KOCH Andrzej
Instytut Matematyki
Akademia Górniczo-Hutnicza, Kraków

## ON CERTAIN COLLINEAR GENERALIZATION CENTRAL-POLAR PROJECTION

Summary. A generalization of the central-polar projection described by W.A. Pieklicz [5] is presented. The generalization consists in projecting apparatus transformation by means of a perspective collineation.

## O PEWNYM KOLINEACYJNYM UOGÓLNIENIU RZUTU ŚRODKOWOBIEGUNOWEGO

Streszczenie. W pracy przedstawiono uogólnienie tzw. rzutu środkowobiegunowego, opisanego przez W.A. Pieklicza [5]. Uogólnienie polega na środkowokolineacyjnym przekształceniu aparatu projekcyjnego.

The fundamental properties of central - polar projection have been described by W.A. Pieklicz in article [5]. In space $\{P\}$ a sphere $K$ with centre $O$ and a plane $\tau$ tangent to $K$ at point $T$ are established, $\tau$ being the plane of projection (fig.1). For each point $A$ of any figure $F \in\{P\}$. where $\{P\}$ is Euclidean space with elements at infinity added, it is possible to find its central projection $A^{S}$ from the centre $O$ on the plane $\tau$.

Set of points $A_{i}^{S}(i=1,2,3, \ldots n)$ determines central projection $F^{s}$ of the figure $F$. Also, for each point $A \in F$ polar plane $\sigma$ with respect to $K$ and straight line $k_{\sigma}=$ $\sigma \cap \tau$ are found. Then points $A_{i}$ are orthogonally projected onto corresponding lines


Fig. 1 Principle of central - polar projection
$K_{\sigma \mathrm{i}}$. Thus polar projections $A_{i}^{b}$ are obtained. Set of points $A_{i}^{b}$ determines polar projection $F^{b}$ of the figure $F$. Therefore:

1. To each point $A \in\{P\}$ correspond two points $A^{s}$ and $A^{b}$ on the plane $\tau$, which are its projections (images), points $A^{s} A^{b}$ perpendicular to the line $k_{\sigma}$ (fig.2),


Fig.2. Projections of point $A$
2. To each straight line $p \in\{P\}$ correspond a straight line $p^{5}$ on the plane $\tau$ as its central projection and a circle $p^{b}$ with diameter TP as its polar projection (fig.3), point $P$ being $P=p^{\prime} \cap \tau$ and the lines $p$ and $p^{\prime}$ - polar lines [7] with regard to the sphere K .


Fig.3. Projections of line $p$

The converse statements lead to theorems that are also true [5].
The aim of this article is to present a certain projective transformation generalizing the above - described method of projection and to carry out preliminary examination of mutual relations of projecting apparatuses and projections (images) that can be obtained as the result of this transformation.

Among numerous possible projective transformations, perspective collineation $\Pi$ between spaces $\{P\}$ and $\{\hat{P}\}$ has been chosen.

Let there be a central - polar projection determined in space $\{P\}$.

Definition 1. Sphere $K$, projection plane $\tau$ tangent to $K$ at point $T$ and previously shown method of finding central $F^{s}$ and polar $F^{b}$ projections (images) of figure $F \in$ $\{P\}$ on the plane $\tau$ will be spoken of as the projecting apparatus $A$ of central - polar projection in space $\{P\}$.

Let in collineation $\Pi$ determined between spaces $\{P\}$ and $\{\hat{P}\}$ by means of centre $S$, plane $\lambda$ of collineation and a pair of homologuos points $A$ and $\hat{A}$, the projection plane $\tau$ passes through $S$ and the vanishining plane $\eta$ of space $\{P\}$ cuts the sphere $K$ in:
a) imaginary circle
b) real circle or
c) is tangent to $K$. Then collineation $\Pi$ transforms:

- the sphere $K$ - into a quadric $\hat{K}$ being: a) ellipsoid b) hyperboloid of two sheets or
d) elliptic paraboloid
- the centre O of the sphere K - into a point $\hat{O}$ being the pole of the vanishing plane $\hat{\xi}$ in space $\{\hat{\mathrm{P}}\}$ with respect to the quadric $\hat{K}$
- the plane $\tau$ of projection tangent to $K$ at point $T$ - into the plane $\hat{\tau}=\tau$ of projection tangent to the quadric $\hat{K}$ at point $\hat{T}$, the planar fields $(\tau)$ and $(\hat{\tau})$ being cobasal and in perspective collineation with centre $S$ and the axis of collineation $\mathrm{k}=\lambda \cap \tau($ fig.4).


