ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1998
Seria: GEOMETRIA I GRAFIKA INŻYNIERSKA z.2 Nr kol. 1383

SULIMA SAMUJŁŁO Tomasz
Instytut Matematyki
Akademia Górniczo-Hutnicza, Kraków

ANALYSIS OF FUNDAMENTAL GEOMETRICAL RELATIONS IN SELECTED DOUBLE-IMAGE PROJECTIONS

Summary. The purpose of the article is the analysis of two typical double-image methods of projection: a representative of quadratic transformations, namely the composition of orthogonal and circular projections and the composition of orthogonal and central projections.

ANALIZA PODSTAWOWYCH RELACJI GEOMETRYCZNYCH W DWÓCH WYBRANYCH DWURZUTOWYCH ODWZOROWANIACH

Streszczenie. W pracy przedstawiono porównanie niektórych podstawowych relacji geometrycznych w odwzorowaniach złożonych alternatywnie z rzutu prostokątnego i tzw. rzutu okręgowego (inwersyjnego) oraz rzutu prostokątnego i środkowego.

In the paper two known in the literature methods, typical of double-image projections, have been considered. It has been resolved to analyze an orthogonal-circular projection as the representative of transformations of the quadratic type [1], [2], being the composition of orthogonal and circular (inversive) projections as well as an orthogonal-central projection as the composition of orthogonal and central projections.

After description of main principles of the above-said projections and after showing resulting from them notations of basic elements of space and their incidence, parallelism and perpendicularity of lines and planes have also been discussed.

1. Principles of the considered projections (fig.1)

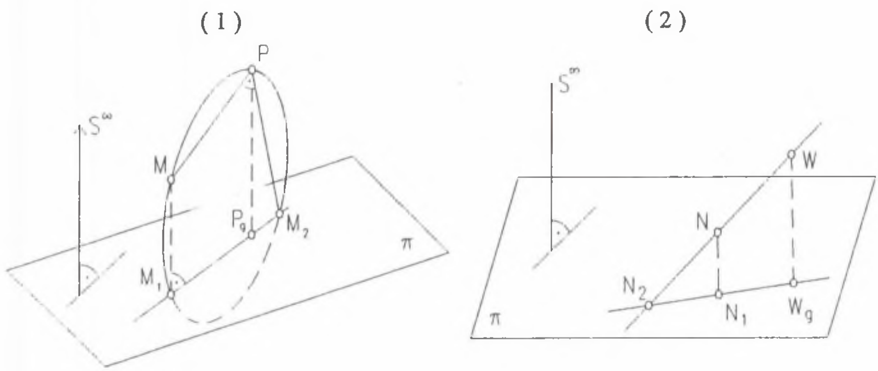
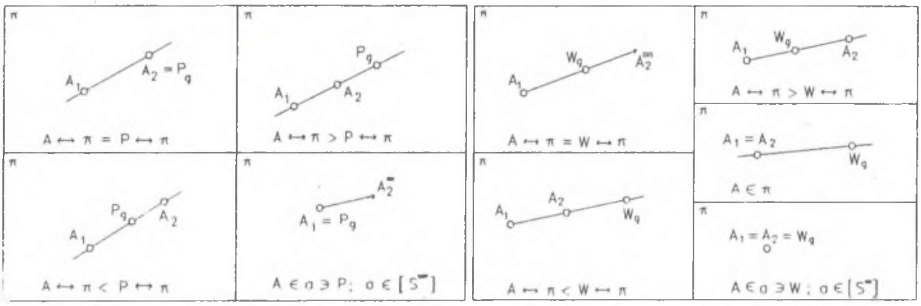


Fig.1. Projection methods: orthogonal-circular (1) and orthogonal-central (2)

For each of the projection we take any plane π of space $\{P^3\}$ as projection plane and for the orthogonal-circular method two points: $P \notin \pi$ and S^∞ as the vertex of the bundle of straight lines perpendicular to π and for orthogonal-central method also two points: $W \notin \pi$ and, as previously, point S^∞ . Any point M (N) of space $\{P^3\}$ not belonging to the line passing through point (W) and being an element of the bundle $[S^\infty]$, is represented on the projection plane π by means of its two images: M_1 (N_1) - the orthogonal projection from point S^∞, and M_2 - the circular projection (in method (1)) being the result of the intersection of the plane π and the circle passing through points P, M, M_1..., or N_2 - the central projection (in method 2)) from point W by means of the line NW belonging to the bundle [W]. Under these assumptions points M_1, M_2 and P_g and N_1, N_2 and W_g are collinear, P_g and W_g being the orthogonal projections of points P and W on π.

Particular positions of a point are shown in both projection methods in fig. 2.



