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A GENERALISATION OF THE RADICAL AXIS

Summary. We can generalise the radical axis as the support of the real or imaginary intersection points of two real non-degenerative second order curves. The new definition is based on projective-properties, which gives a radical axis also in non-constructable cases.

UOGÓLNIENIE OSI POTĘGOWEJ

Streszczenie. Autor prezentuje uogólnienie osi potęgowej opartej o rzeczywiste lub urojone punkty przecięcia z dwoma nie zdegenerowanymi krzywymi stopnia drugiego. Nowa definicja bazuje na własnościach rzutowych, które także umożliwiają wyznaczenie osi potęgowej w nie konstruktywnych przypadkach.

1. Introduction

The definition of the radical axis used in elementary geometry: two circles are given in the Euclidean plane, the locus of those points, which powers with respect to the two given circles are equal, is called a radical axis.

It can be easily seen, that the two real circles generate the same involution on the radical axis. If the Euclidean plane is extended with a line of infinity, then we get another line with same property, on which the circles generate the same involution.

We can see two circles, the two radical axes, and the generated involution on them on the fig. 1. *A* and *A'*, and *B* and *B'* is a corresponding pair of points in the generated involution on h_1 and h_2, respectively. (The arrow on the fig. isn't a direction, instead of they mean B and B' are at infinity.)

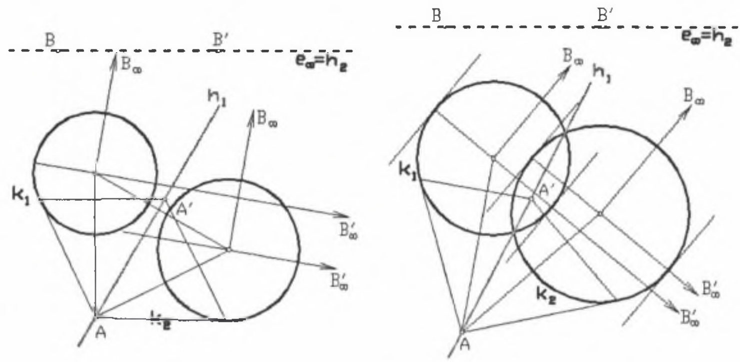

Fig. 1. Radical axes of two real circles
Rys. 1. Oś potęgowa dwu okręgów właściwych

2. The new definition

Accept the following as the new definition! Let two real, non-degenerative second order curves are given. That line, on which both curves generate the same involution is called a radical axis.

Statement: Any two given real or imaginary circles have exactly two radical axes in the projective plane.

Proof: Any two given real or imaginary circles have exactly four intersection points. Two of them are real or imaginary, and two of them are the two absolute (totally) imaginary intersection points. One real line fits the two real or imaginary points, and the generated involution will be hyperbolical or elliptical respectively. The line of infinity fits the two absolute (totally) imaginary intersection points, and the generated involution is elliptical. If there was another radical line, the fix points of the generated involution will be further intersection points of the two circles. This is a contradiction.

Two real, non-degenerative, second order curves have four intersection points, both of which can be real or imaginary. We can distinguish more cases according the number of the real or imaginary intersection points and their multiplicity.

Table 1

Case	Real	Im.	Multiplicity				Rad.axes	Picture
1	4	-	1	1	1	1	6	Fig.2.
2	3	-	2	1	1		3	Fig.3.
3	2	-	2	2			1	Fig.4.
4			3	1			1	Fig.5.
5	1	-	4				0	Fig.6.
6	-	4					2	Fig.7.
7	2	2	1	1			2	Fig.8.
8			2				1	Fig.9.

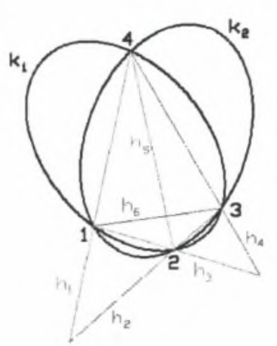

Fig. 2. Case 1 of Table 1
Rys. 2. Przypadek 1 z tabeli 1

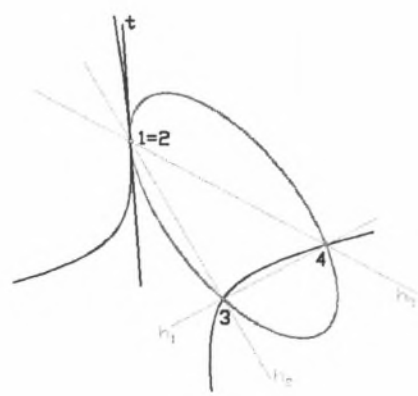

Fig. 3. Case 2 of Table 1
Rys. 3. Przypadek 2 z tabeli 1

In the case of two second order curves with four common points (fig.2) any pair of intersection points determines a radical line, so we get six lines.

