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AUTOMATIC THEOREM PROVING IN ELEMENTARY GEOMETRY

Summary. In the last 20 years several efficient methods for automatic theorem proving in elementary geometry have been developed. First we will introduce some standard notions from automatic proving theorems. Then we will present the methods on the example of the Simson theorem and its generalization. The demonstration is accompanied by the use of dynamic geometry system Cabri II.

AUTOMATYCZNE DOWODZENE TWIERDZENIA W GEOMETRII ELEMENTARNEJ

Streszczenie. W ostatnich 20 latach rozwinięto kilka skutecznych metod automatycznego dowodzenia twierdzeń geometrii elementarnej. Najpierw przedstawiono pewne standardowe nazwy dla automatycznie dowodzonych twierdzeń. Następnie przedstawiono metody oparte na twierdzeniu Simsona i jego uogólnieniach. Do prezentacji wykorzystano dynamiczny system geometryczny Cabri.

1. Introduction

We present here the theory proposed by M.T.Wu and D.Kapur, see [1], [2]. Roughly speaking, *automatic proving* deals with deciding the correctness of statements of the kind $\mathbf{H} \Rightarrow \mathbf{C}$, where $\mathbf{H}, \mathbf{C}$ are, respectively, given sets of hypotheses and conclusions (theses). The expected output is "yes" or "no". In this case some minor modifications of hypotheses can occur for the conclusions to hold (non degenerate conditions). On the other hand, the goal of *automatic deriving* of geometric statements is *to find* conclusion involving prescribed geometric magnitudes which follows from a given set of hypotheses. The third issue we will be concerned with is *automatic discovery*, i.e., the problem of dealing automatically with arbitrary geometric statements, which are not in general true, to find the missing hypotheses so that a given conclusion follows from a given incomplete set of hypotheses.

We will use the technique which is based on the results of commutative algebra of the last third of the 20th century as the Gröbner bases computations and elimination of variables, which are implemented in mathematical software like Axiom, Maple, Mathematica, Reduce, Singular, CoCoA, etc. and even in some calculators. Examples were solved by computer algebra system CoCoA[1] and demonstrated by dynamic geometry system Cabri geometry II.

2. Automatic proving theorems

Let $\mathbb{L}$ be an algebraically closed field, e.g. the field of complex numbers $\mathbb{C}$, containing the field $\mathbb{K}$ of characteristic 0. Let $\mathbb{K}[x_1, x_2, ..., x_n]$ be the ring of polynomials with n variables $x = (x_1, x_2, ..., x_n)$ with coefficients in $\mathbb{K}$. Throughout the paper we will suppose that $\mathbb{K} = \mathbb{Q}$ unless stated otherwise. We will be concerned with statements which are of the form $\mathbf{H} \Rightarrow \mathbf{C}$ where $\mathbf{H}$ is a set of hypotheses which can be expressed in the form of polynomial equations $h_1(x) = 0, h_2(x) = 0, ..., h_s(x) = 0$, the conclusion $\mathbf{C}$ is of the form of a polynomial equation $c(x) = 0$. The algebraic form of a geometric statement is

$$\forall x \in \mathbb{L}^n, \quad h_1(x) = 0, h_2(x) = 0, ..., h_s(x) = 0 \Rightarrow c(x) = 0, \tag{1}$$

where $h_1, h_2, ..., h_s \in \mathbb{Q}[x_1, x_2, ..., x_n]$. The hypothesis variety H is the set of all solutions of the system $h_1 = 0, h_2 = 0, ..., h_s = 0$, the conclusion variety C is the set of all solutions of $c = 0$. The statement (1) is geometrically (or algebraically) true iff H is contained in C. Thus to prove (1) it suffices to show that c vanishes on the set of all solutions of the system of equations $h_1 = 0, h_2 = 0, ..., h_s = 0$. Verification of whether $H \subset C$ or $H \not\subset C$ belongs to the main questions of algebraic geometry and is closely connected with the famous Hilbert Nullstellensatz, see [2]. The statement (1) is true iff 1 belongs to the ideal $I = (h_1, h_2, ..., h_s, ct - 1)$, where t is a new variable, which is equivalent to NormalForm $(1, I) = 0$, see [2,3,4].

3. Examples

As an example we will use the well-known *Simson-Wallace* theorem to show the essence of the technique. Simson-Wallace theorem is as follows, see Fig. 1.

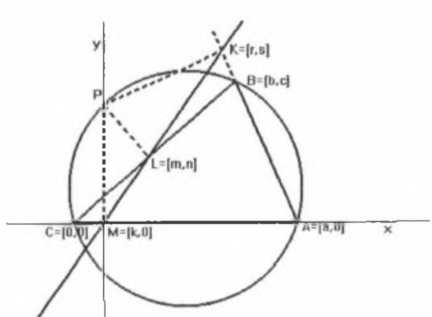

Fig. 1. Simson-Wallace theorem
Rys. 1. Twierdzenie Simsona-Wallace'a

[1] Software CoCoA is freely distributed at cocoa@dima.unige.it

Let ABC be a triangle and P a point of the circumcircle of ABC. Then the feet of perpendiculars from P onto the sides of ABC lie on a straight line.