Projection method (1) Projection method (2)

Fig.2. Particular positions of a point (↔distance)

Let us take such a position of a point in method (1) where A ↔ π > P ↔ π. For this position both projections A_1 and A_2 are situated on the same side of point P_g. Now determine the set of points of space $\{P^3\}$ for which projections A_1 and A_2 coincide.

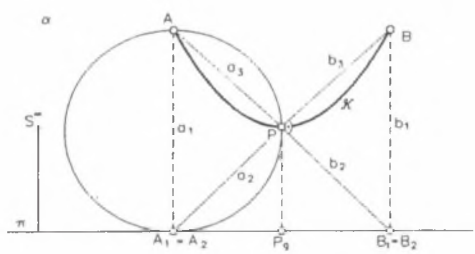


Fig.3. Invariant conic of method (1)

First take any plane α of the pencil (PP_g) (fig.3). Notice, that the circular projections A_2, B_2, ... of points A, B... can be obtained, without using the projecting circle by means of lines A_2P, (B_2P, ...) perpendicular to lines PA (PB, ...) and belonging to planes AP P_g (BP P_g, ...). Thus, we are dealing with three pencils of lines: S^∞ (a_1,b_1,...), P (a_2, b_2, ...) and P (a_3, b_3, ...). The latter two pencils having common centre P are congruent and projective. We have: S^∞ (a_1, b_1, ...) $\stackrel{=}{\wedge}$ P (a_2, b_2, ...) $\overline{\wedge}$ P (a_3, b_3,...) hence (S^∞) $\overline{\wedge}$ P (a_3, b_3, ...). The product of the latter two pencils is a conic $\mathcal{K}$ (P,A,B,...,) on the plane α. To line PS^∞ considered as an element of pencil P (a_3,

b_3, ...) corresponds in pencil (S^∞) the line at infinity of the plane α, which means that the conic $\mathcal{K}$ touches this line at one point. Hence, $\mathcal{K}$ is a parabola. The abovepresented reasoning can be generalized and applied to the rest of the planes of the pencil (PP_g). On this ground we come to conclusion that the sought set of points whose orthogonal and circular projections coincide (another words the invariant of the described projection) is a paraboloid of revolution with PS^∞ as the axis and P as the vertex.

2. Representation of straight line

a) Projection method (1)

Circular projection of a line l of space $\{P^3\}$ in general position with respect to the projection plane π, not belonging to point P and oblique towards the line passing through point P and being an element of the bundle $[S^\infty]$, is a circle l_2 through point P_g [1]. This circle (fig.4) is the intersection of the plane π and the conicoid S^2 of 2nd, order, which is the product of two projective pencils of mutually perpendicular planes with axes PP_g and PL, where $PL \perp \alpha$ (l,P).

Fig.4 shows an example of two pairs of corresponding planes of the pencils: PP_gC_2 - PLC_2, PP_gD_2 - PLD_2, their edges C_2P, D_2P being generating lines of the surface S^2.

Hence, the representation of the line l in the plane π is line l_1 as orthogonal projection and a circle l_2 as circular projection, the line l_1 not belonging to point P_g intersects circle l_2 at two real and distinct points or at two real but double points or at two conjugate imaginary points. It depends on the position of the line l towards the invariant paraboloid. Among the points of the line l (fig.4) we can distinguish the point at infinity and point A, the distance between the latter and the projection plane π being equal to the distance between point P and π.

Particular positions of a line are shown in figure 5.

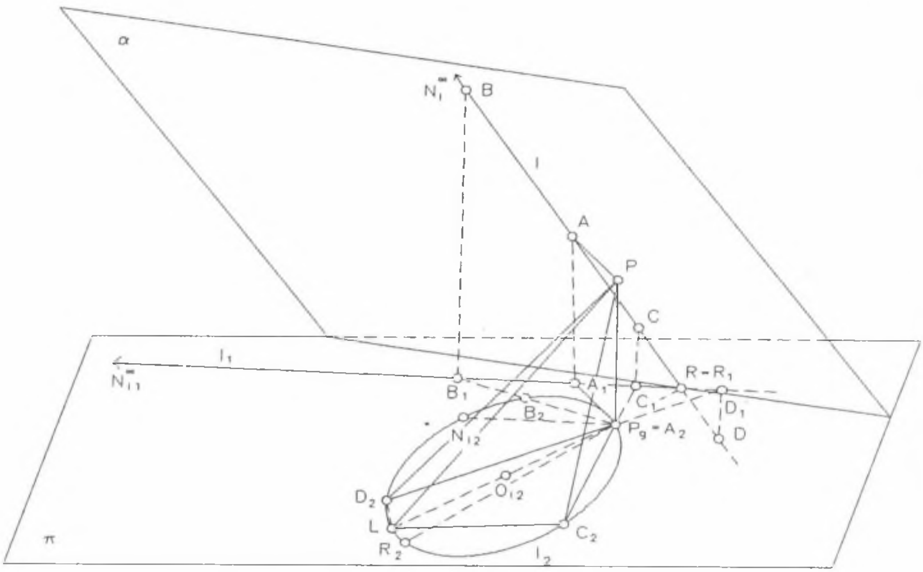


Fig.4. Line in general position in projection method (1)