In the second case (fig.3) there are two second order curves with three common points and they have a common tangent line at one of them. Any pair of points determines a radical line, but on the common tangent line there isn't a generated involution, so we get three radical lines.

In the next case (fig. 4) there are two curves with two common tangent lines. On the fig. 5 there are two curves with a third order tangent point and with an intersection point. In the last two cases there are only one radical line.

On the fig. 6 we can see two second order curves with a four order tangent point and without a radical line.

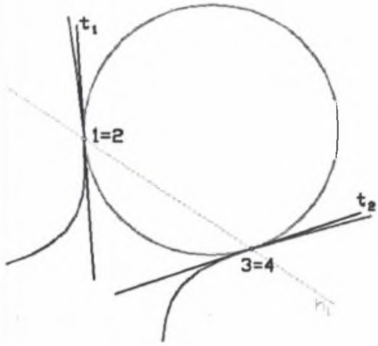

Fig. 4. Case 3 of Table 1
Rys. 4. Przypadek 3 z tabeli 1

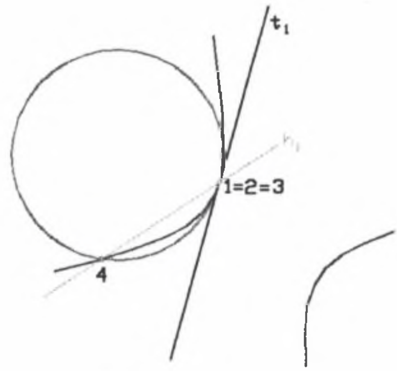

Fig. 5. Case 4 of Table 1
Rys. 5. Przypadek 4 z tabeli 1

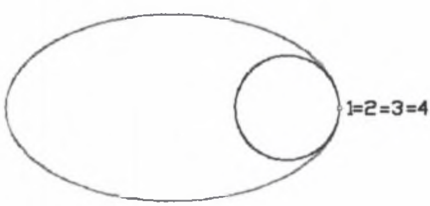

Fig. 6. Case 5 of Table 1
Rys. 6. Przypadek 6 z tabeli 1

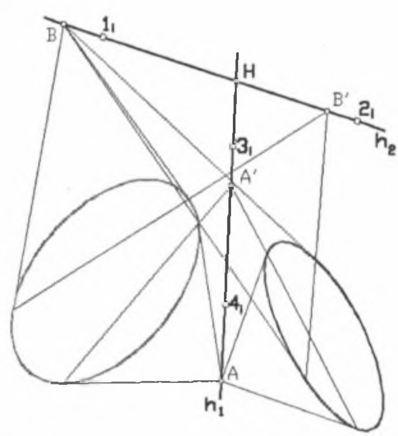

Fig. 7. Case 6 of Table 1
Rys. 7. Przypadek 6 z tabeli 1

On the fig.7 the two second order curves have 2 pair of imaginary intersection points. There is one real line fitting per pairs.

There is only one radical line when the two second order curves have a pair of imaginary intersection points and a tangent point (fig.8).

In the last two cases the imaginary intersection points are only signed on the radical lines.

We can see that any real line can be called a radical line which is fitting on any two real or imaginary intersection points of the curves.

