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RESEARCH OF "PERFECT" OBJECTS DEFINING THE PROJECTION OF 3-DIMENSIONAL SPACE TO PLANE BY PROGRAMMES IN TURBOPROLOG

Summary. The classical projection of 3-dimensional space P^3 to plane π is usually defined as the structure <R, π>, where R is a boundle of lines (projecting rays) with ideal or ordinary point as the centre of projection and π is an ordinary plane as projection surface. It is well known, that the set R may be a congruence of lines in the form $K_{(m,n)}$, for m=1, where m denotes the number of lines of congruence passing by any point of P^3 and n denotes the number of lines lying on plane ([1],[10]). The centre of classical projection is always singular.

POSZUKIWANIE „IDEALNEGO" APARATU RZUTUJĄCEGO PRZESTRZEŃ TRÓJWYMIAROWĄ NA PŁASZCZYŹNIE ZA POMOCĄ PROGRAMÓW W TURBOPROLOGU

Streszczenie. Klasyczny rzut trójwymiarowej przestrzeni rzutowej P^3 na płaszczyznę określany jest jako struktura < R,π>, gdzie R jest wiązką prostych (promieni rzutujących) z punktem właściwym lub niewłaściwym jako środkiem rzutowania i płaszczyzną właściwą π jako rzutnią. Jest dobrze znane, że R może być kongruencją postaci $K_{(m,n)}$, dla m = 1, gdzie m oznacza liczbę prostych kongruencji przechodzących przez dowolny P^3 i n oznacza liczbę prostych leżących na dowolnej płaszczyźnie ([1],[10]). Środek klasycznego rzutu jest zawsze osobliwy.

Użycie innych kongruencji, np. zbioru wszystkich prostych przecinających dwie proste skośne (kongruencja $K_{(1,1)}$ lub zbioru wszystkich bisekant krzywej skośnej 3th rzędu (kongruencja $K_{(1,3)}$ indukuje także punkty osobliwe ([1],[10]). W naturalny, geometryczny sposób kongruencje otrzymujemy jako przecięcia dwóch zdegenerowanych kompleksów i dlatego mamy punkty osobliwe. Interesującym pytaniem jest: „Czy istnieje taka kongruencja prostych, która nie indukuje punktów osobliwych w określonym przez nią rzucie". Przypuszczenie, że kongruencja określona przez dwa niezdegenerowane kompleksy nie dopuszcza punktów osobliwych, jest przedmiotem rozważań niniejszej pracy. Rozwiązanie wspomnianego wyżej problemu otrzymujemy badając skończone przestrzenie rzutowe z wykorzystaniem pakietu programów napisanych w języku TurboPROLOG ([8],[9]).

W pracy formułujemy nowe własności kompleksów i kongruencji prostych w strukturach skończonych i otrzymujemy nowe wyniki kombinatoryczne dotyczące mocy powyższych zbiorów.

1. POLARITIES AND NULL-POLARITIES

For the convenience of the reader we repeat the classic results concerning of the null-polarities. Let F be a field and V be a vector space over F. Let us consider the set of all subspaces of V. Let Sub_k (V) be the set of k-dimensional subspaces of V and let $Sub^l(V)$ be the set of l-codimensional subspaces of V. Then the projective space P=P(V) over V may treated as the structure Sub_l (V), Sub^1 (V),$\in$ > with points u, v, w,... being 1-dimensional subspaces and hyperplanes U, V, W,... being 1-codimensional subspaces of V. Then 2-dimensional subspaces are lines in the obtained projective space. For every point u $\in$ Sub_l (V) there exists x $\in$ V $-\{0\}$ such that

$$U = \{z \in V : \exists a \in F - \{0\}\ (z = ax)\}. \tag{1}$$

Therefore we can write briefly u: =Fx. We may consider the well known transformation

$$\sigma : Sub_1(V) \mapsto Sub^1(V) \tag{2}$$

defined by the condition

$$u \in \upsilon^\sigma \Leftrightarrow \upsilon \in u^\sigma \tag{3}$$

for any u, υ belonging to Sub_l (V). Such mapping is called a polarity ([3]). If p$\in$ u^σ for any u Sub_l (V) the transformation (1) is called null-polarity ([2]). For u $\in$ Sub_l (V) and $u^\sigma \in$ Sub^l (V) such that U = u^σ we say u is a pole of U and U is a polar of u. Any two points u, υ satisfying (3) are said to be conjugate; if a point is conjugate to itself, it is self-conjugate in the polarity.