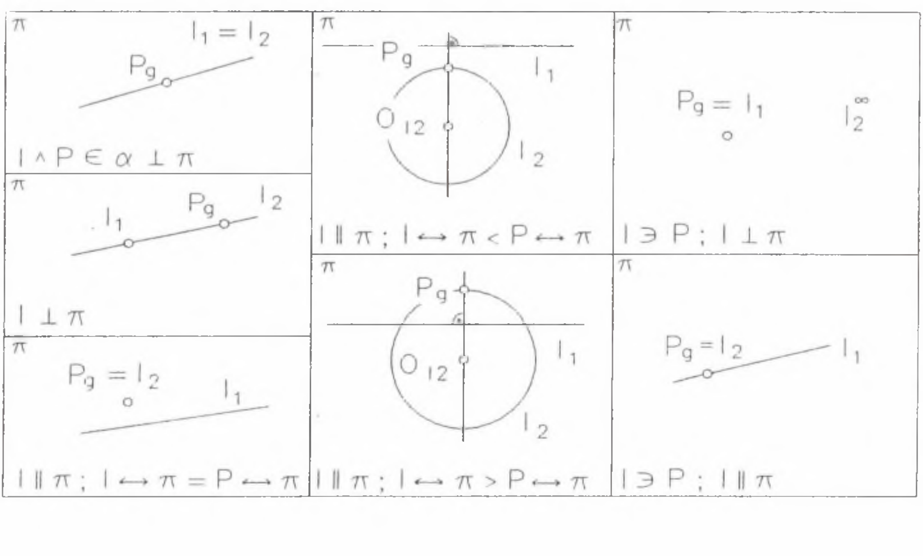


Fig.5. Particular positions of a line in projection method (1)

b) Projection method (2)

Line m in general position with respect to the projection plane π, not belonging to point W and not lying with it on a plane perpendicular to the projection plane, is represented in the plane π by means of its two images (fig.6): line m_1 being orthogo-

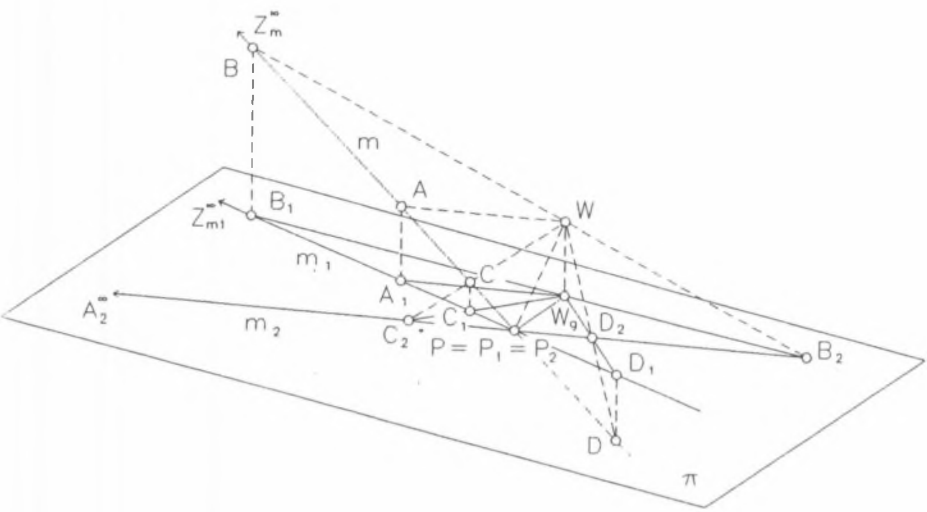


Fig.6. Line in general position in projection method (2)

