ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1997
Seria: GÓRNICTWO z. 236 Nr kol. 1378

David CHAPPELL*

# LECTURE NOTES ON ONE-DIMENSIONAL NON-LINEAR MAPS WITH AN APPLICATION IN MACROECONOMIC THEORY

**Summary.** This paper discusses the behaviour of chosen discrete dynamical systems (maps). We start by considering linear, constant-coefficient, maps and demonstrate that such maps can be readily solved. We also derive stability and boundary conditions for such maps. We then briefly discuss more general, n-dimensional linear maps. We discuss in some detail the behaviour of the logistic map as we vary its parameter. We show that this model can exhibit dynamic behaviour ranging from convergence to a point attractor, through convergence to limit cycles of all orders, and ultimately to so-called deterministic chaos. We conclude by presenting an example of a non-linear mapping from economics (a model of inflation) and derive conditions under which the economy may exhibit chaotic behaviour.

# UWAGI DO WYKŁADU NA TEMAT JEDNOWYMIAROWYCH NIELINIOWYCH MODELI DYNAMICZNYCH Z CZASEM DYSKRETNYM I ICH ZASTOSOWAŃ W MAKROEKONOMII

**Streszczenie.** W artykule jest dyskutowane zachowanie się wybranych modeli dyskretnych systemów dynamicznych. Na początku rozpatrujemy liniowe równanie różnicowe o stałych współczynnikach i pokazujemy, że tego rodzaju równanie może być w sposób czytelny rozwiązane. Wyprowadzamy warunki zbieżności i stabilności takiego ciągu. Następnie, krótko i bardziej ogólnie, omawiamy n-wymiarowy liniowy dynamiczny model dyskretny. W szczegółach omawiamy zachowanie równania logistycznego w zależności od zmian wartości parametru. Pokazujemy, że model ten może prezentować zmieniające się właściwości dynamiczne systemu od zbieżności do punktu, poprzez zbieżność do przebiegów cyklicznych aż do tzw. deterministycznego chaosu. Na zakończenie prezentujemy przykład nieliniowego modelu dyskretnego z zakresu ekonomii (model inflacji) i wyprowadzamy warunki, przy których proces ekonomiczny może charakteryzować się zachowaniem chaotycznym.

* Department of Economics, University of Sheffield, UK

# 1. Introduction

This paper discusses the behaviour of some one-dimensional discrete dynamical systems (maps). We start by considering linear, constant-coefficient, maps and demonstrate that such maps can be readily solved. We also derive stability and boundedness conditions for such maps. We then briefly discuss more general, *n*-dimensional linear maps, but we do not give a fully detailed treatment because our primary purpose in this paper is to consider the behaviour of some one-dimensional *non-linear* maps.

We discuss in some detail the behaviour of the logistic map as we vary its parameter. We show that this model can exhibit dynamic behaviour ranging from convergence to a point attractor, through convergence to limit cycles of all orders, and ultimately to so-called deterministic chaos.

We demonstrate that chaotic behaviour ensues when there is sensitive dependence on initial conditions and that sensitive dependence on initial conditions results in a positive Lyapunov exponent: a necessary and sufficient condition for chaotic behaviour to be possible. Our exposition of this material is in the style of a 'tutorial' and is illustrative rather than mathematically rigorous. We do not claim to present any new results and similar expositions, at various levels of rigour, are to be found in several alternative texts. See, for example, Gulick (1992) - an excellent introduction to the subject - or, at a more advanced level, Devaney (1989). Our aim here is to provide a concise treatment of the subject written in a style that is accessible to non-specialists. We do, however, introduce a simple method of calculating approximations to Lyapunov exponents that we have not seen elsewhere in the literature.

We conclude by presenting an example of a non-linear mapping from economics (a model of inflation) and derive conditions under which the economy may exhibit chaotic behaviour.

