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SOME INCORRECT PROBLEMS OF MATHEMATICAL PHYSICS ARISING IN STRATA MECHANICS

Summary. The problem of the earth surface and surface items protection from the harmful impact of the underground mining is reduced to seeking a solution o the 1D or 2D Fredholm's integral equation from the first kind. It is an incorrect problem of the mathematical physics in the sense of J.S.Hadamard. The solution is achieved by the regularization method.
In that way the foundation of a new branch in the modern earth surface protection theory is laid.

NIEKTÓRE NIEPRAWIDŁOWO POSTAWIONE ZAGADNIENIA FIZYKI MATEMATYCZNEJ POJAWIAJĄCE SIĘ W MECHANICE GÓROTWORU

Streszczenie. Problemy ochrony powierzchni przed niszczącym wpływem eksploatacji górniczej są ograniczone do poszukiwania rozwiązania pojedynczego lub podwójnego równania całkowego Fredholma pierwszego rodzaju. Jest to nieprawidłowo postawione zagadnienie fizyki matematycznej w sensie J.S. Hardamarda. Rozwiązanie zostało znalezione metodą regularyzacji. W ten sposób została położona podstawa nowej gałęzi w nowoczesnej teorii ochrony powierzchni.

НЕКОТОРЫЕ НЕКОРРЕКТНЫЕ ЗАДАЧИ МАТЕМАТИЧЕСКОЙ ФИЗИКИ ВОЗНИКАЮЩИЕ В ГЕОМЕХАНИКЕ

Резюме. Проблема о охране земной поверхности и наземных обьектов от вредного влияния подземных горных работ сведена к поиску решения едномерного и двумерного интегрального уравнения Фредгольма первого рода.

Решение этой некорректной, в смысле Ж. Адамара, задачи получено методом регуляризации. Таким образом положены основы нового раздела современной теории охраны земной поверхности.

1. INTRODUCTION

One of the effects of the underground mining is the occurrence of a subsidence on the Earth surface which creates unfavorable conditions for the surface items (buildings, equipment, bents, etc.). In this connection in the geomechanics arise the following problems: direct problem - when the mining method is given to determine the equation of the mining subsidence; inverse problem - when the equation of the mining subsidence (dictated from the building codes, so to preserve the surface items) is given, to determine the mining operations, which realize the given subsidence. The second problem is of extremely importance in the cases when the mining out has to be done under the already build up areas.

Let us first of all draw our attention to the following facts. In the direct problem, when the cause is given (mining out), the result is searched (the mining subsidence) and in the inverse problem, when the result is given (the equation of the mining subsidence), to determine the cause for it (the mining method). Since one and the same result can be incited by a numerous causes, it becomes clear that we have to face a lot of difficulties for solving the inverse problem.

Let us note, that the difficulties arose also from the fact, that as fare the physicomechanical properties of the rock mass, as the equation of the mining subsidence (in the inverse problem) are given approximately: First are based on measurements (i.e., there is an error), and the equation is dictated, as we noted, from the building codes for different sites (the acceptable differences in the subsidence is given for the different sites in a limits "from..... to.....").

2. FORMULATION AND SOLUTION OF THE DIRECT PROBLEM

Let us now continue with the direct problem. If we accept that the excavated area is known and that the principle of superposition is valid (Fig.1), than the solution of the direct problem, for y=H is given by the formula [5]

$$v(x,z)=\iint\limits_{(F)}\varphi(x-\xi,z-\varsigma)v^0(\xi,\varsigma)d\xi d\varsigma \qquad (1)$$

where $v^0=v^0(\xi,\varsigma)$ is the subsidence of the immediate roof of the excavated area in the point (ξ,ς) [5], F -excavated area, φ -influence function, which according to the authors has different types, for example, according to S.Knothe [5].

