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RELATIONS BETWEEN TWO PLANAR FIELDS OF IMAGES IN CENTRAL–POLAR PROJECTION METHOD

Summary. In central – polar projection method [3] transformations **T** and $\mathbf{T}^{-1}$ convert respectively the field of central projections A^s_i into the field of polar projections A^b_i. In the article have been investigated transformations $*\mathbf{T} = \Pi(\mathbf{T})$ and $*\mathbf{T}^{-1} = \Pi(\mathbf{T}^{-1})$ (Π = collinear transformation) and in the latter 5 cases have been discussed.

ANALIZA NIEKTÓRYCH PRZEKSZTAŁCEŃ W ODWZOROWANIU ŚRODKOWO–BIEGUNOWYM

Streszczenie. Na wstępie przypomniano krótko zasadę odwzorowania środkowo – biegunowego O–SB oraz jego aparat projekcyjny A . Podano również definicje przekształceń T i T^{-1} opisanych w [3]. Przekształcenia te przeprowadzają odpowiednio układ rzutów środkowych A^s_i w układ rzutów A^b_i punktów A_i przynależnych do płaszczyzny α przestrzeni {P} oraz układ rzutów biegunowych A^b_i w układ rzutów A^s_i punktów A_i . Celem artykułu jest zbadanie własności przekształceń $*T = \Pi\ (T)$ i $*T^{-1} = \Pi\ (T^{-1})$ wynikających z $*A = \Pi\ (A)$, gdzie Π jest przekształceniem środkowo–kolineacyjnym [2]. Wykazano, że *T ma charakter liniowo – kwadratowy, a $*T^{-1}$ kwadratowy. W tym ostatnim przekształceniu, w zależności od położenia płaszczyzny granicznej przestrzeni {P} względem pewnego stożka S_1 stopnia 2, wyróżniono i omówiono 5 przypadków dotyczących odpowiedniości $*A^b_i \rightarrow *A_i \in *\alpha$.

Let us briefly recall the principle of Central – Polar Projection (C-PP) and its projecting apparatus **A** which have been described by W.A.Pieklicz in the article [3] . The projection can be accomplished in the following way : the apparatus **A** is determined in the form of a sphere **K** and a plane τ tangent to **K** at point T . Next , points A_i (i = 1,2,3,...n) of space { P } are projected from the centre 0 of the sphere **K** onto the plane τ of projection. Thus , central projections (images) A^s_i of the points A_i are obtained . Then for each point A its polar plane σ with respect to **K** is found as well as the line k_σ =

$\sigma \cap \tau$. The points A_i are projected orthogonally onto corresponding lines $k_{\sigma i}$ .Thus, polar projections (images) A^b_i of the points A_i are achieved .

The truth of the following statements (and converse ones) have been proved [3] :

1. To each point $A \in \{P\}$ of projection plane τ correspond two points A^s and A^b which are collinear with the point T and the line A^sA^b is perpendicular to k_σ.
2. To each line $p \in \{P\}$ of projection plane τ correspond: a line p^s as its central projection and a circle p^b with diameter TP as its polar projection, the point P being $P=p' \cap \tau$ and the lines p and p' – polar lines with regard to **K**.

In the article [2] generalization of C – PP and **A** has been described. It can be executed by means of central –collinear transformation Π determined between spaces {P} and {*P}, which can be written in the symbolic form of the following relations:

$$\Pi\,(C\text{–}PP) = {}^*C\text{–}{}^*P{}^*P \qquad \text{and} \qquad \Pi\,(\mathbf{A}) = {}^*\mathbf{A}$$

The sphere **K** and projection plane τ of the apparatus **A** are transformed into quadric ***K** and projection plane $^*\tau$ of the apparatus ***A** respectivel , the planar fields (τ) and ($^*\tau$) having common base (it has been assumed that the centre S of perspectivity belongs to τ).To the point T of contact between τ and **K** corresponds tangency point *T of $^*\tau$ and ***K** and to the centre O of the sphere **K** – point *O being the pole of the vanishing plane $^*\xi \in$ {*P} with respect to *.**K**.