# 2. One-dimensional linear mappings

Consider the model:

$$X_{t+1} = aX_t + b \qquad (1)$$
$$a, b, X_0 \in R$$

The solution of such a model is a function, $X(t)$, satisfying (1) and with $X(0) = X_0$. The solution may be found by repeated substitution, as follows.

$$X_1 = aX_0 + b$$
$$X_2 = aX_1 + b = a^2X_0 + ab + b$$
$$X_3 = aX_2 + b = a^3X_0 + a^2b + ab + b$$

Repeating this process over and over again we deduce that:

$$X_t = aX_{t-1} + b = a^tX_0 + b(1 + a + a^2 + ........+ a^{t-1})$$

and it now follows that:

$$X_t = \begin{cases} a^t\left(X_0 - \frac{b}{1-a}\right) + \frac{b}{1-a} & ; \text{ if } a \neq 1 \\ X_0 + bt & ; \text{ if } a = 1 \end{cases} \tag{2}$$

It is clear that the behaviour of the solution sequence depends crucially on $|a|$. If $|a| < 1$, the solution converges to the point $b/(1 - a)$: It converges monotonically if $0 < a < 1$ and has damped oscillations if $-1 < a < 0$. If $a = -1$, the soution is a two-point cycle which alternates repeatedly between the values $X_0$ and $b - X_0$. If $|a| > 1$, the solution diverges: monotonically if $a > 1$ and with explosive oscillations if $a < -1$. It is clear that the solution is stable (i.e. approaches a finite limit) if $|a| < 1$ and remains bounded if $-1 \leq a < 1$. Figure 1 (which was produced using the Microsoft Excel spreadsheet) shows a typical solution trajectory.

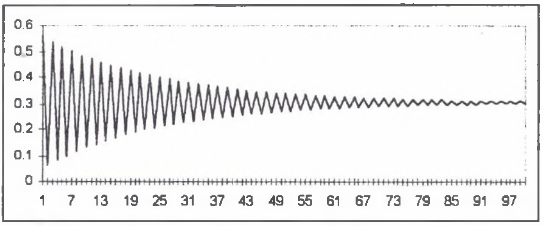


Fig. 1. The solution trajectory of the model (1); a = - 0.96, b = 0.6

Rys. 1. Trajektoria rozwiązania modelu (1); a = - 0.96, b = 0.6

## 3. Multi-dimensional linear maps

One possible generalisation of the models in the previous section are $n$-dimensional linear maps of which a typical example is as follows:

$$\mathbf{X}_{t+1} = \mathbf{A}\mathbf{X}_t + \mathbf{b}$$

where:

$$\mathbf{X}_t = [X_{1t}, X_{2t}, X_{3t}, \ldots\ldots, X_{nt}]^T$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & . & . & . & a_{1n} \\ a_{21} & . & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & . & . & . & a_{nn} \end{bmatrix}$$

$$\mathbf{b} = [b_1, b_2, \ldots\ldots\ldots, b_n]^T$$

Such models are stable if all the eigenvalues of the matrix **A** lie within the unit circle. Such maps can exhibit extremely complex, but nevertheless non-chaotic, dynamic behaviour.

## 4. One-dimensional non-linear maps

Perhaps the most widely studied of the class of one-dimensional non-linear maps is the logistic equation. This has been much discussed in the literature and is a single-parameter quadratic map of the general form:

$$\begin{gathered} X_{t+1} = kX_t(1 - X_t) \\ k \in [1,4],\ X_0 \in (0,1) \end{gathered} \tag{3}$$

This model can exhibit a wide range of dynamic behaviour ranging from the very simple (monotonic convergence to a point) to the very complex (deterministic chaos). Plots of trajectories for two different values of the single parameter, $k$, are given in figures 2 and 3 below. Note that, except for the case of $k = 4$, it is not possible to give a solution of this equation. This is almost invariably the case with non-linear maps: Only in a handful of cases is it possible to derive closed-form solutions. For example, if $k = 4$ in the logistic equation, the solution is:

$$X_t = \sin^2\left(2^t \sin\left(\sqrt{X_0}\right)\right) \tag{4}$$

To gain some insight into the behaviour of this equation for values of $k$ other than $k = 4$, all we can do is carry out numerical simulations using, for example, the Microsoft Excel spreadsheet (which was used to produce figures 2 and 3).

