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Daniela VELICHOVÁ
Department of Mathematics
Mechanical Engineering Faculty, Slovak Technical University
Námestie Slobody 17, 812 31 Bratislava, SLOVAKIA
tel:+ 4217 3596 394, fax:+ 4217 325 749
email: velichov@ dekan.sjf.stuba.sk

SURFACE PATCHES MODELLING AND COMPUTATION OF GEOMETRIC PROPERTIES

Summary. Some of the problems in the classical geometric theories can be successfully solved using Computer Graphics procedures and representations of geometric figures. Geometry of the Creative space offers a new approach to several problems (in [3], [5], [6]), based on the creative representations of geometric figures. Classical way of a geometric figure definition as a subset of the three dimensional extended Euclidean space ${}_{\infty}E^3$ determined by the equation can be substituted by a creative law of the figure represented in the form of a creative representation, from which the point function determining the figure analytically can be expressed. The intrinsic geometric properties of the created figure can be calculated with respect to the Differential Geometry on the base of the partial derivatives of its related point function.

MODELOWANIE I OPIS WŁASNOŚCI GEOMETRYCZNYCH PŁATÓW POWIERZCHNI

Streszczenie. Autorka przedstawia jeden z możliwych sposobów tworzenia płatów różnego rodzaju powierzchni za pomocą metod grafiki komputerowej. Na podstawie znajomości zasad tworzenia płatów w przestrzeni wirtualnej można określić ich formę analityczną w postaci funkcji opisanej za pomocą współrzędnych jednorodnych punktu, który leżąc na powierzchni jest zdefiniowany przez parę współrzędnych krzywoliniowych. Pochodne funkcji określającej płat względem obydwu zmiennych definiują wektory stycznych jednoparametrowych łuków krzywych powierzchni, płaszczyznę styczności i wektor normalny.

W pracy szczegółowo omówiono niektóre rodzaje powierzchni powstałych z translacji, obrotu, ruchu śrubowego i homotetii krzywych. Rozważania zilustrowano przykładami powierzchni utworzonych z prostoliniowego przesunięcia hipocykloidy Steinera i sinusoidalnej translacji lemniskaty Bernoulliego, obrotu spirali Archimedesa i asteroidy oraz ruchu śrubowego walcowego cissoidy, a także ruchu śrubowego stożkowego elipsy. Przedstawiono również przykład powierzchni generowanej przez związek homotetii krzywych podstawowych.

1. INTRODUCTION

Creative space ***K*** is an ordered pair ***K*** = (***U***, ***G***) of ***U*** (a base) as a set of figures in the space (subsets of the extended Euclidean space $_{\infty}E^{3}$) and ***G*** (a generator) as a set of generating principles, ***G*** = *GP* ($_{\infty}E^{3}$) $\cup$ *L*, while *GP* ($_{\infty}E^{3}$) is a group of all projective transformations of $_{\infty}E^{3}$ and *L* is a set of interpolations (according to [4], [5]).

Any figure *F* (a non-empty subset of $_{\infty}E^{3}$) can be in ***K*** synthetically represented by its creative representation, an ordered pair (*U*, *G*) of a basic figure $U \in$ ***U*** and a generating principle $G \in$ ***G*** which are such, that applying *G* on *U* the figure *F* can be created.

The analytic representation of a figure *F* is a point function of one to four variables

$$\mathbf{r}(u,v,w,t) = (x(u,v,w,t), y(u,v,w,t), z(u,v,w,t), h(u,v,w,t))$$

defined at the region Ω (where *x, y, z* and *h* are homogeneous coordinate functions of one to four real variables at least C^1 continuous on Ω), which is a local homeomorphic mapping of the region Ω on the figure *F*. The analytic representation of a figure is the product of the multiplication of matrices, which are analytic representations of the creative representation elements (according to [4], [5]).

A point taken for a basic figure can be moved in a space forming thus a curve segment, while the generating principle - motion can be represented as a class of geometric transformations. Similarly a surface patch φ can be determined by the creative law, in which a curve segment as the basic figure is subdued to a generating principle in the form of a class of geometric transformations. A three-parametric solid cell can be created from a surface patch subduing it to a movement in the space and the movement of a solid cell forms an animation sequence, four-parametric subset of the space determined by the point function of four variables $\mathbf{r}(u,v,w,t)$, where *t* is the time coordinate.