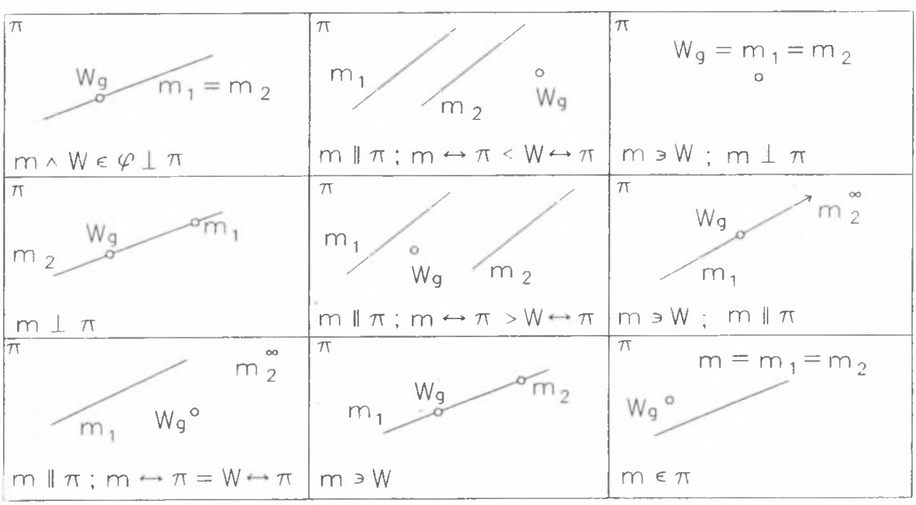


Fig.7. Particular positions of line in projection method (2)

nal projection from point S^{∞}, and line m_2 as central projection from point W. These lines intersect at a point belonging to the line m and the projection plane π. The point Z^{∞}_{m} at infinity of the line m is represented in the plane π as the point Z^{∞}_{m1} at infinity of line m_1 and as point Z^{∞}_{m2} of line m_2. Point A of the line m, the distance of which from plane π is equal to distance from point W to π, is represented by its projections A_1 and by A^{∞}_2 as the point at infinity of line m_2.

Particular positions of a line are shown in figure 7.

3. Representation of planar field

A planar field in general position with respect to the projection plane π and not passing through point P (W), will represent belonging to it perspective pencils of lines and ranges of points.

a) Projection method (1)

Let in space $\{P^3\}$ be a planar field α [E,F,G,...e,f,g,...] in general position towards the plane π and not passing through point P. Let us take two forms of the field (fig.8): $S(a,b,c,...) \overline{\overline{\wedge}}\ l^{\infty} (N^{\infty}_a, N^{\infty}_b, N^{\infty}_c ...)$, l^{∞} being the base of the latter and the line at infinity of the plane α. To these forms on the plane π correspond as orthogonal projections: $S_1 (a_1, b_1, c_1, ...) \overline{\overline{\wedge}}\ l^{\infty}_1(N^{\infty}_{a1}, N^{\infty}_{b1}, N^{\infty}_{c1} ...)$, l^{∞}_1 being the line at infinity of the plane π and as circular projections: pencil of circles P_gS_2 $(a_2, b_2, c_2, ...)$ and circle l_2 as the representation of the line at infinity of plane α, l_2 passing through points P_g, N_{a2}, N_{a2}, N_{b2}, N_{c2}, ...

Let us consider the following forms: $S^{\infty} (a,b,c, ...) \overline{\overline{\wedge}}\ 1(A,B,C,...)$. To the pencil (S^{∞}) corresponds pencil S^{∞}_1 $(a_1, b_1, c_1, ...)$ as orthogonal projection and to the range (1) - range 1_1 $(A_1, B_1, C_1, ...)$, and as circular projection - pencil of circles P_gS_2 $(a_2,b_2,c_2,...)$ and circle 1_2 passing through points P_g, A_2, B_2, C_2, ...

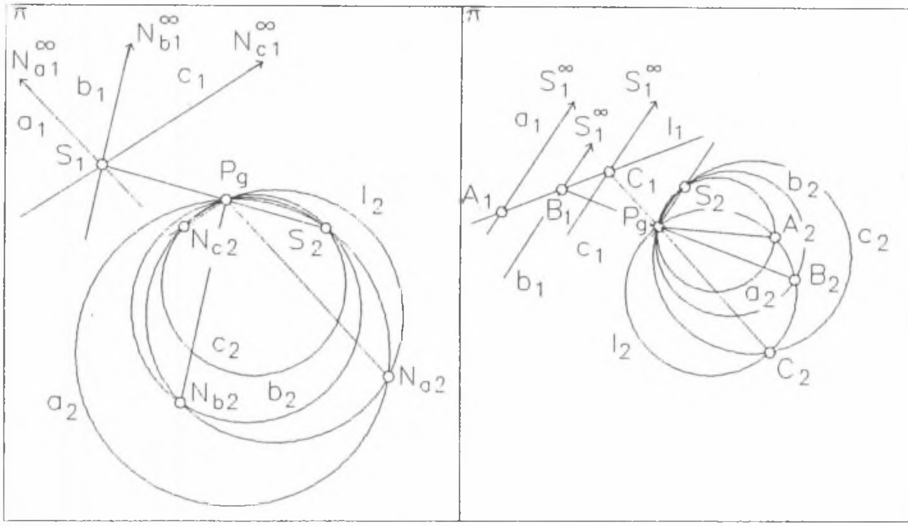


Fig.8. Planar field in projection method (1)

Particular positions of planar field are shown in figure 9.