Fig.4. Collinear transformation of sphere $K$ into quadric $\hat{K}$

Definition 2. The quadric $\hat{\mathrm{K}}$, the projection plane $\hat{\tau}$ tangent to $\hat{K}$ at point $\hat{T}$ and the method of finding central $\hat{F}^{s}$ and polar $\hat{F}^{b}$ projections (images) of the figure $\hat{F} \in\{\hat{P}\}$ on the plane $\hat{\tau}$ will be spoken of as the projecting apparatus $\hat{A}$ of central - polar projection in space $\{\hat{P}\}$. Every component of this apparatus as well as the above - mentioned method of finding $\hat{F}^{s}$ and $\hat{F}^{b}$ result from transformation $\Pi$ of homologous elements of the apparatus $A$ belonging to space $\{P\}$ and from the method of finding projections $F^{s}$ and $F^{b}$ on the plane $\tau$. Hence:

$$
\Pi(\mathrm{A})=\hat{\mathrm{A}} .
$$

We shall also say that apparatuses $A$ and $\hat{A}$ are in perspective collineation.
Let us consider the planar fields $(\tau)$ and $(\hat{\tau})$ and projections $A^{s}$ and $A^{b}$ of point $A$. Let central collineation between these fields be established by means of the centre $S$, the axis $k$ and a pair of homologous points $A$ and $\hat{A}$ (fig.5). On the strength of transformation $\Pi$ there exist three collinear points $\hat{A}^{s}, \hat{T}, \hat{A}^{b}$ corresponding to three
collinear points $A^{s}, T, A^{b}$ and a straight line $\hat{k}_{\sigma}$ corresponding to the line $k_{\sigma}$. Naturally, the following relations of perspectivity take place: $\left(A^{s}, T, A^{b}\right) \stackrel{s}{=}\left(\hat{A}^{s}, \hat{T}, \hat{A}^{b}\right)$ and $\left(\mathbf{k}_{\sigma}\right) \stackrel{s}{=}\left(\hat{K}_{\sigma}\right)$. Projections $\hat{A}^{s}$ and $\hat{A}^{b}$ are the images of point $\hat{A} \in\{\hat{P}\}$ which is regarded as $\Pi(A)=\hat{A}$ and $A \neq \hat{A}$ on the whole.


Fig.5. Collinear transformation of projections of point $A$
Similarly, let us consider projections $p^{s}$ and $p^{b}$ of a line $p$. Central collineation between fields $(\tau)$ and $(\hat{\tau})$ is established as before and point $A \in p$ (fig.6). Through the projecting apparatus $A$ central $p^{5}$. and polar $p^{b}$ projections in the plane $\tau$ correspond to the line $p$. Projection $p^{b}$ is a circle which is the product of two projective pencils $(T)$ and $(P)$. In the field $(\hat{\tau})$ to these projections correspond projections $\hat{p}^{s}$ and $\hat{p}^{b}$ of line $\hat{p} \in\{\hat{P}\}$ which is regarded as $\Pi(p)=\hat{p}$ and $p \neq \hat{p}$ on the whole. The ranges $\left(p^{s}\right)$ and ( $\hat{p}^{s}$ ) are in perspective correspondence and so are the circle $p^{b}$ and the conic $\hat{p}^{b}$, the latter being the product of projective pencils ( $\hat{T}$ ) and where $(\hat{T})^{k}=(T)$ and $(\hat{P})^{k}(P)$. Naturally, the type of the conic $\hat{p}^{b}$ depends on position of circle $p^{b}$ with respect to the vanishing line $g=\eta \cap \tau$ of the field $(\tau)$.