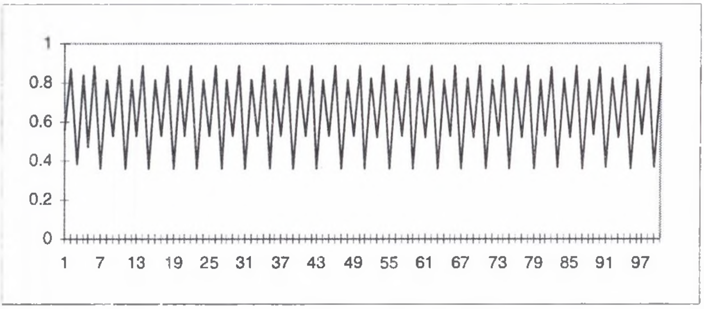


Fig.2. The solution trajectory of the model (3) - an eight point cycle; $k = 3.55$

Rys.2. Trajektoria rozwiązania modelu (3) -cykl ośmiopunktowy; k=3.55

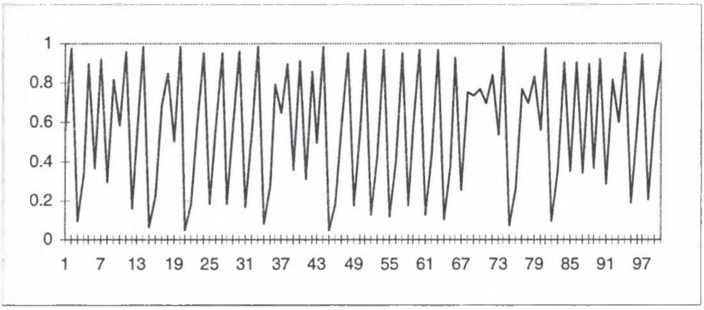


Fig. 3. The solution trajectory of the model (3) - deterministic chaos; $k = 3.95$

Rys. 3. Trajektoria rozwiązania modelu (3) - chaos deterministyczny; k=3.95

Of course at this point we are merely *asserting* that the behaviour illustrated in Figure 3 is chaotic. We will show this a little more convincingly in the following section.

## 5. Chaos: sensitive dependence on initial conditions

It is well known that the trajectories (solution sequences) of some non-linear maps can exhibit chaotic behaviour when there is sensitive dependence on initial conditions. To illustrate this for the logistic map we demonstrate what happens to trajectories which are initially close together for two different values of $k$: $k$ = 2.4 (non-chaotic behaviour) and $k$ = 4 (chaotic). Let the initial points of the two trajectories be $X_0$ and $X_0 + h$, where $h$ is 'small'. Figures 4 and 5 below, which were again produced using Excel, illustrate what happens when for each of these two values of $k$ when $X_0$ = 0.59 and $h$ = 0.05.

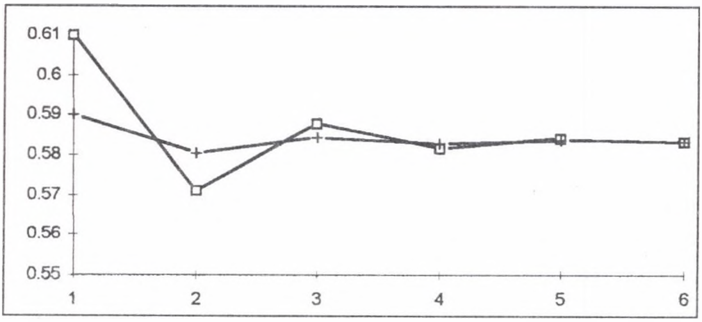


Fig. 4.The solution sequences of the equation (3) for $k$ = 2.4, $X_0$ = 0.59, $h$ = 0.05

Rys. 4. Ciągi rozwiązań równania (3) dla $k$ = 2.4, $X_0$ = 0.59, $h$ = 0.05

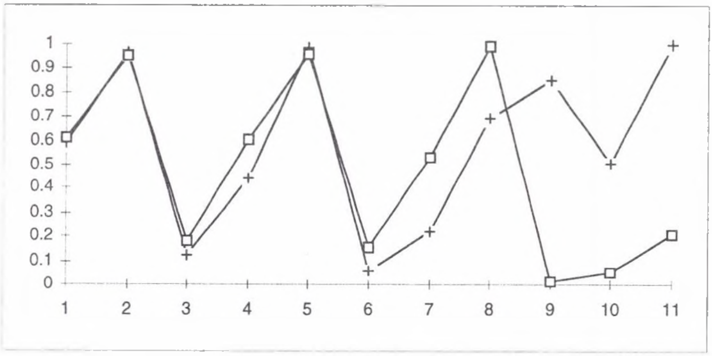


Fig. 5. The solution sequences of the equation (3) for $k$ = 4, $X_0$ = 0.59, $h$ = 0.05

Rys. 5. Ciągi rozwiązań równania (3) dla $k$ = 4, $X_0$ = 0.59, $h$ = 0.05

Two things are immediately apparent. In the non-chaotic case the two trajectories converge after only a few iterations of the map. In the chaotic case, trajectories which were initially close together start to diverge after only a few iterations. It is this feature that makes prediction with a chaotic model impossible.