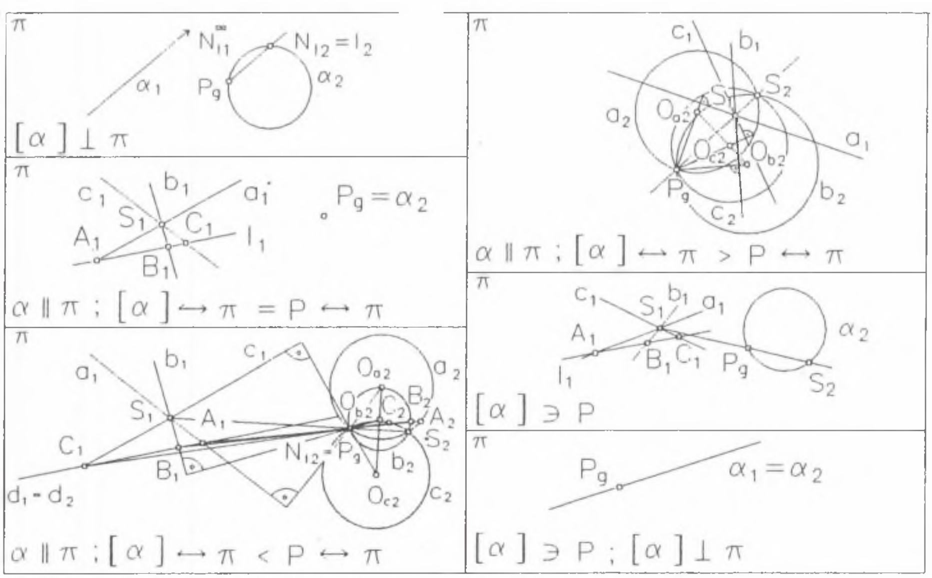


Fig.9. Particular positions of planar field in projection method (1)

b) Projection method (2)

Now let in space $\{P^3\}$ a planar field (φ) be considered, the base of which - plane φ, is in general position with respect to the projection plane π and does not pass through point W. Let us take two forms of this field (fig.10): R (k, l, m, ...) $\stackrel{=}{\wedge}$ z^∞ (Z^∞_k, Z^∞_l, Z^∞_m ...), z^∞ being the base of the latter and the line at infinity of plane φ at the

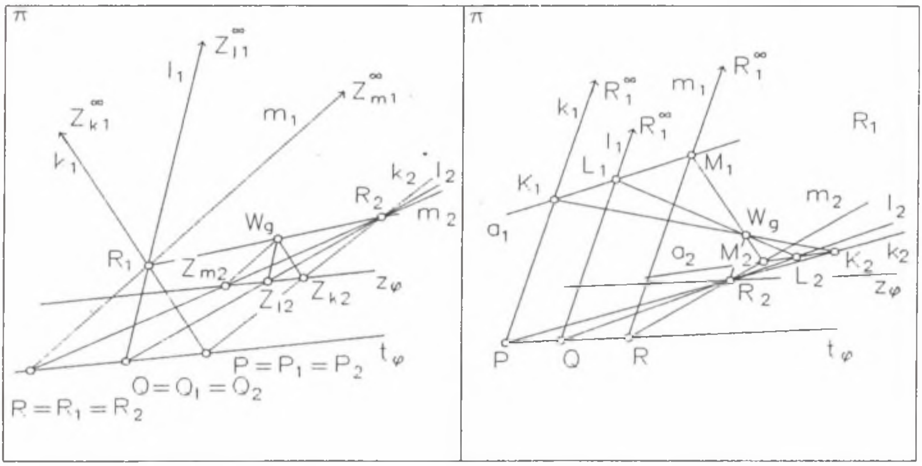


Fig.10. Planar field in projection method (2)

same time. To these forms correspond on the plane π: as orthogonal projection R_1 (k_1, l_1, m_1, ...) $\stackrel{=}{\wedge}$ z^∞_1 (Z^∞_{k1}, Z^∞_{l1}, Z^∞_{m1} ...), z^∞_1 being the line at infinity of the plane π, and, as central projection R_2 (k_2, l_2, m_2, ...) $\stackrel{=}{\wedge}$ Z_φ (Z_{k2}, Z_{l2}, Z_{m2}, ...). The pencils (R_1) and (R_2) are perspective with each other. The line t_φ is both the axis of perspectivity and the trace of plane φ which is the base of the field $[\varphi]$.

In like manner we can consider the following other forms of the field $[\varphi]$: R^∞ (k, l, m, ...) $\stackrel{=}{\wedge}$ r (K, L, M, ...). In the plane π correspond to them as orthogonal projection - R^∞_1 (k_1, l_1, m_1, ...) $\stackrel{=}{\wedge}$ r_1 (K_1, L_1, M_1, ...), and as central projection - pencil of lines R_2 (k_2, l_2, m_2, ...) $\stackrel{=}{\wedge}$ r_2 (K_2, L_2, M_2, ...). It is also clear that (R^∞_1) $\stackrel{=}{\wedge}$ (R_2).