Fig.6. Collinear transformation of projections of line $p$

Let us consider the following objects of space $\{P\}$ : $F, A$, and projections $F^{s}$ and $F^{b}$ on the plane $\tau$. On the strength of transformation $\Pi$ the objects $\hat{F}, \hat{A}$ and projections $\hat{F}^{\mathrm{s}}$ and $\hat{F}^{\mathrm{b}}$ correspond to them in space $(\hat{P})$.

We shall, in general way, examine the restitution problem in projecting apparatu$\operatorname{ses} A$ and $\hat{A}$.

As far as the apparatus $A$ is concerned the problem is related to reciprocally univocal character of the representation, which is expressed by means of the truth of statements 1 and 2 (and their reciprocals) quotated at the beginning. Hence, if we take on $\tau$ three collinear points $A^{5}, T$ and $A^{b}$ and if point $A \in F$ in space $\{P\}$ is to be found, we can apply one of the two following procedures [5]:

1. Let $k_{\sigma} \perp A^{s} A^{b}$ pass through $A^{b}$ and $k_{\sigma}$ be the axis of a pencil of planes (it is easy to notice that one of them must be the polar plane $\sigma$ of point $A$ with respect to sphere $K$ ). The poles of the planes of pencil $\left(k_{\sigma}\right)$ belong to $k_{\sigma}^{\prime}$, lines $k_{\sigma}$ and $k_{\sigma}{ }_{\sigma}$ being the polar lines. The projection plane $\tau$ is one of the elements of the pencil $\left(k_{\sigma}\right)$ and has its pole at the tangent point $T$ of $K$ and $\tau$. Therefore, the line $K_{\sigma}$ passes through $T$ and is perpendicular (skew, in general) to the line $k_{0}$, so it belongs to the plane $A^{s} O A^{b} \perp k_{\sigma}$ and intersects line $A^{s} O$ at the required point $A$.
II. The other method consists on determining the polar plane $\sigma^{c}$ of point $A^{b}$ and finding the point $A$ as $A=A^{5} O \cap \sigma^{c}$. It is possible to do so because point $A^{b}$ belongs to the polar plane $\sigma$ of point $A$, which means that $A$ belongs to the polar $\sigma^{*}$ of point $A^{b}$ and, naturally, to line $A^{s} O$ at the same time.

Thus, through the projecting apparatus $A$ restitution of the figure $F$ composed of points $A_{i}$ can be executed by means of one of the above-described procedures.

As it has already been said (def.2) $\hat{A}$ is the product of central collinear transformation of the apparatus $A$. It means that the reciprocal relation $\Pi(\hat{A})=A$ is also true.

However, the apparatus A is more general and "burdened with defect" of collinear dependence on $A$, which among other things, affects metric properties of the latter. It mainly concerns the way of finding polar projections and, in particular, the set of projecting rays $A_{i} A_{i}^{b}$ where each element is perpendicular to corresponding line $k=$ $\sigma_{i} \cap \tau$. Consequently, it is just the orthogonal projection that enables us to find point $A \in F$ in space $\{P\}($ method $I)$ if $A^{b}$ and $A^{s}$ are given.

It is evident that on the same assumption we cannot find point $\hat{A} \in \hat{F}$ of space $\{\dot{P}\}$ through the apparatus $\hat{A}$.

Method II shows a general principle of restitution and, in case of the appratus $\hat{A}$, can be used only if we have precise data concerning metric properties of this apparatus. But, it is possible to pinpoint the figure $\hat{F} \in\{\hat{P}\}$ by application of transformation $\Pi$. Such operation can be written down in the following symbolic form:

$$
\hat{A} \stackrel{\Pi}{\Rightarrow} A .
$$

In the above-shown procedure the essential point is to determine such central collineation $\Pi$ between spaces $\{P\}$ and $\{\hat{P}\}$ that would transform the quadric $\hat{K} \in\{\hat{P}\}$ into the sphere $K \in\{P\}$ (there exist two such collineations). At the same time the following conditions should be satisfied [4]:

- the curvilinear quadric $\hat{K}$ must be cut in imaginary circle by the vanishing plane $\hat{\xi}$ of space $\{\hat{P}\}$
- the straight line which belongs to the so called polar diameter of the quadric $\hat{K}$ (i.e. the diameter to which the pole of the vanishing plane $\hat{\xi}$ belongs) must be incident with the centre of the polar system uniquely determined on $\hat{\xi}$ by this quadric.
- the straight line belonging to the polar diameter of the sphere $K$ must be perpendicular to the plane $\lambda$ of collineation.

