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GEOMETRIC INTERPRETATION OF DENSITY IN INHOMOGENEOUS SOLID CELLS

Summary. Modelling of free-form solids with the inhomogeneous density of distribution of the set of iso-parametric curves and surfaces and the related solid intrinsic geometry is described in the paper. Interior deformations with respect to the relations of the position of points in the basic grid and the form and density of the solid iso-parametric curve systems are described. Bézier approximation tri-cubic solid cell is presented and its interior deformations performed, by means of the changing position of the eight interior points forming the density core. The density vector determined by the points of the basic grid is expressed from the density function in the arbitrary solid cell point and different forms of the formula for the volumetric charecteristics calculations are given.

GEOMETRYCZNA INTERPRETACJA GĘSTOŚCI W NIEJEDNORODNYCH ELEMENTACH BRYŁ

Streszczenie. Praca dotyczy modelowania brył niejednorodnych. Autorka przedstawia definicję bryły niejednorodnej, omawia pewne jej właściwości, następnie podaje definicje gęstości bryły niejednorodnej w otoczeniu punktu regularnego, bazując na klasycznych charakterystykach tworu geometrycznego, takich jak: I forma różniczkowa, wyznacznik Grama po dokonaniu aproksymacji wielomianowej (wielomianem stopnia trzech zmiennych). Wizualizacja gęstości bryły jest wyrażona poprzez "warstwice powierzchniowe gęstości".

One of the new and interesting parts of Geometry is modelling of geometric figures with preset specific properties concerning attributes of the created models, which can be related to the intrinsic geometric properties of the geometric figures. They can be expressed and calculated with respect to the Differential Geometry and they can offer a wide range of different and startling qualities of the modelled objects.

In the paper we would like to present one possible approach to modeling of inhomogeneous solids. The property of a density in the neighborhood of the solid cell regular point is discussed with respect to the solid cell differential forms and intrinsic properties.

Definition 1. Solid T is a non-empty continuous subset of the extended Euclidean space over the real numbers ${}_{\infty}E^3$, which is a locally homeomorphic mapping Φ of the simple continuous region $\Omega \subset R^3$, $\Phi : \Omega \to {}_{\infty}E^3$, $(u^1, u^2, u^3) \to (x(u^i), y(u^i), z(u^i), w(u^i))$, analytically represented by a point function $\mathbf{s}(u^i) = (x(u^i), y(u^i), z(u^i), w(u^i))$, for i=1, 2, 3. Particular components $x(u^i)$, $y(u^i)$, $z(u^i)$, $w(u^i)$ are homogeneous co-ordinate functions in three real variables differentiable on Ω, and $w(u^i)=0$ for all ideal points on solid, while for all real points the value in the normalised form is $w(u^i)=1$. Numbers $(a, b, c)\in\Omega$ are curvilinear co-ordinates of the solid point $P(a, b, c)\in T$. For $\Omega = [0,1]^3$ we speak about solid cell.

In the solid regular point $P(a, b, c) = \mathbf{s}(a, b, c)$, where $(a, b, c)\in\Omega$, there exist 7 vectors determining interior geometry of the solid, tangent trihedron in the Fig. 1, with edges defined by three tangent vectors to the solid iso-parametric curves intersecting in the point P - values of the solid point function first derivatives for $(a, b, c)\in\Omega$, $\mathbf{s}_i\,(a, b, c)$, i=1, 2, 3, three twist vectors to the solid iso-parametric surfaces sharing the common point P - values of the solid point function second mixed partial derivatives for $(a, b, c)\in\Omega$, $\mathbf{s}_{ij}\,(a, b, c)$, i, j=1, 2, 3, $i\neq j$, and density vector in the point P - value of the solid point function third mixed partial derivative for $(a, b, c)\in\Omega$, $\mathbf{s}_{123}\,(a, b, c)$.

