ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 2004
Seria: ELEKTRYKA z. 191 Nr kol. 1633

Janusz WALCZAK, Dariusz GRABOWSKI
Instytut Elektrotechniki Teoretycznej i Przemysłowej. Zakład Elektrotechniki i Informatyki

TRANSITIVE MAPS FOR TIME-FREQUENCY SIGNAL REPRESENTATIONS

Summary. Time-frequency representations are applied to analysis and synthesis of non-stationary systems and signals. Relations between different groups of these representations are called transitive maps. The paper deals with transitive maps between generalized Wigner representations introduced by Cohen and the others and representations in the form of instantaneous amplitude and frequency as well as wavelets. Theoretical considerations have been illustrated by an example.

TRANSFORMACJE PRZEJŚCIA CZASOWO-CZĘSTOTLIWOŚCIOWYCH REPREZENTACJI SYGNAŁÓW

Streszczenie. Reprezentacje czasowo-częstotliwościowe stosowane są do analizy i syntezy deterministycznych układów i sygnałów niestacjonarnych. Relacje zachodzące pomiędzy różnymi grupami reprezentacji czasowo-częstotliwościowych noszą nazwę transformacji przejścia. W artykule opisano transformacje przejścia pomiędzy uogólnieniami reprezentacji Wignera wprowadzonymi przez Cohena i innych a reprezentacjami sygnałów w postaci amplitudy i częstotliwości chwilowej oraz reprezentacjami falkowymi. Rozważania teoretyczne zilustrowano przykładem.

1. INTRODUCTION

Contemporary methods of non-stationary systems and signals synthesis and analysis are based on time-frequency representations [3, 6, 8, 9]. Three main groups of these representations can be distinguished:

- instantaneous amplitude and frequency introduced for analytical signals [1],
- wavelet representations [7, 11],
- Wigner representation and its generalizations [8].

Relations between different groups of time-frequency representations (Fig. 1) are called transitive maps. They enable direct conversion of one representation into another one and after that comparison of systems or signals in the new feature space. Some of the transitive maps are known. In particular, transformations between wavelet representations and instantaneous amplitude and frequency are given in [2, 10]. Transformations between Wigner representation and instantaneous amplitude and frequency are also known [5, 8].

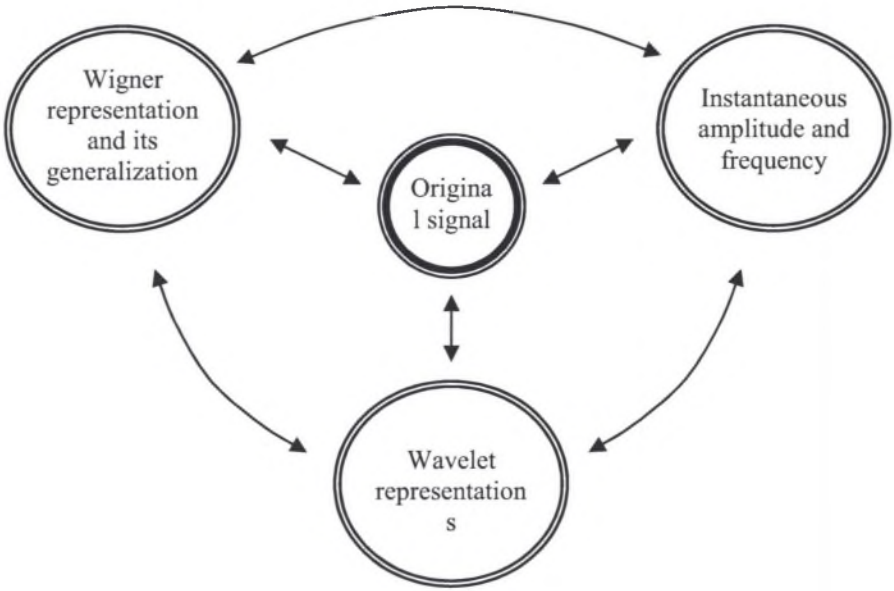

Fig. 1. Time-frequency representations and transitive maps
Rys. 1. Reprezentacje czasowo-częstotliwościowe i transformacje przejścia

The paper deals with transitive maps between generalized Wigner representations introduced by Cohen and the others and representations in the form of instantaneous amplitude and frequency as well as wavelet representations.

