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ENVELOPE SURFACES

Summary. In the differential geometry, an envelope surface is defined as the envelope of a one-parametric system of surfaces satisfying several conditions. Geometry of the Creative space offers a new approach to this problem (in[1], [7]), based on the creative representations of figures. Instead of a one-parametric system of surfaces we speak about a solid, as the three-dimensional connected subset of the extended Euclidean space ${}_{\infty}E^3$. The envelope surface is determined as a set of points satisfying a condition expressing their location on the border of the solid, while the characteristic curve can be found as its subset.

POWIERZCHNIE OBWIEDNIE

Streszczenie. W pracy rozważa się definicję i przykłady tworzenia powierzchni obwiednich w interpretacji geometrii przestrzeni kreatywnej.

1. Introduction

Creative space *K* is an ordered pair *K* = (*U*, *G*) of *U* (a base) as a set of figures in the space (subsets of the extended Euclidean space ${}_{\infty}E^3$) and *G* (a generator) as a set of generating principles ($G = GP({}_{\infty}E^3) \cup L$, while $GP({}_{\infty}E^3)$ is a group projective transformations of ${}_{\infty}E^3$ and *L* is a set of interpolations) (according to [2], [4], [5]).

Solid S (a three-parametric subset of ${}_{\infty}E^3$) is in *K* synthetically represented by its creative representation, an ordered pair (U, G) of a basic figure $U \in U$ and a generating principle $G \in \mathcal{G}$ which are such, that applying G on U the solid S can be created.

The analytic representation of a solid S is a vector function of three variables

$$s(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w), h(u,v,w))$$

defined at the region $\Omega = \langle 0,1 \rangle^3$ (where x, y, z and h are homogeneous coordinate functions of three real variables at least C^3 continuous on Ω), which is a local homeomorphic mapping of the region Ω on the solid S. The analytic representation of a solid is the product of the multiplication of matrices, which are analytic representations of the creative representation elements (according to [3], [6]).

A solid S can be created from a basic figure in a form of a surface patch φ, while the generating principle is a class of geometric transformations, that means, the basic surface patch moves in a space forming thus a one-parametric system of surfaces. The envelope surface can be determined also by the creative law represented by the creative representation with the basic figure in the form of the characteristic curve subdued to the generating principle, which is the same movement in the space - a class of geometric transformations.

Characteristic curve, as a subset of the envelope surface, is the set of those points of the solid, in which there exists a common tangent plane to the envelope surface and the basic surface patch φ. This means, that the partial derivatives of the vector function s(u,v,w) with respect to the variables u,v,w - s_u, s_v s_w, which define tangent vectors to the isoparametric curve segments of the solid in the given point $(a,b,0) \in \langle 0,1 \rangle^3$, are coplanar. This simple condition can be expressed by a triple scalar product of the three vectors that equals to zero. According to this condition homogeneous coordinates of the characteristics' points can be calculated and the resulted curve can be created as the interpolation curve defined by the ordered set of points with respect to the increasing values of the parameter u, resp. v.

2. Basic expressions

A solid S can be created from a basic figure in a form of a surface patch φ analytically represented by the vector function p(u,v) of two variables defined at the unit square region $<0,1>^2$, which is a local homeomorphic mapping of the square region on the surface patch φ (fig.1). The generating principle is a class of geometric transformations expressed analytically by a functional matrix ${}^1T(w)$, a regular square matrix of rank 4 with elements in a form of real functions of one variable defined, and at least C^1 continuous, at the unit interval <0,1>. The basic surface patch moves (and deformes) in a space forming thus a one-parametric system of surfaces. According to the solid creative representation and definition

$$s(u,v,w) = p\,(u,v) \cdot {}^1T(w).$$

The basic surface φ can be created in a similar way - a point M is subdued to a class of geometric transformations expressed by a functional matrix ${}^3T(u)$, while the resulting curve segment is subdued to another class of geometric transformations expressed by a functional matrix ${}^2T(v)$. It follows, that

$$s(u,v,w) = M \cdot {}^3T(u) \cdot {}^2T(v) \cdot {}^1T(w), \text{ for } (u,v,w) \in <0,1>^3$$

and homogeneous coordinates of the solid point can be calculated from the point function for the ordered triple of point curvilinear coordinates $(a,b,c) \in <0,1>^3$

$$s(u,v,w) = M \cdot {}^3T(a) \cdot {}^2T(b) \cdot {}^1T(c).$$

The envelope surface Φ can be determined also by the creative law represented by the creative representation with the basic figure in the form of the characteristic curve, analytically rerpesented by the vector function f(t) defined at the unit interval <0,1>, which is a local homeomorphic mapping of the interval on the characteristics k, subdued to the generating principle, which is the same movement in the space - class of geometric transformations expressed by a functional matrix ${}^3T(w)$

$$p(t,w) = f(t) \cdot {}^3T(w), \text{ for } (t,w) \in <0,1>^2.$$

Characteristic curve k, as a subset of the envelope surface, is the set of those points of the solid, in which there exists a common tangent plane τ to the envelope surface and the basic surface patch φ. Partial derivatives of the vector function s(u,v,w) with respect to the variables u, v, w - vectors s_u, s_v, s_w define tangent vectors