Naturally, collineation $\Pi$ transforms the field $(\hat{\tau})$ into $(\tau) \in\{P\}$ and conics $\hat{p}_{i}^{b}$ into circles $p_{i}^{b}$ of the field $(\tau)[3]$.

Let us consider the following chain of operations:

$$
\hat{F} \Rightarrow\left(\hat{F}^{s}, \hat{F}^{b}\right) \stackrel{\Pi}{\Rightarrow}\left(F^{s}, F^{D}\right) \stackrel{A}{\Rightarrow} F \stackrel{\Pi}{\Rightarrow} \hat{F}
$$

The above shows the procedure in result of which it is possible to find the figure $\hat{F} \in\{\hat{P}\}$ if its projections $\hat{F}^{s}$ and $\hat{F}^{\mathrm{b}}$ are known.

In practice, the problem is reduced to finding projections $F^{s}$ and $F^{b}$ if $\hat{F}^{s}$ and $\hat{F}^{b}$ are given, another words to fixing such central collineation between the fields ( $\hat{\tau}$ ) and $(\tau)$ that would transform conics $\hat{p}_{i}^{b}$ into circles $p_{i}^{b}$ (an example is shown in fig.7).


Fig.7. Collinear transformations of projections of line $\hat{\mathrm{p}}$

In the field ( $\hat{\tau}$ ) take a set of conics $\hat{p}_{i}^{b}$ representing the polar projections of lines $\hat{p}_{i}$ of space $\{\hat{P}\}$. Collineation $\Pi$ will transform the conics into the set $p_{i}^{b}$ of circles of the field $(\tau)$ if [3]:

- each conic of the set $\hat{p}_{i}^{b}$ does not have any real points of contact with the vanishing line $\hat{\mathrm{g}}=\hat{\xi} \cap(\hat{\tau})$
- each conic of the set $\hat{p}_{i}^{b}$ belongs to the established elliptic involution on the line $\hat{\mathrm{g}}$.

Let figure $\hat{F}$ be in the form of a straight line $\hat{p}$, the images $\hat{p}^{s}$ and $\hat{p}^{b}$ of which are given (fig.7).

Conic $\hat{\mathrm{p}}^{\mathrm{b}}$ generates on the vanishing line $\hat{\mathrm{g}}$ elliptic involution of conjugate points $\left(\hat{X} \bar{X}_{1}, \hat{i} \hat{H}_{1}^{\infty}\right)$ for this conic and its polar diameter (i.e. the diameter to which the pole of the vanishing line $\hat{g}$ belongs) passes through the centre $\hat{l}$ of the involution, which is, as we know [3], one of the necessary conditions to transform $\hat{\mathrm{p}}^{\mathrm{b}}$ into circle $\mathrm{p}^{\mathrm{b}}$. The so called orthogonal collineation [3] between the fields ( $\tau$ ) and $(\tau)$ is determined by means of the following elements: centre $S$ which is one of the Laguerre points, an arbitrary chosen axis $k$ of collineation but parallel to $g$ and a pair $\hat{g}$ and $g_{1}{ }^{\infty}$ of corresponding lines. To chords $\hat{T} \hat{P}$ and $\hat{A}^{b} \hat{B}^{b}$ correspond in the field $(\tau)$ two diameters of the circle $\mathrm{p}^{\mathrm{b}}$.

Determination of projections $p^{s}$ and $p^{b}$ is equivalent to establishing the remaining elements of the projecting apparatus $A$ and, consequently, to the restitution of figure $\hat{F} \equiv \hat{p}$ in space $\{\hat{P}\}$.