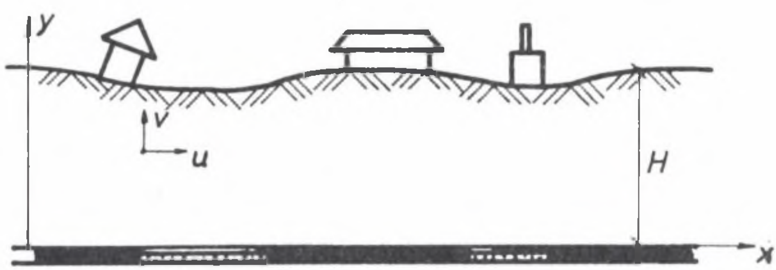

Fig. 1. Basic scheme
Rys. 1. Schemat podstawowy

$$\varphi=\frac{1}{r_0}e^{-\pi\left(\frac{r}{r_0}\right)^2} \qquad (2)$$

where $r=\sqrt{(x-\xi)+(z-\varsigma)^2}$, $r_0=H/tg\beta$, β influence angle which is determined on the base of the geodezical measurements of the given region [5]. If we base our thoughts on the supposal of the incompressibility of the rock mass (J. Litwiniszyn) and if we take into account the experimental dependencies of S.G.Aversin [4], or the empirical dependencies of H. Keinhorst (J. Litwiniszyn) [4] we rich respectively to the Cauchy's problem for the Laplaces equation. The solutions of these problems takes also the form (1), even with a different functions $\varphi(x-\xi)$.

It is noticeable, that the direct problem is reduced to integrating, according to (1) and we are not faced with a big difficulties in solving it.

Let us draw our attention to fact, that in the practice the domain F is limited (the functions φ and v^0 have a local carrier), but they can be easily transformed into infinite, by adding zeros to the functions φ and v^0 over the rest of the axis ξ and ζ.

3. FORMULATION AND SOLUTION OF THE INVERSE PROBLEM

Let us continue with the inverse problem. For facility's sake first we will consider the 2D case, by writing (1) in the form:

$$Av^0 \equiv \int_{+\infty}^{-\infty} \varphi(x-\xi)v^0(\xi)d\xi = v(x)\,, \quad -\infty < x < +\infty \tag{3}$$

or in the form:

$$Av^0 \equiv \int_a^b \varphi(x-\xi)v^0(\xi)d\xi = v(x)\,, \quad c < x < d \tag{4}$$

The inverse problem is reduced to seeking a solution of the Fredholm's integral equations from the first kind (3) or (4), i.e., it is reduced to determining the function $v^0(\xi)$ when the function $\varphi(x-\xi)$ is given and the equation of the mining subsidence is uniformly, approximately given $v = v(x)$ (according to the building codes for protecting the surface items).

Before to consider such a posed problem, let us remind that in order one problem of the mathematical physics to be correctly posed in the sense of J.Hadamard, the following conditions has to be satisfied [10].

1) Existence of the solution.
2) Uniqueness of the solution.
3) Stability of the solution, i.e. small changes in the data to lead to a small changes in the results.

Now we will see, that the inverse problem is incorrect because the third requirement of J.Hadamard is broken. For this the treating of the problem has to be more careful.

Let us consider equation (4) and let us suppose that the real solution $\overline{v}^0(\xi)$ exist and is unique. Let us now consider the function

$$\tilde{v}(\xi) = \overline{v}(\xi) + B\cos n\xi \tag{5}$$

It is easy to see, that for a random fixed B such a N can be find, that for any n ≥ N, the function $\tilde{v}(\xi)$ will satisfy the equation (4) with accuracy δ, because

$$\left|\int_a^b v^0(\xi)\varphi(x-\xi) - v(x)\right| = |B|.\left|\int_a^b \varphi(x-\xi)\cos n\xi d\xi\right| = O\left(\frac{1}{n}\right) \tag{6}$$

In the same time $\tilde{v}^0(\xi)$ differs strongly from $\overline{v}^0(\xi)$ because $\left\|\tilde{v}^0(\xi) - \overline{v}^0(\xi)\right\|_C = |B|$, i.e., the posed problem is unstable and sequently incorrect according to Hadamard. When solving the integral equation (4), the Tichonov's regularization method is used [10] [4]. Its essence we will lay down now.