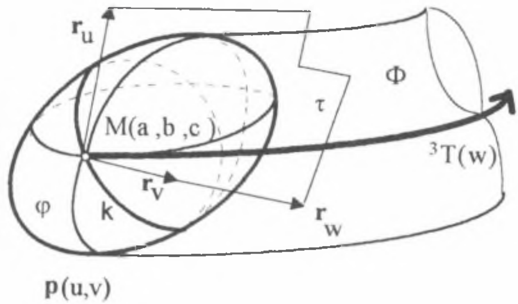

Fig.1

to the isoparametric u-, v- and w- curve segments of the solid in the given point $(a,b,0) \in <0,1>^3$ and can be expressed in the form

$$s_u(a,b,0) = M \cdot {}^3T'(a) \cdot {}^2T(b) \cdot {}^1T(0) = M \cdot {}^3T'(a) \cdot {}^2T(b)$$

$$s_v(a,b,0) = M \cdot {}^3T(a) \cdot {}^2T'(b) \cdot {}^1T(0) = M \cdot {}^3T'(a) \cdot {}^2T'(b)$$

$$r_w(a,b,0) = M \cdot {}^3T'(a) \cdot {}^2T(b) \cdot {}^1T'(0),$$

where matrices ${}^3T'(a)$, ${}^2T'(b)$, ${}^1T'(0)$ are derivatives of the functional matrices ${}^3T(u)$, ${}^2T(v)$ and ${}^1T(w)$ expressed in the values of the variables u=a, v=b, w=0, while ${}^1T(0)=E$ - a unit matrix representing identity. In the points of the solid boundary the three vectors are coplanar (fig.1), their triple scalar product equals to zero.

3. Envelope translation surfaces

Let the envelope surface be created from the basic surface patch analytically represented by the vector function

$$p(u,v) = (x(u,v), y(u,v), z(u,v), 1)$$

satisfying the required conditions on $<0,1>^2$. The generating principle in the form of the class of translations with the trajectory in the curve segment represented by the vector function

$$g(w) = (a(w), b(w), c(w), 1)$$

is represented by the matrix function ${}^1T(w)$ with the derivative in the matrix ${}^1T'(w)$

$$ {}^1T(w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a(w) & b(w) & c(w) & 1 \end{pmatrix} \qquad {}^1T'(w) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a'(w) & b'(w) & c'(w) & 0 \end{pmatrix} $$

required properties on the interval <0,1>. In the case of the movement on the line segment the coordinate functions are in the simple form a(w)=aw, b(w)=bw, c(w)= cw, which derivatives are constants· a,b,c (fig.3). In the fig.2 the trajectory of movement is an arbitrary curve segment and a sinusoid on which the spherical surface moves forming the system of spheres.

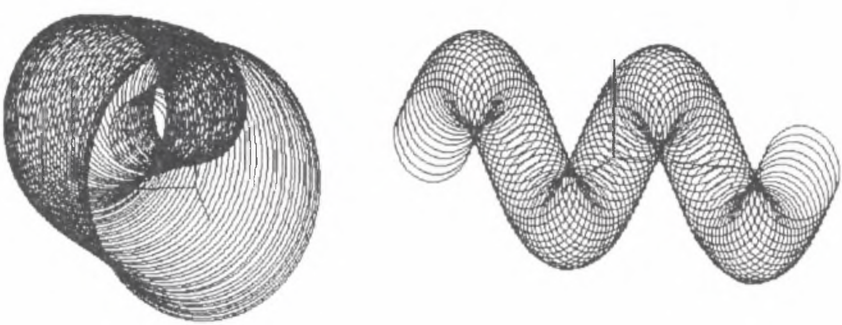

Fig.2

Solid analytic representation is a point function

$$ s(u,v,w) = p(u,v) \cdot {}^1T(w) = (x(u,v), y(u,v), 1) \cdot {}^1T(w), $$

satisfying required conditions on $<0,1>^3$, while the partial derivatives with respect to the variables u, v and w = 0 are in a form

$$ s_u (u,v,w) = p_u (u,v) = (x_u (u,v), y_u(u,v), z_u(u,v), 0) $$
$$ s_v (u,v,w) = p_v (u,v) = (x_v (u,v), y_v(u,v), z_v(u,v), 0) \qquad (1) $$
$$ s_w (u,v,w) = p(u,v) \cdot {}^1T'(w) = (x(u,v), y (u,v), z (u,v) \cdot {}^1T'(0). $$