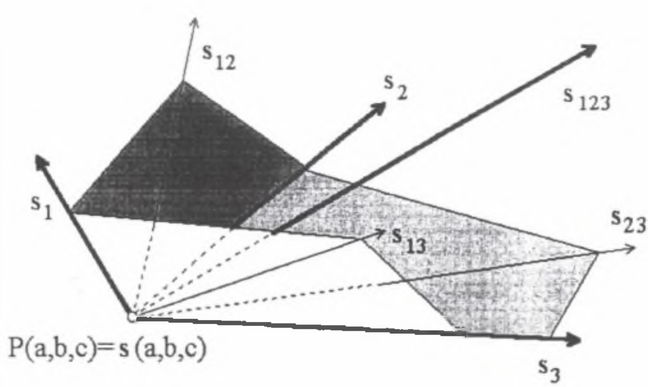

Fig. 1. Seven vectors determining the interior geometry in the solid regular point P
Rys. 1. Siedem wektorów określających geometrię wnętrza elementu bryły w otoczeniu punktu regularnego

The first differential form of the solid $\varphi_1(\overset{1}{u}, \overset{2}{u}, \overset{3}{u})$ is the total of the differential forms of the solid iso-parametric surfaces subtracted by the sum of the first differential forms of the solid iso-parametric curves

$$\Phi_1(\overset{1}{u}, \overset{2}{u}, \overset{3}{u}) = \varphi_1(\overset{1}{u}, \overset{2}{u}) + \varphi_1(\overset{1}{u}, \overset{3}{u}) + \varphi_1(\overset{2}{u}, \overset{3}{u}) - \sum_{i=1}^{3}\left(s_i du^i\right)^2,$$

$$\text{where } \varphi_1(\overset{i}{u}, \overset{j}{u}) = (s_i du^i)^2 + 2s_i s_j du^i du^j + (s_j du^j)^2$$

Let us denote coefficients of the (u^1, u^2) iso-parametric surface patch first fundamental form by $E=s_1^2$, $F=s_1s_2$, $G=s_2^2$, and similarly coefficients of the (u^1, u^3) iso-parametric surface patch first fundamental form by $E=s_1^2$, $F^*=s_1s_3$, $G^*=s_3^2$, and coefficients of the (u^2, u^3) iso-parametric surface patch first fundamental form by $G=s_2^2$, $F^{**}=s_2s_3$, $G^*=s_3^2$. Then we can write the following formula for the discriminant of the solid first differential form, which is a positive number expressed in a solid regular point as the value of the determinant

$$D = \begin{vmatrix} E & F & F^* \\ F & G & F^{**} \\ F^* & F^{**} & G^* \end{vmatrix} = [s_1 s_2 s_3]^2$$

Square of the mixed triple scalar product $s=[s_1s_2s_3]$ of tangent vectors to solid iso-parametric curves in the regular point P can be geometrically represented as the volume, basic metric characteristics of the solid cell equivalent to the surface patch area and the curve segment length, and it is related to the discriminant D of the solid first differential form and calculated as the triple integral over Ω.

Definition 2. Density in the solid regular point P can be expressed as the volumetric element,

$$dV = |s| = \sqrt{\frac{3EGG^* - G^*F^2 - EF^{**2} - GF^{*2}}{h}} = \sqrt{\frac{3A - B}{h}}, h = \frac{1}{\cos^2\Theta} + \frac{1}{\cos^2\Phi} + \frac{1}{\cos^2\Psi}$$

where θ, ϕ, ψ are angles, which normal to one of the iso-parametric surfaces (defined by two iso-parametric curves) forms to the third iso-parametric curve tangent vector.

For values 0, or π in one direction we have $h=3$, the mixed triple scalar product norm is $|s| = \sqrt{A - \frac{1}{3}B}$ and we speak about weakly dense distribution of solid points in the direction of the related tangent vector. When angles are equal, $\theta, \phi, \psi = 0, \pi$ in all three directions, tangent vectors form an orthogonal base, and tangent trihedron in the solid point determined by tangent vectors is right-angled and distribution of points is homogeneous.