Every polarity is a correlation. Any correlation may be expressed as semi-linear map λ: V $\mapsto$ V^* with respect to the isomorphism μ if

$$\lambda(x + y) = \lambda(x) + \lambda(y), \lambda(ax) = a^\mu \lambda(x), \tag{4}$$

where V^* is a dual vector space of a vector space V and μ is an automorfism of F. The equation of polar hyperplane of point Fy can be written as follows

$$\langle y^\sigma, x \rangle = 0, \tag{5}$$

where the symbol „$\langle .,. \rangle$"denotes scalar product defined on cartesian product V^* x V ([3]). Suppose now that dim(V) = n+1. Then for any polarity (null-polarity) σ there exists a symmetric (skew-symmetric) matrix

$$A = (a_{ij})_{i,j=0,1,\dots,n} \tag{6}$$

such that

$$z = Ax, \tag{7}$$

where a vector z is an element of dual space V^* of vector space V and a vector x is an element of V. A hyperplane z and point x are conjugate ([2]). Let z = Ax be the equation of polarity in an allowable coordinate system over F. Two points Fy and Fx are conjugate if $y^T Ax = 0$. If A is skew-symmetric, then any point Fx is self-conjugate, i.e. $x^T Ax = 0$.

2. LINEAR COMPLEX

It is clear, that if two points Fx, Fy are conjugate, so that

$$y^T Ax = -x^T Ay = 0, \tag{8}$$

then any two points ax+by and cx + dy on their join are also conjugate. Let $x = (x_i)_{i=0,1,2,\dots,n}$ and $y = (y_j)_{j=0,1,2,\dots,n}$ with Fx ≠ Fy and A is a skew-symmetric matrix. We also have

$$0 = y^T Ax = \sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij} y_i x_j = \sum_{i<j} a_{ij}(y_i x_j - y_j x_i) = \sum_{i<j} a_{ij} u_{ij}, \tag{9}$$

where $(\dots u_{ij} \dots)$ are the Grassmann (or Plücker) coordinates of the line joining the two conjugate points Fx, Fy ([2]). Hence a null-polarity is associated with a set of lines LC (called a linear complex whose Grassmann coordinates satisfy the relation (8). It is known, that conversely any linear complex is associated with a unique null-polarity ([2]).

3. LINEAR COMPLEX IN 3-DIMENSIONAL SPACE

3.1. Algebraic view point of linear complex

Suppose now that n =3, i.e. dim(V) = 4. Then we can write

$$u_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}, \tag{10}$$

where i,j = 0,1,...,3. We obtain six independent numbers from F:

$$u = (u_{0,1}, u_{0,2}, u_{0,3}, u_{23}, u_{31}, u_{12}), \tag{11}$$

i.e. a vector of space $\wedge^2 V$, where $\wedge^k V$}is the inner k-product of V. Let u: = Fu. Then u is an element of $Sub_1(\wedge^k V)$, i.e. point of 5-dimensional projective space $P(\wedge^2 V)$ over $\wedge^2 V$ ([11]). We introduce the following notation: $u_0 := u_{01}$, $u_1 := u_{02}$, $u_2 := u_{03}$, $u_3 := u_{23}$, $u_4 := u_{31}$, $u_5 := u_{12}$. It is well kwown that the numbers u_0, u_1, u_2, u_3, u_4, u_5 satisfy equation \begin{equation}

$$u_0 u_3 + u_1 u_4 + u_2 u_5 = 0. \tag{12}$$

Let us consider the well known transformation δ : $Sub_k(V) \to Sub_1(\wedge^k V)$, for k=2. The image $\Omega = (Sub_2(V))^\delta$ of the set $Sub_2(V)$ is so-called Plücker' quadric ([11]). From (11) we can write

$$\Omega := \{(u_0, u_1, u_2, u_3, u_4, u_5) \in F^6 : u_0u_3 + u_1u_4 + u_2u_5 = 0\} \tag{13}$$

For the quadric Ω we can define a polarity associated with it. Namely, for $u \in P(\wedge^2 V)$ let us consider a polynomial $\Phi(u_0,u_1,u_2,u_3,u_4,u_5) = u_0u_3 + u_1u_4 + u_2u_5$ and define a polar hyperplane of the point $u^0 = (u_0^0, u_1^0, u_2^0, u_3^0, u_4^0, u_5^0)$ with respect to Ω. The coefficients of equation of this hyperplane are

$$\frac{\partial\Phi}{\partial u}(u^0) = (u_3^0, u_4^0, u_5^0, u_0^0, u_1^0, u_2^0). \tag{14}$$