Triple scalar product of the three vectors is the value of the determinant DT

$$ DT = \begin{vmatrix} s_u (u,v,w) \\ s_v (u,v,w) \\ s_w (u,v,w) \end{vmatrix} = \begin{vmatrix} x_u(u,v) & y_u(u,v) & z_u(u,v) \\ x_v(u,v) & y_v(u,v) & z_v(u,v) \\ a'(0) & b'(0) & c'(0) \end{vmatrix} = 0. \qquad (2) $$

Calculations can be processed with respect to the predefined step for the loop with the variable u=const ∈ <0,1>, while values of the variable v within the interval <0,1> change in the loop with the given step. Ordered pairs of the curvilinear coordinate values $(u,v) \in <0,1>^2$ satisfying the condition (2) define points of the characteristics on the basic surface.

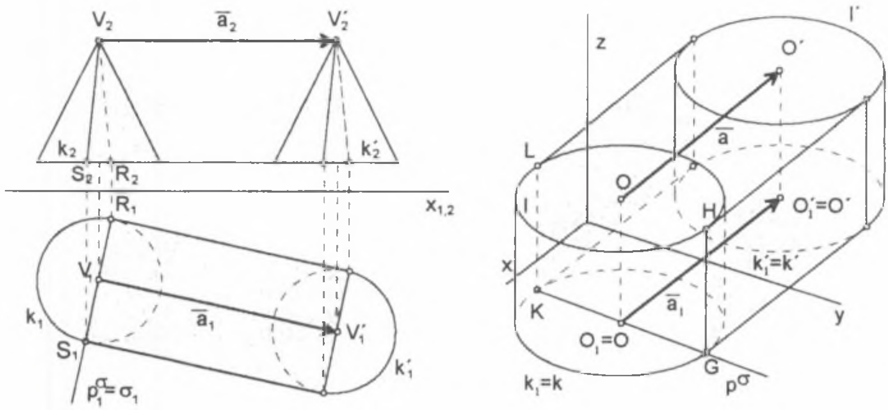

Fig.3

4. Envelope surfaces of revolution

Envelope surface created from the basic surface patch p(u,v) by the class of revolutions about the axis can be analytically represented by the vector function

$$s(u,v,w) = p(u,v) \cdot {}^1T(w) = (x(u,v), y(u,v), z(u,v), 1) \cdot {}^1T(w),$$

while the matrix function representing the class of revolutions (for the easy formulations the axis of revolutions is located into the coordinate axis z) is in a form

$${}^1T(w) = \begin{pmatrix} \cos 2\pi w & \sin 2\pi w & 0 & 0 \\ -\sin 2\pi w & \cos 2\pi w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$ with the first derivative in the form

$${}^1T'(w) = \begin{pmatrix} -2\pi \sin 2\pi w & 2\pi \cos 2\pi w & 0 & 0 \\ -2\pi \cos 2\pi w & -2\pi \sin 2\pi w & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ defined on the interval <0,1>.

Partial derivatives with respect to the variables u, v and w=0 are in a form (1) and

$$ {}^1T'(0) = \begin{pmatrix} 0 & 2\pi & 0 & 0 \\ -2\pi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{therefore} $$

$s_w(u,v,w) = p(u,v)\cdot{}^1T'(0) = (x(u,v), y(u,v), z(u,v), 1)\cdot{}^1T'(0) = (-2\pi\, y(u,v), 2\pi x(u,v), 0, 0).$

Triple scalar product of the three vectors is the value of the determinant DR

$$ DR = \begin{vmatrix} x_u(u,v) & y_u(u,v) & z_u(u,v) \\ x_v(u,v) & y_v(u,v) & z_v(u,v) \\ -2\pi y(u,v) & 2\pi x(u,v) & 0 \end{vmatrix} $$

and the curvilinear coordinates of the points on the characteristics in the basic position on the basic surface can be calculated as in the previous type of envelope surfaces. Moving a basic sphere a torus can be created in the fig. 4.