2. BASIC TIME-FREQUENCY REPRESENTATIONS

For real valued and finite energy signals $x(t) \in L^2(R)$, an associated analytical signal $z(t)$ is introduced as [1]:

$$z(t) = x(t) + j\mathrm{H}\ \{x(t)\} = A(t)e^{j\varphi(t)}, \tag{1}$$

where:

$H\{x(t)\}$ – Hilbert transform of the signal $x(t)$:

$$\mathrm{H}\ \{x(t)\} = (VP)\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau, \tag{2}$$

$A(t)$, $\varphi(t)$ – instantaneous amplitude and phase for an analytical signal $z(t)$.
Instantaneous amplitude, phase and frequency $f_i(t)$ are expressed as follows:

$$A(t) = \sqrt{x^2(t) + \left(\mathrm{H}\ \{x(t)\}\right)^2}, \tag{3}$$

$$\varphi(t) = arctg\frac{\mathrm{H}\ \{x(t)\}}{x(t)}, \tag{4}$$

$$f_i(t) = \frac{1}{2\pi}\frac{d\varphi(t)}{dt}. \tag{5}$$

Therefore, analytical signal can be represented in the Gauss plane by a vector rotating with time-varying angular speed $2\pi f_i(t)$. Its amplitude is also varying with time.

Continuous wavelet transforms can be used for signal decomposition using a set of basis functions called wavelets [11]. The direct W and inverse W^{-1} wavelet transforms are defined as [8]:

$$X(\tau,s) = \mathrm{W}\{x(t)\} = \frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} x(t)\Psi^*\left(\frac{t-\tau}{s}\right)dt, \tag{6}$$

$$x(t) = \mathrm{W}^{-1}\{X(\tau,s)\} = \frac{1}{c_\Psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\tau,s)\frac{1}{\sqrt{s}}\Psi\left(\frac{t-\tau}{s}\right)\frac{ds}{s^2}d\tau, \tag{7}$$

where:

$\Psi(t)$ - a basic wavelet that generates a set of wavelets by scaling with factor s and translation along time axis by τ (* denotes the complex conjugate),

c_Ψ - constant calculated on the base of admissible condition [11].

The time-frequency representation of a signal depends on the choice of basic wavelet $\Psi(t)$.

Discrete wavelet transforms are of great importance if multiresolution analysis has to be performed. It consists in signals decomposition into sum of details obtained using dyadic set of scales $\{2^j\}$, $j\in(-\infty, \infty)$ [7], [11]:

$$x(t) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left(x, \Psi_{jk}\right)_{L^2}\Psi_{jk}(t), \tag{8}$$

where:

$(\cdot,\cdot)$ - inner product in the space $L^2(R)$,

$\Psi_{jk}(t)$ - a wavelet defined by:

$$\Psi_{jk}(t) = \frac{1}{\sqrt{2^j}}\Psi\left(\frac{t-2^j k}{2^j}\right). \tag{9}$$

For the given resolution level function $x(t)$ is a linear combination of fine as well as coarser representations.

The generalized time-frequency representation $C(t,\omega)$, introduced by Cohen and developed later also by Claasen, Mecklenbräuker and Jansen [4], [6], for a signal $x(t)$ is expressed as:

$$C(t,\omega) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^*(u-0.5\tau)\,x(u+0.5\tau)\,\Phi(\Theta,\tau)e^{-j\Theta t - j\tau\omega + j\Theta u}\,du\,d\tau\,d\Theta, \tag{10}$$

where:

$\Phi(\Theta, \tau)$ - a kernel function.

The representation (10) may be considered as Fourier transform of two-dimensional local correlation function of a signal $x(t)$ multiplied by weight function $\Phi(\Theta, \tau)$.

Properties and what follows applications of the representation (10) depend on the kernel function $\Phi(\Theta, \tau)$. If $\Phi(\Theta, \tau) \equiv 1$ then formula (10) expresses classical Wigner transform. The other well-known transforms that can be obtained from (10) were developed by Margenau-Hill, Rihaczek, Borne-Jordan, Page, Choi-Williams, Zhao and the others [5].