The partial derivatives of the figure point function $\mathbf{r}(u,v,w,t)$ with respect to the variables *u, v, w, t* , vector functions $\mathbf{r}_u$, $\mathbf{r}_v$, $\mathbf{r}_w$, $\mathbf{r}_t$, that define tangent vectors to the isoparametric curve segments of the figure in the given point $(a,b,c,d) \in [0,1]^4$, and the mixed second partial derivatives $\mathbf{r}_{uv}$, $\mathbf{r}_{vw}$, $\mathbf{r}_{uw}$, $\mathbf{r}_{ut}$, $\mathbf{r}_{vt}$, $\mathbf{r}_{wt}$ that define twist vectors to the isoparametric surface patches of the figure, and the mixed third partial derivatives $\mathbf{r}_{uvw}$, $\mathbf{r}_{uwt}$, $\mathbf{r}_{uvt}$, $\mathbf{r}_{vwt}$ that define density vectors to the isoparametric solid cells, respectively the mixed fourth derivative $\mathbf{r}_{uvwt}$, determine the intrinsic geometric properties of the figure.

2. BASIC RELATIONS

Let the surface patch *S* be created from a basic figure in the form of a curve segment subdued to a generating principle that is a class of geometric transformations. Analytic representation of the basic curve segment is the point function $\mathbf{p}(u)$ of one variable defined at the unit interval $I=[0,1]$, which is a local homeomorphic mapping of the interval *I* onto the curve segment. Class of geometric transformations can be expressed analytically by a functional matrix $T(v)$, a regular square matrix of rank 4 with elements in the form of real functions of one variable defined and at least C^1 continuous at the unit interval *I*, while the determinant $|T(v)|=1$ for any $v \in I$. The basic curve segment moves (and deforms) in the space forming thus a one-parametric system of curve segments. From the surface creative representation and definition the surface point function can be obtained in the form

$$\mathbf{s}(u,v) = \mathbf{p}(u) \,.\, \mathrm{T}(v)\,,$$

and homogeneous coordinates of the surface point can be calculated from the point function for the ordered pair of point curvilinear coordinates $(a,b) \in [0,1]^2$

$$\mathbf{s}(a,b) = \mathbf{p}(a) \,.\, \mathrm{T}(b)$$

Partial derivatives of the surface point function $\mathbf{s}(u,v)$ with respect to the variables u and v that are vectors $\mathbf{s}_u$, $\mathbf{s}_v$ define tangent vectors to the isoparametric u-, and v-curve segments located on the surface patch in the given point $(a,b) \in [0,1]^2$. Tangent vectors $\mathbf{s}_u$, $\mathbf{s}_v$ determine the tangent plane τ the created surface patch and can be expressed in the form

$$\mathbf{s}_u(a,\, b) = \mathbf{p}'(a) \,.\, \mathrm{T}(b),$$

$$\mathbf{s}_v(a,\, b) = \mathbf{p}(a) \,.\, \mathrm{T}'(b)$$

Matrix $\mathrm{T}'(b)$ is the value of the first derivative of the functional matrix $\mathrm{T}(v)$ expressed for the variable $v = b$. The vector product of the two tangent vectors is the normal vector to the surface patch in the given point

$$\mathbf{n}(a,\, b) = \mathbf{s}_u(a,\, b) \times \mathbf{s}_v(a,\, b)$$

Choosing special classes of geometric transformations for generating principles we can create classical types of surfaces wellknown and used in engineering geometry and we can find analytic representations of these surfaces in the form of point functions representing the oriented surface patches. Intrinsic geometric properties can be calculated on computers on the base of the presented formulas. Created surface patches can be easily visualised in some of the available graphical systems and programmes, their views in some of the projection methods can be drawn as suitably dense nets of the isoparametric curve segments.
Illustrative figures are included for separate surface types.

3. TRANSLATION SURFACES

Let the analytical representation of the basic curve segment be the point function

$$\mathbf{p}(u) = (x(u),\ y(u),\ z(u),\ 1)$$

satisfying the required conditions on the unit interval $I=[0,1]$.