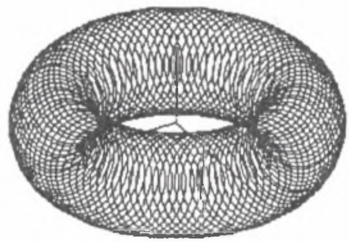

Fig.4

5. Envelope helical surfaces

Envelope surface created from the basic surface patch p(u,v) by the helical movement about the axis can be analytically represented by the vector function

$$ s(u,v,w) = p(u,v)\cdot{}^1T(w) = (x(u,v), y(u,v), z(u,v), 1)\cdot{}^1T(w), $$

while the matrix function representing the helical movement and its derivative (for the easy formulations the axis of cylindrical helical movement is located into the axis z) are in forms

$$ {}^1T(w) = \begin{pmatrix} \cos 2\pi w & \sin 2\pi w & 0 & 0 \\ -\sin 2\pi w & \cos 2\pi w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & aw & 1 \end{pmatrix}, \qquad {}^1T'(w) = \begin{pmatrix} -2\pi\sin 2\pi w & 2\pi\cos 2\pi w & 0 & 0 \\ -2\pi\cos 2\pi w & -2\pi\sin 2\pi w & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \end{pmatrix} $$

Partial derivative with respect to the variable w for the value w=0 equals

$$ s_w(u,v,0) = p(u,v)\cdot{}^1T'(0) = (x(u,v), y(u,v), z(u,v), 1)\cdot{}^1T'(0)= $$

$$ (x(u,v), y(u,v)\ z(u,v), 1)\cdot \begin{pmatrix} 0 & 2\pi & 0 & 0 \\ -2\pi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \end{pmatrix} =(-2\pi y(u,v), 2\pi\, x(u,v), a, 0). $$

The triple scalar product of the tangent vectors to the isoparametric curve segments can be calculated as the value of the determinant DH, solutions of the equation DH = 0 in the form of the ordered pairs are curvilinear coordinates (u,v) of the characteristics points.

$$DH = \begin{vmatrix} x_u(u,v) & y_u(u,v) & z_u(u,v) \\ x_v(u,v) & y_v(u,v) & z_v(u,v) \\ -2\pi y(u,v) & 2\pi x(u,v) & a \end{vmatrix} = 0$$

In the fig.5 there are illustrations of the envelope helical surfaces created from the spherical surface by cylindrical and conical helical movement.

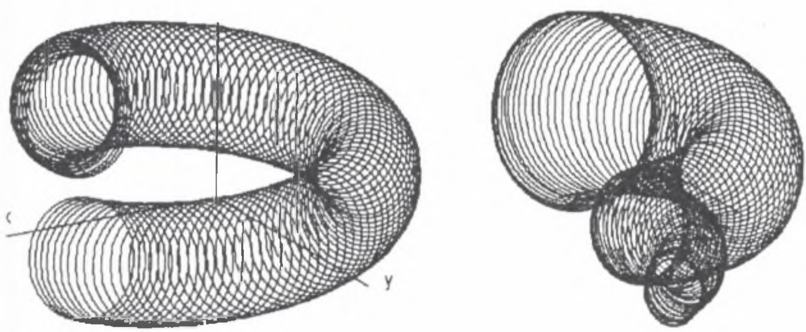

Fig.5

8. Conclusions

Homogeneous coordinates of the points on the characteristics of the basic surface determining the envelope surface by the „movement in the space" can be calculated on computers on the base of an easy algorithm. For simple basic surfaces the characteristics can be determined by the classical geometric construction in some of the projection method, but for more complex basic surfaces this problem must be solved with the help of the Computer Graphics. There are simple programmes available (in the QBasic) in which the basic surface can be created and subdued to the „movement in the space", while the characteristics can be calculated and the envelope surface created on the base of the presented algorithms.

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Streszczenie

W geometrii różniczkowej powierzchnię obwiednią definiuje się jako obwiednię jednoparametrowych rodzin powierzchni spełniających określone warunki. Geometria kreatywna [1], [7] oferuje inne podejście do tej definicji. Zamiast jednoparametrowych rodzin powierzchni rozważa się trójwymiarową bryłę S, będącą podzbiorem poszerzonej przestrzeni euklidesowej $_\infty E^3$.Powierzchnię obwiednią definiuje się jako

brzeg bryły S. Bryła S określona jest funkcją wektorową trzech zmiennych. Można ją również otrzymać poprzez działanie odpowiednimi przekształceniami geometrycznymi na płat powierzchniowy φ lub krzywą charakterystyczną pojmowaną jako zbiór takich punktów, w których istnieją płaszczyzny wspólnie styczne do powierzchni obwiedniej i płata φ. W pracy omówiono szczegółowo przykłady powierzchni utworzonych przez translację, obrót i ruch helikoidalny płata φ. Podano wzory wyrażające S za pomocą analitycznej reprezentacji płata i macierzy przekształceń.