Now consider the general class of (non-linear) models:

$$X_{t+1} = F(X_t)$$

where $F$ is some non-linear function. The separation of two trajectories, initially 'close' together, after $n$ iterations of the map is given by:

$$\left|F^{[n]}(X_0 + h) - F^{[n]}(X_0)\right| \quad (5)$$

where $h$ is 'small'. We make the following assumption:

$$\left|F^{[n]}(X_0 + h) - F^{[n]}(X_0)\right| \approx \left(e^{\lambda_n(X_0)}\right)^n h \quad (6)$$

Taking the limit as $h \to 0$ it follows that:

$$e^{n\lambda_n(X_0)} \approx \left|\frac{dF^{[n]}(X_0)}{dX_0}\right|$$

$$\Rightarrow \lambda_n(X_0) \approx \frac{1}{n}\ln\left|\frac{dF^{[n]}(X_0)}{dX_0}\right|$$

Finally, taking the limit as $n \to \infty$, the Lyapunov exponent is defined as:

$$\lambda(X_0) = \lim_{n\to\infty} \frac{1}{n}\ln\left|\frac{dF^{[n]}(X_0)}{dX_0}\right| \quad (7)$$

Suppose $D$ is a finite closed interval of $R$ and suppose that $F$: $D \to D$. Then $F$ is said to be chaotic at $X_0$ if $\lambda(X_0) > 0$. In these circumstances (bounded trajectories) a positive Lyapunov exponent is a necessary and sufficient condition for chaotic behaviour.

## 6. Calculation of Lyapunov exponents

For any one-dimensional map $F$ an approximation to the Lyapunov exponent can be found by using the fact that:

$$\ln\left|\frac{dF^{[n]}(X_0)}{dX_0}\right|$$

$$\equiv \ln\left|\frac{dF^{[n]}(X_0)}{dX_{n-1}}\frac{dF^{[n-1]}(X_0)}{dX_{n-2}}\cdots\cdots\frac{dF^{[1]}(X_0)}{dX_0}\right| \tag{8}$$

$$= \sum_{i=1}^{n}\ln\left|\frac{dF^{[i]}(X_0)}{dX_{i-1}}\right|$$

It follows that:

$$\lambda_n(X_0) = \frac{1}{n}\sum_{i=1}^{n}\ln\left|\frac{dF^{[i]}(X_0)}{dX_{i-1}}\right| \tag{9}$$

Note that in order to calculate the Lyapunov exponent directly as defined in (7) above it is necessary to be able to derive a closed-form expression for $F^{[n]}(X_0)$ and this is, of course, not possible in the vast majority of cases. Hence the need for an approximation, as given in (9). However, for illustration, we can calculate the Lyapunov exponent exactly in the following two cases.

1. For the *linear* map given in (1) above we know from (2) that, for $a \neq 1$ ,

$$F^{[n]}(X_0) = X_n = a^n\left(X_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$$

from which:

$$\lambda(X_0) = \underset{n\to\infty}{Lim.}\ \frac{1}{n}\ln\left|\frac{dF^{[n]}(X_0)}{dX_0}\right| = \ln(a)$$

which is non-positive for a bounded map. Hence, as of course is well-known, bounded *linear* maps *cannot* be chaotic.

2. For the logistic map with $k = 4$ we know from (4) above that:

$$F^{[n]}(X_0) = X_n = \sin^2\left(2^n \sin\left(\sqrt{X_0}\right)\right)$$

from which

$$\lambda(X_0) = \underset{n\to\infty}{Lim.}\ \frac{1}{n}\ln\left|\frac{dF^{[n]}(X_0)}{dX_0}\right| = \ln(2) > 0$$

Hence, the logistic map with $k = 4$ is, as is well-known, chaotic.

We conclude this section by using the expression in (9) to calculate the finite approximation to the Lyapunov exponent of the logistic map as a function of *n*: (i) For $k = 1.5$ (which is non-chaotic) and (ii) for $k = 4$

(chaotic). The results are shown in figures (6) and (7) below. Note that in each case $n$, which ranges from 1 to 1,000, is on the horizontal axis and the value of the approximate Lyapunov exponent is on the vertical axis.