Particular positions of a planar field are shown in figure 11.

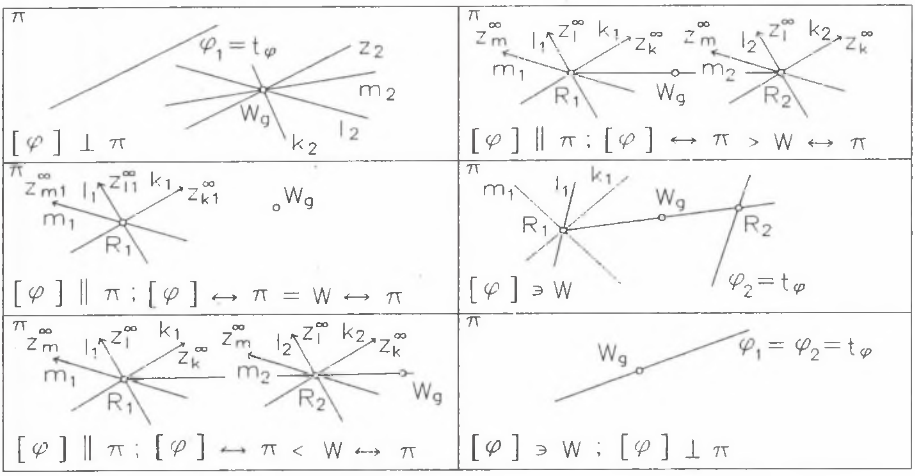


Fig.11. Particular positions of a planar field in projection method (2)

4. Perpendicularity

Next, perpendicularity between a straight line and a plane is going to be considered. It means that an arbitrary chosen corresponding pair of elements, namely: a straight line and a plane of the bundle of orthogonal lines and planes, is taken under consideration, and then after generalization, relations in two presented double - image projection methods will also be discussed.

a) Projection method (1)

Let a pencil of planes α, α_1, α_2, ... with axis n^∞ be in general position towards the projection plane π. Also, let us take a bundle of lines m, m_1, m_2, ... perpendicular to (n^∞) and passing through vertex M^∞. Any plane α of the pencil (n^∞) has been regarded as the base of pencil of lines S (a, b, c). Its line n^∞_α at infinity determined by points N^∞_a, N^∞_b, N^∞_c has also been considered. The orthogonal projection of the plane α (fig.12) is represented by a pencil of lines S_1 (a_1, b_1, c_1) and the line $n^\infty_{\alpha 1}$, whereas circular projection - by a pencil of circles P_g S_2 (a_2, b_2, c_2) and circle $n_{\alpha 2}$.

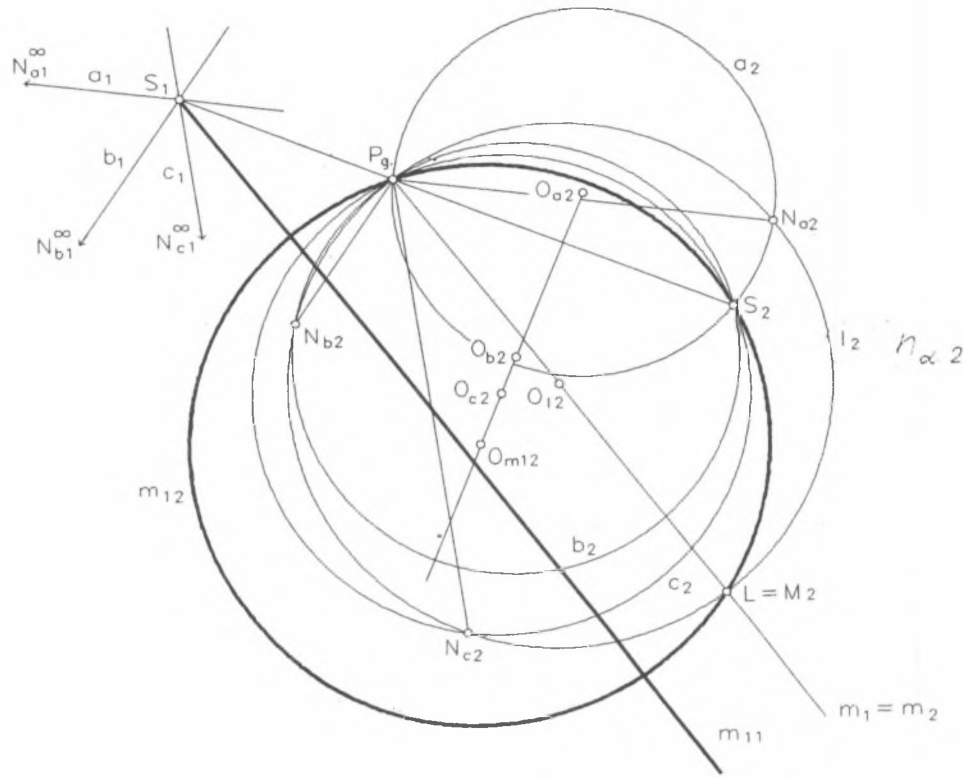


Fig.12. Perpendicularity of a line and a plane in projection method (1)