The purpose of this article is to investigate what happens to the properties of the transformation **T** (as well as the converse one $\mathbf{T}^{-1}$) described for C–PP if they are conveyed onto *C –*P*P, another words to characterize the transformations $^*\mathbf{T} = \Pi\,(\mathbf{T})$ and $^*\mathbf{T}^{-1} = \Pi\,(\mathbf{T}^{-1})$.

ANALYSIS OF TRANSFORMATIONS T AND *T

Definition 1. The transformation which converts the field of projections (images) A_i^s into the field of projections (images) A_i^b of points A_i belonging to an arbitrary plane α of space {P} will be named *transformation* **T** .

Let the projecting apparatus **A** be given as well as a plane α and a point A^s moving along a line p^s of the projections plane τ.To subsequent positions A_i^s of the point A correspond projecting rays OA_i^s generating pencil (O), the base of which is projecting plane intersecting α along the line p which in turn is the the base of range of points (A_i) .

The correspondence $A_i^s \rightarrow A_i$ is central projection , hence it is linear.

The polar projections A_i^b of points $A_i \in p$ determine a circle $p^b \in \tau$ being the polar projections of the line p.

Therefore, the correspondence $A_i \rightarrow A_i^b$ is quadratic.Thus , the transformation **T** such that $A_i^s\,(\mathbf{T}) = A_i^b$, which is the product of the above –described correspondences is of linear–quadratic nature [3] .

Now let us consider the nature of the transformation ***T** . We can apply similar reasoning as the previous one i.e. we can seek the results through the apparatus ***A** = Π (A) . It is sufficient, however , to take the following relations between the fields of the pairs of projections in the plane τ [2] :

$$({}^*A_i^s) = \Pi\,(A_i^s) \qquad \text{and} \qquad ({}^*A_i^b) = \Pi\,(A_i^s)$$

From the above it can be immediately concluded that $*p^s$ is a line and $*p^b$ – a conic and that the correspondence $*A_i^s \rightarrow *A_i$ is linear (central projection in *C–*P*P) and the correspondence $*A_i \rightarrow *A_i^b$ –quadratic .

Therefore, the product of these two correspondences, another words the transformation ***T** , is also of linear –quadratic nature .

ANALYSIS OF TRANSFORMATIONS T^{-1} AND $*T^{-1}$

Definition 2. *The transformation* $\mathbf{T^{-1}}$ is such a transformation which converts the field of projections (images) A_i^b into the field of projections (images) A_i^s of points A_i belonging to an arbitrary plane α of space { P}.

Let again be given the projecting apparatus **A** , an arbitrary plane α and point A^b moving along a line p^b of the projections plane τ. It is known that in the field (A_i^b) of polar projections to each line TA^b is attached at point A^b a line k_σ perpendicular to TA^b .

It means that the set of the lines $k_{\sigma i} = \sigma_i \cap \tau$ forms a pencil (k^2) of 2[nd] order. The envelope of (k^2) is a parabola **s** with its focus at the point T and the vertex at the point of tangency with the line p^b (fig.1).

Let us consider the correspondence : $A^b_i \rightarrow A_i \in \alpha$. As it has been proved [3] this is a quadratic correspondence. It results from the following reasoning : let the polar transformation with respect to the sphere **K** be applied to the set of lines $k_{\sigma i}$ .Then a new set $k'_{\sigma i}$ is obtained, the lines $k_{\sigma i}$ and $k'_{\sigma i}$