The generating principle in the form of the class of translations is represented analytically by the matrix function $\mathrm{T}(v)$ with the derivative in the matrix form $\mathrm{T}'(v)$

$$\mathrm{T}(v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a(v) & b(v) & c(v) & 1 \end{pmatrix} \qquad \mathrm{T}'(v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a'(v) & b'(v) & c'(v) & 0 \end{pmatrix}$$

of the required properties on the unit interval I. The trajectory of movement of the basic segment points is the curve segment represented by the point function

$$\mathbf{g}(v) = (a(v), b(v), c(v), 1)$$

defined on I and satisfying required conditions.

In the case of the movement on a line segment (in the direction of a vector (a,b,c)) the coordinate functions are in the simple forms $a(v)=av$, $b(v)=bv$, $c(v)=cv$, and their derivatives are constants a, b, c.

Surface patch analytic representation is a point function

$$\mathbf{s}(u,v) = \mathbf{p}(u) \,.\, \mathrm{T}(v) = (x(u), y(u), z(u), 1) \,.\, \mathrm{T}(v)$$

$$= (x(u) + a(v), y(u) + b(v), z(u) + c(v), 1) \qquad (u, v) \in [0,1]^2$$

satisfying required conditions on region $[0,1]^2$.
The partial derivatives with respect to the variables u, v are in forms

$$\mathbf{s}_{u(u,v)} = \mathbf{p}'(u) \,.\, \mathrm{T}(v) = (x'(u), y'(u), z'(u), 0) \,.\, \mathrm{T}(v) = (x'(u), y'(u), z'(u), 0)$$

$$\mathbf{s}_{v(u,v)} = \mathbf{p}(u) \,.\, \mathrm{T}'(v) = (x(u), y(u), z(u), 1) \,.\, \mathrm{T}'(v) = (a'(v), b'(v), c'(v), 0) \qquad \text{(I)}$$

Normal vector can be calculated as the vector product of the first partial derivatives, which is the value of the determinant DT

$$\mathbf{n}(u, v) = \mathbf{s}_{u(u,v)} \,.\, \mathbf{s}_{v(u,v)} = \mathrm{DT} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ x'(u) & y'(u) & z'(u) \\ a'(v) & b'(v) & c'(v) \end{vmatrix}$$

Calculations can be processed with respect to the predefined values of the two variables u, $v \in [0,1]$. In the Fig.1 there is the illustration of the translation surface created by the movement of the Lemniscate of Bernoulli on the sinusoidal trajectory and the surface created from the Steiner hypocycloid translated on the linear trajectory.

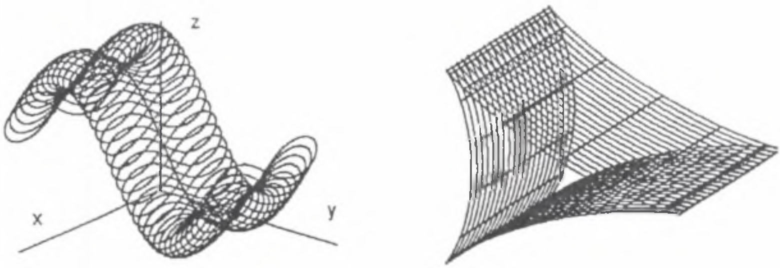

Fig.1
Rys.1

4. SURFACES OF REVOLUTION

Surface of revolution can be created from the basic curve segment analytically represented by the point function $\mathbf{p}(u)$ that revolves about an axis.
The matrix function representing the class of revolutions (for the easy formulations the axis of revolutions is located into the coordinate axis z) is a square functional matrix $T(v)$ of rank 4 in the form

$$T(v) = \begin{pmatrix} \cos 2\pi v & \sin 2\pi v & 0 & 0 \\ -\sin 2\pi v & \cos 2\pi v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

, with the first derivative in the form

$$T'(v) = \begin{pmatrix} -2\pi \sin 2\pi v & 2\pi \cos 2\pi v & 0 & 0 \\ -2\pi \cos 2\pi v & -2\pi \sin 2\pi v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

defined on the interval $[0,1]$.