One of the lines of the bundle [m^{∞}], namely m passing through point P, has piercing point L with the plane π (fig.4). With this point coincide its orthogonal and circular projections as well as the circular projection M_2 of point M^{∞}. Therefore, the orthogonal and circular projections m_1 and m_2 of the line m coincide, too. Orthogonal projection of the bundle of lines M^{∞} (m, m_1, m_2, ...) perpendicular to the pencil of planes (n^{∞}) is pencil of lines M^{∞}_1 (m_1, m_{1l}, m_{2l}), and its circular projection (with exception of the line m) - pencil of circles P_g M_2 (m_{l2}, m_{22}, ...). In fig.12 one of the lines of the bundle [M^{∞}], namely line m_1 passing through point S of the plane α, has been shown as the result.

b) Projection method (2)

Similarly as in the previous case, we intend to find a bundle of lines Q^{∞} [q,q_1,q_2,...) perpendicular to a generally positioned pencil of planes z^{∞} (α, α_1, α_2, ...). An arbitrary taken plane α of the pencil (z^{∞}) has been established by means of a pencil of lines R (k, l, m). Its line z^{∞}_{α} defined by points Z^{∞}_{k}, Z^{∞}_{l}, Z^{∞}_{m} has also been considered. Orthogonal projection of the plane α is shown in the form of pencil of lines R_1 (k_1, l_1, m_1) and the line $z^{\infty}_{\alpha 1}$, whereas its central projection - by pencil of lines R_2 (k_2, l_2, m_2) and line $z_{\alpha 2}$. Let us find the line q of the bundle [Q^{∞}] incident with point W and piercing the plane α at point S (fig.13 a and 13b).

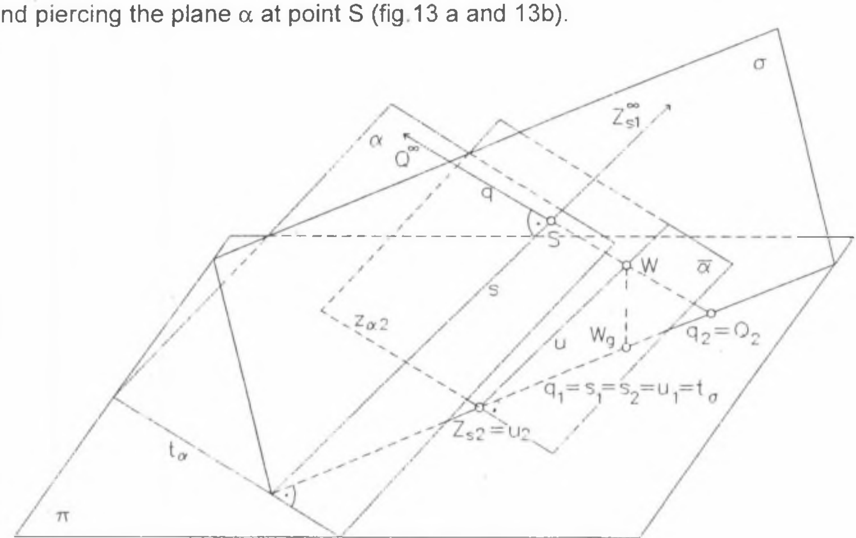


Fig.13a. Perpendicularity of a line and plane (pictorial view)

Orthogonal projection of q is line $q_1 \perp z_{\alpha 2}$ passing through point W_g and its central projection - a double point q_2 lying on the line q_1. With point q_2 coincides the central projection Q_2 of the vertex Q^{∞} of the bundle of lines [Q^{∞}]. In order to find point Q_2 (fig.13a) a plane σ perpendicular to π and incident with the line q has been shown. The edge line s between planes σ and α has also been found. The central projection z_{s2} of the point Z^{∞}_{s} at infinity of the line s is the point of intersection of the lines $z_{\alpha 2}$ and t_{σ}. Points Z_{s2} and W have been joined by the line u||s. Naturally, s, u $\perp$ q. Next,