The signal $x(t)$ reconstruction on the base of $C(t, \omega)$ can be carried out by means of the following formula [5]:

$$x(t) = \frac{1}{2\pi x(0)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{C(t', \omega)}{\Phi(\Theta, t)} e^{j\omega t + j\Theta(t' - 0.5t)} dt' \, d\omega \, d\Theta \,. \tag{11}$$

Thus, unambiguous reconstruction of the signal $x(t)$ requires information about its value for $t = 0$ ($x(0) \neq 0$) and is possible for regions where the kernel function $\Phi(\Theta, \tau)$ takes non-zero values.

3. TRANSITIVE MAPS BETWEEN INSTANTANEOUS AMPLITUDE AND FREQUENCY AND GENERALIZED TIME-FREQUENCY REPRESENTATIONS OF SIGNALS

3.1. Particular case $\Phi(\Theta,\tau) \equiv 1$

In that case equation (10) defines classical Wigner-Ville representation of an analytical signal $z(t)$ expressed by formula (1):

$$Z(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z^*(t - 0.5\tau) \, z(t + 0.5\tau) \, e^{-j\omega\tau} d\tau \,, \tag{12}$$

and subsequently [5], [8]:

$$A(t) = \sqrt{\int_{-\infty}^{\infty} C(t, \omega) \, d\omega} = \sqrt{\int_{-\infty}^{\infty} Z(t, \omega) \, d\omega} \,, \tag{13}$$

$$f_i(t) = \frac{\int_{-\infty}^{\infty} \omega \, C(t, \omega) \, d\omega}{\int_{-\infty}^{\infty} C(t, \omega) \, d\omega} = \frac{\int_{-\infty}^{\infty} \omega \, Z(t, \omega) \, d\omega}{A^2(t)} \,. \tag{14}$$

3.2. General case

Transitive map $(A(t), \varphi(t)) \rightarrow C(t, \omega)$ can be developed by direct substitution of formulae defining analytical signal (1) into equation (10):

$$C(t, \omega) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(u - 0.5\tau) A(u + 0.5\tau) \, \Phi(\Theta, \tau) e^{j[\varphi(u+0.5\tau) - \varphi(u-0.5\tau)] - j\Theta t - j\tau\omega + j\Theta u} du \, d\tau \, d\Theta \tag{15}$$

Inverse transitive map $C(t, \omega) \rightarrow (A(t), \varphi(t))$ can be obtained by substitution of given representation $C(t, \omega)$ of the signal $x(t)$ as well as the representation of Hilbert transform $H\{C(t, \omega)\}$ into equation (11):

$$A(t) = \sqrt{\alpha^2(t) + \beta^2(t)} \,, \tag{16}$$

$$\varphi(t) = arctg \frac{\beta(t)}{\alpha(t)} \,, \tag{17}$$

where:

$$\alpha(t) = \frac{1}{2\pi x(0)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{C(t',\omega)}{\Phi(\Theta,\tau)} e^{j\omega t + j\Theta(t'-0.5t)} dt'\, d\omega\, d\Theta\,, \tag{18}$$

$$\beta(t) = (VP)\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(\tau)}{t-\tau} d\tau\,. \tag{19}$$

Instantaneous amplitude and phase determination on the base of $C(t, \omega)$ representation requires information about the value of the signal $x(t)$ for $t = 0$.

4. TRANSITIVE MAPS BETWEEN GENERALIZED TIME-FREQUENCY REPRESENTATIONS AND WAVELET REPRESENTATIONS

4.1. Continuous wavelet transforms

Maps $X(\tau, s) \to C(t, \omega)$ are obtained as a result of superposition of eq. (7) and (10):

$$C(t,\omega) = \frac{1}{4\pi^2} \frac{1}{c_\Psi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Omega(u,\tau,\Theta,s,v,\varsigma,\xi) du\, d\tau\, d\Theta\, ds\, dv\, d\varsigma\, d\xi\,, \tag{20}$$

where:

$$\Omega(u,\tau,\Theta,s,v,\varsigma,\xi) = X(v,s)\Psi^*\left(\frac{u-0.5\tau-v}{s}\right) X(\xi,\varsigma)\Psi\left(\frac{u+0.5\tau-\xi}{\varsigma}\right)\Phi(\Theta,\tau)e^{-j\Theta t - j\tau\omega + j\Theta u} \frac{1}{\sqrt{s\varsigma s^2\varsigma^2}} \tag{21}$$