Created patch of the surface of revolution can be analytically represented by the point function

$$\mathbf{s}(u, v) = \mathbf{p}(u) \,.\, T(v) = (x(u), y(u), z(u), 1) \,.\, T(v) =$$

$$= (x(u)\cos 2\pi v - y(u) \sin 2\pi v, x(u) \sin 2\pi v + y(u)\cos 2\pi v, z(u), 1) , \quad (u, v)\in [0,1]^2$$

Partial derivatives of the point function $\mathbf{s}(u, v)$ with respect to the variables u, v are in forms (I) with the corresponding matrices $T(v)$ and $T'(v)$ and the normal vector can be calculated as the value of the determinant DR,

$$\mathbf{s}_{u(u,v)} = \mathbf{p}'_{(u)} \,.\, T(v) = (x'(u), y'(u), z'(u), 0) \,.\, T(v) =$$

$$= (x'(u)\cos 2\pi v - y'(u)\sin 2\pi v, x'(u)\sin 2\pi v + y'(u)\cos 2\pi v, z'(u), 0) =$$

$$= (\xi'(u,v), \zeta'(u,v), z'(u), 0)$$

$$\mathbf{s}_{v(u,v)} = \mathbf{p}(u) \,.\, T'(v) =$$

$$= (-2\pi.(x(u)\sin 2\pi v + y(u)\cos 2\pi v), 2\pi.(-x(u)\cos 2\pi v + y(u)\sin 2\pi v), 0, 0) =$$

$$= -2\pi\, (x(u)\sin 2\pi v + y(u)\cos 2\pi v, x(u)\cos 2\pi v - y(u)\sin 2\pi v), 0, 0) =$$

$$= -2\pi\, (\xi(u,v), \zeta(u,v), 0, 0)$$

$$DR = 2\pi \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \xi'(u,v) & \zeta'(u,v) & z'(u) \\ -\xi(u,v) & \zeta(u,v) & 0 \end{vmatrix}$$

while the curvilinear coordinates of the points located on the surface can be calculated as in the previous type of surfaces.

Surface of revolution created from the Archimedean spiral by the class of revolutions about the coordinate axis y and another one generated from the Asteroid by revolving it about the coordinate axis z is presented in the Fig. 2.

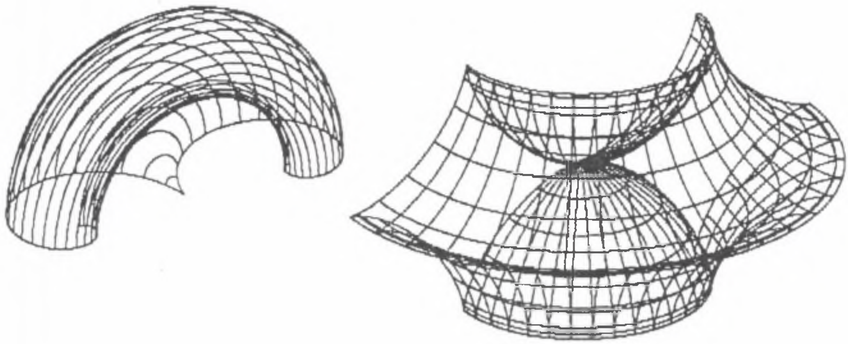

Fig.2
Rys.2

5. HELICAL SURFACES

Helical surface patch can be created from the basic curve segment determined by the point function $\mathbf{p}(u)$ by the helical movement about an axis.
The matrix function representing the helical movement and its derivative (for the easy formulations the axis of cylindrical helical movement is located into the coordinate axis z) are in form

$$T(v) = \begin{pmatrix} \cos 2\pi v & \sin 2\pi v & 0 & 0 \\ -\sin 2\pi v & \cos 2\pi v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & av & 1 \end{pmatrix}, \quad T'(v) = \begin{pmatrix} -2\pi \sin 2\pi v & 2\pi \cos 2\pi v & 0 & 0 \\ -2\pi \cos 2\pi v & -2\pi \sin 2\pi v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \end{pmatrix},$$

where $a=2\pi r$, while r is the reduced pitch defining the helical movement.
Analytic representation of the helical surface patch is the point function

$$\mathbf{s}(u, v) = \mathbf{p}(u)\,.\,T(v) = (x(u), y(u), z(u), 1)\,.\,T(v) =$$

$$= (x(u)\cos 2\pi v - y(u) \sin 2\pi v,\ x(u) \sin 2\pi v + y(u)\cos 2\pi v,\ z(u) + 2\pi r v,\ 1), \quad (u, v)\in[0,1]^2$$