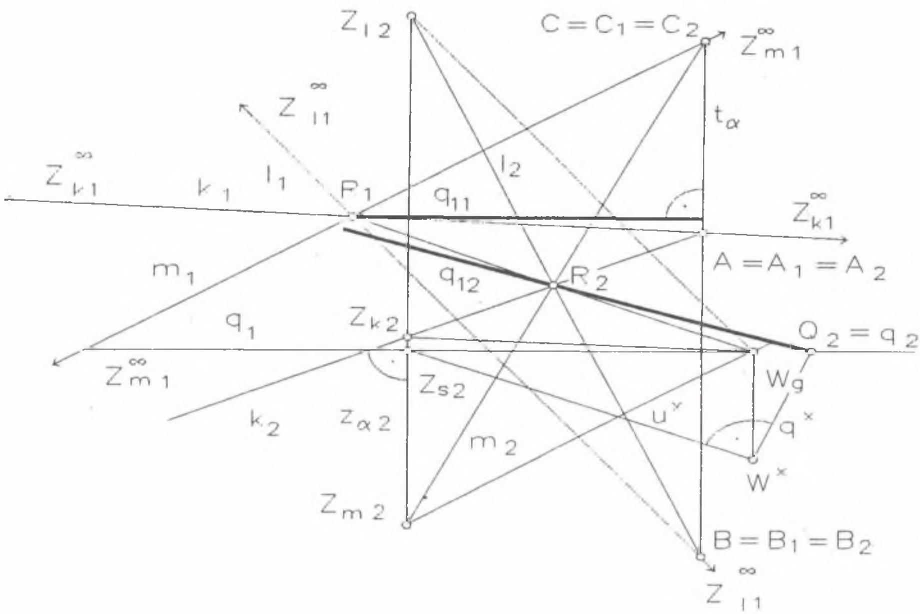


Fig.13b. Perpendicularity of a line and plane in projection method (2)

the plane σ along with the lines $u \cap q = W$ (fig.13b) has been brouhgt to coincidence with plane π by means of revolution about right angle where line t_σ is the axis of the revolution. The location of point W^x - coincident position of point W with π - determines (np signs attached) the distance between point W and the plane π. Orthogonal projection of the sought bundle of lines Q^∞ [q, q_1, q_2, ...] perpendicular to the pencil of planes z^∞ (α, α_1, α_2, ...) is a pencil of lines $Q^\infty{}_1$ (q_1, q_{11}, q_{21} ...), whereas its central projection (with exception of line q) is a pencil of lines Q_2 (q_{12}, q_{22}, q_{32} ...). In fig.13b one of the lines of the bundle [Q^∞] has been shown as the result. This is the line q_1 passing through point R of the plane α.

The relations that have been discussed so far can be used for analysis of further geometrical problems of the above - shown double - image projections. They are mainly metric problems: shortest distances, revolutions, angles and transformations of affinity. It would also be of interest to compare the already obtained results with relations occuring in other important double - image projections.

REFERENCES

1. CZECH L.: Involutory Quadratic Transformations Formulated in Constructional Geometry. Monografia PK nr 133A, Kraków 1996
2. MOROZOWA I.M., PIEKLICZ W.A.: Ortogonalnoje i krugowoje projektirowanije. Prikładnaja gieometrija i inżienierna grafika. Omsk 1972
3. PLAMITZER A.: Elementy geometrii rzutowej. Lwów 1927
4. PLAMITZER A.: Geometria rzutowa układów płaskich i powierzchni stopnia drugiego. Warszawa 1938
5. ŚLUSARCZYK B.: Geometria rzutowa i wykreślna w zakresie krzywych i powierzchni stopnia drugiego. Warszawa 1976
6. WALKOW K.I.: Lekcji po osnowam gieometriczieskowo modielirowanija. Leningrad 1970

Recenzent: Dr hab. inż. Ludmiła Czech
Profesor Politechniki Częstochowskiej

Streszczenie

Zadaniem, które postawiono sobie w pracy, jest analiza dwóch typowych odwzorowań dwuobrazowych: przedstawiciela przekształceń kwadratowych jako złożenia rzutu prostokątnego i okręgowego oraz złożenie rzutów prostokątnego i środkowego. Zatem odwzorowaniami, które wzięto pod uwagę, są: 1) ortogonalno-okręgowe i 2) ortogonalno-środkowe.

Dla każdego z przyjętych odwzorowań omówiono podstawowe założenia, a następnie rozpatrzono wszystkie możliwe położenia punktu oraz prostej względem rzutni π. Następnie przeanalizowano odwzorowanie układu płaskiego przyjmowanego poprzez różne pary jego utworów perspektywicznych oraz pokazano jego szcze-

gólne położenia. W kolejnej części pracy badano prostopadłość. W tym celu przyjęto ortogonalną wiązkę prostych i płaszczyzn oraz rozpatrzono w przyjętych odwzorowaniach dowolną odpowiadającą sobie parę elementów tej wiązki: prostą i płaszczyznę. Autor zamierza rozszerzyć swoje badania o inne ważne odwzorowania dwuobrazowe.