Consider an arbitrary triangle ABC and assigne K,L,M to the feet of the perpendiculars dropped from a point P of the cimcumcircle of ABC to the sides AB,BC,CA respectively. We are to prove that the points K,L,M are collinear. Choose the system of coordinates so that $A=[a,0], B=[b,c], C=[0,0], P=[p,q]$, $K=[r,s], L=[m,n], M=[k,0]$. The hypotheses are as follows:

H1: $PM \perp AC$: $p-k=0$
H2: $L \in BC$: $cm-bn=0$
H3: $PL \perp BC$: $(p-m)b+(q-n)c=0$
H4: $K \in AB$: $(b-a)s-(r-a)c=0$
H5: $PK \perp AB$: $(p-r)(b-a)+(q-s)c=0$
H6: $P \in$ circumcenter of ABC: $-acp+cp^2+abq-b^2q-c^2q+cq^2=0$.

The conclusion is expressed by

C: K,L,M are collinear: $(m-k)s-(r-k)n=0$.

In CoCoA we enter

```
I:=Ideal(p-k,cm-bn,(p-m)b+(q-n)c,(b-a)s-(r-a)c,(b-a)(p-r)+(q-
s)c,-acp+cp^2+abq-b^2q-c^2q+cq^2,((m-k)s-(r-k)n)t-1);
NF(1,I);
1
```

The normal form equals 1 which means that the statement is not true. The following command eliminates dependent variables p,q,k,m,n,r,s and the slack variable t.

```
I:=Ideal(p-k,cm-bn,(p-m)b+(q-n)c,(b-a)s-(r-a)c,(b-a)(p-r)+(q-
s)c,-acp+cp^2+abq-b^2q-c^2q+cq^2,((m-k)s-(r-k)n)t-1);
Elim(p..t,I);
```

We get the non degenerate condition $(b^2+c^2)((a-b)^2+c^2)\neq 0$, which means that for the vertices of the triangle $B\neq C$ and $B\neq A$ must hold. In the next step we verify the normal form

```
I:=Ideal(p-k,cm-bn,(p-m)b+(q-n)c,(b-a)s-(r-a)c,(b-a)(p-r)+(q-
s)c,-acp+cp^2+abq-b^2q-c^2q+cq^2,(b^2+c^2)u-1,((a-b)^2+c^2)v-
1,((m-k)s-(r-k)n)t-1);
NF(1,I);
0;
```

Thus the theorem is true.

In the next part we generalize the Simson-Wallace theorem, see [1]. Instead of a collinearity of the points K,L,M we will demand the constant area of the triangle KLM. Let K,L,M be the feet of perpendiculars dropped from a point P to the sides AB,BC,CA of the triangle ABC respectively. Let the (oriented) area of the triangle KLM be a constant f. We will look for the locus of points P. With the same notation as in the Fig. 1. we describe this situation in a similar way as we did in the previous example. Notice that for the area f of the triangle KLM the formula $f=\frac{1}{2}(kn+ms-rn-ks)$ holds. First assume that for a point P there is no restriction, i.e, we suppose that P is an arbitrary point. We find out that the statement is not true, NF=1 as expected (we omit this). Further we look for additional assumptions.

Eliminating dependent variables k,m,n,r,s from the ideal $I=(p-k,cm-bn,(p-m)b+(q-n)c,(b-a)s-(r-a)c,(b-a)(p-r)+(q-s)c,kn+ms-rn-ks-2f)$ we obtain the only equation $-a^2c^3p+ac^3p^2+a^2bc^2q-ab^2c^2q-ac^4q+ac^3q^2-2a^2b^2f+4ab^3f-2b^4f-2a^2c^2f+4abc^2f-4b^2c^2f-2c^4f=0$. A close inspection shows that it is a circle with the center $O=[q/2,(b^2-ab+c^2)/(2c)]$ and radius

$$r=\sqrt{(b^2+c^2)((a-b)^2+c^2)(ac+8f)/(4ac^3)},$$

see Fig. 2., which is concentric with the circumcircle of ABC.
Assume that a point P fulfils the above equation. Then K,L,M form a triangle of the area f. Let us verify it, preserving the condition $(b^2+c^2)((a^2-b^2)+c^2)\neq 0$.

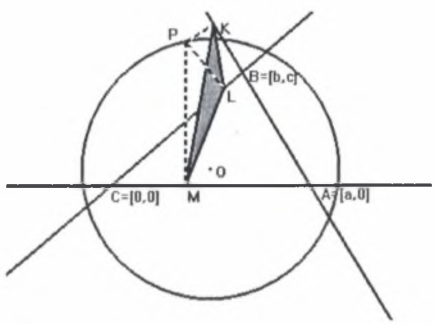

Fig. 2. Generalization of the Simson-Wallace theorem
Rys. 2. Uogólnienie twierdzenia Simson-Wallace

```
I:=Ideal(p-k,cm-bn,(p-m)b+(q-n)c,(b-a)s-(r-a)c,(b-a)(p-r)+(q-
s)c,-a^2c^3p+ac^3p^2+a^2bc^2q-ab^2c^2q-ac^4q+ac^3q^2-2a^2b^2f
+4ab^3f-2b^4f-2a^2c^2f+4abc^2f-4b^2c^2f-2c^4f, (b^2+c^2)((a-
b)^2+c^2)v-1,(kn+ms-rn-ks-2f)t-1);
NF(1,I);
0
```

We have stated the following theorem:
Let ABC be a triangle and P a point of a circle which is concentric with the circumcircle of ABC. Then the feet of perpendiculars from P onto the sides of ABC form a triangle with the constant area.

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Abstract

In the last 20 years several efficient methods for automatic theorem proving in elementary geometry have been developed. First we will introduce some standard notions from automatic proving and discovering theorems. Then we will present the methods on the example of the Simson-Wallace theorem and its generalization. The demonstration is accompanied by the use of dynamic geometry system Cabri II.