t) = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\sqrt{\left(\dot{x}(t)^2 + \dot{y}(t)^2\right)^3}} \qquad \text{and} \qquad r(t) = \frac{1}{k(t)},$$

where $\kappa(t)$ and $r(t)$ stand for the curvature and radius of the corresponding osculating circle.

Let us denote the length of the tangent vectors by α and β, i.e. $\alpha^2 = (\alpha u_1)^2 + (\alpha v_1)^2$, and $\beta^2 = (\beta u_2)^2 + (\beta v_2)^2$; $u_1^2 + v_1^2 = 1$, $u_2^2 + v_2^2 = 1$.

From equations (1), (2) we get the conditions

$$x(-1) = -a_{11} + a_{12} - a_{13} + a_{14} \tag{17}$$

$$y(-1) = -a_{21} + a_{22} - a_{23} + a_{24} \tag{18}$$

$$x(+1) = a_{11} + a_{12} + a_{13} + a_{14} \tag{19}$$

$$y(+1) = a_{21} + a_{22} + a_{23} + a_{24} \tag{20}$$

$$\dot{x}(-1) = 3a_{11} - 2a_{12} + a_{13} \tag{21}$$

$$\dot{y}(-1) = 3a_{21} - 2a_{22} + a_{23} \tag{22}$$

$$\dot{x}(+1) = 3a_{11} + 2a_{12} + a_{13} \tag{23}$$

$$\dot{y}(+1) = 3a_{21} + 2a_{22} + a_{23} \tag{24}$$

$$\ddot{x}(-1) = -6a_{11} + 2a_{12} \tag{25}$$

$$\ddot{y}(-1) = -6a_{21} + 2a_{22} \tag{26}$$

$$\ddot{x}(+1) = 6a_{11} + 2a_{12} \tag{27}$$

$$\ddot{y}(+1) = 6a_{21} + 2a_{22} \tag{28}$$

from which

$$\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t) = Det(t) = \begin{vmatrix} 0 & 0 & 1 \\ \dot{x} & \dot{y} & 0 \\ \ddot{x} & \ddot{y} & 0 \end{vmatrix}$$

The denominator at the initial point is α^3, at the end point it is β^3.

Now

$$Det(-1) = \begin{vmatrix} 0 & 0 & 1 \\ 3a_{11} - 2a_{12} + a_{13} & 3a_{21} - 2a_{22} + a_{23} & 0 \\ -6a_{11} + 2a_{12} & -6a_{21} + 2a_{22} & 0 \end{vmatrix}$$

$$Det(-1) = 2\begin{vmatrix} -1 & -3 & -3 \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}$$

from which

$$r_1 = r(-1) = \frac{-\alpha^2}{\begin{vmatrix} 0 & 1.5 & \beta \\ u_1 & u_2 & d_1 \\ \upsilon_1 & \upsilon_2 & d_2 \end{vmatrix}}$$

and

$$r_2 = r(+1) = \frac{-\beta^2}{\begin{vmatrix} 1.5 & 0 & \alpha \\ u_1 & u_2 & d_1 \\ \upsilon_1 & \upsilon_2 & d_2 \end{vmatrix}}$$

where $d_1 = x_2 - x_1$, $d_2 = y_2 - y_1$.

We got two equations for α and β:

$$-1.5\, r_1(u_1 d_2 - d_1 \upsilon_1) + r_1\beta(u_1\upsilon_2 - u_2\upsilon_1) = -\alpha^2 \quad (29)$$

$$1.5\, r_2(u_2 d_2 - d_1 \upsilon_2) + r_2\alpha(u_1\upsilon_2 - u_2\upsilon_1) = -\beta^2 \quad (30)$$

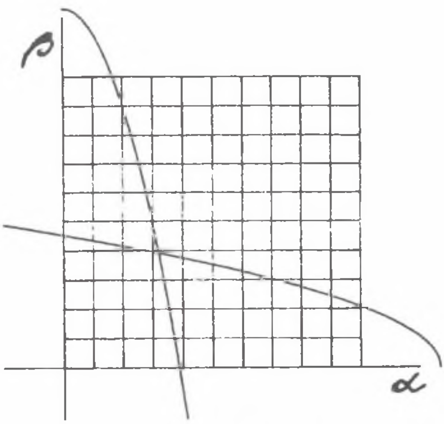

Fig. 2

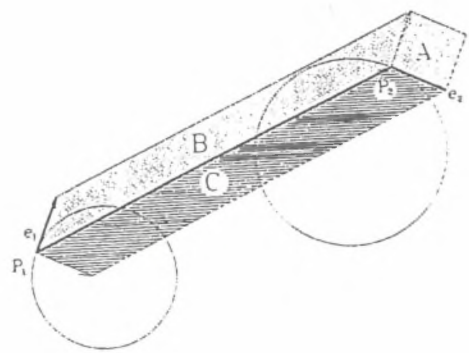

Fig. 3

Let us introduce the notations

$$A = (u_1\upsilon_2 - u_2\upsilon_1) \quad (31)$$

$$B = (u_1 d_2 - d_1\upsilon_1) \quad (32)$$

$$C = (u_2 d_2 - d_1\upsilon_2) \quad (33)$$

which have a definite geometric meaning.