It worth to note, that the solution of the integral equation (3) (from convolutionary type) can be written in the type of the classical formula for exact data:

$$v^0(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\hat{v}(\omega)}{\hat{\varphi}(\omega)}.e^{-i\omega\xi}d\omega \tag{7}$$

where

$$\hat{v}(\omega) = \int_{-\infty}^{+\infty} v(x)e^{i\omega x}dx, \quad \hat{\varphi}(\omega) = \int_{-\infty}^{+\infty} \varphi(x)e^{i\omega x}dx \tag{8}$$

are the Fourier's transforms of the right part of the kernel of (4).

The usage of (7) for calculation of $v^0(\xi)$ for inexact data $\tilde{v}(x)$ (see the conclusion) is inadmissible. Really if we suppose that $v(x) \in L_2(-\infty,+\infty)$, $\varphi(x) \in L_1(-\infty,+\infty)$, $v^0(\xi) \in L_1(-\infty,+\infty)$ and we apply the Fouier's transform over (3) in the case, when the right part is given approximately, i.e. $v(x) = \overline{v}(x) + v^{*}(x)$, where $\overline{v}(x)$ is an exact part and $v^{*}(x)$ is the noise, then after retransformation we will receive

$$v^{0}(x) = \overline{v}(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\hat{v}(\omega)}{\hat{\varphi}(\omega)}\exp(-i\omega x)dx \tag{9}$$

This function, however, is posible not to exist, since the last integral is divergent. How to proceed in this case we will explain down.

Now we will debate as generally as possible.

We saw that our main problem is reduced to seeking a solution of the, generally sad, operator equation

$$Av^{0} = v\,,\quad v \in V\,,\quad v^{0} \in V_{o} \tag{10}$$

where A: $D_A \subseteq V_o \to V$ is an operator with a not empty domain of determination over D_A, acting from the metrical space V_o in the analogical space V. The setting of V and V_o here is important.

Now we can say, that one problem is correctly posed by J.Hadamard if: 1) The domain of the values Q_A of the operator A coincide with V_o (solvability); 2) From the equality $Au_1 = Au_2$ for a random $u_1, u_2 \in D_A$ it follows the equality $u_1 = u_2$ (uniqueness); 3) The inverse operator A^{-1} is continuous over V (stability).

The question "is the inverse operator A^{-1} continuous?" is basic. If the answer is negative, then the problem is unstable. Such an unstable problem is the problem for solving the operator equation, given with completely continuous operator. For example: the operator (4) is completely continuous when mapping the continuous over [a, b] functions and consequently the problem for solving the integral equation (4) is incorrect by Hadamard.

Let us draw our attention to the fact, that formally we can transform the problem (10) into correctly posed if we limit ourselves in considering the right parts v(x), of some closer class V_1, but this approach is inconstructive. In the practice v(x) is given with an error (see the conclusion) and that's why for V we are forced to choose the Chebishev or Hilbert's space and the solution $v^{0}(x)$ has to be received in Chebishev's space.

So what we have to do when solving incorrect problems?

Let us first of all introduce the concept conventionally correct problem (Tichonov). One problem is conventionally correct if: 1') It is appriory known that the solution of the problem (10) exist for some class of data from V and belongs to an appriory given set $M \subset D_A$; 2') the solution is unique in the class M; 3') to a infinitely small variations in the right part which does not take out the solution from the class M, corresponds infinitely small variations in the solution. The set M is called set of correctness. In the considered theory extremely important role plays the theorem for stability of the inverse operator: If the not empty set $M \subseteq D_A$ is a compact and satisfies the conditions 1') and 2'), then for the continuous operator A, its inverse A^{-1}, considered over the image of the set M is continuous.