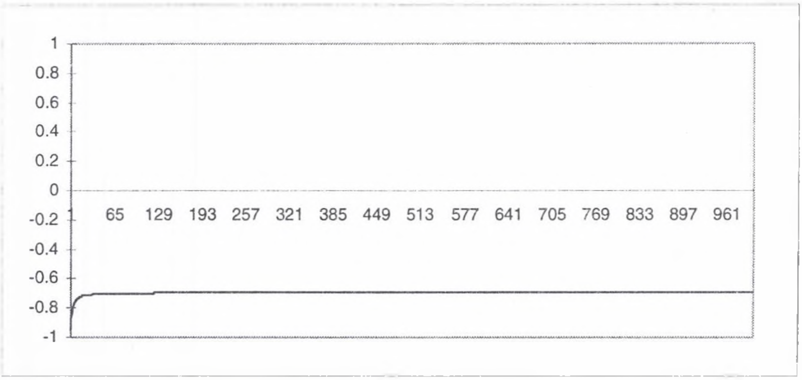


Fig. 6. Lyapunov exponent when $k$ = 1.5

Rys. 6. Wykładnik Lapunowa dla k = 1.5

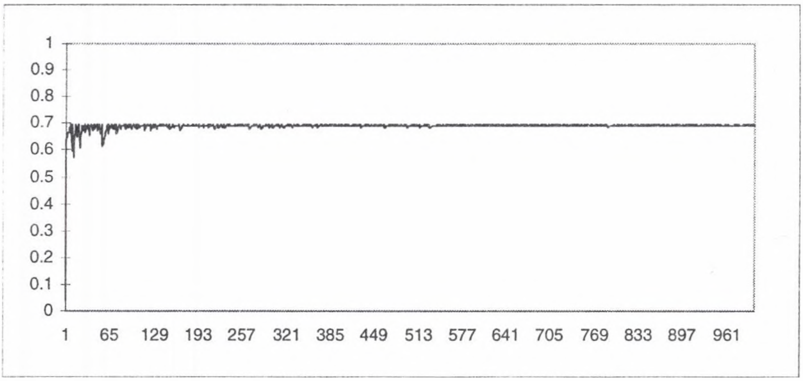


Fig. 7. Lyapunov exponent when $k$ = 4

Rys. 7. Wykładnik Lapunowa dla k = 4

We observe that in the non-chaotic case the Lyapunov exponent quickly converges to a negative value (approximately -0.7), whereas in the chaotic case it rapidly converges to its (known) true value of ln(2). It appears that this is a useful way of calculating approximations to the Lyapunov exponent of a one-dimensional map when it cannot be calculated exactly.

## 7. An economic application

In this section we consider a simple macroeconomic model of a closed economy. (The model is described in considerably more detail in Chappell (1997)).

The model consists of two equations; a demand function for real balances (purchasing power) and a Government budget constraint. The demand for real balances as a proportion of real disposable income depends negatively on the nominal rate of interest, which here is equal to the expected inflation rate since we are assuming a zero real rate of interest. Hence:

$$\ln\left(\frac{M_t}{P_t Y_t (1-v)}\right) = \alpha - \gamma E_t \pi_{t+1} \tag{10}$$

where $M_t$ is the stock of money, $P_t$ is the price level, $Y_t$ is real income, all at time $t$; $v$ is the income tax rate, $E_t\pi_{t+1}$ is the expectation at time $t$ of the inflation rate, $\pi$, at time $t$+1 and $\alpha$ ,$\gamma$ and $v$ are positive constants with $0 \leq v < 1$. Government purchases are financed partly by tax revenues and partly by printing money. Hence:-

$$G_t = \frac{M_t - M_{t-1}}{P_t} + vY_t \tag{11}$$

where $G_t$ is real government purchases of goods and services at time $t$. We assume that the growth rate of real income is exogenous and constant (possibly zero). We also assume that real government purchases are a constant proportion of real income and that the government adjusts its rate of expansion of the nominal money supply in order to maintain this proportion. Finally, we assume that economic agents form their expectations of the future inflation rate by means of the simple rule:

$$E_t \pi_{t+1} = \beta \pi_t \tag{12}$$

where $\beta$ is a positive constant. Substituting from (10) and (12) and rearranging, it follows that equation (11) may be written:-

