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## BITMAP TRANSFORMATIONS BASED ON PROJECTIVE INVARIANTS

Summary. The purpose of the research is to create a program which is able to transform bitmaps in real time. The algorithms developed so far are able to accomplish the transformation of an $\mathrm{M}^{*} \mathrm{~N}$ bitmap to either a convex polygon or a spherical surface or a cone-patch or a cylinder-patch in real time. We do a simple projective mapping to the surfaces. In the sphere case, we work with projective coordinates of the planes, which is very comfortable and fast.

## PRZEKSZTAŁCENIA MAP BITOWYCH BAZUJACE NA NIEZMIENNIKACH RZUTOWOŚCI

Streszczenie. Celem badań jest opracowanie programu umożliwiającego przeksztalcanie map bitowych. W pracy zanalizowano algorytmy realizujace przekształcanie map na wielokąt wypukły oraz na sferę. Opisano również możliwość przekształceń na powierzchnię stożkową i walcowa.

## 1. Transforming to a convex polygon

Let us indicate the vertices of the $M^{*} N$ bitmap with $A, B, C$ and $D$. Let $O_{1}$ be the point at infinity of the straight line $\overline{A B}$. Let $\mathrm{O}_{2}$ be the point at infinity of the straight line $\overline{A D}$

After the transformation of the bitmap to a convex polygon, we can obtain one of the following cases:


Fig. 1


Fig. 2

We have to find a quadrilateral for each pixel of the original bitmap. When we know the position of the four vertices of the quadrilateral then we have to make a scan line conversion.


Fig. 3

The four vertices are points of intersections of lines which are defined by four pairs of points. The first pair of these points fit on the $\overline{A^{\prime} B^{\prime}}$ section, the second pair of these points fit on the $\overline{C^{\prime} D^{\prime}}$ section, the third pair of these points fit on the $\overline{A^{\prime} D^{\prime}}$ section and the forth pair of points fit on the $\overline{\mathrm{B}^{\prime} \mathrm{C}^{\prime}}$ section.

Let us indicate the actual grid-point of a border section with $P$ and the transformed grid-point with $\mathrm{P}^{\prime}$. Since we can divide into N pieces the sections independently, we have to examine only two cases:
a) Opposite sides are parallel to each other.

Let $P$ a grid-point of the $\overline{A B}$ section. In this case we have to divide the $\overline{A^{\prime} B^{\prime}}$ and the $\overline{C^{\prime} D^{\prime}}$ section into $N$ pieces according to the following:

$$
\begin{aligned}
& \left(A B P O_{1}\right)=\left(A^{\prime} B^{\prime} P^{\prime} O_{1}^{\prime}\right) \\
& (A B P)=\left(A^{\prime} B^{\prime} P^{\prime}\right) \\
& \overline{\overline{P A}} \overline{\overline{\mathrm{~PB}}}=\frac{\overline{P^{\prime} A^{\prime}}}{\overline{P^{\prime} B^{\prime}}}
\end{aligned}
$$

We get this result because $O_{1}$ and $O_{1}{ }^{\prime}$ are points at infinity. This means that the length of the sections, defined by the transformed grid-points, will be the same.
b) Opposite sides are not parallel to each other.

Let $P$ a grid-point of the $\overline{A B}$ section. The location of $P^{\prime}$ has to be defined according to the invariance of the crossratio. We get the following result from

