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Natalya PANKRATOWA, Anatoli WASILENKO
Institute of Mechanics of the Ukrainian Academy of Sciences, Kijów

THE STRESS STATE OF ANISOTROPIC INHOMOGENEOUS ENVIRONMENT WITH SPHERICAL CAVITY

Summary. An approach to analytic-numerical solution of problems about distribution of stresses in elastic transversal-isotropic environment near spherical cavity or nonhomogeneous spherical inclusion is proposed. The resultant solution is written as the sum of two subsolutions: the first of them corresponds to the main stress state, the another - the disturbed stated caused by cavity or inclusion. The results of solution of the real problem for nonhomogeneous transversally isotropic medium with aspherical cavity is given.

STAN NAPRĘŻENIA ANIZOTROPOWEGO NIEJEDNORODNEGO OŚRODKA Z PUSTKĄ KULISTĄ

Streszczenie. Przedstawiono podejście do analityczno-numerycznego rozwiązania zadań rozkładu naprężeń w sprężystym ośrodku transwersalnie izotropowym w sąsiedztwie pustki lub niejednorodnej inkluzji kulistej. Rozpatrywane rozwiązanie zapisywane jest jako suma dwóch rozwiązań: pierwsze z nich odpowiada podstawowemu stanowi naprężenia, drugie - stanowi zaburzonemu, uwarunkowanemu obecnością pustki lub inkluzji. Przytacza się wyniki rozwiązania konkretnego zadania dla ośrodka transwersalnie izotropowego ze swobodną pustką kulistą.

НАПРЯЖЕННОЕ СОСТОЯНИЕ АНИЗОТРОПНОЙ НЕОДНОРОДНОЙ СРЕДИ СО СФЕРИЧЕСКОЙ ПОЛОСТЬЮ

Резюме. Предлагается подход к аналитическо-численному решению задач о распределении напряжений в упругой трансверсально изотропной среде вблизи сферического вкючения. Рассматриваемое решение залисывается как сумма двух решений: первое из них соответствует основному напряженному состоянию, второе - возмущенному состоянию, обусловленному полостью или включением. Приводятся результаты решения конкретной задачи для трансверсально изотропной среды со свободной сферической полостью.

1. INTRODUCTION

The problems about distribution of stress in elastic environment in the vicinity of cavities and inclusions are a subject of actual investigations in mechanics, having important theoretical and applied value.The results of decision of such problems are applied in mechanics of composite materials for definition of their effective characteristics. This problem has also important value in mining mechanics for estimation of concentmodulus of stress near reinforced and unreinforced output, for calculation of underground constructions. While deciding problems of considering class it is necessary to take in account such important factors as structure of environment, its unhomogen eit and anisotropy, the characteristics of interaction with inclusions, varieties of mining pressure and jet repulse. In order to consider the influence of mentioned above factors the equations of three- dimension problems of elastic theory of unhomogeneous anisotropic body should be used. It allows to put aside a whole set of simplifications.

2. THE METHOD OF INVESTIGATION

This article proposes the approach for analytic- numerical solution of the set of problems about stress state of elastic transversal-isotropic environment, containing spherical cavity or unhomogeneous spherical inclusion [3]. It is considered that the environment is unhomogeneous in the cavity vicinity. That's why its elastic characteristics are known functions of radial coordinate. Inclusion presents itself as a cavity or continuous elastic sphere made from transversal-isotropic layers, rigidly connected with each other. Simulation of the inclusion interaction with elastic environment takes into account two types of contacts: rigid clamp, excluding the slip and alienation, and ideal slip. Other conditions can be considered as those among stated above.

It is supposed that environment is subjected to action of pressure stresses σ_0, τ_0 and bending stress σ_1 on the sufficient distance from inclusion (on "infinity"), and it can be written in Cartesian coordinates X,Y,Z as following:

$$\sigma_z = \sigma_0, \qquad \sigma_x = \sigma_y = \tau_0 + \sigma_1 z, \tag{1}$$

where $\sigma_0, \tau_0, \sigma_1$ - known constants.