Hence we obtain equation:

$$\left\langle \frac{\partial\Phi}{\partial u}(u^0), v \right\rangle = 0, \tag{15}$$

i.e.

$$u_3^0 \upsilon_0 + u_4^0 \upsilon_1 + u_5^0 \upsilon_2 + u_0^0 \upsilon_3 + u_1^0 \upsilon_4 + u_2^0 \upsilon_5 = 0 \tag{16}$$

It is easy to see that the quadric Ω consists of the self-conjugate elements of polarity with respect to itself, namely

$$\left\langle \frac{\partial\Phi}{\partial u}(u), u \right\rangle = 0. \tag{17}$$

i.e.

$$u_3u_0 + u_4u_1 + u_5u_2 + u_0u_3 + u_1u_4 + u_2u_5 = 0. \tag{18}$$

Hence (in field of characteristic not equal to 2) we have

$$u_0u_3 + u_1u_4 + u_2u_5 = 0. \tag{19}$$

Let's assume $\omega(u,v) := \left\langle \frac{\partial\Phi}{\partial w}(u), v \right\rangle$. It is a bilinear form in $\wedge^2 V$ over the field F. Moreover, notice that $\omega(u,v) = \omega(v,u)$. Now the linear complex determined by (8) may be defined as the set

$$K(u) = \{x \in \Omega: \omega(u,x) = 0\}. \tag{20}$$

Every complex is induced by any point of or by any nonzero vector of $\wedge^2 V$ If $u \in \Omega$ then we say that the complex K(u) is degenerate. For field F having the characteristic not equal to 2 the complex K(u) is degenerate iff $\omega(u,) = 0$.

3.2. Axiomatic approach to line geometry

Every line can be treated as a subset of P^n (i.e. as 2-dimensional subspace of V) and as an element of Plücker' quadric, i.e. as a sequence of six elements of the field F. In the former the line will be denoted by a, b, c,...,k, l, m and the latter by u, v, w,..., x, y, z. u, v, w,...,x, y, z. will denote the elemnets of $P(\wedge^2 V)$, which do not belong to Ω.

The map δ is one-to-one correspondence from $Sub_2(V)$ to Ω ([6]). The elements of Plücker'quadric Ω may be treated as lines from $Sub_2(V)$. [5],[7] present an axiomatic approach to $Sub_2(V)$ as PLS - Partial Line Space, exactly as LS - Line Space ([12]).

The theory presented in [5], [7] has been considered as a one-sorted structure < L, -> with L as an universum of lines and one primitive notion "-" as an intersection of lines. Supposition that the elements of L are a,b,c,...and $-(a_1a_2...a_k)$ denotes an intersection of every pair of lines (a_i,a_j) and $a \div b$ denotes a nointersection of lines a,b [5] has introduced two relations:
the Tri (Tri - triangle/tripod) relation defined as follows

$$\mathrm{Tri}(abc) \Leftrightarrow (\neq (abc) \wedge -(abc) \wedge \exists d(-(abd) \wedge c \div d)). \tag{21}$$

which induces the set of the so-called variety} i.e. plane system} of lines or boundle of lines

$$[abc] := \{k \in L: -(abck) \wedge \mathrm{Tri}\,(abc)\} \tag{22}$$

and
the Pen (Pen - pencil) relation defined as follows

$$\mathrm{Pen}(abc) \Leftrightarrow (a \neq b \wedge -(abc) \wedge \forall d(-(abd) \Leftrightarrow d - c)). \tag{23}$$

which induces the set of the so-called pencil of lines

$$< ab > := \{k \in L : \mathrm{Pen}(abk) \wedge a \neq b\}. \tag{24}$$

3.3. Axiomatic approach to linear complex in projective space

The description, given here, is a continuation of axiomatization of line geometry presented in [5], [7].

Consider in any projective space P (dim(P) $\geq$ 3) a set of points P, a set of lines L. Let the symbol "$-_{PL}$" now denotes an intersection of lines (i.e. $-_{PL} \subset L \times L$) and an incidence relation the points and the lines (i.e. $-_{PL} \subset P \times L$) simultaneously. We can define the set C as follows. From algebraic properties of linear complex we have the following axioms for non-degenerate complex:

A1. $\forall_{p\in P} \forall_{k\in L} \exists_{m\in C} (p, k -_{PL} m)$.