Partial derivatives with respect to the variables u and v for the values from the interval I are calculated according to (I), with the corresponding matrices $T(v)$ and $T'(v)$ and the normal vector can be calculated as the value of the determinant DH

$$\mathbf{s}_u(u, v) = \mathbf{p}'(u) \,.\, \mathrm{T}(v) = (x'(u), y'(u), z'(u), 0) \,.\, \mathrm{T}(v) =$$

$$= (x'(u)\cos 2\pi v - y'(u)\sin 2\pi v,\ x'(u)\sin 2\pi v + y'(u)\cos 2\pi v,\ z'(u)\ ,\ 0) =$$

$$= (\ \xi'(u,v),\ \zeta'(u,v),\ z'(u),\ 0)$$

$$\mathbf{s}_v(u, v) = \mathbf{p}(u)\,\mathrm{T}'(v) = (-2\pi(x(u)\sin 2\pi v + y(u)\cos 2\pi v),\ 2\pi(x(u)\cos 2\pi v - y(u)\sin 2\pi v),\ a,\ 0) =$$

$$= 2\pi(-\xi(u, v),\ \zeta(u, v),\ r,\ 0)$$

$$\mathrm{DH} = 2\pi \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \xi'(u,v) & \zeta'(u,v) & z'(u) \\ -\xi(u,v) & \zeta(u,v) & r \end{vmatrix}$$

In the Fig. 3 there is an illustrations of the helical surface patch created from the basic curve segment in the form of a cissoid by a cylindrical helical movement and a patch created from the ellipse by the conical helical movement.

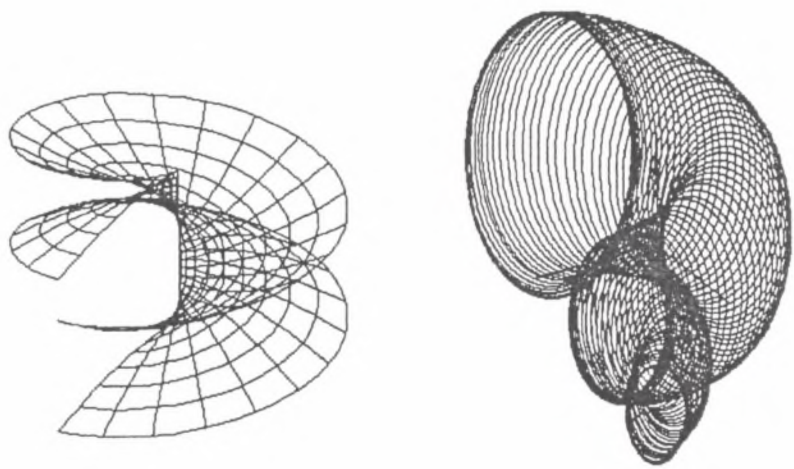

Fig. 3

Rys.3

6. HOMOTHETICAL SURFACES

Homothetical surface can be created from the basic curve segment $\mathbf{p}(u)$ by the class of homotheties (scalings) to the centre in the given point $S=(a,b,c,1)$ determined by a nonzero coefficient h.
The matrix function representing the class of homotheties and its derivative are in forms

$$T(v) = \begin{pmatrix} 1-vh & 0 & 0 & 0 \\ 0 & 1-vh & 0 & 0 \\ 0 & 0 & 1-vh & 0 \\ ahv & bhv & chv & 1 \end{pmatrix} \quad T'(v) = \begin{pmatrix} -h & 0 & 0 & 0 \\ 0 & -h & 0 & 0 \\ 0 & 0 & -h & 0 \\ ah & bh & ch & 0 \end{pmatrix}$$