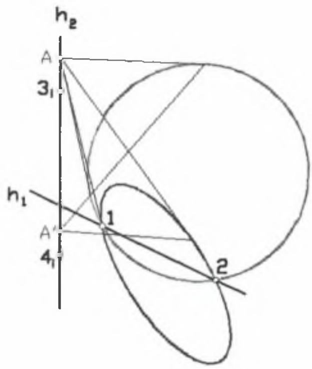

Fig. 8. Case 7 of Table 1
Rys. 8. Przypadek 7 z tabeli 1

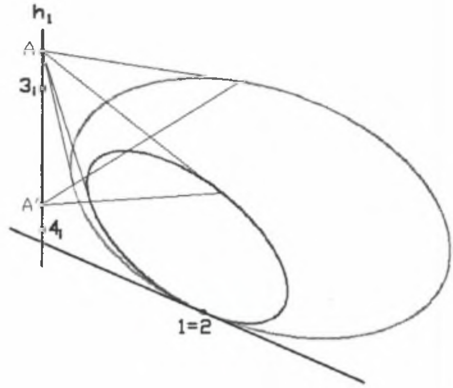

Fig. 9. Case 8 of Table 1
Rys. 9. Przypadek 8 z tabeli 1

We can't construct the radical axes in a general case, because it leads a fourth order problem. However we can do it with some other information. If we know two intersection

3. Constructional methods

We can't construct the radical axes in a general case, because it leads a fourth order problem. However we can do it with some other information. If we know two intersection points – it can be real or imaginary (in this case we know the fitting line) – then we know one of the radical axes, we can construct the others, if they exist.

Let us see it in general case. Let be given two non-degenerative second order curves and two of the intersection points: *A* and *B*. We must find a central collineation which maps one of the curves into the other. Because *A* and *B* are common points, we choose the fitting line as the axis of the collineation. Now we must find the centre of the collineation, and a pair of corresponding points.

In the construction to create the tangents, we can use the Pascal-theorem. It will be not noticed at each occurrence.

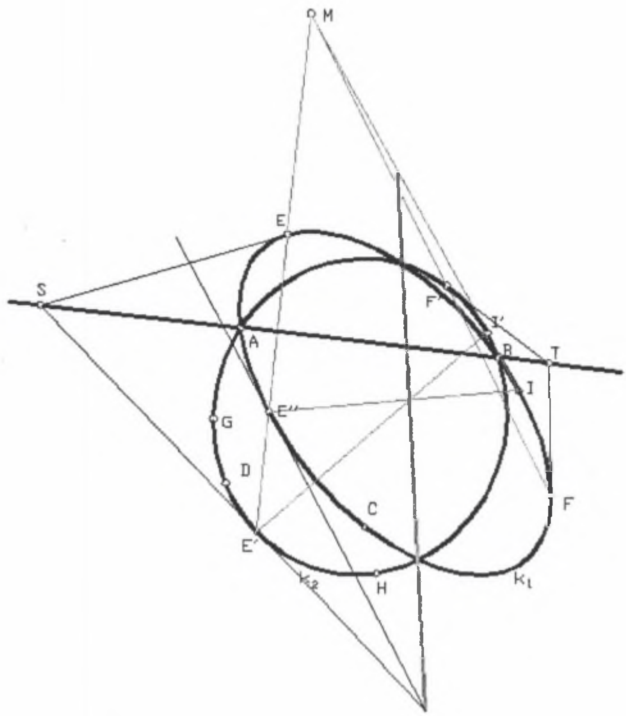

Fig. 10. Construction in the Case 1
Rys. 10. Konstrukcja w pierwszym przypadku

Follow the next steps.

0. Let be given two non-degenerative second order curves with the points *A*, *B*, *C*, *E*, *F* – curve k_1 and *A*, *B*, *D*, *G*, *H* – curve k_2.
1. Create the tangent lines at *E* and *F*.
2. Denote *T* and *S* the intersection point of *AB* with the tangents at *E* and at *F* respectively.
3. Create the tangent lines to curve k_2 from *T* and *S*. Denote the tangent points *E'* and *F'* respectively.
4. In the searched collineation *E*, *E'* and *F*, *F'* will be the corresponding pairs of points.
5. The intersection point of EE' and FF' lines will be the centre of collineation. Denote it *M*.

6. There will be a common tangent line from *M* to curve k_1 and curve k_2. Denote the tangent points *I* and *I'* respectively. We determined three pairs of points in the collieneation.
7. Let us denote *E"* the other intersection point of the line *EE'* and curve k_1.
8. If we choose the following pairs, I – I', and E" – E', then we get another collineation with centre *M*, which maps curve k_1 into curve k_2.
9. The intersection point of the lines *E"F* and *E'F'*, and the intersection point of the tangents at *E"* and at *E'*, fit to the axis of the collineation.
10. Getting the searched points we must construct the intersection points between the line of the second collineation and one of the given curves. See fig.10.