Similarly, superposition of eq. (6) and (11) gives:

$$X(v,s) = \frac{1}{2\pi x(0)\sqrt{s}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{C(t',\omega)}{\Phi(\Theta,t)} \Psi^*\left(\frac{t-v}{s}\right) e^{j\omega t + j\Theta(t'-0.5t)} dt'\, d\omega\, d\Theta\, dt\,, \tag{22}$$

4.2. Discrete wavelet transforms

As in the previous section, the superposition of maps (8) and (10) results in the map $(x, \psi_{jk})_{j,k\in N} \to C(t, \omega)$:

$$C(t,\omega) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left(x, \Psi_{jk}\right)_{L^2} \Gamma_{jk}(t,\omega)\,, \tag{23}$$

where:

$$\Gamma_{jk}(t,\omega) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_{jk}^*(u-0.5\tau)\Psi_{jk}(u+0.5\tau)\Phi(\Theta,\tau)e^{-j\Theta t - j\tau\omega + j\Theta u} du\, d\tau\, d\Theta\,. \tag{24}$$

Taking into account (8) and (11), the inverse map $C(t, \omega) \to (x, \psi_{jk})_{j,k\in N}$ may be expressed as follows:

$$x(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \zeta_{jk} \Psi_{jk}(t)\,, \tag{25}$$

where:

$$\zeta_{jk} = \frac{1}{2\pi x(0)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_{jk}(t) \frac{C(t',\omega)}{\Phi(\Theta,t)} e^{j\omega t + j\Theta(t'-0.5t)} dt'\, d\omega\, d\Theta\, dt\,. \tag{26}$$

5. EXAMPLE

Let's apply proposed in the paper conversion formulae to an EEG signal presented in Fig. 2. It belongs to a healthy object and was acquired using standard EEG system of electrodes called 10-20 which was introduced by International Federation of Electroencephalography and Clinical Neurology. The sampling frequency f_s was set to 200 Hz. The length of the analyzed signal segment is equal to 256 points.

First the instantaneous amplitude and frequency have been calculated using formulae (3) and (5), Fig.3. Next, the Wigner-Ville and Cohen's representation for cone-shaped kernel function [12] have been obtained, Fig. 4. Cohen's representation for cone-shaped kernel function, which has the form $\Phi(\Theta,\tau) = g(\tau)\,|\tau|\,(a\Theta\tau)^{-1}\sin(a\Theta\tau)$, is called Zhao-Atlas-Marks representation. Finally, the wavelet transform (Fig. 5) has been calculated using mexican hat basic wavelet [11]. The results have been presented using 3D graphs as well as x-y projections, Fig. 4-5. Representations shown in Fig. 4 and 5 have been obtained on the base of formulae developed in chapters 3.2 and 4.2.

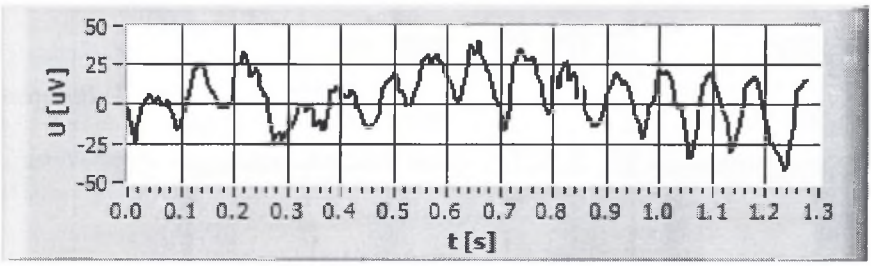

Fig. 2. EEG signal from electrode F4 (system 10-20)
Rys. 2. Sygnał EEG uzyskany z elektrody F4 (system 10-20)