A2.
$\forall_{p\in P} \forall_{k,m_1,m_2,m_3\in L} (((p, k -_{PL} m_1, m_2, m_3) \wedge (m_1 \neq m_2) \wedge (p \div_{PL} k) \wedge (m_1, m_2 \in C)) \Rightarrow m_3 \in C)$.

A3.
$\forall_{m_1,m_2,m_3\in L} \forall_{p_1,p_2,p_3\in P} [((p_i -_{PL} m_j, m_k) \wedge \neq (i,j,k))] \Rightarrow (p_i = p_j \vee p_i = p_k \vee p_j = p_k)]$.

We can formulate

Theorem 1. $((m_1 \neq m_2 \neq m_3 \neq m_1) \wedge A_3) \Rightarrow (p_1 = p_2 = p_3)$
Proof. Suppose, that $p_1 = p_2 \neq p_3$. When $m1 = (p_2p_3)$, $m_2 = (p_1p_3)$. Therefore $m_1 = m_2$. We obtain a contradiction.
This theorem says that no triangle belongs to C.

Theorem 2. $\forall_{p\in P}\exists_{m_1,m_2\in C}[(p -_{PL} m_1, p -_{PL} m_2)\wedge(m_1 \neq m_2)]$
Proof. Take a point p and a line l such that $p \div_{PL} l$. By A1 it exists m_1 such that $p,l_{PL}m_1$. Take a line k skew with respect to m_1 and such that $p \div_{PL} k$. On the basis of A1 there exists a line $m_2 -_{PL} p,k$. Naturally $m_1 \neq m_2$.

For any $p \in P$ let us take the set $b(p) = \{k: p -_{PL} k\}$, i.e. a complete boundle of lines in L. We can formulate

Theorem 3. $\forall_{p\in P}(b(p) \not\subset C)$
Proof. Suppose that $b(p) \subset C$. Take $q \neq p$. According to theorem 2 there exist two lines $m_1,m_2 \in C$ such that $q -_{PL} m_1,m_2$, $m_1 \neq m_2$.One of two cases: $m_1 \div_{PL} p$, $m_2 \div_{PL} p$ holds. Let $m_1 \div_{PL} p$. Because every line from the boundle b(p) belongs to C, take such lines l_{11}, l_{12}, which intersect m_1 in two different points q_{11}, q_{12}. Then we obtain a triangle l_{11}, l_{12}, m_1. It is in conflict with A3.

Now suppose that (dim(P) = n). Take the set Hip_k (P) of all k-dimensional hyperplanes of P, treated as the sets of points and as sets of lines. For any $p \in P$ and $p \in H_{n-1} \in Hip_{n-1}(P)$ let us define the set $b(p)_{n-1} := b(p) \cap H_{n-1}$. There is

Theorem 4. $\forall_{p\in P} b(p)_{n-1} \subset C)$
Proof. Let us take a point $p \in P$ and two lines $m_1,m_2 \in C$, such that $m_1,m_2 -_{PL} p$ and $m_1 \neq m_2$. The inclusion $b(p)_2 \subset C$ holds. Consider now a hyperplane $H_2(m_1,m_2)$ and a line l_1 such that $l_1 \cap H_2 = \emptyset$. There exists a line m_3 such that $m_3 -_{PL} p$, $m_3 -_{PL} l_1$. Naturally $m_3 \not\subset H_2$. Next, let us consider a hyperplane $H_3(m_1,m_2,m_3)$. The set C contains $b(p)_3$. By induction we proceed to H_{n-1} (a boundle $b(p)_{n-1}$). There does not exist a line $m_n \in C$ such that $m_n -_{PL} p$ and $m_n \not\subset H_{n-1}$. In the oposite case we would construct the complete boundle $b(p)_n \subset C$. It is impossible. Moreover, notice that there do not exist a line l_{n-2} such that $l_{n-2} \cap H_{n-1} = \emptyset$. And a continuation of the reasoning above is not impossible. The theorems, presented above, play an important role in formulation of the cardinal numbers' properties of complexes and congruences. This is important in construction of PROLOG' predicates.