The next important step, made by A.Tichonov is the constructing of the regularization method. It is based on the idea for stabilizing the minimum of the shift of Av^0 from $\bar{v}$ by the smoothing functional $\Omega(v^0)$, determined over some part of $V_{00} \subseteq D_A$, which is a metrical space. It is necessary the sets $M_C = \{v^0 \in V_{00};\ \Omega(v) \le C$ to be compacts in V for every $C \ge 0$. It is supposed, that the solution of (10) for some value $C < +\infty$, consists in M_C. Now we are solving the problem for minimization by $v^0 \in V_{00}$ of the Tichonov's parametrical functional

$$M_\alpha\left[v^0;\bar{v}\right] = \rho_V^2\left(Av^0,\bar{v}\right) + \alpha\Omega^2\left(v^0\right),\ \alpha > 0 \tag{11}$$

The solution $\bar{v}_\alpha^0$ of the problem for a known regularization parameter $\alpha = \alpha(\delta)$ is accepted as a approximate solution of the problem. The parameter of the regularization α is determined from the condition

$$\rho(\alpha) \equiv \rho_V\left(A\bar{v}_\alpha^0,\bar{v}\right) = \delta \tag{12}$$

where δ is the error in the giving of the right part (see [8]).

Before to apply the above lyed theory, by considering a concrete integral equation, let us draw our attention to the following important fact. In the kernel of (1) are involved the characteristics of the rock mass, which are determined on the base of field measurements, i.e., they always are given with some error. It means, that the operator in (3) and (4) is also given approximately. All this in not an obstacle the described theory to be applied in these cases too [1].

Now we will consider the integral equation (3) [11]. Let $\varphi(y) \in L_1(-\infty,+\infty) \cap L_2(-\infty,+\infty)$, $\bar{v}^0(x) \in L_2(-\infty,+\infty)$, $\bar{v}^0(\xi) \in W_2^1(-\infty,+\infty)$, i.e., $A: W_2^1 \to L_2$. We will suppose, that the kernel $\varphi(y)$ is closed, i.e., A is operator and we will consider the problem (3) without limits. Let to the function $\bar{v}(x)$ correspond a unique solution of (3) - the function $\bar{v}^0(\xi) \in W_2^1$, besides it is known not the function $\bar{v}(x)$ and the operator A, but a function
$v_\delta(x)$ and an operator A_h from a convolutionary type with a kernel $\varphi_h(y)$, given with some error $\delta, h > 0$ and such, that

$$\|v_\delta(x) - \bar{v}(x)\|_{L_2} \le \delta \ , \ \|A - A_h\|_{W_2^1 \to L_2} \le h \tag{13}$$

Let us consider the smoothing functional

$$M_\alpha[v_0] = \left\|A_h v^0 - v_\delta\right\|_{L_2}^2 + \alpha\left\|v^0\right\|_{W_2^1}^2 \tag{14}$$

Since $W_2^1(-\infty,+\infty)$ is a Hilbert space, than for any $\alpha > 0$ and a random $v_\delta \in L_2$ it exist an unique element $v_\eta^{0\alpha}(\xi)$ with realize the minimum of the functional $M_\alpha\left[v^0\right]$.

On the base of the convolutionary theorem and the Plansherel's equality, after varying the functional $M_\alpha\left[v^0\right]$ over the set of functions from W_2^1 we receive:

$$v_\eta^{0\alpha}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\tilde{\varphi}_h^*(\omega)\tilde{v}_\delta(\omega)e^{-i\omega\xi}}{L(\omega) + \alpha(\omega^2+1)} d\omega \tag{15}$$

where $\tilde{\varphi}_h^*(\omega) = \tilde{\varphi}_h(-\omega)$, $L(\omega) = |\tilde{\varphi}_h(\omega)|^2 = \tilde{\varphi}_h^*(\omega)\tilde{\varphi}_h(\omega)$ and $\tilde{\varphi}_h(\omega)$, $\tilde{v}_\delta(\omega)$ are Fourier's images of the functions $\varphi_h(y)$, $v_\delta(x)$, f.e.