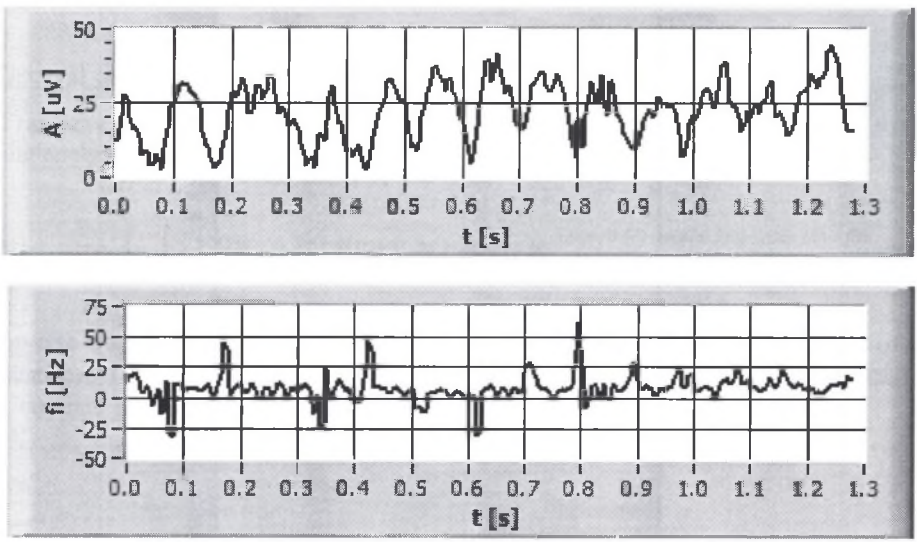

Fig. 3. Instantaneous amplitude and frequency of the EEG signal
Rys. 3. Amplituda i częstotliwość chwilowa analizowanego sygnału EEG

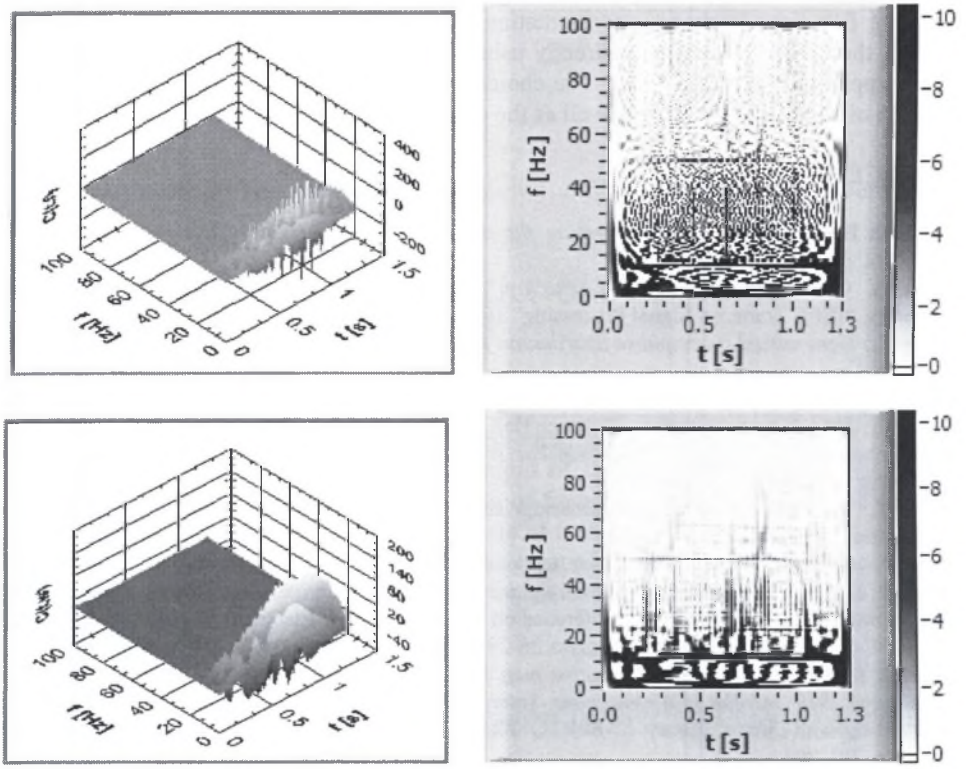

Fig. 4. Wigner-Ville and Zhao-Atlas-Marks distributions of the EEG signal
Rys. 4. Rozkłady Wignera-Ville'a oraz Zhao-Atlas-Marksa analizowanego sygnału EEG