$$u_t = \frac{ae^{\beta\gamma\pi_t}}{1 - ae^{\beta\gamma\pi_t}} \tag{13}$$

where $a \equiv \frac{(g-v)e^{-\alpha}}{1-v} \geq 0$ (by assumption), $u_t \equiv \frac{M_t - M_{t-1}}{M_{t-1}}$ and $g \equiv \frac{G_t}{Y_t}$ and assumed to be constant. Substituting from (13) into the first difference of equation (10) and re-arranging gives (in implicit form) the following non-linear first order difference equation for the rate of inflation:

$$e^{\beta\gamma\pi_t} - (1+y)(1+\pi_t)(1 - ae^{\beta\gamma\pi_t})e^{\beta\gamma\pi_{t-1}} = 0 \tag{14}$$

where $\pi_t \equiv \frac{P_t - P_{t-1}}{P_{t-1}}$ and $y \equiv \frac{Y_t - Y_{t-1}}{Y_{t-1}}$ and is assumed to be constant.

It is clear that equation (14) is defined only for $-1 < \pi_t < (\beta\gamma)^{-1}\ln(a^{-1})$. Because of its structure, it cannot be solved directly for $\pi_t$ and, in the simulations which follow, it is necessary to run the model 'backwards' through time: In effect, we generate the 'history' of the inflation rate rather than its 'forwards' path. This is done merely for mathematical convenience, however, and is taken into account in our conclusions about the dynamic behaviour of the inflation rate which are based on forward paths; all diagrams also show forward trajectories. Let $\beta\gamma\pi_t = x_{-t}$; then, from equation (14):

$$x_{t+1} = x_t + \ln\left(\frac{b}{1+y}\right) - \ln\left((b + x_t)(1 - ae^{x_t})\right) = F(x_t) \tag{15}$$

where $b \equiv \beta\gamma$. The mapping defined in (15) above is capable of exhibiting a range of different types of dynamic behaviour; convergence to a point attractor, convergence to cycles of various lengths and deterministic chaos. Each of these patterns will be associated with particular ranges of values of the parameters *a*, *b* and *y*. Suppose the mapping defined in equation (15):

(i) Has two fixed points, both of which are locally repelling.

(ii) Is bounded in the sense that it maps a subset, *S*, of the real line into itself.

(iii) Has a positive Lyapunov exponent.

Suppose $x_0 \in S$. Then the mapping will generate trajectories that *appear* to be chaotic. Note that condition (i) can only be satisfied if the map has a turning-point. It is straightforward to show that the map has a turning-point (which is a minimum) when $x_t = \theta$, where $\theta$ satisfies $ae^{\theta} + \theta = 1 - b$. It is also easy to show (by considering the gradient of (15) and using the definition of $a$) that the mapping has two fixed points iff

$$g < v + (1 - v)\left[1 - \sqrt{\frac{b}{1+y}}\right] e^{\left(a + b - \sqrt{\frac{b}{1+y}}\right)} \tag{16}$$

We have already made the assumption that $a \geq 0$; if $a = 0$ the government balances its budget but if $a > 0$ at least some of its expenditure is financed by printing money. For $a$ to be negative we would have the somewhat curious, and we believe less-interesting, case of the government taking money out of circulation. If our assumption is to hold, then, referring to (16), we need $\beta\gamma \leq 1 + y$; the demand for real money balances must not be too interest elastic.

Now let us consider the boundedness and stability properties of the map. Let $F^m(x_0)$ denote the $m$'th iterate of the map starting from some $x_0 \in S$. Then it is easy to see that the trajectory will be bounded iff $F^3(\theta)) \leq F^2(\theta)$ and, if this is the case, the set $S$ is the interval $[F(\theta), F^2(\theta)]$. It should also be clear that the upper fixed point, $x_u$, is always repelling (since $F'(x_u) > 1$ ) and the lower fixed point, $x_l$, is also repelling if $F'(x_l) < -1$. The Lyapunov exponent (as defined in (7)) cannot be calculated exactly and so we use the method of approximation discussed in the previous section. Of course the *exact* value of the Lyapunov exponent for this map is unknown. However, based on our experiments with the logistic equation, we feel that our approximations (which are based on iterating the map 2,000 times) will be fairly accurate. Figures 8 and 9 show, respectively, the last 100 iterated values of the variable $x_t$ for a chaotic map (where $y$ = 0.02, $a$ = 0.13, $b$= 0.3 and $x_0$ = 0.25) and the behaviour of the Lyapunov exponent as $m$ is increased from 1000 to 2,000.