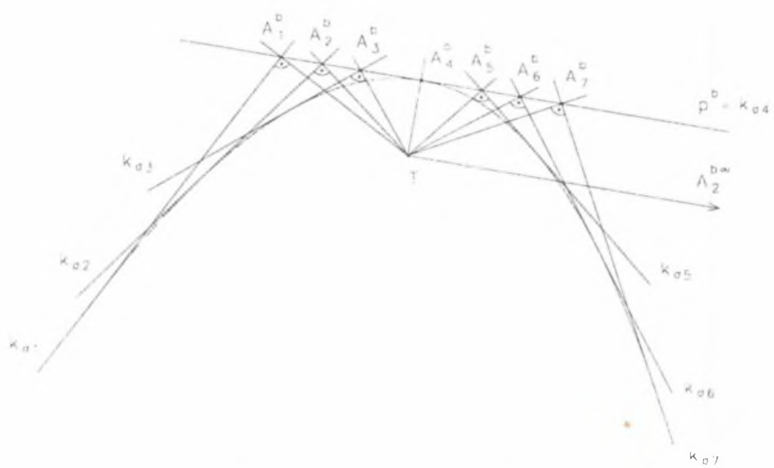

Fig.1. Parabola as the result of point A^b moving along line p^b
Rys.1. Parabola jako wynik poruszającego się punktu A^b wzdłuż prostej p^b

being polar lines. Note that the lines $k'_{\sigma i}$ pass through the point T which is the pole of the projection plane τ . Thus, the above – mentioned polar transformation converts the pencil (k^2) of 2nd order into the bundle (T), the elements $k'_{\sigma i}$ of which are the generators of a cone $\mathbf{S_1}$

of 2nd degree. In the correspondence $A^b_i \to A_i$ the conic $p = \mathbf{S_1} \cap \alpha$ is the image of the line p^b, which means that this correspondence is of quadratic nature.

Naturally, the line p^b of the plane τ is the projection (image) of the conic p.

Now let us take the correspondence $A_i \to A^s_i$. It is clear that this is central projection. The conic p and the centre O of the sphere **K** also determine a cone $\mathbf{S_2}$ of 2nd degree. In the correspondence $A_i \to A^s_i$ the conic $p^s = \mathbf{S_2} \cap \tau$ is the image of the conic p of the projection plane τ, which means that the correspondence is also quadratic.

Therefore, the transformation $\mathbf{T^{-1}}$ such that $A^b_i\,(\mathbf{T^{-1}}) = A^s_i$ being the product of the above – described correspondences, is of the quadratic nature [3].

In like manner let us consider the transformation $^*\mathbf{T^{-1}}$. Let this time, as the results of transformation Π, be given : the projecting apparatus *$\mathbf{A}$, plane $^*\alpha$ and point $^*A^b$ moving along the line $^*p^b$ belonging to the projecting plane $^*\tau$. It is clear that the above – listed objects of space {*P } correspond in perspective collineation to the objects of space { P }.

Similarly to the previous case, to each line $^*T^*A^b$ a line $^*k_\sigma$ at point $^*A^b$ is attached. This line, however, is not perpendicular to $^*T^*A^b$ because the transformation Π does not preserve the right angle. It denotes that the set of lines $^*k_{\sigma i} = {}^*\sigma_i \cap {}^*\tau$ is a pencil of 2nd order cicumscribing a conic **s** which, unlike the previous case, can be any of three types resulting from the affine classification of the conic sections. Naturally, the type of the conic ***s** depends on the position of the vanishing line g of the field (τ) with respect to parabola **s**.

Next, let us consider the correspondence $^*A^b \to {}^*A_i \in {}^*\alpha$. The polar transformation with respect to the quadric *$\mathbf{K}$ converts the set of lines $^*k_{\sigma i}$ into the set of lines $^*k'_{\sigma i}$. The latter pass through the point *T being the polar of the projection plane $^*\tau$. It means that the pencil ($^*k^2$) has been transformed into bundle (*T), the elements of which are the generators of a cone $^*\mathbf{S_1}$ of 2nd order. The intersection of this cone and the plane $^*\alpha$ is a conic *p which in the correspondence $^*A^b_i \to {}^*A_i$ is the image of the range $^*p^b$ on $^*\alpha$. Therefore, it means that this correspondence is of the quadratic nature as well.