For extremal values of angles in one direction, $\theta, \phi, \psi = \pi/2$, number h streams to infinity, and we speak about dense distribution of the solid points in the direction of the related tangent vector.

Definition 3. Density function of the solid cell can be determined as the function of three variables defined on the region $\Omega = [0,1]^3$ in the form

$$h(u^1, u^2, u^3) = \frac{D + 2\left(EGG^{*} - FF^{*}F^{**}\right)}{\left[\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_3\right]}$$

with the value in the density of the solid cell in the solid regular point P(a, b, c), $(a, b, c) \in \Omega$.

The possibility to describe and to control the feature of the non-uniform distribution of points in a solid is provided in the case of its modelling as an interpolated figure, while the generating principle is an interpolation. The intrinsic geometric properties of the created solid are explicitly predetermined in the basic figure, which is in a form of the ordered set of separate geometric figures (points, curve segments or surface patches) related to the solid, or vectors that can be tangent vectors to iso-parametric curve segments, twist vectors to iso-parametric surface patches and density vectors in the given solid points. These geometric figures are represented analytically by their vector functions appearing as the elements of the analytic representation of the entire basic figure, a map of the created solid. In the map - matrix, there are all elements distributed in the appropriate order and predetermine the intrinsic geometric properties of the created solid, the curvature of iso-parametric curve segments and iso-parametric surface patches, or the inhomogeneity of the interior points' distribution and density.

The interpolation tri-cubic solid cell analytic representation is in the form

$$\mathbf{s}(u,v,w) = \mathbf{a}_{333}u^3v^3w^3 + \mathbf{a}_{332}u^3v^3w^2 + \ldots + \mathbf{a}_{100}u + \mathbf{a}_{000} = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} F_i(u)F_j(v)F_k(w)\,\mathbf{b}_{ijk}$$

for $(u,v,w) \in [0,1]^3$, where $F_i(u)$, $F_j(v)$, $F_k(w)$, are cubic interpolation polynomials. Geometric coefficients $\mathbf{b}_{ijk}$ define explicitly geometric properties of the solid and they form a three-dimensional matrix of the type 4x4x4, the map of the solid. Illustration of the change in the solid cell interior geometry is given in the Fig. 2.

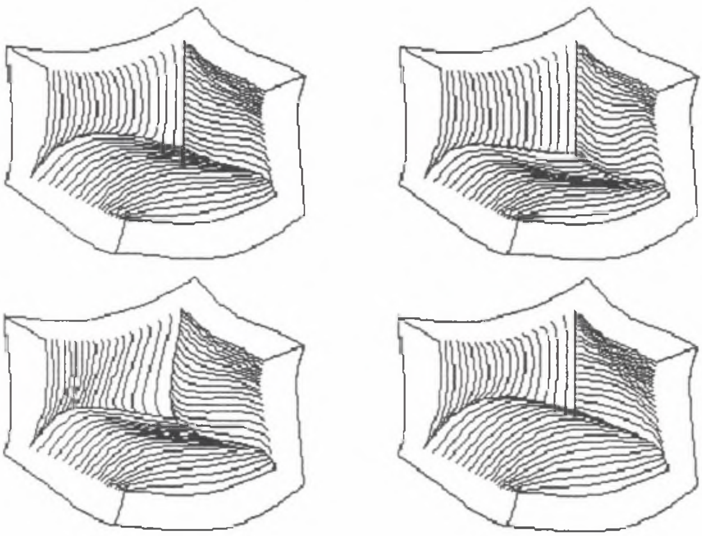

Fig. 2. Changing interior geometry in the solid cell regular point
Rys. 2. Zmiana geometrii wnętrza elementu bryły w otoczeniu punktu regularnego

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Recenzent: Dr Edwin Koźniewski

Abstract

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