Solution of given problems is built by means of the method, which is used in the problems of stress concentmodulus near cavities [2]. It allows to introduce solution as a sum sum of two items ,the first of which corresponds to main stressed state (1) and the second - disturbed one, is caused by the presence of cavity or inclusion.

The main stressed state can be written in spherical system of coordinates r, φ, θ

$$\sigma_r(r,\varphi) = 1/3(\sigma_0 + 2\tau_0)P_0 + 2/5r\sigma_1 P_1(\cos\varphi) + 2/3(\sigma_0 - \tau_0)P_2(\cos\varphi) - 2/5r\sigma_1 P_3(\cos\varphi),$$

$$\tau_{r\varphi}(r,\varphi) = -1/5r\sigma_1 dP_1(\cos\varphi)/d\varphi + 1/3(\sigma_0 - \tau_0)dP_2(\cos\varphi)/d\varphi - 2/15r\sigma_1 dP_3(\cos\varphi)/d\varphi, \tag{2}$$

$$\sigma_\varphi(r,\varphi) = 1/3(2\sigma_0 + \tau_0)P_0 + r\sigma_1 P_1(\cos\varphi) - 2/3(\sigma_0 - \tau_0)P_2(\cos\varphi),$$

$\sigma_0(r,\varphi) = \tau_0 + 3/5r\sigma_1 P_1(\cos\varphi) + 2/5r\sigma_1 P_3(\cos\varphi)$, where $Pn = Pn(\cos\varphi)$ - Lezandr's polynomials.

Generally, laminated cavity sphere is distinguished from considering environment, and its internal radius coincides with the cavity radius $r = r_0$, and externalone $r = rn$ is much greater than the radius of cavity and inclusion thickness. Thus, disturbed stress state is determined by solving the problem about stress state of mentioned sphere, the external surface $r = rn$ of which is free from stresses, i.e. $\sigma_r(r_n, \varphi) = \tau_n(r_n, \varphi) = 0$ (3)

But internal one $r = r_0$ is loaded in the following manner:

$$\sigma_r(r_0,\varphi) = -\sigma_{r0}(r_0,\varphi) + p_0(\varphi), \tag{4}$$

$\tau_{r\varphi}(r_0,\varphi) = -\tau_{\varphi_0}(r_0,\varphi) + g_0(\varphi)$, where p_0, g_0 are self-equilibated loads, acting on the cavity surface. If the cavity is free, then $p_0 = g_0 = 0$.

Strain-stress state of radially-unhomogeneous transversal-isotropic spheres is determined by the system of partial differential equations [1]

$$\partial\bar{\sigma}^{(i)}/r = A_0^{(i)}(r,\varphi)\bar{\sigma}^i + A_1^{(i)}(r,\varphi)\partial\bar{\sigma}^i/\partial\varphi + A_2^{(i)}(r,\varphi)\partial^2\bar{\sigma}^i/\partial\varphi^2$$

$$r_{i-1} \le r \le r_i, i = 1,2,\ldots,n \tag{5}$$

$$\bar{\sigma}^i = \{\sigma_r^i, \tau_{r\varphi}^i, u_r^i, u_\varphi^i\}$$

Here u_r^i, u_φ^i are displacement in the layer i.The matrix elements $A_0^{(i)}, A_1^{(i)}, A_2^{(i)}$ depend on the properties of material of the layer i.

Except boundary conditions (3),(4), solutions of system (5) must satisfy conditions of adjacent layers contact. For rigidly connected layers it is necessary to implement the following:

$$\sigma_r^{(i)}(r_i,\varphi)=\sigma_r^{(i+1)}(r_i,\varphi);\tau_{r\varphi}^{(i)}(r_i,\varphi)=\tau_{r\varphi}^{(i+1)}(r_i,\varphi)$$
$$u_r^{(i)}(r_i,\varphi)=u_r^{(i+1)}(r_i,\varphi);u_\varphi^{(i)}(r_i,\varphi)=u_\varphi^{(i+1)}(r_i,\varphi) \quad (6)$$

The decision of equation system (5) is conducted by separating coordinate in system with decomposition of searched functions in rows by Lezhandr's polynomials and it's derivatives. It is allows to get the system of ordinary differential equations for n-th member of decomposition

$$d\bar{\sigma}_n^i/dr=B_n^i(r)\bar{\sigma}_n^i;\bar{\sigma}_n^i=\{\sigma_{rn}^i,\tau_{\varphi n}^i,u_{rn}^i,u_{\varphi n}^i\} \quad (7)$$

with boundary conditions in points $r=r_0$ and $r=rn$ and conditions in separate points of integmodulus interval, formulated according to relations (6).