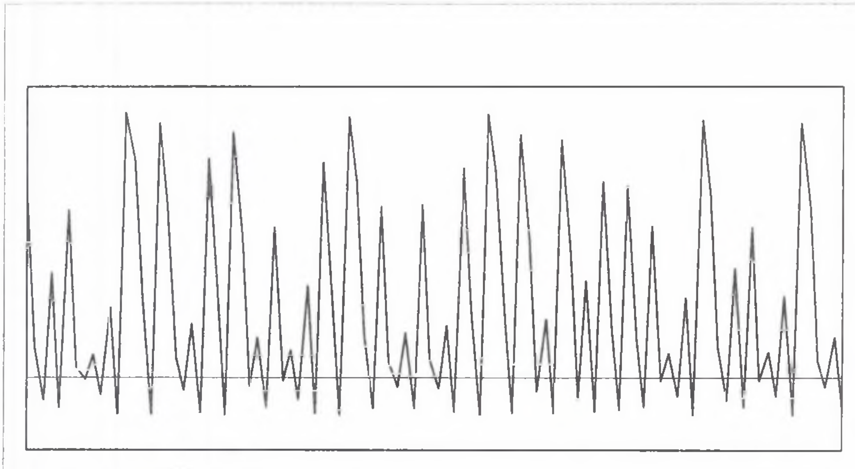

Fig. 8. Plot of x against time

Rys. 8. Wykres zmiennej x w funkcji czasu

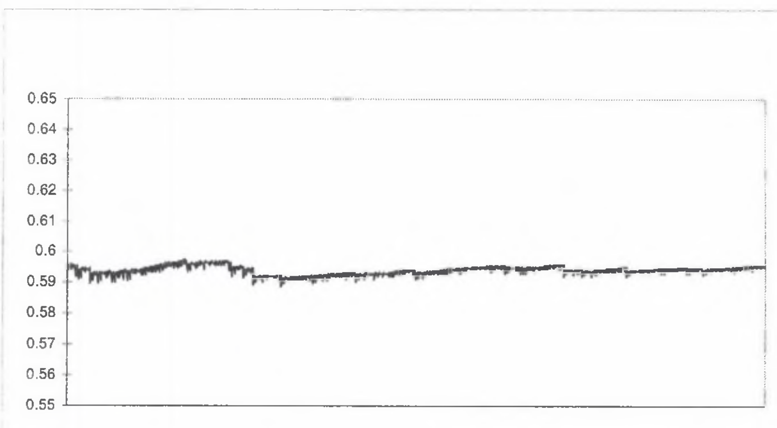


Fig. 9. Plot of Lyapunov exponent against time

Rys.9. Wykładnik Lapunowa w funkcji czasu

In conclusion, we have demonstrated that deterministic chaos is a real possibility in this simple, discrete-time, model of inflation. Based on numerical experiments, there is a fairly wide range of combinations of plausible parameter values that can give rise to chaotic behaviour. This gives rise to some interesting possibilities. For example, suppose that the parameter values are as follows:

$$\alpha = 0.025,\ \beta = 1.0,\ \gamma = 0.3,\ g = 0.5, y = 0.02 \text{ and } v = 0.25.$$

Using these values in the spreadsheet we find, that for $x_0 \in S$, all trajectories converge to the stable fixed point $x_t = 0.18223$; (note that this corresponds to a steady state inflation rate of 60.74%). Suppose the economy is in this steady state and the Government decides to cut the deficit by reducing $g$ to 0.4. This results (asymptotically) in the inflation rate following a regular cyclical pattern (a four-point cycle). Alternatively, suppose the Government makes an even bigger cut in $g$; to 0.35, say. This results in the inflation rate exhibiting chaotic behaviour. Other experiments are, of course, possible.

## References

1. Chappell, D. (1997), "Chaotic Behaviour in a Simple Model of Inflation", *The Manchester School of Economic and Social Studies*, Vol. LXV, No. 3, pp. 235 - 243.
2. Devaney, R.L. (1989), *An Introduction to Chaotic Dynamical Systems*, Second Edition, Addison-Wesley, Menlo Park, California.
3. Gulick, D. (1992), *Encounters with Chaos*, McGraw-Hill Inc., New York.