This construction works in both of that case, if the curves have two real intersection points, and we know the two real, or the two imaginary points. See fig.11 and fig. 12. In fig. 12 we can't see the points *A* and *B*, because they are the imaginary intersection points, we can know only, that they are fitting on the given radical line.

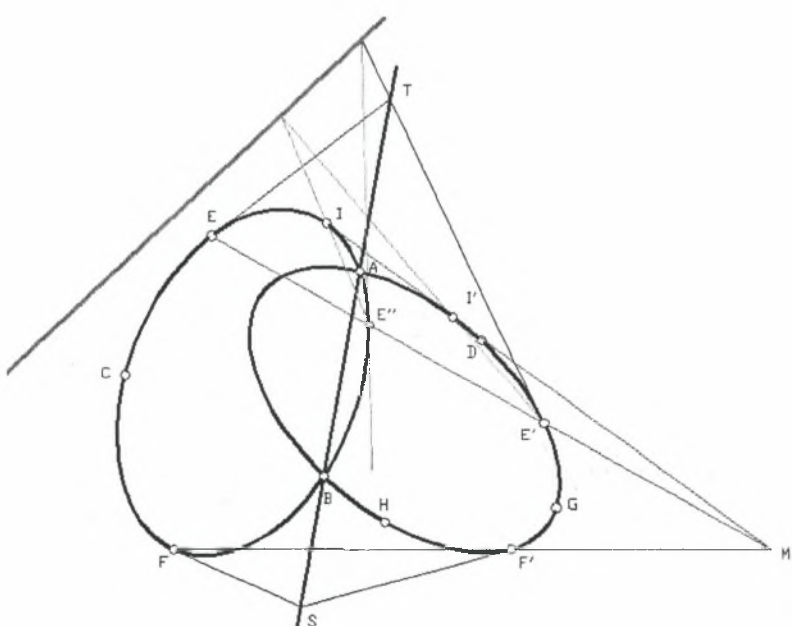

Fig. 11. Construction in the Case 7
Rys. 11. Konstrukcja w przypadku 7

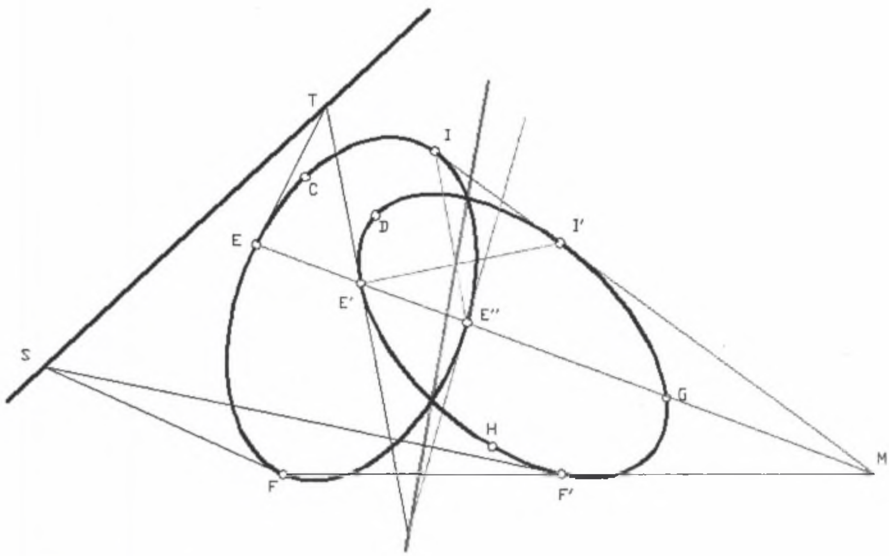

Fig. 12. Construction in the Case 7
Rys. 12. Konstrukcja w przypadku 7

4. Conclusion

The following definition is equivalent with the one mentioned in the introduction. There are two given second order curves. We call a line a radical line, which remains fix as object in the Steiner-relationship.

Recenzent: Dr inż. Witold M. MIRSKI

Abstract

The radical axis is a notion in elementary geometry. It's generalised form takes place in cyclography, where it is mentioned as the support of the real or imaginary cross points of two real or imaginary circles. Following this thought a question is risen, can be given a more general radical axis definition, which is based on projective-properties, which gives a radical axis also in case of two real non-degenerative second order curves.