4. PROLOG'S IMPLEMENTATION OF GEOMETRIC OBJECTS

The axiomatization described above allows define the virtual geometry in TurboPROLOG program. We will use an algebraic description (9) of linear complex LC and Plücker' quadric Ω (12), (13) too. It is worth noting that the definition (15) is not easy for implementation in Prolog logic([8],[9]). Therefore we introduce the other in following way:

$$\mathrm{Pen}\ (abc) \Leftrightarrow (a \neq b \wedge \exists kl(-(abck) \wedge -(abcl) \wedge k \div l)). \tag{25}$$

In Prolog implementation it is explained in the following manner: Let k, l be two lines such that $k \div l$

$$\mathrm{Pen}\ (abckl) \Leftrightarrow (a \neq b \wedge -(abck) \wedge -(abcl) \wedge k \div l)). \tag{26}$$

Generally, in Prolog' implementation we may use the predicates from axiomatic characterization and from image δ. Formerly we introduced in virtual geometry the computer definition of field with characteristics for q = 2,3,4,5. The relation "$-_\Omega$" (intersection of lines, important in PLS) in image δ for $u = (u_0, u_1, u_2, u_3, u_4, u_5)$, $v = (v_0, v_1, v_2, v_3, v_4, v_5)$ has a form:

$$u -_\Omega v :\Leftrightarrow u_0v_3 + u_1v_4 + u_2v_5 + u_3v_0 + u_4v_1 + u_5v_2 = 0. \tag{27}$$

Obviously $u -_\Omega u$.

The condition (27) for the lines is equivalent to the condition (15) which for the lines and for the objects no belonging to quadric Ω holds.

4.1. Examples of predicates describing the select geometric notion in PROLOG

For example in PROLOG' implementation we show the following predicates: intersec (intersection), prod (product in field), pointPluck (predicate defining a point of P^5), notpointPluck (predicate defining a point lying out of Ω, abel (commutativity of field), equ_w_coo (equality with coordinate in field), equal (equal), notequ (not equal), tri (Tri - relation), pen (Pen - relation), var (variety - the set [...])

```
intersec(X1,X2,X3,X4,X5,X6,Y1,Y2,Y3,Y4,Y5,Y6):-
  pointPluck(X1,X2,X3,X4,X5,X6),
  pointPluck(Y1,Y2,Y3,Y4,Y5,Y6),
  not(notpointPluck(X1,X2,X3,X4,X5,X6)),
  not(notpointPluck(Y1,Y2,Y3,Y4,Y5,Y6)),
  prod(X1,Y4,P1), prod(X2,Y5,P2), prod(X3,Y6,P3),
  prod(X4,Y1,P4), prod(X5,Y2,P5), prod(X6,Y3,P6),
  add(P1,P2,PP1), add(PP1,P3,PP2), add(PP2,P4,PP3),
  add(PP3,P5,PP4), add(PP4,P6,"0").

abel(X,Y,Z):-prod(X,Y,Z), prod(Y,X,Z).

equ_w_coo(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6,Z):-
  coordinate(Z), abel(Z,A1,B1), abel(Z,A2,B2), abel(Z,A3,B3),
  abel(Z,A4,B4), abel(Z,A5,B5), abel(Z,A6,B6).

equal(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6):-
  equ_w_coo(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6,"1");
  equ_w_coo(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6,"2");
  equ_w_coo(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6,"3");
  equ_w_coo(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6,"4").
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```
notequ(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6):-
  not(equal(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6)).

tri(A1,A2,A3,A4,A5,A6,
   B1,B2,B3,B4,B5,B6,
   C1,C2,C3,C4,C5,C6,
   D1,D2,D3,D4,D5,D6):-
  intersec(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6),
  intersec(A1,A2,A3,A4,A5,A6,C1,C2,C3,C4,C5,C6),
  intersec(B1,B2,B3,B4,B5,B6,C1,C2,C3,C4,C5,C6),
  intersec(A1,A2,A3,A4,A5,A6,D1,D2,D3,D4,D5,D6),
  intersec(B1,B2,B3,B4,B5,B6,D1,D2,D3,D4,D5,D6),
  not(intersec(C1,C2,C3,C4,C5,C6,D1,D2,D3,D4,D5,D6)),
  notequ(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6),
  notequ(A1,A2,A3,A4,A5,A6,C1,C2,C3,C4,C5,C6),
  notequ(B1,B2,B3,B4,B5,B6,C1,C2,C3,C4,C5,C6).

pen(A1,A2,A3,A4,A5,A6,
   B1,B2,B3,B4,B5,B6,
   C1,C2,C3,C4,C5,C6,
   D1,D2,D3,D4,D5,D6,
   X1,X2,X3,X4,X5,X6):-
  tri(A1,A2,A3,A4,A5,A6,
     B1,B2,B3,B4,B5,B6,
     C1,C2,C3,C4,C5,C6,
     D1,D2,D3,D4,D5,D6),
  intersec(A1,A2,A3,A4,A5,A6,X1,X2,X3,X4,X5,X6),
  intersec(B1,B2,B3,B4,B5,B6,X1,X2,X3,X4,X5,X6),
  intersec(C1,C2,C3,C4,C5,C6,X1,X2,X3,X4,X5,X6),
  intersec(D1,D2,D3,D4,D5,D6,X1,X2,X3,X4,X5,X6).

var(A1,A2,A3,A4,A5,A6,
   B1,B2,B3,B4,B5,B6,
   C1,C2,C3,C4,C5,C6,
   D1,D2,D3,D4,D5,D6,
   X1,X2,X3,X4,X5,X6):-
  intersec(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6),
  intersec(A1,A2,A3,A4,A5,A6,C1,C2,C3,C4,C5,C6),
  intersec(B1,B2,B3,B4,B5,B6,C1,C2,C3,C4,C5,C6),
  intersec(A1,A2,A3,A4,A5,A6,D1,D2,D3,D4,D5,D6),
  intersec(B1,B2,B3,B4,B5,B6,D1,D2,D3,D4,D5,D6),
  not(intersec(C1,C2,C3,C4,C5,C6,D1,D2,D3,D4,D5,D6)),
  intersec(A1,A2,A3,A4,A5,A6,X1,X2,X3,X4,X5,X6),
  intersec(B1,B2,B3,B4,B5,B6,X1,X2,X3,X4,X5,X6),
  intersec(C1,C2,C3,C4,C5,C6,X1,X2,X3,X4,X5,X6),
  notequ(A1,A2,A3,A4,A5,A6,B1,B2,B3,B4,B5,B6).
```