$$\tilde{v}_\delta(\omega) = \int_{-\infty}^{+\infty} v_\delta(x) e^{i\omega x} dx \tag{16}$$

If we substitute (16) in (15), than $v_\eta^{0\alpha}(\xi)$ accepts the form

$$v_{\eta}^{0\alpha}(\xi) = \int_{-\infty}^{+\infty} \varphi^{\alpha}(x-\xi) v_{\delta}(x) dx \tag{17}$$

where the kernel of the inversion $\varphi^{\alpha}(t)$ has the form

$$\varphi^{\alpha}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\bar{\varphi}_h^{*}(\omega) e^{i\omega t}}{|\bar{\varphi}_h(\omega)|^2 + \alpha(\omega^2+1)} d\omega \tag{18}$$

Since in the practice normally $v_{\delta}(x)$ has a limited carrier, than the interpretation in (17) is over the area where $v_{\delta}(x)$ differs from zero. Thus for receiving of $v_{\eta}^{0\alpha}(\xi)$ for fixed α it is sufficient to be determined numerically the Fourier's image of the kernel $\bar{\varphi}_h(\omega)$ and after that to determine the kernel of the inversion $\varphi^{\alpha}(t)$, using the standart programs for calculations of integrals with fast oscillating functions and finally to apply the formula (17) (for more detail see [11])

If we want to attack the 3D inverse problem, then we have to solve the Fredholm's integral equation from the first kind

$$Av^0 \equiv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k(x-\xi, z-\varsigma) v^0(\xi,\varsigma) d\xi d\varsigma = v(x,z), \quad -\infty < x, z < +\infty \tag{19}$$

for v^0, when v is given. Let $v^0 \in L_2(R^2)$ is the unique, exact solution, which corresponds to some exact right part $v \in L_2(R^2)$ .Then the searched solution v^0 of (19), according to the theorem for the convolution, we can formally write in the form

$$v^0(\xi,\varsigma) = F^{-1}V^0 = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{V(\lambda,\omega)}{K(\lambda,\omega)} . e^{i(\lambda\xi+\omega\varsigma)} d\lambda d\omega, \quad i = \sqrt{-1} \tag{20}$$

where F is an operator of Fourier- Planscherel, V^0, K, V are the 2D Fourier's transforms (Fourier's images), respectively to the functions v^0, k , v, determined by the formula (f.e. for v^0):

$$V^0(\lambda,\omega) = Fv^0 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v^0(\xi,\varsigma)e^{-i(\lambda\xi+\omega\varsigma)}d\xi d\varsigma\,, \quad -\infty < \lambda,\ \omega < +\infty \tag{21}$$

Unfortunately the formal solution (20) in the considered case can not be used, because for an approximately given operator and right part of (19), it is not stabile and the problem of solving the Fredholm's integral equation in this case is incorrectly posed [4] [10]. The formulated inverse problem is exactly like this - the properties of the rock mass are determined on the base of measurements, consequently are approximately given and the equation of the mining subsidence being dictated by the building codes is approximately known.

In this case we will apply the regularization method [1], [11].

It reduces the problem to minimization by v^0 of the smoothing functional

$$M^\alpha\left[\bar{v}^0, \bar{v}\right] = \left\|\bar{A}\bar{v}^0 - \bar{v}\right\|^2_{L_2} + \alpha\Omega\left[\bar{v}^0\right] \tag{22}$$

where

$$\left\|\bar{A}\bar{v}^0 - \bar{v}\right\|^2_{L_2} = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\bar{K}(\lambda,\omega)\bar{V}^0(\lambda,\omega) - \bar{V}(\lambda,\omega)\right|^2 d\lambda d\omega$$

is the square of the disjunction, $\alpha > 0$ is a regularization parameter,

$$\Omega\left[\bar{v}^0\right] = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[1 + \left(\lambda^2 + \omega^2\right)^p\right]\left|\bar{v}^0(\lambda,\omega)\right|^2 d\lambda d\omega$$

- stabilizing functional from power $p \geq 0$ (p is not suppose to be integral), $\bar{K}, \bar{V}^0, \bar{V}$ - Fourier's images of the functions $\bar{k}, \bar{v}^0, \bar{v}$, which are determined, according to (21).