## REFERENCES

1. CZETWIERUCHIN N.A.: Projektiwnaja gieometria. Moskwa 1969
2. FILON L.N.G.: An Introduction to Projective Geometry. London 1947
3. KOCH A., SULIMA SAMUJŁŁO T.: Przekształcenie kolineacyjne stożkowej w okrag. Geometria WykreśIna i Grafika Inżynierska. Z. 2, Wroclaw 1996
4. KOCH A., SULIMA SAMUJヒŁO T.: Przekształcenie kwadryk w sfery za pomoca kolineacji ortogonalnej. Geometria Wykreślna i Grafika Inżynierska, Wrocław (paper in printing)
5. PIEKLICZ W.A.: Sojedinienje centralnowo i polarnowo projektirowania. Woprosy Geometriczeskowo Modelirowania, Sbornik Naucznych Trudow No.71. Leningrad 1972
6. PLAMITZER A.: Elementy geometrii rzutowej. Lwów 1927
7. PLAMITZER A.: Geometria rzutowa układów płaskich i powierzchni stopnia drugiego. Warszawa 1938

Recenzent: Dr hab.inż. Ludmiła Czech
Profesor Politechniki Częstochowskiej

## Streszczenie

Celem niniejszego artykułu jest przedstawienie metody uogólniajacej odwzorowanie środkowo-biegunowe opisane przez W.A. Pieklicza w pracy [5]. Aparat projekcyjny A tego odwzorowania można opisać w następujący sposób: w przestrzeni $P$ przyjmuje się sferę $K$ o środku $O$ oraz płaszczyznę ז styczną do $K$ w punkcie $T$. Płaszczyzna $\tau$ stanowi rzutnię. Dla każdego punktu A znajduje się jego rzut środkowy $A^{s}$ ze środka O. Następnie wyznacza się dla tego punktu jego płaszczyznę biegunową $\sigma$ względem $K$ oraz prostą $k_{\sigma}=\sigma \cap \tau$. Punkt $A$ rzutuje się prostokątnie na $\mathrm{k}_{\text {o }}$, otrzymując jego rzut biegunowy $A^{\text {b }}$. Zatem każdemu punktowi $A$ odpowiadaja dwa rzuty (obrazy) $A^{s}$ i $A^{b}$ wspólliniowe z punktem $T$, przy czym prosta $A^{s} A^{b}$ jest prostopadła do $k_{\sigma}$. Każdej prostej $p$ odpowiadają rzuty środkowy $p^{s}$ i biegunowy $p^{b}$, przy czym ten ostatni jest okręgiem o średnicy TP, gdzie $P=p^{\prime} \cap \tau$, a proste p i p' sq wzajemnie biegunowe względem sfery K.

Kolineacja środkowa \I została wybrana w celu uogólnienia powyżej opisanego odwzorowania.

Kolineacja П określona między przestrzeniami $\{P\}$ i $\{\hat{P}\}$ przekształca aparat A w aparat $\hat{A}$, a w szczególności sferę $K$ w krzywoliniową kwadrykę $\hat{K}$ i w konsekwencji rzuty $p^{s} i p^{b}$ prostej $p$ odpowiednio w prostą $\hat{p}^{s} i$ stożkowa $\hat{p}^{b}$.

Rozważono także problem restytucji zarówno dla $A$, jak i dla $\hat{A}$ oraz przekształcenie kolineacyjne odwrotne, przeprowadzajace aparat $\hat{A} w A$, a mianowicie kwadrykę $\hat{K}$ w sferę K, a rzuty na ( $\hat{\tau}$ ) w obrazy na ( $\tau$ ). Podano warunki, które muszą być spełnione, aby przekształcić kwadrykę $\hat{K}$ oraz zbiór stożkowych $\hat{p}_{i}^{b}$ odpowiednio w sferę K izbiór okręgów pi.

Na koniec pokazano przykład przekształcenia rzutów $\hat{p}^{\text {s } i} \hat{p}^{b}$ za pomocą tak zwanej kolineacji ortogonalnej.