Calculating algorithm and program package for decision of one-dimension boundary problems were made and realized for system (7) with implementation of stable numeric method.

The solution of problem can be received, adding the complement of main and disturbed stress states for each of layers.

$$\begin{aligned}
x_1&=\sigma_{r0}^0(r,\varphi)+\sum_{n=0}^{\infty}\sigma_{rn}(r)P_n(\cos\varphi),\\
x_2&=\tau_{\varphi}^0(r,\varphi)+\sum_{n=1}^{\infty}\tau_{r\varphi n}(r)dP_n(\cos\varphi)/d\varphi,\\
x_3&=u_r^0(r,\varphi)+\sum_{n=0}^{\infty}u_{rn}(r)P_n(\cos\varphi),\\
x_4&=u_{\varphi 0}^0(r,\varphi)+\sum_{n=1}^{\infty}u_{\varphi n}(r)dP_n(\cos\varphi)/d\varphi
\end{aligned} \quad (8)$$

Here u_r^0,u_φ^0 - displacements, which correspond to main state.

If the conditions of the rigid contact of adjacent layers are not fulfiled,their slip takes place. In this case the jump of displacement takes place at the surface of layer conjunction, for instance, in the area of inclusion's contact with environment. Boundary conditions for disturbed state, as distinct from rigid contact, should be formulated for taking account of summary condition.

$$\sigma_r^i(r_i,\varphi)=\sigma_r^{i+1}(r_i,\varphi); u_r^i(r_i,\varphi)=u_r^{i+1}(r_i,\varphi),$$
$$x_2^i(r_i,\varphi)=x_2^{i+1}(r_i,\varphi)=0. \qquad (9)$$

Conditions (9) are related to polynomial boundary edge conditions. The solution of the problem of such type can be received by reducing it to row of two-pointone.

The realization of proposed approach in program package gave a possibility to implement a decision of set of concrete problems.

During the problem deciding it is necessary to control the value of external spherical radius *rn*. When *rn*$>>r_0$, the distribution of stresses in the cavity vicinity, presented by solution (8), is expected to be the same as in infinite environment. It allows to adopt the change of the solution depending on r_n as criterion of estimating the solution correctness. The accuracy of solution can be taken from the modulus of stresses ranges in different values r_n.

3. CALCULATION OF THE NONHOMOGENEOUS ENVIRONMENT WITH SPHERICAL CAVITY

Lets observe the problem about distribution of stresses in isotropic unhomogeneous environment under the influence of tensile stresses . The modulus of elasticity changes by law: E = E*o*(b + a*r* $(1/r_0)$), E*o* =const. Poisson ratio is constant and it equals 0.3. Decision of problem was implemented for three variants ofa, b values:

1)a=0, b=1 2)a=b=0.5 3)a= -1,b=2. In the first variant we have homogeneous environment. In all variants the modulus of environment elasticity on the cavity surface is noticed to be the same. The analysis of decided problem results has shown that the results do not practically depend on the range of environment unhomogenity with the distance from the cavity near to its radius. It was received as the result of implemented calculations that the changing of stresses on the cavity surface for considered variants a and b is given in the following formulas:

1) $\sigma_\varphi / \sigma_0 = 2.045 - 2.722\cos^2\varphi$; $\sigma_\theta / \sigma_0 = 0.136 - 0.818\cos^2\varphi$

2) $\sigma_\varphi / \sigma_0 = 2.214 - 2.912\cos^2\varphi$; $\sigma_\theta / \sigma_0 = 0.123 - 0.812\cos^2\varphi$

3) $\sigma_\varphi / \sigma_0 = 1.852 - 2.519\cos^2\varphi$; $\sigma_\theta / \sigma_0 = 0.137 - 0.805\cos^2\varphi$

Unhomogenity of environment influences affects the values of stresses σ_θ. Decreasing the modulus of elasticity and removing from cavity increases stresses σ_φ . Decreasing E , the reverse maximum values σ_φ take place, and σ_φ are changing on 8.2% and 9.4% as compared with the quantity for homogeneous material.