4.2. Combinatoric formulas for cardinal numbers of complexes

We suppose now that the number of elements of a field F is q. Then the number of all k-dimension hyperplanes of projective space P^n (q) n,k are projective dimensions) is equal to $\binom{n+1}{k+1}_q$, where

$$\binom{n}{k}_q = \frac{(q^n-1)(q^n-q)(q^n-q^2)..(q^n-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)..(q^k-q^{k-1})}. \tag{28}$$

Since, for k = 0 we obtain the number of all points in $P^n(q)$ ([4]). It is equal to

$$\binom{n+1}{1}_q = \frac{q^{n+1}-1}{q-1} = q^n + q^{n-1} + q^{n-2} + ... + q + 1. \tag{29}$$

The number of all lines in $P^n(q)$ is equal to

$$\binom{n+1}{2}_q = \frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)} = \frac{(q^n+q^{n-1}+...+q+1)(q^{n-1}+q^{n-2}+...+q+1)}{q+1}. \tag{30}$$

On the basis on the theorem 4 we can determine the cardinal number of non-degenerate linear complex. From (29) it follows that we have $q^n + q^{n-1} + ... + q + 1$ points and in each point according to theorem 4 we have $q^{n-2} + q^{n-3} + ... + q + 1$ lines. Together we have $(q^n + q^{n-1} + ... + q + 1)(q^{n-2} + q^{n-3} + ... + q + 1)$. The above number must be divided by q + 1 (the number points on the line). Thus we obtain the cardinal number of every non-degenerate linear complex

Theorem 5. The cardinal number of the non-degenerate linear complex equal

$$\frac{(q^n+q^{n-1}+...+q+1)(q^{n-2}+q^{n-3}+...+q+1)}{q+1}. \tag{31}$$

Determining of the cardinal number of the degenerate complex (i.e. the set of all lines intersecting the fixed line) is simplier than in the above case. Let any line k be given. In every point p of line k, by theorem 4, we have a boundle $b(p)_{n-1}$ of lines, which includes $q^{n-2} + q^{n-3} + ... + q$ lines without one line (a line k). The line k contains q + 1 points, then we obtain $(q^{n-2} + q^{n-3} + ... + q)(q+1)$. Adding one line k, previously excluded, we obtain $(q^{n-2} + q^{n-3} + ... + q)(q+1) + 1$ lines. Thus we have

Theorem 6. The cardinal number of the degenerate linear complex equal

$$(q^{n-2} + q^{n-3} + ...q)(q+1) + 1. \tag{32}$$

Moreover, notice that in (9) the number of all linear complexes in $P^n(q)$ equals $\dfrac{q^{\frac{(n+1)n}{2}}}{q-1}$ and the number of all degenerate complexes, by (30) equals $\binom{n+1}{2}_q$.