$$
\begin{aligned}
& \left(A B P O_{1}\right)=(A B P)=\frac{\overline{P A}}{\overline{P B}}=\left(A^{\prime} B^{\prime} P^{\prime} O_{1}^{\prime}\right)=\frac{\overline{P^{\prime} A^{\prime}}}{\overline{P^{\prime} B^{\prime}}}: \frac{\overline{O_{1}^{\prime} A^{\prime}}}{O_{1}^{\prime} B^{\prime}} \text { and } \\
& \overline{P^{\prime} B^{\prime}}=\overline{A^{\prime} B^{\prime}}-\overline{P^{\prime} A^{\prime}} \text { equalities: } \\
& \frac{\overline{P A}}{\overline{P B}} \cdot \frac{\overline{O_{1}^{\prime} A^{\prime}}}{\overline{O_{1}^{\prime} B^{\prime}}}=\frac{\overline{P^{\prime} A^{\prime}}}{\overline{A^{\prime} B^{\prime}-\overline{P^{\prime} A^{\prime}}}} \\
& \overline{P^{\prime} A^{\prime}}=\frac{\overline{P A}}{\overline{P B}} \cdot \overline{\frac{O_{1}^{\prime} A^{\prime}}{O_{1}^{\prime} B^{\prime}}} \cdot\left(\overline{A^{\prime} B^{\prime}}-\overline{P^{\prime} A^{\prime}}\right) \\
& \overline{P^{\prime} A^{\prime}} \cdot\left(1+\frac{\overline{P A}}{\overline{P B}} \cdot \frac{\overline{O_{1}^{\prime} A^{\prime}}}{\overline{O_{1}^{\prime} B^{\prime}}}\right)=\overline{A^{\prime} B^{\prime}} \cdot \frac{\overline{P A}}{\overline{P B}} \cdot \frac{\overline{O_{1}^{\prime} A^{\prime}}}{\overline{O_{1}^{\prime} B^{\prime}}} \\
& \overline{P^{\prime} A^{\prime}}=\frac{\overline{A^{\prime} B^{\prime}} \cdot \overline{\overline{P A}} \cdot \overline{\overline{P B}} \cdot \frac{\overline{O_{1}^{\prime} A^{\prime}}}{\overline{O_{1}^{\prime} B^{\prime}}}}{1+\frac{\overline{\mathrm{PA}}}{\overline{\mathrm{~PB}}} \cdot \overline{\frac{\mathrm{O}_{1}^{\prime} \mathrm{A}^{\prime}}{\overline{O_{1}^{\prime} B^{\prime}}}}} .
\end{aligned}
$$

So we can define a $P^{\prime}$ point to each $P$ point on the $\overline{A^{\prime} B^{\prime}}$ section. In the same way we can define the projected slide of the points of the $\overline{C D}$ section.

If we call the intersecting function, which in fact does a vectorial multiplication between two points having projective co-ordinates, with the appropriate points of $\overline{A^{\prime} B^{\prime}}$ and $\overline{C^{\prime} D^{\prime}}$ sections, we get equalities of a line sequence. Intersecting these equalities with those we defined from $\overline{A^{\prime} B^{\prime}}$ and $\overline{C^{\prime} D^{\prime}}$ the same way, we get the projected slides of the points of the $\mathrm{M}^{*} \mathrm{~N}$ grid.

## 2. Transforming to a sphere

Let us indicate the vertices of the $M^{*} N$ bitmap with $A, B, C$ and $D$ like in the polygon case. Let $\mathrm{O}_{1}$ be the point at infinity of the straight line $\overline{\mathrm{AB}}$. Let $\mathrm{O}_{2}$ be the point at infinity of the straight line $\overline{A D}$.


Fig. 4

The situation of the transformed bitmap is defined by four points which have to be placed on the same half sphere. Thus segments of main circles of the sphere will represent the paralel sides of the original rectangle. Indeed it is the projective mapping of the plane to a half sphere. We will indicate the transformed points with $A^{\prime}, B^{\prime}, C^{\prime}, D^{\prime}, O_{1}{ }^{\prime}$ and $O_{2}{ }^{\prime} \cdot r$ will be the radius of the sphere. Let a be the vector that points to $A^{\prime}$ from $O$ which is the center of the sphere and the co-ordinate system too. If we apply this notation to $\mathrm{B}^{\prime}, \mathrm{C}^{\prime}$ and $\mathrm{D}^{\prime}$ points as well, we can obtain $\mathrm{O}_{1}{ }^{\prime}$ and $\mathrm{O}_{2}{ }^{\prime}$ for given $A^{\prime}, B^{\prime}, C^{\prime}$ and $D^{\prime}$ the following way:

$$
\begin{aligned}
\underline{o_{1}}=(\underline{a} \times \underline{b}) \times(\underline{d} \times \underline{c}) & \underline{o_{2}}=(\underline{a} \times \underline{d}) \times(\underline{b} \times \underline{c}) \\
\underline{O O_{1}^{\prime}}=r \frac{o_{1}}{\underline{o_{1}}} & \underline{O O_{2}^{\prime}}=r \frac{o_{2}}{\underline{o_{2}}}
\end{aligned}
$$

The spherical quadrilateral, on which the projection is accomplished, is defined by those segments of the spherical main circle which is the intersection of the plane determined by the centre of the sphere and the proper vertical pairs and the sphere. Let us apply the following notation:


Fig. 5

Let us consider that intersection of the sphere which is received so that it be perpendicular to the line of intersection of the $A^{\prime} D^{\prime} O$ and $B^{\prime} C^{\prime} O$ planes and let the $O$ point, which is the centre of the sphere, fit into this intersection.