Let's make the investigation of stressed state of elastic isotropic environment with hole, reinforced by spherical shell with thickness h. The modulus of elasticity of shell material E is d time as much as environment one Eo. Calculations were made for h/r_0=0.1;0.5 and d=2,4,6,8,10.

The distribution of stresses σ_φ and σ_θ can be investigated in environment on the surface of its contact with reinforcement.

$$\sigma_\varphi = (\alpha + \beta\cos^2\varphi)\sigma_0, \sigma_\theta = (\delta + \gamma\cos^2\varphi)\sigma_0.$$

The values $\alpha, \beta, \delta, \gamma$ are given in the Table 1.

Table 1

d	α	β	δ	γ
h/r_0=0.1				
2	-1.593	1.905	-0.046	0.358
4	-1.477	1.718	-0.021	0.262
6	-1.404	1.603	-0.007	0.206
8	-1.359	1.524	0.001	0.170
10	-1.315	1.466	0.006	0.144
h/r_0=0.5				
2	-1.125	-1.132	0.013	-0.103
4	-1.078	-1.075	0.016	-0.020
6	-1.057	-1.051	0.016	-0.022
8	-10.45	-1.038	0.015	-0.022
10	-1.037	-1.030	0.011	-0.020

Analysis of given data shows that the presence of reinforcement decreases concentmodulus of stresses essentially. If thickness of reinforcement h=0.1r_0 maximum values σ_φ are decreasing on the 22% when α=2 and on 35% when α=10.

Increasing of reinforcement thickness leads to more essential σ_φ decrease. So, for h/r_0=0.5 when α >6 maximum values σ_φ are greater on several percents the values of themselves, according to main stress state.

There is an interest to consider problem, when isotropic spherical layer is between reinforcing isotropic shell with modulus of elasticity E and transversal isotropic elastic environment, and elastic properties of layers is distinct from material and shell properties.

The modulus of elasticity of intermediate layer equals E'=0.025E, Poisson ratioγ''=0.45. The environment material has following characteristics in the plane of isotropy: the modulus of elasticity equals E''=0.1E, Poisson ratio equals γ''=0.3. In the directions which are perpendicular to plane of isotropy the modulus of elasticity equals E"=0.1E, the modulus of shear G''=0.0025E'' , Poisson ratio equals γ'' =0.25. The radius of conjugate surface of shell with intermediate layer equals r_1=R, its thickness equals R. Calculations were made for the case of environment bend by stresses $\sigma_1(\sigma_0=\tau_0=0)$ for different values of cavity radius r_0=0.1R,0.2R,...,0.9R. The stresses in reinforcing shell on the surface of its conjunction with intermediate layer, carried away to maximum values σ^0 in homogeneous environment without cavity can be presented as

$$\bar{\sigma}_\varphi = \sigma_\varphi(R,\varphi)/\max|\sigma_\varphi^0(R,\varphi)| = A\cos\varphi + B\cos^3\varphi$$

$$\bar{\sigma}_\theta = \sigma_\theta(R,\varphi)/\max|\sigma_\theta^0(R,\varphi)| = C\cos\varphi + D\cos^3\varphi$$

The values of coefficients are given in Table 2.