5. CONGRUENCES

By congruence we understand the intersection of two complexes. Accordingly to (20) the congruence is determined by two points u, v of the projective space $P(\wedge^2V)$ or by two vectors u, v from vector space $\wedge^2V$ can be expressed in the following way

$$K(u,v)=\{x\in\Omega:\omega(u,x)=0,\omega(v,x)=0\}. \tag{33}$$

It is equivalence to

$$K(u,v)=\{x\in\Omega:\alpha\omega(u,x)+\beta\omega(v,x)=0\}. \tag{34}$$

for any $\alpha, \beta \in F$. The we have equality $\alpha\omega(u,x) + \beta\omega(v,x) = \omega(\alpha u + \beta v,x)$.Thus we can write

$$K(u,v)=K(u)\cap K(v)=\cap_{\alpha,\beta\in F}K(\alpha u+\beta v). \tag{35}$$

This denote that the congruence is an intersection of all complexes of the pencil of complexes $K(\alpha u + \beta v)$. This pencil may contain 0, 1, 2 or card F + 1 degenerate complexes. Indeed for any two elements α, β, the complex $K(\alpha u + \beta v)$ is degenerate iff $\omega(\alpha u + \beta v, \alpha u + \beta v) = 0$. Because ω is a bilinear form, then we obtain the equation

$$\omega(\alpha u+\beta v,\alpha u+\beta v)=a^2\omega(u,u)+2\alpha\beta\omega(u,v)+\beta^2\omega(v,v)=0. \tag{36}$$

5.1. Combinatoric formulas for cardinal numbers of congruences

First we shall prove in P^n, for n = 3 the following lemma.

Lemma 1. The congruence being an intersection of two nondegenerate complexes does not consist of a pencil of lines.

Proof. Indeed. Let two complexes K(u) and K(v) such that $K(u) \neq K(v)$ be given. Let $k \in K(u) \cap K(v)$. Then a line k_1 does not exist that $k_1 \in K(u) \cap K(v)$, $k\text{-}_{PL}k_1$ and $k \neq k_1$. Suppose that we have the opposite option. Then by A1 with two lines k, k_1 both complexes contain the pencil $< kk_1 >$ of lines determined by these lines. The lines k, k_1 determine 2-dimensional hyperplane H_2, which contains $q^2 + q + 1$ points. Out of this hyperplane we have q^3 points. Since (n = 3), then through every point, from theorem 4, there passes at least one line m which belongs to every complex. A line m intersects the hyperplane H_2. Hence a line m intersects a certain line k_2 belonging to the pencil of lines $<kk_1>$. A line k_2 is a common line for q points lying outside of the H_2. Hence we have $\frac{q^3}{q}=q^2$ pencils of lines. Each pencil has q + 1 lines. Then we have $q^2 (q + 1)$ lines. Let

$$H_4(u):=\left\{x\in P^5:\sum_{i=0}^{5}u_ix_i=0\right\}. \tag{37}$$

Then $K(u) = H_4(u) \cap \Omega$, $K(u) = H_4(u) \cap \Omega$. Next $K(u) \cap K(v) = (H_4(u) \cap H_4(v)) \cap \Omega$. For two different complexes $K(u)$, $K(v)$ the set $H_3(u,v)$: $= H_4(u) \cap H_4(v)$ is 3-dimensional hyperplane. Thus the set $H_3(u,v) \cap \Omega$ is at least a quadric in 3-dimensional space, which includes $(q+1)^2$ (cf. [7]) or $2q^2 + q + 1$ lines, if there is a pair planes. Notice that $(q + 1)^2 > 2q^2 + q + 1$ for $q \geq 2$. And next $q^3 + q^2 > 2q^2 + q + 1$ for $q \geq 2$, because a function $f(q) = q^3 - q^2 - q - 1$ is monothonic on the set $(1, +\infty)$ and $f(2) = 1$. The cardinal number of the set $K(u) \cap K(v)$ is greater than the cardinal number of the set $H_3(u,v) \cap \Omega$. Therefore according to theorem 5, the complexes K(u) i K(v) are identical. It is in conflict with our assumption.