The functional (22), when α is fixed can be minimized by the function

$$\bar{v}^0_\alpha(\xi,\varsigma) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\bar{K}^*(\lambda,\omega)\bar{V}(\lambda,\omega)}{\left|\bar{K}(\lambda,\omega)\right|^2 + \alpha\left[1 + \left(\lambda^2 + \omega^2\right)^p\right]} \cdot e^{i(\lambda\xi+\omega\varsigma)}d\lambda d\omega \tag{23}$$

(* is a sign of a complex correlation), which is considered as a regularized solution of the problem.

A desecrate analogize and a numerical algorithm for solving this problem are given in [1].

On (Fig. 2.) the results of the solution of one inverse example problem are given. In the direct problem the theory of Knothe (H = 150 m ; β = 70° ; v_o = 1,2 m ; a = 50 m) is used and on the base of the received by the regularization method solution (the program "Mudle") v = v(x), the initially condition is restored, i.e. it is determined the subsidence of the immediate roof for y = 0, which realize the subsidence v = v(x). The received results are encouregable.

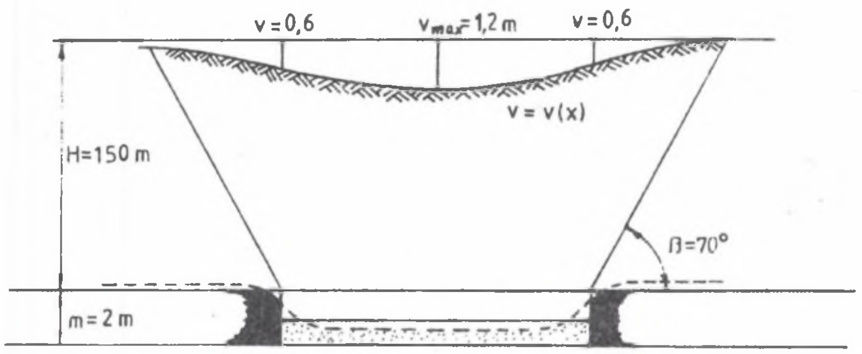

Fig. 2. An example solution of inverse problem
Rys. 2. Przykład rozwiązania problemu odwrotnego

3. CONCLUSION

When the mining has to be done under the build up areas and the problem for protecting the surface items appears, then the inverse problem in the theory of mining subsidence appears too. Its essence is: For approximately uniformly given equation of the mining subsidence (dictated from the building codes), to determine the method of mining which has caused the given subsidence. This problem is reduced to seeking a solution of the Fredholm's integral equation from the first kind, which is incorrectly posed in the sense of Hadamard. For solving it, the regularization method (A.N.Tichonov) , which has been described, is applied. The results from the numerical experiments are encouregable.

Let us also note, that in the known discussion between Berry [2] and Litwiniszyn [6] the last was claming (see also [7]), that the inverse problem can be solved by "inversing the operator" in (3). The reasonings made in chapter 2 shows that it is impossible.

We can meet the following reasonings too: The building codes for every object dictates the acceptable inclination (i.e. the differences in the subsidence) and other characteristics of the subsidence, which shows the "limits from...... to.......". Then isn't it possible to take a fixed data in-between the acceptable for the codes and to "inverse" the operator in (3). The answer of the question is negative because if other investigator, independently choose again fixed data and again in-between the acceptable from the codes and again "inverse" the operator in (3), then the two solutions would not have nothing common because the problem is not stabile. Then which one of the two to choose? Simply we have to apply the method described above.

For the first time in 1985 [3] we posed and show the way for solving the inverse problem for the mining subsidence. In that way the foundations of the modern theory for the Earth surface protection in the build up areas as a branch of the modern theory of the environmental protection has been layed.

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