In order to determine the transformed grid-points, we need two plane-sequences: one between the $A^{\prime} D^{\prime} O$ and $B^{\prime} C^{\prime} O$ planes and one between the $A^{\prime} B^{\prime} O$ and $C^{\prime} D^{\prime} O$ planes. The transformed grid-points will be the intersection points of the sphere and the two plane-sequences. This means a projective mapping between two first degree projective basic forms: between a plane-sequence and a point-sequence.

We can calculate a plane that will be element of the first plane-sequence and contains $P^{\prime}$ point for every $P$ grid-point of the $\overline{\mathrm{AB}}$ section. The same way we can calculate a plane that will be element of the second plane-sequence and contains $\mathrm{P}^{\prime}$ point for every $P$ grid-point of the $\overline{A D}$ section.

Let the basic planes of a projective co-ordinate system be the $A^{\prime} D^{\prime} O$ and $B^{\prime} C^{\prime} O$ planes. A'D'O will have $(1,0)$ projective co-ordinate values and $B^{\prime} C^{\prime} O$ will have $(0,1)$ projectiv coordinate values in the new co-ordinate system. The line of intersection of these basic planes will be the holder of the plane-sequence with the help of which we can define the locations of the points transformed to the surface of the sphere.


Fig. 6

Let us express the $\mathrm{O}_{1}^{\prime} \mathrm{O}_{2}^{\prime} \mathrm{O}$ plane having $(\lambda, \mu)$ co-ordinate values in the new coordinate system in terms of the basic planes. Let $\mu$ be 2. (We can do this because pairs having the same ratio define the same plane.) According to the equalities of the planes, $\lambda$ can also be determined. We will get the proper $\lambda$ value as well, if we use the normal vectors of the planes instead of the equalities of the planes.

Since $\underline{n_{A^{\prime} D^{\prime} O}}$ and $\underline{n_{B^{\prime} C^{\prime} O}}$ differ from each other at least in one co-ordinate value and $\mu$ is 2 , the following three equations will contain at least two independent equations. There are two unknown values: $k$ and $\lambda$.

$$
\begin{aligned}
& k\left(\underline{n_{O_{1}} O_{2}^{\prime} O}\right)(x)=\lambda\left(\frac{\left.n D_{D^{\prime} O}\right)}{A^{\prime}}(x)+\underline{\mu\left(n_{B^{\prime} C^{\prime} O}\right)}(x)\right. \\
& k\left(\underline{n_{0} O_{2} O_{2}^{\prime O}}\right)(y)=\lambda\left(n_{A^{\prime}} D^{\prime} O\right)(y)+\mu\left(n_{B^{\prime} C^{\prime} O}\right)(y) \\
& \mathrm{k}\left(\underline{\left.\mathrm{n}_{\mathrm{O}_{1} \mathrm{O}_{2}{ }^{\prime} \mathrm{O}}\right)}(\mathrm{z})=\lambda \underline{\mathrm{n}_{\mathrm{D}} \mathrm{D}^{\prime} \mathrm{O}}(\mathrm{z})+\mu\left(\underline{\mathrm{n}_{\mathrm{B}^{\prime} \mathrm{C}^{\prime} \mathrm{O}}}(\mathrm{z})\right.\right.
\end{aligned}
$$

From these equations $k$ and $\lambda$ can be determined easily. We are only interested in the value of $\lambda$.

Let an element of the plane-sequence defined by $A^{\prime} B^{\prime} O$ and C'D'O planes be indicated by $\left(x 1_{i}, y 1_{i}\right)$. We can produce this $\left(x 1_{i}, y 1_{i}\right)$ sequence if we take all $P_{i}$ gridpoint on the $\overline{\mathrm{AB}}$ section, because the crossratio is invariant facing this type of projective projection. Thus:

$$
\begin{gathered}
\left(A B P_{i} O_{1}\right)=\left(A B P_{i}\right)=\frac{\overline{P_{i} A}}{\overline{P_{i} B}}=\frac{\lambda-1}{\lambda-0}: \frac{x 1_{i}-1}{x 1_{i}-0} \\
\frac{\overline{P_{i} A}}{\overline{P_{i} B}} \cdot \frac{x 1_{i}-1}{x 1_{i}}=\frac{\lambda-1}{\lambda} \\
\frac{x 1_{i}-1}{x 1_{i}}=\frac{\lambda-1}{\lambda} \cdot \frac{\overline{P_{j} B}}{P_{i} A} \\
x 1_{i} \cdot\left(1-\frac{\lambda-1}{\lambda} \cdot \frac{\overline{P_{i} B}}{\overline{P_{i} A}}\right)=1 \\
x 1_{i}=\frac{1}{1-\frac{\lambda-1}{\lambda} \cdot \frac{\overline{P_{B} B}}{\overline{P_{i} A}}}
\end{gathered}
$$