Let us consider two complexes K(u), K(v) and the equation (36) formulated for them. The equation (36) has 0, 1, 2 and q + 1 solutions. Consider thess cases:

Case 1 (no solution). In this case a degenerate complex does not belong to pencil. Then for any pair of lines $k,l \in K(u) \cap K(v)$ we have $k \div_{PL} l$. Take a line $k \in K(u,v) = K(u) \cap K(v)$. Out of the line k we have $q^3 + q^2$ points. Exactly one line of congruence K(u,v) passes through every point. Then we obtain $q^3 + q^2$ lines. But every line passes through q + 1 points. Therefore we have $\frac{q^3 + q^2}{q+1} = q^2$ lines. Finally we must add one line k and we obtain $q^2 + 1$ lines.

Case 2 (1 solution). In this case exactly one degenerate complex belongs to pencil of complexes. We assume, that K(u) is degenerate complex and K(v) is non-degenerate complex. Suppose that a line k, having the coordinates u, is an axis of complex K(u). Therefore the congruence K(u,v) can be treated as the set of all lines of the complex K(v) intersecting line k. Out of the line k in P^3 we have $q^3 + q^2$ points. Through any such point there passes exactly one line intersecting of the line k. Then we obtain $q^3 + q^2$ of lines. But to every line belong q points out of the line k. Therefore we have $\frac{q^3 + q^2}{q} = q^2 + q$ lines. Finally we must add one line k and we obtain q2 + q + 1 lines.

Case 3 (2 solutions). In this case exactly two degenerate complexes belongs to pencil of complexes. Let the lines k, l be the axis of this complexes. The lines k, l are skew. Then we obtain as the congruence K(u,v) the set of all lines intersecting both lines k, l. The number of these lines equal $(q + 1)^2$ because through every point of line k pass the pencil of lines intersecting a line l, i.e. q + 1 lines.

Case 4 (q + 1 solutions). In this case the axes k, l of degenerate complexes intersect. Then we obtain as the congruence K(u,v) the set of all lines created by the union of the plane system of lines ($q^2 + q + 1$ lines) and of the boundle of lines ($q^2 + q + 1$ lines). Q + 1 lines is doubled. Therefore we obtain $2q^2 + q + 1$ lines.

The obtained theorem can be formulated as follows

Theorem7. The cardinal number of the linear congruences in P^3 depends on the number of solutions of equation (36) and it equals:

a) $q^2 + 1$, if the equation (36) has not solution,
b) $q^2 + q + 1$, if the equation (36) has 1 solution,
c) $(q + 1)^2$, if the equation (36) has 2 solutions,
d) $2q^2 + q + 1$, if the equation (36) has q + 1 solutions.

5.2. Final remarks

The results obtained one above by reasoning above aided by PROLOG' programs from the theorem 7 are shown in Table 1.

The projection realized by all congruences, discussed above, have the singular points and it is worse than classic central projection realized by a congruence $K_{(1,0)}$ (complete boundle of lines). The above considerations allow the assumption that these does not exist the perfect projection of objects not having singular points.

Table 1

Classification of singular points in projection by K(u,v)

Solution of equation (36)	Card (K(u,v))	Number of singular points	Sets of singular points
0	$q^2 + 1$	$q + 1$	$K(u,v) \cap \pi = \{l\}$
1	$q^2 + q + 1$	$q^2 + q + 1$	π
2	$(q + 1)^2$	$q^2 + 3q + 1$	$\pi \cup l \cup k$
q + 1	$2q^2 + q + 1$	$q^2 + q + 1$	π

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Abstract

The classical projection of 3-dimensional space P^3 to plane π is usually defined as the structure <R, π>, where R is a boundle of lines (projecting rays) with ideal or ordinary point as the centre of projection and π is an ordinary plane as projection surface. It is well known, that the set R may be a congruence of lines in the form $K_{(m,n)}$, for m=1, where m denotes the number of lines of congruence passing by any point of P^3 and n denotes the number of lines lying on plane ([1],[10]). The centre of classical projection is always singular.

The employement of other congruence e.g. the set of all lines intersecting two skew lines (congruence $K_{(1,1)}$ or the set of all bisecantes of the skew curve of the 3th order (congruence $K_{(1,3)}$ induces the singular points too ([1],[10]). The natural geometric congruences are obtained as the intersection of two degenerated complexes of lines and therefore the singular points are obtained. The interesting question is: "Does such congruence of lines exist which doesn't induce the singular points in a projection it defines?". A supposition, that the congruence defined by two non-degenerated complexes eliminate the singular points, leads to a solution formulated in this work problem. The solution of above problem is obtained by an investigation of finite projective spaces on the basis of the package of programmes in TurboPROLOG language ([8],[9]).

In the paper we formulate some properties of linear complexes and congruences in finite structures and we obtain the new combinatorial results concerning above sets.