Table 2

r_0/R	A	B	C	D
0.1	-0.376	2.210	-0.128	1.963
0.2	-0.382	2.219	-0.128	1.966
0.3	-0.415	2.266	-0.121	1.979
0.4	-0.519	2.415	-0.120	2.016
0.5	-0.753	2.751	-0.088	2.085
0.6	-1.185	3.370	0.026	2.159
0.7	-1.864	4.339	0.381	2.093
0.8	-2.669	5.489	1.442	1.378
0.9	-2.885	6.086	4.583	-1.394

Data given in the Table 2 , shows that concentration of stresses growth with increase of r_o/R. Coefficient of concentmodulus equals 1 .834 when $r_o/R=0.1$, grows to 3 .183 when $r_o/R=0.9$. The above mentioned coefficient doesn't actually change in the interval $0.1<= r_o/R <=0.4$. By this, values σ_ϕ and σ_θ are close to orbitrary angle ϕ and fixed modulus r_o/R. Distinction between σ_ϕ and σ_θ grows with increase of r_o/R. So, there is a range in interval $0<=\phi <= \pi/2$, where σ_ϕ and σ_θ are distinct by signs. Thus, the change of thickness of reinforcing shell gives a considerable influence on the value of stresses and character of its distribution in the cavity vicinity.

The approach and solutions of concrete problems, received on its basis, proposed in this work, can be used to control the influence of mountains pressure on the cavities in elastic environment and selection of parameters of construction elements reinforcement.

REFERENCES

[1] Grigorenko Y.M. , Vasilenko A.T. , Pankratova N.D. Problems ofthe elasticity theory for nonhomogeneous bodies, p. 2 16. Naukova dumka, Kiev 199 I . In Russian.

[2] Timoshenko S.P. Course theory of elasticity, p. 575. Naukova dumka, Kiev 1975 In Russian.

[3] Vasilenko A.T. , Pankratova N. D. Numerical-analytic solution of the problem of elasticity theory for nonhomogeneous environment with spherical cavity or inclusion . I. Mech. of Solid Body, N1, 82-88, 1993 In Russian.

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Streszczenie

Proponowane jest pewne podejście do numeryczno-analitycznego rozwiązywania zadań rozkładu naprężeń w transwersalnie izotropowym ośrodku sprężystym w otoczeniu pustki lub niejednorodnej inkluzji kulistej. Zakłada się, że w otoczeniu pustki lub inkluzji ośrodek jest osiowo-niejednorodny o charakterystykach sprężystych opisanych zadanymi funkcjami. Inkluzja jest pełną lub pustą kulą składającą się z warstw transwersalnie izotropowych. Inkluzja może być sztywno złączona z ośrodkiem, w wyniku czego wykluczone jest jej oderwanie się i przemieszczanie; możliwy jest także wariant idealnego poślizgu. W dostatecznej odległości od pustki ośrodek poddany jest działaniu naprężeń rozciągających lub ściskających. Szukane rozwiązanie zapisywane jest sumą dwóch rozwiązań; jedno z nich odpowiada podstawowemu stanowi naprężenia, drugie - stanowi zaburzonemu, wywołanemu obecnością pustki lub inkluzji. Zaburzony stan naprężenia formowany jest jako stan naprężenia kuli pełnej lub pustej, której promień zewnętrzny jest dostatecznie duży w porównaniu z rozmiarami pustki lub inkluzji. Do rozwiązania zadania stanu naprężenia uwarstwionej niejednorodnej kuli wykorzystywane są związki teorii sprężystości niejednorodnego ciała anizotropowego. Bierze się przy tym pod uwagę, że kula może składać się z dowolnej liczby warstw, z których każda charakteryzuje się własną anizotropią i niejednorodnością własności sprężystych. Zastosowanie przekształceń analitycznych, obejmujących rozdzielenie zmiennych, daje możliwość sprowadzenia wyjściowego zadania przestrzennego do zadania jednowymiarowego, które rozwiązywane jest za pomocą metody numerycznej. Wiarygodnośc wyników zadania rozwiązanego takim sposobem potwierdzona jest poprzez porównanie z rozwiązaniem analitycznym dla szczególnego przypadku ośrodka jednowymiarowego z pustką niezamocowaną. Podane są wyniki rozwiązania konkretnych zadań dla niejednorodnego ośrodka transwersalnie izotropowego ze swobodną pustką kulistą.