Surface patch analytical representation is the point function

$$\mathbf{s}(u, v) = \mathbf{p}(u) \,.\, T(v) = (x(u), y(u), z(u), 1) \,.\, T(v) =$$

$$= ((1-vh)x(u) + ahv, (1-vh)y(u) + bhv, (1-vh)\, z(u) + chv, 1), \quad (u, v)\in[0,1]^2$$

Partial derivatives with respect to the variables u and v for the values from the interval [0,1] are calculated according to (I)

$$\mathbf{s}_u(u, v) = p'(u)\,.\, T(v) = (x'(u), y'(u), z'(u), 0) \,.\, T(v)= (1-vh)(x'(u), y'(u), z'(u), 0)$$

$$\mathbf{s}_v(u, v) = \mathbf{p}(u)\,.\, T'(v)= (x(u), y(u), z(u), 1) \,.\, T'(v)= - h(x(u)+a, y(u)+b, z(u)+c, 0)$$

The normal vector can be calculated as the value of the determinant DS.

$$DS = \begin{vmatrix} i & j & k \\ x'(u) & y'(u) & z'(u) \\ x(u)+a & y(u)+b & z(u)+c \end{vmatrix}$$

Homogeneous coordinates of the points located on the surface can be calculated on computers from the point function determining the point curvilinear coordinates.

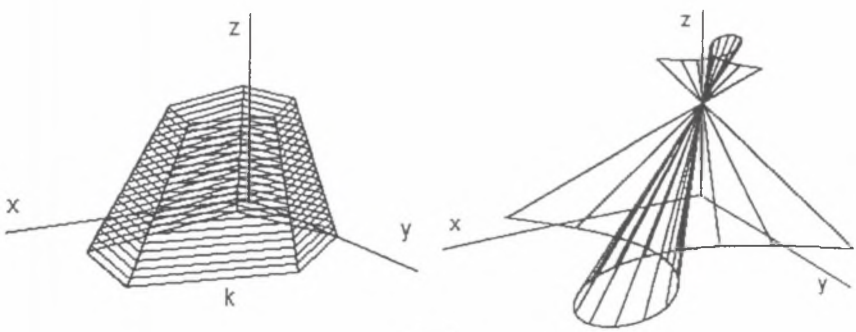

Fig. 4
Rys.4

Illustrations are in the Fig. 4, truncated pyramidal surface created from the basic pentagon subdued to the class of homotheties with the coefficient $h<1$ and a double conical surface created from the Folium of Descartes by the class of homotheties with the coefficient $h>1$, while centres of these homotheties are in the surface vertices.

REFERENCES

1. Quilin, D., Davies, B. J.: Surface Engineering Geometry for Computer-aided Design and manufacture, Ellis Horwood Limited, 1987, England
2. Szarková, D, Velichová, D.: Computation of Intrinsic Geometric Properties of Surfaces, Proceedings of the Special SEFI European Seminar on Geometry in Engineering Education, Bratislava - Smolenice 1997, Slovak republic, pp. 157 - 163
3. Velichová, D.: Learn and Play!, Proceedings of the 8th SEFI European Seminar on Mathematics in Engineering Education, Prague 1995, Czech republic, pp. 28 - 33
4. Velichová, D.: Computer Aided Geometry, Zeszyty Naukowe Politechniki 1 skiej, Geometria i Grafika Inżynierska, z.1, Gliwice 1996, Poland, pp. 79 - 87
5. Velichová, D.: A New Formalism in Teaching Geometry: Creative Geometry, Proceedings of COMPUGRAPHICS´96, GRASP , Paris 1996, France, pp. 293 - 297
6. Velichová, D.: Envelope Surfaces, Zeszyty Naukowe Politechniki Śląskiej, Geometria i Grafika Inżynierska, z.2, Gliwice 1996, Poland, pp. 79 – 87

Recenzent: Dr Edwin Koźniewski
Politechnika Białostocka

Abstract

The paper deals with some of the possible strategies for generating different kinds of surface patches using computer graphics methods. On the base of the surface patch creative law there can be derived its analytical representation in the form of a point function representing homogeneous coordinates of any point located on the surface patch and defined by a pair of the curvilinear coordinates. Derivatives of the surface patch point function with respect to both variables define tangent vectors to the surface patch isoparametric curve segments, tangent plane and a normal vector.