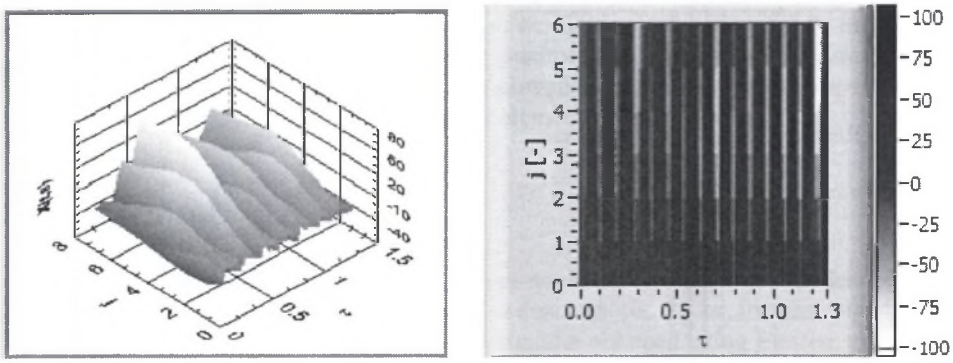

Fig. 5. Wavelet transform of the EEG signal
Rys. 5. Transformata falkowa analizowanego sygnału EEG

6. CONCLUSIONS

The transitive maps between time-frequency signal representations, including instantaneous amplitude and phase, wavelets and generalized Wigner representations, have been given in the paper. They enable comparison of signals and systems properties from different points of view.

Having first time-frequency representation calculated, the second one can be calculated directly on the base of signal or indirectly using formulae proposed in the paper. Generally, these two approaches are equivalent. The choice, which one should be used, may be made by comparison of computation times as well as the other specific numerical requirements.

REFERENCES

1. Boashash B.: *Estimating and interpreting the instantaneous frequency of a signal.* "Proc. IEEE" 1992, Vol. 80, No 4, p. 520-570.
2. Bovik A. C., Maragos P.: *AM-FM detection and separation in noise using multiband energy operators.* "IEEE Trans. on Signal Processing" 1999, Vol. 41, No 12, p. 32-35.
3. Cohen L.: *Generalized space-phase distribution functions.* "Jour. Math. Phys." 1996, Vol. 7, p. 781-786.
4. Cohen L.: *Time-frequency distribution – a review.* "Proc. IEEE" 1989, Vol. 77, No 7, p. 941-981.
5. Cohen L.: *Time-frequency analysis.* Prentice Hall, Englewood Cliffs 1995.
6. Mecklenbrauker W., Hlawatsch F. (eds.): *The Wigner distribution theory and applications in signal processing.* Elsevier, Amsterdam 1997.
7. Meyer Y.: *Wavelets and operators.* Cambridge studies in advanced mathematics, Vol. 37, Cambridge Univ. Press, London 1992.
8. Poularikas A.D.: *The transforms and applications handbook.* CRC Press, Florida 1995.
9. Walczak J., Grabowski D.: *Application of instantaneous amplitude and frequency to electrical systems description.* International Conference on Fundamentals of Electrical Engineering and Circuit Theory IC-SPETO 2004. Gliwice-Niedzica 26-29.05.2004. Gliwice 2004, Vol. II, p.461-464.
10. Walczak J., Lewandowski M.: *Transitive map between instantaneous amplitude and frequency of signals and their wavelet representations.* International Conference on Fundamentals of Electrical Engineering and Circuit Theory IC-SPETO 2004. Gliwice-Niedzica 26-29.05.2004. Gliwice 2004, Vol. II, p. 201-204.
11. Wojtaszczyk P.: *Teoria falek.* PWN, Warszawa 2000.
12. Zhao Y., Atlas L. E., Marks R. J.: *The use of cone-shaped kernels for generalized time-frequency representations or nonstationary signals.* "IEEE Trans. Acoust., Speech, Signal Processing" 1990, Vol. 38, p. 1084-1091.

Recenzent: Prof. dr hab. inż. Tadeusz Łobos

Wpłynęło do Redakcji dnia 4 maja 2004 r.