However, the above – described general relations can in *C–*P*P assume a few particular forms because the succession of various cases may occur. Depending on the position of the vanishing plane η of space {P} towards the cone $\mathbf{S_1}$ and assumption that α is generally positioned with respect to it ,the following cases are posible :

a) The plane η intersects $\mathbf{S_1}$ in non-degenerate conic
b) The plane η intersects $\mathbf{S_1}$ in two real and different generating lines
c) The plane η intersects $\mathbf{S_1}$ in two coincident generating lines (η is tangent to $\mathbf{S_1}$)
d) The plane η intersects $\mathbf{S_1}$ at point T but is not tangent to **K**
e) The plane η intersects $\mathbf{S_1}$ at point T and is tangent to **K**.

Ad a) It is a general case .To the cone $\mathbf{S_1}$ in *C – *P*P correspond : a cone $^*\mathbf{S_1}$ with point *T as the vertex, to the conic p – a conic $^*p = \Pi(p)$ of the plane $^*\alpha$ and to the parabola **s** – a conic $^*\mathbf{s} = \Pi(\mathbf{s})$.

Ad b) .To the cone $\mathbf{S_1}$ in *C – *P*P corresponds a hyperbolic cylinder $^*\mathbf{S_1}$ whose generators $^*k'_{\sigma i}$ pass through point $^*T^\infty$, whereas to the conic p–a hyperbola *p of the plane $^*\alpha$.

Since the vanishing line g of the field (τ) passes throught point T being the focus of the parabola **s** , then this line is secant towards **s**, which means that in the field ($^*\tau$) to this parabola corresponds a hyperbola ***s**.

Ad c) . To the cone S_1 in $*C - *P*P$ corresponds a parabolic cylinder $*S_1$ with vertex $*T^{\infty}$, whereas to the conic p - a parabola $*p$.
Similarly to the previous case and on account of the same reason, to the parabola s corresponds a hyperbola $*s$ in the field $(*\tau)$.

Ad d) . To the cone S_1 in $*C - *P*P$ corresponds an elliptic cylinder $*S_1$ with vertex $*T^{\infty}$, to the conic p - an ellipse $*p$, and to the parabola s - a hyperbola $*s$.

Ad e) . In this case we have $\eta \equiv \tau$. For practical reasons (the images on the projection plane τ are transformed into elements at infinity) this case is not considered .

Lastly let us take the case where the planes η and α are parallel . Then the homologous conics p and $*p$ are centrally similar .
Finally, let us take the correspondence $*A_i \rightarrow *A^s_i$ which is central projection . The conic $*p$ and the point $*O$ of the quadric $*K$ determine a cone $*S_2$ of 2nd order (a cylinder – in particular case where $O \in \eta$) . The conic $*p^s = *S_2 \cap *\tau$ is the image of the conic $*p \in *\alpha$ and the correspondence in question is a quadratic one .

Therefore, the transformation $*T^{-1}$ such that $*A_i^b\ (*\mathbf{T}^{-1}) = *A^s_i$ and being a composition of the last – considered two correspondences, is also of the quadratic character .

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Abstract

At the beginning of the article the principle of central – polar projection (C-PP) as well as its projecting apparatus **A** have been recalled. Also definitions of transformations **T** and $\mathbf{T}^{-1}$ described in [3] have been given. The transformations convert respectively : the planar field of central projections A^s_i into the field of polar projections A^b_i of points A_i belonging to plane α of space $\{\mathbf{P}\}$ and the field of projections A^b_i into the field of projections A^s_i of points A_i . The purpose of the article is to investigate the properties of transformations $*\mathbf{T} = \Pi\ (\mathbf{T})$ and $*\mathbf{T}^{-1} = \Pi\ (\mathbf{T}^{-1})$ resulting from relation $*\mathbf{A} = \Pi\ (\mathbf{A})$, where Π is central – collinear transformation. It has been proved that $*\mathbf{T}$ is of linear quadratic nature and $*\mathbf{T}^{-1}$ is quadratic. In the latter, depending on the position of the vanishing plane of space $\{\mathbf{P}\}$ towards a certain cone of 2^{nd} order, five cases concerning correspondence $*\mathbf{A}^b_i \rightarrow *\mathbf{A}_i \in *\alpha$ have been singled out and discussed.