Finally, we get the two equations

$$\alpha^2 + r_1 A\beta - 1.5 r_1 B = 0 \quad (34)$$

$$r_2 A\alpha + \beta^2 + 1.5 r_2 C = 0 \quad (35)$$

Equation (34) describes a parabola with an axis that coincides with the axis β in the coordinate system (O, α,β); while the graph of (35) is a parabola with an axis coinciding with the axis α, see Fig 2.

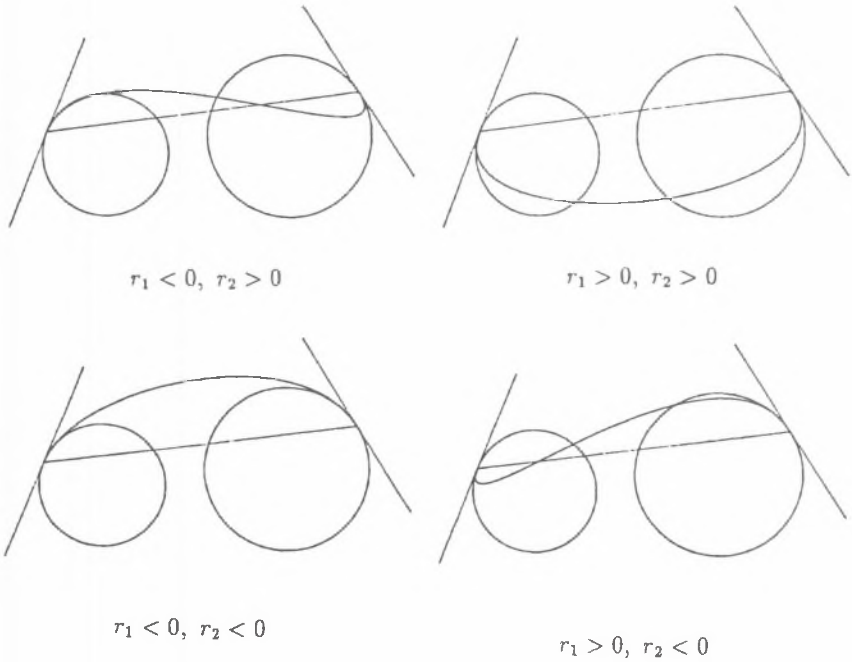

Fig. 4

In this case the problem has a solution; the length of the first tangent is 3.4 and the lenght of the second one is 3.9. The numerical solution of the system of equation causes no problem.

Open problem 1:

We get a system of equations of two nice equations which inspires us to solve the system the traditional way (the programs Mathematica and Maple have both found the solutions after almost half an hour's hard work, one of the solutions fills seven pages on the screen in TEX source code).

Open problem 2:

The conditions of the existence of a solution are not clear yet, although the conditions can easily be demonstrated geometrically. (Fig. 2, Fig. 3 and equations 34-35)

The figure 4 has been obtained by combining the signed radii of the osculating circles. The sign of the radius determines the direction of the tangent.

Finally we give two illustrations.

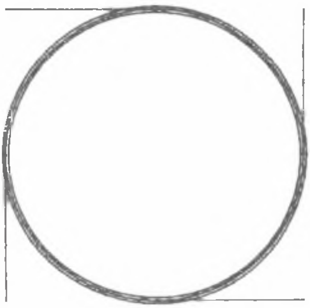

Fig. 5

The difference between the radii of the two concentric circles between which the four arcs run is 1mm. (Fig. 5). The arcs, the equations of which have the form (1), (2), are given by four points and the corresponding tangents. The curvatures of the approximating arcs are also given.

It is known that the only closed curve with a constant curvature is the circle on the euclidean plane. On Fig. 6 a number of points of an ellipse are given with the corresponding tangents.

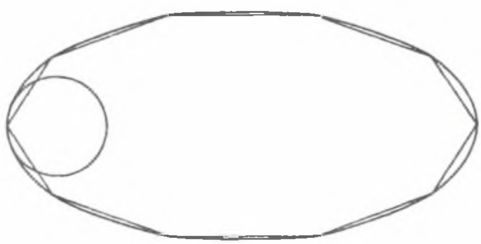

Fig. 6

We want to construct an interpolating curve through the given points such that the radii of the osculating circles of the interpolating curve at the given points are equal to a certain constant value. This value, however, cannot be chosen arbitraryly. Indeed, the problem has a solution if this value is not greater than the radius of the osculating circle of the ellipse at the endpoint of its major axis. This fact has a geometric interpolation: the inscribed small circle can be rolled inside the ellipse such that it concides with the osculating circles of the interpolating arcs it at the given points.

Streszczenie

W pracy rozpatruje się konstrukcję łuku krzywej Beziera dla danych dwóch punktów łuku (początkowego i końcowego) oraz prostych stycznych i krzywizn łuku w tych punktach.

Rozważania autora dotyczą rozwiązywalności układu równań, w których niewiadomymi są długości wektorów stycznych. Wyłaniają się przy tym dwa problemy: posłużenie się w rozwiązywaniu odnośnych równań programami Mathematica and Maple, których realizacja wymaga około półgodzinnej pracy urządzenia i 7-stronicowych obliczeń dla jednego rozwiązania oraz problem

warunków do zaistnienia rozwiązań. Problem ten jest dotąd niejasny, aczkolwiek warunki takie łatwo można przedstawić geometrycznie (rys. 2 i 3 oraz równania (34), (35)).