Recenzent: Dr hab. inż. Anna Walaszek-Babiszewska

Wpłynęło do Redakcji 18.11.1997 r.

## Streszczenie

W artykule jest dyskutowane zachowanie się wybranych modeli systemów dynamicznych z czasem dyskretnym, przedstawionych w postaci równań różnicowych.

Jako pierwsze jest rozpatrywane liniowe równanie różnicowe (1) o stałych, rzeczywistych współczynnikach oraz z rzeczywistym warunkiem początkowym:

$$X_{t+1} = aX_t + b$$
$$a, b, X_0 \in R$$

gdzie t= 0,1,2,...

Podstawiając do wzoru ogólnego $X_t = X_0$, otrzymujemy wyrażenie na $X_1$. Rozpisując kolejne iteracje i stosując wzór na sumę częściową szeregu geometrycznego, otrzymuje się ogólną postać rozwiązania w postaci wzoru (2). Zachowanie się ciągu rozwiązań zależy w sposób decydujący od wartości $|a|$ i może mieć charakter monotoniczny lub oscylacyjny o charakterze tłumionym lub nieograniczonym. Warunkiem stabilności systemu dynamicznego (zbieżności szeregu) jest $|a|<1$.

Uogólnieniem powyższego jednowymiarowego modelu liniowego z czasem dyskretnym jest model n-wymiarowy

$\mathbf{X}_{t+1} = \mathbf{A}\,\mathbf{X}_t + \mathbf{b}$

gdzie:

$\mathbf{X}_t$, $\mathbf{X}_{t+1}$, $\mathbf{b}$ - macierze kolumnowe n-wymiarowe,

$\mathbf{A}$ - macierz kwadratowa nxn.

Równanie to posiada rozwiązanie stabilne, jeśli wszystkie wartości własne macierzy $\mathbf{A}$ leżą wewnątrz okręgu jednostkowego.

Do szeroko dyskutowanych w literaturze jednowymiarowych nieliniowych modeli dynamicznych należy tzw. równanie logistyczne

$$X_{t+1} = kX_t(1 - X_t)$$
$$k \in [1,4],\ X_0 \in (0,1)$$

Równanie to, przy różnych wartościach parametru $k$, może opisywać szeroki zakres zachowań systemu dynamicznego, od monotonicznej zbieżności do punktu aż do trajektorii bardzo złożonych, zaliczanych do tzw. chaosu deterministycznego. Z wyjątkiem przypadku, gdy parametr $k$ jest równy 4, nie można podać rozwiązania tego równania. Pozostaje możliwość poszukiwania rozwiązań metodami numerycznymi. Przykłady trajektorii przedstawiają rysunki. Porównano zachowanie się ciągów iteracji startujących z niezbyt odległych warunków początkowych. Jeśli są to przebiegi o charakterze chaotycznym, to po kilku iteracjach trajektorie rozbiegaja się. Ta cecha wyklucza stosowanie ciągów chaotycznych do predykcji.

Dla formuły ogólnej równania nieliniowego

$$X_{t+1} = F(X_t)$$

przyjęto oszacowanie odległości dwóch trajektorii po $n$ krokach, startujących z dwóch „bliskich" warunków początkowych $X_0$ oraz $X_0+h$, za pomocą pewnej funkcji wykładniczej (6):

$$\left|F^{[n]}(X_0+h)-F^{[n]}(X_0)\right|\approx\left(e^{\lambda_n(X_0)}\right)^n h$$

gdzie $\lambda_0$ jest tzw. wykładnikiem Lapunowa.

Warunkiem koniecznym i wystarczającym chaotycznych zachowań ciągu iteracji jest dodatnia wartość wykładnika Lapunowa. Wzory (7) i (9) podają sposób obliczania współczynnika Lapunowa, a przykłady i kolejne rysunki stanowią ich ilustrację.

W ostatnim rozdziale jest rozważany nieliniowy model inflacji (15), którego postać uzasadniają podstawowe prawa makroekonomii

$$x_{t+1}=x_t+\ln\left(\frac{b}{1+y}\right)-\ln\left((b+x_t)(1-ae^{x_t})\right)$$

Parametry a, b, y oraz punkt początkowy decydują o zachowaniu trajektorii. Szczególnie interesujące jest uzależnienie charakteru zmienności modelu od ograniczeń wprowadzanych przez rząd, wynikających np. z deficytu budżetowego.