and:

$$
\begin{gathered}
\frac{\overline{P_{i} A}}{\overline{P_{i} B}}=\frac{\mu-0}{\mu-1}: \frac{y 1_{i}-0}{y 1_{i}-1} \text { and } \mu=2 \text {, so: } \\
\frac{\overline{P_{i} A}}{\overline{P_{i} B}} \cdot \frac{y 1_{i}}{y 1_{i}-1}=2 \\
y 1_{i}=2 \cdot\left(y 1_{i}-1\right) \cdot \frac{\overline{P_{B}}}{\overline{P_{i} A}}=2 \cdot \overline{\frac{\overline{P B}}{\overline{P_{i} A}}} \cdot y 1_{i}-2 \cdot \frac{\overline{P_{B} B}}{\overline{P_{i} A}} \\
y 1_{i}=\frac{2 \cdot \frac{\overline{P_{i}}}{\overline{P_{i} A}}}{2 \cdot \frac{\overline{P_{B}}}{\overline{P_{i} A}}} .1
\end{gathered}
$$

If we consider $A^{\prime} D^{\prime} O$ and $B^{\prime} C^{\prime} O$ planes the basic planes, then the line of intersection of these planes will also be the holder of a plane-sequence. Let us indicate an element of this plane-sequence by $\left(x 2_{j}, y 2_{j}\right)$ parameters. So the $P_{i j}$ ' points adequate to the $\mathrm{P}_{\mathrm{ij}}$ grid-points of the original bitmap can be calculated as the point of intersection of the sphere and the line of intersection of the plane having
$\left(x 1_{i}, y 1_{i}\right)$ parameters related to the first basic plane pair and the plane having ( $x 2_{j}$, $y 2_{j}$ ) parameters related to the second basic plane pair. During the producing of the algorithm, it is worth calculating with the normal vectors of the planes, because we can get the projected points of the grid-points of the bitmap as the $r$ lengthed vectorial product of the normal vectors of the appropriate planes of the planesequences. We have to fill up the area having the vertices BITMAP_T[I][J], BITMAP_T[I][J+1], BITMAP_T[I+1][J+1] and BITMAP_T[I+1][J] $(I \in[0 \ldots m-1], J \in[0 \ldots$ $\mathrm{n}-1]$ ), which are projected grid-points, with the proper BITMAP [II[J] colour, where the BITMAP_T is an array of POINT_3D elements and stores the transformed gridpoints. At this point, if we have accomplished the previous operation on each I and $J$ pairs that are different from each other and which fulfil the conditions, then we get the slide of the grid of the original bitmap projected to spherical surface.

Consequently, we could transform to spherical surface taking advantage of the invariance of the crossratio without using trigonometrical functions in the algorithm, which are too time-consumming for the computer.

## 3. Transformation to cone-superficies and cylinder-superficies

First we have to transform the four points defining the location of the transformed bitmap to the plane (on the basis of the lay-out of the surface), where we perform the definition of the points of the bitmap-grid with the help of the above-expounded method. Afterwards we transform the points of the grid back to the surface. (The function of the lay-out of the surface is bijective, so that we can do backtransformation without any difficulty). We have to fill up the quadrilaterals bordered by four points calculated this way with the proper colours of the bitmap. Thus these quadrilaterals give us the slide of the bitmap projected to either cone-superficies or cylinder-superficies.

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## Streszczenie

Celem pracy jest stworzenie programu komputerowego, realizujacego przekształcenia map bitowych. Przedstawiono szczegółowe algorytmy umożliwiające alternatywne przekształenie mapy bitowej na. wielokąt wypukły bądź sferę. Przekształcenie na wielokąt wypukły realizowane jest za pomocą rzutu środkowego, tj. z zachowaniem wartości dwustosunku podziału, natomiast przy przekształcaniu na sferę posłużono się współrzędnymi rzutowymi płaszczyzn. Opisano również działania związane z przekształcaniem map bitowych na powierzchnie stożkowe i walcowe. Ogólnie, działania te polegaja na rozwinięciu tych powierzchni, powtórzeniu konstrukcji z przekształcenia na wielokąt wypukły, a następnie odniesieniu siatki do „zwiniętych" na powrót powierzchni.

